

Optimization methods.

Seminar 6. Optimality conditions, vol. 2

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Equality constrained minimization problem

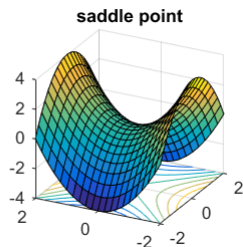
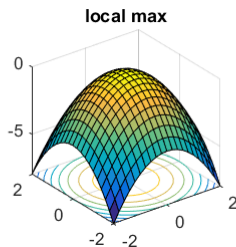
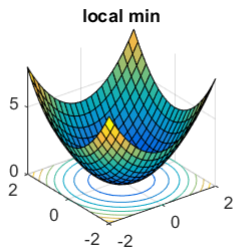
Minimization problem

$$\begin{aligned} f(x) &\rightarrow \min_{x \in \mathbb{R}^n} \\ \text{s.t. } g_i(x) &= 0, \quad i = 1, \dots, m \end{aligned}$$

Lagrangian

$$L(x, \lambda) = f(x) + \sum_{i=1}^m \lambda_i g_i(x)$$

Possible options



Plot is from

<http://www.offconvex.org/2016/03/22/saddlepoints/>

Examples

- ▶ Quadratic problem with linear constraints

$$\begin{aligned} \min_{\mathbf{x}} \quad & \frac{1}{2} \|\mathbf{x}\|_2^2 \\ \text{s.t.} \quad & \mathbf{Ax} = \mathbf{b} \end{aligned}$$

- ▶ Minimum eigenvalue

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{x}^\top \mathbf{Ax} \\ \text{s.t.} \quad & \|\mathbf{x}\|_2 = 1 \end{aligned}$$

Equality and inequality constrained minimization problem

Minimization problem

$$\begin{array}{ll}\min_{\mathbf{x} \in \mathbb{R}^n} & f(\mathbf{x}) \\ \text{s.t.} & g_i(\mathbf{x}) = 0, \quad i = 1, \dots, m \\ & h_j(\mathbf{x}) \leq 0, \quad j = 1, \dots, p\end{array}$$

Lagrangian

$$L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\mathbf{x}) + \sum_{i=1}^m \lambda_i g_i(\mathbf{x}) + \sum_{j=1}^p \mu_j h_j(\mathbf{x})$$

Optimality conditions

Necessary condition (Karush - Kuhn - Tucker)

- ▶ $g_i(\mathbf{x}^*) = 0, i = 1, \dots, m$
- ▶ $h_j(\mathbf{x}^*) \leq 0, j = 1, \dots, p$
- ▶ $\mu_j^* \geq 0, j = 1, \dots, p$
- ▶ $\mu_j^* h_j(\mathbf{x}^*) = 0, j = 1, \dots, p$
- ▶ $L'_x(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) = 0$

If the minimization problem is convex, then the necessary optimality condition is also sufficient.

Examples

- ▶ Toy linear programming problem

$$\begin{aligned} \min \quad & x_1 - 2x_2 + x_3 \\ \text{s.t.} \quad & -x_1 + x_2 + x_3 \leq -2 \\ & x_1 + 2x_2 + x_3 \leq 10 \\ & x_1 + x_2 - x_3 = 4 \\ & x_i \geq 0 \end{aligned}$$

- ▶ Entropy maximization

$$\begin{aligned} \min_{\mathbf{x}} \quad & \sum_{i=1}^n x_i \log x_i \\ \text{s.t.} \quad & x_i \geq 0 \\ & \sum_{i=1}^n x_i = 1 \end{aligned}$$

- Quadratic problem with quadratic constraints

$$\begin{aligned} \min_{\mathbf{x}} \quad & \frac{1}{2} \mathbf{x}^\top \mathbf{Q} \mathbf{x} + \mathbf{p}^\top \mathbf{x} \\ \text{s.t.} \quad & \frac{1}{2} \mathbf{x}^\top \mathbf{A}_i \mathbf{x} + \mathbf{c}_i^\top \mathbf{x} \leq 0 \end{aligned}$$

Recap

- ▶ Optimality conditions for
 - ▶ equality constrained minimization problem
 - ▶ equality and inequality constrained minimization problem