The Ziegler spectrum of tubular canonical algebras

FD Seminar

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- E.g.: Decidability problem (for $A = \mathbb{Z}$ goes back to Szmielew, 1955).
- Closed sets of $\operatorname{Zg} A$ bijectively correspond to subcategories of *Mod-A*, which are additive and first-order axiomatizable.
- Points of $\operatorname{Zg} A$ are the so-called indecomposable pure-injective (aka algebraically compact) modules.

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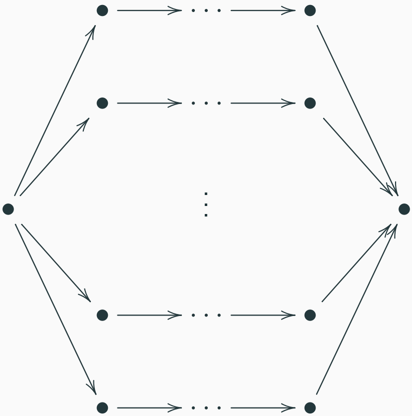
3. $\mathrm{Zg}\, A$ is also known for domestic string algebras (Laking, Prest, Puninski, proving Ringel's conjecture).

Pure-injectives over tubular canonical algebras

Tubular canonical algebra

These are path algebras with quivers



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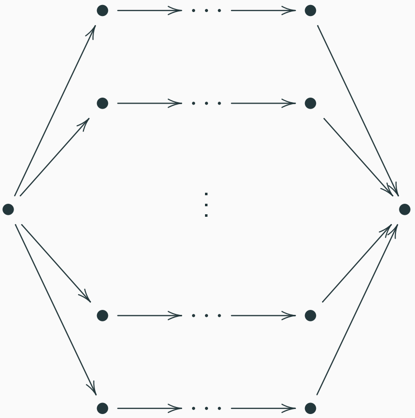

with branches of lengths (2,2,2,2), (3,3,3), (4,4,2), (6,3,2), modulo suitable relations ($\operatorname{gldim} A = 2$).

Large cotilting modules

- If A is a ring, a module $C \in \mathit{Mod}\text{-}A$ is cotilting if
  (C1) $\operatorname{inj.dim.} C < \infty$,
  (C2) $\operatorname{Ext}^{>0}(C^I, C) = 0$ for every set I,
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- Key facts:
  1. (Riccardo-Gregorio-Mantese 2007, Š. 2014): There is a Grothendieck category $\mathcal{C}$ and a derived equivalence between $\mathit{Mod}\text{-}A$ and $\mathcal{C}$ identifying $\operatorname{Prod} C$ with the injective objects in $\mathcal{C}$.

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- Example: If A is a tubular canonical algebra, then $\mathit{Mod}\text{-}A$ is derived equivalent to $\operatorname{Qcoh} \mathbb{X}$, where $\mathbb{X}$ is a tubular weighted projective line of the corresponding type.

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Relation to elliptic curves

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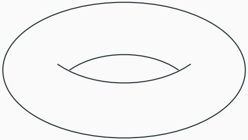

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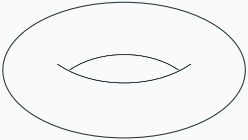

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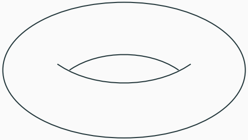

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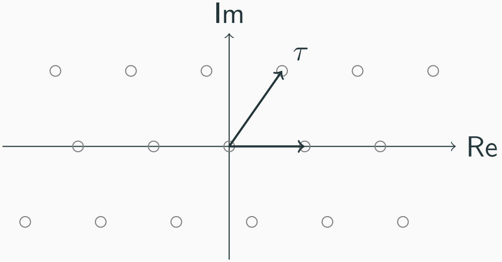

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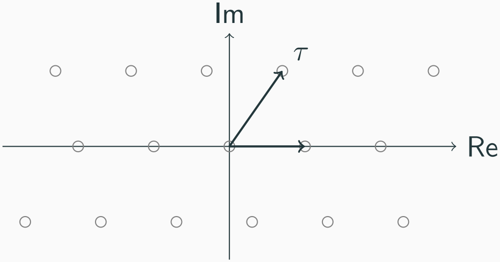

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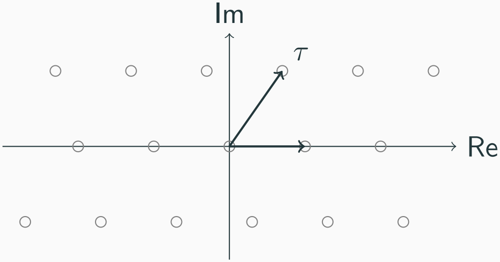

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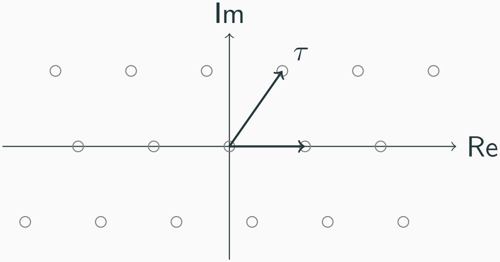

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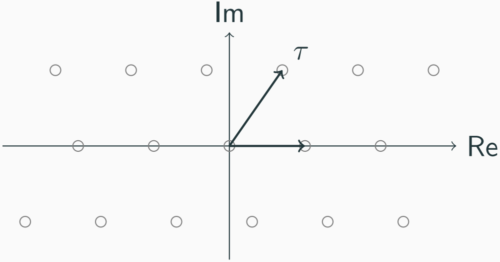

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- Fact (Bundgaard-Nielsen 1951; Fox 1952): Complex tubular weighted projective lines are such quotients of elliptic curves, when viewed as orbifolds.

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- If we put $\mathbb{X} := \mathbb{E}/G$ (whatever it precisely means), we may just define (thanks to Chen-Chen-Zhou 2015)

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- If G acts on an algebra, then $(Mod\text{-}A)^G \cong Mod\text{-}(G \ltimes A)$, where $G \ltimes A$ is the skew-group algebra.

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Simple sheaves on non-commutative tori

(Quasi-)coherent sheaves on an elliptic curve

Continued fractions

Construction of simples on a non-commutative torus