3.52 Consider the random variable X with cumulative distribution function

$$F(x) = \begin{cases} 0 & x < -1 \\ x/5 + 3/10 & -1 \le x < 0 \\ 2/5 & 0 \le x < 1 \\ x/5 + 2/5 & 1 \le x < 3 \\ 1 & x \ge 3. \end{cases}$$

Find $E[X]$.

3.53 Find the population mean $\mu = E[X]$ and the population variance $\sigma^2 = V[X]$ for the discrete random variable X defined by the probability mass function

$$f(x) = \begin{cases} 1-p & x = -1 \\ p & x = 1 \end{cases}$$

for some fixed real parameter p satisfying $0 < p < 1$.

3.54 Consider the discrete random variable X with probability mass function

$$f(x) = \frac{1}{3} \qquad x = -2, 0, 1.$$

(a) Find $E\left[|X| + 4\right]$.

(b) Find $V\left[|X| + 4\right]$.

3.55 Peter visits St. Petersburg and discovers a paradox in the following gambling game. Peter flips a fair coin repeatedly until a head appears. If the first head appears on flip x, then Peter wins 2^x dollars.

(a) What are Peter's expected winnings?

(b) If it costs Peter $17 to play this game, should he play? Comment.

3.56 A spinner yields three equally-likely outcomes: 1, 2, 3.

(a) If the random variable X denotes the sum of the outcomes of two spins, find $E[X]$.

(b) If the random variable Y denotes the product of the outcomes of two spins, find the 99th percentile of Y.

3.57 The continuous random variable X has a probability distribution that is described by the probability density function

$$f(x) = 6x(1-x) \qquad 0 < x < 1.$$

Compute the population mean $\mu = E[X]$.

3.58 For a continuous random variable X with probability density function $f(x)$, what constant c minimizes $E\left[|X - c|\right]$?

3.59 The monthly cost for gasoline and oil for an automobile is a random variable with population mean $45 and population standard deviation $10. The government is considering placing a 10% tax on retail purchases of gasoline and oil. Give the population mean and population variance of the monthly cost of gasoline and oil after the tax is implemented.

3.60 A *Shuttles-R-Us* van can transport 10 passengers from the airport to the convention center at a price of \$40 per passenger. Because of no-shows, management is deciding whether to sell 10 or 11 passenger tickets per van. They enlist your services as an applied probabilist. Assume that each ticketed passenger shows up with probability p independent of all other passengers, where $0 < p < 1$, and a ticketed passenger who is turned away due to the lack of a seat is refunded their ticket price and also receives an additional \$100 from *Shuttles-R-Us*. Ignoring customer goodwill or ill will, for what values of p is overbooking an effective strategy for *Shuttles-R-Us*?

3.61 The probability distribution of the random variable X is defined by the probability density function

$$f(x) = c\,|x| \qquad -1 < x < 3,$$

for some positive constant c.

(a) Find c.

(b) Find $E[X]$.

3.62 Mike spins a spinner three times. The spinner has three equally-likely outcomes: 0, 1, and 2, The random variable X is the product of the three outcomes. Find $E[X]$ and $V[X]$.

3.63 The random variable X has probability density function

$$f(x) = 2xe^{-x^2} \qquad x > 0.$$

(a) Find the 95th percentile, $x_{0.95}$, of this distribution.

(b) Find the population mode of this distribution.

3.64 Let X be a continuous random variable whose first three moments exist. Furthermore, the probability distribution of X is described by a probability density function $f(x)$ which is an even function, that is, $f(x) = f(-x)$ for all values of x. Find $E\left[X^3\right]$.

3.65 Let the random variable X have probability mass function

$$f(x) = \begin{cases} 1-p & x = -1 \\ p & x = 1, \end{cases}$$

where p is a real-valued parameter satisfying $0 < p < 1$. Find $E\left[X^3\right]$.

3.66 A biased coin shows heads when flipped with probability p, where p is a real number satisfying $0 < p < 1$. Let the random variable X be the number of tosses of this biased coin required to observe the second head or the second tail, whichever occurs first.

(a) Find the probability mass function of X.

(b) Find an expression for $E[X]$ as a function of p.

3.67 Let X be a random variable with moment generating function $M_X(t)$. Show that

$$M_{a+bX}(t) = e^{at} M_X(bt)$$

for real-valued constants a and b.

3.68 Let the random variable X have probability density function

$$f(x) = \frac{1}{2} + m(x-1) \qquad 0 < x < 2,$$

where m is a parameter.

(a) What are the restrictions on the value of the parameter m?

(b) Find $E[X]$.

3.69 Let $x_1, x_2, \ldots, x_n$ be real-valued constants satisfying $x_1 < x_2 < \cdots < x_n$. A piecewise-linear cumulative distribution function $F(x)$ for an associated random variable X can be formed by connecting the points

$$(x_1, 0), \left(x_2, \frac{1}{n-1}\right), \left(x_3, \frac{2}{n-1}\right), \ldots, \left(x_{n-1}, \frac{n-2}{n-1}\right), (x_n, 1)$$

with line segments.

(a) Find the associated probability density function $f(x)$.

(b) Find $E[X]$. Simplify your formula as much as possible.

(If $x_1, x_2, \ldots, x_n$ are data values, then the cumulative distribution function constructed in this manner is known as the *piecewise linear empirical cumulative distribution function*.)

3.70 What is the expected number of tosses of a fair coin required to observe three consecutive heads?

3.71 Consider the random variable X with moment generating function $M_X(t) = \cosh t$, for $-\infty < t < \infty$.

(a) Find all of the moments of X.

(b) Find the probability mass function of X.

3.72 There are 15 billiard balls numbered $1, 2, \ldots, 15$ in a bag. Noah draws balls sequentially at random and without replacement until either two even-numbered balls or two odd-numbered balls have been drawn. Let X be the number of draws that are required. Find $f(x)$, $E[X]$, and $V[X]$.

3.73 Find $E[X]$ for the discrete random variable X with probability mass function

$$f(x) = \begin{cases} 1/3 & x = 0 \\ 2/3 & x = 3. \end{cases}$$

3.74 Find $E[X(3-X)]$ for the discrete random variable X with probability mass function

$$f(x) = \begin{cases} 1/3 & x = 0 \\ 2/3 & x = 3. \end{cases}$$

3.75 Find $E\left[X^{\pi}\right]$ for the continuous random variable X with probability density function

$$f(x) = \frac{1}{4} \qquad 2 < x < 6.$$

3.76 The expected value of a random variable X, denoted by $E[X]$, is defined by two separate formulas in Definition 3.3. One formula is for discrete random variables and involves a summation; the other formula is for continuous random variable and involves an integral. A single formula for handling discrete random variables, continuous random variables, and mixed discrete–continuous random variables in terms of the cumulative distribution function of X, that is, $F(x)$, is

$$\begin{aligned} E[X] &= -\int_{-\infty}^{0} P(X \le x)\, dx + \int_{0}^{\infty} (1 - P(X \le x))\, dx \\ &= -\int_{-\infty}^{0} F(x)\, dx + \int_{0}^{\infty} (1 - F(x))\, dx. \end{aligned}$$

Apply this formula to

(a) a discrete random variable X with probability mass function

$$f(x) = \frac{1}{3} \qquad x = -1, 0, 1,$$

(b) a continuous random variable X with probability density function

$$f(x) = \frac{1}{3} \qquad -1 < x < 2,$$

(c) a mixed discrete–continuous random variable X that is an equally-likely mixture of the two distributions given in parts (a) and (b).

3.77 Mainak rolls a fair die that has 1 and 6 on opposite sides, 2 and 5 on opposite sides, and 3 and 4 on opposite sides. Find the expected *product* of the number of spots on the up and down sides.

3.78 Let the continuous random variable X have probability density function

$$f(x) = cx \qquad 0 < x < 4,$$

where c is a positive real constant.

(a) Find the value of c so that $f(x)$ is a legitimate probability density function.

(b) Find $P(X > 1)$.

(c) Find $E[X]$.

3.79 Consider the random variable X with cumulative distribution function

$$F(x) = \begin{cases} 0 & x \le 0 \\ x^2/36 & 0 < x < 6 \\ 1 & x \ge 6. \end{cases}$$

What is the expected value of X?

3.80 Anh chooses three points at random on the circumference of a circle of radius 1. Use Monte Carlo simulation to estimate the expected area of the triangle created by his three points. Report your estimate to three-digit accuracy. *Hint:* Heron's formula can be used to calculate the area of a triangle from the lengths of its three sides.

3.81 The random variable X has moment generating function

$$M(t) = 0.2e^{4t} + 0.7e^{7t} + 0.1e^{9t} \qquad -\infty < t < \infty.$$

Find $P(X = 7)$.

3.82 Write APPL code to find the population kurtosis of the random variable X with probability density function

$$f(x) = c \cdot \sin x \qquad 0 < x < \pi,$$

where c is a real constant that is chosen to satisfy the existence conditions for a probability density function. Key these APPL statements into Maple and write the resulting population kurtosis as an exact value and the floating point value to four digits. Write a Monte Carlo simulation in R to support the analytic solution. *Hint:* Use `Pi` rather than `pi` for π in your APPL code.

3.83 The probability of winning a game of craps is 244/495. If Vladi places \$1 bets on two consecutive games of craps, find his expected winnings. (Note: if he wins a game, his winnings are \$1; if he loses a game, his winnings are −\$1.)

3.84 Let X be the number of *black jacks* (that is, jack of ♣ or jack of ♠) in a five-card poker hand dealt from a well-shuffled deck. Find $E[X]$.

3.85 Let the distribution of the random variable X be described by the probability density function

$$f(x) = \frac{1}{5} \qquad 0 < x < 5.$$

Find the population mean and variance of

$$3\lceil 2X \rceil + 4.$$

3.86 Let X be a continuous random variable. Find the population mean and variance of $Y = \lceil X \rceil - \lfloor X \rfloor$.

3.87 For the probability mass function defined by

$$f(x) = \begin{cases} 4/5 & x = 0 \\ 1/5 & x = 2, \end{cases}$$

what is the coefficient of variation, σ/μ?

3.88 Consider a discrete random variable X with cumulative distribution function $F(x)$ defined on the positive integers. If $F(17) < 1/2$ and $F(18) > 1/2$, what is the population median of the distribution?

3.89 Anke chooses two points at random on the circumference of a circle of radius 1.

(a) Find the 50th percentile (that is, the population median) of the length of the chord connecting her two points (this can be done by inspection—no math needed).

(b) Write a Monte Carlo simulation using 10,001 random pairs of points that supports your solution to part (a).

3.105 Find the population mean of the distribution defined by the probability density function

$$f(x) = \theta x^{\theta - 1} \qquad 0 < x < 1,$$

where θ is a positive parameter.

3.106 Consider the random variable X with probability density function

$$f(x) = \theta x^{\theta - 1} \qquad 0 < x < 1,$$

where θ is a positive parameter. For what value of θ is the population variance of X maximized?

3.107 A bag contains 100 bills, 80 of which are authentic bills and 20 of which are counterfeit bills. Bills are drawn sequentially and without replacement from the bag. Let X be the number of counterfeit bills drawn prior to the *third* authentic bill being drawn.

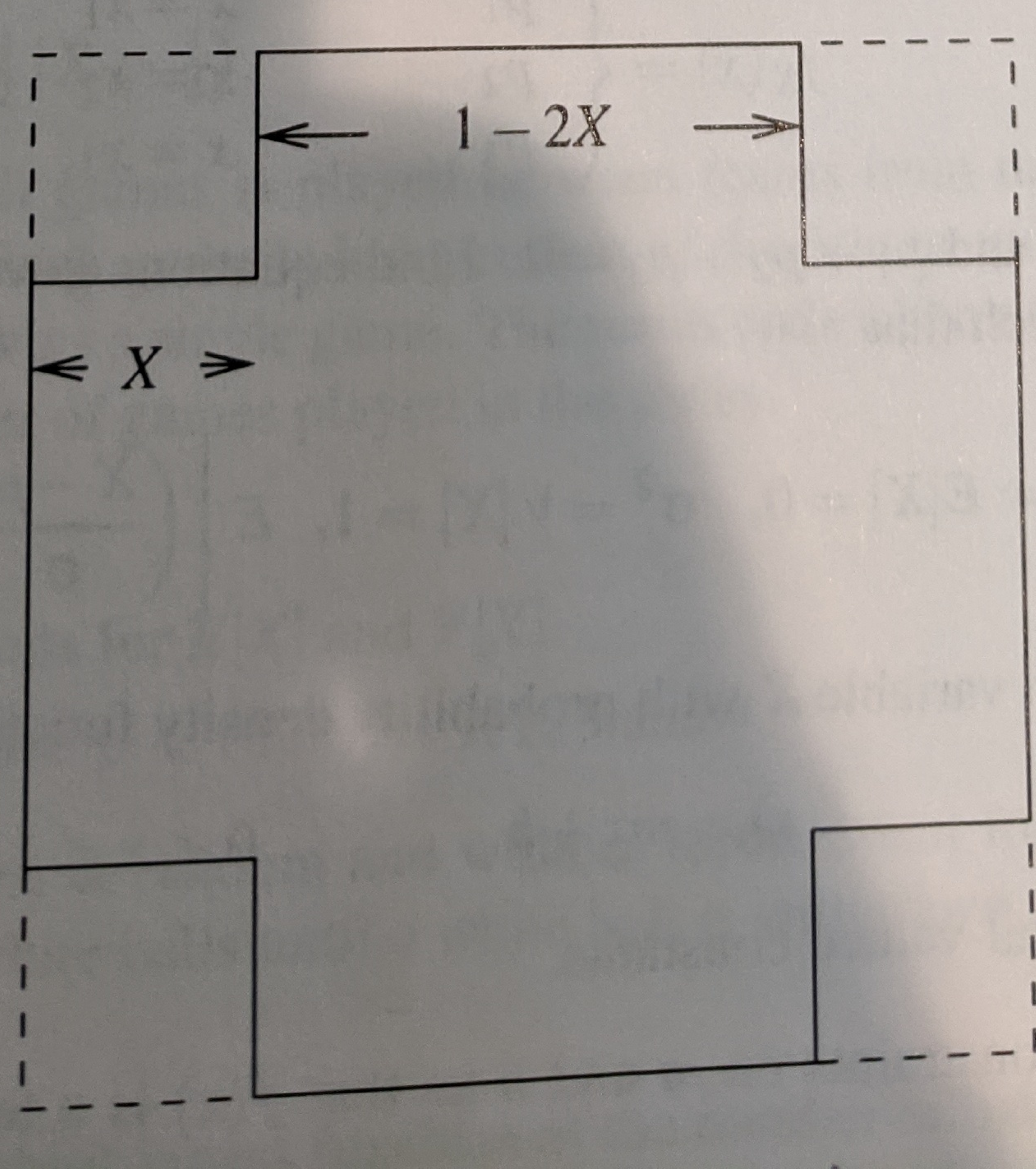
(a) Find the probability mass function of X.

(b) Write and execute a computer program that calculates the population mean of X.

3.108 Tanujit spies a square sheet of paper that measures 1 foot by 1 foot. He cuts four identical squares out of each corner with a random side length X (measured in feet), where X is a continuous random variable with probability density function

$$f(x) = 8x \qquad 0 < x < \frac{1}{2}.$$

Find the expected volume of the box, illustrated in Figure 3.33, that Tanujit has created when he folds up the sides.



3.117 If $E[X]=2$, $E[X^2]=5$, $E[X^3]=0$, and $E[X^4]=30$, find

(a) $E[(X-\pi)^3]$,

(b) $V[17-4X]$,

(c) $V[X^2]$.

3.118 Consider the continuous random variable X with probability density function

$$f(x)=\frac{3}{2}x^2 \qquad -1<x<1.$$

(a) Find the population median of this distribution.

(b) Find the interquartile range of this distribution.

(c) Find $P(X>1/3)$.

3.119 Consider the continuous random variable X with probability density function

$$f(x)=cx^2 \qquad -1<x<1,$$

where c is a real-valued constant.

(a) Find c.

(b) Find $F(x)$. Plot $f(x)$ and $F(x)$.

(c) Find $E[X]$ and $V[X]$.

(d) Find $E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right]$.

(e) Find $P\left(X>\frac{1}{2} \mid X>\frac{1}{4}\right)$.

3.120 A series of baseball games is played between teams from the American League and the National League. Each game is independent of the others and p is the probability that the American League wins a single game. The series ends when one of the teams wins n games. Let X be the number of games played in the series.

(a) Find $f(x)$.

(b) Give expressions for $E[X]$ and $V[X]$.

(c) Plot $E[X]$ on $0<p<1$ for $n=1$, 2, 3 and 4.

3.121 Marco removes balls at random and without replacement from an urn that contains three red balls and two white balls until a white ball is withdrawn. Let X be the number of balls removed.

(a) Find the probability mass function of the number of balls removed, $f(x)$.

(b) Find the expected number of balls removed, $E[X]$.

...umber of balls removed, $V[X]$.

3.129 The waiting time in a queue is a mixed discrete–continuous random variable X with cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - 0.7e^{-x} & x \ge 0. \end{cases}$$

Find $E[X]$ and $V[X]$.

3.130 Find the population mean of the random variable X with probability density function

$$f(x) = \begin{cases} \dfrac{2(x-a)}{(b-a)(m-a)} & a < x < m \\ \dfrac{2(b-x)}{(b-a)(b-m)} & m \le x < b, \end{cases}$$

where $a < m < b$ are real numbers.

3.131 For a random variable X with cumulative distribution function $F(x)$, identify each of the following as a *constant*, *random variable*, *event*, *undefined*, or *none of the above*.

(a) $\cos X < 0.3$

(b) $E[\cos X < 0.3]$

(c) $V\left[\sqrt{X}\right]$

(d) X^2

(e) $F(3)$

3.132 A bag contains three balls numbered -1, 0, and 2. Two balls are sampled at random. Find the expected value of the product of the numbers on the two balls that are sampled when

(a) sampling is performed without replacement,

(b) sampling is performed with replacement.

3.133 A continuous random variable X has probability density function

$$f(x) = c\left(x^2 + 2x\right) \qquad 0 < x < 1.$$

(a) Find the constant c so that $f(x)$ is a legitimate probability density function.

(b) Find $E[X]$.

3.134 Consider a random variable X with moment generating function

$$M(t) = e^{3t} \qquad -\infty < t < \infty.$$

(a) Find $E[X]$.

(b) Find $V[X]$.

(c) Find $f(x)$.

3.135 Consider a random variable X with moment generating function

$$M_X(t) = \frac{1}{2}e^{-ct} + \frac{1}{2}e^{ct} \qquad -\infty < t < \infty,$$

for some positive constant c. Find the value of c so that $V[X] = 1$.