Home assignment 3

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The strict deadline to submit your solution is November, 7 20:00 MSK. Template to compose a solution is available at
`https://www.overleaf.com/read/vknkchxdwsmk`
The submission form is `https://forms.gle/LcEQ1s72Xa5AQNar5`

1. Soft-thresholding (2 pts)

Solve the following optimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{x}\|_1 + \frac{1}{2\alpha}\|\mathbf{x}-\mathbf{y}\|_2^2,$$

where $\alpha > 0$ is a given scalar and $\mathbf{y}$ is a known vector.

The main steps for the solving of this problem:

1. Analyze the convexity
2. Analyze smoothness
3. Compute the gradient or subdifferential
4. Find the point in which gradient is zero or zero belongs to subdifferential
5. Explain why this point is a solution or provide additional analysis

Bonus: plot the solution in the case of $n = 1$ and explain why the problem is called "soft thresholding". The plot should be inserted as a TikZ[1] picture.

2. Projection onto the unit ball in euclidean norm (1 pts)

Solve the following optimization problem based on the KKT conditions

$$\begin{aligned} &\min \frac{1}{2}\|\mathbf{x}-\mathbf{y}\|_2 \\ \text{s.t. } &\|\mathbf{x}\|_2 \le 1. \end{aligned}$$

Do not forget to explain why every step in your solution is correct from the theory claims.

[1] `https://overleaf.com/learn/latex/TikZ_package`

3. First-order optimality condition as a system of non-linear equations (3 pts)

Solve the following optimization problem

$$\min_{(x_1,x_2)\in\mathbb{R}^2} x_1^2 + 2x_2^2 - x_1x_2 + e^{x_1+x_2}$$

1. (0.5 pts) Analyze convexity
2. (0.5 pts) Compute the gradient and derive the FOOC
3. (1.3 pts) To get stationary point you should use some method for numerical solution of non-linear equation or system of equations, e.g. Newton method[2] or bisection method[3]. Implement one of such method and plot the convergence plot to demonstrate that it works and gives as a result the stationary point. Attach the Jupyter Notebook with implementation and plot.
4. (0.2 pts) Make a conclusion on the solution of the stated problem.

4. How to use KKT conditions if the constraint is non-smooth? (1.5 pts)

Solve the following optimization problem using KKT conditions

$$\begin{aligned} &\min_{\mathbf{x}\in\mathbb{R}^2}(x_1-2)^2 + (x_2+1)^2 \\ \text{s.t. } &\max(x_1+x_2+1, x_1-x_2-1)+1 \leq 0 \end{aligned}$$

Do not forget to explain why every step in your solution is correct from the theory claims.

[2] https://tutorial.math.lamar.edu/classes/calci/newtonsmethod.aspx
[3] https://en.wikipedia.org/wiki/Bisection_method