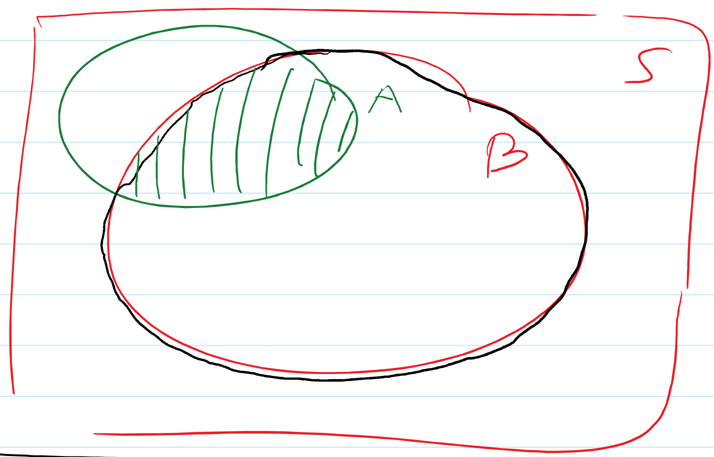
Defn: Conditional Probability

If $A, B \subset S$ then

$$\underbrace{\mathbb{P}(A \mid B)}_{\text{probability of } A \text{ given } B} = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)} \quad \left(\text{so long as } \mathbb{P}(B) > 0\right)$$

probabilities are now proportions of area of B



Facts: $\mathbb{P}(B) > 0$ and $\mathbb{P}(A) > 0$

(1) $\boxed{\mathbb{P}(B \mid B) = 1}$

Pf. $\mathbb{P}(B \mid B) = \frac{\mathbb{P}(BB)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1.$

(2) If $AB = \emptyset$ then $\boxed{\mathbb{P}(A \mid B) = 0.}$

Pf. $\mathbb{P}(A \mid B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)} = \frac{\mathbb{P}(\emptyset)}{\mathbb{P}(B)} = \frac{0}{\mathbb{P}(B)} = 0.$

Ex. Roll two dice.

What is the prob. of the first being 2 given the sum of the two is ≤ 5.

$$S = \{(i,j) \text{ when } 1 \leq i \leq 6 \text{ and } 1 \leq j \leq 6\}$$

$$|S| = 36$$

Let D = "the first is two" $\triangle$

E = "the sum ≤ 5" $\circledast$ = 10 outcomes given in table

Want: $\mathbb{P}(D \mid E) = \dfrac{\mathbb{P}(DE)}{\mathbb{P}(E)} = \dfrac{3/36}{10/36} = 3/10$

E

DE

die 1

die 2	1	2	3	4	5	6
1	$\circledast$	$\circledast$ $\triangle$	$\circledast$	$\circledast$		
2	$\circledast$	$\circledast$ $\triangle$	$\circledast$			
3	$\circledast$	$\circledast$ $\triangle$				
4	$\circledast$	$\triangle$				
5		$\triangle$				
6		$\triangle$				

= Pct of outcomes in E where the first is a 2

Theorem: Conditional Prob. defines a new prob. fn on B.

(Idea: instead of sample space S, sample space is now B)

Given $B \subset S$ and $P(B) > 0$

define a prob. fn $P_B : P(B) \to \mathbb{R}$

defined by for $E \subset S$

$$P_B(E) = P(EB)/P(B)$$ (conditional prob. formula)

Theorem says: P_B satisfies the Kolmogorov axioms (on B^B).

Theorem: Compound Probability

$$P(AB) = P(A|B)P(B) = P(B|A)P(A).$$

pf.

$$P(A|B) = \frac{P(AB)}{P(B)} \quad \text{(rearrange)}$$

or

$$P(B|A) = \frac{P(BA)}{P(A)} \quad \text{(rearrange)}$$

notice if $P(B) = 0$ then $AB \subset B$

$$\text{so } 0 \le P(AB) \le P(B) = 0$$

so $P(AB) = 0$.

So

$$P(AB) = 0 = P(A|B)P(B)$$

somewhat undefined

so $\underbrace{\mathbb{P}(AB)}_{0} = 0 = \mathbb{P}(A|B)\underbrace{\mathbb{P}(B)}_{0}$

Extension:

If $\{A_i\}_{i=1}^{n}$ where $\mathbb{P}(A_i) > 0$

then

$$\mathbb{P}(A_1 A_2 A_3 \cdots A_n)$$

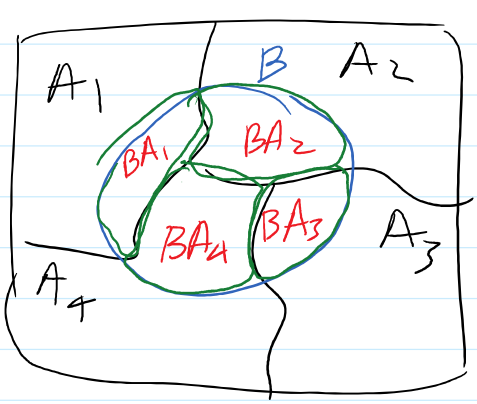
$$= \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3|A_1A_2)\mathbb{P}(A_4|A_1A_2A_3) \cdots \mathbb{P}(A_n|A_1 \cdots A_{n-1})$$

pf. iteratively apply prev. theorem.

Theorem: Law of Total Probability

If $\{A_i\}_{i=1}^{n}$ is a partition of S

then for any $B \subset S$



$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B|A_i)\mathbb{P}(A_i).$$

= weighted sum of $\mathbb{P}(B|A_i)$ weighted by $\mathbb{P}(A_i)$.

pf.

$\mathbb{P}(B) = \overbrace{\sum^{n} \mathbb{P}(BA_i)}^{\text{see notes 2}}$ ... compound

Pf.

$$P(B) = \sum_{i=1}^{n} P(BA_i)$$

$$= \sum_{i=1}^{n} P(B|A_i)P(A_i)$$

compound probability

Ex. Basket 1 Basket 2

Game: (1) Randomly (equally likely) choose a ball from basket 1 and put in basket 2

(2) Choose a ball from basket 2

Q: what is prob. of choosing a black ball on step 2?

Let W = choose white on first step

W^c " black "

and B = choose black on second step

B^c " white "

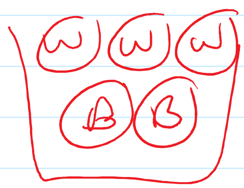
Want $\mathbb{P}(B)$.

Notice $\{W, W^c\}$ partition S.

$$\mathbb{P}(B) = \mathbb{P}(B|W)\mathbb{P}(W) + \mathbb{P}(B|W^c)\mathbb{P}(W^c).$$

$\mathbb{P}(W) = 3/5$ Basket 1

$\mathbb{P}(W^c) = 2/5$



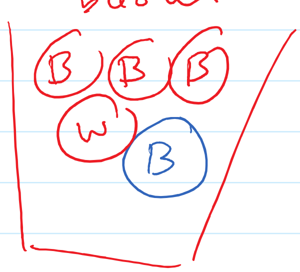
If I choose white first

Basket 2

$\mathbb{P}(B|W) = 3/5$

If I choose black first

basket 2



$\mathbb{P}(B|W^c) = 4/5$

Recall total prob. says

$$\mathbb{P}(B) = \mathbb{P}(B|W)\mathbb{P}(W) + \mathbb{P}(B|W^c)\mathbb{P}(W^c)$$

$$= \left(\tfrac{3}{5}\right)\left(\tfrac{3}{5}\right) + \left(\tfrac{4}{5}\right)\left(\tfrac{2}{5}\right)$$

$$\approx .68$$

Theorem: Bayes' Theorem

If $A, B \subset S$ and $\mathbb{P}(A), \mathbb{P}(B) > 0$

$$\mathbb{P}(A|B) = \mathbb{P}(B|A)\,\frac{\mathbb{P}(A)}{\mathbb{P}(B)}$$

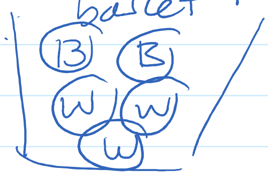
pf.

$$\mathbb{P}(A|B) \overset{\text{defn}}{=} \frac{\mathbb{P}(AB)}{\mathbb{P}(B)} = \frac{\mathbb{P}(BA)}{\mathbb{P}(B)} \overset{\text{compound prob. formula}}{=} \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}.$$

Ex. Consider prev. example

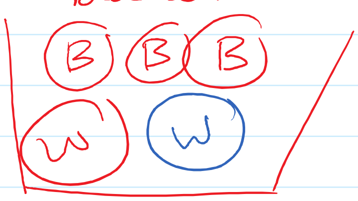
If I choose a black ball on second step, what is the prob. I orig. chose a white ball?

basket 1



$$\mathbb{P}(W|B) = \mathbb{P}(B|W)\,\frac{\mathbb{P}(W)}{\mathbb{P}(B)}$$

basket 2



$$\approx \left(\tfrac{3}{5}\right)\frac{\left(\tfrac{3}{5}\right)}{(.68)}$$

law of total prob.

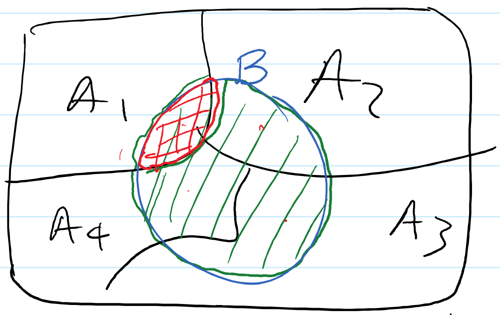
$$\approx .53$$

Note: $\mathbb{P}(W) = 3/5 = .6 > .53$.

Theorem: Law of Total Prob. + Bayes Theorem

If $\{A_i\}$ is a partition of S and $B \subset S$ then

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(B|A_i)\mathbb{P}(A_i)}{\sum_{j=1}^{n}\mathbb{P}(B|A_j)\mathbb{P}(A_j)}.$$



pf.

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(B|A_i)\mathbb{P}(A_i)}{\mathbb{P}(B)}$$

Law of tot. prob. says

$$\mathbb{P}(B) = \sum_{j=1}^{n}\mathbb{P}(B|A_j)\mathbb{P}(A_j)$$

all together:

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(B|A_i)\mathbb{P}(A_i)}{\sum_{j=1}^{n}\mathbb{P}(B|A_j)\mathbb{P}(A_j)}.$$

Ex. I have a disease w/ a prevalence rate of 1% $\mathbb{P}(D) = .01$ $\mathbb{P}(D^c) = .99$

(D: have disease)

We test for the disease and get a result of "+" or "−" $\mathbb{P}(+|D) = .95$

we test for the disease and get a result of "+" or "−" $\mathbb{P}(+|D) = .95$

The test accurately reports a "+" for 95% of people

The test accurately reports a "−" for 99% of people. $\mathbb{P}(-|D^c) = .99$

If I get a + what is prob. it is correct?

$$\mathbb{P}(D|+)?$$

Partition is $\{D, D^c\}$.

According to the theorem:

$$\mathbb{P}(D|+) = \frac{\mathbb{P}(+|D)\mathbb{P}(D)}{\mathbb{P}(+|D)\mathbb{P}(D) + \mathbb{P}(+|D^c)\mathbb{P}(D^c)}$$

$(1-\mathbb{P}(-|D^c))$ ↑ Know $= .99$

$1-\mathbb{P}(D) = .99$

need: $\mathbb{P}(+|D^c) = 1 - \mathbb{P}(-|D^c)$

$\mathbb{P}(+) = 1 - \mathbb{P}(-)$

$$= \frac{(.95)(.01)}{(.95)(.01) + (.01)(.99)}$$

$$\simeq .49$$

notice $\mathbb{P}(A|B) \neq \mathbb{P}(A)$.