Optimization methods
Lecture 4: Introduction to duality theory

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Brief reminder of the previous lecture

- Optimality conditions for unconstrained optimization problems

Brief reminder of the previous lecture

- Optimality conditions for unconstrained optimization problems
- Saddle points

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- Optimality conditions for unconstrained optimization problems
- Saddle points
- Optimality conditions for constrained optimization problem in the general form

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- Karush-Kuhn-Tucker conditions

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- Optimality conditions for constrained optimization problem in the general form
- Karush-Kuhn-Tucker conditions
- Slater regularity

Optimization problem with functional constraints

$$\begin{aligned}
&\min_{\mathbf{x}\in\mathcal{D}} f_0(\mathbf{x}) \\
\text{s.t. } & g_i(\mathbf{x}) = 0,\ i = 1,\dots,m \\
& h_j(\mathbf{x}) \le 0,\ j = 1,\dots,p
\end{aligned}$$

dom $f_0 = \mathcal{D} \subseteq \mathbb{R}^n$, $f_0(\mathbf{x}^*) = p^*$

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Lagrangian $L : \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$

$$
L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f_0(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x}) + \sum_{j=1}^{p} \mu_j h_j(\mathbf{x})
$$

- λ_i – Lagrange multipliers for constraints $g_i(\mathbf{x}) = 0, \; i = 1, \dots, m$
- μ_j – Lagrange multipliers for constraints $h_j(\mathbf{x}) \leq 0, \; j = 1, \dots, p$

Dual function

Definition

A function $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ such that

$$g(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \inf_{\mathbf{x} \in \mathcal{D}} \left(f_0(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x}) + \sum_{j=1}^{p} \mu_j h_j(\mathbf{x}) \right)$$

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Properties

- Concave
- Can be equal to $-\infty$ for some pairs $(\boldsymbol{\lambda}, \boldsymbol{\mu})$

Lower bound for the optimal objective function value

Claim

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Proof

- If $\hat{\mathbf{x}} \in \mathcal{D}$ and belongs to the feasible set and $\boldsymbol{\mu} \geq 0$, then

$$f_0(\hat{\mathbf{x}}) \geq L(\hat{\mathbf{x}}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \geq \inf_{\mathbf{x} \in \mathcal{D}} L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = g(\boldsymbol{\lambda}, \boldsymbol{\mu})$$

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Q: what can we do now to get the best approximation of the optimal p^*?

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& \max g(\boldsymbol{\lambda}, \boldsymbol{\mu}) \\
\text{s.t. } & \boldsymbol{\mu} \geq 0
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- It is the best lower bound of p^* that can be given by dual function
- Vectors $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ are called dual feasible if $\boldsymbol{\mu} \geq 0$ and $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \text{dom } g$

Weak and strong duality

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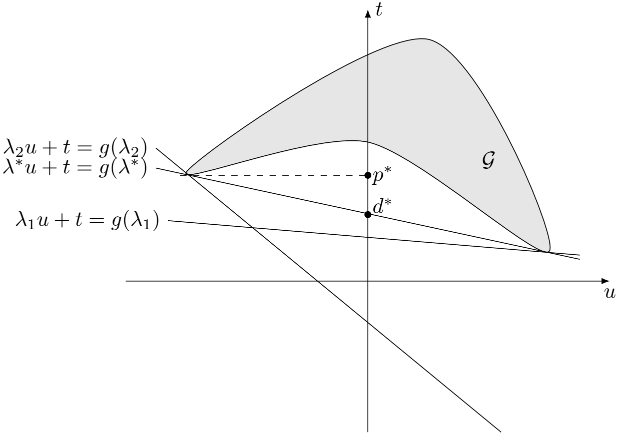
Duality gap: $f_0(\mathbf{x}) - g(\boldsymbol{\lambda}, \boldsymbol{\mu})$

- How current approximations $(\mathbf{x}_k, \boldsymbol{\lambda}_k, \boldsymbol{\mu}_k)$ are accurate
- Proof correctness and convergence of numerical methods

Geometric interpretation

$$\min_{\mathbf{x}\in\mathcal{D}} f_0(\mathbf{x}) \qquad g(\lambda) = \inf_{(u,t)\in\mathcal{G}} (t + \lambda u)$$

$$\text{s.t. } f_1(\mathbf{x}) \le 0 \qquad \mathcal{G} = \{(f_1(\mathbf{x}), f_0(\mathbf{x})) \mid \mathbf{x} \in \mathcal{D}\}$$



Slater condition and strong duality

Slater condition: reminder

The Slater condition holds if $\exists \bar{\mathbf{x}} \in \text{int } \mathcal{D}: \; f_i(\bar{\mathbf{x}}) < 0, \; \mathbf{A}\bar{\mathbf{x}} = \mathbf{b}$

Theorem

Strong duality holds for the convex optimization problem

$$\begin{gathered} \min f_0(\mathbf{x}) \\ \textit{s.t. } f_i(\mathbf{x}) \leq 0, \; i = 1, \ldots, m \\ \mathbf{A}\mathbf{x} = \mathbf{b}, \; \mathbf{A} \in \mathbb{R}^{p \times n} \end{gathered}$$

if the Slater condition holds.

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&\leq f_0(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i^* g_i(\mathbf{x}^*) + \sum_{j=1}^{p} \mu_j^* h_j(\mathbf{x}^*) \\
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- $\mathbf{x}^*$ is a minimizer of $L(\mathbf{x}, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$
- Complementary slackness conditions
  $\mu_j^* h_j(\mathbf{x}^*) = 0,\ j = 1, \ldots, p$

$$
\mu_j^* > 0 \Rightarrow h_j(\mathbf{x}^*) = 0, \quad h_j(\mathbf{x}^*) < 0 \Rightarrow \mu_j^* = 0
$$

Karush-Kuhn-Tucker conditions: convex and non-convex problems

If $(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ are solutions of primal and dual problems and the strong duality holds, then

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5. $f_0'(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i^* g_i'(\mathbf{x}^*) + \sum_{j=1}^{p} \mu_j^* h_j'(\mathbf{x}^*) = 0$

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The last equality holds since

$$\min_{\mathbf{x}} L(\mathbf{x}, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) = L(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$$

and first-order necessary condition.

KKT conditions for convex problems

Claim 1

Assume that the primal problem is convex (f_0, h_j are convex, g_i are affine) and for $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\mu}})$ the KKT conditions hold, then

- the strong duality holds
- $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{\mu}})$ are primal and dual solutions

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- Thus, the strong duality holds
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Claim 2

Assume that the Slater condition holds for convex optimization problem. Then $\mathbf{x}$ is a primal solution iff there exist $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ such that the KKT conditions hold for this triplet.

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- Neccesity of the KKT follows from the general fact.
- Sufficiency follows from the claim 1

Summary on duality theory

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- KKT for convex optimization problem with Slater regularity