# Optimization methods.
# Seminar 1. Convex sets and cones

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# Convex set

## Convex set

A set $C$ is called convex if

$$\forall x_1,\ x_2 \in C, \theta \in [0,1] \to \theta x_1 + (1-\theta)x_2 \in C.$$

$\emptyset$ and $\{x_0\}$ are also convex.

Examples: $\mathbb{R}^n$, ray, segment.

## Convex combination of points

Assume $x_1, \ldots, x_k \in G$, then a point $\theta_1 x_1 + \ldots + \theta_k x_k$ such that $\sum\limits_{i=1}^{k} \theta_i = 1,\ \theta_i \geq 0$ is called convex combination of points $x_1, \ldots, x_k$.

## Convex hull

A set $\left\{ \sum\limits_{i=1}^{k} \theta_i x_i \mid x_i \in G, \sum\limits_{i=1}^{k} \theta_i = 1, \theta_i \geq 0 \right\}$ is called convex hull of $G$ and is denoted as $\mathrm{conv}(G)$.

# Operations that preserve convexity

- Intersection of any number of convex sets is a convex set
- Image convex set under any affine map is convex set
- Linear combination of convex sets is a convex set
- Cartesian product of convex sets is a convex set

# Examples

Check the following sets on the convexity:

1. Half-space: $\{\mathbf{x} \mid \mathbf{a}^\top \mathbf{x} \leq c\}$
2. Polytope: $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b},\ \mathbf{C}\mathbf{x} = 0\}$
3. Norm ball in $\mathbb{R}^n$: $B(r, \mathbf{x}_c) = \{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_c\| \leq r\}$
4. Ellipsoid: $\mathcal{E}(\mathbf{x}_c, \mathbf{P}, r) = \{\mathbf{x} \mid (\mathbf{x} - \mathbf{x}_c)^\top \mathbf{P}^{-1}(\mathbf{x} - \mathbf{x}_c) \leq r^2\}$
5. Set of PSD matrix $\mathbf{S}_+^n = \{\mathbf{X} \in \mathbb{R}^{n \times n} \mid \mathbf{X}^\top = \mathbf{X},\ \mathbf{X} \succeq 0\}$
6. $\{\mathbf{X} \in \mathbb{R}^{n \times n} \mid \mathrm{trace}(\mathbf{X}) = const\}$
7. Hyperbolic set $\{\mathbf{x} \in \mathbb{R}_+^n \mid \prod\limits_{i=1}^n x_i \geq 1\}$