

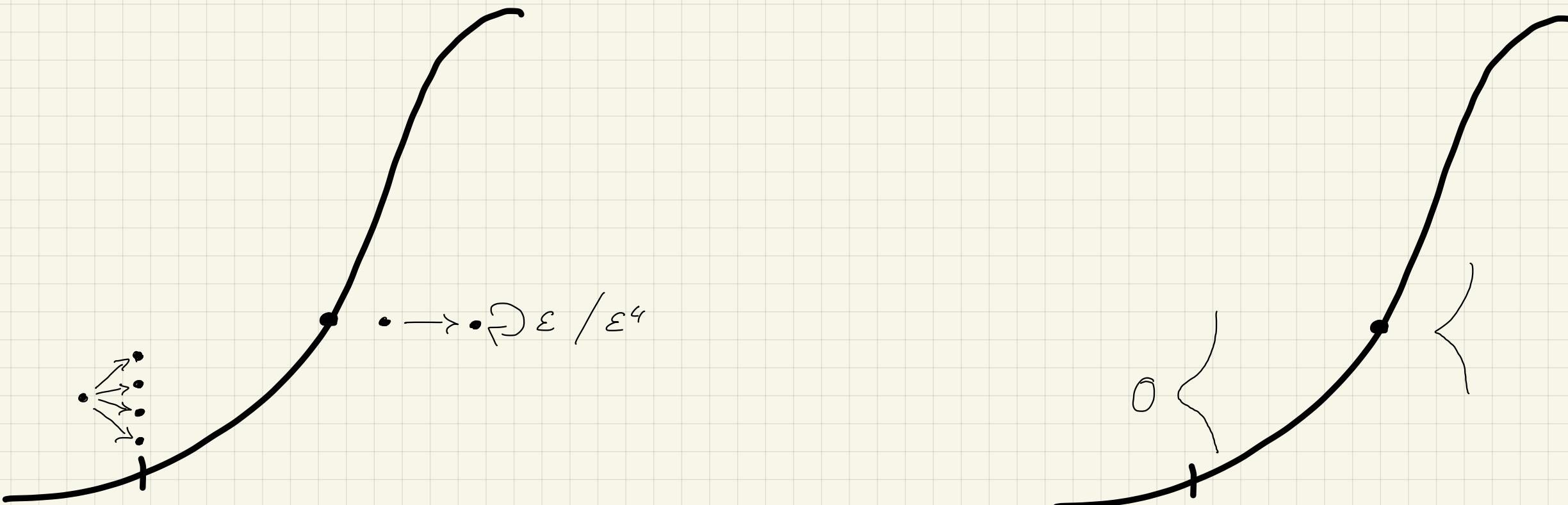
On \mathfrak{I} -representation types
with examples from
the representation theory of valued quivers

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FD Seminar

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Setting

K : algebraically closed field of characteristic 0.

$A = \frac{KQ}{I}$ basic finite-dimensional associative and unital K -algebra.
↑ ↓
quiver admissible ideal

$\text{mod}(A) := \underset{\text{ul}}{\text{category}}$ of finite-dimensional left A -modules.

$\text{proj}(A) :=$ full subcategory of projective A -modules.

\mathfrak{T}_A : Auslander-Reiten translation for A .

$K_0(A) := \underset{\text{ul}}{K_0(\text{mod } A)} = \mathbb{Z} Q_0$. Grothendieck group of $\text{mod}(A)$.

$K_0(A)^+ = \mathbb{N} Q_0$. submonoid of dimension vectors.

$\text{Rep}(A, d) :=$ affine variety of K -linear representations of Q
with dimension vector d and satisfying relations in I .

$\text{Irr}(A) := \{ \text{irreducible components } \mathcal{Z} \subseteq \text{Rep}(A, d), d \in K_0(A)^+ \}$

$K_0(\text{proj } A)_\mathbb{R} := K_0(\text{proj } A) \otimes_{\mathbb{Z}} \mathbb{R}$

$K_0(A)_\mathbb{R}^* := \text{Hom}_{\mathbb{Z}}(K_0(A), \mathbb{R})$

Euler pairing: $\langle -, - \rangle_A : K_0(\text{proj } A) \times K_0(A) \rightarrow \mathbb{Z}$, $\langle P, V \rangle := \dim_K \text{Hom}_A(P, V)$.

Σ-tilting theory

Def: [AIR] $(V, P) \in \text{mod}(A) \times \text{proj}(A)$

is Σ -rigid : if $\begin{aligned} * & \text{Hom}_A(P, V) = 0 \\ * & \text{Hom}_A(V, \Sigma_A V) = 0 \end{aligned}$

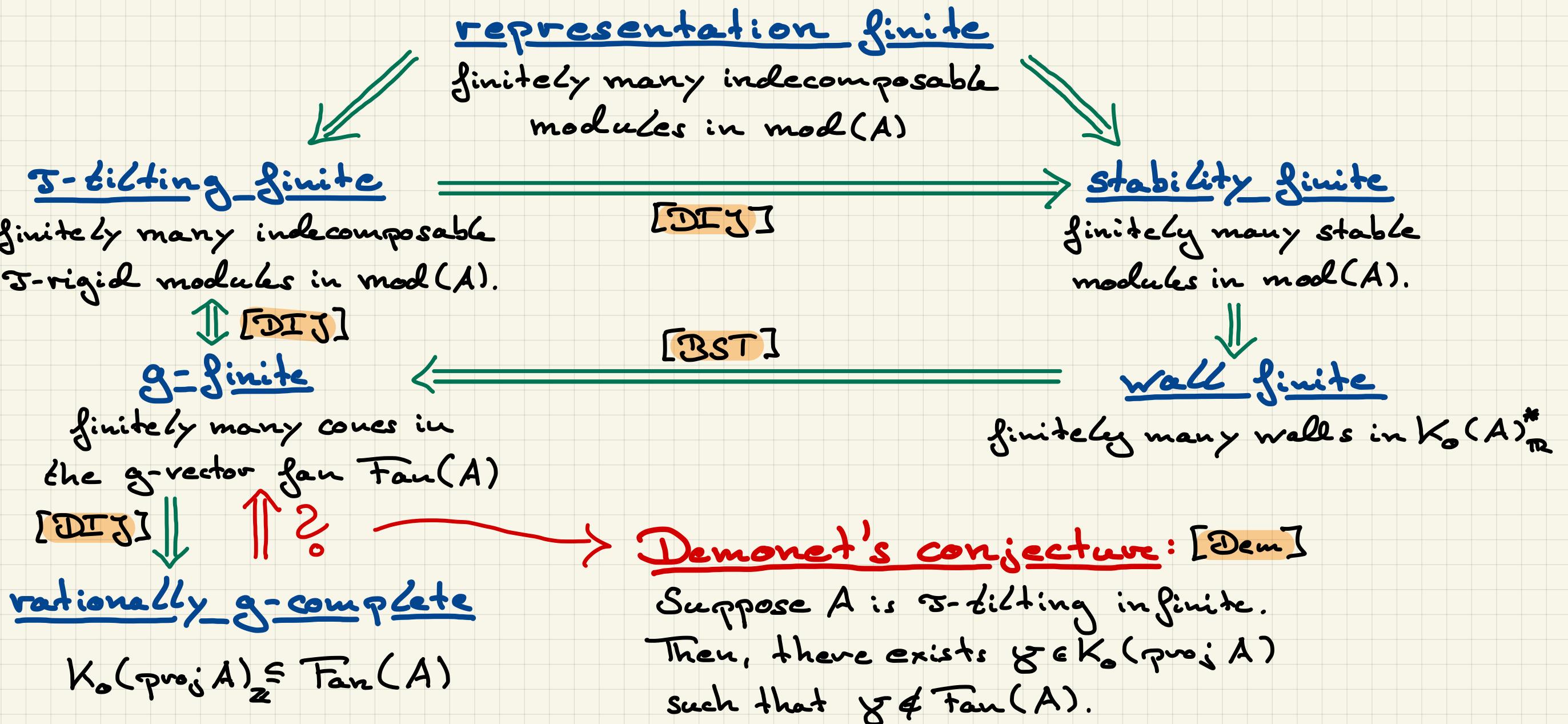
Stability

Def: [King] $\Theta \in K_0(A)^*_\mathbb{R}$. $V \in \text{mod}(A)$

is Θ -semistable if $\begin{aligned} * & \Theta(V) = 0 \\ * & \Theta(U) \leq 0 \quad \forall 0 \neq U \in V. \end{aligned}$

it is Θ -stable if strict inequality holds.

Interplay of Σ -tilting theory and stability: [DIJ], [BST], [Asai], [AI], ...



Brauer-Thrall I

Thm: [Roiter] Suppose A is representation infinite.

For every $d > 0$ exists an indecomposable $V \in \text{mod}(A)$ with $\dim_K(V) \geq d$.

σ -rigid BTI

Prop: Suppose A is σ -tilting infinite.

For every $d > 0$ exists an indecomposable σ -rigid $V \in \text{mod}(A)$ with $\dim_K(V) \geq d$.

Pf: Let $d \in K_0(A)^+$.

$\text{Rep}(A, d) :=$ affine variety of representations of A with dimension vector d .

$GL(K, d) := \prod_{i \in Q_0} GL(K, d_i)$

$\forall V \in \text{Rep}(A, d) \rightsquigarrow \mathcal{O}(V) := GL(K, d).V$

$\forall V \text{ } \sigma\text{-rigid} \xrightarrow{\text{Voigt}} \mathcal{O}(V) \subseteq \text{Rep}(A, d) \text{ open}$
 $\xrightarrow{} \overline{\mathcal{O}(V)} \subseteq \text{Rep}(A, d) \text{ irreducible component.}$

$\hookrightarrow \exists i < \infty \text{ } \sigma\text{-rigid modules with dimension vector } d. \quad \square$

stable BTI

Thm: [ST] Suppose A is σ -tilting infinite.

For every $d > 0$ exists a stable $V \in \text{mod}(A)$ with $\dim_K(V) \geq d$.

Pf: [ST] using g- & c-vectors
works over arbitrary fields.

[MPT] using similar strategy as for σ -rigid BTI over $K = \bar{K}$.

Notation: $\text{Irr}(A) := \bigcup_{d \in K_0(A)^+} \{Z \mid Z \text{ is an irreducible component of } \text{Rep}(A, d)\}$

Brauer-Thrall II

Thm: [B] Suppose A is representation infinite.

There exist $d \in K_0(A)^+$ and ∞ -many indecomposable $V \in \text{mod}(A)$ with $\dim(V) = d$.

Def: [GLS'12] $\mathcal{Z} \in \text{Irr}(A)$ is \mathfrak{S} -reduced if

$$c_A(\mathcal{Z}) = \text{hom}_A^{\mathfrak{S}}(\mathcal{Z}) := \min_{V \in \mathcal{Z}} \dim_K \text{hom}_A(V, \mathfrak{S}_A V)$$

Rk: " \leq " holds by a lemma of Voigt

$$\begin{array}{ccc} \checkmark & \{\mathcal{Z} \in \text{Irr}(A) \text{ } \mathfrak{S}\text{-rigid}\} / \cong \\ \downarrow & \uparrow 1:1 \\ \overline{\mathcal{O}(V)} & \{\mathcal{Z} \in \text{Irr}(A) \text{ } \mathfrak{S}\text{-reduced with } c_A(\mathcal{Z}) = 0\}. \end{array}$$

\mathfrak{S} -reduced BT II

Conj: Suppose A is representation infinite.

There exist $d \in K_0(A)^+$ and $\mathcal{Z} \in \text{Irr}(A)$ such that \mathcal{Z} is \mathfrak{S} -reduced with $c_A(\mathcal{Z}) \geq 1$.

Remarks on the history:

- BT I & II < 1950 first published in [Janus]
- brick BT II first introduced in [Mo2] stated with this name in [STV].
- Conj: [Mo2] Dense orbit property $\Rightarrow \mathfrak{S}$ -tilting finite.

Thm: [King] Let $\Theta \in K_0(A)^*$, $d \in K_0(A)$.

There exists a quasi-projective variety $\text{Mod}(A, d, \Theta)^{\text{st}}$

parametrizing isomorphism classes of Θ -stable A -modules $V \in \text{Rep}(A, d)$.

stable BT II

Conj: Suppose A is representation infinite.

There exist $d \in K_0(A)^+$ and $\Theta \in K_0(A)^*$ such that $\dim \text{Mod}(A, d, \Theta)^{\text{st}} \geq 1$.

Verified for:

- hereditary algebras. ✓
- minimal rep.-infinite special biserial and " " " " non-distributive [Mol18z]
- special biserial algebras [STV]
- biserial algebras [allP]

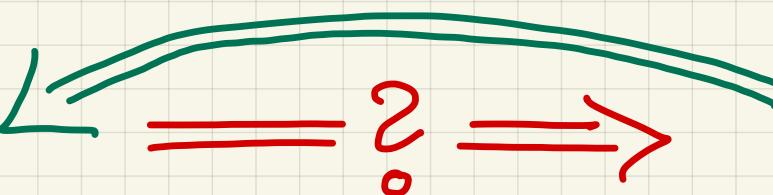
Brauer-Thrall II

Thm: [Bass] Suppose A is representation infinite.

There exist $d \in K_0(A)^+$ and ∞ -many indecomposable $V \in \text{mod}(A)$ with $\underline{\dim}(V) = d$.

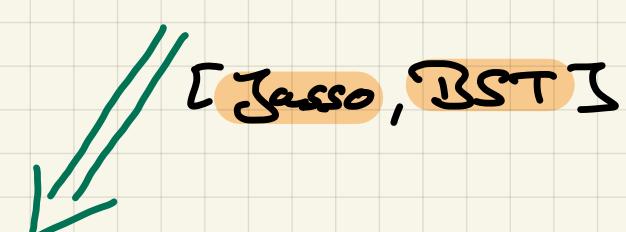
\mathfrak{T} -reduced BT II

Conj: Suppose A is \mathfrak{T} -tilting infinite.
There exist $d \in K_0(A)^+$ and $Z \in \text{Irr}(A)$
such that Z is \mathfrak{T} -reduced with $c_A(Z) > 1$.



stable BT II

Conj: Suppose A is \mathfrak{T} -tilting infinite.
There exist $d \in K_0(A)^+$ and $\Theta \in K_0(A)^*$
such that $\dim \text{Mod}(A, d, \Theta)^{\text{st}} \geq 1$.



Demonet's conjecture

Conj: Suppose A is \mathfrak{T} -tilting infinite.
Then, there exists $\gamma \in K_0(\text{proj } A)$
such that $\gamma \notin \text{Fan}(A)$.

Representation tame

Indecomposables of any fixed dimension are parametrized by finitely many points and rational curves.

[CC]

generically σ -reduced tame

For all generically indecomposable and σ -reduced $\mathbb{Z} \text{Covr}(A)$ is

$$c_A(\mathbb{Z}) \leq 1.$$

↓
[GLFS]

E-tame [DF, AI]

For all σ -reduced $\mathbb{Z} \text{Covr}(A)$ is

$$\hom_A^\sigma(\mathbb{Z}, \mathbb{Z}) = 0$$

[PY]

[CC]

stability tame

For all $\theta \in K_0(A)^*$, $d \in K_0(A)^+$ is
 $\dim \text{Mod}(A, d, \theta) \leq 1$

g-tame [AY]

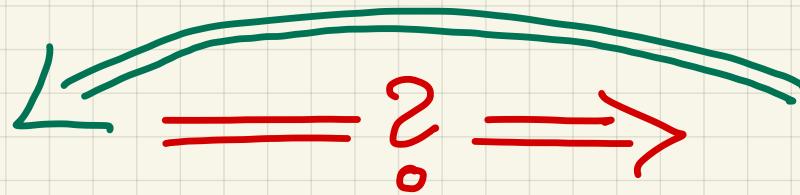
$$K_0(\text{Proj } A) \subseteq \overline{\text{Fan}(A)}$$

[A, PY]

wall tame [BST]

$$\overline{\bigcup_{V \neq 0} W(V)} \subseteq K_0(A)_\pi^* \text{ has finite measure.}$$

\mathfrak{S} -reduced BT II



stable BT II

Conj: Suppose A is \mathfrak{S} -tilting infinite.
There exist $d \in K_0(A)^+$ and $Z \in \text{Irr}(A)$
such that Z is \mathfrak{S} -reduced with $c_A(Z) \geq 1$.

Conj: Suppose A is \mathfrak{S} -tilting infinite.
There exist $d \in K_0(A)^+$ and $\Theta \in K_0(A)^*$
such that $\dim \text{Mod}(A, d, \Theta)^{\text{st}} \geq 1$.

[Pla]

[Jasso, BST]

Demonet's conjecture

Conj: Suppose A is \mathfrak{S} -tilting infinite.
Then, there exists $g \in K_0(\text{proj } A)$
such that $g \notin \text{Fan}(A)$.

Thm: [P'23] Suppose A is \mathfrak{S} -tame. Then, the following implication holds:

A satisfies \mathfrak{S} -reduced BT II $\implies A$ satisfies the stable BT II

Pf Idea: Given \mathfrak{S} -reduced $Z \in \text{Irr}(A)$ with $c_A(Z) \geq 1$

Consider $\Theta := \langle g(Z), - \rangle_A \in K_0(A)^*$

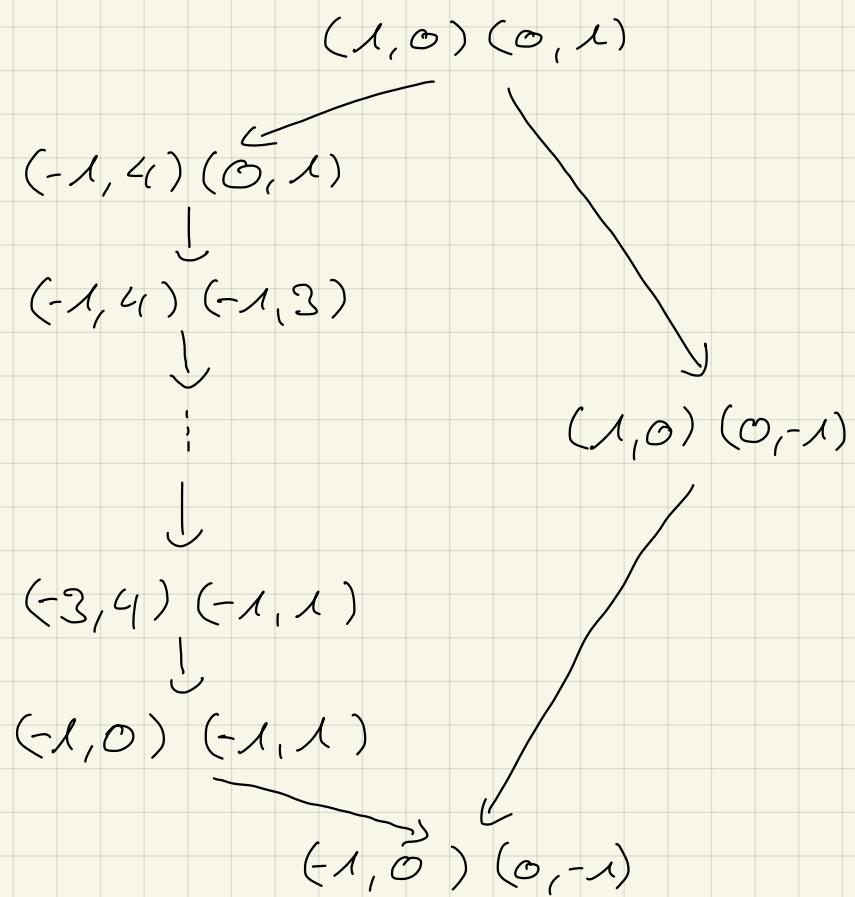
Construct Θ -stable modules as factors of generic elements of Z . \square

Rk: It is enough to prove Demonet's conjecture for \mathfrak{S} -tame algebras.

An example of affine type \widetilde{BC}_1

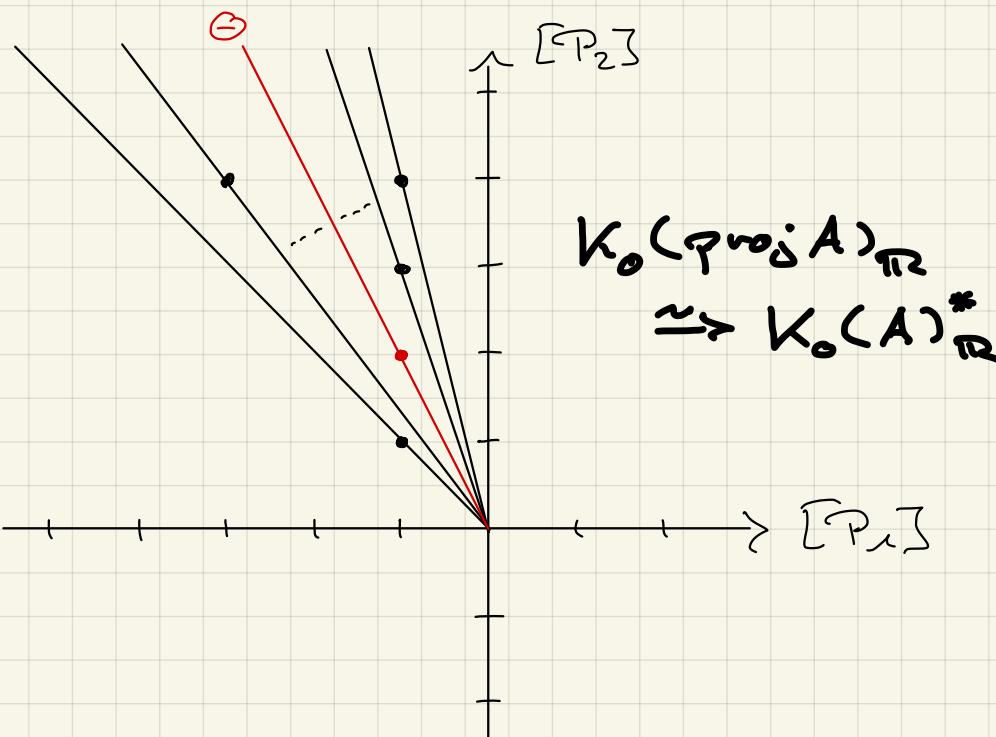
$A := KQ/I$, $Q : 2 \rightarrow 1 \vdash \varepsilon$, $I := \langle \varepsilon^4 \rangle$. is representation wild

\mathfrak{I} -tilting exchange quiver



wall & chamber structure

$$\Theta := (-1, 2) \in K_0(A)^*$$



$$K_0(\text{proj } A)_\mathbb{R} \cong K_0(A)_\mathbb{R}^*$$

- ⇒ * minimal \mathfrak{I} -tilting infinite [Ito, Wa]
- * wall & g-tame
- * stability tame
- * generically \mathfrak{I} -reduced tame

Θ -stable modules: $\bar{V}_\infty := K \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} K^2 \rightleftharpoons \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $V_\lambda := K^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} K^4 \rightleftharpoons \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ for $\lambda \in K$.

Thm: [P'23] These are pairwise flow-orthogonal with $\varpi_\lambda(V_\lambda) \cong V_\lambda \quad \forall \lambda \in K$. Moreover, every Θ -stable A -module is isomorphic to \bar{V}_∞ or V_λ for some $\lambda \in K$.

Valued quivers & GLS algebras

A valued quiver Γ consists of

- * $\Gamma_0 : \text{a finite set of vertices}$
- * $\Gamma_1 \subseteq \Gamma_0 \times \Gamma_0$ a set of **half edges** s.t. $(i, j) \in \Gamma_1 \Rightarrow (j, i) \in \Gamma_1$ and $i \neq j$.
- * $\Omega \subseteq \Gamma_1$ an **orientation** s.t. $(i, j) \in \Omega \Rightarrow (j, i) \notin \Omega$.
- * $\nu : \Gamma_1 \rightarrow \mathbb{N}_{\geq 1}$ a **valuation**
- * $\subseteq : \Gamma_0 \rightarrow \mathbb{N}$ a **symmetrizer** s.t. $c_i \nu_{ij} = c_j \nu_{ji} \quad \forall (j, i) \in \Gamma_1$.

$$i \xrightarrow[c_i]{\nu_{ii} \mid \nu_{ji}} j$$

Rk: Equivalent to the data (C, Ω, \mathcal{D}) of a generalized Cartan matrix C with orientation Ω and symmetrizer \mathcal{D} .

↪ [GLS'17] Quiver $Q = Q(\Gamma)$

- * $Q_0 := \Gamma_0$,
- * $Q_1 := \left\{ \alpha_{ji}^{(g)} : i \rightarrow j \mid (j, i) \in \Omega, 1 \leq g \leq g_{ji} := \gcd(\nu_{ji}, \nu_{ij}) \right\}$
 $\cup \{ \varepsilon_i : i \rightarrow i \mid i \in \Gamma_0 \}$.

with ideal $I \trianglelefteq KQ$ generated by relations

- * **nilpotency**: $\varepsilon_i^c = 0 \quad \forall i \in \Gamma_0$,
- * **commutativity**: $\varepsilon_j \delta_{ij} \alpha_{ji}^{(g)} = \alpha_{ji}^{(g)} \varepsilon_i \delta_{ji} \quad \forall (i, j) \in \Omega$.

$$\varepsilon_i : G_i \xrightarrow{\begin{array}{c} \alpha_{ii}^{(1)} \\ \vdots \\ \alpha_{ii}^{(g_{ii})} \end{array}} \bigoplus_j \varepsilon_j$$

↪ **GLS algebra**

$$H(\Gamma) := KQ / I.$$

Valued quivers & GLS algebras

↪ [GLS'17] Quiver $Q = Q(\Gamma)$

- * $Q_0 := \Gamma_0^*$,
- * $Q_1 := \{ \alpha_{ji}^{(g)} : i \rightarrow j \mid (j,i) \in \Omega, 1 \leq g \leq g_{ji} := \text{gcd}(v_{ji}, v_{ij}) \}$
 $\cup \{ e_i : i \rightarrow i \mid i \in \Gamma_0^* \}$.

with ideal $I \trianglelefteq KQ$ generated by relations

- * nilpotency: $e_i^c = 0 \quad \forall i \in \Gamma_0^*$,
- * commutativity: $e_j f_{ii} \alpha_{ji}^{(g)} = \alpha_{ji}^{(g)} e_i f_{ii} \quad \forall (i,j) \in \Omega$. when $f_{ii} := v_{ji} / g_{ji}$.

Thm: Let Γ be a valued quiver and $H := H(\Gamma)$.

- * [GLS'17] H is **1-tilting - Gorenstein**
- * [Pla'13] $\xrightarrow{\text{[Pla'13]}}$ $Z \in \text{Irr}(H)$ is **\mathfrak{S} -reduced** iff Z is **generically locally free**

$$\begin{aligned} [\text{GLS'18}]: \quad \{ Z \in \text{Irr}(H) \text{ gen. loc. fr.} \} &\xrightarrow{\cong} \text{IN} \Gamma_0^* \\ Z(v) &\longleftrightarrow v \end{aligned}$$

- * [GLS'20] H is a (formal) **degeneration of a species over $K((\varepsilon))$**

\Leftarrow H is **\mathfrak{S} -tilting finite** if and only if Γ is of **finite type**

\Leftarrow H is **g-tame** if and only if Γ is of **affine type**

$$e_i: G_i \xrightarrow{\alpha_{ji}^{(g)}} j \cap \mathcal{E}_j$$

$$\alpha_{ji}^{(g)}$$

$$\alpha_{ji}^{(g)}$$

↪ **GLS algebra**

$$H(\Gamma) := KQ / I.$$

Valued quivers & GLS algebras

Example: * Γ simply laced (i.e. $c_i = \lambda \forall i \in \Gamma_0^+$) $\Rightarrow H$ is ordinary path algebra.

* $\Gamma: 2 \xrightarrow{\lambda^{14}} 1$ affine type \widetilde{BC}_1 $\hookrightarrow H = K(2 \rightarrow 1) / \langle \varepsilon^4 \rangle$ $\xrightarrow[\varepsilon^4 = t c_1]{\text{defn.}}$ path alg. of $2 \begin{array}{c} \nearrow \\ \searrow \\ \swarrow \end{array} 1$

* $\Gamma: 3 \xrightarrow{\lambda^2} 2 \xrightarrow{\lambda^2} 1$ affine type \widetilde{C}_2 $\hookrightarrow H = K(3 \rightarrow 2 \rightarrow 1) / \langle \varepsilon_3^2, \varepsilon_1^2 \rangle$
 $\xrightarrow[\varepsilon_i^2 = t c_i]{\text{defn.}}$ path algebra of $3 \begin{array}{c} \xrightarrow{\varepsilon_3} 2 \xrightarrow{\varepsilon_2} 1 \\ \searrow \\ 3 \xrightarrow{\varepsilon_1} 1 \end{array}$

* $\Gamma: 4 \xrightarrow{\lambda^2} 3 \rightarrow 2 \xrightarrow{\lambda^2} 1$ affine type \widetilde{B}_3 $\hookrightarrow H = K(4 \rightarrow 3 \xleftarrow{\varepsilon_3} 2 \rightarrow 1) / \langle \varepsilon_3^2, \varepsilon_2^2, \varepsilon_2 \alpha = \alpha \varepsilon_3 \rangle$
 $\xrightarrow[\varepsilon_i^2 = t c_i]{\text{defn.}}$ path algebra of $4 \begin{array}{c} \xrightarrow{\varepsilon_3} 3 \xrightarrow{\varepsilon_2} 2 \rightarrow 1 \\ \searrow \\ 4 \xrightarrow{\varepsilon_1} 3 \xrightarrow{\varepsilon_2} 1 \end{array}$

* $\Gamma: 3 \xrightarrow{\lambda^2} 2 \xrightarrow{\lambda^2} 1$ affine type \widetilde{BC}_2 $\hookrightarrow H = K(3 \rightarrow 2 \xleftarrow{\varepsilon_2}, 1) / \langle \varepsilon_2^2, \varepsilon_1^2, \varepsilon_1 \alpha = \alpha \varepsilon_2 \rangle$

$\xrightarrow[\varepsilon_i^2 = t c_i]{\text{defn.}}$ path algebra of $3 \begin{array}{c} \xrightarrow{\varepsilon_2} 2 \rightarrow 1 \\ \searrow \\ 3 \xrightarrow{\varepsilon_1} 2 \xrightarrow{\varepsilon_1} 1 \end{array}$

Affine GLS algebras

Thm: [P'23] Suppose Γ is of affine type with minimal symmetrizer.

Then $\mathfrak{h} := \mathfrak{h}(\Gamma)$ is generically \mathfrak{G} -reduced tame.

More precisely, every $v \in \mathbb{N}\Gamma_0^+$ can be decomposed as

$$v = m \cdot \gamma + w, \quad (\text{"generalized Kac decomposition"})$$

for some $w \in \mathbb{N}\Gamma_0^+, m \in \mathbb{N}$ and $\gamma \in \mathbb{N}\Gamma_0^+ \setminus \{0\}$ the primitive null root

such that

$$\mathcal{Z}(v) = \overline{\mathcal{Z}(\gamma)^m \oplus \mathcal{Z}(w)},$$

$$\mathcal{Z}(w) = \overline{\mathcal{O}(w)},$$

$$\mathcal{Z}(\gamma) = \overline{\bigcup_{\lambda \in \mathbb{R}^\times} \mathcal{O}(\lambda \gamma)}$$

where $W \in \text{mod}(A)$ is \mathfrak{G} -rigid

and $V_\lambda \in \text{mod}(A)$ is an explicit 1-parameter family of locally free modules

Moreover, for almost all $\lambda \in \mathbb{R}^\times$ is

- * $\mathfrak{I}_H(V_\lambda) \cong V_\lambda$

- * V_λ is ∂ -stable for the defect $\partial := \langle \gamma, - \rangle_H : K_0(A)^*$.

Question: Are the V_λ together with the stable modules associated to \mathfrak{G} -rigid modules all stable \mathfrak{h} -modules i.e. is \mathfrak{h} stability tame?

c.f. [BDG] on plane degenerations of elliptic curves.

hand

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