Lecture 11 - Expectation and Variance

Tuesday, March 3, 2020 11:00 AM

Defn: Expectation / Expected Value / Mean

(i) Discrete : $\mathbb{E}[X] = \sum_x x f(x)$

(ii) Continuous: $\mathbb{E}[X] = \int_{\mathbb{R}} x f(x)\,dx$

Ex. $X \sim \text{Bin}(n, p)$

= flip n independent coins w/ prob p of H,

X = # of heads

$\text{Support}(X) = \{0, 1, 2, \dots, n\}$

the pmf is

$$f(x) = \mathbb{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

Note: to show this is a valid pmf

① $f(x) \geq 0$

② $\sum_x f(x) = 1$ (Binomial Theorem)

$$(x+y)^n = \sum_{x=0}^{n} (\dots)$$

$$\mathbb{E}[X] = \sum_{x=0}^{n} x f(x)$$

$$= \sum_{x=\cancel{0}1}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

aside:

$$x \binom{n}{x} = x \frac{n!}{x!\,(n-x)!}$$

$$= \frac{n!}{(x-1)!\,(n-x)!}$$

$\underline{n} \quad (n-1) \quad x, \quad {}^{n-x}$

$x = \cancel{0}\,1$

define $y = x - 1 \Leftrightarrow x = y + 1$

$$= \sum_{y=0}^{n-1} n \binom{n-1}{y} p^{y+1} (1-p)^{n-y-1}$$

$$= n \frac{(n-1)!}{(x-1)!((n-1)-(x-1))!} = n \binom{n-1}{x-1}$$

$$= \underbrace{np \sum_{y=0}^{n-1} \underbrace{\binom{n-1}{y} p^{y} (1-p)^{(n-1)-y}}_{\text{pmf of } Bin(n-1,p)}}_{\substack{\text{Sum of pmf of } Bin(n-1,p) \\ \text{over support} \\ = 1}}$$

$Y \sim Bin(n-1, p)$

$f(y) = \underbrace{\binom{n-1}{y} p^{y} (1-p)^{(n-1)-y}}$

$Support(Y) = \{0, 1, \ldots, n-1\}$

$\boxed{= np}$ = (# of trials)(prob. of success for each)

doesn't have to be in the support (integer)

General trick:

Convert tricky sums / integrals to sums / integrals of pmf/pdfs.

Theorem: Law of the Unconscious Statistician

$$E[g(X)] = \begin{cases} \sum_x g(x) f(x) & \text{discrete} \\ \ldots \end{cases}$$

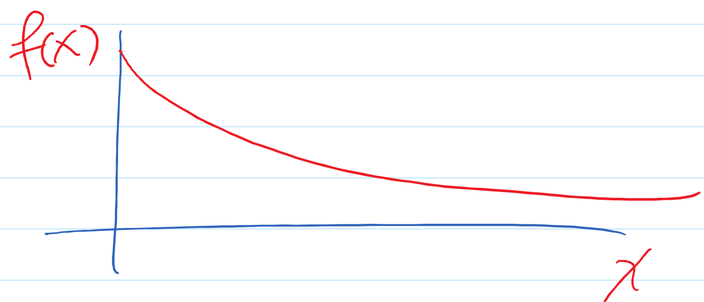
$$\int_{\mathbb{R}} g(x) f(x)\,dx$$

Ex. $X \sim \mathrm{Exp}(\lambda)$

$$f(x) = \lambda e^{-\lambda x} \quad \text{for } x > 0$$

Saw

$$\mathbb{E}[X] = 1/\lambda$$



Q: $$E[X^2] = \int x^2 f(x)\,dx$$

$$= \int_0^\infty x^2 \lambda e^{-\lambda x}\,dx = \int u\,dv$$

by Parts: $u = x^2$, $du = 2x\,dx$ $\qquad$ $v = -e^{-\lambda x}$, $dv = \lambda e^{-\lambda x}\,dx$

$$= uv - \int v\,du$$

$$= \left[-x^2 e^{-\lambda x}\right]_0^\infty - \int_0^\infty -e^{-\lambda x}\, 2x\,dx$$

$$= 0 + \frac{2}{\lambda}\int_0^\infty \lambda x e^{-\lambda x}\,dx$$

$$E[X] = \int x f(x)\,dx$$

$$= \frac{2}{\lambda} \cdot \frac{1}{\lambda}$$

$$= 2/\lambda^2$$

Q: Does expected value always exist? No.

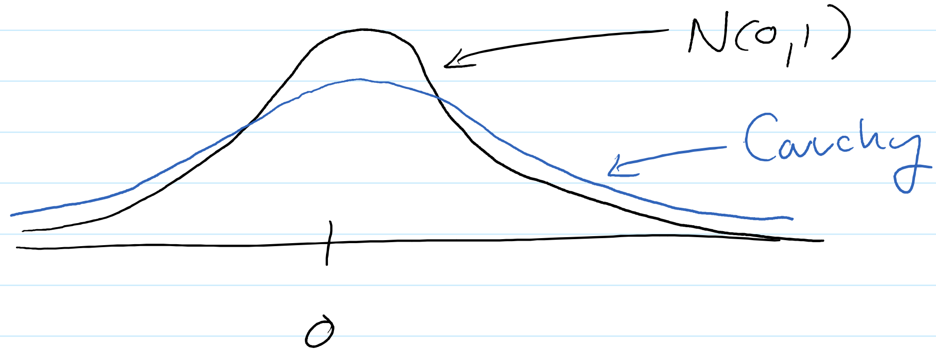
improper integral:

$$\int_0^\infty \frac{1}{x^2}\,dx \quad \text{has a meaningful value}$$

$$\int_0^\infty \frac{1}{x}\,dx \quad \text{no meaningful value}$$

Ex. X has a Cauchy distribution

$$f(x) = \frac{1}{\pi} \frac{1}{1+x^2} \quad \text{for } x \in \mathbb{R}$$



$$E[X] = \int^\infty x \frac{1}{1+x^2}\,dx$$ doesn't converge $= \infty$

$$E[X] = \int_{-\infty} x \underbrace{\frac{1}{1+x^2}} dx$$ doesn't converge $= \infty$

$$\frac{x}{1+x^2} \underset{\text{asymptotically}}{\approx} 1/x$$

Theorem: Properties of $\mathbb{E}$

(a) $E[aX + b] = aE[X] + b$ $\quad a, b \in \mathbb{R}$

(linearity)

pf.

$$E[aX+b] = \int (ax+b) f(x)\, dx$$
$$= a\int x f(x)\, dx + \int b f(x)\, dx$$
$$= aE[X] + b \underbrace{\int f(x)\, dx}_{1}$$
$$= aE[X] + b$$

(b) If $X \geq 0$ (Support (X) is non-neg.)

then $E[X] \geq 0$.

pf. $E[X] = \int_0^\infty x f(x)\, dx \geq 0$

≥ 0 $\quad f(x) \geq 0$

(c) If g_1 and g_2 are functions and

$$g_1(x) \le g_2(x) \quad \forall x$$

then

$$\mathbb{E}[g_1(X)] = \mathbb{E}[g_2(X)]$$

Pf. Combine (a) and (b)

$$\mathbb{E}[g_1(X)] \le \mathbb{E}[g_2(X)]$$

$$\Updownarrow$$

$$\mathbb{E}[g_1(X)] - \mathbb{E}[g_2(X)] \le 0$$

$$\Updownarrow \text{ (linearity)}$$

$$\mathbb{E}[\underbrace{g_1(X) - g_2(X)}_{\text{call this } Y}] \le 0$$

$$\mathbb{E}[Y] \le 0 \quad \text{where } Y \le 0$$

(d) Furthermore if $a \le X \le b$ then

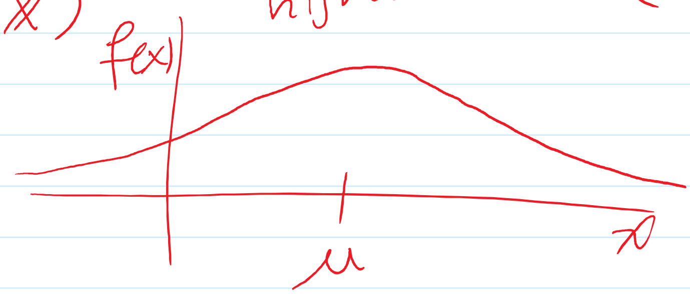
$$a \le \mathbb{E}[X] \le b.$$

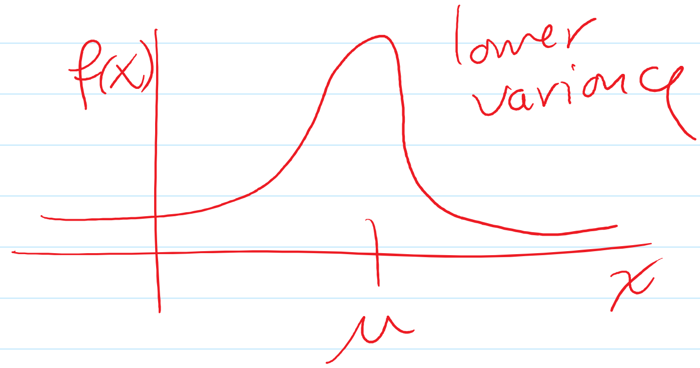
Pf use previous Theorems.

Defn : Variance

Variance of a r.v. tells you how spread the mass/density is around the mean.

Var(X) higher Variance





Mathematically,

$$\boxed{Var(X) = E\left[\left(X - E[X]\right)^2\right]}$$

Note:

$$Y = X - \underbrace{E[X]}_{\text{same number}}$$

$\mu = E[X]$

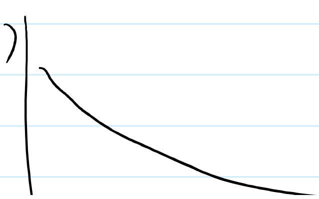
$$E[Y] = E[\underbrace{X - E[X]}_{X - \mu}] = E[X] - E[\underbrace{E[X]}_{\mu}]$$

$$= \underbrace{E[X]}_{\mu} - \underbrace{E[X]}_{\mu}$$

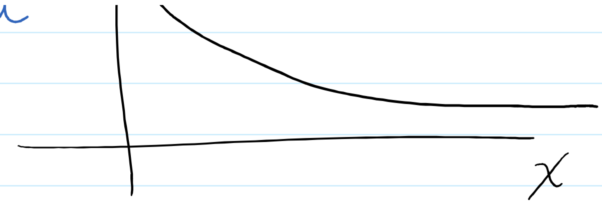
$$= 0$$

Ex. $X \sim Exp(\lambda)$ here $\lambda > 0$

recall: $E[X] = 1/\lambda = \mu$



recall: $\mathbb{E}[X] = 1/\lambda = \mu$



$$\mathrm{Var}(X) = \mathbb{E}[(X-\mu)^2]$$

$$= \int (x-\mu)^2 f(x)\,dx$$

Law of ...

$$= \int (x - 1/\lambda)^2 \lambda e^{-\lambda x}\,dx$$

$$= \int \left(x^2 - \frac{2x}{\lambda} + \frac{1}{\lambda^2}\right) \lambda e^{-\lambda x}\,dx$$

$$= \underbrace{\int x^2 \lambda e^{-\lambda x}\,dx}_{\mathbb{E}[X^2]} - \frac{2}{\lambda} \underbrace{\int x \lambda e^{-\lambda x}\,dx}_{\mathbb{E}[X]} + \frac{1}{\lambda^2} \underbrace{\int \lambda e^{-\lambda x}\,dx}_{1}$$

$$= \frac{2}{\lambda^2} - \frac{2}{\lambda}\frac{1}{\lambda} + \frac{1}{\lambda^2}$$

$$= 1/\lambda^2 = \mathrm{Var}(X).$$

Theorem: Short-cut Theorem For Variance

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

pf.

$$\mathrm{Var}(X) = \mathbb{E}[(X-\mu)^2]$$

$$= \mathbb{E}[X^2 - 2X\mu + \mu^2]$$

$\mathbb{E}[c] = c$ when c is

$$= \mathbb{E}[X^2] - \mathbb{E}[2X\mu] + \mathbb{E}[\mu^2] \quad \text{a const.}$$

$$= \mathbb{E}[X^2] - 2\mu \underbrace{\mathbb{E}[X]}_{\mu} + \mu^2$$

$$= \mathbb{E}[X^2] - 2\mu^2 + \mu^2$$

$$\mathbb{E}\,|\,[X^2] - \mu^2$$

$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

Ex. $X \sim \text{Exp}(\lambda)$

$$\mathbb{E}[X^2] = 2/\lambda^2 \quad \text{and} \quad \mathbb{E}[X] = 1/\lambda$$

so

$$\text{Var}(X) = \frac{2}{\lambda^2} - \left(1/\lambda\right)^2 = 1/\lambda^2$$

Theorem:

$$\text{Var}(aX + b) = a^2\,\text{Var}(X)$$

pf.

$$\text{Var}(aX + b)$$

$$= \mathbb{E}[(aX+b)^2] - \mathbb{E}[aX+b]^2 \quad (\text{short-cut})$$

$$= \mathbb{E}[a^2X^2 + 2abX + b^2] - (a\,\mathbb{E}[X] + b)^2$$

$$= a^2\,\mathbb{E}[X^2] + \cancel{2ab\,\mathbb{E}[X]} + \cancel{b^2} - (a^2\,\mathbb{E}[X]^2 + \cancel{2ab\,\mathbb{E}[X]} + \cancel{b^2})$$

$$= a^2 E[X^2] - a^2 E[X]^2$$

$$= a^2 \left(E[X^2] - E[X]^2\right) \quad \text{(short-cut)}$$

$$= a^2 \, Var(X)$$