Lecture 6 - Conditional Probability

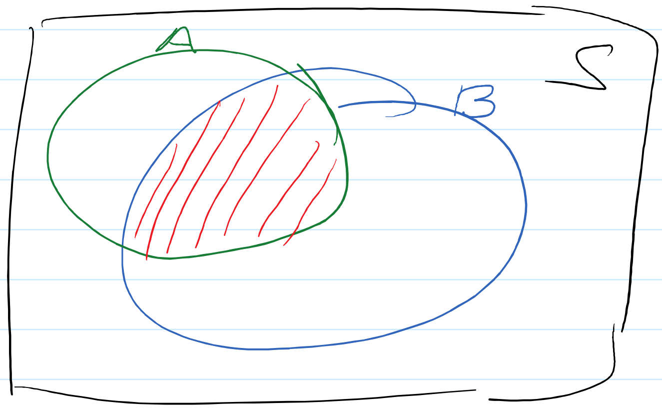
Tuesday, February 11, 2020 1:59 PM

Defn: Conditional Probability

If $A, B \subset S$ then

$$P(A|B) = \frac{P(AB)}{P(B)} \quad \left(\text{req'} \; P(B) > 0\right)$$

bar read: "given" or "conditioned on"



Facts: $P(B) > 0$ and $P(A) > 0$.

① $\boxed{P(B|B) = 1}$

pf: $P(B|B) = \frac{P(BB)}{P(B)} = \frac{P(B)}{P(B)} = 1$.

② If $AB = \emptyset$ then $\boxed{P(A|B) = 0}$

pf. $P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(\emptyset)}{P(B)} = \frac{0}{P(B)} = 0$.

Ex. Roll two dice.

Q: what is the prob. of the first die being 2 given the sum of the dice is ≤ 5.

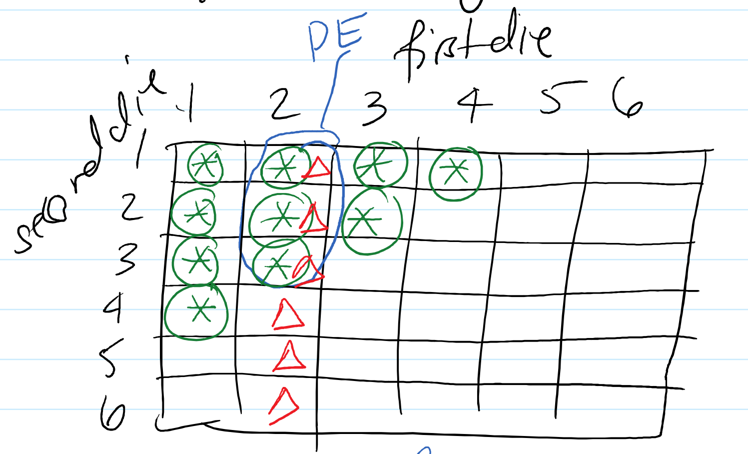
$$S = \{(i,j) \;\; 1 \leq i \leq 6 \text{ and } 1 \leq j \leq 6\}$$

assume outcomes equally likely

$$|S| = 36$$

$\triangle$ D = "first is 2"

$\circledast$ E = "Sum ≤ 5"



represents S

Want: $\mathbb{P}(D|E)$?

$$= \frac{\mathbb{P}(DE)}{\mathbb{P}(E)}$$

So $\mathbb{P}(E) = {}^{10}/_{36}$ and $\mathbb{P}(DE) = {}^{3}/_{36}$

hence $\mathbb{P}(D|E) = \dfrac{3/36}{10/36} = {}^{3}/_{10}$

Theorem: Conditional Probability defines a new prob. fn on B.

Note:

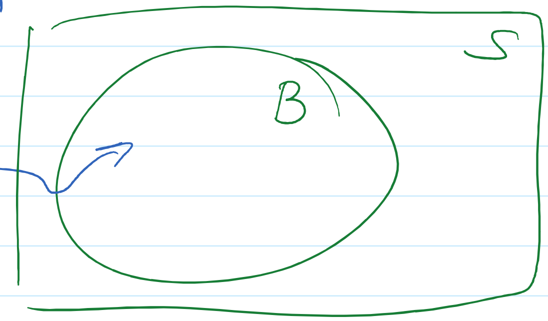
$$\mathbb{P} : \mathcal{P}(S) \longrightarrow \mathbb{R}$$

Conditional prob. fn defines a new prob. fn

$P_B(B) = 1$

$$P_B : \mathcal{P}(S) \to \mathbb{R}$$

where $P_B(E) = \dfrac{P(EB)}{P(B)}$.



this P_B is a valid prob. fn.

To prove: Show it satisfies the Kolmogorov Axioms.

Ex. $P(A^c \mid B) = 1 - P(A \mid B)$.

Theorem: Compound Probability

$$P(AB) = P(A \mid B)\,P(B) = P(B \mid A)\,P(A).$$

Pf. $P(A \mid B) = \dfrac{P(AB)}{P(B)}$ ... move $P(B)$ to other side of Eq.

Similarly

$$P(B \mid A) = \frac{P(AB)}{P(A)}.$$

Extension:

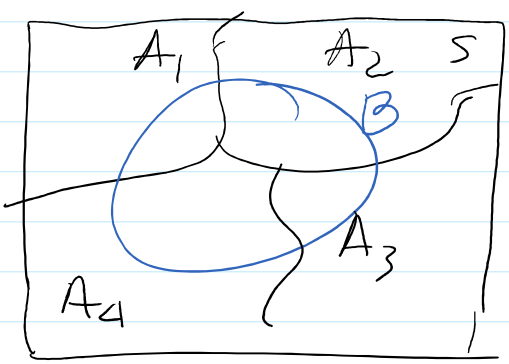
If $\{A_i\}_{i=1}^{n}$ are events then

$$P(A_1 A_2 A_3 \cdots A_n)$$
$$= P(A_1)P(A_2|A_1)P(A_3|A_1A_2)P(A_4|A_1A_2A_3) \cdots P(A_n|A_1A_2 \cdots A_{n-1})$$

<u>pf.</u> Iteratively apply prev. theorem.

Theorem: Law of Total Probability

If $\{A_i\}_{i=1}^{n}$ is a partition of S then for any $B \subset S$



$$\boxed{P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i).}$$

a weighted sum of $P(B|A_i)$ weighted by $P(A_i)$.

see lec. 2

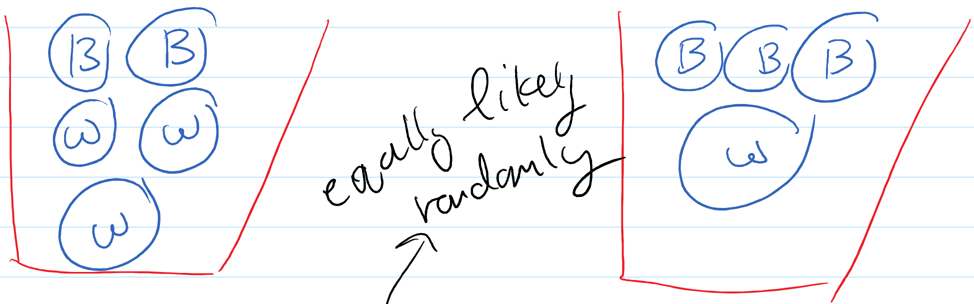
<u>pf.</u> $$P(B) = \sum_{i=1}^{n} P(BA_i)$$

compound prob. rule $$= \sum_{i=1}^{n} P(B|A_i)P(A_i).$$

compound prob. rule $= \sum_{i=1}^{n} \mathbb{P}(B|A_i)\mathbb{P}(A_i)$.

Ex.

Basket 1 Basket 2



equally likely randomly

Game: (1) Choose a ball from basket 1 and put in basket 2.

(2) Choose a ball from basket 2.

Q: what is the prob. of choosing a black ball on step (2).

Let W is the event I choose white in step (1)

W^c 〃 black 〃

and B is the event I choose black in step (2)

B^c 〃 white 〃

I want $\mathbb{P}(B)$.

Notice that $\{W, W^c\}$ partition S.

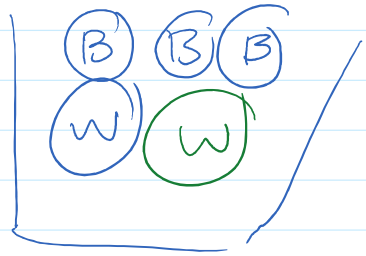
Law of total prob. says

$$\mathbb{P}(B) = \mathbb{P}(B \mid W)\,\mathbb{P}(W) + \mathbb{P}(B \mid W^c)\,\mathbb{P}(W^c)$$

$$\mathbb{P}(W) = 3/5$$

$$\mathbb{P}(W^c) = 2/5$$

Basket 2

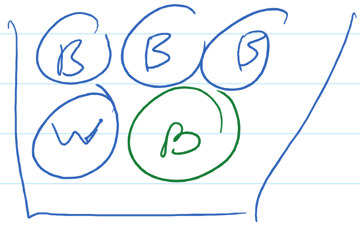


If I choose white in step 1

$$\mathbb{P}(B \mid W) = 3/5$$

If I choose black in step 1

Basket



$$\mathbb{P}(B \mid W^c) = 4/5$$

putting this all together:

$$\begin{aligned}\mathbb{P}(B) &= \mathbb{P}(B/W)\,\mathbb{P}(W) + \mathbb{P}(B \mid W^c)\,\mathbb{P}(W^c) \\ &= (3/5)(3/5) + (4/5)(2/5) \\ &\approx .68\end{aligned}$$

Theorem: Bayes' Theorem

If A, B and $\mathbb{P}(B) > 0$ $\mathbb{P}(A) > 0$.

Then

$$\mathbb{P}(A \mid B) = \mathbb{P}(B \mid A)\,\frac{\mathbb{P}(A)}{\mathbb{P}(B)}.$$

$$\mathbb{P}(A \mid B) = \mathbb{P}(B \mid A)\,\frac{\mathbb{P}(A)}{P(B)}.$$

Pf. $\mathbb{P}(A \mid B) = \dfrac{P(AB)}{P(B)} = \dfrac{\mathbb{P}(B \mid A)\,P(A)}{\mathbb{P}(B)}.$

Ex. Consider the previous example.

If I choose a black ball on the second step, what is the prob. a white ball was chosen on step 1.

$$\mathbb{P}(w \mid B) = \frac{\mathbb{P}(B \mid w)\,\mathbb{P}(w)}{\mathbb{P}(B)}$$

$$\approx \frac{(3/5)(3/5)}{(.68)} \approx .53$$

Note: $P(w) = .6 = 3/5$

Theorem: Law of Total Prob. + Bayes'.

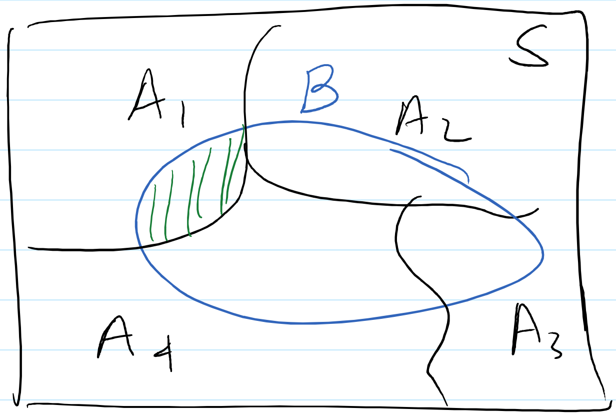
If $\{A_i\}_{i=1}^{n}$ partition S, and $B \subset S$.

Then

$$\mathbb{P}(A_i \mid B) = \frac{P(B \mid A_i)\,P(A_i)}{\quad}$$

Then

$$\mathbb{P}(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{n} \mathbb{P}(B|A_j)\mathbb{P}(A_j)}$$



Pf.

$$\mathbb{P}(A_i|B) \overset{\text{Bayes'}}{=} \frac{\mathbb{P}(B|A_i)\mathbb{P}(A_i)}{\mathbb{P}(B)}$$

$$\underset{\text{Law of Total Prob.}}{=} \frac{\mathbb{P}(B|A_i)\mathbb{P}(A_i)}{\sum_{j=1}^{n} \mathbb{P}(B|A_j)P(A_j)}.$$

Ex. I have a disease (D and D^c = no disease) w/ a prevalence rate of 1% in the population

$\mathbb{P}(D) = .01$ and $\mathbb{P}(D^c) = .99$

We test for the disease and get a + or a − result.

The test accurately reports a + 95% of the time

$P(+|D) = .95$

The test accurately reports − 99% of the time

$$\boxed{\mathbb{P}(-|D^c) = .99}$$

Q: I get a + result.

what is the prob. it is correct?

$$\mathbb{P}(D \mid +)$$

partition on D and D^c

$$\mathbb{P}(D \mid +) = \frac{\mathbb{P}(+ \mid D)\mathbb{P}(D)}{\mathbb{P}(+ \mid D)\mathbb{P}(D) + \mathbb{P}(+ \mid D^c)\mathbb{P}(D^c)}$$

$P(+ \mid D^c)$
$= 1 - \mathbb{P}(- \mid D^c)$
$= 1 - .99$
$= .01$

$$= \frac{(.95)(.01)}{(.95)(.01) + (.01)(.99)}$$

$$= \frac{.95}{.95 + .99} \approx .49$$

Defn: Independence

If $A, B \subset S$ we say that "A is independent of B" denoted $A \perp\!\!\!\perp B$ if

$$\mathbb{P}(AB) = \mathbb{P}(A)\mathbb{P}(B).$$

Theorem: If $A \perp\!\!\!\perp B$ then

$$\mathbb{P}(A \mid B) = \mathbb{P}(A).$$

Pf. $\mathbb{P}(A \mid B) = \frac{P(AB)}{P(B)} = \frac{\mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(B)} = \mathbb{P}(A).$
