Seminar 3. Optimality Conditions

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Seminar 3

$A \in \mathbb{R}^{n\times n}$ - positive-definite

if $\forall z \neq 0 \quad \underbrace{z^T A z \geq 0}$

$\underbrace{f''(z) \geq 0}$ $\forall z$

① $f(x_1, x_2) = e^{x_1} x_1 - (1+e^{x_1}) \underbrace{\cos x_2}_{\to \min}$

1) convex / non-convex ? no!

2) $f'(x_1,x_2) = \begin{pmatrix} e^{x_1}(1+x_1-\cos x_2) \\ \sin x_2 (1+e^{x_1}) \end{pmatrix} = 0$

$\underbrace{\sin x_2} = 0 \Rightarrow \boxed{\cos x_2 = \lambda} \; \boxed{|\lambda| = 1}$

$\Downarrow$

$\left[\begin{array}{l} x_2 = \pi n, \; n \in \mathbb{Z} \\ x_1 = \cos x_2 - 1 = \lambda - 1 \end{array}\right.$

3) $f''(x) = \begin{pmatrix} \boxed{e^{x_1}(2+\sin x_2 - \cos x_2 + x_1)} & e^{x_1}\sin x_2 \\ e^{x_1}\sin x_2 & \cos x_2 (1+e^{x_1}) \end{pmatrix}$

a) $\boxed{\begin{matrix} x_2 = \pi n \\ x_1 = \lambda - 1 \end{matrix}}$ $\boxed{\begin{cases} x_2 = 2\pi n, & n \in \mathbb{Z} \\ x_1 = 0 & f''(\ldots) > 0 \end{cases}}$ $\rightarrow$ local minimum

$$f''(x) = \begin{pmatrix} & 0 \\ 0 & \end{pmatrix}$$

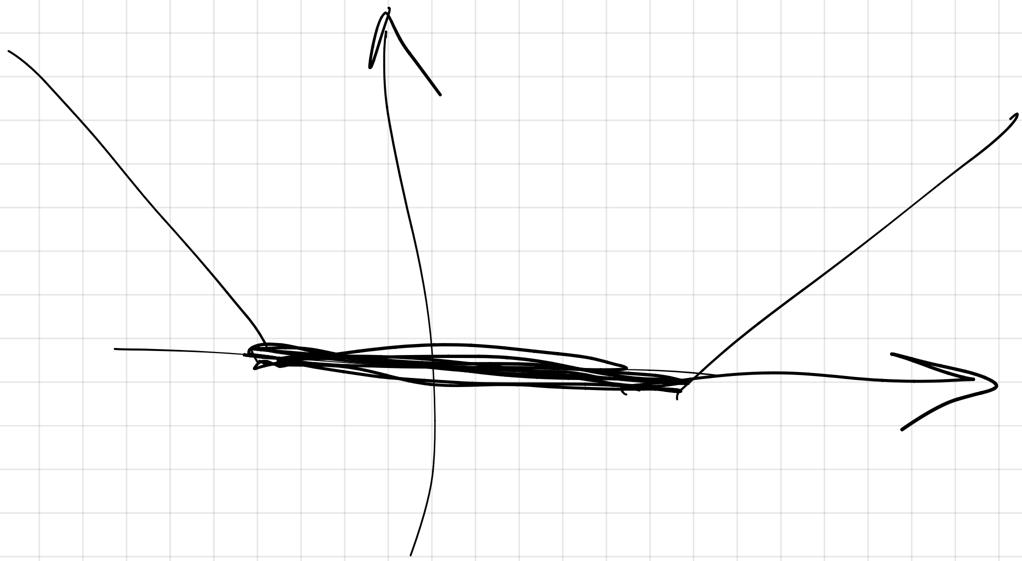
$$e^{x_1}(2 + x_1 + \sin x_2 - \cos x_2) =$$

$$= e^{\lambda - 1}(2 + \lambda - 1 + 0 - \boxed{\cos 2\pi n}) =$$

$$= \frac{e^{\lambda-1} > 0}{\lambda(1+e^{x_1})} \sim \lambda$$

b) $\begin{cases} x_2 = \pi + 2\pi n = \underbrace{(2n+1)}\pi & n \in \mathbb{Z} \\ x_1 = -2 & \text{— saddle-points} \end{cases}$

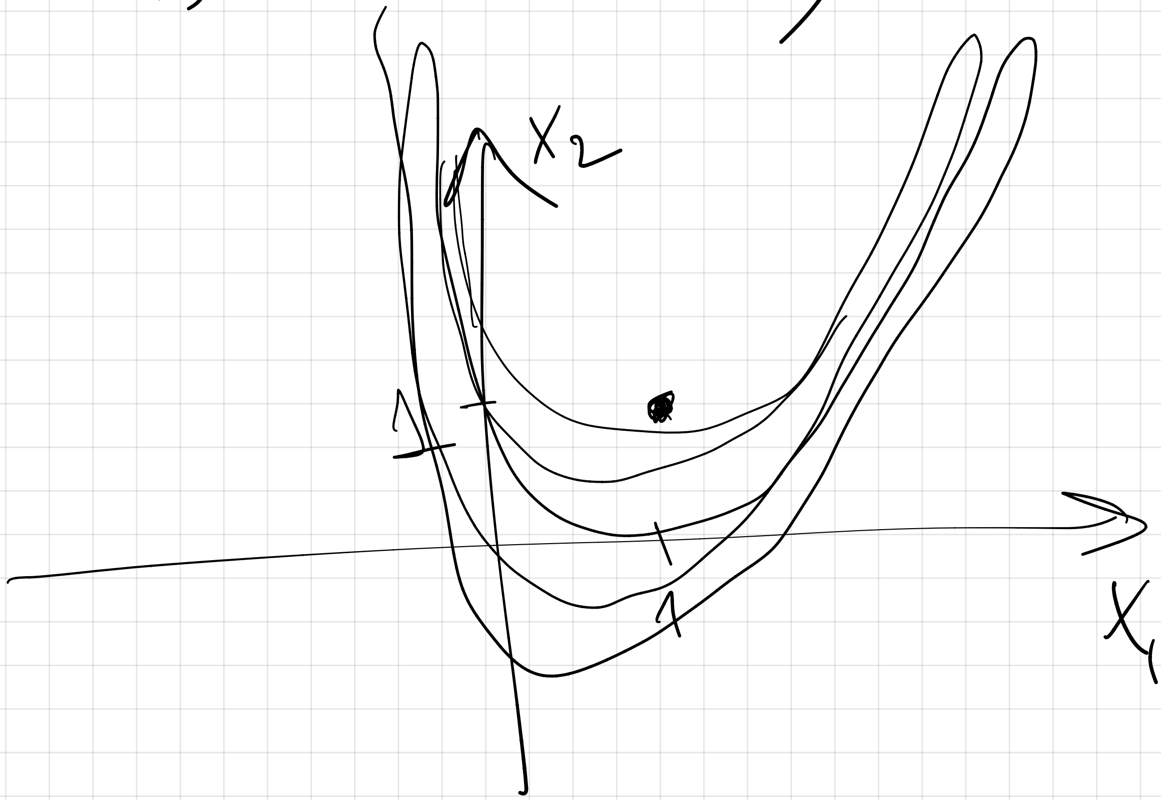
$$f'' = \begin{pmatrix} >0 & 0 \\ 0 & <0 \end{pmatrix} \text{ — indefinite}$$



$$\underbrace{f''(x) \succeq 0 \quad \forall x \in \mathrm{dom}(f)}$$

$$\boxed{f(x) \cdot \underline{\cos(x_2)}} \quad x \in \mathbb{R}$$

$$(1-x_1)^2 + \alpha(x_2 - x_1^2)^2 = c$$



$$\begin{pmatrix} -2(1-x_1) + 2\alpha(x_2 - x_1^2)(-2x_1) \\ 2\alpha(x_2 - x_1^2) \end{pmatrix}$$

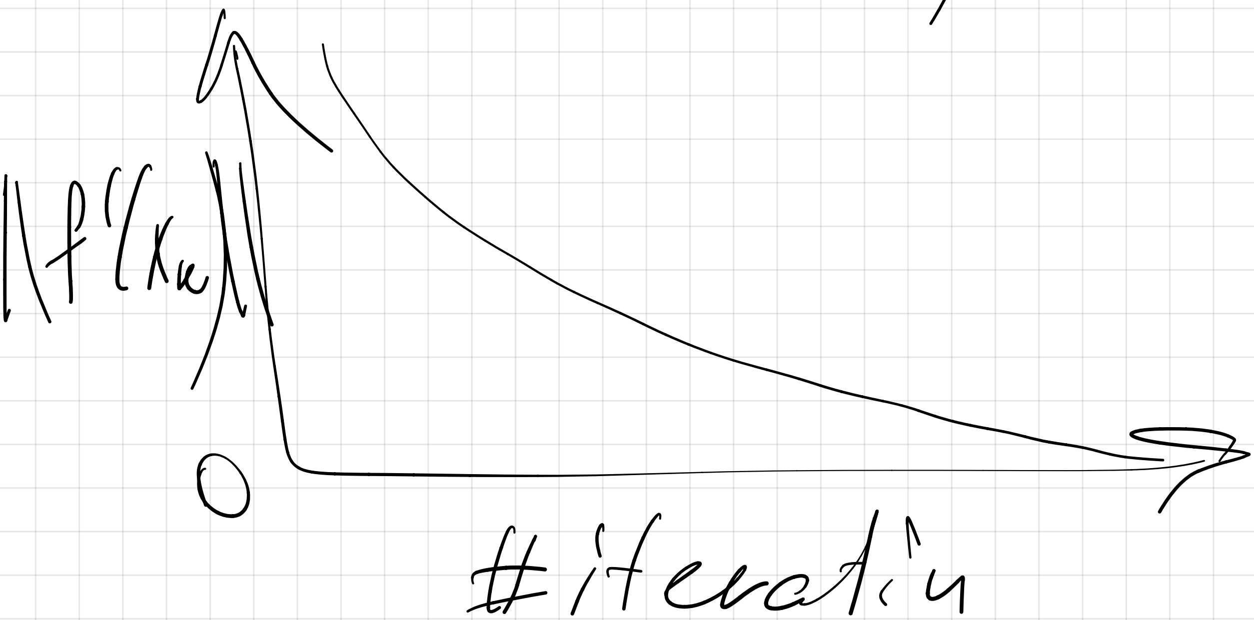
$$\underline{2\,4 - 4\alpha\left(x_2 - x_1^2 + x_1(x_2 - x_1^2)(-2x_1)\right)}$$

$$-4\alpha x_1 \qquad\qquad 2\alpha$$

$$\boxed{f'(x) = 0}$$

$$x_0 \to x_1 \to x_2 \to \dots \to \boxed{x_N} \approx x^*$$

$$f'(x_0) \quad f'(x_1) \dots f'(x_N) \to 0$$



(4) $\min_{x} \|Ax-b\|_2^2$ $\quad A \in \mathbb{R}^{m\times n}$

$m \geq n$

$b \in \mathbb{R}^m$

1) $f'(x) = 2A^T(Ax-b)$

$$\boxed{x^* = (A^TA)^{-1}A^Tb}$$

$$\cancel{2}A^TAx = \cancel{2}A^Tb$$

$$\boxed{x^* = (A^TA)^{-1}A^Tb}$$ — stationary point

$$f''(x) = 2A^TA \qquad \cancel{2}\,z^TA^TAz \geq 0$$

$$\|Az\|_2^2 \geq 0$$

$$f'(x) = 2A^T(Ax - b) =$$

$$(Az)^TAz =$$

$$= 2A^TAx - 2A^Tb$$

$$= z^TA^TAz$$

$$\frac{\partial^2 f}{\partial x_k \partial x_j} = \frac{\partial}{\partial x_k}\left(\frac{\partial f}{\partial x_j}\right) =$$

$$\exists z \neq 0 \quad \|Az\|_2^2 = 0$$

$$= \frac{\partial}{\partial x_k}\left(2\sum_{p,i} a_{pj}a_{pi}x_i\right) =$$

$$A z = 0$$

$$= 2\sum_p a_{pj}a_{pk} \Rightarrow 2A^TA$$

$$(AB)^T = B^TA^T$$

$$\boxed{x^* = (A^T A)^{-1} A^T b} \quad A^+ b$$

$$\boxed{(A^T A) x = A^T b}$$

$$\underbrace{(A^T A + \varepsilon I) x = A^T b.}$$

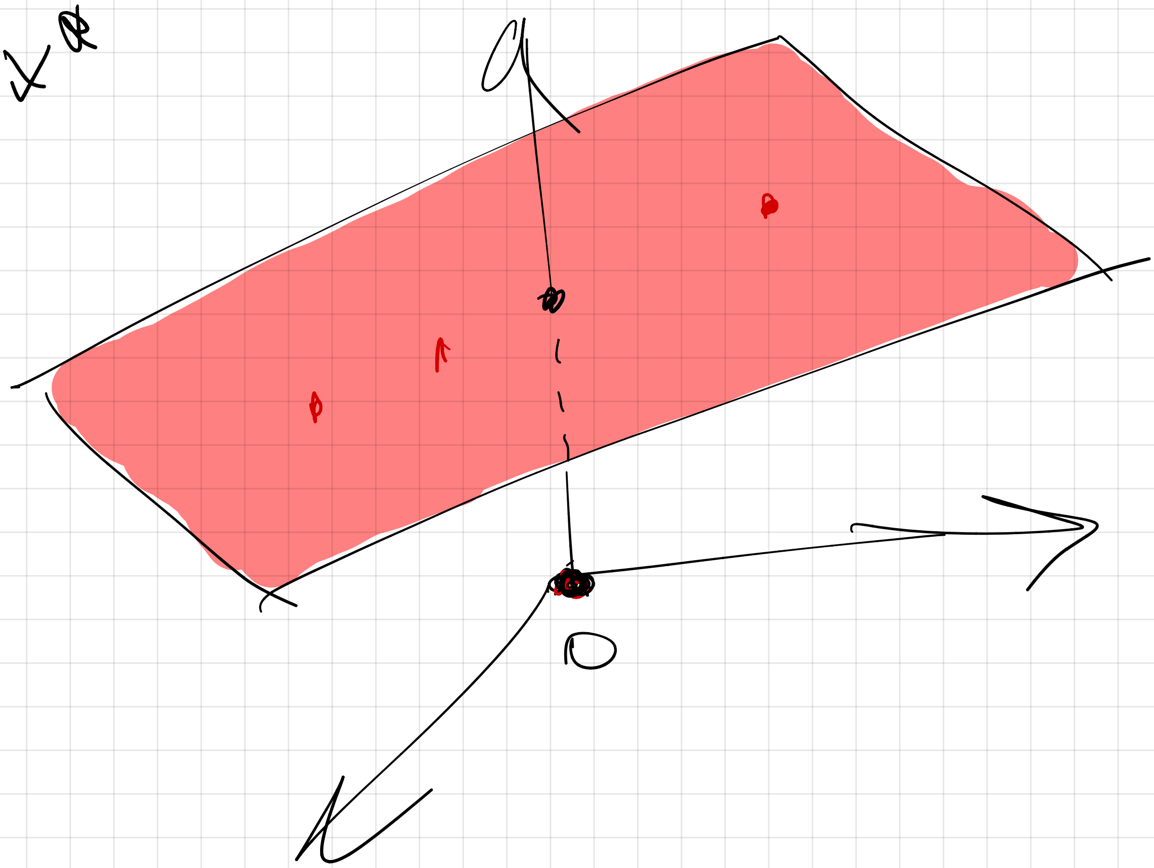
$$\min_x \boxed{\|Ax - b\|_2^2 + \frac{\varepsilon}{2} \|x\|_2^2}$$

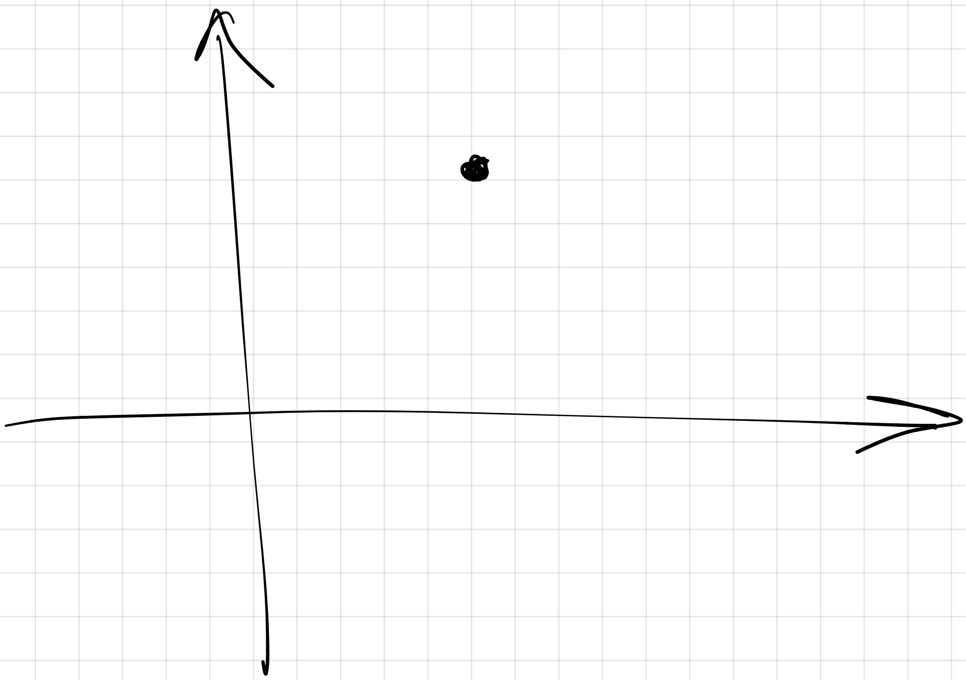
$$A^+ = \lim_{\varepsilon \to 0} \underbrace{(A^T A + \varepsilon I)^{-1} A^T b.}$$

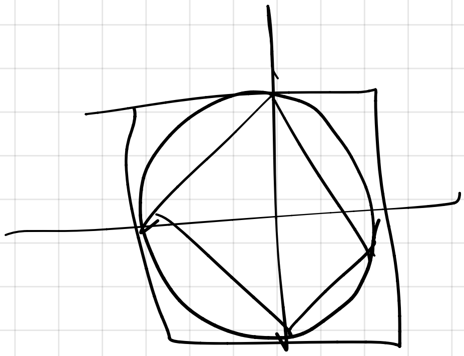
$$\min \frac{1}{2}\|x\|_2^2 \qquad \frac{1}{2}\|x\|_1$$

$$\text{s.t. } Ax = b, \quad A \in \mathbb{R}^{n \times m}, \quad n < m$$

x^*

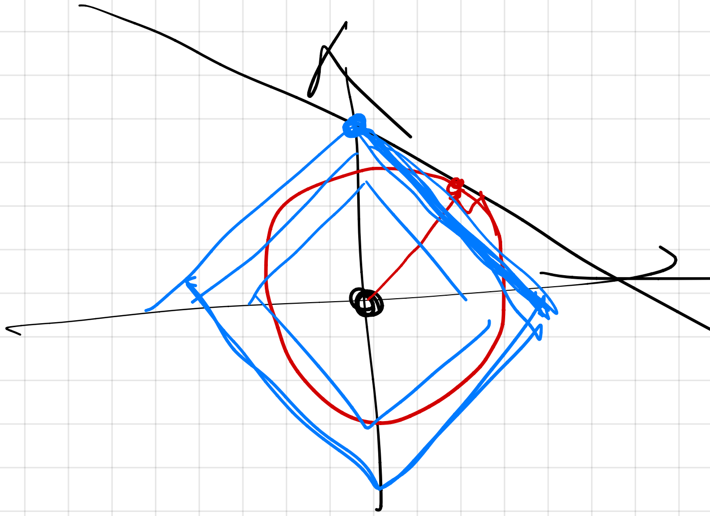






$\min f(x)$

s.t. $\boxed{Ax=b}$



$x^3+x^2+x-1=0$

$x^2-4=0 \Rightarrow x=\{2;-2\}$

$L=\frac{1}{2}\|x\|^2+\lambda^T(Ax-b)$

$(A^T\lambda)^T x - b^T\lambda$

1) $L'_x = 0$

2) $\boxed{Ax=b}$

$\Rightarrow$ station points.

$\boxed{x^*=-A^T\lambda}$

$L'_x = x + \cancel{\lambda^T A}\, A^T\lambda = 0$

$f(x)=c^Tx=\sum c_i x_i \quad f'(x)=c$

$$-AA^T\lambda = b$$

$$\lambda = -(AA^T)^{-1} b$$

$$x^* = A^T(AA^T)^{-1} b$$

$$A + \lambda I \to$$

$$\lambda(A + \hat{\lambda} I) =$$

$$\lambda(A) + \hat{\lambda}$ — exercise

(2)

$$\min \boxed{x^T A x}$$

$$\text{s.t. } \|x\|_2^2 = 1$$

$$A \in S^n$$

$$\begin{cases} L'_x = 0 \\ \|x\|_2 = 1 \end{cases}$$

stationary

$$\boxed{L''_{xx} \succ 0}$$

$$L''_{xx} = A + \lambda I \succ 0$$

$$\lambda(A)$$

$$-\lambda_{min}(A)$$

$$x^T A x_2$$

$$L = x^T A x + \lambda(\|x\|_2^2 - 1)$$

$$= -\lambda \|x^*\|_2^2 =$$

$$= -\lambda$$

$$L' = \boxed{2Ax} + 2\lambda x = 0$$

$$Ax^* = -\lambda x^* + \|x^*\|_2 = 1$$

$$\min \sum x_i \log x_i \qquad f''(x) = \mathrm{diag}(\,\vdots\,)$$

$$\text{s.t.} \quad x_i \geq 0 \rightarrow -x_i \leq 0$$

$$\sum x_i = 1 \qquad x^* = \begin{pmatrix} 1/n \\ \vdots \\ 1/n \end{pmatrix}$$

$$L = \sum x_i \log x_i + \lambda\left(\sum x_i - 1\right) - \sum \mu_i x_i$$

$$\mu_i \geq 0$$

$$\boxed{\mu_i x_i = 0}$$

$$L_k = 1 + \log x_k + \lambda - \mu_k = 0$$

$$\log_e x_k = e^{\mu_k - \lambda - 1} > 0$$

$$\Downarrow$$

$$\mu_k = 0 \qquad \boxed{e^{-\lambda - 1} = \frac{1}{n} = x_k}$$

$$x_k = e^{-\lambda - 1} \qquad \sum_{i=1}^{n} x_i = 1$$

$$\sum_{i=1}^{n} e^{-\lambda - 1} = 1 \Rightarrow e^{-\lambda - 1}\left(\sum 1\right) = 1$$