

Seminar 3. Optimality

Conditions

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Seminar 3

$A \in \mathbb{R}^{n \times n}$ - positive-definite

if $\forall z \neq 0 \quad z^T A z \geq 0$

$f''(z) \geq 0 \quad \forall z$

$$① \quad f(x_1, x_2) = e^{x_1} x_1 - (1 + e^{x_1}) \cos x_2 \Big|_{\min}$$

1) convex / non-convex ? no!

$$2) \quad f'(x_1, x_2) = \begin{pmatrix} e^{x_1} (1 + x_1 - \cos x_2) \\ \sin x_2 (1 + e^{x_1}) \end{pmatrix} = 0$$

$$\sin x_2 = 0 \Rightarrow \cos x_2 = \lambda \quad (\lambda = 1)$$

$$x_2 = \pi n, n \in \mathbb{Z}$$

$$x_1 = \cos x_2 - 1 = \lambda - 1$$

$$3) f''(x) = \frac{e^{x_1}(2 + \sin x_2 - \cos x_2 + x_1)}{(e^{x_1} \sin x_2)^2 \cos x_2 (1 + e^{x_1})}$$

a)

$$\begin{cases} x_2 = 2\pi n \\ x_1 = \lambda - 1 \end{cases}$$

$$\begin{cases} x_2 = 2\pi n, n \in \mathbb{Z} \\ x_1 = 0 \quad f''(\cdot) > 0 \end{cases}$$

$$f''(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

0

local minimum

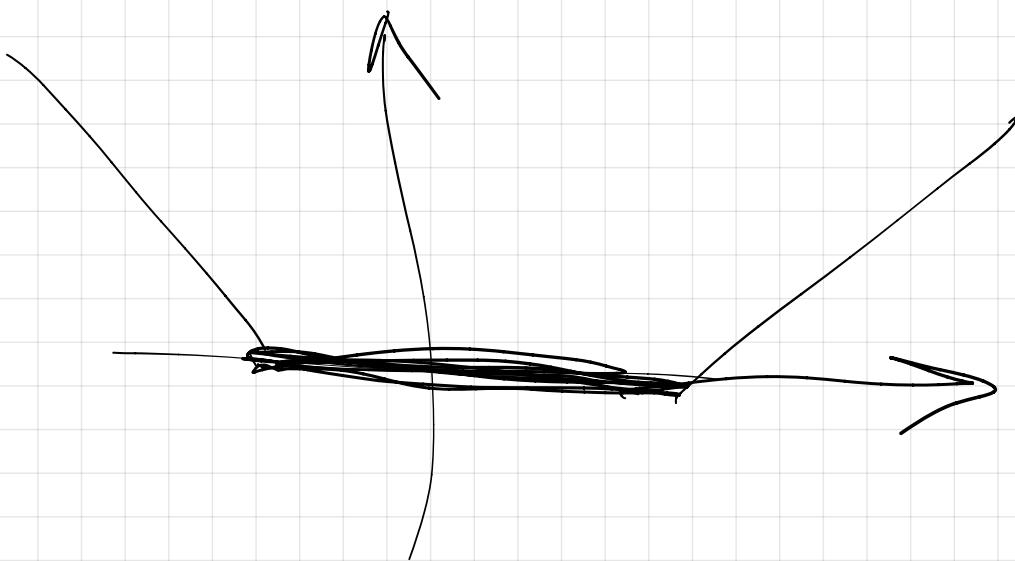
$$e^{x_1}(2 + x_1 + \sin x_2 - \cos x_2) =$$

$$= e^{\lambda-1}(2 + \lambda - 1 + 0 - \underbrace{\cos \cancel{2\pi n}}_{= 1})$$

$$\begin{aligned} \text{2)} \quad e^{\lambda-1} &> 0 \\ \lambda(1 + e^{x_1}) &\sim \text{circle} \end{aligned}$$

b) $\begin{cases} x_2 = \pi + 2\pi n = \underbrace{(2n+1)\pi}_{n \in \mathbb{Z}} \\ x_1 = -2 \end{cases}$ - saddle-points

$$f'' = \begin{pmatrix} > 0 & 0 \\ 0 & < 0 \end{pmatrix} \text{ - indefinite}$$

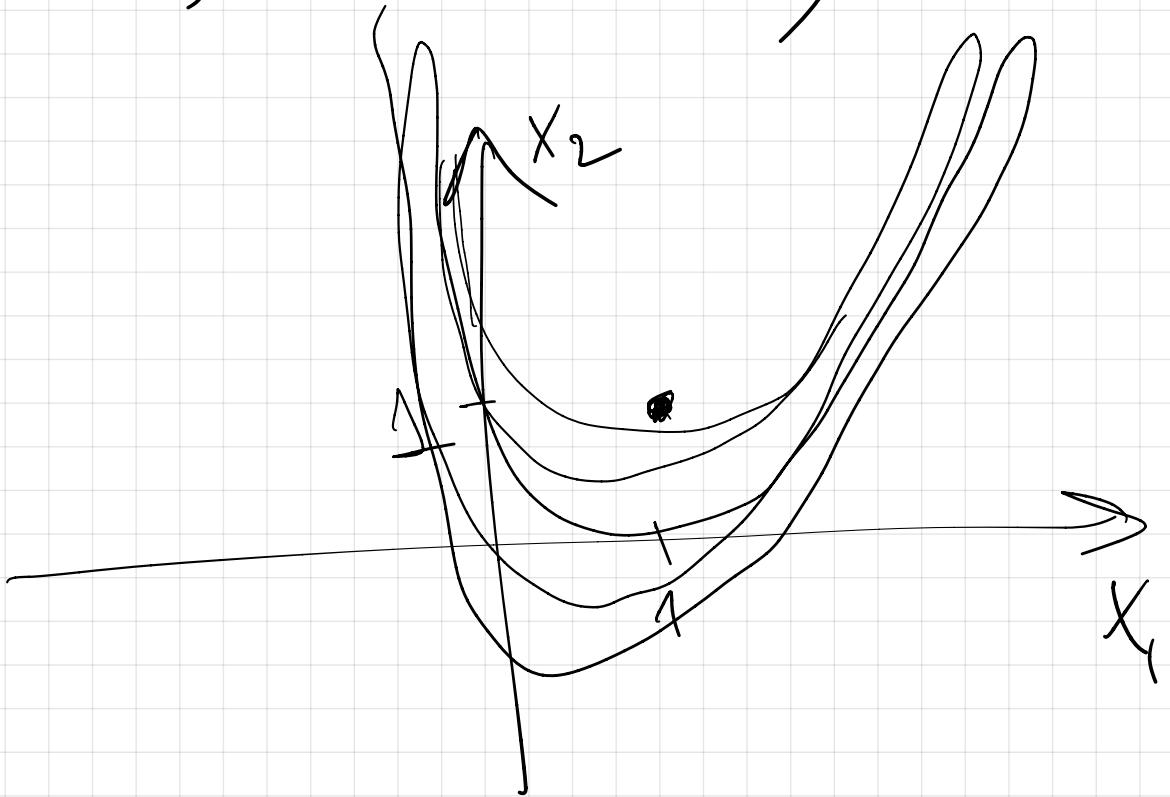


$$f''(x) \geq 0 \quad \forall x \in \text{dom}(f)$$

$$f(x) \cdot \cos(x)$$

$$x \in \mathbb{R}$$

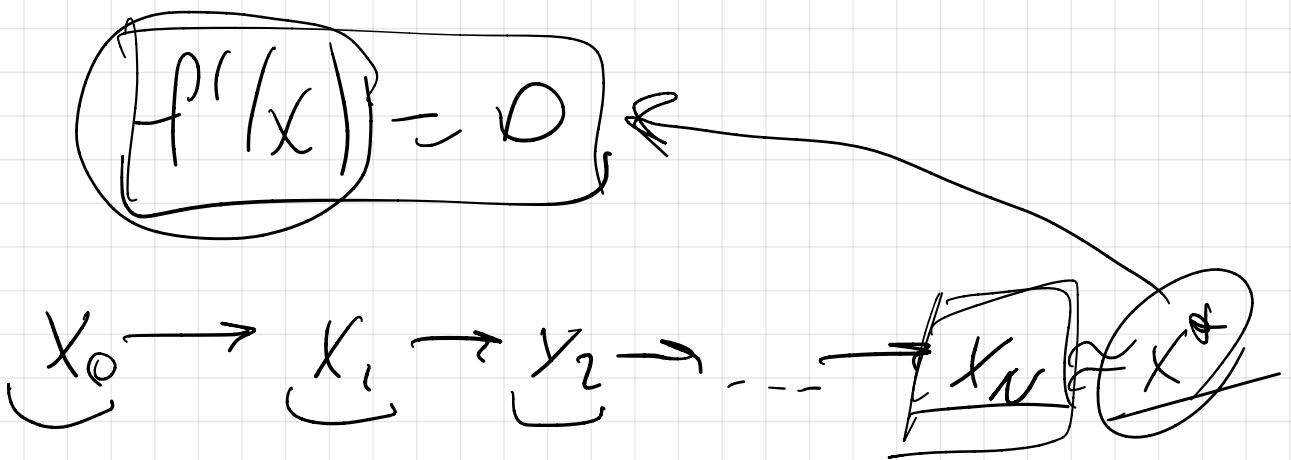
$$(1-x_1)^2 + \alpha(x_2 - x_1^2)^2 = C$$



$$\begin{aligned}
 & \left(-2(1-x_1) + 2\alpha(x_2 - x_1^2)(-2x_1) \right) \\
 & 2\alpha(x_2 - x_1^2) \\
 & \underbrace{2 - 4\alpha(x_2 - x_1^2) + x_1(x_2 - x_1^2)(-2x_1)}_{-4\alpha x_1}
 \end{aligned}$$

$-4\alpha x_1$

2α



$$f'(x_0) - f'(x_1) \dots - f'(x_N) \rightarrow 0$$



iteration

① $\min_{\mathbf{x}} \|\mathbf{Ax} - \mathbf{b}\|_2^2$ $\mathbf{A} \in \mathbb{R}^{m \times n}$

1) $f'(\mathbf{x}) = 2\mathbf{A}^T(\mathbf{Ax} - \mathbf{b})$ $\mathbf{b} \in \mathbb{R}^m$

$$\mathbf{x}^* = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b}$$

$$2A^T A x = 2A^T b$$

$$x^* = (A^T A)^{-1} A^T b \quad \text{stationary point}$$

$$f^g(x) = 2A^T A x$$

$$z^T A^T A z \geq 0$$

$$f'(x) = 2A^T (Ax - b)$$

$$\|Az\|_2^2 \geq 0$$

$$= 2(A^T A x) - 2A^T b$$

$$(Az)^T A z =$$

$$\frac{\partial^2 f}{\partial x_k \partial x_j} = \frac{\partial}{\partial x_k}$$

$$\left(\frac{\partial f}{\partial x_j} \right) = \underline{z^T A^T A z}$$

$$= \frac{\partial}{\partial x_n} \left(2 \sum_{p,i} a_p a_{pi} x_i \right) = \|Az\|_2^2 = 0$$

$$= \sum_p a_p a_{pj} \Rightarrow 2A^T A$$

$$(AB)^T = B^T A^T$$

$$X = (A^T A)^{-1} A^T b$$

$$(A^T A) X = A^T b$$

$$(A^T A + \epsilon I) X = A^T b.$$

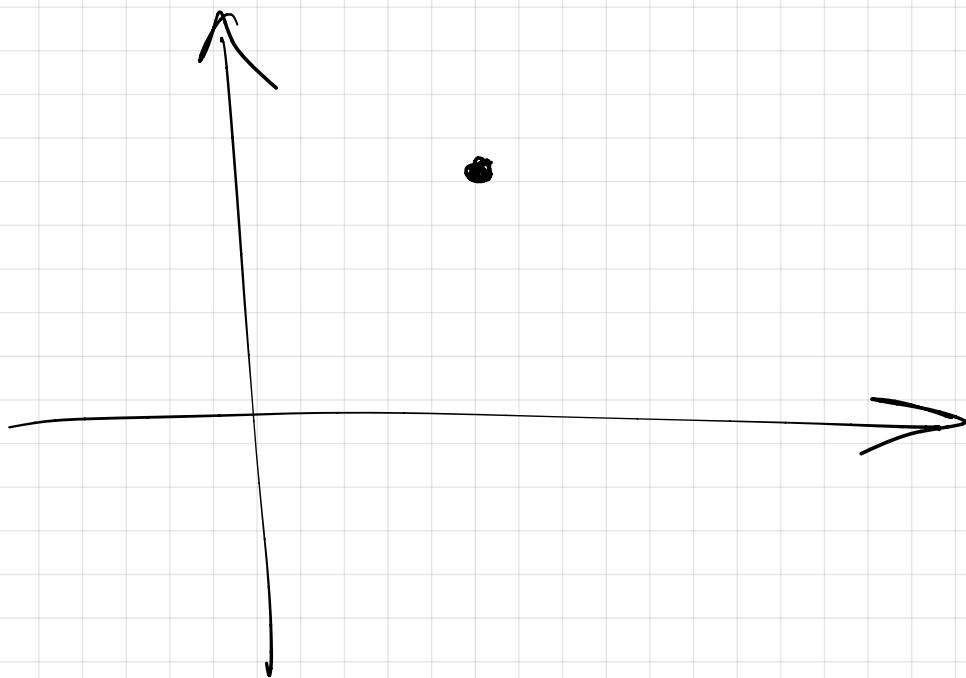
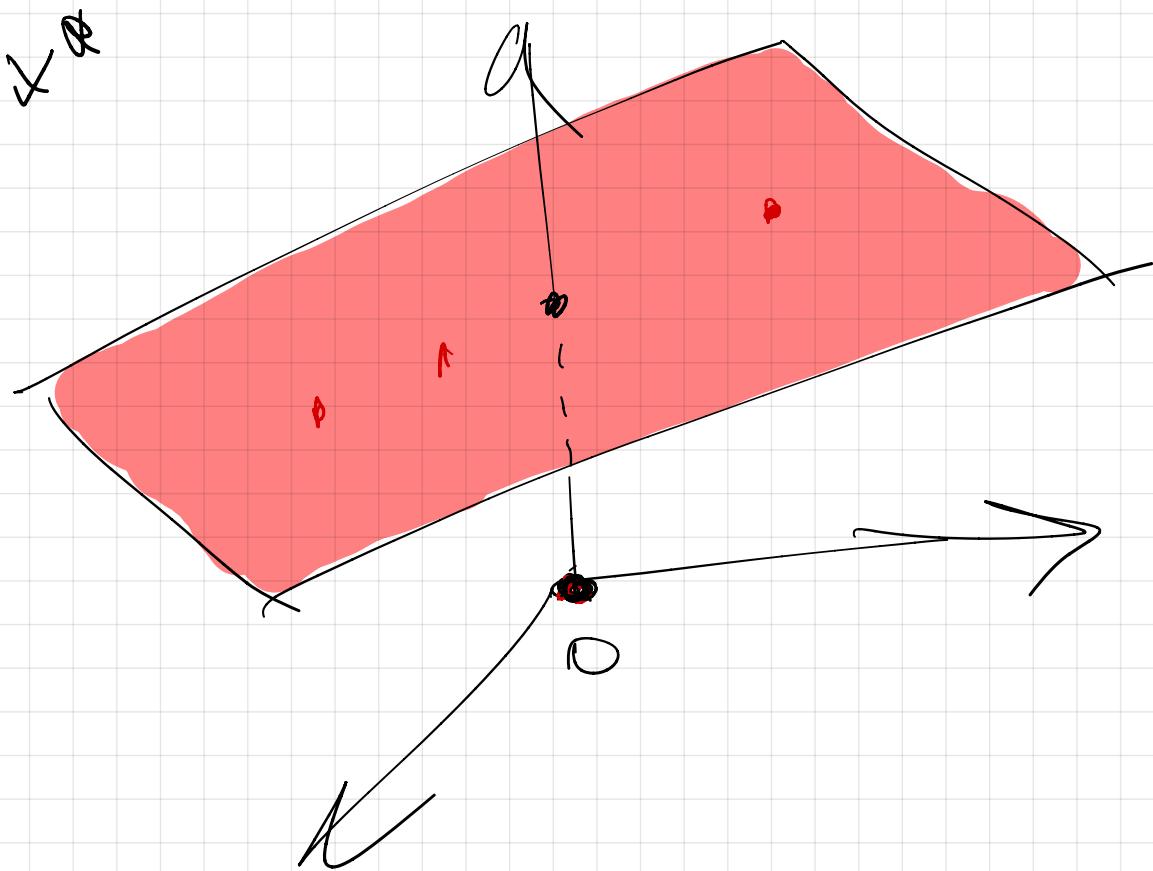
$$\min_x \|Ax - b\|_2^2 + \frac{\epsilon}{2} \|x\|_2^2$$

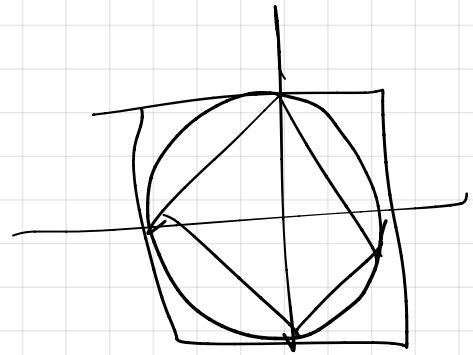
$$A^+ = \lim_{\epsilon \rightarrow 0} (A^T A + \epsilon I)^{-1} A^T b.$$

$$\min \frac{1}{2} \|x\|_2^2$$

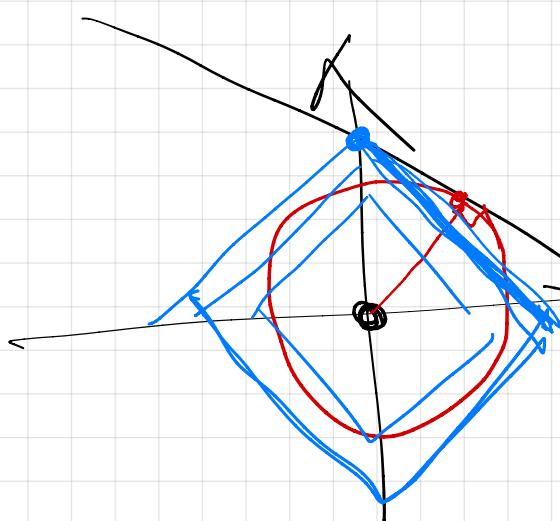
$$\frac{1}{2} \|x\|_1$$

$\text{S.t. } Ax = b, A \in \mathbb{R}^{n \times m}, n < m$





$$\begin{aligned} & \min f(x) \\ \text{s.t. } & Ax = b \end{aligned}$$



$$\begin{cases} x^3 + x^2 + x - 1 = 0 \\ x^2 - 4 = 0 \end{cases}$$

$$x = \{2; -2\}$$

$$L = \frac{1}{2} \|x\|^2 + d^T (Ax - b)$$

1) $L'_x = 0$

2) $\boxed{Ax = b}$

$\boxed{x^* = x + \cancel{\lambda} A^T A^{-1} b} \geq 0$

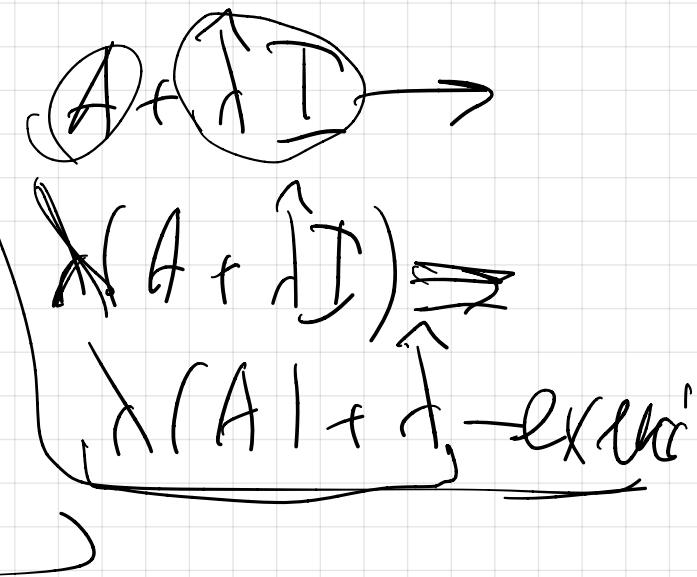
$\boxed{x^* = -A^T d}$

$$f(x) = \underbrace{c^T x}_{\text{linear part}} = \sum c_i x_i \quad f'(x) = c$$

$$-AAT^T \lambda = b$$

$$\lambda = -(AA^T)^{-1} b$$

$$x^* = A^T(AA^T)^{-1} b$$



(2)

$$\min_{A \in S^n} x^T Ax$$

s.t. $\|x\|_2 = 1$

$$\begin{cases} L'_x = 0 \\ \|x\|_2 = 1 \end{cases}$$

stationary

$$L''_{xx} \geq 0$$

$$L_{yx} = A + JI \geq 0$$

$$x^T Ax \geq \lambda_{\min}(A)$$

$$L = x^T Ax + \lambda (\|x\|_2^2 - 1) = -\lambda \|x^2\|_2^2 - \lambda$$

$$L' = 2Ax + 2\lambda x = 0$$

$$Ax^* = -\lambda x^*$$

$$+ \|x^*\|_2^2 = 1$$

$$\min \sum x_i \log x_i$$

$$f^Q(x) \sim \text{diag}(\cdot)$$

s.t. $x_i \geq 0 \rightarrow -x_i \leq 0$

$$\sum x_i = 1$$

$$x = \begin{pmatrix} 1/n \\ \vdots \\ 1/n \end{pmatrix}$$

$$L = \sum x_i \log x_i + \lambda \left(\sum x_i - 1 \right) - \mu_i$$

$$-\sum \mu_i x_i$$

$$\begin{cases} \mu_i \geq 0 \\ \mu_i x_i = 0 \end{cases}$$

$$L_k = 1 + \log x_k + \lambda - \mu_k = 0$$

$$\log \sum_{i=1}^{n-t-1} x_i = e^{\mu_n - t - 1} \geq 0$$

$$\downarrow$$

$$x_n = e^{-t-1} = \sum_{i=1}^n$$

$$e^{-t-1} = \frac{1}{n} = x_n$$

$$\sum_{i=1}^n e^{-t-1} = 1 \rightarrow e^{-t-1} (\sum_{i=1}^n 1) - 1 = 1$$