Atoms in Singularland

Hankyung Ko

Based on joint works with Ben Elias, Nicolas Libedinsky, Leonardo Patimo

FD Seminar, 8 february 2024

Regularland

Regularland

A *Coxeter group* is a group W, together with a finite generating set S, presented by relations of the following form:

Regularland

A *Coxeter group* is a group W, together with a finite generating set S, presented by relations of the following form:

Regularland

A *Coxeter group* is a group W, together with a finite generating set S, presented by relations of the following form:

- the *braid relation*

$$\underbrace{st\cdots}_{m} = \underbrace{ts\cdots}_{m}, \text{ for each } s,t \in S \text{ with } m = m_{st} = m_{ts} < \infty;$$

Regularland

A *Coxeter group* is a group W, together with a finite generating set S, presented by relations of the following form:

- the *braid relation*

$$\underbrace{st\cdots}_{m} = \underbrace{ts\cdots}_{m}, \text{ for each } s,t \in S \text{ with } m = m_{st} = m_{ts} < \infty;$$

- the *reflection relation*

$$ss = e$$

  for each $s \in S$;

Regularland

A *Coxeter group* is a group W, together with a finite generating set S, presented by relations of the following form:

- the *braid relation*

$$\underbrace{st\cdots}_{m} = \underbrace{ts\cdots}_{m}, \text{ for each } s,t \in S \text{ with } m = m_{st} = m_{ts} < \infty;$$

- the *reflection relation*

$$ss = e$$

  for each $s \in S$;

Example. Symmetric groups S_n with $S = \{s_i := (i, i+1)\}_{1 \leqslant i \leqslant n-1}$,

Regularland

A *Coxeter group* is a group W, together with a finite generating set S, presented by relations of the following form:

- the *braid relation*

$$\underbrace{st\cdots}_{m} = \underbrace{ts\cdots}_{m}, \text{ for each } s, t \in S \text{ with } m = m_{st} = m_{ts} < \infty;$$

- the *reflection relation*

$$ss = e$$

  for each $s \in S$;

Example. Symmetric groups S_n with $S = \{s_i := (i, i+1)\}_{1 \leqslant i \leqslant n-1}$, with the braid relations

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad \text{and} \quad s_i s_j = s_j s_i \quad \text{for } |i - j| > 1.$$

Regular expressions

- An *expression* of $w \in W$ is a string $[s_1, s_2, \ldots, s_r]$ of elements in S such that $w = s_1 s_2 \cdots s_r$.

Regular expressions

- An *expression* of $w \in W$ is a string $[s_1, s_2, \ldots, s_r]$ of elements in S such that $w = s_1 s_2 \cdots s_r$.
- The *length function*: for $w \in W$, the length $\ell(w)$ is the smallest number of letters in an expression of w.

Regular expressions

- An *expression* of $w \in W$ is a string $[s_1, s_2, \ldots, s_r]$ of elements in S such that $w = s_1 s_2 \cdots s_r$.
- The *length function*: for $w \in W$, the length $\ell(w)$ is the smallest number of letters in an expression of w.
- An expression $[s_1, \ldots, s_r]$ of $w \in W$ is *reduced* if $r = \ell(w)$.

Regular expressions

- An *expression* of $w \in W$ is a string $[s_1, s_2, \ldots, s_r]$ of elements in S such that $w = s_1 s_2 \cdots s_r$.
- The *length function*: for $w \in W$, the length $\ell(w)$ is the smallest number of letters in an expression of w.
- An expression $[s_1, \ldots, s_r]$ of $w \in W$ is *reduced* if $r = \ell(w)$.

Matsumoto's Theorem

Any two reduced expressions of $w \in W$ are related by braid relations.

Regular expressions

- An *expression* of $w \in W$ is a string $[s_1, s_2, \dots, s_r]$ of elements in S such that $w = s_1 s_2 \cdots s_r$.
- The *length function*: for $w \in W$, the length $\ell(w)$ is the smallest number of letters in an expression of w.
- An expression $[s_1, \dots, s_r]$ of $w \in W$ is *reduced* if $r = \ell(w)$.

Matsumoto's Theorem

Any two reduced expressions of $w \in W$ are related by braid relations.

Fact: if $|W| < \infty$, there's a unique longest element in W.

Regular expressions

- An *expression* of $w \in W$ is a string $[s_1, s_2, \ldots, s_r]$ of elements in S such that $w = s_1 s_2 \cdots s_r$.
- The *length function*: for $w \in W$, the length $\ell(w)$ is the smallest number of letters in an expression of w.
- An expression $[s_1, \ldots, s_r]$ of $w \in W$ is *reduced* if $r = \ell(w)$.

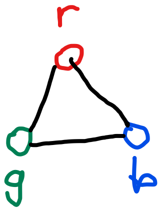
Matsumoto's Theorem

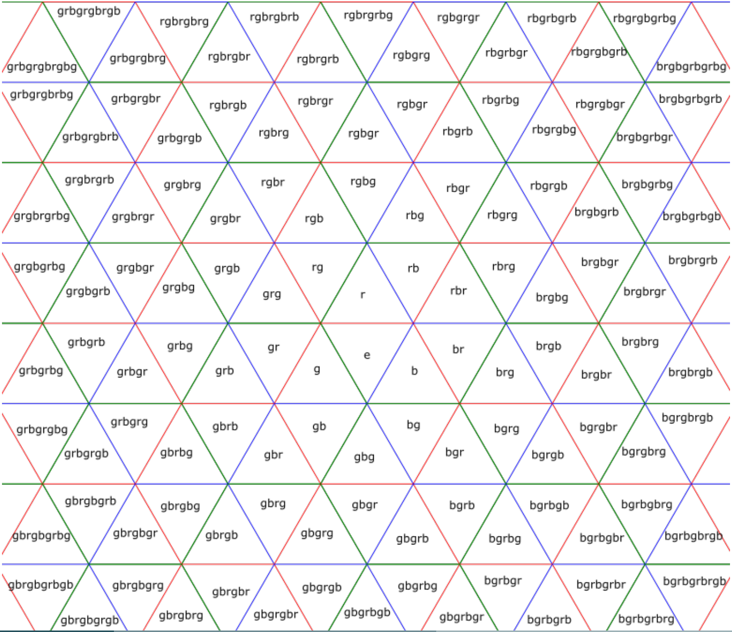
Any two reduced expressions of $w \in W$ are related by braid relations.

Fact: if $|W| < \infty$, there's a unique longest element in W.

Fact: the finite Coxeter groups are classified into the types $ABDEFH$ and the dihedral groups.

Example: affine symmetric group $\widetilde{S}_3$





Demazure operators

Demazure operators

Let $(W, S) = (S_n, \{s_1, \cdots, s_{n-1}\})$, for simplicity, and consider its natural action on $R = K[x_1, \ldots, x_n]$.

Demazure operators

Let $(W, S) = (S_n, \{s_1, \cdots, s_{n-1}\})$, for simplicity, and consider its natural action on $R = K[x_1, \ldots, x_n]$.

Demazure operators

Let $(W, S) = (S_n, \{s_1, \cdots, s_{n-1}\})$, for simplicity, and consider its natural action on $R = K[x_1, \ldots, x_n]$.

The *Demazure operator* for s_i is the map $D_{s_i} : R \to R$

$$D_{s_i}(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}.$$

Demazure operators

Let $(W, S) = (S_n, \{s_1, \cdots, s_{n-1}\})$, for simplicity, and consider its natural action on $R = K[x_1, \ldots, x_n]$.

The *Demazure operator* for s_i is the map $D_{s_i} : R \to R$

$$D_{s_i}(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}.$$

Theorem [Demazure, 1973]

The Demazure operators satisfy

Demazure operators

Let $(W, S) = (S_n, \{s_1, \cdots, s_{n-1}\})$, for simplicity, and consider its natural action on $R = K[x_1, \ldots, x_n]$.

The *Demazure operator* for s_i is the map $D_{s_i} : R \to R$

$$D_{s_i}(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}.$$

Theorem [Demazure, 1973]

The Demazure operators satisfy

$$D_{s_i} D_{s_{i+1}} D_{s_i} = D_{s_{i+1}} D_{s_i} D_{s_{i+1}}, \quad D_{s_i} D_{s_j} = D_{s_j} D_{s_i} \text{ for } |i - j| > 1 \quad \text{(braid relations)}$$

Demazure operators

Let $(W, S) = (S_n, \{s_1, \cdots, s_{n-1}\})$, for simplicity, and consider its natural action on $R = K[x_1, \ldots, x_n]$.

The *Demazure operator* for s_i is the map $D_{s_i} : R \to R$

$$D_{s_i}(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}.$$

Theorem [Demazure, 1973]

The Demazure operators satisfy

$$D_{s_i} D_{s_{i+1}} D_{s_i} = D_{s_{i+1}} D_{s_i} D_{s_{i+1}}, \quad D_{s_i} D_{s_j} = D_{s_j} D_{s_i} \text{ for } |i-j| > 1 \quad \text{(braid relations)}$$

$$D_{s_i} D_{s_i} = 0. \qquad (\textit{nilquadratic relations})$$

Demazure operators

Let $(W, S) = (S_n, \{s_1, \cdots, s_{n-1}\})$, for simplicity, and consider its natural action on $R = K[x_1, \ldots, x_n]$.

The *Demazure operator* for s_i is the map $D_{s_i} : R \to R$

$$D_{s_i}(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}.$$

Theorem [Demazure, 1973]

The Demazure operators satisfy

$$D_{s_i} D_{s_{i+1}} D_{s_i} = D_{s_{i+1}} D_{s_i} D_{s_{i+1}}, \quad D_{s_i} D_{s_j} = D_{s_j} D_{s_i} \text{ for } |i - j| > 1 \quad \text{(braid relations)}$$

$$D_{s_i} D_{s_i} = 0. \qquad \text{(\textit{nilquadratic relations})}$$

Moreover, this gives a presentation of the algebra generated by D_{s_i},

Demazure operators

Let $(W, S) = (S_n, \{s_1, \cdots, s_{n-1}\})$, for simplicity, and consider its natural action on $R = K[x_1, \ldots, x_n]$.

The *Demazure operator* for s_i is the map $D_{s_i} : R \to R$

$$D_{s_i}(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}.$$

Theorem [Demazure, 1973]

The Demazure operators satisfy

$$D_{s_i} D_{s_{i+1}} D_{s_i} = D_{s_{i+1}} D_{s_i} D_{s_{i+1}}, \quad D_{s_i} D_{s_j} = D_{s_j} D_{s_i} \text{ for } |i - j| > 1 \quad \text{(braid relations)}$$

$$D_{s_i} D_{s_i} = 0. \qquad (\textit{nilquadratic relations})$$

Moreover, this gives a presentation of the algebra generated by D_{s_i}, called the *nilCoxeter algebra* (or nilHecke algebra).

Demazure operators

Let $(W, S) = (S_n, \{s_1, \cdots, s_{n-1}\})$, for simplicity, and consider its natural action on $R = K[x_1, \ldots, x_n]$.

The *Demazure operator* for s_i is the map $D_{s_i} : R \to R$

$$D_{s_i}(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}.$$

Theorem [Demazure, 1973]

The Demazure operators satisfy

$$D_{s_i} D_{s_{i+1}} D_{s_i} = D_{s_{i+1}} D_{s_i} D_{s_{i+1}}, \quad D_{s_i} D_{s_j} = D_{s_j} D_{s_i} \text{ for } |i-j| > 1 \quad \text{(braid relations)}$$

$$D_{s_i} D_{s_i} = 0. \qquad (\textit{nilquadratic relations})$$

Moreover, this gives a presentation of the algebra generated by D_{s_i}, called the *nilCoxeter algebra* (or nilHecke algebra). It has a basis $\{D_w\}_{w \in W}$, where $D_w = D_s D_t \cdots D_u$ if $w = st \cdots u$ is a reduced expression.

Demazure operators

Let $(W, S) = (S_n, \{s_1, \cdots, s_{n-1}\})$, for simplicity, and consider its natural action on $R = K[x_1, \ldots, x_n]$.

The *Demazure operator* for s_i is the map $D_{s_i} : R \to R$

$$D_{s_i}(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}.$$

Theorem [Demazure, 1973]

The Demazure operators satisfy

$$D_{s_i} D_{s_{i+1}} D_{s_i} = D_{s_{i+1}} D_{s_i} D_{s_{i+1}}, \quad D_{s_i} D_{s_j} = D_{s_j} D_{s_i} \text{ for } |i - j| > 1 \quad \text{(braid relations)}$$

$$D_{s_i} D_{s_i} = 0. \qquad (\textit{nilquadratic relations})$$

Moreover, this gives a presentation of the algebra generated by D_{s_i}, called the *nilCoxeter algebra* (or nilHecke algebra). It has a basis $\{D_w\}_{w \in W}$, where $D_w = D_s D_t \cdots D_u$ if $w = st \cdots u$ is a reduced expression.

Demazure operators, general

Let (W, S) be a Coxeter group and consider its natural action on $R = Sym(V)$, where V is a reasonable (faithful etc) realization.

The *Demazure operator* for $s \in S$ is the map $D_s : R \to R$

$$D_s(f) = \frac{f - s(f)}{\alpha_s}.$$

Theorem [Demazure, 1973]

The Demazure operators satisfy

$$\underbrace{D_s D_t \cdots}_{m} = \underbrace{D_t D_s \cdots}_{m}, \text{ for each } s, t \in S \qquad \text{(braid relations)}$$

$$D_s D_s = 0. \qquad \text{(\textit{nilquadratic relations})}$$

Moreover, this gives a presentation of the algebra generated by D_s, called the *nilCoxeter algebra* (or nilHecke algebra). It has a basis $\{D_w\}_{w \in W}$, where $D_w = D_s D_t \cdots D_u$ if $w = st \cdots u$ is a reduced expression.

Zoom-in on Demazure operators

Note $D_{s_i}(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}$ is s_i-invariant, i.e., belongs to

$$R^{s_i} = \{f \in R \mid s_i(f) = f\}.$$

Zoom-in on Demazure operators

Note $D_{s_i}(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}$ is s_i-invariant, i.e., belongs to

$$R^{s_i} = \{f \in R \mid s_i(f) = f\}.$$

Decompose the Demazure operator as

$$D_{s_i} : R \xrightarrow{\delta_{s_i}} R^{s_i} \xhookrightarrow{\iota_{s_i}} R.$$

Zoom-in on Demazure operators

Note $D_{s_i}(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}$ is s_i-invariant, i.e., belongs to

$$R^{s_i} = \{f \in R \mid s_i(f) = f\}.$$

Decompose the Demazure operator as

$$D_{s_i} : R \xrightarrow{\delta_{s_i}} R^{s_i} \xhookrightarrow{\iota_{s_i}} R.$$

Then for example the nilquadratic relation factors as

$$D_{s_i} D_{s_i} = \iota_{s_i} \circ \delta_{s_i} \circ \iota_{s_i} \circ \delta_{s_i} = 0$$

and follows from the zoomed-in relation

$$\delta_{s_i} \circ \iota_{s_i} = 0.$$

Zoom-in on Demazure operators

Note $D_{s_i}(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}$ is s_i-invariant, i.e., belongs to

$$R^{s_i} = \{f \in R \mid s_i(f) = f\}.$$

Decompose the Demazure operator as

$$D_{s_i} : R \xrightarrow{\delta_{s_i}} R^{s_i} \xhookrightarrow{\iota_{s_i}} R.$$

Then for example the nilquadratic relation factors as

$$D_{s_i} D_{s_i} = \iota_{s_i} \circ \delta_{s_i} \circ \iota_{s_i} \circ \delta_{s_i} = 0$$

and follows from the zoomed-in relation

$$\delta_{s_i} \circ \iota_{s_i} = 0.$$

Geometrically: Considering the algebraic groups $GL_n \supset P_i \supset B(orel)$, the ring R is the B-equivariant cohomology; R^{s_i} is the P_i-equivariant cohomology; δ_{s_i} is the pushforward; ι_{s_i} is the pullback.

Parabolic cosets

Parabolic cosets

Let (W, S) be a Coxeter group. For a subset $I \subseteq S$, denote by W_I the subgroup generated by $I \subseteq W$.

Parabolic cosets

Let (W, S) be a Coxeter group. For a subset $I \subseteq S$, denote by W_I the subgroup generated by $I \subseteq W$. Then (W_I, I) is a Coxeter group.

Parabolic cosets

Let (W, S) be a Coxeter group. For a subset $I \subseteq S$, denote by W_I the subgroup generated by $I \subseteq W$. Then (W_I, I) is a Coxeter group.

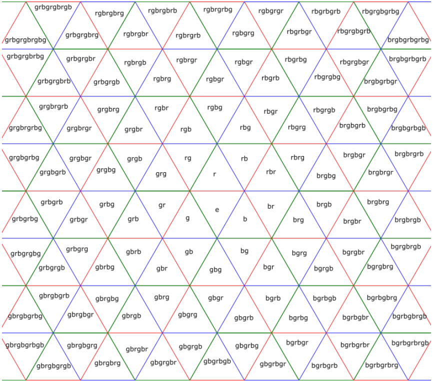
Example. Symmetric groups have $W_I = S_a \times S_b \times \cdots \times S_c \subseteq S_n$.

Parabolic cosets

Let (W, S) be a Coxeter group. For a subset $I \subseteq S$, denote by W_I the subgroup generated by $I \subseteq W$. Then (W_I, I) is a Coxeter group.

Example. Symmetric groups have $W_I = S_a \times S_b \times \cdots \times S_c \subseteq S_n$.

The Coxeter complex consists of the left cosets for W_I in W for all $I \subsetneq S$.



Parabolic cosets

Let (W, S) be a Coxeter group. For a subset $I \subseteq S$, denote by W_I the subgroup generated by $I \subseteq W$. Then (W_I, I) is a Coxeter group.

Example. Symmetric groups have $W_I = S_a \times S_b \times \cdots \times S_c \subseteq S_n$.

The Coxeter complex consists of the left cosets for W_I in W for all $I \subsetneq S$.

We assume: $I \subseteq S$ is *finitary* i.e., W_I is finite; so W_I has the longest element w_I.

Parabolic cosets

Let (W, S) be a Coxeter group. For a subset $I \subseteq S$, denote by W_I the subgroup generated by $I \subseteq W$. Then (W_I, I) is a Coxeter group.

Example. Symmetric groups have $W_I = S_a \times S_b \times \cdots \times S_c \subseteq S_n$.

The Coxeter complex consists of the left cosets for W_I in W for all $I \subsetneq S$.

We assume: $I \subseteq S$ is *finitary* i.e., W_I is finite; so W_I has the longest element w_I.

For $I, J \subseteq S$ finitary, consider the **finitary (parabolic) double cosets**:

$$p = W_I w W_J = \{xwy \in W \mid x \in W_I, y \in W_J\}$$

Parabolic cosets

Let (W, S) be a Coxeter group. For a subset $I \subseteq S$, denote by W_I the subgroup generated by $I \subseteq W$. Then (W_I, I) is a Coxeter group.

Example. Symmetric groups have $W_I = S_a \times S_b \times \cdots \times S_c \subseteq S_n$.

The Coxeter complex consists of the left cosets for W_I in W for all $I \subsetneq S$.

We assume: $I \subseteq S$ is *finitary* i.e., W_I is finite; so W_I has the longest element w_I.

For $I, J \subseteq S$ finitary, consider the **finitary (parabolic) double cosets**:

$$p = W_I w W_J = \{xwy \in W \mid x \in W_I, y \in W_J\}$$

- Each p contains a unique maximal element $\overline{p}$

Parabolic cosets

Let (W, S) be a Coxeter group. For a subset $I \subseteq S$, denote by W_I the subgroup generated by $I \subseteq W$. Then (W_I, I) is a Coxeter group.

Example. Symmetric groups have $W_I = S_a \times S_b \times \cdots \times S_c \subseteq S_n$.

The Coxeter complex consists of the left cosets for W_I in W for all $I \subsetneq S$.

We assume: $I \subseteq S$ is *finitary* i.e., W_I is finite; so W_I has the longest element w_I.

For $I, J \subseteq S$ finitary, consider the **finitary (parabolic) double cosets**:

$$p = W_I w W_J = \{xwy \in W \mid x \in W_I, y \in W_J\}$$

- Each p contains a unique maximal element $\overline{p}$
- Each p contains a unique minimal element $\underline{p}$

Zoom-in further on Demzure operators

Zoom-in further on Demzure operators

Take for $I \subset S$ the invariant ring

$$R^I = \{f \in R \mid s_i(f) = f \text{ for } s_i \in I\} = R^{W_I}.$$

Zoom-in further on Demzure operators

Take for $I \subset S$ the invariant ring

$$R^I = \{f \in R \mid s_i(f) = f \text{ for } s_i \in I\} = R^{W_I}.$$

Given $I \subset J \subset S$, the operator $D_{w_J w_I^{-1}} : R \to R$ is shown to restrict to

$$\partial_{[I,J]} : R^I \to R^J.$$

Zoom-in further on Demzure operators

Take for $I \subset S$ the invariant ring

$$R^I = \{f \in R \mid s_i(f) = f \text{ for } s_i \in I\} = R^{W_I}.$$

Given $I \subset J \subset S$, the operator $D_{w_J w_I^{-1}} : R \to R$ is shown to restrict to

$$\partial_{[I,J]} : R^I \to R^J.$$

Also let $\partial_{[J,I]} : R^J \hookrightarrow R^I$ be the inclusion, where $I \subset J \subset S$.

Zoom-in further on Demzure operators

Take for $I \subset S$ the invariant ring

$$R^I = \{f \in R \mid s_i(f) = f \text{ for } s_i \in I\} = R^{W_I}.$$

Given $I \subset J \subset S$, the operator $D_{w_J w_I^{-1}} : R \to R$ is shown to restrict to

$$\partial_{[I,J]} : R^I \to R^J.$$

Also let $\partial_{[J,I]} : R^J \hookrightarrow R^I$ be the inclusion, where $I \subset J \subset S$.

Example: $\partial_{[\varnothing,J]} : R \to R^J$ is given by

$$\partial_{[\varnothing,J]}(f) = D_{w_J}(f) = \frac{\sum_{w \in W_J} (-1)^{\ell(w)} w(f)}{\prod_{i<j}(x_i - x_j)}.$$

Zoom-in further on Demzure operators

Take for $I \subset S$ the invariant ring

$$R^I = \{f \in R \mid s_i(f) = f \text{ for } s_i \in I\} = R^{W_I}.$$

Given $I \subset J \subset S$, the operator $D_{w_J w_I^{-1}} : R \to R$ is shown to restrict to

$$\partial_{[I,J]} : R^I \to R^J.$$

Also let $\partial_{[J,I]} : R^J \hookrightarrow R^I$ be the inclusion, where $I \subset J \subset S$.

Example: $\partial_{[\varnothing,J]} : R \to R^J$ is given by

$$\partial_{[\varnothing,J]}(f) = D_{w_J}(f) = \frac{\sum_{w \in W_J} (-1)^{\ell(w)} w(f)}{\prod_{i<j}(x_i - x_j)}.$$

Remarks. These too come from geometry.

Zoom-in further on Demzure operators

Take for $I \subset S$ the invariant ring

$$R^I = \{f \in R \mid s_i(f) = f \text{ for } s_i \in I\} = R^{W_I}.$$

Given $I \subset J \subset S$, the operator $D_{w_J w_I^{-1}} : R \to R$ is shown to restrict to

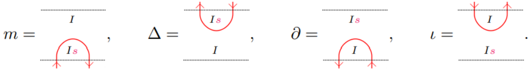
$$\partial_{[I,J]} : R^I \to R^J.$$

Also let $\partial_{[J,I]} : R^J \hookrightarrow R^I$ be the inclusion, where $I \subset J \subset S$.

Example: $\partial_{[\varnothing,J]} : R \to R^J$ is given by

$$\partial_{[\varnothing,J]}(f) = D_{w_J}(f) = \frac{\sum_{w \in W_J} (-1)^{\ell(w)} w(f)}{\prod_{i<j}(x_i - x_j)}.$$

Remarks. These too come from geometry. The inclusion $R^J \subset R^I$ is a Frobenius extension with trace $\partial_{[I,J]}$.



Singular Demazure operators

Definition. Let $\mathcal{D}$ be the K-linear category

- whose objects are R^I where $I \subset S$ are finitary subsets;
- whose morphisms are generated by the singular Demazure operators $\partial_{[I,J]}, \partial_{[J,I]}$, for $I \subset J \subset S$ finitary.

Singular Demazure operators

Definition. Let $\mathcal{D}$ be the K-linear category

- whose objects are R^I where $I \subset S$ are finitary subsets;
- whose morphisms are generated by the singular Demazure operators $\partial_{[I,J]}, \partial_{[J,I]}$, for $I \subset J \subset S$ finitary.

Notation.

$$\partial_{[I_0,I_1,\cdots,I_d]} = \partial_{[I_0,I_1]}\partial_{[I_1,I_2]}\cdots\partial_{[I_{d-1},I_d]}.$$

Singular Demazure operators

Definition. Let $\mathcal{D}$ be the K-linear category

- whose objects are R^I where $I \subset S$ are finitary subsets;
- whose morphisms are generated by the singular Demazure operators $\partial_{[I,J]}, \partial_{[J,I]}$, for $I \subset J \subset S$ finitary.

Notation.

$$\partial_{[I_0, I_1, \cdots, I_d]} = \partial_{[I_0, I_1]} \partial_{[I_1, I_2]} \cdots \partial_{[I_{d-1}, I_d]}.$$

We want to present $\mathcal{D}$ by (the above) generators and relations

Singular Demazure operators

Definition. Let $\mathcal{D}$ be the K-linear category
- whose objects are R^I where $I \subset S$ are finitary subsets;
- whose morphisms are generated by the singular Demazure operators $\partial_{[I,J]}, \partial_{[J,I]}$, for $I \subset J \subset S$ finitary.

Notation.

$$\partial_{[I_0,I_1,\cdots,I_d]} = \partial_{[I_0,I_1]}\partial_{[I_1,I_2]}\cdots\partial_{[I_{d-1},I_d]}.$$

We want to present $\mathcal{D}$ by (the above) generators and relations

Example. if $I \subset J \subset K$ then $\partial_{[I,J,K]} = \partial_{[I,K]}$ and $\partial_{[K,J,I]} = \partial_{[K,I]}$.

Singular Demazure operators

Definition. Let $\mathcal{D}$ be the K-linear category

- whose objects are R^I where $I \subset S$ are finitary subsets;
- whose morphisms are generated by the singular Demazure operators $\partial_{[I,J]}, \partial_{[J,I]}$, for $I \subset J \subset S$ finitary.

Notation.

$$\partial_{[I_0,I_1,\cdots,I_d]} = \partial_{[I_0,I_1]}\partial_{[I_1,I_2]}\cdots\partial_{[I_{d-1},I_d]}.$$

We want to present $\mathcal{D}$ by (the above) generators and relations

Example. if $I \subset J \subset K$ then $\partial_{[I,J,K]} = \partial_{[I,K]}$ and $\partial_{[K,J,I]} = \partial_{[K,I]}$.
Note. It is enough to have $\partial_{[I,Is]}, \partial_{[Is,I]}$ as generators, where $Is := I \sqcup \{s\}$.

Singular Demazure operators

Definition. Let $\mathcal{D}$ be the K-linear category
- whose objects are R^I where $I \subset S$ are finitary subsets;
- whose morphisms are generated by the singular Demazure operators $\partial_{[I,J]}, \partial_{[J,I]}$, for $I \subset J \subset S$ finitary.

Notation.

$$\partial_{[I_0,I_1,\cdots,I_d]} = \partial_{[I_0,I_1]}\partial_{[I_1,I_2]}\cdots\partial_{[I_{d-1},I_d]}.$$

We want to present $\mathcal{D}$ by (the above) generators and relations

Example. if $I \subset J \subset K$ then $\partial_{[I,J,K]} = \partial_{[I,K]}$ and $\partial_{[K,J,I]} = \partial_{[K,I]}$.
Note. It is enough to have $\partial_{[I,Is]}, \partial_{[Is,I]}$ as generators, where $Is := I \sqcup \{s\}$.

and give a basis.

Expressions in Singularland

Expressions in Singularland

A (singular) *expression* is a string

$$I_\bullet = [I_0, I_1, I_2, \dots, I_r]$$

of finitary subsets of S such that,

Expressions in Singularland

A (singular) *expression* is a string

$$I_\bullet = [I_0, I_1, I_2, \dots, I_r]$$

of finitary subsets of S such that, for each i, either

$$I_i = I_{i-1} \backslash s \qquad \text{('additive' notation : "}-s\text{")}$$

or

$$I_i = I_{i-1} s. \qquad \text{('additive' notation : "}+s\text{")}$$

Expressions in Singularland

A (singular) *expression* is a string

$$I_\bullet = [I_0, I_1, I_2, \ldots, I_r]$$

of finitary subsets of S such that, for each i, either

$$I_i = I_{i-1} \backslash s \qquad \text{('additive' notation : "}-s\text{")}$$

or

$$I_i = I_{i-1} s. \qquad \text{('additive' notation : "}+s\text{")}$$

Expressions in Singularland

A (singular) *expression* is a string

$$I_\bullet = [I_0, I_1, I_2, \ldots, I_r]$$

of finitary subsets of S such that, for each i, either

$$I_i = I_{i-1} \backslash s \qquad \text{('additive' notation : "}-s\text{")}$$

or

$$I_i = I_{i-1} s. \qquad \text{('additive' notation : "}+s\text{")}$$

An expression $I_\bullet$ is *reduced* if $\partial_{I_\bullet} \neq 0$. (There's a more natural definition.)

Expressions in Singularland

A (singular) *expression* is a string

$$I_\bullet = [I_0, I_1, I_2, \ldots, I_r]$$

of finitary subsets of S such that, for each i, either

$$I_i = I_{i-1} \backslash s \qquad \text{('additive' notation : "}-s\text{")}$$

or

$$I_i = I_{i-1} s. \qquad \text{('additive' notation : "}+s\text{")}$$

An expression $I_\bullet$ is *reduced* if $\partial_{I_\bullet} \neq 0$. (There's a more natural definition.)

'Regular' example: $[\varnothing, \{s\}, \varnothing, \{t\}, \varnothing, \{s\}, \varnothing, \{u\}, \varnothing, \{t\}, \varnothing, \{s\}, \varnothing]$

Expressions in Singularland

A (singular) *expression* is a string

$$I_\bullet = [I_0, I_1, I_2, \ldots, I_r]$$

of finitary subsets of S such that, for each i, either

$$I_i = I_{i-1} \backslash s \qquad\qquad (\text{`additive' notation : ``}-s\text{''})$$

or

$$I_i = I_{i-1} s. \qquad\qquad (\text{`additive' notation : ``}+s\text{''})$$

An expression $I_\bullet$ is *reduced* if $\partial_{I_\bullet} \neq 0$. (There's a more natural definition.)

'Regular' example: $[\varnothing, \{s\}, \varnothing, \{t\}, \varnothing, \{s\}, \varnothing, \{u\}, \varnothing, \{t\}, \varnothing, \{s\}, \varnothing]$
in additive notation: $[\varnothing + s - s + t - t + s - s + u - u + t - t + s - s]$

Expressions in Singularland

A (singular) *expression* is a string

$$I_\bullet = [I_0, I_1, I_2, \ldots, I_r]$$

of finitary subsets of S such that, for each i, either

$$I_i = I_{i-1} \backslash s \qquad \text{('additive' notation : "}-s\text{")}$$

or

$$I_i = I_{i-1} s. \qquad \text{('additive' notation : "}+s\text{")}$$

An expression $I_\bullet$ is *reduced* if $\partial_{I_\bullet} \neq 0$. (There's a more natural definition.)

'Regular' example: $[\varnothing, \{s\}, \varnothing, \{t\}, \varnothing, \{s\}, \varnothing, \{u\}, \varnothing, \{t\}, \varnothing, \{s\}, \varnothing]$
in additive notation: $[\varnothing + s - s + t - t + s - s + u - u + t - t + s - s]$

Other examples: $[\varnothing, \{s\}, \{s,t\}, \{s,t,u\}, \{s,u\}, \{u\}, \varnothing]$, $[\{s,t,u\}, \{s,u\}]$

Expressions in Singularland

A (singular) *expression* is a string

$$I_\bullet = [I_0, I_1, I_2, \ldots, I_r]$$

of finitary subsets of S such that, for each i, either

$$I_i = I_{i-1} \backslash s \qquad \text{('additive' notation : "}-s\text{")}$$

or

$$I_i = I_{i-1} s. \qquad \text{('additive' notation : "}+s\text{")}$$

An expression $I_\bullet$ is *reduced* if $\partial_{I_\bullet} \neq 0$. (There's a more natural definition.)

'Regular' example: $[\varnothing, \{s\}, \varnothing, \{t\}, \varnothing, \{s\}, \varnothing, \{u\}, \varnothing, \{t\}, \varnothing, \{s\}, \varnothing]$
in additive notation: $[\varnothing + s - s + t - t + s - s + u - u + t - t + s - s]$

Other examples: $[\varnothing, \{s\}, \{s,t\}, \{s,t,u\}, \{s,u\}, \{u\}, \varnothing]$, $[\{s,t,u\}, \{s,u\}]$
in additive notation: $[\varnothing + s + t + u - t - s - u]$, $[\{s,t,u\} - t]$

What does a reduced expression express?

What does a reduced expression express?

Rewrite expressions as

$$[[J_0 \supseteq K_1 \subsetneq J_1 \supsetneq K_2 \subsetneq \cdots \supsetneq K_d \subseteq J_d]]. \qquad (\star)$$

What does a reduced expression express?

Rewrite expressions as

$$[[J_0 \supseteq K_1 \subsetneq J_1 \supsetneq K_2 \subsetneq \cdots \supsetneq K_d \subseteq J_d]]. \qquad (\star)$$

An alternative formulation of regular reduced expression:

A *reduced expression* of $w \in W$ is a string $[s, t, \cdots, u]$ in S such that

$$w = st \cdots u$$

and

$$\ell(st \cdots u) = \ell(s) + \ell(t) + \cdots + \ell(u).$$

What does a reduced expression express?

Rewrite expressions as

$$[[J_0 \supseteq K_1 \subsetneq J_1 \supsetneq K_2 \subsetneq \cdots \supsetneq K_d \subseteq J_d]]. \qquad (\star)$$

Definition [Williamson 2008, Elias-K. 2021]

Given $I, J \subset S$ finitary, a *reduced expression of a double coset* $p = W_I \backslash W / W_J$ is a string $(\star)$, with $J_0 = I$ and $J_d = J$, such that

$$\overline{p} = w_{J_0} w_{K_1}^{-1} w_{J_1} w_{K_2}^{-1} w_{J_2} \ldots w_{K_d}^{-1} w_{J_d}$$

and

$$\ell(\overline{p}) = \ell(w_{J_0}) - \ell(w_{K_1}) + \ell(w_{J_1}) - \ell(w_{K_2}) + \ldots - \ell(w_{K_d}) + \ell(w_{J_d}).$$

Definition [Williamson 2008, Elias-K. 2021]

A *reduced expression* of $p = W_I w W_J$ is a string

$$[[J_0 \supset K_1 \subset J_1 \supset K_2 \subset \cdots \supset K_d \subset J_d]],$$

with $J_0 = I$ and $J_d = J$, such that

$$\overline{p} = w_{J_0} w_{K_1}^{-1} w_{J_1} w_{K_2}^{-1} w_{J_2} \cdots w_{K_d}^{-1} w_{J_d}$$

and

$$\ell(\overline{p}) = \ell(w_{J_0}) - \ell(w_{K_1}) + \ell(w_{J_1}) - \ell(w_{K_2}) + \ldots - \ell(w_{K_d}) + \ell(w_{J_d}).$$

Example. A 'regular' singular expression

$$\{w\} \leftrightharpoons [\varnothing + s - s + t - t + \cdots + u - u]$$

Definition [Williamson 2008, Elias-K. 2021]

A *reduced expression* of $p = W_I w W_J$ is a string

$$[[J_0 \supset K_1 \subset J_1 \supset K_2 \subset \cdots \supset K_d \subset J_d]],$$

with $J_0 = I$ and $J_d = J$, such that

$$\overline{p} = w_{J_0} w_{K_1}^{-1} w_{J_1} w_{K_2}^{-1} w_{J_2} \cdots w_{K_d}^{-1} w_{J_d}$$

and

$$\ell(\overline{p}) = \ell(w_{J_0}) - \ell(w_{K_1}) + \ell(w_{J_1}) - \ell(w_{K_2}) + \ldots - \ell(w_{K_d}) + \ell(w_{J_d}).$$

Example. A 'regular' singular expression

$$\{w\} \leftrightharpoons [\varnothing + s - s + t - t + \cdots + u - u]$$

is reduced if and only if $w = st \cdots u$ is reduced.

Singular braid relations

As noted above, we have the relations

$$[J+s+t] \leftrightharpoons [J+t+s] \qquad (\textit{upup relation})$$

$$[J-s-t] \leftrightharpoons [J-t-s] \qquad (\textit{downdown relation})$$

between reduced expressions.

Singular braid relations

As noted above, we have the relations

$$[J + s + t] \leftrightharpoons [J + t + s] \qquad \text{(\textit{upup relation})}$$

$$[J - s - t] \leftrightharpoons [J - t - s] \qquad \text{(\textit{downdown relation})}$$

between reduced expressions.

Some nontrivial relations

The *switchback relations* are of the form

$$[J + u_0 - u_d] \leftrightharpoons [J - u_1 + u_0 - u_2 + u_1 - u_3 + u_2 \ \cdots - u_{d-1} + u_{d-2} - u_d + u_{d-1}]$$

where u_i is a certain periodic sequence in (finitary) $J' = J \sqcup \{u_0\} \subseteq S$.

Singular braid relations

As noted above, we have the relations

$$[J+s+t] \leftrightharpoons [J+t+s] \qquad (\textit{upup relation})$$

$$[J-s-t] \leftrightharpoons [J-t-s] \qquad (\textit{downdown relation})$$

between reduced expressions.

Some nontrivial relations

The *switchback relations* are of the form

$$[J+u_0-u_d] \leftrightharpoons [J-u_1+u_0-u_2+u_1-u_3+u_2 \ \cdots -u_{d-1}+u_{d-2}-u_d+u_{d-1}]$$

where u_i is a certain periodic sequence in (finitary) $J' = J \sqcup \{u_0\} \subseteq S$.

The u_i sequence and the number d depend on $u_0 \neq u_1$ and the type of J'.

Singular braid relations

As noted above, we have the relations

$$[J+s+t] \leftrightharpoons [J+t+s] \qquad (\textit{upup relation})$$

$$[J-s-t] \leftrightharpoons [J-t-s] \qquad (\textit{downdown relation})$$

between reduced expressions.

Some nontrivial relations

The *switchback relations* are of the form

$$[J+u_0-u_d] \leftrightharpoons [J-u_1+u_0-u_2+u_1-u_3+u_2 \ \cdots -u_{d-1}+u_{d-2}-u_d+u_{d-1}]$$

where u_i is a certain periodic sequence in (finitary) $J' = J \sqcup \{u_0\} \subseteq S$.

The u_i sequence and the number d depend on $u_0 \neq u_1$ and the type of J'.

We call these (singular) *braid relations*.

Singular Coxeter presentation

Singular Matsumoto Theorem [Elias-K.]

For finitary $I, J \subseteq S$ and a double coset $p \in W_I \backslash W / W_J$, any two reduced expressions of p are related by braid relations (upup, downdown, and switchback).

Singular Coxeter presentation

Singular Matsumoto Theorem [Elias-K.]

For finitary $I, J \subseteq S$ and a double coset $p \in W_I \backslash W / W_J$, any two reduced expressions of p are related by braid relations (upup, downdown, and switchback).

It follows that $\partial_p := \partial_{I_\bullet}$, where $I_\bullet$ is a reduced expression for p, is well-defined.

Singular Coxeter presentation

Singular Matsumoto Theorem [Elias-K.]

For finitary $I, J \subseteq S$ and a double coset $p \in W_I \backslash W / W_J$, any two reduced expressions of p are related by braid relations (upup, downdown, and switchback).

It follows that $\partial_p := \partial_{I_\bullet}$, where $I_\bullet$ is a reduced expression for p, is well-defined.

Non-reduced expressions? Recall the following zoomed-in version of $D_s D_s = 0$.

$$\partial_{[Js,J,Js]} = \partial_{[Js-s+s]} = 0. \qquad (\textit{nilquadratic relation})$$

Singular Coxeter presentation

Singular Matsumoto Theorem [Elias-K.]

For finitary $I, J \subseteq S$ and a double coset $p \in W_I \backslash W / W_J$, any two reduced expressions of p are related by braid relations (upup, downdown, and switchback).

It follows that $\partial_p := \partial_{I_\bullet}$, where $I_\bullet$ is a reduced expression for p, is well-defined.

Non-reduced expressions? Recall the following zoomed-in version of $D_s D_s = 0$.

$$\partial_{[Js,J,Js]} = \partial_{[Js-s+s]} = 0. \qquad (\textit{nilquadratic relation})$$

Presentation Theorem [Elias-K., Elias-K.-Libedinsky-Patimo]

The braid and nilquadratic relations generate all relations beteewn compositions of Demazure operators. That is, $\mathcal{D}$ has a presentation by the generators $\partial_{[I,Is]}, \partial_{[Is,I]}$ and the above relations.

Singular Coxeter presentation

Singular Matsumoto Theorem [Elias-K.]

For finitary $I, J \subseteq S$ and a double coset $p \in W_I \backslash W / W_J$, any two reduced expressions of p are related by braid relations (upup, downdown, and switchback).

It follows that $\partial_p := \partial_{I_\bullet}$, where $I_\bullet$ is a reduced expression for p, is well-defined.

Non-reduced expressions? Recall the following zoomed-in version of $D_s D_s = 0$.

$$\partial_{[Js,J,Js]} = \partial_{[Js-s+s]} = 0. \qquad (\textit{nilquadratic relation})$$

Presentation Theorem [Elias-K., Elias-K.-Libedinsky-Patimo]

The braid and nilquadratic relations generate all relations beteewn compositions of Demazure operators. That is, $\mathcal{D}$ has a presentation by the generators $\partial_{[I,Is]}, \partial_{[Is,I]}$ and the above relations. Moreover, the category $\mathcal{D}$ has a basis

$$\{\partial_p \mid p \in W_I \backslash W / W_J, I, J \subset S \text{ finitary}\}.$$

Core double cosets

Let $p \in W_I \backslash W / W_J$ be a finitary double coset.

Core double cosets

Let $p \in W_I \backslash W / W_J$ be a finitary double coset.

Definition. The *left (resp. right) redundancy* of p is the subset

$$\mathrm{LR}(p) = I \cap \underline{p} J \underline{p}^{-1} \subset I, \quad \text{resp.,} \quad \mathrm{RR}(p) = \underline{p}^{-1} I \underline{p} \cap J \subset J,$$

Core double cosets

Let $p \in W_I \backslash W / W_J$ be a finitary double coset.

Definition. The *left (resp. right) redundancy* of p is the subset

$$\mathrm{LR}(p) = I \cap \underline{p} J \underline{p}^{-1} \subset I, \quad \text{resp.,} \quad \mathrm{RR}(p) = \underline{p}^{-1} I \underline{p} \cap J \subset J,$$

Definition. We call p a *core coset* if $\mathrm{LR}(p) = I$ and $\mathrm{RR}(p) = J$.

Core double cosets

Let $p \in W_I \backslash W / W_J$ be a finitary double coset.

Definition. The *left (resp. right) redundancy* of p is the subset

$$\mathrm{LR}(p) = I \cap \underline{p} J \underline{p}^{-1} \subset I, \quad \text{resp.,} \quad \mathrm{RR}(p) = \underline{p}^{-1} I \underline{p} \cap J \subset J,$$

Definition. We call p a *core coset* if $\mathrm{LR}(p) = I$ and $\mathrm{RR}(p) = J$.

Remark. A coset $W_I w W_J$ is core if and only if $W_I w W_J = w W_J = W_I w$.

Core double cosets

Let $p \in W_I \backslash W / W_J$ be a finitary double coset.

Definition. The *left (resp. right) redundancy* of p is the subset

$$\mathrm{LR}(p) = I \cap \underline{p} J \underline{p}^{-1} \subset I, \quad \text{resp.,} \quad \mathrm{RR}(p) = \underline{p}^{-1} I \underline{p} \cap J \subset J,$$

Definition. We call p a *core coset* if $\mathrm{LR}(p) = I$ and $\mathrm{RR}(p) = J$.

Remark. A coset $W_I w W_J$ is core if and only if $W_I w W_J = w W_J = W_I w$.

Example. Any $W_\varnothing w W_\varnothing = \{w\}$ is core.

Core double cosets

Let $p \in W_I \backslash W / W_J$ be a finitary double coset.

Definition. The *left (resp. right) redundancy* of p is the subset

$$\mathrm{LR}(p) = I \cap \underline{p} J \underline{p}^{-1} \subset I, \quad \text{resp.,} \quad \mathrm{RR}(p) = \underline{p}^{-1} I \underline{p} \cap J \subset J,$$

Definition. We call p a *core coset* if $\mathrm{LR}(p) = I$ and $\mathrm{RR}(p) = J$.

Remark. A coset $W_I w W_J$ is core if and only if $W_I w W_J = w W_J = W_I w$.

Example. Any $W_\emptyset w W_\emptyset = \{w\}$ is core.

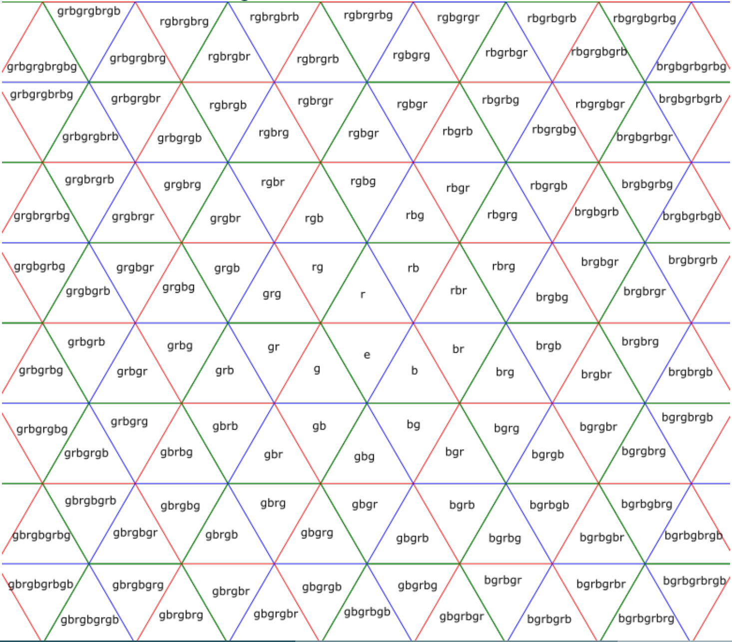
The low road theorem [Elias-K. 2021]

Given $p \in W_I \backslash W / W_J$, the double coset $p^{core} := W_{\mathrm{LR}(p)} \underline{p} W_{\mathrm{RR}(p)}$ is a core coset. Moreover, if $p^{core} \leftrightharpoons M_\bullet$ is a reduced expression then

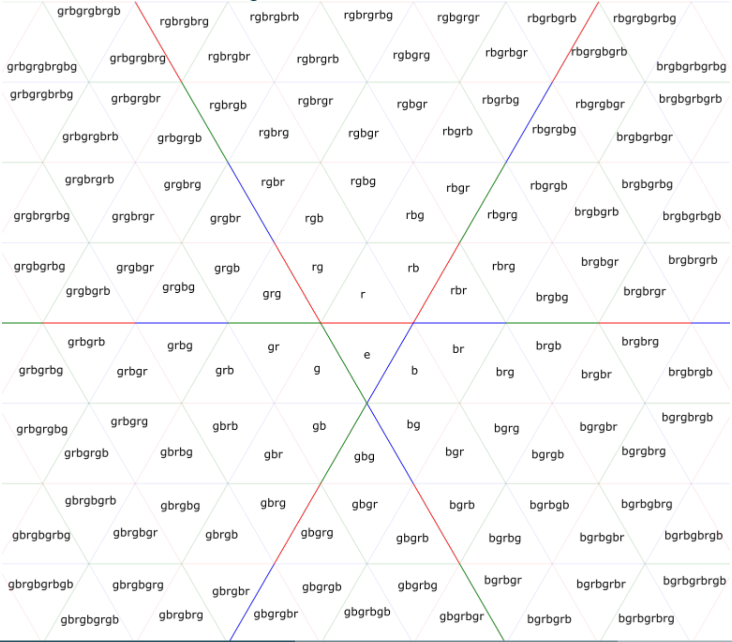
$$p \leftrightharpoons [[I \supset \mathrm{LR}(p)]] \circ M_\bullet \circ [[\mathrm{RR}(p) \subset J]]$$

is a reduced expression.

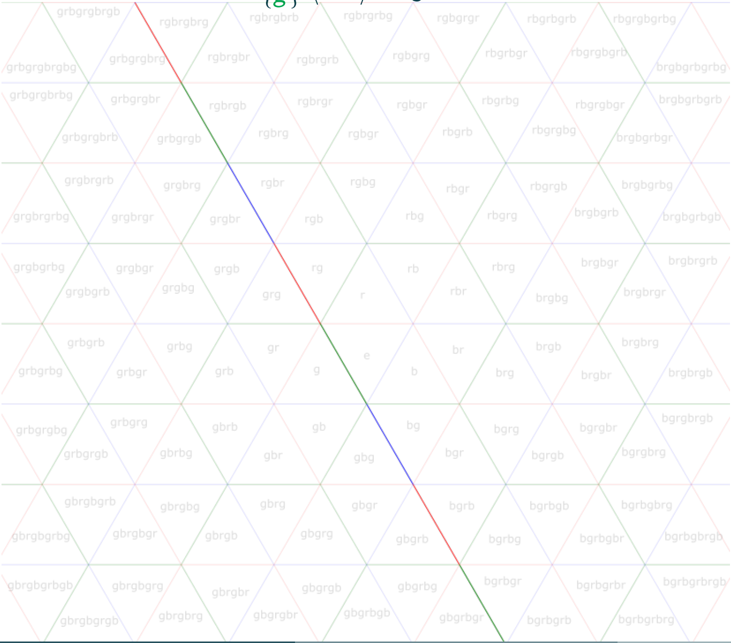
Left cosets in $\widetilde{S}_3$



Core cosets in $\widetilde{S}_3$

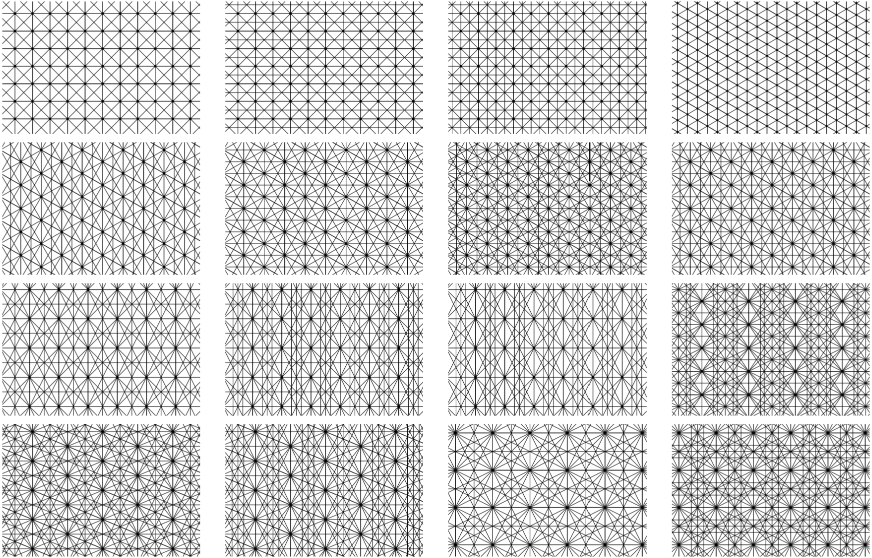


Core cosets in $W_{\{g\}}\backslash W/W_J$, where $J \subset S$ runs



Core cosets in $W_I \backslash W / W_J$, where $J \subset S$ runs

is called *Tits cone intersection* by Iyama-Wemyss whose pictures I paste:



Atomic expressions

Definition. An *atom* is a core coset with a (unique) reduced expression

$$\mathtt{a} \leftrightharpoons [I + s - t].$$

Atomic expressions

Definition. An *atom* is a core coset with a (unique) reduced expression

$$\mathtt{a} \leftrightharpoons [I + s - t].$$

Example. For each $s \in S$ the coset $\{s\} \leftrightharpoons [\varnothing + s - s]$ is an atom.

Atomic expressions

Definition. An *atom* is a core coset with a (unique) reduced expression

$$\mathsf{a} \leftrightharpoons [I + s - t].$$

Example. For each $s \in S$ the coset $\{s\} \leftrightharpoons [\varnothing + s - s]$ is an atom.

Definition. An *atomic (reduced) expression* is a(n reduced) expression of the form

$$p \leftrightharpoons \mathsf{a} \circ \mathsf{a}' \circ \cdots \circ \mathsf{a}'' := [I + s - t + s' - t' + \cdots + s'' - t'']$$

where $\mathsf{a}, \mathsf{a}', \cdots, \mathsf{a}''$ are atoms. (Or think of

$$\partial_p = \partial_{\mathsf{a}} \partial_{\mathsf{a}'} \cdots \partial_{\mathsf{a}''} = \partial_{[I+s-t+s'-t'+\cdots+s''-t'']}.)$$

Atomic expressions

Definition. An *atom* is a core coset with a (unique) reduced expression

$$\mathrm{a} \leftrightharpoons [I + s - t].$$

Example. For each $s \in S$ the coset $\{s\} \leftrightharpoons [\varnothing + s - s]$ is an atom.

Definition. An *atomic (reduced) expression* is a(n reduced) expression of the form

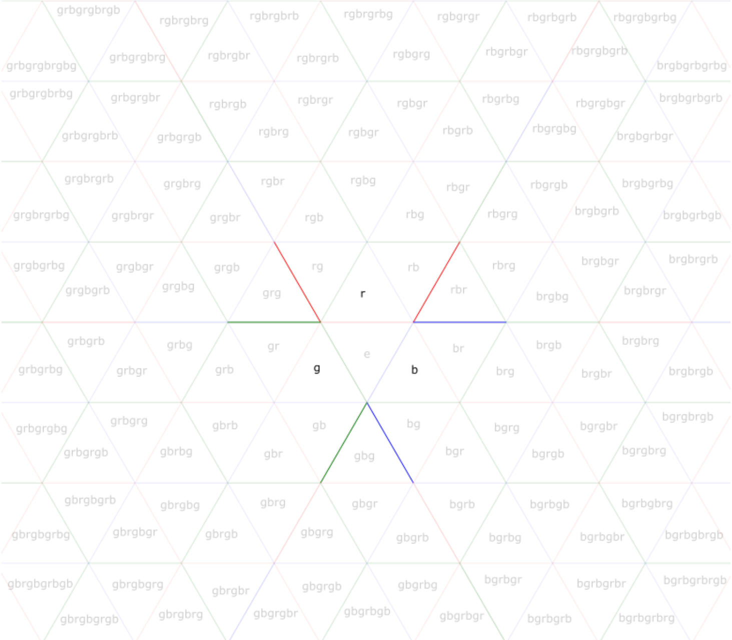
$$p \leftrightharpoons \mathrm{a} \circ \mathrm{a}' \circ \cdots \circ \mathrm{a}'' := [I + s - t + s' - t' + \cdots + s'' - t'']$$

where $\mathrm{a}, \mathrm{a}', \cdots, \mathrm{a}''$ are atoms. (Or think of

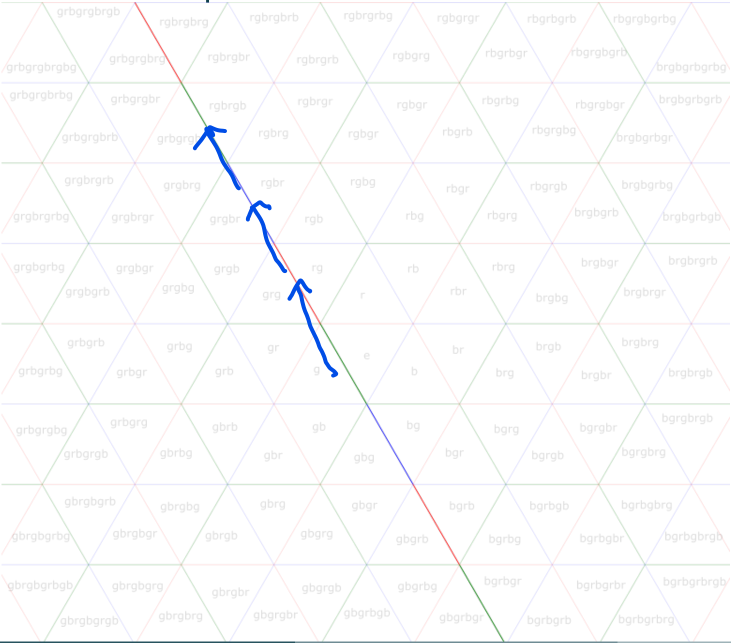
$$\partial_p = \partial_{\mathrm{a}} \partial_{\mathrm{a}'} \cdots \partial_{\mathrm{a}''} = \partial_{[I+s-t+s'-t'+\cdots+s''-t'']}.)$$

Example. For a regular (reduced) expression $w = st \cdots u$ in W, the expression $\{w\} \leftrightharpoons [\varnothing + s - s + t - t + \cdots + u - u]$ is an atomic (reduced) expression.

Atoms in $\widetilde{S}_3$



An atomic expression



Atomic Coxeter presentation

Atomic Coxeter presentation

Let $\mathcal{D}^{at}$ be the subcategory of $\mathcal{D}$ generated by the atoms, i.e., the morphisms $\partial_{\mathtt{a}}$ where $\mathtt{a}$ are atoms.

Atomic Coxeter presentation

Let $\mathcal{D}^{at}$ be the subcategory of $\mathcal{D}$ generated by the atoms, i.e., the morphisms $\partial_{\mathtt{a}}$ where $\mathtt{a}$ are atoms.

Theorem [Elias-K.-Libedinsky-Patimo, K.]

The category $\mathcal{D}^{at}$ has a basis

$$\{\partial_p \mid p \text{ is a core double coset}\}$$

Atomic Coxeter presentation

Let $\mathcal{D}^{at}$ be the subcategory of $\mathcal{D}$ generated by the atoms, i.e., the morphisms $\partial_{\mathtt{a}}$ where $\mathtt{a}$ are atoms.

Theorem [Elias-K.-Libedinsky-Patimo, K.]

The category $\mathcal{D}^{at}$ has a basis

$$\{\partial_p \mid p \text{ is a core double coset}\}$$

and a presentation by generators (the atoms $\partial_{\mathtt{a}}$) and the following relations:

- *atomic braid relations*

$$\underbrace{\partial_{\mathtt{a}}\partial_{\mathtt{b}'}\partial_{\mathtt{a}''}\cdots}_{m} = \underbrace{\partial_{\mathtt{b}}\partial_{\mathtt{a}'}\partial_{\mathtt{b}''}\cdots}_{m},$$

  where $\mathtt{a}', \mathtt{a}'', \cdots$ and $\mathtt{b}', \mathtt{b}'', \cdots$ are certain twists of the atoms $\mathtt{a}$ and $\mathtt{b}$; and $m = m_{\mathtt{a},\mathtt{b}} \geqslant 2$ is an integer determined by a switchback relation.

- *atomic nilquadratic relations*

$$\partial_{\mathtt{a}}\partial_{\mathtt{a}^{-1}} = 0.$$

Tilting modules for preprojective algebras

Let $Q = (Q_0, Q_1)$ be a quiver and let $\overline{Q} = (Q_0, Q_1 \sqcup Q_1^{op})$ be the double quiver.

Tilting modules for preprojective algebras

Let $Q = (Q_0, Q_1)$ be a quiver and let $\overline{Q} = (Q_0, Q_1 \sqcup Q_1^{op})$ be the double quiver.

We will assume that Q is extended Dynkin, i.e., the Dynkin diagram of a (simply-laced) affine Coxeter type.

Tilting modules for preprojective algebras

Let $Q = (Q_0, Q_1)$ be a quiver and let $\overline{Q} = (Q_0, Q_1 \sqcup Q_1^{op})$ be the double quiver.

We will assume that Q is extended Dynkin, i.e., the Dynkin diagram of a (simply-laced) affine Coxeter type. Let (W, S) be this Coxeter group.

Tilting modules for preprojective algebras

Let $Q = (Q_0, Q_1)$ be a quiver and let $\overline{Q} = (Q_0, Q_1 \sqcup Q_1^{op})$ be the double quiver. We will assume that Q is extended Dynkin, i.e., the Dynkin diagram of a (simply-laced) affine Coxeter type. Let (W, S) be this Coxeter group.

Definition. The associated *preprojective algebra* Π is the completion of the path algebra for $\overline{Q}$ modulo the ideal $(\sum_{\alpha \in Q_1} \alpha\alpha^* - \alpha^*\alpha)$.

Tilting modules for preprojective algebras

Let $Q = (Q_0, Q_1)$ be a quiver and let $\overline{Q} = (Q_0, Q_1 \sqcup Q_1^{op})$ be the double quiver. We will assume that Q is extended Dynkin, i.e., the Dynkin diagram of a (simply-laced) affine Coxeter type. Let (W, S) be this Coxeter group.

Definition. The associated *preprojective algebra* Π is the completion of the path algebra for $\overline{Q}$ modulo the ideal $(\sum_{\alpha \in Q_1} \alpha\alpha^* - \alpha^*\alpha)$.

Theorem [Iyama-Reiten, Buan-Iyama-Reiten-Scott].

The ideals

$$I_s = (1 - e_s), \qquad \text{for } s \in Q_0 = S$$

and their products are tilting modules. Moreover, we have a bijection

$$W \to \{\text{(basic) tilting modules for } \Pi\}$$

given by $w \mapsto I_w = I_s I_t \cdots I_u$ where $w = st \cdots u$ is a reduced expression in (W, S).

Tilting theory for contracted preprojective algebras

In the same setting, for $J \subset S = Q_0$ consider the idempotent $e_J = 1 - \sum_{s \in J} e_s$ and the *contracted preprojective algebra*

$$\Gamma_J = e_J \Pi e_J.$$

Tilting theory for contracted preprojective algebras

In the same setting, for $J \subset S = Q_0$ consider the idempotent $e_J = 1 - \sum_{s \in J} e_s$ and the *contracted preprojective algebra*

$$\Gamma_J = e_J \Pi e_J.$$

Theorem [Iyama-Wemyss, *Tits Cone Intersections and Applications*].

Let Q be extended Dynkin. Then we have a bijection

$$\{\text{core double cosets in } (W, S)\} \to \bigsqcup_{J \subsetneq S} \{\text{tilting modules for } \Gamma_J\}$$

given by $W_J x W_I \longmapsto e_J I_x e_I$.

Tilting theory for contracted preprojective algebras

In the same setting, for $J \subset S = Q_0$ consider the idempotent $e_J = 1 - \sum_{s \in J} e_s$ and the *contracted preprojective algebra*

$$\Gamma_J = e_J \Pi e_J.$$

Theorem [Iyama-Wemyss, *Tits Cone Intersections and Applications*].

Let Q be extended Dynkin. Then we have a bijection

$$\{\text{core double cosets in } (W,S)\} \to \bigsqcup_{J \subsetneq S} \{\text{tilting modules for } \Gamma_J\}$$

given by $W_J x W_I \longmapsto e_J I_x e_I$. Moreover, a reduced composition $- \circ a$ by an atom corresponds to a tilting mutation on the right side.

Theorem [Iyama-Wemyss, *Tits Cone Intersections and Applications*]

Let Q be extended Dynkin and fix $J \subsetneq S$. Then we have a bijection

$$\{\text{core double cosets in } W_J \backslash W / W_I\} \to \{\text{tilting modules for } \Gamma_J\}.$$

Theorem [Iyama-Wemyss], original version.

Let Q be extended Dynkin and fix $J \subsetneq S$. Then we have a bijection

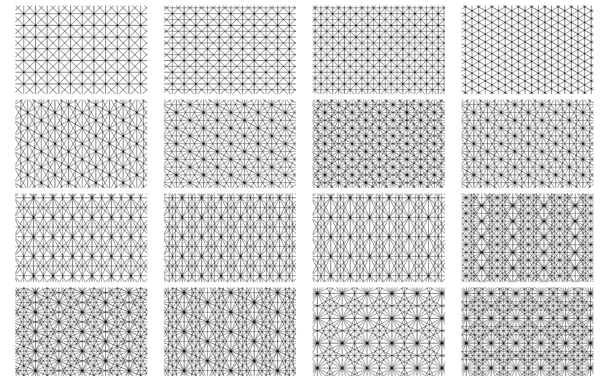
$$\{\text{chambers in the Tits } J\text{-cone}\} \to \{\text{tilting modules for } \Gamma_J\}.$$

Theorem [Iyama-Wemyss], original version.

Let Q be extended Dynkin and fix $J \subsetneq S$. Then we have a bijection

$$\{\text{chambers in the Tits } J\text{-cone}\} \to \{\text{tilting modules for } \Gamma_J\}.$$

If $|S \backslash J| = 3$ then the Tits J-cone is one of the following types:



An atomic Leibniz rule (in progress with ELP)

Recall the regular Demazure operator $D_s : R \to R$

$$D_s(f) = \frac{f - s(f)}{\alpha_s}, \qquad \text{(or, for } s = s_i \in S_n,\text{)} \quad D_{s_i}(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}.$$

An atomic Leibniz rule (in progress with ELP)

Recall the regular Demazure operator $D_s : R \to R$

$$D_s(f) = \frac{f - s(f)}{\alpha_s}, \qquad \text{(or, for } s = s_i \in S_n,\text{)} \quad D_{s_i}(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}.$$

The *(twisted) Leibniz rule* is well-known:

$$D_s(f \cdot g) = s(f)D_s(g) + D_s(f)g.$$

An atomic Leibniz rule (in progress with ELP)

Recall the regular Demazure operator $D_s : R \to R$

$$D_s(f) = \frac{f - s(f)}{\alpha_s}, \qquad \text{(or, for } s = s_i \in S_n,\text{)} \quad D_{s_i}(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}.$$

The *(twisted) Leibniz rule* is well-known:

$$D_s(f \cdot g) = s(f)D_s(g) + D_s(f)g.$$

Singular analogue.

An atomic Leibniz rule (in progress with ELP)

Recall the regular Demazure operator $D_s : R \to R$

$$D_s(f) = \frac{f - s(f)}{\alpha_s}, \qquad \text{(or, for } s = s_i \in S_n,\text{)} \quad D_{s_i}(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}.$$

The *(twisted) Leibniz rule* is well-known:

$$D_s(f \cdot g) = s(f) D_s(g) + D_s(f) g.$$

Singular analogue. An *atomic Leibniz rule* (for an atom $\mathtt{a}$) is

$$\partial_{\mathtt{a}}(f \cdot g) = \underline{p}(f)\partial_{\mathtt{a}}(g) + \sum_{p < \mathtt{a}} \partial_p(T_p(f) \cdot g) \tag{$\star$}$$

where $T_p(f)$ is some element depending on the coset $p < \mathtt{a}$ and we use the Bruhat order on double cosets.

An atomic Leibniz rule (in progress with ELP)

Recall the regular Demazure operator $D_s : R \to R$

$$D_s(f) = \frac{f - s(f)}{\alpha_s}, \qquad \text{(or, for } s = s_i \in S_n,\text{)} \quad D_{s_i}(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}.$$

The *(twisted) Leibniz rule* is well-known:

$$D_s(f \cdot g) = s(f)D_s(g) + D_s(f)g.$$

Singular analogue. An *atomic Leibniz rule* (for an atom $\mathtt{a}$) is

$$\partial_{\mathtt{a}}(f \cdot g) = \underline{p}(f)\partial_{\mathtt{a}}(g) + \sum_{p<\mathtt{a}} \partial_p(T_p(f) \cdot g) \qquad (\star)$$

where $T_p(f)$ is some element depending on the coset $p < \mathtt{a}$ and we use the Bruhat order on double cosets.

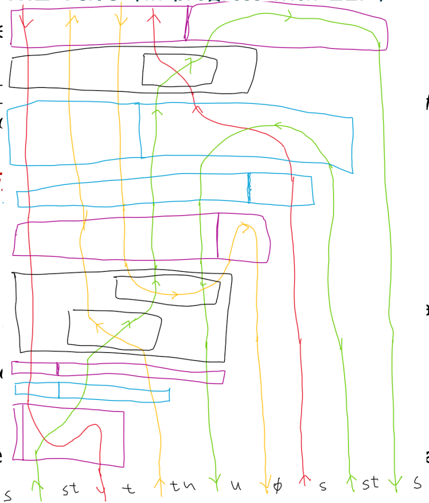
Remark. The summands in $(\star)$ have interpretation in terms of singular light leaves, a basis of the singular Hecke category, aka singular Soergel bimodules, and $(\star)$ is equivalent to an essential property of singular Bott-Samelson bimodules for singular Soergel calculus.

An atomic Leibniz rule (in progress with ELP)

Recall the regular De

$$D_s(f) = \frac{f - \cdots}{\cdots} \quad \cdots f) = \frac{f - s_i(f)}{x_i - x_{i+1}}.$$

The *(twisted) Leibni*



Singular analogue. ... a) is

$$\cdots g) \qquad (\star)$$

where $T_p(f)$ is some ... a and we use the Bruhat order on dou

Remark. The summands in $(\star)$ have interpretation in terms of singular light leaves, a basis of the singular Hecke category, aka singular Soergel bimodules, and $(\star)$ is equivalent to an essential property of singular Bott-Samelson bimodules for singular Soergel calculus.

Papers in Singularland (boldface for cited)

Elias-K., **A Singular Coxeter presentation**, arXiv:2105.08563 (2023)

The Singular Land (with Elias, Libedinsky, Patimo), Season 1 (2023-2024)

EKLP, **Demazure operators for double cosets**, arXiv:2307.15021

EKLP, Subexpressions and the Bruhat order for double cosets, arXiv:2307.15726

EKLP, **On reduced expressions for core double cosets** (to be posted)

EKLP, Singular Light Leaves, arXiv:2401.03053

K, **An Atomic Coxeter presentation**, arXiv:2312.16666

The Singular Land Season 2 is coming:

Atomic Leibniz rule, Singular Soergel calculus, ...