

Optimization methods.

Seminar 1. Convex sets and functions

Aleksandr Katrutsa

Moscow Institute of Physics and Technology

October 12, 2020

Convex set

Convex set

A set C is called convex if

$$\forall x_1, x_2 \in C, \theta \in [0, 1] \rightarrow \theta x_1 + (1 - \theta)x_2 \in C.$$

\emptyset and $\{x_0\}$ are also convex.

Examples: \mathbb{R}^n , ray, segment.

Convex combination of points

Assume $x_1, \dots, x_k \in G$, then a point $\theta_1 x_1 + \dots + \theta_k x_k$ such that

$\sum_{i=1}^k \theta_i = 1, \theta_i \geq 0$ is called convex combination of points x_1, \dots, x_k .

Convex hull

A set $\left\{ \sum_{i=1}^k \theta_i x_i \mid x_i \in G, \sum_{i=1}^k \theta_i = 1, \theta_i \geq 0 \right\}$ is called convex hull of G and is denoted as $\text{conv}(G)$.

Operations that preserve convexity

- ▶ Intersection of any number of convex sets is a convex set
- ▶ Image convex set under any affine map is convex set
- ▶ Linear combination of convex sets is a convex set
- ▶ Cartesian product of convex sets is a convex set

Examples

Check the following sets on the convexity and affinity:

1. Half-space: $\{\mathbf{x} | \mathbf{a}^\top \mathbf{x} \leq c\}$
2. Polytope: $\{\mathbf{x} | \mathbf{A}\mathbf{x} \leq \mathbf{b}, \mathbf{C}\mathbf{x} = 0\}$
3. Norm ball in \mathbb{R}^n : $B(r, \mathbf{x}_c) = \{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_c\| \leq r\}$
4. Ellipsoid: $\mathcal{E}(\mathbf{x}_c, \mathbf{P}, r) = \{\mathbf{x} | (\mathbf{x} - \mathbf{x}_c)^\top \mathbf{P}^{-1}(\mathbf{x} - \mathbf{x}_c) \leq r^2\}$
5. Set of PSD matrix $\mathbf{S}_+^n = \{\mathbf{X} \in \mathbb{R}^{n \times n} | \mathbf{X}^\top = \mathbf{X}, \mathbf{X} \succeq 0\}$
6. $\{\mathbf{X} \in \mathbb{R}^{n \times n} | \text{trace}(\mathbf{X}) = \text{const}\}$
7. Hyperbolic set $\{\mathbf{x} \in \mathbb{R}_+^n | \prod_{i=1}^n x_i \geq 1\}$