Defn: Independence (Events)

If $A, B \subset S$ we say "A is independent of B" denoted $A \perp\!\!\!\perp B$, if

$$\mathbb{P}(AB) = \mathbb{P}(A)\mathbb{P}(B).$$

Theorem: If $A \perp\!\!\!\perp B$ then

$$\mathbb{P}(A|B) = \mathbb{P}(A).$$

pf.

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)\cancel{\mathbb{P}(B)}}{\cancel{\mathbb{P}(B)}} = \mathbb{P}(A).$$

Ex. What is the prob. we roll at least one 6 in two independent rolls of dice.

$\mathbb{P}(\text{at least one } 6)$

$= 1 - \mathbb{P}(\text{no } 6\text{s})$ "no 6 on roll 1" $) A_1$ $\cap$ "no 6 on roll 2" $) A_2$

$= 1 - \mathbb{P}(A_1 A_2)$

$= 1 - \mathbb{P}(A_1)\mathbb{P}(A_2)$ $A_1 \perp\!\!\!\perp A_2$

$= 1 - (5/6)(5/6)$

$= 11/36$

Counting perspective:

Unordered: sample w/ repl. from $\underbrace{1, \ldots, 6}_{n=6}$ $\underbrace{\text{two}}_{r=2}$ times:

$$\binom{6+2-1}{2} = \binom{7}{2} = \frac{7!}{2!5!} = \frac{7 \cdot 6}{2} = 21 \checkmark$$

$\{1,1\}, \{2,2\}, \{3,3\}, \ldots, \{6,6\}$

$\{1,2\}, \{1,3\}, \ldots$ — $|S|$

one or more 6

$\{1,6\}, \{2,6\}, \{3,6\}, \{4,6\}, \{5,6\}, \{6,6\}$

Ordered: $n=6$, $r=2$ w/repl : $n^r = 6^2 = 36$ — $|S|$

$(1,6)$ $(6,1)$ $(2,6)$ $(6,2), \ldots$

$6+5 = 11$ total

Takeaway: Ordered counting gives same answer as assuming independence.

Theorem: Complementary Independence

If $A \perp\!\!\!\perp B$

1. $A \perp\!\!\!\perp B^c$
2. $A^c \perp\!\!\!\perp B$

Pf. Case 1:

$\mathbb{P}(AB^c) = \mathbb{P}(A) - \mathbb{P}(AB)$

$= \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B)$

② $A^c \perp\!\!\!\perp B$ $= \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B)$

③ $A^c \perp\!\!\!\perp B^c$ $= \mathbb{P}(A)(1-\mathbb{P}(B))$

$= \mathbb{P}(A)\mathbb{P}(B^c)$.

Defn: Mutual Independence

generalizes independence to multiple events

If $\{A_i\}_{i=1}^n$ are a seq. of events, we say they are mutually independent if

for any subsequence $\underbrace{A_{i_1}, A_{i_2}, A_{i_3}, \ldots, A_{i_k}}_{\text{subsequence of length } k}$

$$\mathbb{P}\left(\bigcap_{j=1}^{k} A_{i_j}\right) = \prod_{j=1}^{k} \mathbb{P}(A_{i_j})$$

Ex. $\mathbb{P}(A_1 A_3 A_4) = \mathbb{P}(A_1)\mathbb{P}(A_3)\mathbb{P}(A_4)$

$\mathbb{P}(A_2 A_7 A_{11} A_{12}) = \mathbb{P}(A_2)\mathbb{P}(A_7)\mathbb{P}(A_{11})\mathbb{P}(A_{12})$

etc... for all possible subsequences.

Q: is this the same as

$$\mathbb{P}(A_1 A_2 A_3 \cdots A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2) \cdots \mathbb{P}(A_n)?$$

No. Need to check all subsequences.

Ex. Roll two dice.

$|A|=6$ $A =$ "doubles" $= \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$

$|B|=18$ $B =$ "sum is between 7 and 10 (inclusive)"

$= \{(2,5),(1,6),(3,4),(4,3),(5,2),(6,1),$
$(2,6),(3,5),(4,4),(5,3),(6,2),(3,6),(4,5),(5,4),(6,3),$
$(6,4),(5,5),(4,6)\}$

$|C|=12$ $C =$ "sum is 2, 7 or 8"
$= \{(1,1),\ldots\}$

Q: Mutually independent?

$$\underbrace{\mathbb{P}(ABC)}_{1/36} = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = \left(\tfrac{6}{36}\right)\left(\tfrac{18}{36}\right)\left(\tfrac{12}{36}\right) = \left(\tfrac{1}{6}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{3}\right) = \tfrac{1}{36}$$

double fours

Consider: $BC =$ "sum is 7 or 8" ; $|BC| = 11$

need: $\mathbb{P}(BC) = \mathbb{P}(B)\mathbb{P}(C)$

$$\tfrac{11}{36} \neq \left(\tfrac{1}{2}\right)\left(\tfrac{1}{3}\right) = \tfrac{1}{6}$$

fails for this subsequence.
Hence A, B, C aren't mutually independent.

Defn: Pairwise Independence

$\{A_i\}_{i=1}^{n}$ are pairwise independent if

$$\mathbb{P}(A_iA_j) = \mathbb{P}(A_i)\mathbb{P}(A_j) \quad \forall i \neq j$$

aside:

$$\underset{\mathbb{P}(A)}{\mathbb{P}(AA)} = \mathbb{P}(A)\mathbb{P}(A) = \mathbb{P}(A)^2$$

Q: Equiv. to mutual independence? No.

Ex. $S = \{aaa, bbb, ccc, \underbrace{abc, bca, acb, cba, bac, cab}_{\text{permutations of } abc}\}$

Assume all outcomes are equally likely.

$A_i = \{i^{th}$ place in the triplet is an "a"$\}$

e.g. $A_1 = \{aaa, abc, acb\}$

$A_2 = \{bac, aaa, cab\}$

$A_3 = \{aaa, bca, cba\}$

pairwise independent?

mutually independent?

Check: $\mathbb{P}(A_iA_j) = \mathbb{P}(A_i)\mathbb{P}(A_j) \quad \forall i \neq j$

$\frac{1}{9} = (\frac{1}{3})(\frac{1}{3})$ ✓

$A_iA_j = \{a$ in i^{th} and j^{th} position$\}$

$= \{aaa\}$

hence, these are pairwise independent.

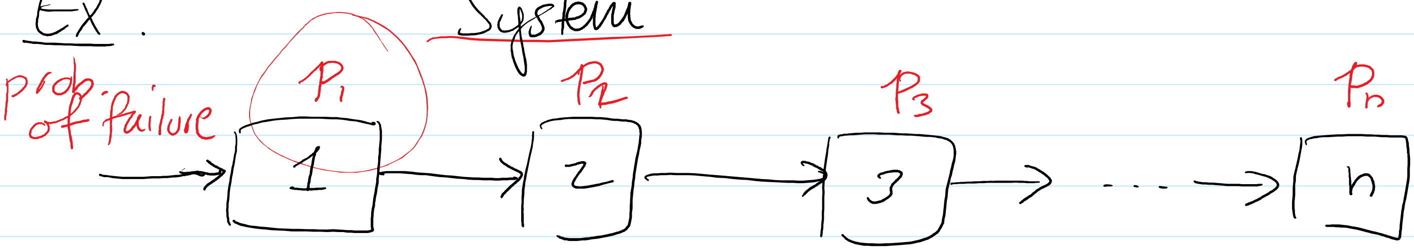
Mutual independence?

$$\underbrace{\mathbb{P}(A_1A_2A_3)}_{\{aaa\}} = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3)$$

$$\tfrac{1}{9} \neq \left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right) = \tfrac{1}{27}$$

No. not mutually independent.

Ex. System



System only works if all components work.

Let F_i = event that the i^{th} component works.

If the $\{F_i\}_{i=1}^{n}$ are (mutually) independent.

What's the prob. the system works?

$\mathbb{P}(\text{system works})$

$= \mathbb{P}(F_1 \cap F_2 \cap F_3 \cap \cdots \cap F_n)$

$= \mathbb{P}(F_1)\mathbb{P}(F_2)\mathbb{P}(F_3)\cdots\mathbb{P}(F_n)$

recall that $p_i = \mathbb{P}(i^{th} \text{ component fails})$

$= \mathbb{P}(F_i^c)$

$= 1 - \mathbb{P}(F_i)$

or $\mathbb{P}(F_i) = 1 - p_i$

or $\mathbb{P}(F_i) = 1 - P_i$

$= (1-P_1)(1-P_2)(1-P_3)\cdots(1-P_n).$

Exam 1 materials up to this point

Random Variables

Often we want to summarize outcomes in S.

Ex. Flip a coin 3 times.

$S = \{HHH, HHT, \ldots\}$

Let X = # of heads in outcome

For each $s \in S$ we get a number $X(s) \in \mathbb{R}$ in this

Outcomes (s)	$X(s)$
HHH	3
HHT	2
HTH	2
HTT	1
THH	2
THT	1
TTH	1
TTT	0

Defn: Random Variable

a random variable X is a function
r.v.

$$X : S \longrightarrow \mathbb{R}$$

Ex. Experiment	R.V.
(1) toss two dice	X = sum of dice
(2) toss a coin 20 times	X = longest chain of consecutive Hs.
(3) observe rainfall	X = yeild of crops.

Idea: go from asking about prob. of events to questions about the r.v.s

e.g. "$P(X=5)$"