

## Relative Calabi-Yau completions and higher preprojective algebras

### Prehistory

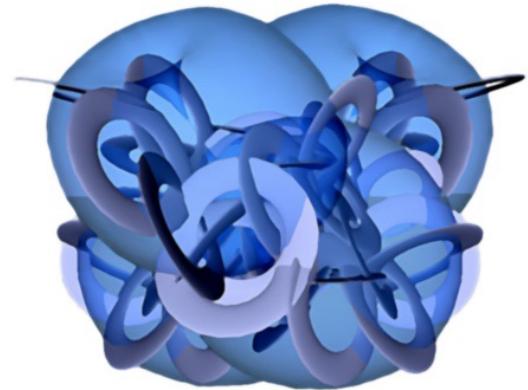
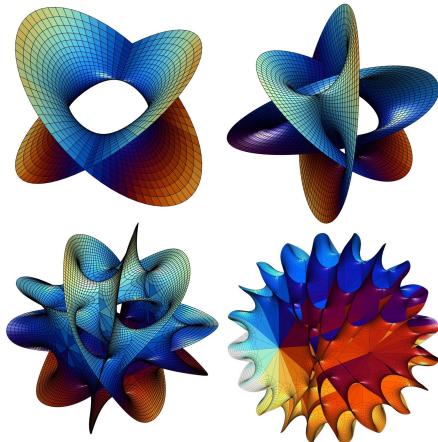
(Absolute) Calabi-Yau varieties



Eugenio Calabi (now 97) conjectured in 1957 that CY-varieties admit Ricci-flat metrics.



Shing-Tung Yau 丘成桐 (now 71) proved Calabi's conjecture in 1977.



## Present times



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Report on part of Yilin Wu's ongoing Ph. D. thesis.

**Plan:** 0. Brief history and overview in pictures

1. Absolute CY-completions
2. Relative CY-completions
3. A key equivalence :  $S_{A,B}^- \sim \tau_n^-$
4. Higher preproj. algebras

## O. Brief history and overview in pictures

### History



Bertrand Toën

- (Right) relative CY-structures were invented by Bertrand Toën in 2014.



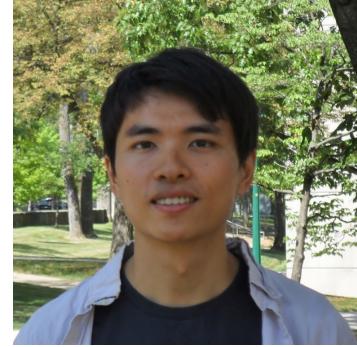
Chris Brav



Tobias Dyckerhoff

- (Right and left) relative CY-structures were elaborated by Chris Brav and Tobias Dyckerhoff (2016 and 2018). In particular: glueing.

- Wai-kit Yeung introduces relative CY-completions in 2016 and advocates the idea that they are non-commutative conormal bundles.



Wai-kit Yeung 楊偉傑 杨伟杰



Tristan Bozec



Damien Calaque



Sarah Scherotzke

- Bozec-Calaque-Scherotzke prove (06/20) that Yeung's idea is justified by Kontsevich-Rosenberg's criterion.

## Overview in pictures

dg algebra

+

CY-structure  
(absolute)

dg algebra morphism

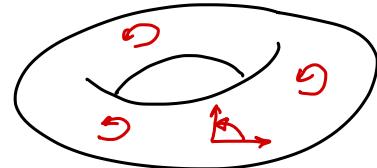
$B \rightarrow A$

+

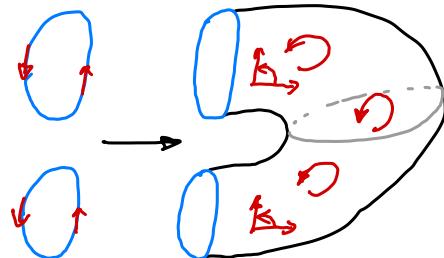
relative CY-structure

BOUNDARY

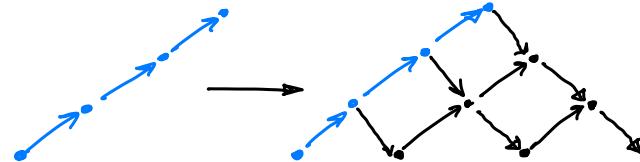
AUSLANDER



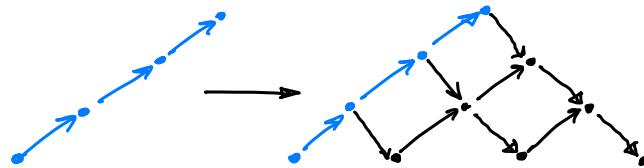
manifold  
+  
orientation



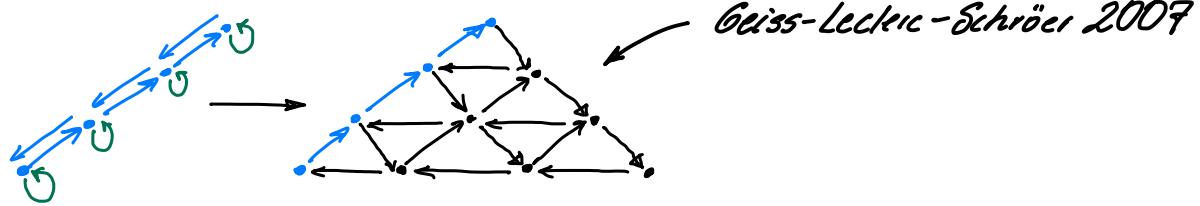
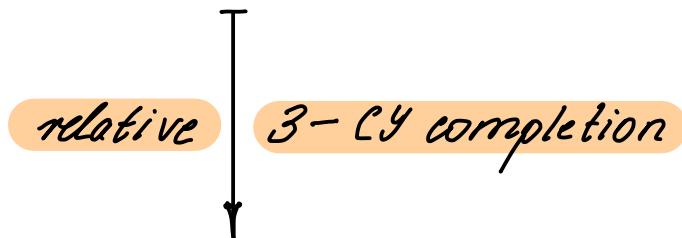
manifold with  
boundary  
+  
orientation



NO relative CY-structure here!



NO relative CY-structure here!



2-dim. (abs.) Ginzburg alg.

$H^0$ : preproj. alg.

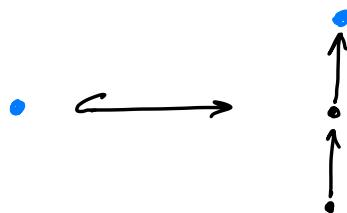
$\dim H^* = \infty$

3-dim. relative Ginzburg alg.

$$W = \sum \text{double arrows} - \sum \text{single arrows}$$

surprise!  $\begin{cases} H^* = H^0: \quad \partial_\alpha W = 0, \quad \alpha \text{ not frozen} \\ \dim H^0 < \infty \end{cases}$

*Exercise:* Compute the rel. 2-CY completion of



*Hint:* The 2-dim. rel. Ginzburg algebra is  
the Auslander algebra of a 4-dim. self-inj. algebra.

## 1. Absolute CY-completions (2011)

$k$  a perfect field (for simplicity)

$A$  a dg (=differential graded) algebra (assoc., with 1, non com.)

$\mathcal{C}A = \{ \text{dg right } A\text{-modules} \} \quad (A = A^\circ : \text{complexes of right } A\text{-mod.})$

$\mathcal{D}A = \text{derived category} = (\mathcal{C}A)[Qis'] : \text{triang. with arb. coprod.}$

$\text{per}(A) = \{ C \in \mathcal{D}A \mid C \text{ compact} \} = \text{thick}(A_A) \quad (A = A^\circ : \mathcal{H}^b(\text{proj}(A)))$

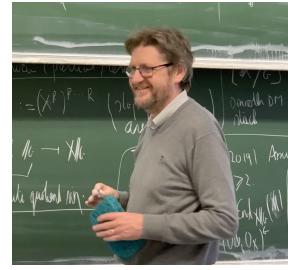
$A^c = A \otimes A^{op} : \mathcal{C}A^c = \{ \text{dg } A\text{-}A\text{-bimodules} \}$

$A$  is *smooth*  $\iff {}_A A_A \in \text{per}(A^c)$

$M \in \mathcal{D}A^c : M^\vee = R\text{Hom}_{A^c}(M, A^c) \in \mathcal{D}((A^c)^{op}) \cong \mathcal{D}(A^c)$

$\Omega_A = \text{inverse dualizing complex} = A^\vee$

Fix  $n \in \mathbb{Z}$ .



Def. (Ginzburg '06, VdB '11):  $A$  is  $n$ -CY  $\Leftrightarrow A^\vee \cong \sum^{-n} A$  in  $\mathcal{D}A^c$ .

Suppose  $B$  is a smooth dg algebra.

Def. ('11):  $T_{n+1}B = (\text{abs.}) (n+1)\text{-CY-completion}$

$$= T_B(\omega) = B \otimes \omega \otimes (\omega \otimes \omega) \otimes \dots, \omega = \text{cofibrant res. of } \sum^n \Omega_A$$

Thm ('11):  $T_{n+1}B$  is smooth and canonically  $(n+1)$ -CY.

Example:  $B: \overset{\text{adj}}{\circ} \rightarrow \underset{2}{\pi} B = 2\text{-dim. Ginzburg algebra}:$

$$\begin{array}{l} t_1 \circlearrowleft \begin{matrix} \overset{\bar{a}}{\nearrow} \\ \downarrow a \end{matrix} \begin{matrix} \nearrow a \\ \searrow \bar{a} \end{matrix} \begin{matrix} \nearrow \bar{a} \\ \downarrow a \end{matrix} \begin{matrix} \nearrow a \\ \searrow \bar{a} \end{matrix} \end{array} \quad \begin{array}{l} |t_1| = -1 \\ dt_1 = -\bar{a}a \\ dt_2 = a\bar{a}. \end{array}$$

$$\Rightarrow \dim H^* T_2^\wedge B = \infty \quad (\Leftarrow Q_1 \neq \emptyset)$$

$$\Rightarrow H^0(T_2^\wedge B) = \text{preproj. alg. : } \begin{array}{c} \text{a} \\ \swarrow \curvearrowright \circlearrowleft \end{array} \text{a}$$

## 2. Relative CY-completions

$B \rightarrow A$  morphism of dg algebras,  $A, B$  smooth.

Def.: ①  $\mathbb{Q}_{A,B} = \text{relative inverse dualizing complex} = (\text{cone}(A \xrightarrow{\wedge} B \rightarrow A))^\vee \in \mathfrak{D}A^e$   
 (Young '16)

②  $T_{n+2}(A, B) = \text{rel. } (n+2)\text{-dim. Ginzburg alg. of } B \rightarrow A$

$= T_A(\omega), \omega = \text{cofib. ms. of } \Sigma^{n+1} \mathbb{Q}_{A,B}.$

③ (Rel.  $(n+2)$ -CY completion of  $B \rightarrow A$ ) = ( $T_{n+1}B \xrightarrow{\text{can}} T_{n+2}(A, B)$ )

Example:  $n=1$ :

$$(B \rightarrow A) = \begin{array}{ccc} \nearrow & \hookrightarrow & \nearrow \\ & \curvearrowright & \searrow \end{array}$$

Rel. 3-CY-completion:

$$H^0 \left( \begin{array}{c} \text{blue arrow} \\ \text{red arrow} \end{array} \right) \xrightarrow{\sim} \begin{array}{c} \text{blue arrow} \\ \text{red arrow} \end{array}$$

abs. 2-CY completion

= abs. 2-dim. Ginzburg alg.

$\dim H^* = \infty$ ,  $H^0$  = preproj. alg.

rel. 3-dim. Ginzbg. alg.:

$$\begin{aligned} d(\bar{a}) &= bc, d(\bar{c}) = ab, \\ d(t) &= a\bar{a} - \bar{c}c. \end{aligned}$$

$H^* = H^0$  is fin. dim. (!!)

$\curvearrowright H^0$

*Thm (Y16, BCS20):*  $(T_{n+1}B \rightarrow T_{n+2}(B, A))$  is smooth and can. rel.  $(n+2)$ -CY.

Not.:  $S_{A,B}^-$  = "rl. inverse Serre functor"

$$= ? \otimes_A^L S_{A,B} : \mathcal{D}^b A \longrightarrow \mathcal{D}^b A$$

Rk: Thus,  $T_{n+2}(A, B) \cong \bigoplus_{i \geq 0} (\sum'' S_{A,B}^-)^i(A)$  in  $\mathcal{D}^b A$ .

3. A key equivalence:  $S_{A,B}^- \sim \mathbb{E}_n^-$

$n \geq 1$  an integer

$B$  a fin. dim. algebra. Suppose  $B$  is  $n$ -repns. finite [IOT11], i.e.

a)  $\text{mod } B$  contains an  $n$ -cluster tilting object  $M$ , i.e.  $M = \text{add } M$  satisfies

$$M = \{X \in \text{mod } B \mid \text{Ext}_B^{0 < i < n}(X, M) = 0\} = \{X \in \text{mod } B \mid \text{Ext}^{0 < i < n}(M, X) = 0\}.$$

b)  $\text{gldim } B \leq n$ .

Let  $\mathcal{S} = ? \otimes_B^L \mathcal{D}B : \mathcal{D}B \rightarrow \mathcal{D}^b B$  be the Serre functor and define

$$\begin{array}{ccccc} \text{mod}B & \xhookrightarrow{\text{can}} & \mathcal{D}^b B & \xrightarrow{\sum^n \mathcal{S}^{-1}} & \mathcal{D}^b B \\ & & & & \xrightarrow{H^0} \text{mod}B \end{array}$$

$\tau_n^- := \text{higher inverse AR-translation (Iyama '07)}$

We have  $M = \text{add}(\tau_n^{-i}B)_{i \geq 0}$  [I07].

Put  $A = (\text{n-Auslander alg. of } B) = \text{End}_B(M)$ .

Abuse of notation:

$$\begin{array}{ccccc} \text{add}M & \xrightarrow{\sim} & \text{proj}A & \xrightarrow{\quad} & \text{mod}A \\ \tau_n^- \downarrow & \cong & \downarrow \tau_{n,B}^- & \cong & \downarrow \tau_{n,B}^- \text{ right ex. extension} \\ \text{add}M & \xrightarrow{\sim} & \text{proj}A & \xrightarrow{\quad} & \text{mod}A \end{array}$$

**KEY LEMMA:** Up to shift and derivation  $S_{A,B}^- : \mathcal{D}^b A \rightarrow \mathcal{D}^b A$  is  $\tau_{n,B}^-$ !

More precisely, as functors  $\mathcal{D}^b A \rightarrow \mathcal{D}^b A$ :

$$\sum_{i=0}^{n+1} S_{A,B}^- = \underline{\tau}_{n,B}^-$$

**Consequence:**

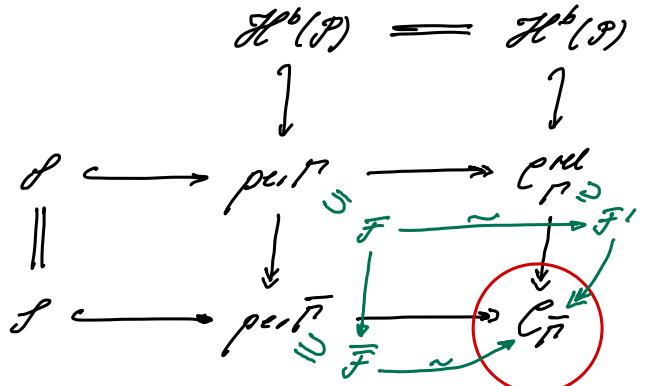
$$\begin{aligned} T\Gamma_{n+2}(A, B) &\xrightarrow[\text{in } \mathcal{D}^b A]{\sim} \bigoplus_{i \geq 0} \left( \sum_{j=0}^{n+1} S_{A,B}^- \right)^i(A) \\ &\sim \bigoplus_{i \geq 0} \underbrace{\tau_{n,B}^{-i}(A)}_{\in \text{proj } A!} \end{aligned}$$

**Cor.**: a)  $T\Gamma_{n+2}(A, B)$  is conc. in deg. 0, fin. dim., of finite gldim. and

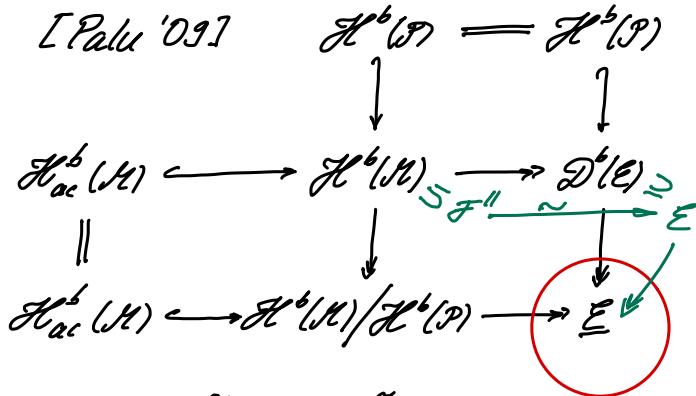
the functor  $H^0 T\Gamma_{n+1} B \rightarrow T\Gamma_{n+2}(A, B)$  is fully faithful

- b)  $T\Gamma_{n+2}(A, B)$  is internally bimodule  $(n+2)-\text{CY}$  w.r.t.  $\text{supp}(B)$  in the sense of Pressland '17.

Rk: Using the Cor. and the fact that  $H^0 \mathcal{T}_{n+1} B = (n+1)$ -preproj. alg.  $= \tilde{B}$  is selfinjective [IJO13], we obtain a new proof of the fact that  $\text{mod } \tilde{B}$  contains an  $(n+1)$ -cluster tilting object (first proved in [IJO13]). The proof consists in showing the equivalence between the following two diagrams:



$$\mathcal{P} = \mathcal{T}_{n+2}(A, B), \quad \bar{\mathcal{P}} = \mathcal{T}_{n+2}(A/\langle B \rangle), \quad \mathcal{S} = \text{thick}(\mathcal{S}_1) / \text{id supp } \mathcal{B}$$



$$\begin{aligned}
 \mathcal{E} &= \text{mod } \tilde{B}, \quad \mathcal{P} = \text{proj } \tilde{B}, \\
 \mathcal{H}_{ac}^b(M) &= \{X \in \mathcal{H}^b(M) \mid X \text{ acyclic in } \mathcal{E}\}
 \end{aligned}$$

$\mathcal{F}$  = rel. fund. domain  $\subseteq \text{perf}^{\Gamma}$

$= \{X \in \text{perf}^{\Gamma} \mid X = (\dots 0 \rightarrow x_n \rightarrow \dots \rightarrow x_0 \rightarrow 0 \rightarrow \dots),$

$H_i \text{Hom}(X, F) = 0, \forall 1 \leq i \leq n, \forall F \in \text{add}^{\Gamma}, F \text{"frozen"}$

$\mathcal{F}'' = \{X \in \mathcal{D}^b(\mathcal{M}) \mid \text{similarly with } F \in \mathcal{P}\}$

Rhs: 1) The algebra  $\Gamma = \overline{\mathcal{O}_{n+2}}(A, B)$  is isom. to the end. alg. of the can.  $(n+1)$ -cto in  $\mathcal{E}$ -mod $\mathcal{B}$ .

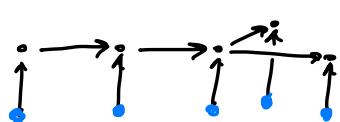
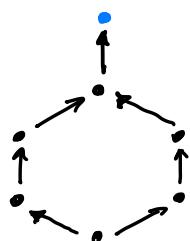
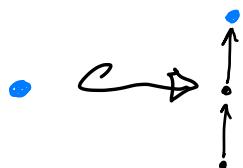
$\Gamma$  does not appear in [I013]. The algebra  $\widehat{\Gamma}$  is isom. to the corresp. stable end. alg.

2) The cat.  $\mathcal{U} = t\left(\sum_{i=1}^n S^{-1}\right)^{\widehat{\mathcal{C}}}(B) / i \in \mathbb{Z} \} \subseteq \mathcal{D}^b B$  of [I013] does not appear in our set up.

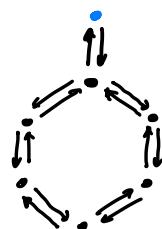
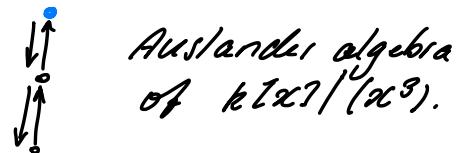
3) The contents of p. 13 adapt well to the setup of  $n$ -complete algebras [I11] (and probably many other settings of higher AR-theory).

## Appendix: Uniqueness of 2-dim. rel. Ginzburg algebras

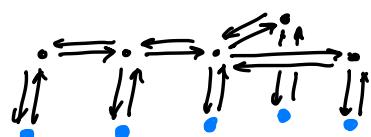
### Morphism



### 2-dim. rel. G. alg.



Framed Auslander algebra of  
the simple sing. of type  $P_5$



Nakajima algebra  
of type  $D_5$