Algebras of amenable representation type and (dimensional) expansion

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Outline

Hyperfiniteness and Amenability

Definition

Let k be a field, A be a finite dimensional k-algebra and let $\mathcal{M}$ be a set of A-modules. $\mathcal{M}$ is called **hyperfinite** provided for every $\varepsilon > 0$ there exists $L_\varepsilon > 0$ such that for every $M \in \mathcal{M}$ there exists a submodule $P \subseteq M$ such that

$$\dim_k P \geq (1 - \varepsilon) \dim_k M, \tag{1}$$

and modules $N_1, N_2, \ldots N_t \in \operatorname{mod} A$, with $\dim_k N_i \leq L_\varepsilon$, such that $P \cong \bigoplus_{i=1}^t N_i$.

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The k-algebra A is said to be of **amenable representation type** provided the set of all finite dimensional A-modules (or more specific, a set which meets any isomorphism class of finite dimensional A-modules) is hyperfinite.

Motivation

Conjecture (Elek '17)

Let k be a countable algebraically closed field and A be a finite dimensional algebra of infinite representation type over k. Then A is of tame representation type if and only if A is of amenable representation type.

Some (non-)examples

Example (finite representation type)

An algebra A of finite representation type is amenable.

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Theorem (Elek '17)

Let k be a countable field. Any string algebra R is of amenable representation type.

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The wild Kronecker quiver algebras are not of amenable representation type.

Some observations

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It is enough to check for hyperfiniteness on indecomposable modules.

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Proposition

Left-exact functors with bounds on dimensions of the image preserve hyperfiniteness.

The 2-Kronecker quiver

Let us look at an example to see how one may prove amenable representation type.

Example

$$1 \underset{b}{\overset{a}{\rightrightarrows}} 2$$

Let k be any field. Then the path algebra of the 2-Kronecker quiver is of amenable representation type.

Representations of the Kronecker quiver

Question

Given any ε, can we find L_ε such that for all finite dimensional Kronecker-modules M there is a submodule P with $\dim P \geq (1-\varepsilon) \dim M$ which decomposes into summands of dimension bounded by L_ε?

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Well-known classification of indecomposable Kronecker-modules:

$$P_n\colon k^n \underset{\left[\begin{smallmatrix}0\\ \mathrm{id}\end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix}\mathrm{id}\\ 0\end{smallmatrix}\right]}{\rightrightarrows}} k^{n+1}, \qquad Q_n\colon k^{n+1} \underset{[0\ \mathrm{id}]}{\overset{[\mathrm{id}\ 0]}{\rightrightarrows}} k^n, \qquad R_n(\phi,\psi)\colon k^n \underset{\psi}{\overset{\phi}{\rightrightarrows}} k^n,$$

where $\forall\, n \in \mathbb{N}$ either

- $\phi = \mathrm{id}$ and ψ is companion matrix of power of monic irreducible over k, or
- $\psi = \mathrm{id}$ and ϕ is given by companion matrix of polynomial λ^m.

Finding a large submodule

- For preprojective P_n:

$$
\begin{tikzcd}
e_1 \arrow[d] \arrow[dr] & e_2 \arrow[d] \arrow[dr] & e_3 \arrow[d] \arrow[dr] & e_4 \arrow[d] \arrow[dr] & e_5 \arrow[d] \arrow[dr] & \cdots & \cdots & e_{n-2} \arrow[d] \arrow[dr] & e_{n-1} \arrow[d] \arrow[dr] & e_n \arrow[d] \arrow[dr] & \\
f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & \cdots & f_{n-2} & f_{n-1} & f_n & f_{n+1}
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Do this by considering submodule generated by subspace at source with basis all but every $\left\lceil \frac{1}{2\varepsilon} \right\rceil + 1$-th basis element.

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  Leaves us with submodule decomposing into summands of dimension bounded by $\frac{1}{\varepsilon} + 3$.
  But: have removed less than $\varepsilon \dim P_n$ basis elements, so dimension of submodule is nearly as big as $\dim P_n$.

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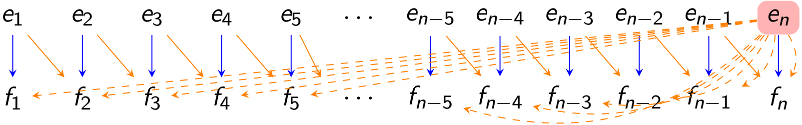
  But: have removed less than $\varepsilon \dim P_n$ basis elements, so dimension of submodule is nearly as big as $\dim P_n$.

Recall

A amenable $\iff \forall \varepsilon > 0\ \exists L_\varepsilon > 0\colon \forall M \in \operatorname{mod} A\ \exists N \subseteq M\colon$
$\dim N \geq (1-\varepsilon)\dim M \quad \wedge \quad \forall S \mid N\colon\ \dim S \leq L_\varepsilon.$

Finding a large submodule



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$$

- For the regular modules, consider the submodule generated by deleting the last basis element in vector space at source: it is preprojective, since $\psi = \begin{bmatrix} 0 & 0 & \dots & 0 & * \\ 1 & 0 & \dots & 0 & * \\ 0 & 1 & \dots & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & * \end{bmatrix}$ is replaced by $\left[\begin{smallmatrix} 0 \\ \mathrm{id} \end{smallmatrix} \right]$.

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e_1 \arrow[d, blue] \arrow[dr, orange] & e_2 \arrow[d, blue] \arrow[dr, orange] & & e_4 \arrow[d, blue] \arrow[dr, orange] & e_5 \arrow[d, blue] \arrow[dr, orange] & \cdots & & e_{n-4} \arrow[d, blue] \arrow[dr, orange] & e_{n-3} \arrow[d, blue] \arrow[dr, orange] & & e_{n-1} \arrow[d, blue] \arrow[dr, orange] & \\
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- For the postinjective indecomposables, use the surjective map to the simple injective to find a submodule without postinjective summands.

Tame hereditary path algebras

Proposition

Let Q be a quiver of tubular type (p, q, r), where $p > 1$. Let all extended Dynkin quivers of type $(p-1, q, r)$ be amenable. If T is an inhomogeneous simple regular module belonging to a tube of rank p in Γ_{kQ}, then $T^{\perp}$ is hyperfinite.

Theorem

Let Q be an acyclic quiver of extended Dynkin type. Let k be any field. Then the path algebra kQ of Q is of amenable representation type.

Sketch of the proof

Pick a tube $\mathbb{T}$ of rank $p \geq 2$ (or maximal rank)

- Preprojective X either is in $S^{\perp}$ for regular simple $S \in \mathbb{T}$ or $\exists Y$ with $0 \to Y \to X \to T \to 0$ exact and $Y \in S^{\perp}$ for regular simples $S, T \in \mathbb{T}$.

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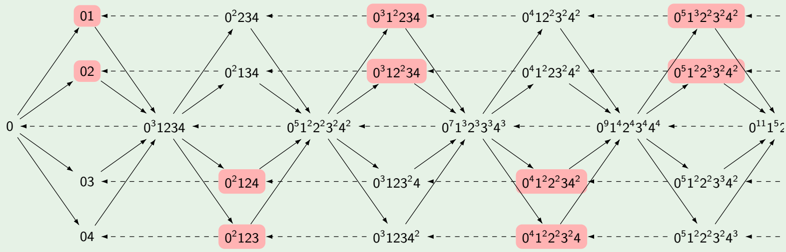
Example (Four subspace quiver)

Choose $S = 012,\ T = 034,$



visualized using Jan Geuenich's knitting algorithm applet

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- Indecomposable regular modules: either in $S^{\perp}$ (via orthogonality) or have submodule in $T^{\perp}$ for some regular-simple $T \in \mathbb{T}$.
- For indecomposable postinjectives: induction on the defect, showing hyperfiniteness of $\mathcal{N}_d := \{\text{indecomposable modules of defect} \leq d\}$.

Going further

With similar methods, we show the analogue result for all finite dimensional, tame hereditary algebras.

- Tame concealed works okay.
- There are partial results for tubular canonical algebras: preprojective, postinjective and integral slope modules (using classification of [DMM14])
- One might do it for clannish algebras, as Elek did it for string algebras.

Input from graph theory

Problem

How to approach the wild/non-amenable part of the conjecture?

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Hyperfiniteness for modules based on notion from graph theory:

Definition (Elek)

Collection $\mathcal{G}$ of finite graphs is **hyperfinite** if $\forall \varepsilon > 0\ \exists K_\varepsilon$ finite s.t. $\forall G \in \mathcal{G}\ \exists S \subset E(G)$ s.t. $|S| \leq \varepsilon |V(G)|$ and every connected component of $G \setminus S$ has at most K_ε vertices.

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Remark

Related notion of fragmentability ([EM94]) can be used to show that preprojective and postinjective component of wild Kronecker quivers are hyperfinite.

Expander Graphs

Definition

$G = (V, E)$, k-regular is an ε-**expander** if $\forall A \subset V$ with $|A| \leq \frac{|V|}{2}$,

$|N(A)| \geq (1 + \varepsilon)|A|$, where $N(A) = \{y \in V : \mathit{distance}(y, A) \leq 1\}$.

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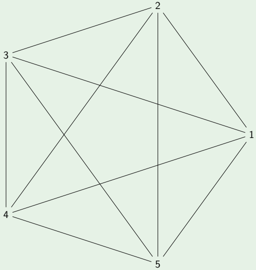

The complete graph K_n on $n \geq 2$ vertices is a 1-expander.

Expander Graphs

Given a group G and S a finite, symmetric set of generators of G, the *Cayley graph* $Cay(G,S)$ is the graph with vertex set G and edges connecting x to sx for $s \in S$, thus each vertex $x \in G$ is connected to the $|S|$ elements sx, so $Cay(G,S)$ is a regular graph. Now, the above condition becomes

$$|N(A)| = |A \cup \bigcup_{i=1}^{k} s_i A| \geq (1+\varepsilon)|A|.$$

Dimension expanders and non-hyperfinite families

Definition (Barak-Impagliazzo-Shpilka-Wigderson)

k a field, $d \in \mathbb{N}$, $\alpha > 0$, V k-vector space, and $T_1, \dots, T_d$ k-linear endomorphisms of V. The pair $(V, \{T_i\}_{i=1}^d)$ is an α-**dimension expander of degree** d if $\forall W \subset V$ with $\dim W \leq \frac{\dim_k V}{2}$, we have $\dim_k \left(W + \sum_{i=1}^d T_i(W)\right) \geq (1+\alpha) \dim_k W$.

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Proposition

k be a field, $d \in \mathbb{N}$ and $\alpha > 0$. If $\{(V_i, \{T_l^{(i)}\}_{l=1}^d)\}_{i \in I}$ is a sequence of α-dimension expanders of degree d s.t. $\dim V_i$ is unbounded, then the induced family of $k\Theta(d+1)$-modules $M_i = \left((V_i, V_i), \left(\mathrm{id}, T_1^{(i)}, \ldots, T_d^{(i)}\right)\right)$ is not hyperfinite.

Sketch of proof

$$
\begin{tikzcd}
M = V \arrow[rr, bend left=50, "\mathrm{id}"] \arrow[rr, bend left=25, "T_1"'] \arrow[rr, bend right=30, "T_d"'] & \vdots & V
\end{tikzcd}
$$

All small summands of M, say $W_I \overset{\to}{\underset{\to}{\cdot\cdot}} Z_I$, must have $\dim Z_I \leq (1+\alpha) \dim W_I$. But in the source vertex, we also need $\sum_I W_I \geq (1-2\varepsilon) \dim V$. A contradiction.

Constructing an example

Problem (Wigderson '04)

For fixed field k, fixed d, fixed α, find α-dim. expanders of degree d of arbitrarily large dimension.

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Solutions

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Corollary

Let k a field, char $k = 0$*. Then the wild Kronecker algebra $K\Theta(3)$ is not of amenable representation type.*

A construction

Proposition

If $\rho\colon \Gamma \to U_n(\mathbb{C})$ *is an irreducible unitary representation, then* $(\mathbb{C}^n, \rho(S))$ *is an* α*-dimension expander of degree* $|S|$ *where* $\alpha = \frac{\kappa^2}{12}$*,* $\kappa = K_\Gamma^S(S\ell_n(\mathbb{C}), \operatorname{adj}\rho)$*, where* $S\ell_n(\mathbb{C})$ *denotes the subspace of all linear transformations of zero trace, and* $\operatorname{adj}\rho$ *is the adjoint representation on* $\mathrm{End}(\mathbb{C}^n)$ *induced by conjugation.*

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Now,

- find representations of $\mathrm{SL}(2, p)$ of arbitrarily large dimension (Steinberg)
- $SL(2, \mathbb{Z})$ has property (τ) (inspired by property (T))
- proved via an application of Selberg's $\frac{3}{16}$ Theorem

An example

$\{((k^p, k^p), (\mathrm{id}, T_p, S_p))\}_{p\in\mathbb{P}}$, where

$$T_p = \begin{pmatrix} 0 & \dots & 0 & -1 & -1 \\ 1 & & & -1 & -1 \\ & \ddots & & \vdots & \vdots \\ & & 1 & -1 & -1 \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_p(\mathbb{Q}),$$

$$S_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, S_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \dots$$

Strictly wild algebras are not amenable

Definition

A f.d. k-algebra. A is **strictly wild** if $\exists$ orthogonal pair (X, Y) of f.d., f.p. modules, s.t. $\mathrm{End}(X)$, $\mathrm{End}(Y)$ are division rings and

$$p = \dim_{\mathrm{End}_A(Y)} \mathrm{Ext}^1_A(X, Y) \cdot \dim_{\mathrm{End}_A(X)} \mathrm{Ext}^1_A(X, Y) \geq 5.$$

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Theorem

Let A be a finite dimensional k-algebra. If A is strictly wild, then A is not of amenable representation type.

Tools

Proposition

$\{M_i\}_{i\in I} \subseteq \operatorname{mod} A$ non-hyperfinite family of modules. Let $K_1, K_2 > 0$. Functors $F_i\colon \operatorname{mod} A \to \operatorname{mod} B$, $G_i\colon \operatorname{mod} B \to \operatorname{mod} A$ s.t.

- *$G_iF_i(M_i) \cong M_i$ for all $i \in I$,*
- *all G_i are left exact,*
- *$K_1 \dim_k F_i(M_i) \leq \dim_L G_iF_i(M_i)$ for all $i \in I$,*
- *$\dim_L G_i(X) \leq K_2 \dim_k X$ for all $X \in \operatorname{mod} B$ and $i \in I$,*

preserve these counterexamples to hyperfiniteness.

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- *all* G_i *are left exact,*
- $K_1 \dim_k F_i(M_i) \leq \dim_L G_iF_i(M_i)$ *for all* $i \in I$,
- $\dim_L G_i(X) \leq K_2 \dim_k X$ *for all* $X \in \operatorname{mod} B$ *and* $i \in I$,

preserve these counterexamples to hyperfiniteness.

Idea

Use suitable tensor product functor $\operatorname{mod} L\Theta(d) \to \operatorname{mod} A$ for F_is.

A locally wild example

Theorem

The local wild algebra $A = k\langle x_1, x_2, x_3\rangle / M_2$, where M_2 is the ideal generated by all monomials of degree two, is not of amenable representation type.

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Proof.

The functor $F\colon \operatorname{mod} A \to \operatorname{mod} k\Theta(3)$, with

$F(M) = \operatorname{top} M \overset{x_1\cdot -}{\underset{x_3\cdot -}{\xrightarrow{x_2\cdot -}}} \operatorname{rad} M,$ is exact and preserves monomorphisms if we ignore simple modules. $\square$

A problem?

Here, we use that A is a radical square zero algebra.
What functor should one use in general?
If the (restricted) functor is not left exact, can we preserve submodules?

Modify the definition

Definition

k a field, A f.d. k-algebra, $\mathcal{M} \subseteq \operatorname{mod} A$ a family of f.d. A-modules. $\mathcal{M}$ is **weakly hyperfinite** if $\forall \varepsilon > 0 \exists L_\varepsilon > 0$ s.t. $\forall M \in \mathcal{M}\ \exists \theta\colon N \to M$ for some $N \in \operatorname{mod} A$ s.t.

$$\dim_k \ker \theta \leq \varepsilon \dim M, \quad \dim_k \operatorname{coker} \theta \leq \varepsilon \dim M, \tag{2}$$

and $\exists N_1, \ldots, N_t \in \operatorname{mod} A$ with $\dim_k N_i \leq L_\varepsilon$ s.t. $N \cong \bigoplus_{i=1}^t N_i$.

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Remarks

- hyperfinite $\Rightarrow$ weakly hyperfinite
- Kronecker representations induced by dimension expanders are not even weakly hyperfinite

Finitely controlled wild algebras are not amenable

Let k be alg. closed.

Definition (Ringel)

An algebra A is **(finitely) controlled wild** if for any f.d. algebra B $\exists F : \operatorname{mod} B \to \operatorname{mod} A$ faithful exact and $C \in \operatorname{mod} A$ s.t.

1. $\mathrm{Hom}_A(FM, FN) = F(Hom_B(M, N)) \oplus \mathrm{Hom}_A(FM, FN)_{\operatorname{add} C}$, and
2. $\mathrm{Hom}_A(FM, FN)_{\operatorname{add} C} \subseteq \operatorname{rad} \mathrm{End}_A(FM)$.

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2. $\mathrm{Hom}_A(FM, FN)_{\operatorname{add} C} \subseteq \operatorname{rad} \mathrm{End}_A(FM)$.

Theorem

Let A be a finite dimensional k-algebra. If A is finitely controlled wild, then A is not of weakly amenable representation type.

Sketch of proof

Proof.

Use the functor $F\colon \operatorname{mod} k\Theta(d) \to A$ from the definition of controlled wildness. By [GP16, Theorem 4.2], $\exists G\colon \operatorname{mod} A \to \operatorname{mod} k\Theta(d)$ s.t. $(G \circ F)(M) \cong M$ for all $M \in \operatorname{mod} k\Theta(d)$. Indeed, on objects this functor is given by

$$G(X) = {}^{\mathrm{Hom}_A(F(K), X)}\!\big/\!_{\mathrm{Hom}_A(F(K), X)_{\mathcal{C}}},$$

where $\mathrm{Hom}_A(X, Y)_{\mathcal{C}} = \{A\text{-homs } X \to Y \text{ factoring through } \mathcal{C}\}$. Remains to check estimates on dimensions. $\square$

Thank you!

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