Seminar 6. Review of the first part of the course

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Seminar 6

$$\min f_0(x)$$
$$\text{s.t. } g_i(x) = 0 \longrightarrow \text{s.t. } Ax = b$$
$$h_j(x) \leq 0$$

$$\min f_0(x)$$
$$\text{s.t. } Ax = b$$
$$h_j(x) \leq 0$$

a) f_0 – convex

b) $h_j(x)$ – convex

1. Convex sets, cones, affine sets.

$$\underbrace{\alpha x_1 + (1-\alpha) x_2}_{} \in C$$
$$\forall x_1, x_2 \in C \quad \alpha \in [0, 1].$$

– operations that preserve convexity

– criteria for affinity.

2. Convex function

$$f(\alpha x_1 + (1-\alpha) x_2) \le \alpha f(x_1) + (1-\alpha) f(x_2), \quad \alpha \in [0;1]$$

$x_1, x_2 \in \text{dom } f$

- criteria of convexity
- operations that preserve convexity.

3. Subgradient, subdifferential + gradient and hessian.

$$\underbrace{f(y) \ge f(x) + \langle a, y-x \rangle}_{}, \quad \underline{\forall y \in \text{dom } f}$$

a - subgradient of f at x.

$$f(x) = \max_{i=1,\dots,k} f_i(x)$$

- Th. Moreau - Rockafellar

$f_i(x)$ - convex, $d_i \geq 0$.

$f(x) = \sum_{i=1}^{p} d_i f_i(x) \Rightarrow \partial f(x) =$

$= \sum_{i=1}^{p} d_i \partial f_i(x)$

- subdiff. of max.

$f(x) = \max_{i=1,\dots,p} f_i(x)$ $f_i(x)$ are convex

$$\partial f(x) = \operatorname{conv}\left(\bigcup_{i \in J} \partial f_i(x) \right)$$

$J = \{ i = 1, \dots, p \mid f_i(x) = f(x) \}$.

$g_i(x)$ - concave

$g(x) = \min g_i(x)$ - concave function

$g(x) = - \max_{i=1,\dots,m} (-g_i(x))$

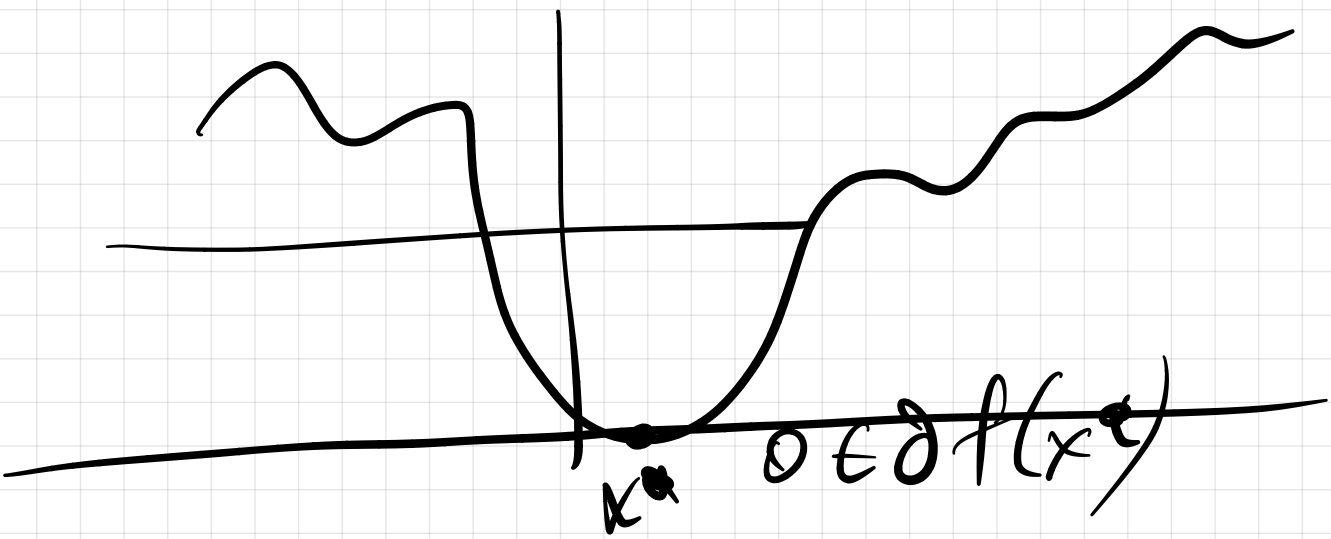
$g(x,y)$ - convex for both x and y

$f(x) = \inf_{y \in C} g(x,y)$ - convex.

3. Optimality conditions.

- FOOC: if x^* is a local min $\Rightarrow$ $f'(x^*) = 0$.

- general $0 \in \partial_x f(x^*) \Leftrightarrow x^*$ is global optimal point



- KKT

$$\boxed{\begin{aligned} &\min f_0(x) \\ \text{s.t. } &g_i(x) = 0 \\ &h_j(x) \le 0 \end{aligned}}$$

$$L = f_0(x) + \sum \lambda_i g_i(x) + \sum \mu_j \cdot h_j(x)$$

if x^* is a solution and strong dual. holds: for (λ^*, μ^*)

1) $L'_x(x^*, \lambda^*, \mu^*) = 0$

2) $\mu_j^* \ge 0$ — dual feasibility

3) $\mu_j^* h_j(x^*) = 0$ — complementary slackness

4) $g_i(x^*) = 0$

5) $h_j(x^*) \le 0$

$$\begin{cases} \mu_j > 0 \Rightarrow h_j(x^*) = 0 \\ h_j(x^*) < 0 \Rightarrow \mu_j = 0 \end{cases}$$

$$\boxed{p^* = f_0(x^*) \geq g(\lambda, \mu)} = \inf_x L(x, \mu)$$

$\downarrow$

$-\infty$

$\boxed{\mu \geq 0}$

- Slater regularity

a) convex problem

i) convex objective

ii) affine equality constraints

iii) convex inequality

b) $\exists\ \hat{x}$: $A\bar{x} = b$

$h_i(\bar{x}) < 0$

Slater regularity $\Rightarrow$ strong duality.

Convex problem + KKT $\Rightarrow$ x^*

x^*

convex dual problem

$$\max g(\lambda, \mu)$$
$$\text{s.t. } \mu \geq 0 \longrightarrow d^* = g(\lambda^*, \mu^*)$$

1) primal problem is not convex

$$p^* \geq d^*$$

2) convex problem :

a) Slater regularity holds

b) otherwise

strong duality $p^* = d^*$

DCP and CVXPy

Assignment 5 – 17. 11. 2020.

Resubmit from week 1–5 $\sim$ 4–5 days.

December, 27 – pre-exam

January, ... – exam