Seminar 4. Introduction to duality

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Seminar 4

$$\min f(x)$$
s.t. $g_i(x) = 0$
$h_j(x) \le 0$

$f(x^*) = p^*$

primal problem

1) $L(x, \lambda, \mu) = f(x) + \sum_i \lambda_i g_i(x) + \sum_j \mu_j h_j(x)$

$\langle \lambda, g(x) \rangle$

$\langle \mu, h(x) \rangle$

2) $g(\lambda, \mu) = \inf_x L(x, \lambda, \mu)$

3) $$\max_{\lambda, \mu} g(\lambda, \mu)$$
s.t. $\mu \ge 0$

convex optim. problem

$p^* \ge g(\lambda, \mu)$

1) $$\boxed{\begin{aligned} &\min \frac{1}{2}\|x\|_2^2 \\ &\text{s.t. } \boxed{Ax=b} \end{aligned}}$$ $A \in \mathbb{R}^{m\times n}$, $m<n$

m equailies $\langle a_i, x\rangle = b_i$

$$L(x,\lambda) = \frac{1}{2}\|x\|_2^2 + \lambda^T(Ax-b)$$

$$g(\lambda) = \inf_x \boxed{\left(\underbrace{\frac{1}{2}\|x\|_2^2}\ + \underbrace{\lambda^T(Ax-b)}\right)}$$

1) check convexity - yes!

2) $L'(x^*,\lambda) = x^* + A^T\lambda = 0$

$$\boxed{x^* = -A^T\lambda}$$ ✓

$$g(\lambda) = \frac{1}{2}\underbrace{\|A^T\lambda\|_2^2} + \lambda^T(-AA^T\lambda - b) =$$

$$= \frac{1}{2}\underbrace{\lambda^T AA^T\lambda} - \underbrace{\lambda^T AA^T\lambda} - \lambda^T b =$$

$$= \underbrace{-\frac{1}{2}\|A^T\lambda\|_2^2 - \lambda^T b} - \text{concave}$$

$$\boxed{\max_{\lambda} \left(-\frac{1}{2}\|A^T\lambda\|_2^2 - \lambda^T b\right)}$$

$$\boxed{p^* = d^* - \text{dual optimal value}}$$

(2) $\min c^T x$ (n)

s.t. $Ax = b$, $A \in \mathbb{R}^{m\times n}$, $m < n$

$x \geq 0$ ~~$m > n$~~

$$L(x,\lambda,\mu) = c^T x + \lambda^T(Ax - b) - \mu^T x =$$

$$= (c + A^T\lambda - \mu)^T x - \lambda^T b$$

$$g(\lambda,\mu) = \inf_x L(x,\lambda,\mu) = \begin{cases} -\infty, & \text{otherwise} \\ -\lambda^T b, & \boxed{c + A^T\lambda - \mu = 0} \end{cases}$$

$\max -\lambda^T b$

s.t. $\mu \geq 0$

$c + A^T\lambda - \mu = 0$

$\min \lambda^T b$

s.t. $c + A^T\lambda \geq 0$

(m)

Q: how to recover x^* from λ^* and μ^* - ?

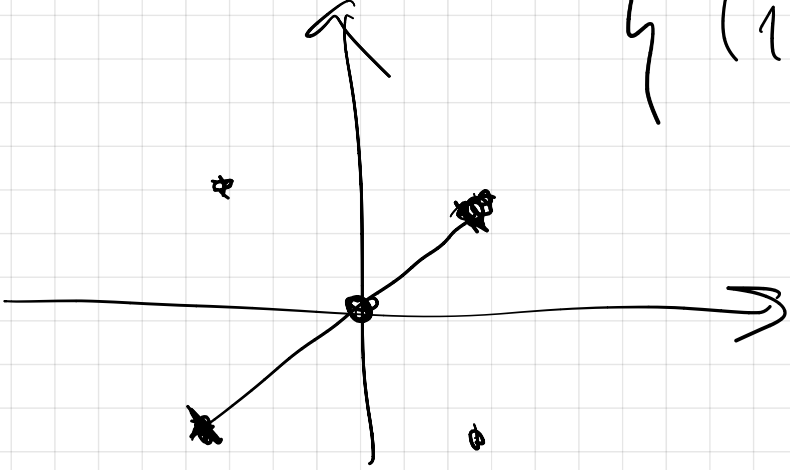
③ $\min\limits_x x^T W x$ $\quad \underbrace{W \in S^n}$ - symmetric matrix

s.t. $x_i^2 = 1, \ i = 1, \dots, n$

$x_i = \pm 1$ - nonconvex problem

2^n possible solutions

$\{(1;1)\ (1;-1)\ (-1;1)\ (-1;-1)\}$



$$L = \underbrace{x^T W x + \sum_i \lambda_i (x_i^2 - 1)}$$

$$g(\lambda) = \inf_x \left(\quad \right)$$

$$L(x,\lambda) = x^T W x + x^T \mathrm{diag}(\lambda) x - \mathbb{1}^T \lambda =$$

$$= \boxed{x^T (W + \mathrm{diag}(\lambda)) x - \mathbb{1}^T \lambda}$$

$$\sum x_i^2 \lambda_i = \underbrace{x^T \mathrm{diag}(\lambda) x}$$

$$g(\lambda) = \begin{cases} -\infty & \text{, otherwise} \\ \underline{-\mathbb{1}^T \lambda} & , \boxed{W + \mathrm{diag}(\lambda) \succeq 0} \end{cases}$$

$$\underbrace{W + \mathrm{diag}(\lambda)} \not\succeq 0$$

$\hat{v}$ - eigenvector for negative eigenvalue

$$\underbrace{\alpha \hat{v}^T (W + \mathrm{diag}(\lambda)) \alpha \hat{v}} \xrightarrow[\alpha \to +\infty]{} -\infty$$

$$\wedge$$

$$0$$

$$\left.\begin{array}{l} \max\limits_{\lambda} -\mathbb{1}^T \lambda \\ \text{s.t. } \underline{W + \mathrm{diag}(\lambda) \succeq 0} \end{array}\right] \text{SDP}$$

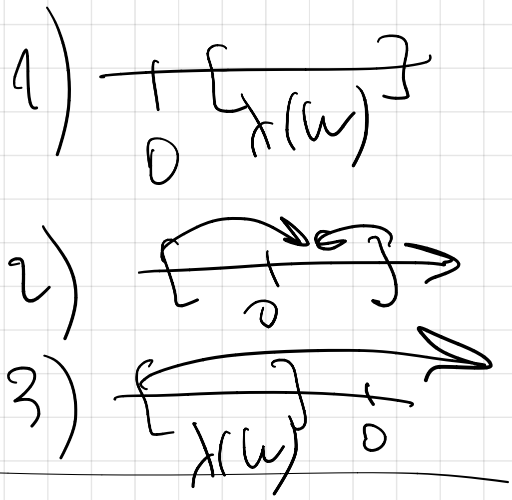
$$p^* \geq \underline{g(\lambda)} \quad , \boxed{W + \mathrm{diag}(\lambda) \succeq 0}$$

$-\lambda_{min}(W)$ – scalar $\qquad W \in S^n$

$$diag(\lambda) \equiv -\lambda_{min}(W)\cdot I \qquad \boxed{\lambda = -1\cdot\lambda_{min}(W)}$$

$$W - \lambda_{min}(W)\cdot I$$

$$\lambda(W - \lambda_{min}(W)\cdot I)$$



$A \in \mathbb{R}^{n\times n} \qquad A \in S^n$

$\lambda_1(A), \dots, \lambda_n(A)$ – eigenvalues of A

$v_1 \qquad\qquad v_n$ – eigenvectors of A

$\lambda(A - \mu\cdot I)$ – ?

$$(A-\mu I)v_i = Av_i - \mu v_i =$$

$$= \boxed{(\lambda_i(A) - \mu)v_i}$$

$$p^* \geq -b^T \lambda = n \lambda_{min}(W)$$

4)

$$\min_{X} \ \mathrm{trace}(CX)$$

s.t. $\mathrm{trace}(A_i X) = b_i$

$$X \succeq 0$$

$$L = \mathrm{trace}(CX) + \sum \lambda_i [\mathrm{trace}(A_i X) - b_i] - \langle \Lambda, X \rangle =$$

$\mu \geq 0$

$\Lambda \succeq 0$

$$= \mathrm{trace}((C + \sum \lambda_i A_i - \Lambda) X) - \sum \lambda_i b_i \rightarrow \inf_X$$

$$g(\lambda, \Lambda) = \begin{cases} -\infty & C + \sum \lambda_i A_i - \Lambda \neq 0 \\ -\sum \lambda_i b_i & \text{otherwise.} \end{cases}$$

$$\max \lambda^T b$$

$$\text{s.t. } C + \sum \lambda_i A_i = \Lambda$$

$$\Lambda \succeq 0$$

$$\Downarrow$$

$$\boxed{\begin{aligned} &\max_{\lambda} \lambda^T b \\ &\text{s.t. } C + \sum \lambda_i A_i \succeq 0 \end{aligned}} \text{ — LMI}$$

$$\operatorname{diag}(\lambda) = \sum \lambda_i A_i$$

$$A_i - ?$$

$$A_i = \begin{pmatrix} \cancel{0} & & 0 \\ & 1 & \\ 0 & & \cancel{0} \end{pmatrix}$$

with i written above the middle column of the matrix.