THE CATEGORY OF LOCAL REPRESENTATIONS OF A FINITE GROUP

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The category of local representations of a finite group

Based on joint work with

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Isle of Skye, 2015

Inspiration for this project:

n^{th} Morava K-theory

structure of the $K(n)$-local category

of spectra in stable homotopy theory

Setting for this talk

or a finite group scheme

G a finite group, k a field of characteristic $p>0$

kG = group algebra (self-injective algebra)

$H^*(G,k) = \mathrm{Ext}^*_{kG}(k,k)$ group cohomology

(graded comm. noetherian algebra)

$\mathrm{Mod}\, kG$ = cat. of all kG-modules

$\mathrm{StMod}\, kG$ = stable category (modulo projectives)

- compactly generated triangulated category
- admits a tensor product $- \otimes_k -$ (diagonal action) and function objects $\mathrm{Hom}_k(-,-)$

$\rightsquigarrow$ tensor triangulated category

— " —

$\mathfrak{p} \in \mathrm{Proj}\, H^*(G,k)$ = set of homogeneous prime ideals different from $H^{>0}(G,k)$

$\Gamma_{\mathfrak{p}} : \mathrm{StMod}\, kG \longrightarrow \mathrm{StMod}\, kG$ local cohomology functor

$X \longmapsto X \otimes (\Gamma_{\mathfrak{p}} k)$ (exact and idempotent, preserving all $\oplus$s)

Rickard idempotent module

$\Gamma_{\mathfrak{p}}(\mathrm{StMod}\, kG)$ = minimal tensor ideal localising subcategory of $\mathrm{StMod}\, kG$

(= category of local representations of G)

Aim of this talk: Discuss the structure of $\Gamma_{\mathfrak{p}}(\mathrm{StMod}\, kG)$ as a tensor triangulated category (= smallest building block of $\mathrm{StMod}\, kG$).

Plan of this talk

- an analogy (highest weight categories)
- the local cohomology functor $\Gamma_{\mathfrak{p}}$
- compact/dualising objects in $\Gamma_{\mathfrak{p}}(\mathrm{StMod}\, kG)$
- an example (Klein four group)

Digression: FD Seminar:

finite dimensional algebras

versus finite dimensional representations

Highest weight categories via recollements

Recall: a recollement of abelian/triangulated categories is a diagram of functors

$$\begin{tikzcd} \mathcal{C}' \arrow[r, tail, "i"] & \mathcal{C} \arrow[l, bend right, two heads, "i_\lambda"'] \arrow[l, bend left, two heads, "i_q"] \arrow[r, two heads, "p"] & \mathcal{C}'' \arrow[l, bend right, tail, "p_\lambda"'] \arrow[l, bend left, tail, "p_q"] \end{tikzcd}$$

- $\operatorname{Ker} p = \operatorname{Im} i$
- (i_λ, i, i_q) and (p_λ, p, p_q) are adjoint triples
- i, p_λ, p_q are fully faithful

$\mathcal{A}$ an (abelian) length category

$\{L_\lambda\}_{\lambda \in \Lambda}$ a representative set of simple objects

$\Lambda = (\Lambda, \leq)$ finite poset (weights)

$\mathcal{A} \ni X \mapsto \operatorname{Supp} X := \{\lambda \in \Lambda \mid L_\lambda \text{ compos. factor of } X\}$
support

For $\mathcal{U} \subseteq \Lambda$

$$\mathcal{A}_\mathcal{U} := \{X \in \mathcal{A} \mid \operatorname{Supp} X \subseteq \mathcal{U}\} \subseteq \mathcal{A}$$

(Serre subcategory)

Suppose: There is a proj. cover $\Delta_\lambda \to L_\lambda$ in $\mathcal{A}_{\leq\lambda}$ $\lambda \in \Lambda$.

Cline-Parshall-Scott, 1988

<u>Theorem</u> $\mathcal{A}$ is a highest weight category with standard objects $\{\Delta_\lambda\}_{\lambda\in\Lambda}$ $\iff$

For each $\lambda \in \Lambda$ there is a recollement of abelian categories

$$\begin{tikzcd} \mathcal{A}_{<\lambda} \arrow[r] & \mathcal{A}_{\leq\lambda} \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[r, "p"] & \operatorname{mod} K_\lambda \arrow[l, shift right=2] \arrow[l, shift left=2] \end{tikzcd}$$

with $p = \operatorname{Hom}(\Delta_\lambda, -)$ and $K_\lambda = \operatorname{End}(\Delta_\lambda)$ a division ring, inducing a recollement of derived categories

$$\begin{tikzcd} D^b(\mathcal{A}_{<\lambda}) \arrow[r] & D^b(\mathcal{A}_{\leq\lambda}) \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[r] & D^b(\operatorname{mod} K_\lambda) \arrow[l, shift right=2] \arrow[l, shift left=2] \end{tikzcd}.$$

<u>Idea</u> Standard objects Δ_λ are building blocks of $\mathcal{A}$, glued via localisation sequences

$$\operatorname{Filt}\{\Delta_\mu \mid \mu < \lambda\} \rightleftarrows \operatorname{Filt}\{\Delta_\mu \mid \mu \leq \lambda\} \rightleftarrows \operatorname{Filt}\{\Delta_\lambda\}.$$

(given by exact functors)

$\operatorname{Filt}(\mathcal{X})$:= smallest extension closed subcategory containing $\mathcal{X}$

Representations of finite groups via recollements (analogy)

$$\mathrm{StMod}\, kG \ni X \longmapsto \operatorname{supp} X := \{ \mathfrak{p} \in \operatorname{Proj} H^*(G,k) \mid \Gamma_{\mathfrak{p}} X \neq 0 \}$$

support

$\operatorname{Proj} H^*(G,k)$ poset (ordered by inclusion)

Benson – Iyengar – K – Pevtsova, 2017

Theorem For each $\mathfrak{p} \in \operatorname{Proj} H^*(G,k)$ there is a recollement

functor $\Gamma_{\mathfrak{p}}$

$$\begin{tikzcd}[column sep=large]
(\mathrm{StMod}\, kG)_{<\mathfrak{p}} \arrow[r] & (\mathrm{StMod}\, kG)_{\leq \mathfrak{p}} \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \Gamma_{\mathfrak{p}}(\mathrm{StMod}\, kG) \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

Goal Identify the local building blocks (analogues of standard objects Δ_λ).

- The local cohomology functor Γ_p
- Compact/dualising objects in Γ_p (StMod kG)

Example

Klein four group $G = \mathbb{Z}/2 \times \mathbb{Z}/2$

char $k = 2$

$$kG \cong k[x,y]/(x^2,y^2)$$

$$\bullet \underset{y}{\overset{x}{\rightrightarrows}} \bullet \qquad\qquad x \circlearrowleft \bullet \circlearrowright y \Big/ \begin{matrix} x^2, y^2 \\ xy - yx \end{matrix}$$

$$X : \; V \rightrightarrows W \quad \longmapsto \quad \bar{X} : \; \circlearrowleft V \oplus W \circlearrowright$$

Kronecker repres. group repres.

$$H^*(G,k) \cong k[\zeta_0, \zeta_1] \qquad \zeta_i \in \mathrm{Ext}^1(k,k)$$

$$\mathrm{Proj}\, H^*(G,k) \cong \mathbb{P}^1_k$$

$\mathfrak{p} = (0)$ generic point

$$Q : \; k(t) \underset{t}{\overset{1}{\rightrightarrows}} k(t) \quad \longmapsto \quad \bar{Q} : \; \circlearrowleft k(t) \oplus k(t) \circlearrowright$$

generic repres.

$$\Gamma_{\mathfrak{p}}(\mathrm{StMod}\, kG) = \mathrm{Add}\, \bar{Q}$$

$\mathfrak{p} \in \mathrm{Proj}\, H^*(G,k)$ closed point

tube $\{R_{\mathfrak{p}}[n] \mid n \geq 1\}$ $\qquad R_{\mathfrak{p}}[\infty] = \varinjlim R_{\mathfrak{p}}[n]$

regular repres. Prüfer module

$$\Gamma_p(\mathrm{StMod}\, kG)^c = \mathrm{add}\left\{ \overline{R}_p[n] \;\middle|\; 1 \le n < \infty \right\}$$ compacts

$$\Gamma_p(\mathrm{stmod}\, kG) = \mathrm{add}\left\{ \overline{R}_p[n] \;\middle|\; 1 \le n \le \infty \right\}$$ dualising objects