# Recollements and Injective Generation of the Derived Category

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ICRA 2020/fd-seminar

# Injective Generation

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**Definition**

Let $A$ be a ring. If $\mathcal{D}(A)$ is generated, as a triangulated category with infinite coproducts, by the injective $A$-modules then we say that **injectives generate for** $A$.

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## Examples

- Rings with finite global dimension.
- Monomial algebras (Rickard 2019) (FDC: Green, Kirkman, Kuzmanovich 1991).
- Commutative noetherian rings (Rickard 2019, Neeman 1992).

# Recollements

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$$(R) := \begin{tikzcd} \mathcal{D}(B) \arrow[r, "i_*"] & \mathcal{D}(A) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{D}(C) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

Let $A$, $B$ and $C$ be rings. Let $(i^*, i_*, i^!)$ and $(j_!, j^*, j_*)$ be triples of adjoint functors satisfying the properties defined by Beĭlinson, Bernstein and Deligne (1982).

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Idea - $\mathcal{D}(A)$ is 'built from' $\mathcal{D}(B)$ and $\mathcal{D}(C)$.

# Recollements

## Example (Triangular Matrix Algebra)

Let $A$ be a quiver algebra with quiver $Q_A$.

$$Q_A \coloneqq \begin{tikzcd} \boxed{Q_C} \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \boxed{Q_B} \end{tikzcd}$$

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Let $B$ be the quiver algebra associated to $Q_B$ and $C$ be the quiver algebra associated to $Q_C$.

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Let $B$ be the quiver algebra associated to $Q_B$ and $C$ be the quiver algebra associated to $Q_C$.
Then there exists a recollement $(R)$.

$$(R) := \begin{tikzcd} \mathcal{D}(B) \arrow[r, "i_*"] & \mathcal{D}(A) \arrow[l, bend right=60, "i^*"'] \arrow[l, bend left=60, "i^!"] \arrow[r, "j^*"] & \mathcal{D}(C) \arrow[l, bend right=60, "j_!"'] \arrow[l, bend left=60, "j_*"] \end{tikzcd}$$

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Suppose that $i_*$ preserves **compact objects**.

1. If injectives generate for both $B$ and $C$ then injectives generate for $A$.
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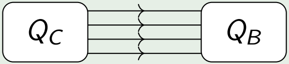
Note that $(R)$ restricts to a recollement of bounded above derived categories (Angeleri Hügel, Koenig, Liu, Yang 2017).

# Recollements

## Example (Triangular Matrix Algebra)

Let $A$ be the quiver algebra with quiver $Q_A$.

$Q_A :=$ 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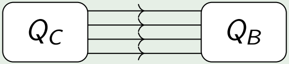
$Q_A :=$ 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1. If injectives generate for both $B$ and $C$ then injectives generate for $A$.
2. If injectives generate for $A$ then injectives generate for $B$.

# Vertex Removal

(Fuller, Saorín 1992), (Green, Psaroudakis, Solberg 2018).

$$Q_A := \begin{tikzcd} & 2 \arrow[dr, bend left] & \\ 1 \arrow[ur, bend left] \arrow[rr, bend right] & & 3 \arrow[ll, bend right] \arrow[loop right] \end{tikzcd}$$

Projective $A$-modules

$$\begin{tikzcd}[column sep=small] & 1 \arrow[dl, dash] \arrow[dr, dash] & & \\ 2 \arrow[dr, dash] & & 3 \arrow[dl, dash] \arrow[dr, dash] & \\ & 3 & & 1 \end{tikzcd} \qquad \begin{tikzcd} 2 \arrow[d, dash] \\ 3 \arrow[d, dash] \\ 3 \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & 3 \arrow[dl, dash] \arrow[dr, dash] & \\ 1 \arrow[d, dash] & & 3 \\ 2 & & \end{tikzcd}$$

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Let $e = e_2 + e_3$.

$$Q_{eAe} := \begin{tikzcd} 2 \arrow[dr, bend left] & \\ & 3 \arrow[ul, bend left] \arrow[loop right] \end{tikzcd}$$

Projective $eAe$-modules

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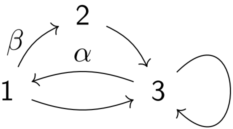
$Q_A :=$ 


Projective $A$-modules

$$\begin{array}{ccccc} & 1 & & \\ / & & \backslash & \\ 2 & & 3 & \\ \backslash & / & & \backslash \\ & 3 & & & 1 \end{array} \qquad \begin{array}{c} 2 \\ | \\ 3 \\ | \\ 3 \end{array} \qquad \begin{array}{ccc} & 3 & \\ \alpha\, / & & \backslash \\ 1 & & 3 \\ \beta\, | & & \\ 2 & & \end{array}$$

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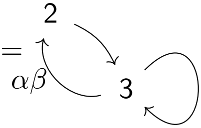
$Q_{eAe} :=$ 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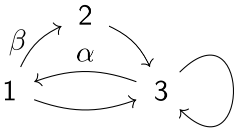


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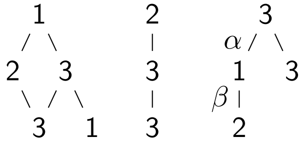


$$S(1) = (1-e)A/\mathrm{rad}\,((1-e)A)$$

Let $e = e_2 + e_3$.

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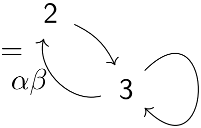


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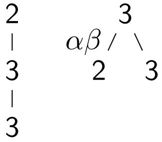

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**Example**

Let $A$ be a finite dimensional algebra and $e \in A$ be an idempotent. Let $S := (1-e)A/\mathrm{rad}\,((1-e)A)$.

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Let $A$ be a finite dimensional algebra and $e \in A$ be an idempotent. Let $S := (1-e)A/\mathrm{rad}\,((1-e)A)$. Suppose that $\mathrm{inj.dim}_A\,(S) = 1$.

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Let $A$ be a finite dimensional algebra and $e \in A$ be an idempotent. Let $S := (1-e)A/\mathrm{rad}\left((1-e)A\right)$. Suppose that $\mathrm{inj.dim}_A(S) = 1$. Then there exists a recollement $(R)$.
(Green, Psaroudakis, Solberg 2018)

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(R) := \quad
\begin{tikzcd}
\mathcal{D}(A/AeA) \arrow[r, "i_*"] & \mathcal{D}(A) \arrow[l, bend right=60, "i^*"'] \arrow[l, bend left=60, "i^!"] \arrow[r, "j^*"] & \mathcal{D}(eAe) \arrow[l, bend right=60, "j_!"'] \arrow[l, bend left=60, "j_*"]
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Moreover, $(R)$ restricts to a recollement of bounded below derived categories.

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**Example**

Let $A$ be a finite dimensional algebra and $e \in A$ be an idempotent. Let $S := (1-e)A/\mathrm{rad}\,((1-e)A)$. Suppose that $\mathrm{inj.dim}_A\,(S) = 1$.

$$(R) := \begin{tikzcd} \mathcal{D}\,(A/AeA) \arrow[r, "i_*"] & \mathcal{D}\,(A) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{D}\,(eAe) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

# Vertex Removal

**Example**

Let $A$ be a finite dimensional algebra and $e \in A$ be an idempotent. Let $S := (1-e)A/\mathrm{rad}\,((1-e)A)$. Suppose that $\mathrm{inj.dim}_A\,(S) = 1$.

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1. If injectives generate for both $A/AeA$ and $eAe$ then injectives generate for $A$.
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\end{tikzcd}
$$

Note that $A/AeA$ has finite global dimension.

Injectives generate for $A$ if and only if injectives generate for $eAe$.

# Recollements of dg algebras

Let $A$ be a finite dimensional algebra over a field and $e \in A$ be an idempotent.

# Recollements of dg algebras

Let $A$ be a finite dimensional algebra over a field and $e \in A$ be an idempotent.
Then there exists a dg algebra $B$ such that $(R)$ is a recollement.

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Suppose that $Ae \otimes^{L}_{eAe} eA$ is bounded in cohomology i.e. $\mathrm{Tor}_i^{eAe}(Ae, eA) = 0$ for $i \gg 0$.

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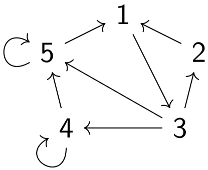
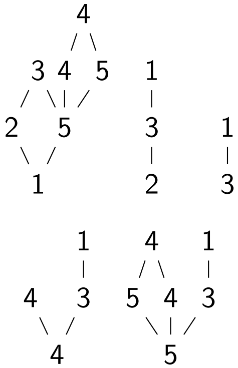
# Example

$Q_A :=$




Injective $A$-modules



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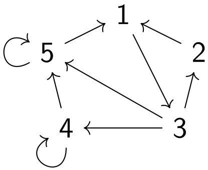
$Q_A :=$ 


| | $S(2)$ | $S(1)$ |
|---|---|---|
| $\text{inj.dim}_A$ | 1 | 2 |

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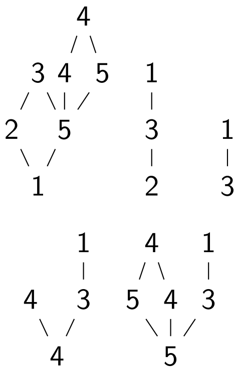

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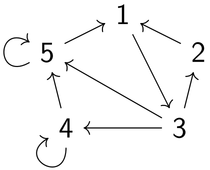

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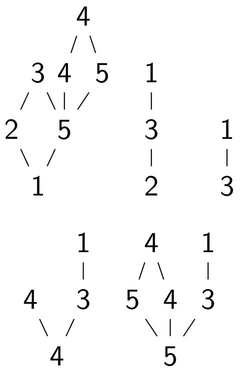
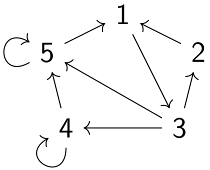
- $e := e_3 + e_4 + e_5$,

Injective $A$-modules



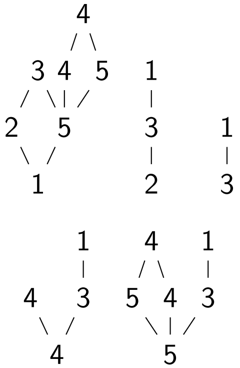
# Example

$Q_A :=$ 

| | $S(2)$ | $S(1)$ |
|---|---|---|
| inj.dim$_A$ | 1 | 2 |

- $e := e_3 + e_4 + e_5$,
- $Ae \otimes^L_{eAe} eA$ is bounded in cohomology.

Injective $A$-modules



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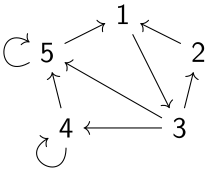
$Q_A :=$ 

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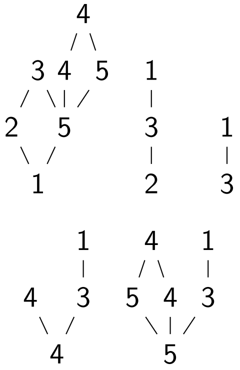
Injectives generate for $A$ if injectives generate for $eAe$.

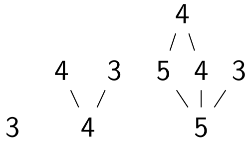
Injective $A$-modules



# Example

$$Q_{eAe} := \begin{tikzcd} & 5 \arrow[loop left] & \\ & 4 \arrow[u] \arrow[loop below] & 3 \arrow[l] \arrow[ul] \end{tikzcd}$$

Injective $eAe$-modules



# Example

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Injective $eAe$-modules

$$3 \qquad \begin{tikzcd}[column sep=small] 4 \arrow[dr, dash] & & 3 \arrow[dl, dash] \\ & 4 & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & 4 \arrow[dl, dash] \arrow[d, dash] & \\ 5 \arrow[dr, dash] & 4 \arrow[d, dash] & 3 \arrow[dl, dash] \\ & 5 & \end{tikzcd}$$

---

Apply triangular matrix algebra result.

$$Q_C := \begin{tikzcd} 4 \arrow[loop below] & 3 \arrow[l] \end{tikzcd}$$

Injective
$C$-modules

$$3 \qquad \begin{tikzcd}[column sep=small] 4 \arrow[dr, dash] & & 3 \arrow[dl, dash] \\ & 4 & \end{tikzcd}$$

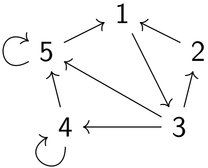
$$Q_B := \begin{tikzcd} 5 \arrow[loop left] \end{tikzcd}$$

Injective
$B$-module

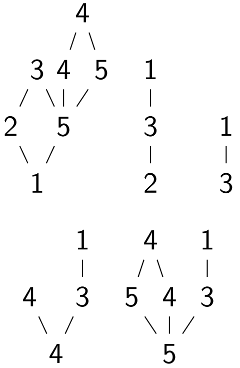
$$\begin{tikzcd} 5 \arrow[d, dash] \\ 5 \end{tikzcd}$$

# Example

So injectives generate for $A$.

$Q_A :=$ 


Injective $A$-modules



# Projective Cogeneration

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**Definition**

Let $A$ be a ring. If $\mathcal{D}(A)$ is generated, as a triangulated category with infinite **products**, by the **projective** $A$-modules then we say that **projectives cogenerate for** $A$.

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Let $A$ be a finite dimensional algebra over a field.

Projectives cogenerate for $A^{\mathrm{op}}$ $\xRightarrow[\text{Rickard}]{}$ Finitistic dimension conjecture holds for $A$

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# Thank you


- Lidia Angeleri Hügel, Steffen Koenig, Qunhua Liu, and Dong Yang. Ladders and simplicity of derived module categories. 2017
- Hong Xing Chen and Changchang Xi. Recollements of derived categories III: finitistic dimensions. 2017
- Charley Cummings. Ring constructions and generation of the unbounded derived module category. 2020
- Robert M. Fossum, Phillip A. Griffith, and Idun Reiten. Trivial extensions of abelian categories. 1975
- K. R. Fuller and M. Saorín. On the finitistic dimension conjecture for Artinian rings. 1992
- Edward L. Green, Chrysostomos Psaroudakis, and Øyvind Solberg. Reduction techniques for the finitistic dimension. 2018
- Jeremy Rickard. Unbounded derived categories and the finitistic dimension conjecture. 2019