Lecture 5 - Unordered Counting

Thursday, February 6, 2020 1:57 PM

	w/ repl.	w/o repl.
ordered	n^r	$\frac{n!}{(n-r)!}$
unordered		

Theorem: Sampling Unordered w/o Repl.

The number of ways to choose a sample of r items from a total of n when we care about ordering and w/o replacement is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \text{"}n \text{ choose } r\text{"}$$

aside:
- binomial coefficient: c.f. $(x+y)^n$ = polynomial w/ coefficients rel. to $\binom{n}{r}$
- e.g. $(x+y)^2 = x^2 + 2xy + y^2$

Pf.

Simple example:

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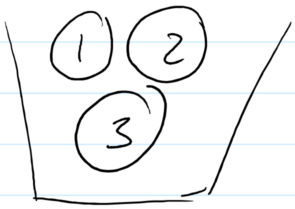


draw $r=2$ from the $n=3$
w/o replacement

If $\boxed{\text{order}}$ matters:

$2!=2$

(1, 2)	(1, 3)	(2, 3)
(2, 1)	(3, 1)	(3, 2)

6 possibilities

$$\frac{n!}{(n-r)!} = \frac{3!}{(3-2)!} = 6$$

If order doesn't matter:

$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$

3 possibilities

In general if I have an unordered sample of
r items $\longrightarrow \{a, b, c, d, \ldots\}$

I can use this unordered sample to create
$r!$ ordered samples

$$\underbrace{\left(\#\text{ of ordered samples}\right)}_{n!} = r!\left(\#\text{ of unordered samples}\right)$$

$$\frac{}{(n-r)!} = r!\,(\underbrace{\#\text{ of unordered}}_{\text{what I want}})$$

$$\#\text{ of unordered samples w/o repl} = \frac{n!}{r!\,(n-r)!} = \binom{n}{r}$$

Ex. I have 10 profs., how many co-equal committees can I form of 4 members?

unordered

ans: $\binom{10}{4} = \boxed{\frac{10!}{4!\,(10-4)!}}$ accept for exam

$$= \frac{10\cdot \cancel{9}^{3}\cdot \cancel{8}\cdot 7}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}} = 10\cdot 3\cdot 7$$

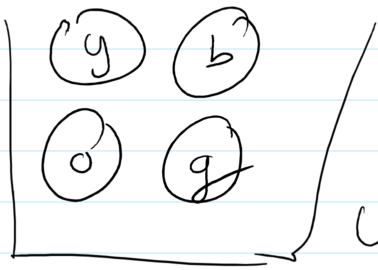
$$= 210$$

Ex. How many 5-card poker hands are possible in a deck of 52 cards?

Sampling w/o ordering or replacement:

$$\binom{52}{5} = \frac{52!}{5!\,(52-5)!} \approx 2.5\text{ mil}$$

Ex. I have a Jar w/ a yellow, blue, orange and green marble.



I choose 3 at random (w/o repl.)
(all samples of 3 are equally likely)

What is the prob. I choose a sample containing yellow and blue?

$S = \{$all 3-samples from 4 w/o repl.$\}$

$$E = \{\{y, b, o\}, \{y, b, g\}\}$$

$$P(E) = \frac{|E|}{|S|} = \frac{2}{\binom{4}{3}} = \frac{2}{4} = 1/2.$$

Theorem: Unordered w/ Replacement

The number of size r samples that may be drawn from a collection of n items w/o caring about ordering and w/ replacement is

$$\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!}$$

$$\Big(n+r-1\Big)$$

$$= \binom{n+r-1}{n-1}$$

Temptation:

true for sampling w/o replacement

$$(\#\text{ ordered}) = r!\,(\#\text{ unorded})$$

not true when sampling w/ repl.

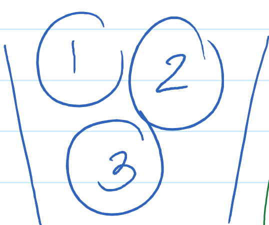
$$\cancel{n^r = r!\,(\#\text{ unorded})}$$

$$\cancel{(\#\text{ unorded}) = n^r/r!}$$

Wrong.

Ex.

$r=2 \quad n=3$



Ordered samples w/ replacement

$(1,2)$ $(1,3)$ $(2,3)$ $(1,1)$ $(3,3)$

$(2,1)$ $(3,1)$ $(3,2)$ $(2,2)$

b/c we allow replacement

Unordered Samples w/ replacement

$\{1,2\}, \{1,3\}, \{2,3\}, \{1,1\}, \{2,2\}, \{3,3\}$

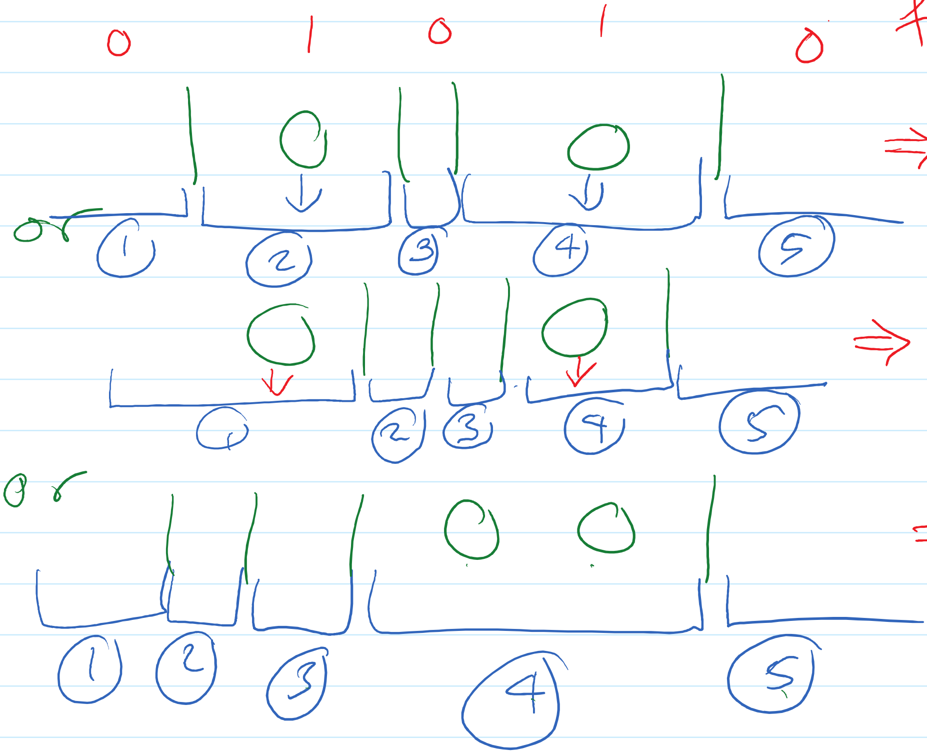
So our previous route to a proof fails when we allow replacement.

Game of Partitioning

Ex. $n=5 \quad r=2$

How many ways can I partition 2 objects using 4 walls

each "partition" corresp. to an unordered sample of $r=2$ from $n=5$ (vice versa)



$\Rightarrow \{2,4\}$

$\Rightarrow \{1,4\}$

$\Rightarrow \{4,4\}$

# of unordered samples w/ repl. $\Longleftrightarrow$ # ~~of~~ distinct partitions I can make

want this

want this

make

count this

Q: how many ways can I partition r objects w/ $n-1$ walls?

I have: $\underbrace{\circ\ \circ\ \circ \cdots \circ}_{r \text{ objects}}$ $\underbrace{|\ |\ | \cdots |}_{n-1 \text{ walls}}$

in total I have $\boxed{r+n-1}$ things to put in a certain order

Each "partition" is a permutation of these $r+n-1$ things

swap

$\boxed{\circ\circ|\circ||\circ \cdots |\circ}$

swap

notice: I can swap any two objects or any two walls and get the same picture

there are $r!$ ways to permute the objects
" $n-1!$ " walls
and get the same partition (picture)

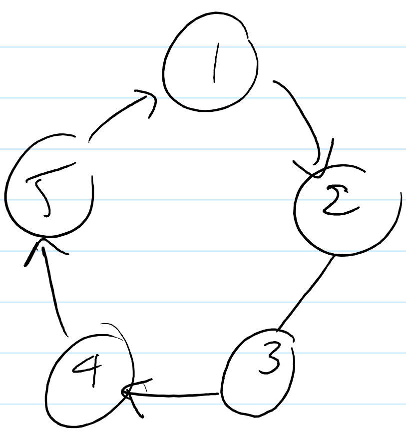
So I have, really,

So I have, really,

$$\binom{r+n-1}{r} = \frac{(r+n-1)!}{r!\,(n-1)!} = \#\text{ of distinct partitions.}$$

Ex. 10 passengers on a bus on a route w/ 5 hotels.

The bus driver counts how many people get off at each hotel.



record

Stop	# get off
1	0
2	3
3	1
4	2
5	4

Can convert to sample w/ repl. w/o ordering:

$\{2,2,2,3,4,4,5,5,5,5\}$

sample from $1,\dots,5$ of size 10 w/ repl.

How many possible records are there?

Formula: $\binom{r+n-1}{r} = \binom{10+5-1}{10}$

$$= \binom{14}{10} = \frac{14!}{10!(14-10)!} = 1001$$

Ex. Jar contains 1 yellow, blue, orange, green marble.
draw a sample of size 3 w/ repl.
What is the prob. of the sample contain y and b?
(all such samples are equally likely)

$E = \{\{y,b,y\}, \{y,b,o\}, \{y,b,b\}, \{y,b,g\}\}$

$S = \{$all unord. samp. of 3 from 4 w/ rep.$\}$

$$P(E) = \frac{|E|}{|S|} = \frac{4}{\binom{3+4-1}{3}} = \frac{4}{\binom{6}{3}} = \frac{4}{20} = 1/5.$$

	w/ rep	w/o rep
ordered	n^r	$\frac{n!}{(n-r)!}$
unordered	$\binom{n+r-1}{r}$	$\binom{n}{r} = \frac{n!}{r!(n-r)!}$

Conditional Probability

Ex. Survey all W&M students

pol. party affil.

gender

	A	B	
men	501	238	739
women	782	123	905
	1283	361	1644

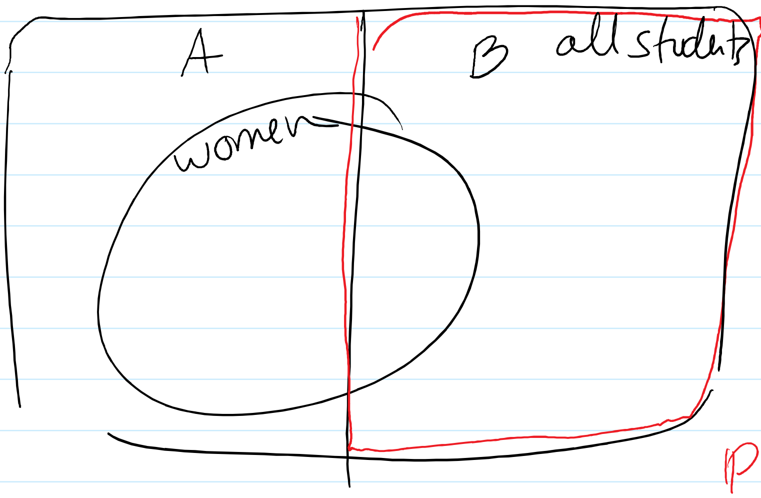
Q: If I randomly select a student what is prob. they are a woman?

$$\mathbb{P}(\text{woman}) = 905/1644 \approx 55\%$$

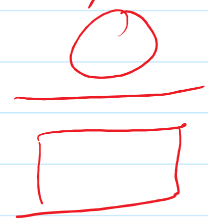
Q: GIVEN the student is part of party B, what is the prob. they are a woman?

$$\mathbb{P}(\text{woman GIVEN in } B) = \frac{123}{361} \approx 34\%$$

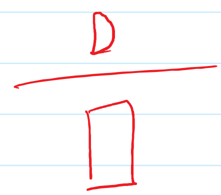
called conditional prob.



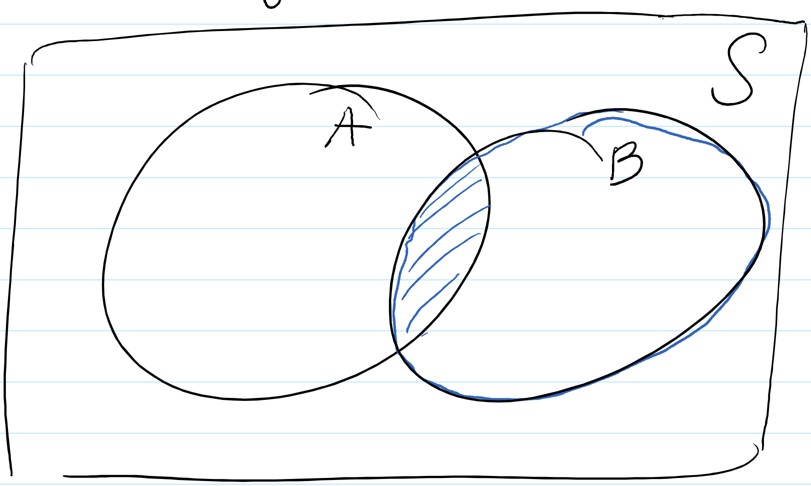
$\mathbb{P}$(woman)

= 

$\mathbb{P}$(woman given B)

= 

Generically:



$\mathbb{P}$(A given B)

$= \dfrac{\text{area of } AB}{\text{area of } B}$

Defn: Conditional Probability

If $A, B \subset S$ then

$$P(A \mid B) \stackrel{\text{def}}{=} \frac{P(AB)}{P(B)} \quad \left(\begin{array}{c}\text{so long as} \\ P(B) > 0\end{array}\right)$$

read: prob. of A given B

prob. of A given B

Facts: Assume $P(B) > 0$

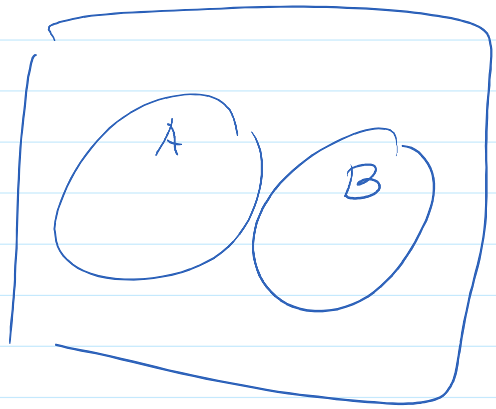
(1) $\boxed{P(B|B) = 1}$

Pf. $P(B|B) = \dfrac{P(BB)}{P(B)} = \dfrac{P(B)}{P(B)} = 1.$

(2) If A and B are disjoint

then

$$P(A|B) = 0$$



$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(\emptyset)}{P(B)}$$

$$= \frac{0}{P(B)} = 0.$$