Optimization methods
Lecture 5: Convex modeling

Alexandr Katrutsa

Modern State of Artificial Intelligence Masters Program
Moscow Institute of Physics and Technology

Brief reminder of the previous lecture

- Dual function and dual problem

Brief reminder of the previous lecture

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- Weak and strong duality

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- Dual function and dual problem
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- Slater regularity condition

Plan for today

- General approaches to deal with convexity

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- Standard forms of convex optimization problems

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- Standard forms of convex optimization problems
- Convex calculus
- Disciplined convex programming

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\text{s.t. } & f_i(\mathbf{x}) \leq 0,\ i = 1, \ldots, m \\
& h_j(\mathbf{x}) = 0,\ j = 1, \ldots, p
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- The possibility of efficient solving depends on the properties f_0, f_i, h_j

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- The possibility of efficient solving depends on the properties f_0, f_i, h_j
- If f_0, f_i, h_j are affine, then we have a linear programming problem (LP), that can be solved extremely fast
- Simple problems with non-linear inequality constraints f_i, h_j may be very difficult to solve

Convex optimization problem

$$\begin{aligned} &\min_{\mathbf{x}\in\mathbb{R}^n} f_0(\mathbf{x}) \\ \text{s.t. } &f_i(\mathbf{x}) \leq 0,\ i = 1, \ldots, m \\ &\mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$

- f_0, f_i are convex functions: for all $\mathbf{x}, \mathbf{y}$ and $\alpha \in [0, 1]$

$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leq \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y})$$

- Equality constraints are affine

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- Many applications

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  - Given a set of simple convex functions which is known to be convex
  - Given a set of composition rules and a proper transformations that are preserved the convexity of result

Examples of simple convex functions

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- $\log \det \mathbf{X}^{-1}$ for $\mathbf{X} \in \mathbb{S}^n_+$

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- And many others...

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- Maximum eigenvalue of $\mathbf{A} \in \mathbb{S}^n$:

$$\lambda_{\max}(\mathbf{A}) = \sup_{\|\mathbf{x}\|_2 = 1} (\mathbf{x}^\top \mathbf{A}\mathbf{x})$$

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  - Transformation may be not simple

Three main classes of problems

- Linear programming

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- Second-order cone programming

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- Semidefinite programming problem

$$\begin{aligned} & \min_{\mathbf{X}} \operatorname{trace}(\mathbf{C}\mathbf{X}) \\ \text{s.t. } & \operatorname{trace}(\mathbf{A}_i \mathbf{X}) = b_i \\ & \mathbf{X} \succeq 0 \end{aligned}$$

Interior point method (IPM) for convex problems

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- Extremely efficient method: some dozens of iterations are sufficient to convergence independently on the problem dimension
- Every iteration requires solving some linear system

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- Even if the transformed problem has more variables, it can be efficiently solved by IPM

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- Original problem: n variables, no constraints

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- **Important point**: problems are equivalent! If you solve one of them, you can derive solution of the other one

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From convexity proof to applicability of IPM

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- Rules of construction f_i give the proof of convexity
- The same parsing leads to a standard form of problem appropriate for IPM

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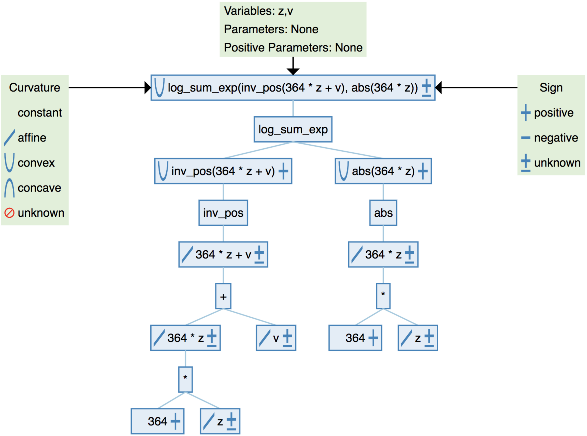
- The problem is convex by construction
- It is automatically parsed by elements
- It is reduced to the form suitable for IPM
- It is solved by some standard IPM solver
- The solution of the original problem is reconstructed

Example of parsing the convex function expression and convexity verification



More examples you can find in `http://dcp.stanford.edu/`

Take home message on DCP

Pro:

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- Convexity verification and generation of the problem transformation for IPM

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- Problem generation: elementary convex functions + composition rules and transformations

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Contra:

Take home message on DCP

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- Convexity verification and generation of the problem transformation for IPM
- Problem generation: elementary convex functions + composition rules and transformations
- It is very similar to standard math notation

Contra:

- It is not about «plug & play» or «try my code»

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Contra:

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- You can not write arbitrary problem and hope that it will be convex

Solvers for the general optimization problems

- ipopt
- Pyomo
- Gurobi

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- Composition rules to construct convex functions

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- Reduce problems to some standard forms

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- Other solvers for solving optimization problems