Optimization methods.
Seminar 3. Optimality conditions

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Reminder

- Subgradient and subdifferential
- Matrix calculus
- Autodiff

Unconstrained minimization problem

Problem: $f(x) \to \min\limits_{x \in \mathbb{R}^n}$.

Optimality criterion for convex functions

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then x^* is a solution of the unconstrained minimization problem iff $0 \in \partial f(x^*)$.

Corollary

If $f(x)$ is convex and differentiable, then x^* is a solution of the problem iff $\nabla f(x^*) = 0$.

Sufficient condition for non-convex functions

Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable and x^* such that $\nabla f(x^*) = 0$. If $\nabla^2 f(x^*) \succ 0$, then x^* is a strict local minimizer.

Examples

- $x_1 e^{x_1} - (1 + e^{x_1}) \cos x_2 \to \min$
- Rosenbrock function:
  $(1 - x_1)^2 + \alpha \sum_{i=2}^{n} (x_i - x_{i-1}^2)^2 \to \min, \ \alpha > 0$
- $x_1^2 + x_2^2 - x_1 x_2 + e^{x_1 + x_2} \to \min$
- $\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$
- $\min_{\mathbf{x}} \|\mathbf{x}\|_1 + \lambda \|\mathbf{x} - \mathbf{y}\|_2^2$

Equality constrained minimization problem

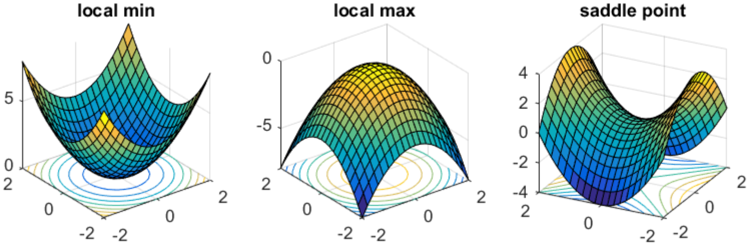
Minimization problem

$$
\begin{aligned}
&f(x) \to \min_{x \in \mathbb{R}^n} \\
&\text{s.t. } g_i(x) = 0, \; i = 1, \dots, m
\end{aligned}
$$

Lagrangian

$$
L(x, \boldsymbol{\lambda}) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)
$$

Possible options



Plot is from
`http://www.offconvex.org/2016/03/22/saddlepoints/`

Examples

- Quadratic problem with linear constraints

$$\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{x}\|_2^2$$
$$\text{s.t. } \mathbf{A}\mathbf{x} = \mathbf{b}$$

- Minimum eigenvalue

$$\min_{\mathbf{x}} \mathbf{x}^\top \mathbf{A}\mathbf{x}$$
$$\text{s.t. } \|\mathbf{x}\|_2 = 1$$

Equality and inequality constrained minimization problem

Minimization problem

$$\begin{aligned} &\min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}) \\ \text{s.t. } & g_i(\mathbf{x}) = 0, \ i = 1, \dots, m \\ & h_j(\mathbf{x}) \leq 0, \ j = 1, \dots, p \end{aligned}$$

Lagrangian

$$L(x, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x}) + \sum_{j=1}^{p} \mu_j h_j(\mathbf{x})$$

Optimality conditions

Necessary condition (Karush - Kuhn - Tucker)

- $g_i(\mathbf{x}^*) = 0,\ i = 1, \ldots, m$
- $h_j(\mathbf{x}^*) \leq 0,\ j = 1, \ldots, p$
- $\mu_j^* \geq 0,\ j = 1, \ldots, p$
- $\mu_j^* h_j(\mathbf{x}^*) = 0,\ j = 1, \ldots, p$
- $L'_{\mathbf{x}}(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) = 0$

If the minimization problem is convex, then the necessary optimality condition is also sufficient.

Examples

- Toy linear programming problem

$$\begin{aligned} &\min\ x_1 - 2x_2 + x_3 \\ \text{s.t.}\ & -x_1 + x_2 + x_3 \le -2 \\ & x_1 + 2x_2 + x_3 \le 10 \\ & x_1 + x_2 - x_3 = 4 \\ & x_i \ge 0 \end{aligned}$$

- Entropy maximization

$$\begin{aligned} &\min_{\mathbf{x}}\ \sum_{i=1}^{n} x_i \log x_i \\ \text{s.t.}\ & x_i \ge 0 \\ & \sum_{i=1}^{n} x_i = 1 \end{aligned}$$

- Quadratic problem with quadratic constraints

$$\begin{aligned} \min_{\mathbf{x}} \; & \frac{1}{2}\mathbf{x}^\top \mathbf{Q}\mathbf{x} + \mathbf{p}^\top \mathbf{x} \\ \text{s.t.} \; & \frac{1}{2}\mathbf{x}^\top \mathbf{A}_i \mathbf{x} + \mathbf{c}_i^\top \mathbf{x} \leq 0 \end{aligned}$$

Recap

- Optimality conditions for
  - general minimization problem
  - unconstrained minimization problem
  - equality constrained minimization problem
  - equality and inequality constrained minimization problem