7.4 Exercises

7.1 For the table below, place YES, NO, or the appropriate number into each entry.

mathematical expression	domain	function?	one-to-one function?	inverse exists?	$g(9)$	$g^{-1}(9)$
$y = g(x) = 3x - 1$	$\{x \mid x \in \mathcal{R}\}$					
$y = g(x) = x^2$	$\{x \mid x \in \mathcal{R}\}$					
$y = g(x) = \pm\sqrt{x}$	$\{x \mid x \in \mathcal{R}, x \geq 0\}$					

7.2 Let X be a strictly continuous random variable with cumulative distribution function $F_X(x)$. Write an expression for the cumulative distribution function of $Y = X^4$ in terms of $F_X(x)$.

7.3 Let $X \sim \text{exponential}(\lambda_1)$ and $Y \sim \text{exponential}(\lambda_2)$ be independent random variables. Use the *convolution formula*

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\, dy$$

to determine the probability density function of $Z = X + Y$.

7.4 Let $X_1 \sim \text{exponential}(\lambda_1)$ and $X_2 \sim \text{exponential}(\lambda_2)$ be independent random variables. Use the cumulative distribution function technique to find the cumulative distribution function of the sample mean $\bar{X} = (X_1 + X_2)/2$.

7.5 Let X and Y be uniformly distributed over a disk with radius 1 centered at the origin. Find the cumulative distribution function of $Z = \max\{X, Y\}$.

7.6 Consider the pair of random variables (X, Y) which is the solution to the 2×2 set of linear equations

$$Ax + y = 2$$

$$x + y = 3$$

where the coefficient A is a random variable with probability density function

$$f_A(a) = 2(1-a) \qquad 0 < a < 1.$$

What is the cumulative distribution function, 99th percentile, and expected value of the rectilinear distance (or Manhattan distance or L_1 norm) from the origin to the random solution to the set of equations?

7.7 Use APPL to find the population mean of the difference between the two real roots (larger root less the smaller root) of the quadratic equation

$$x^2 + Bx + C = 0,$$

where $B \sim U(2, 3)$ and $C \sim U(0, 1)$ are independent random variables. Verify your result by Monte Carlo simulation.

7.8 Find the population mean, population variance, and 95th percentile of the distance between two random points in the interior of the unit square using APPL. Use Monte Carlo simulation in R to check the results.

7.9 Let the random variable X be uniformly distributed between 0 and 1. Find the probability density function of $Y = X^n$, where n is a positive integer.

7.10 Let X be the number of fours in five tosses of a fair die. Find the probability mass function of $Y = X^2$.

7.11 Let X have probability density function

$$f_X(x) = e^{-x} \qquad x > 0$$

and Y have probability density function

$$f_Y(y) = 2e^{-2y} \qquad y > 0.$$

Assuming that X and Y are independent, use APPL to find $V\left[|X - Y|\right]$, and support your result by Monte Carlo simulation.

7.12 Use APPL to find the population mean area of the triangle created by the origin and the two complex roots of the quadratic equation

$$x^2 + x + C = 0$$

in the complex plane, where $C \sim U(1, 2)$. Support your result by Monte Carlo simulation.

7.13 The continuous random variable X has probability density function

$$f_X(x) = \begin{cases} x & 0 < x < 1 \\ 1/4 & 2 < x < 4. \end{cases}$$

Find the probability density function of $Y = \sqrt{X}$.

7.14 Let X_1 and X_2 be independent and identically distributed Bernoulli(p) random variables. What is the probability mass function of $X_1 - X_2$?

7.15 A homeowner's insurance policy reimburses damages, in dollars, modeled by the random variable X, up to a maximum reimbursement of \$2000. The probability distribution of damages X has probability density function

$$f(x) = 0.001e^{-0.001x} \qquad x > 0.$$

Find the expected reimbursement for a policy that makes an insurance claim.

7.16 The probability of winning a single game of craps is 244/495. Brian has \$10. He plays craps ten times, betting \$1 each time. If he wins one of the games, he gets the \$1 that he originally bet back, plus an additional \$1. If he loses one of the games, he loses the \$1 that he originally bet.

(a) Find the probability that Brian has exactly \$10 after playing playing craps ten times.

(b) Find the probability mass function of Y, the amount of money that Brian has after playing craps ten times.

7.17 Let $Z_1 \sim N(0, 1)$ and $Z_2 \sim N(0, 1)$ be independent random variables. Use the cumulative distribution function technique to find the probability density function of $X = Z_1^2 + Z_2^2$.

7.18 Let X and Y be independent, continuous random variables with marginal probability density functions $f_X(x)$ and $f_Y(y)$, respectively. Let $Z = X + Y$. Use the cumulative distribution function technique to find the probability density function of Z.

7.19 Let $X \sim N(\mu, \sigma^2)$. The random variable $Y = e^X$ is known as a *lognormal* random variable. Find the probability density function of a lognormal random variable.

7.20 Let $X \sim \text{beta}(\alpha, \beta)$. The support of the beta distribution, $\mathcal{A} = \{x \mid 0 < x < 1\}$, limits its applicability. Some have suggested that by adding a positive location parameter c_1 and a positive scale parameter c_2 and transforming X by

$$Y = c_1 + c_2 X,$$

the resulting probability distribution has wider applicability. What is the probability density function of Y?

7.21 Let X have probability density function

$$f_X(x) = e^{-x} \qquad x > 0.$$

Find the probability density function of $Y = 1/X$ using

(a) the cumulative distribution function technique,

(b) the transformation technique.

7.22 Let Z_1 and Z_2 be independent standard normal random variables.

(a) Find the joint probability density function of $Y_1 = Z_1 + Z_2$ and $Y_2 = Z_1 - Z_2$.

(b) Are Y_1 and Y_2 independent?

(c) Find the marginal probability density function of Y_1.

7.23 Let $X \sim \text{Bernoulli}(p)$ and $Y \sim \text{Poisson}(\lambda)$ be independent random variables.

(a) Find the probability mass function of $X + Y$.

(b) Find the probability mass function of $X \cdot Y$.

7.24 Let $X \sim \text{Bernoulli}(p)$ and $Y \sim \text{Poisson}(\lambda)$ be independent random variables.

(a) Find the probability mass function of $\min\{X, Y\}$.

(b) Find the probability mass function of $\max\{X, Y\}$.

7.25 Let X_1, X_2, X_3 be mutually independent and identically distributed random variables, each with probability mass function

$$f(x) = \begin{cases} 1-p & x = -1 \\ p & x = 1, \end{cases}$$

where p is a real-valued parameter satisfying $0 < p < 1$.

(a) Find the probability mass function of $X_1 + X_2 + X_3$.

(b) Find the probability mass function of X_1 / X_2.

7.39 Let X_1 and X_2 be continuous random variables with support on the entire first quadrant and joint probability density function $f_{X_1,X_2}(x_1, x_2)$. Use the transformation technique to find the probability density function of X_1/X_2.

7.40 Let the continuous random variable X have probability density function $f_X(x)$. What is the probability density function of $Y = mX + b$, where m and b are real constants?

7.41 Let $X \sim U(0, 8)$. Write the function $f_Y(y)$, and its associated support, for the following transformations:

(a) $Y = X/2$,

(b) $Y = \lceil X \rceil$,

(c) $Y = \lceil X \rceil - \lfloor X \rfloor$,

(d) $Y = |X - 2|$,

(e) $Y = \max\{|X - 2|, 1\}$.

Hint: Use reasoning rather than mathematics to write $f_Y(y)$.

7.42 Let $X \sim U(-k, 2k)$ for some positive real constant k. Find the probability density function of $Y = X^2$.

7.43 A single spin of a spinner can result in three equally likely outcomes: 1, 2, and 3. Let X_1 be the result of one spin of the spinner. Let X_2 be the result of a second spin of the spinner. Find the probability mass function of $X_1^2 - X_2^2$.

7.44 Let $X_1 \sim N(0, 1)$, $X_2 \sim \chi^2(1)$, $X_3 \sim \chi^2(n)$ be mutually independent random variables. Find the probability distribution (name and parameter values) of

(a) $-7X_1$,

(b) $X_1^2 + X_3$,

(c) $X_3/(nX_1^2)$,

(d) $X_1/\sqrt{X_2}$,

(e) $\sqrt{n}X_1/\sqrt{X_3}$.

No mathematics is required on this problem; simply write down the solution.

7.45 Mike and Laurel choose two points at random on the *perimeter* of a unit square.

(a) Find the probability density function, population mean, and population variance of the distance between the two points using APPL. Report the population mean and population variance to ten-digit accuracy.

(b) Use Monte Carlo simulation to check your population mean and population variance from part (a).

7.46 Let X_1, X_2, X_3, be mutually independent exponential(λ) random variables. Find the 96th percentile of the random variables:

(a) $3\min\{X_1, X_2, X_3\}$,

(b) $X_1 + X_2 - X_3$.

- $X_3 \sim$ exponential(μ) because X_3 is the service time for customer 3.

Theorem 6.19 can now be invoked to determine the joint probability density function of T_1 and T_2 in Scenario 2. Substituting $\lambda_1 = \mu$, $\lambda_2 = \lambda + \mu$, and $\lambda_3 = \mu$ into the mixture of Scenarios 1 and 2 yields the joint probability density function of T_1 and T_2 as

$$f_{T_1, T_2}(t_1, t_2) = \begin{cases} \dfrac{\mu^2 \left(\lambda e^{-\mu t_2} + \mu e^{-\lambda t_1 - \mu t_1 - \mu t_2}\right)}{\lambda + \mu} & 0 < t_1 \le t_2 \\[2ex] \dfrac{\mu^2 \left(\lambda e^{-\lambda t_1 - \mu t_1 + \lambda t_2} + \mu e^{-\lambda t_1 - \mu t_1 - \mu t_2}\right)}{\lambda + \mu} & 0 < t_2 < t_1. \end{cases}$$

Using this joint probability density function, the population covariance between the sojourn times of customers 1 and 2 is

$$\text{Cov}(T_1, T_2) = \frac{\lambda(\lambda + 2\mu)}{(\lambda + \mu)^2 \mu^2}.$$

6.6 Exercises

6.1 A bag contains three balls numbered 1, 2, and 3. Two balls are sampled *with replacement.* Let X_1 be the number on the first ball sampled and let X_2 be the number on the second ball sampled. Find the probability mass function of the sample mean

$$\frac{X_1 + X_2}{2}.$$

6.2 A bag contains three balls numbered 1, 2, and 3. Two balls are sampled *without replacement.* Let X_1 be the number on the first ball sampled and let X_2 be the number on the second ball sampled. Find the probability mass function of the sample mean

$$\frac{X_1 + X_2}{2}.$$

6.3 The continuous random variables X and Y have joint probability density function

$$f(x, y) = \frac{1}{100} \qquad 0 < x < 10, 0 < y < 10.$$

Find $P(2X + Y < 12)$.

6.4 The continuous random variables X and Y have joint probability density function

$$f(x, y) = \frac{1}{2} \qquad |x| + |y| < 1.$$

Find $P(X > 1/2)$.

6.5 The random variables X and Y are uniformly distributed over the *interior* of a circle of radius 1 centered at the origin. Find the marginal probability density function $f_X(x)$.

6.17 Let X_1 be uniformly distributed between zero and one. Let x_1 be the realization of X_1 (that is, x_1 is a sample value of X_1). Let X_2 be uniformly distributed between zero and x_1.

(a) Find the marginal probability density function of X_2.

(b) Find the population mean of X_2 and execute a Monte Carlo simulation that supports the value that you derive analytically.

6.18 What is the probability that there are more spades than hearts in a five-card poker hand?

6.19 Kent leaves for work every day between 5:00 AM and 5:10 AM. He needs to be at work by 6:00 AM. He can take the highway or cut through town. The travel time to work via the highway is between 45 and 55 minutes. The travel time to work by cutting through town is between 40 and 60 minutes. Assuming that his departure and travel times are uniform on the given ranges, find the probability he will be on time by the two routes.

6.20 Let X be a Poisson random variable with a random parameter Λ, which has probability density function

$$f_\Lambda(\lambda) = e^{-\lambda} \qquad \lambda > 0.$$

Find the probability mass function of X. *Hint*: The conditional probability mass function of X given λ is $f_{X|\Lambda=\lambda}(x|\Lambda=\lambda) = \lambda^x e^{-\lambda}/x!$ for $x = 0, 1, 2, \ldots$, and $\lambda > 0$.

6.21 Let X_1 and X_2 be the numbers on two balls drawn randomly from a bag of billiard balls numbered $1, 2, \ldots, 15$. Find the joint probability mass function of $Y_1 = \min\{X_1, X_2\}$ and $Y_2 = \max\{X_1, X_2\}$ and the marginal probability mass function of Y_1 when

(a) sampling is without replacement,

(b) sampling is with replacement.

6.22 For the random variables X and Y with joint probability density function

$$f(x, y) = 2 \qquad 0 < x < y < 1,$$

find the joint cumulative distribution function for all real values of x and y.

6.23 Let X and Y be uniformly distributed on the unit disk with joint probability density function

$$f(x, y) = \frac{1}{\pi} \qquad x^2 + y^2 < 1.$$

(a) Find $P(|X| + |Y| > 1)$ without using calculus.

(b) How many separate regions is the joint cumulative distribution function $F(x, y)$ defined on? *Hint* for part (b): Do not find the joint cumulative distribution function here—just find the number of regions it is defined on. Here is an example of the number of regions where the cumulative distribution function is defined: the joint cumulative distribution function of X and Y is

$$F(x, y) = \begin{cases} 0 & x \le 0 \text{ or } y \le 0 \\ 1 - e^{-(x+y)} & x > 0, y > 0 \end{cases}$$

is defined on *two* separate regions.

6.24 Let the random variables X and Y have joint probability density function

$$f(x, y) = \frac{1}{\pi} e^{-\frac{1}{2}(x^2+y^2)} \qquad x > 0, y < 0 \text{ or } x < 0, y > 0.$$

Find the marginal probability density functions. *Hint*: consider $\mathcal{A}$ carefully.

6.25 Consider the random variables X and Y with joint probability density function

$$f(x, y) = \frac{1}{\pi} \qquad x^2 + y^2 < 1.$$

(a) Find $P\left(X^2 + Y^2 \le r^2\right)$ for any real $r \ge 0$.

(b) Set up, but do not evaluate, expressions for $P(X + Y \le r)$ for any real r.

6.26 Let X_1 and X_2 be continuous random variables with joint probability density function

$$f(x_1, x_2) = \frac{1}{9} \qquad 0 < x_2 < x_1^2 < 9, x_1 > 0.$$

Find the marginal distribution for X_1.

6.27 Consider the Cartesian coordinate system. Li-Hsing selects a point that is uniformly distributed between the origin and (1, 0). Huarng selects a point that is uniformly distributed between the origin and (0, 1), independently of Li-Hsing's choice. Find the population median of the distribution of the distance between the two points.

6.28 Consider the random variables X and Y with joint probability density function

$$f(x, y) = cxy \qquad 0 < y < x < 1.$$

(a) What is the value of the constant c?

(b) What is $P(X > 1/2)$?

(c) What is $P\left(Y > X^2\right)$?

6.29 There are five balls in a bag numbered 1, 2, 3, 4, and 5. Two balls are drawn at random without replacement. Let X be the smaller number chosen and let Y be the larger number chosen.

(a) What is the joint probability mass function of X and Y?

(b) What is the probability mass function of $Z = Y - X$?

6.30 Let (x_i, y_i), for $i = 1, 2, 3$, denote the vertices of a triangle on the Cartesian coordinate system. Let the random ordered pair (X, Y) be uniformly distributed over the triangle (that is, the joint probability density function of X and Y is the reciprocal of the area of the triangle). Find the marginal distribution of X, $f_X(x)$. *Hint 1*: You may assume, without loss of generality, that $x_1 \le x_2 \le x_3$, although you must explicitly consider the cases $x_1 = x_2$ and $x_2 = x_3$. *Hint 2*: For a triangle with side lengths a, b, and c, and semi-perimeter $s = (a+b+c)/2$, Heron's formula gives the area as $\sqrt{s(s-a)(s-b)(s-c)}$.

6.31 Let X and Y be uniformly distributed on the support $|x| + |y| < 3$. Find $F(1, 1)$. *Hint:* Solve this problem geometrically rather than using integration.

6.32 Let X and Y be continuous random variables with joint probability density function

$$f(x, y) = \frac{1}{3\pi} \qquad 1 < x^2 + y^2 < 4.$$

Find the marginal probability density function of X.

6.33 The random variables X and Y have joint probability density function

$$f(x, y) = 2 \qquad x \ge 0,\ y \ge 0,\ x + y \le 1.$$

Find the point in the support of X and Y that maximizes $F(x, y)$.

6.34 The random variables X and Y have a joint probability distribution. Let

$$W = 2(\lceil X \rceil - \lfloor X \rfloor) + 3(\lceil Y \rceil - \lfloor Y \rfloor).$$

(a) If X and Y are continuous random variables, find the probability mass function of W.

(b) If X and Y are discrete random variables defined only on the integers, find the probability mass function of W.

6.35 Let X and Y have joint probability density function

$$f(x, y) = 2e^{-2x} \qquad x > 0,\ -x < y < x.$$

Find $P(2 < X < 3)$.

6.36 Let X and Y have joint probability density function

$$f(x, y) = 2e^{-2x} \qquad x > 0,\ -x < y < x.$$

Find $P(-3 < Y < 3 \mid X = 4)$.

6.37 Let the random variables X and Y have the joint probability density function

$$f(x, y) = \frac{3}{2}xy \qquad 0 < x < 2,\ 0 < y < 2 - x.$$

(a) Find $f_X(x)$.

(b) Find $P(2Y < X \mid X = 1)$.

6.38 The independent random variables T_1 and T_2 denote the lifetimes of two components arranged in parallel. The probability density function of T_1 is

$$f_{T_1}(t_1) = e^{-t_1} \qquad t_1 > 0.$$

The probability density function of T_2 is

$$f_{T_2}(t_2) = 2e^{-2t_2} \qquad t_2 > 0.$$

The cost of power to operate the system is $X = 3T_1 + 4T_2$. Find the cumulative distribution function of X.

6.39 Let $X_1 \sim U(0, 1)$ and $X_2 \sim U(0, 1)$ be independent random variables. Find the exact value of $P\left(X_1^2 + X_2^2 > 1\right)$ using a purely geometric argument (that is, use no calculus).