Abelian envelopes of exact categories

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§ Exact categories

$\mathcal{E}$ - exact category is an additive category $\mathcal{E}$ with a class $\mathcal{S}$ of

conflations, i.e. sequences

$$X \xrightarrow{i} Y \xrightarrow{d} Z \qquad i = \ker d \quad d = \operatorname{coker} i$$

i - inflation, d - deflation

s.t. 1) $0 \to X \xrightarrow{id_X} X$ conflation

2) composition of 2 deflations is a deflation

3) $\forall$
$$\begin{tikzcd} X \arrow[r] & Y \arrow[r, "d"] & Z \\ X \arrow[u, equal] \arrow[r] & Y' \arrow[u] \arrow[r, "\exists d'"] & Z' \arrow[u, "\forall f"'] \end{tikzcd}$$
$\operatorname{Ext}^1(Z, X) \underset{\operatorname{Hom}(Z', Z)}{\times} \to \operatorname{Ext}^1(Z', X)$

4)
$$\begin{tikzcd} X'' \arrow[r] & Y'' \arrow[r] & Z \\ X \arrow[u, "\forall g"] \arrow[r] & Y \arrow[u] \arrow[r] & Z \arrow[u] \end{tikzcd}$$
$\forall$

Examples: $\mathcal{A}$ additive $\quad$ $\mathcal{S}$ - split sequences $X \to X \oplus Y \to Y$

$\mathcal{A}$ abelian $\quad$ $\mathcal{S}$ - short exact sequences

$\mathcal{E} \subset \mathcal{A}$ - fully exact subcategory of an abelian category

$\mathcal{E} \subset \mathcal{A}$ full, closed under extensions

$\mathcal{S}$: short ex seq in $\mathcal{A}$ with all terms in $\mathcal{E}$

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Z \arrow[r] & 0 \\ & X \arrow[u, "\wr"] \arrow[r] & Y' \arrow[u] \arrow[r] & Z' \arrow[u, "f"] & \end{tikzcd}$$
$Y' \in \mathcal{E}$

Thm (Gabriel-Quillen): Any exact category $\mathcal{E}$ is a fully exact subcategory of the abelian category $fp(\mathcal{E}^{op}, Ab)$.

$F \in fp(\mathcal{E}^{op}, Ab) \iff \operatorname{Hom}(-, E_1) \to \operatorname{Hom}(-, E_2) \to F(-) \to 0$

§ Examples of exact categories

$(\mathcal{A}, \Lambda)$ - higest weight category $\qquad$ Λ - partial order on simples

$\{S_\lambda\}_{\lambda \in \Lambda} \subset \mathcal{A}$ iso-classes of simple

$\{P_\lambda\}_{\lambda \in \Lambda} \subset \mathcal{A}$ projective covers, $\quad \{I_\lambda\}_{\lambda \in \Lambda} \subset \mathcal{A}$ injective hulls

$\exists \{\Delta_\lambda\}_{\lambda \in \Lambda}$ standard

(1) $0 \to \{S_{<\lambda}\} \to \Delta_\lambda \to S_\lambda \to 0$

(2) $0 \to \{\Delta_{>\lambda}\} \to P_\lambda \to \Delta_\lambda \to 0$

$\{\nabla_\lambda\}_{\lambda \in \Lambda}$ costandard

$S_\lambda \to \nabla_\lambda \to \{S_{<\lambda}\}$

$\nabla_\lambda \to I_\lambda \to \{\nabla_{>\lambda}\}$

$\mathcal{F}(\Delta), \mathcal{F}(\nabla) \subset \mathcal{A}$ fully exact subcategories

$$\mathcal{F}(\Delta) = \{M \in \mathcal{A} \mid \exists\ 0 = M_0 \subset \dots \subset M_n = M \quad M_i/M_{i-1} \simeq \Delta_{j(i)}\}$$

A- directed algebra

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r, shift left] \arrow[r, shift right] \arrow[from=1-1, to=1-3, bend left] & 3 \arrow[r] & 4 \end{tikzcd}$$

$(\text{mod-}A,\ 1 \leq 2 \leq \dots \leq n)$ $\quad \Delta_\lambda = S_\lambda$

$(\text{mod-}A,\ n \leq n-1 \leq \dots \leq 1)$ $\quad \Delta_\lambda = P_\lambda$

$\mathcal{F}(\Delta_\lambda) = \text{mod-}A$ or $\mathcal{F}(\Delta_\lambda) = \text{Proj}(A)$

$$\begin{tikzcd} 1 \arrow[r, shift left, "a"] & 2 \arrow[l, shift left, "b"] \end{tikzcd} \qquad ba = 0$$

$$P_1 = \begin{array}{c} 1 \\ \downarrow b \\ 2 \\ \downarrow a \\ 1 \end{array} \quad \begin{array}{l} \Delta_1 \\ \\ \Delta_2 \end{array} \qquad P_2 = \begin{array}{c} 2 \\ \downarrow a \\ 1 \end{array} \ \Delta_2$$

$$\mathcal{F}(\Delta) = \text{add}(\underset{S_1}{\overset{\|}{\Delta_1}}, \Delta_2, P_1)$$

$$S: \quad \Delta_2 \to P_1 \to S_1$$

X - normal CM surface $\qquad \omega_X$ - dualizing sheaf

$F \in Coh(X)$ maximal Cohen-Macaulay $\qquad \mathcal{E}xt^{>0}(F, \omega_X) = 0 \Rightarrow CM(X) \subset Coh(X)$
fully exact subcat.

$F \in CM(X) \Leftrightarrow F$ is reflexive $\qquad F \xrightarrow{\simeq} F^{**} \qquad F^* = \mathcal{H}om(F, \mathcal{O}_X)$

Canonical exact structure on $CM(X)$.

$f: F_0 \longrightarrow F_1 \qquad F_0, F_1 \in CM(X)$

$K = \ker f \quad K \in CM(X) \qquad K \to F_0 \xrightarrow{f} F_1$ kernel in $CM(X)$

$Q = (\operatorname{coker} f)^{**} \in CM(X) \qquad F_0 \to F_1 \to \operatorname{coker} f \to Q$ cokernel in $CM(X)$

$\mathcal{A}$ - additive category with kernels and cokernels

$f: A \to A'$ is strict if $\operatorname{coker}(\ker f) \xrightarrow{\simeq} \ker(\operatorname{coker} f)$

A morphism $f: F_1 \to F_2$ in $CM(X)$ is a strict monomorphism

$\Leftrightarrow$ f is a monomorphism in $Coh(X)$ with torsion-free cokernel.

$\operatorname{coker}(\ker f) = \operatorname{coker}(0 \to F_1) = F_1$

$\ker(\operatorname{coker} f) = \ker(F_2 \to (\operatorname{coker} f)^{**}) = F_1 \Leftrightarrow T = 0$

$$\begin{tikzcd}
 & T \arrow[r] & 0 & & \\
 & T \arrow[r] \arrow[u] & \operatorname{coker} f \arrow[r] & \operatorname{coker} f^{**} & \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] & F_2 \arrow[r] \arrow[u] & F_2 \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[r] & F_1 \arrow[r] \arrow[u] & \ker(\operatorname{coker}(f)) \arrow[u] \arrow[uuull, bend right=60] &
\end{tikzcd}$$

A morphism $f: F_1 \to F_2$ in $CM(X)$ is a strict epimorphism

$\Leftrightarrow$ the cokernel of f in $Coh(X)$ is an Artinian sheaf.

$f: F_1 \to F_2$ epi $\Leftrightarrow (\operatorname{coker} f)^{**} = 0 \Leftrightarrow \operatorname{coker} f$ – torsion

$\operatorname{coker}(\ker f) = \operatorname{coker}(K \to F_1) = (\operatorname{Im} f)^{**}$

$\ker(\operatorname{coker} f) = \ker(F_2 \to 0) = F_2$

$$\begin{tikzcd}
 & \operatorname{coker}(f) \arrow[r] & Q \arrow[r] & 0 & \\
0 \arrow[r] & F_2 \arrow[r] & F_2 \arrow[u] \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Im}(f) \arrow[u] \arrow[r] & \operatorname{Im}(f)^{**} \arrow[u] \arrow[r] & C \arrow[r] & 0 \\
 & 0 \arrow[u] & K \arrow[u] \arrow[r] & C \arrow[u] \arrow[uuull, bend left] &
\end{tikzcd}$$

$(\operatorname{Im} f)^{**} \simeq F_2 \Leftrightarrow C \simeq \operatorname{coker}(f)$

$\Downarrow$

$\operatorname{coker}(f)$ is Artinian

(every cotorsion is)

$\operatorname{coker}(f)$-Artinian $\Rightarrow \operatorname{Ext}^1(\operatorname{coker}(f), \mathcal{O}_X) = 0 \Rightarrow$
$0 \to \operatorname{Im}(f) \to F_2 \to \operatorname{coker}(f) \to 0$
$0 \to F_2^* \xrightarrow{\simeq} \operatorname{Im}(f)^* \to 0 \Rightarrow (\operatorname{Im} f)^{**} \simeq F_2^{**} \simeq F_2$

A-additive category with kernels and cokernels.

A is quasi-abelian if

- pull-back of a strict epimorphism is a strict epimorphism
- push-out of a –"– monomorphism –"– mono

X-normal CM surface. The category $CM(X)$ is quasi-abelian.

A- quasi-abelian $\Rightarrow$ A has canonical exact structure

$S: A' \xrightarrow{i} A \xrightarrow{d} A''$

$i = \ker d$ - strict mono

$d = \operatorname{coker} i$ - strict epi

<u>Cor:</u> X-normal CM surface. $CM(X)$ has exact structure with <u>conflations</u>

$F_1 \xrightarrow{f} F \xrightarrow{g} F_2$

such that $0 \to F_1 \xrightarrow{f} F \xrightarrow{g} F_2 \to Q \to 0$ is exact in $Coh(X)$ with

Q-Artinian.

Example: $\mathbb{C}[x,y] \circlearrowleft \mathbb{Z}_n$ $\quad 1 \le a \le n \quad (n,a)=1$

$\frac{1}{n}(1,a) \quad \sigma \cdot x = \varepsilon x \quad \sigma \cdot y = \varepsilon^a y \qquad \varepsilon$ - n'th primitive root of unity

$\sigma \in \mathbb{Z}_n$ generator

$R_i = \{ f \in \mathbb{C}[x,y] \mid \sigma f = \varepsilon^i f \}$

$\mathbb{C}[x,y] = R_0 \oplus R_1 \oplus \dots \oplus R_{n-1}$ – all isomorphism classes of indecomposable modules in $CM(R_0)$

$\frac{1}{2}(1,-1) \qquad R_0 = \langle 1, x^2, xy, y^2 \dots \rangle \qquad R_1 = \langle x, y, x^3, \dots \rangle$

$R_0 \xrightarrow{\binom{-y}{x}} R_1 \oplus R_1 \xrightarrow{(x\ y)} R_0$ conflation in the canonical exact str.

§ Abelian envelopes of exact categories

$\mathcal{E}, \mathcal{E}'$ - exact categories $\qquad F : \mathcal{E} \to \mathcal{E}'$ additive functor

F is exact if $\forall$ confl. $X \to Y \to Z$, $F(X) \to F(Y) \to F(Z)$ is a conflation

F is right exact if $\forall$ confl $X \xrightarrow{i} Y \xrightarrow{d} Z \qquad F(d)$ - deflation

$F(i) = i' \cdot \tilde{d} \qquad i' = \ker F(d)$

$$\begin{tikzcd} & K \arrow[r, "i'"] & F(Y) \arrow[r, "F(d)"] & F(Z) \\ & & F(X) \arrow[ul, "\tilde{d}"] \arrow[u, "F(i)"'] & \end{tikzcd}$$

deflation $\to \tilde{d}$

F is left exact if $F^{op} : \mathcal{E}^{op} \to \mathcal{E}'^{op}$ is right exact

If $\mathcal{E}' \simeq \mathcal{A}$ is abelian $\quad F$ exact $\iff 0 \to F(X) \to F(Y) \to F(Z) \to 0$

F right exact $\iff F(X) \to F(Y) \to F(Z) \to 0$

F left exact $\iff 0 \to F(X) \to F(Y) \to F(Z)$

are exact

$\mathrm{Rex}(\mathcal{E},\mathcal{E}')$ - category of right exact functors and natural transformations

The right abelian envelope of an exact category $\mathcal{E}$ is an abelian category $\mathcal{A}_r(\mathcal{E})$ and right exact functor $i_R' : \mathcal{E} \to \mathcal{A}_r(\mathcal{E})$ which induces an equivalence

$$\mathrm{Rex}(\mathcal{A}_r(\mathcal{E}),\mathcal{B}) \xrightarrow{(-)\circ i_R'} \mathrm{Rex}(\mathcal{E},\mathcal{B})$$

for any abelian category $\mathcal{B}$.

$$\begin{tikzcd} \mathcal{E} \arrow[r, "i_R"] \arrow[dr, "F"'] & \mathcal{A}_r(\mathcal{E}) \arrow[d, dashed, "\bar{F}"] \\ & \mathcal{B} \end{tikzcd}$$

Left abelian envelope $\quad i_L' : \mathcal{E} \to \mathcal{A}_l(\mathcal{E}) \qquad \mathrm{Lex}(\mathcal{A}_l(\mathcal{E}),\mathcal{B}) \simeq \mathrm{Lex}(\mathcal{E},\mathcal{B})$

$$\mathcal{A}_l(\mathcal{E}^{op}) \simeq \mathcal{A}_r(\mathcal{E})^{op}$$

Abelian hull $\mathcal{A}(\mathcal{E})$ - universal for exact functors

[Adelman-Stein] The abelian hull $\mathcal{A}(\mathcal{E})$ always exists.

Construction: $\mathcal{E}$ - additive $\quad$ Adelman category $\mathrm{Adel}(\mathcal{E})$

Objects: $E_0 \xrightarrow{e_0} E_1 \xrightarrow{e_1} E_2$ $\qquad$ Morphisms (f_0,f_1,f_2):

$$\begin{tikzcd} E_0 \arrow[r, "e_0"] & E_1 \arrow[r, "e_1"] & E_2 \\ D_0 \arrow[u, "f_0"] \arrow[r, "d_0"] \arrow[ur, phantom, "\circlearrowleft"] & D_1 \arrow[u, "f_1"] \arrow[r, "d_1"] \arrow[ur, phantom, "\circlearrowleft"] & D_2 \arrow[u, "f_2"'] \end{tikzcd} \Big/ \sim$$

$(f_0,f_1,f_2) \sim (g_0,g_1,g_2) \iff \exists$

$$\begin{tikzcd} E_0 \arrow[r, "e_0"] & E_1 & \\ & D_1 \arrow[ul, "h_1"] \arrow[r, "d_1"'] & D_2 \arrow[ul, "h_2"'] \end{tikzcd}$$

$$f_1 - g_1 = e_0 h_1 + h_2 d_1$$

Adelman '73: $\mathrm{Adel}(\mathcal{E})$ is abelian. It is the abelian hull of $\mathcal{E}$ with split exact structure.

$\mathcal{A}(\mathcal{E}) = \mathrm{Adel}(\mathcal{E})/\mathcal{C}$

$\mathcal{C}$ - minimal Serre subcategory containing

$$Y \xrightarrow{d} Z \to 0 \qquad X \xrightarrow{i} Y \xrightarrow{d} Z \qquad 0 \to X \xrightarrow{i} Y$$

for any conflation $\quad X \xrightarrow{i} Y \xrightarrow{d} Z$

Adelman: $\mathcal{A}(\mathcal{E})$ - abelian hull of $\mathcal{E}$

Stein: $\mathcal{E} \to \mathcal{A}(\mathcal{E})$ exact, fully faithful, detects exactness

Assume $\mathcal{A}_r(\mathcal{E})$ exists.

$$\begin{tikzcd}
\mathrm{Hom}(-,\tilde{E}_1) \arrow[r] & \mathrm{Hom}(-,E_1) \arrow[r] & F_2 \arrow[r] & 0 \\
\mathrm{Hom}(-,E_1) \arrow[r] & \mathrm{Hom}(-,E_1) \arrow[r] \arrow[u, dashed] & F_1 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

$\mathcal{A}_r(\mathcal{E}) \subset \mathrm{Lex}(\mathcal{E}^{op}, Ab)$ fully exact subcategory

$i_R : \mathcal{E} \hookrightarrow \mathcal{A}_r(\mathcal{E})$ faithful and right exact

i_R - full $\Rightarrow$ i_R - exact and detects exactness.

Examples:

(1) $\mathcal{A}$ - additive category with weak kernels and split exact structure

$$\begin{tikzcd}
X' \arrow[d, dashed, "j"'] \arrow[dr, "g"] & & \\
K \arrow[r, "i"] & X \arrow[r, "f"] & Y
\end{tikzcd}
\qquad
\begin{matrix} fi = 0 \\ \forall g \;\; fg = 0 \;\; \exists j \;\; g = ij \end{matrix}$$

$$fp(\mathcal{A}^{op}, Ab) = \mathcal{A}_r(\mathcal{E})$$

(2) $\mathcal{E} \subset \mathcal{A}$ fully exact subcategory of an abelian category

- $\mathcal{E}$ is closed under kernels of epimorphisms
- Any object of $\mathcal{A}$ is a quotient of an object of $\mathcal{E}$

[Kashiwara-Schapira]: $\mathcal{E} \hookrightarrow \mathcal{A}$ is the right abelian envelope

X - scheme with enough locally free sheaves

$$\mathrm{Coh}(X) = \mathcal{A}_r(\mathrm{Bun}(X))$$

$(\mathcal{A}, \Lambda)$ - highest weight $\mathcal{A}_r(\mathcal{F}(\Delta)) = \mathcal{A}$

(3) $\mathcal{E}$ - quasi-abelian with canonical exact structure

Schneiders: $\mathcal{E}$ has the right and the left abelian envelopes

hearts of t-structures on $\mathcal{D}^b(\mathcal{E})$.

§ Monad description

$\mathcal{E}x_r$ - exact categories which have the right abelian envelope + right exact functors + natural transformations

$\mathcal{A}b_r$ - abelian categories + right exact functors + natural transformations

There exist functor $A_r : \mathcal{E}x_r \longrightarrow \mathcal{E}x_r$,

natural transformation: $\jmath : \mathrm{Id} \to A_r$,

A_r - idempotent $\mu : A_r^2 \xrightarrow{\simeq} A_r$

$(A_r, \jmath, \mu)$ - monad

$\mathcal{A}b_r$ - the category of algebras over $(A_r, \jmath, \mu)$

§ Highest weight categories as abelian envelopes of thin categories

Thin categories: exact categories with a 'full exceptional collection'

Torsion pair on $\mathcal{E}$: $(\mathcal{T}, \mathcal{F})$

- $\mathcal{T} \subset \mathcal{E}$, $\mathcal{F} \subset \mathcal{E}$ full, closed under extensions
- $\mathrm{Hom}(T, F) = 0$ $\forall$ $T \in \mathcal{T}$ $F \in \mathcal{F}$
- $\forall E \in \mathcal{E}$ $\exists$ conf $T \to E \to F$, $T \in \mathcal{T}$, $F \in \mathcal{F}$

no nonsplit conf $F \to \quad \to T$

$(\mathcal{T}, \mathcal{F})$ is a perpendicular torsion pair if $\underline{\underline{\mathrm{Ext}^1(T, F) = 0}}$ for $T \in \mathcal{T}$, $F \in \mathcal{F}$.

<u>Thm:</u> If $(\mathcal{T}, \mathcal{F})$ is a perpendicular torsion pair on an exact category $\mathcal{E}$ then $D^b(\mathcal{E})$ has a semi-orthogonal decomposition $\langle D^b(\mathcal{F}), D^b(\mathcal{T}) \rangle$

$(\mathcal{T}, \mathcal{F})$ - torsion pair $\Rightarrow$ $i : \mathcal{T} \to \mathcal{E}$ has right adjoint $i^! : \mathcal{E} \to \mathcal{T}$

$\delta : \mathcal{F} \to \mathcal{E}$ has left adjoint $\delta^* : \mathcal{T} \to \mathcal{F}$

Thm: $(\mathcal{T},\mathcal{F})$ perpendicular torsion pair $\Leftrightarrow$ $i^!$ is exact $\Leftrightarrow$ δ^* is exact.

$\mathcal{T}\subset\mathcal{E}$ right admissible subcategory if $\mathcal{T}$ is a torsion part of a perpendicular torsion pair $(\mathcal{T},\mathcal{T}^\perp)$ $\Leftrightarrow$ $\exists\, i^!:\mathcal{E}\to\mathcal{T}$ and $ii^!\to \mathrm{Id}$ is an inflation when applied to any $E\in\mathcal{E}$.

$\mathcal{T}^\perp=\{E\in\mathcal{E}\mid \mathrm{Hom}(\mathcal{T},\mathcal{E})=0=\mathrm{Ext}^1(\mathcal{T},\mathcal{E})\}$

$\mathcal{E}$ k-linear exact category

is a thin category if $\exists$ $0=\mathcal{T}_0\subset\mathcal{T}_1\subset\ldots\subset\mathcal{T}_n\simeq\mathcal{E}$ $\quad\mathcal{T}_i\subset\mathcal{E}$ right admissible

$$\underset{\substack{\Psi\\ E_i}}{\mathcal{T}_i^\perp\cap\mathcal{T}_{i+1}}\simeq\underset{\substack{\Psi\\ k}}{k\text{-mod}}$$

Thm: $(\mathcal{A},\Lambda)$-highest weight category $\Rightarrow$ $\mathcal{F}(\Delta),\mathcal{F}(\nabla)$ are thin $\quad \mathcal{A}=\mathcal{A}_r(\mathcal{F}(\Delta))=\mathcal{A}_l(\mathcal{F}(\nabla))$

$\mathcal{E}$-thin category $\Rightarrow$ $(\mathcal{A}_r(\mathcal{E}),\Lambda^{op})$, $(\mathcal{A}_l(\mathcal{E}),\Lambda)$ are Ringel dual highest weight categories.

Ringel duality for thin: $\quad\mathcal{E}\simeq\mathcal{F}(\Delta)$ in $\mathcal{A}_r(\mathcal{E})$ $\quad RD(\mathcal{E})=\mathcal{F}(\nabla)\subset\mathcal{A}_r(\mathcal{E})$
$RD(\mathcal{E})=\mathcal{A}_l(\mathcal{E})\cap S^{-1}\mathcal{A}_r(\mathcal{E})$

Λ - canonical poset of $\mathcal{E}$ minimal s.t. $\mathrm{Hom}(E_i,E_j)\neq0$ or $\mathrm{Ext}^1(E_i,E_j)\neq 0$ $\Rightarrow i\leq j$

§ Right abelian envelope of MCM(X)

X - normal CM surface

$\mathcal{T}_0(X)\subset \mathrm{Coh}(X)$ Serre subcategory of sheaves with 0-dimensional support.

Thm: $\mathrm{Coh}(X)/\mathcal{T}_0(X)$ is the right abelian envelope of $\mathrm{CM}(X)$.

Calabrese-Pirisi: $\quad X_{\geq 1}\subset X$ subset of points of dimension ≥ 1.

X, Y - schemes of finite type over k. $\quad \mathrm{Coh}(X)/\mathcal{T}_0(X)\simeq\mathrm{Coh}(Y)/\mathcal{T}_0(Y)\Leftrightarrow X_{\geq1}\simeq Y_{\geq1}$

Abstract point of view:

$\mathcal{S} \subset \mathcal{A}$ Serre subcategory $\mathcal{S} = \mathcal{T}_0(X)$

$\mathcal{E} \subset \mathcal{A} = \{A \in \mathcal{A} \mid \mathrm{Hom}(\mathcal{S}, A) = 0 = \mathrm{Ext}^1(\mathcal{S}, A)\}$ - the category of S-closed objects $\mathcal{E} = \mathrm{CM}(X)$

$\mathcal{T} \subset \mathcal{A}$ $\mathcal{T} = \{T \in \mathcal{A} \mid \mathrm{Hom}(T, \mathcal{E}) = 0\}$ $\mathcal{T} = \mathcal{T}_1(X)$ – torsion sheaves

$\mathcal{F}$ – torsion-free sheaves

Assume that

(1) $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{A}$.

(2) Any $F \in \mathcal{F}$ fits into a short exact sequence $0 \to F \to E \to S \to 0$

$E \in \mathcal{E}$ $S \in \mathcal{S}$

$0 \to F \to F^{**} \to \mathrm{ctor}(F) \to 0$

$\mathrm{ctor}(F) \in \mathcal{T}_0(X)$

(3) Any object of $\mathcal{A}$ is a quotient of an object in $\mathcal{F}$

Then $\mathcal{A}/\mathcal{S}$ has a torsion pair $(q\mathcal{T}, \mathcal{E})$ and $\mathcal{A}/\mathcal{S} \simeq \mathcal{A}_r(\mathcal{E})$.

$q : \mathcal{A} \to \mathcal{A}/\mathcal{S}$