

Stable Invariance of Structures on Hochschild Cohomology

Joint with
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and (partly)
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Part 1 Stable invariance of the "p power structure" on $\text{HH}^*(A)$

Part 2 Maximal tori in $\text{HH}^*(A)$ and the fundamental group

A is finite dim algebra over a field $k = \bar{k}$ (not needed)

$$A^{\text{ev}} = A^{\text{op}} \otimes A \quad \text{and} \quad \text{HH}^*(A) = \text{Ext}_{A^{\text{ev}}}^*(A, A)$$

↪ the Hochschild cohomology of A .

Fact $\text{HH}'(A) \cong \frac{\text{Der}_k(A, A)}{(\text{inner derivations})}$ is a finite dimensional Lie algebra over k

$[a, -]$

bracket: $[\alpha, \beta] = \alpha\beta - \beta\alpha$

Recent interest in the structure of $\text{HH}'(A)$ (Rydh, Linckelmann, Schroll, Solotar, Chaparro, Benson, Kessar, Eisele, Raedschelders ...)

If $\text{char } k = p > 0$ then $\text{HH}'(A)$ is a restricted Lie algebra

i.e. there is a p -power operation $\text{HH}'(A) \rightarrow \text{HH}'(A)$: $\alpha \mapsto \alpha^p$

check:

$$\alpha^p(xy) = \sum \binom{p}{i} \alpha^i(x) \alpha^{p-i}(y) = \alpha^p(x)y + x\alpha^p(y) \text{ is a derivation}$$

(and this respects inner derivations)

In fact all of $H\mathcal{H}^{>0}(A)$ is a graded restricted Lie algebra.

But focus on $H\mathcal{H}^1(A)$:

Example $A = \frac{k(x)}{x^p}$ $\partial_x(x^p) = px^{p-1} = 0$ $[x^i \partial_x, x^j \partial_x] = (j-i)x^{i+j-1} \partial_x$

$$\Rightarrow H\mathcal{H}^1(A) = \text{Span} \{ \partial_x, x\partial_x, \dots, x^{p-1}\partial_x \}$$

$$(x^i \partial_x)^p = \begin{cases} 0 & i \neq 1 \\ x\partial_x & i=1 \end{cases}$$

This is the Jacobson-Witt Lie algebra

(see RyD-Lindemann '18)

Motivation: Need the p power operation to do Lie theory in positive characteristic. e.g. $T \subseteq L$ is called a torus

If $[T, T] = 0$ and T is generated by elements with $x^p = x$
 $\text{if } k \text{ perfect}$

Eg (x_{λ}) is the only torus in the JW Lie algebra above.
rank = max dim of torus = 1 in this case.

↑
use this in second half.

Assume A is self injective: then the stable module category
 $\underline{\text{mod}} A$ is triangulated

cat of
 fg A -modules with $\underline{\text{Hom}}_A(M, N) = \text{Hom}_A(M, N)$
 (maps which factor
 through a projective)

Def'n (Broué 94) A stable equivalence of Morita type $A \xrightarrow{\sim_{\text{SEM}}^{\text{S}}} B$

is a pair of bimodules ${}_A^M{}_B$ ${}_B^N{}_A$ each projective on either side

$$M \otimes_B N \simeq A \text{ in } \underline{\text{mod}} A^{\text{op}} \quad \& \quad N \otimes_A M \simeq B \text{ in } \underline{\text{mod}} B^{\text{op}}.$$

This induces an equivalence of $\Delta^{\text{od cat}}$

$$\underline{\text{mod } A} \xrightleftharpoons[-\otimes N]{-\otimes M} \underline{\text{mod } B}$$

problem: does every equivalence come from a SEMT?

These are more general than derived equivalences

Example $A_4 \subseteq A_5$, then $k = 2$

Induces a SEMT $kA_4 \sim kA_5$ using $M = kA_5$

"because A_4 and A_5 have the same Sylow 2 subgroups" (Lückelmann's book) ^{See}

Big problem classify algebras (or groups) up to SEMT.

\Rightarrow want invariants

(as because Auslander Reiten bijection)

Hochschild cohomology is a derived Marita invariant

but $\mathrm{HH}^0(A) = Z(A)$ not stably invariant:

$$\text{eg } Z(kA_4) \not\cong Z(kA_5)$$

Theorem (Xi '02, König, Liu, Zhou '12)

A, B fd symmetric algebras and $A \xrightarrow{\sim_{\mathrm{SEM}}} B$, there is an iso

$$\mathrm{tr}_n: \mathrm{HH}^{>0}(A) \xrightarrow{\cong} \mathrm{HH}^{>0}(B)$$

"transfer map, Bouc"

also Lickelman

respecting the

cup product and
Gerstenhaber bracket

KLZ

- proof uses BV operator $\Delta : HH^* \rightarrow HH^{*-1}$

$$\Delta(xy) - \Delta(x)y - \textcolor{brown}{\Delta}^{(x)} \times \Delta(y) = [x, y]$$

- But it is impossible to write the p-power operation in terms of Δ (RgD thesis)

(And transfer maps do not respect p-power map in general)
— RgD v/z

So:

Question (Lückemann) Does the transfer map associated to a SEMT respect the p-power map ?

(RyD $\cap \mathbb{Z}$) Yes for $H\Gamma'_{int}(A) \subseteq H\Gamma'(A)$ the "integrable derivations" interesting but a different story

Theorem (- RyD '20)

If A, B are fd. self-injective algebras, $\text{char } k = p$, and $A \underset{\text{SENT}}{\sim} B$
 then $\text{tr}_n : \text{HH}^{>0}(A) \xrightarrow{\cong} \text{HH}^{>0}(B)$ isom of restricted graded Lie algebras.

So the restricted Lie algebra $H^1(A)$ is a stable invariant.

will
use in
part 2

- the proof
- uses the B_∞ structure on the Hochschild cochain complex $C^*(A)$
 - the p -power operation can be expressed in terms of the B_∞ structure (work of Turchin '06
also appendix of our paper)

Compare later work of Chen, Li, Wang

Note: B_∞ algebras defined by Jones - Getzler

But Gerstenhaber defined essentially the same thing in "On the def's of rings and algebras III" 68 → cool stuff.
Called them composition complexes.

Part 2 the fundamental group:

let Q be a quiver, $I \subseteq kQ$ admissible ideal,

$$A = \frac{kQ}{I} \text{ fd basic algebra}$$

- kQ has a basis of paths $\{p_i\}$
- I has a basis of minimal relations $\{r = \sum a_i p_i\}$
 - no proper sum of r is in I
- say $p_i \sim p_j$ if they both occur in a minimal relation.

- a walk is a path in $Q \cup Q^{-1}$
up to the equivalence relation generated by
 - $x \alpha \alpha^{-1} y \sim xy$
 - $x p_i y \sim x p_j y \text{ if } p_i \sim p_j$
- $\pi_1(Q, I) = \frac{\{ \text{walks } v \rightarrow v' \}}{\sim}$ this is a fg group using concatenation.

Pick $v \in Q_0$ vertex : doesn't depend on e upto isom

This definition is due to Martínez-Villa and de la Peña.

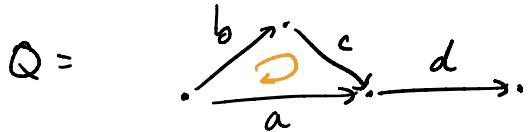
Example

$$\text{if } I = (0) \quad \pi_1(Q, I) = \pi_1(|Q|)$$

Note: the
same holds if
 I is monomial

the topological fundamental group of the
underlying graph of Q

Example (from Le Meur '05)



$$I = (da) \quad J = (da - dc, b)$$

$$\pi_1(Q, I) \cong \mathbb{Z}$$

$$\pi_1(Q, J) \cong 0$$

But

$$kQ/I \cong kQ/J$$

So the fundamental group depends on the choice of presentation...

We want to use π_1 as a stable invariant that tells us about the shape of the quiver, but it isn't even an invariant of A . **Don't worry!**

As (Q, I) varies (moduli space of presentations of A)

you different $\pi_1(Q, I)$ \rightarrow generically zero

\rightarrow special pts get maximal
fundamental groups (to mean)

Theorem (Assem - de la Peña '96, de la Peña Saenz '00)

For any presentation $A = kQ/I$ there is a canonical embedding

$$\text{Hom}_k(\pi_1(QI), k) \hookrightarrow \text{HH}^1(A)$$

and the image is a torus in $\text{HH}^1(A)$.

Question: which tori do you get?

(de Meur '10) You get all the maximal tori if either

- k has char 0 and Q has no double bypasses
- A is monomial and Q has no oriented cycles and no parallel arrows

A unique
maximal
 π_1

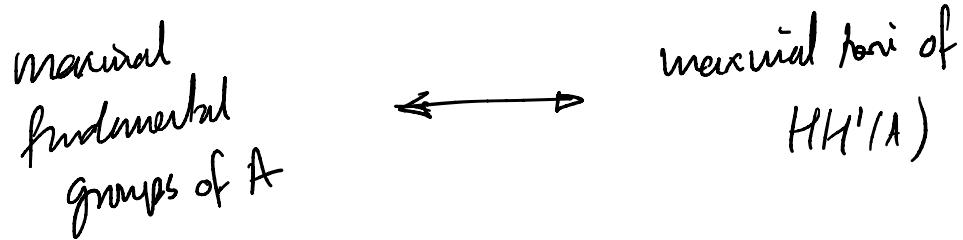
π

}

Theorem (- Ry D Samin 21)

For any A , every maximal torus in $\text{HH}^1(A)$ is the image of some $\pi_1(Q, I)^*$

Get a correspondence:



Proof uses Lagrange interpolation.

Cor the maximal rank* of $\pi_1(Q, I)$, for any presentation $A = \frac{RQ}{I}$
is a stable invariant uses part ①

* Note if $\dim k=0$ use $\text{rank} = \dim \mathbb{Q} \otimes_{\mathbb{Z}} \pi_1(Q, I)$
 if $\dim k=p$ use $p\text{-rank} = \dim \mathbb{F}_p \otimes_{\mathbb{Z}} \pi_1(Q, I)$

could be bigger \uparrow if π_1 has p torsion!

Fact $\max \text{rank } \pi_1(Q, I) \leq *$ of holes in $Q = |\text{connected components}| - |Q_0| + |Q_1|$

↑ equality for monomial algebras

You can tell how many "holes" A has from its derived / stable equivalence class

Cor Derived equivalent monomial algebras have the same number of arrows $\rightarrow \left\{ \begin{array}{l} \text{Compare Avella-Alaminos} \\ \text{-Gillesse gentle algs} \\ \text{and Artin-Zvonareva} \\ \text{Brauer graph algebras.} \end{array} \right.$

Final application

A simply connected if $\pi_1(Q/I) = 0$ for all presentations $A \cong kQ/I$

Is this equivalent to $\text{HH}^1(A) = 0$?

Yes in lots of cases

Buchweitz Lin,

Cohen -

Lanzilotta -

Saviooli

Assam Lanzilotta

Le Meur

Bustamante

but \exists counterexample due to Buchweitz Lin.

Cor A is simply connected $\Leftrightarrow \text{HH}^1(A)$ has no tori

In particular, being simply connected is closed under SEMT

✓ uses both parts