# MATH 451/551 Midterm 1

Name: ____________________________________________

This exam contains 12 pages (including this cover page) and 11 questions. Total of points is 42.

Grade Table (for instructor use only)

| Question | Points | Score |
|---|---|---|
| 1 | 3 | |
| 2 | 3 | |
| 3 | 4 | |
| 4 | 3 | |
| 5 | 5 | |
| 6 | 3 | |
| 7 | 3 | |
| 8 | 5 | |
| 9 | 5 | |
| 10 | 5 | |
| 11 | 3 | |
| Total: | 42 | |

1. (3 points) Show that $(A \cup B \cup C)^c = A^c B^c C^c$. You may use DeMorgan's Law.

$$(A \cup B \cup C)^c = A^c (B \cup C)^c = A^c B^c C^c$$

applying DeMorgan's Law twice.

2. (3 points) Show that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC).$$

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B \cup C) - P(AB \cup AC)) \\ &= P(A) + P(B) + P(C) - P(BC) - P(AB) - P(AC) + P(ABC) \end{aligned}$$

applying our rule for unions twice.

3. (a) (2 points) Let $P(A) = .5$ and $P(B^c) = .3$ and $P(A \cup B) = .8$. What is $P(AB)$?

$$P(AB) = -P(A \cup B) + P(A) + P(B) = -.8 + .5 + .7 = .4.$$

(b) (1 point) What is $P(A^cB)$?

$$P(A^c \cup B) = P(B) - P(AB) = .7 - .4 = .3$$

(c) (1 point) What is $P(AB^c)$?

$$P(AB^c) = P(A) - P(AB) = .5 - .4 = .1$$

4. (3 points) Let $P$ be a probability measure. Show that if $A \subset B$ then $P(A) \leq P(B)$.

$$0 \leq P(BA^c) = P(B) - P(AB) = P(B) - P(A).$$

hence $P(A) \leq P(B)$.

5. (5 points) You are dealt 3 cards from a deck of 52. What is the probability that they all have the same suit?

   There are 4 ways to choose the suit and then $\binom{13}{3}$ ways to choose the cards out of a total of $\binom{52}{3}$ ways. Hence the probability is

$$\frac{4 * \binom{13}{3}}{\binom{52}{3}}$$

6. (3 points) A box contains eight balls. Four of the balls have an "A" written on them and four of them have a "B" written on them. You draw out the balls one by one (without replacement) and place them in a row. How many distinct strings of "A"s and "B"s are there?

   We simply need to choose out of the 8 possible places, the four places for the As to go. So $\binom{8}{4} = 70$.

7. (3 points) A group of n friends contains Larry and Rex. If the friends line up randomly, what is the probability Rex is standing next to Larry?

   Consider two possibilities of orderings: (Rex, Larry, ...) and (Larry, Rex, ...). There are $(n-2)!$ ways to arrange each of the friends in the remaining spots and so $2*(n-2)!$ ways total. However Rex and Larry don't need to stand at the start of the line. The first of them can be in any of $n-1$ positions. Thus there are $2*(n-1)!$ ways for Rex and Larry to stand next to each other. Since there are a total of $n!$ ways to arrange the friends then this probability is $2/n$.

8. (5 points) Of students at W&M, 55% are women and 45% are men. 1 in 100 men have the flu and 2 in 100 women have the flu. If a randomly chosen student has the flu, what is the probability they are a man?

   Let $M$ be the event of being a man and $F$ be the event of getting the flu. Then

   $$P(M|F) = \frac{P(F|M)P(M)}{P(F)} + = \frac{P(F|M)P(M)}{P(F|M)P(M) + P(F|M^c)P(M^c)} = \frac{(.01)(.45)}{(.01)(.45) + (.02)(.55)}$$

9. (5 points) A car factory has three machines labeled A, B, and C. A makes 20% of the cars, B makes 30% of the cars, and C makes 50% of the cars. 6% of the cars made by A are defective, 7% of the cars made by B are defective, and 8% of the cars made by C are defective. What is the probability that a car is defective if it is made by this factory?

   Let $D$ be the event that the car is defective.

   $P(D) = P(D|A)P(A)+P(D|B)P(B)+P(D|C)P(C) = (.2)(.06)+(.3)(.07)+(.5)(.08) = .073.$

10. (5 points) Let $p$ be the probability that I find a pearl when I open an oyster. How many oysters should I open to make sure I have at least a 50% chance of finding at least one pearl? We can assume that finding a pearl in one oyster is independent of finding one in another.

Let $A_i$ be finding a pearl in the $i^{th}$ oyster, then

$$\begin{aligned} P(\text{finding at least one pearl}) = 1 - P(\text{finding no pearls}) &= 1 - P(A_1^c A_2^c A_3^c \cdots A_n^c) \\ &= 1 - \prod_{i=1}^{n} P(A_i) \\ &= 1 - (1-p)^n \end{aligned}$$

and so if $1/2 \leq 1-(1-p)^n$ then $(1-p)^n \leq 1/2$ and so $n \log(1-p) \leq \log(1/2)$ or $n \geq -log(2)/log(1-p)$.

11. (3 points) Let $A$ and $B$ be both disjoint and independent. Let $P(A) = 1/2$. What is $P(B)$?

    $P(AB) = P(\emptyset) = 0$ and so $0 = P(AB) = P(A)P(B) = 1/2P(B)$ hence $P(B) = 0$.