Lecture 11 - Expectation and Variance

Tuesday, March 3, 2020 1:58 PM

Defn: Expected Value / Expectation / Mean

$$\mathbb{E}[X] = \begin{cases} \sum_x x f(x) & \text{(discrete)} \\ \int_{\mathbb{R}} x f(x)\, dx & \text{(continuous)} \end{cases}$$

Balancing Point

Ex. $X \sim Bin(n, p)$

If I flip a coin n times (independently) w/ a prob. p of H, $X = \#$ of heads

$Support(X) = \{0, 1, 2, \ldots, n\}$

$$f(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

aside: $f(x) \geq 0$, $\sum_x f(x) = 1$? Binomial Theorem $(x+y)^n$

$\mathbb{E}[X]$

$$\sum^n \qquad \sum^n x \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=\cancel{0}1}^{n} x f(x) = \sum_{x=\cancel{0}1}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

look at $x\binom{n}{x} = x \dfrac{n!}{x!(n-x)!}$

$$= \frac{n!}{(x-1)!(n-x)!}$$

$$= n \underbrace{\frac{(n-1)!}{(x-1)!((n-1)-(x-1))!}}_{\binom{n-1}{x-1}}$$

$$= n\binom{n-1}{x-1}$$

$$= \sum_{x=1}^{n} n \binom{n-1}{x-1} p^x (1-p)^{n-x}$$

$y = x-1 \Leftrightarrow x = y+1$

$$= \sum_{y=0}^{n-1} n \binom{n-1}{y} p^{y+1} (1-p)^{n-y-1}$$

$$= np \underbrace{\sum_{y=0}^{n-1} \underbrace{\binom{n-1}{y} p^y (1-p)^{(n-1)-y}}_{\text{pdf of Bin}(n-1,p)}}_{\text{Sum of a Bin}(n-1,p) \text{ over the support}} = 1$$

$Y \sim \text{Bin}(n-1, p)$

$f(y) = \binom{n-1}{y} p^y (1-p)^{(n-1)-y}$

Support$(Y) = \{0, \dots, n-1\}$

$\boxed{= np}$ = (# of trials)(prob of a success)

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Note: If X then any function of that random variable is also a r.v.

e.g. if X = # of successes

Could consider X^2 = Square of # of successes $\quad g(x) = x^2$

So $Y = g(X)$ is a r.v. for any function g.

Theorem: Law of the Unconscious Statistician

$$E[g(X)] = \begin{cases} \int g(x) f(x)\, dx & \text{(continuous)} \\ \sum g(x) f(x) & \text{(discrete)} \end{cases}$$

Ex. $X \sim Exp(\lambda) \quad \lambda > 0$

$$f(x) = \lambda e^{-\lambda x} \text{ for } x > 0$$

Recall: $E[X] = 1/\lambda$

Q: $E[X^2] \neq E[X]^2$

$g(x) = x^2$

$$E[X^2] = \int_0^\infty x^2 f(x)\,dx = \int_0^\infty x^2 \lambda e^{-\lambda x}\,dx = \int u\,dv$$

by parts $\quad u = x^2 \qquad v = -e^{-\lambda x}$

$du = 2x\,dx \qquad dv = \lambda e^{-\lambda x}\,dx$

$$= uv - \int v\,du$$

$$= \left[x^2(-e^{-\lambda x})\right]_0^\infty - \int_0^\infty -e^{-\lambda x}(2x\,dx)$$

$$= [0 - 0] + \frac{2}{\lambda}\underbrace{\int_0^\infty x \lambda e^{-\lambda x}\,dx}_{\int_0^\infty x f(x)\,dx = E[X]}$$

$$= \frac{2}{\lambda} E[X]$$

$$= \frac{2}{\lambda}\frac{1}{\lambda} \boxed{= \frac{2}{\lambda^2}} = E[X^2] \neq \frac{1}{\lambda^2} = E[X]^2$$

Recall Improper Integrals $\qquad$ Harmonic series

$\int^\infty \frac{1}{x}\,dx$ doesn't converge $\Longleftrightarrow$ $\sum^\infty 1/k$ doesn't

$\int_0^\infty \frac{1}{x}\,dx$ doesn't converge $\iff$ $\sum_{k=0}^{\infty} 1/k$ doesn't converge

$\int_0^\infty \frac{1}{x^2}\,dx$ has a value $\iff$ $\sum_{k=0}^{\infty} 1/k^2$ converge

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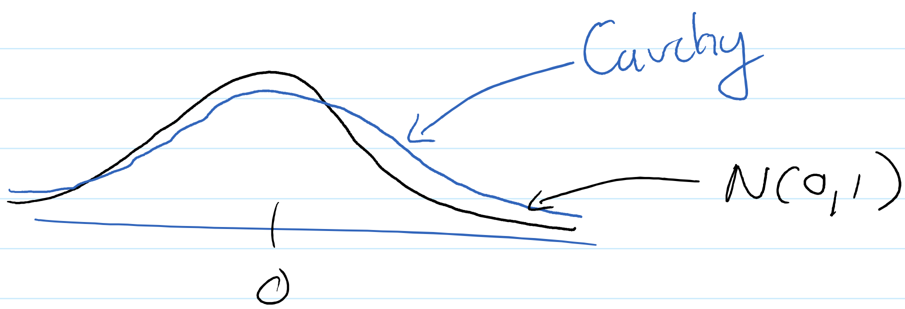
Do expected values always exist? No.

Ex. Cauchy distribution

$$f(x) = \frac{1}{\pi}\,\frac{1}{1+x^2} \quad \text{for all } x \in \mathbb{R}$$

For $N(0,1)$

$f(x) \propto e^{-x^2}$



$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} x \frac{1}{\pi}\,\underbrace{\frac{1}{1+x^2}}\,dx$$

$$\frac{x}{1+x^2} \approx \frac{1}{x}$$ for large x (asymptotically)

So this integral doesn't converge.

Theorem: Properties of Expectation

(a) Expectation is linear

$$\boxed{E[aX+b] = a\,E[X] + b.}$$

pf. (Cts case)

$$E[aX+b] = \int_{\mathbb{R}} (ax+b)f(x)\,dx$$

Integration is linear

$$= \int_{\mathbb{R}} \left(axf(x) + bf(x)\right) dx = a\int xf(x)dx + b\int f(x)dx$$
$$= a\,E[X] + b$$

(b) If $X \geq 0$ then $E[X] \geq 0$

pf. $X \geq 0$ I mean support is non-negative

So $E[X] = \int_0^\infty \underbrace{x}_{\geq 0}\underbrace{f(x)}_{\geq 0}dx \geq 0$ b/c Integrating non-neg. fn.

(c) If g_1 and g_2 are function then

$$E[g_1(X) + g_2(X)] = E[g_1(X)] + E[g_2(X)]$$

and if $g_1(x) \leq g_2(x)$ then

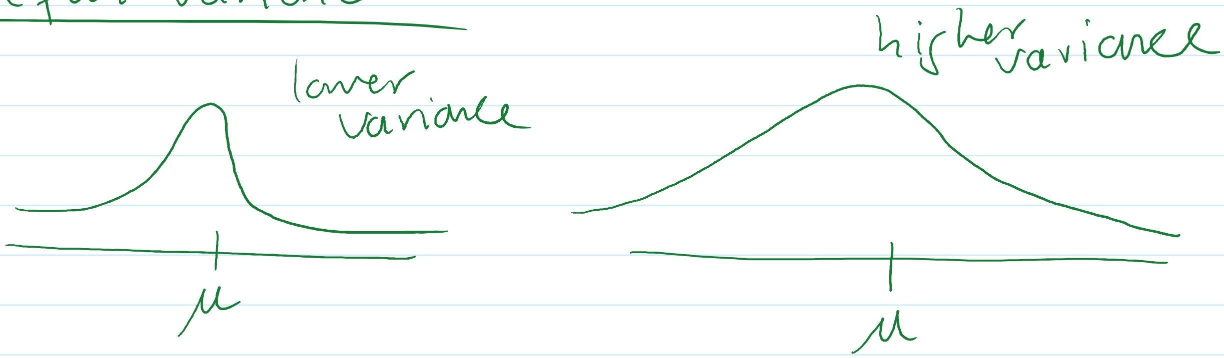
$$E[g_1(X)] \leq E[g_2(X)]$$

pf. $g_1(X)$ and $g_2(X)$ are themselves r.v.s.
So we can apply (a) and (b).

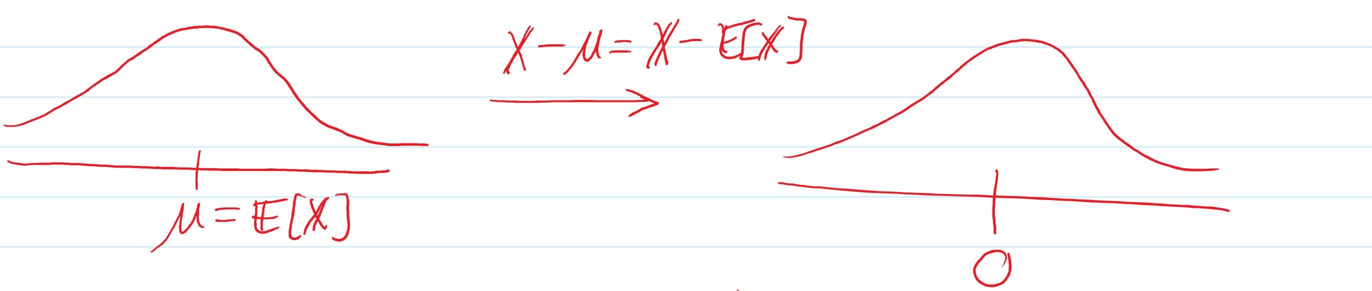
(d) If $a \le X \le b$ then

$$a \le E[X] \le b.$$

Defn: Variance



$$\mathrm{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$$



$Y = X - \mu$ then $\mathbb{E}[Y] = \mathbb{E}[X - \mu]$

$$= \mathbb{E}[X] - \mathbb{E}[\mu]$$

$$= \mathbb{E}[X] - \mu$$

$$= \mu - \mu = 0$$

$\mathbb{E}[c] = c$

$= \int c f(x) dx$

$= c(1)$

Ex. $X \sim \mathrm{Exp}(\lambda)$

$\mu = \mathbb{E}[X] = 1/\lambda$, $\mathbb{E}[X^2] = 2/\lambda^2$

$X \sim Exp(\lambda)$

Recall: $\mathbb{E}[X] = 1/\lambda$ and $\mathbb{E}[X^2] = 2/\lambda^2$

$$Var(X) = \mathbb{E}[(X-\mu)^2] = \int_0^\infty (x-\mu)^2 f(x)\,dx$$

$$= \int_0^\infty (x^2 - 2\mu x + \mu^2) f(x)\,dx$$

$$= \underbrace{\int_0^\infty x^2 f(x)\,dx}_{\mathbb{E}[X^2] = 2/\lambda^2} - 2\mu \underbrace{\int_0^\infty x f(x)\,dx}_{\mathbb{E}[X] = 1/\lambda} + \mu^2 \underbrace{\int_0^\infty f(x)\,dx}_{1}$$

recall $\mu = 1/\lambda$

$$= 2/\lambda^2 - 2\mu \, 1/\lambda + \mu^2$$

$$= \cancel{\frac{2}{\lambda^2} - \frac{2}{\lambda^2}} + 1/\lambda^2$$

$$\boxed{= 1/\lambda^2.}$$

Theorem: Short-cut formula for Variance

$$\boxed{Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2}$$

pf

$$Var(X) = \mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2 - 2\mu X + \mu^2]$$

defn: $\mu = \mathbb{E}[X]$

$$= \mathbb{E}[X^2] - 2\mu \mathbb{E}[X] + \mu^2$$

$$= \mathbb{E}[X^2] - 2\mu^2 + \mu^2$$

$$= \mathbb{E}[X^2] - \mu^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

Ex. $X \sim Exp(\lambda)$

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$
$$= 2/\lambda^2 - (1/\lambda)^2$$
$$= 1/\lambda^2$$

Theorem:

$$\boxed{Var(aX+b) = a^2 Var(X)}$$

short-cut formula

pf.

$$Var(aX+b) = \mathbb{E}[(aX+b)^2] - \mathbb{E}[aX+b]^2$$
$$= \mathbb{E}[a^2X^2 + 2abX + b^2] - (a\mathbb{E}(X)+b)^2$$
$$= a^2\mathbb{E}[X^2] + \cancel{2ab\mathbb{E}[X]} + \cancel{b^2} - (a^2\mathbb{E}[X]^2 \cancel{-2ab\mathbb{E}[X]} \cancel{+b^2})$$
$$= a^2(\mathbb{E}[X^2] - \mathbb{E}[X]^2)$$
$$= a^2 Var(X).$$