Ex. Flip a coin 3 times (independently)

X = # of heads

s	$X(s)$
H H H	3
H H T	2
H T H	2
H T T	1
T H H	2
T H T	1
T T H	1
T T T	0

Defn: Random Variable

a r.v. X is a function

$$X : S \longrightarrow \mathbb{R}.$$

Ex.

(1) toss two dice ; X = sum of two dice

(2) toss a coin 25 times

X = longest chain of consecutive Hs

③ observe rainfall

X = yield of crop

<u>Idea:</u> We want to say things like

$$\mathbb{P}(X = 1).$$

"the prob. that X is 1"

recall: $\mathbb{P} : \mathcal{P}(S) \to \mathbb{R}$

If X = # heads in 3 coin flips.

$$\mathbb{P}(X = 1) = \mathbb{P}(\{HTT, THT, TTH\}) = 3/8$$

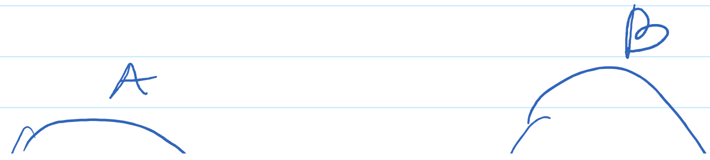
So "$X = 1$" short-hand for $\{s \in S \mid X(s) = 1\}$

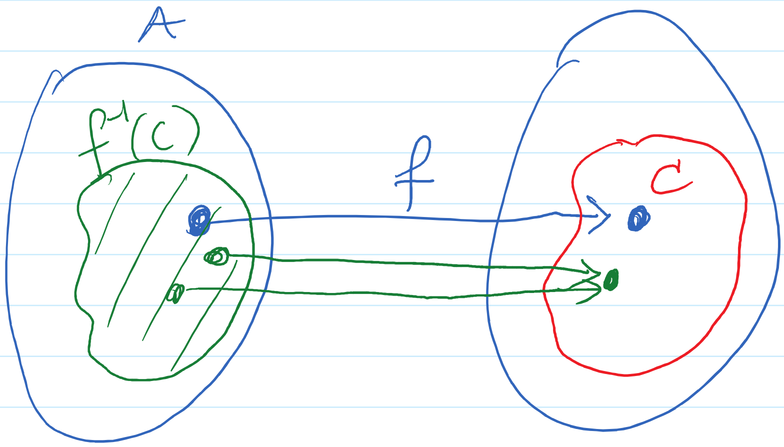
which outcomes $s \in S$ give $X(s) = 1$.

called the inverse image

<u>Aside: Inverse Image of A under f</u>

$$f : A \to B$$





For $C \subset B$,

$$f^{-1}(C) = \{a \in A \mid f(a) \in C\}.$$

If f is bijective (invertible) then f^{-1} is basically the usual function inverse.

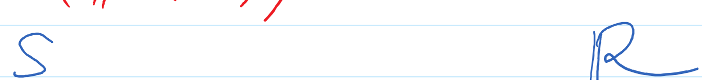
Sloppy notation: for $b \in B$ write $f^{-1}(b)$ to mean $f^{-1}(\{b\})$.

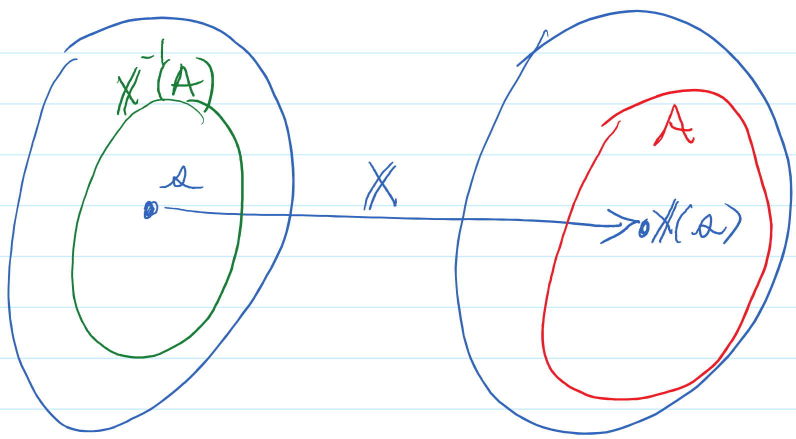
Notation:

X is a r.v., we write

$P(X \in A)$ where $A \subset \mathbb{R}$

means $= \mathbb{P}(X^{-1}(A))$





Ex. $X = \#$ heads

$$\begin{aligned}
&\mathbb{P}(X = 1) \\
&= \mathbb{P}(X \in \{1\}) \\
&= \mathbb{P}(X^{-1}(\{1\})) \\
&= \mathbb{P}(\{HTT, THT, TTH\}) = 3/8
\end{aligned}$$

$$\begin{aligned}
&\mathbb{P}(X = 1 \text{ or } 2) \\
&= \mathbb{P}(X \in \{1, 2\}) \\
&= \mathbb{P}(X^{-1}(\{1, 2\})) \\
&= \mathbb{P}(\{ \ldots \}) = 6/8
\end{aligned}$$

Defn: Support of a R.V.

If X is a r.v. then the support of X is the set of possible values:

$$\mathrm{Im}(X) = X(S)$$

image of S under X.

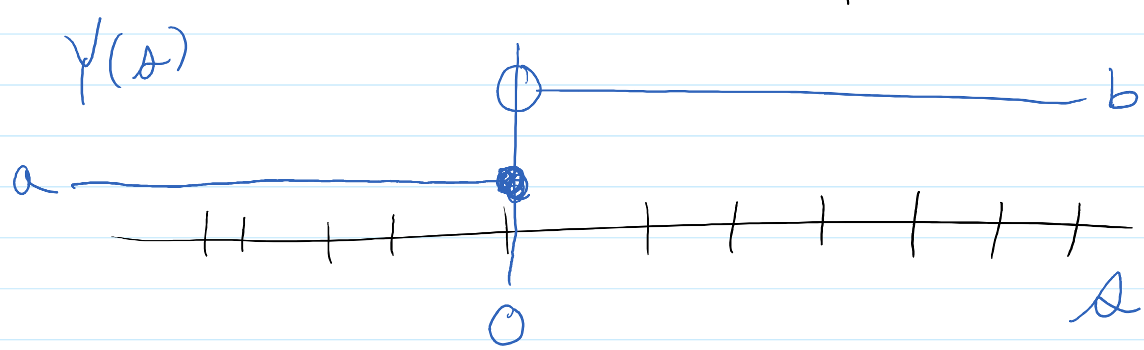
Ex. $X = \#$ heads.

$$\text{Support}(X) = \{0, 1, 2, 3\}.$$

Ex. Let $S = \{0, \pm 1, \pm 2, \dots\}$

and define Y so that

$$Y(s) = \begin{cases} a & \text{if } s \le 0 \\ b & \text{if } s > 0 \end{cases}.$$



$$\text{Support}(X) = \{a, b\}.$$

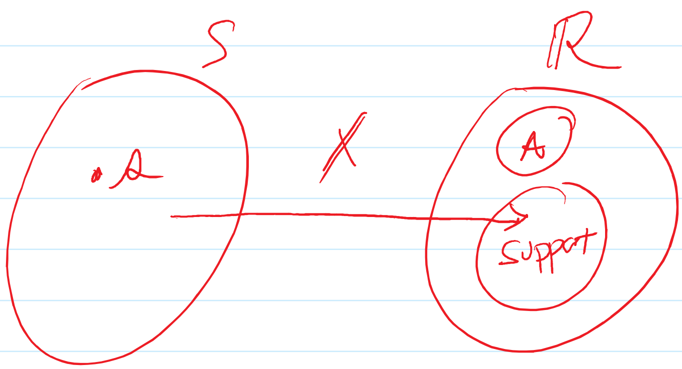
$$P(Y = a) = P(Y^{-1}(a)) = P(\{0, -1, -2, -3, \dots\})$$

$$P(Y = b) = P(Y^{-1}(b)) = P(\{1, 2, 3, 4, \dots\}).$$

$$P(Y = c) = 0 \quad \text{where } c \ne a,\ c \ne b.$$

If A is disjoint w/ Support(X)
then
$$\mathbb{P}(X \in A) = 0.$$

Pf. $\mathbb{P}(X^{-1}(A)) = \mathbb{P}(\emptyset) = 0.$



Types of Random Variables

Heuristic: (Informal Def)

discrete: if the support is "discrete" (finite or countably infinite)

Ex. Flip a coin 3 times
X = # heads
support $\{0, 1, 2, 3\}$.

Ex. X = # of customers arriving at restaurant.
Support $= \mathbb{N}$

Continuous : the support is uncountably infinite.

ex. X = waiting time for bus

Support $= [0, \infty)$

Defn: Cumulative Distribution Function (CDF)

The CDF of a r.v. X is a function

$$F : \mathbb{R} \to \mathbb{R}$$

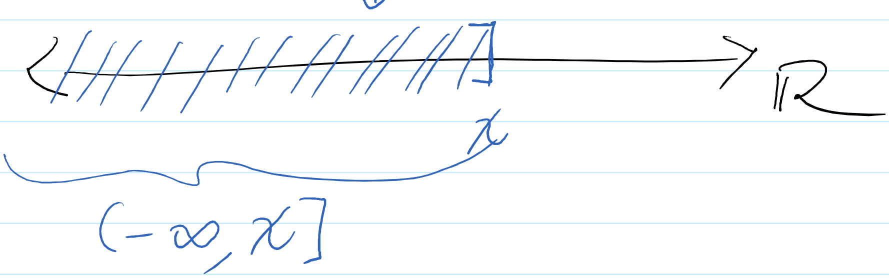
defined for $x \in \mathbb{R}$ as

$$F(x) = \mathbb{P}(\mathbb{X} \le x)$$

r.v. capital bold

little x is a real number

$F(x) = \mathbb{P}(\ ?\ )$ = prob of being here



Notation:

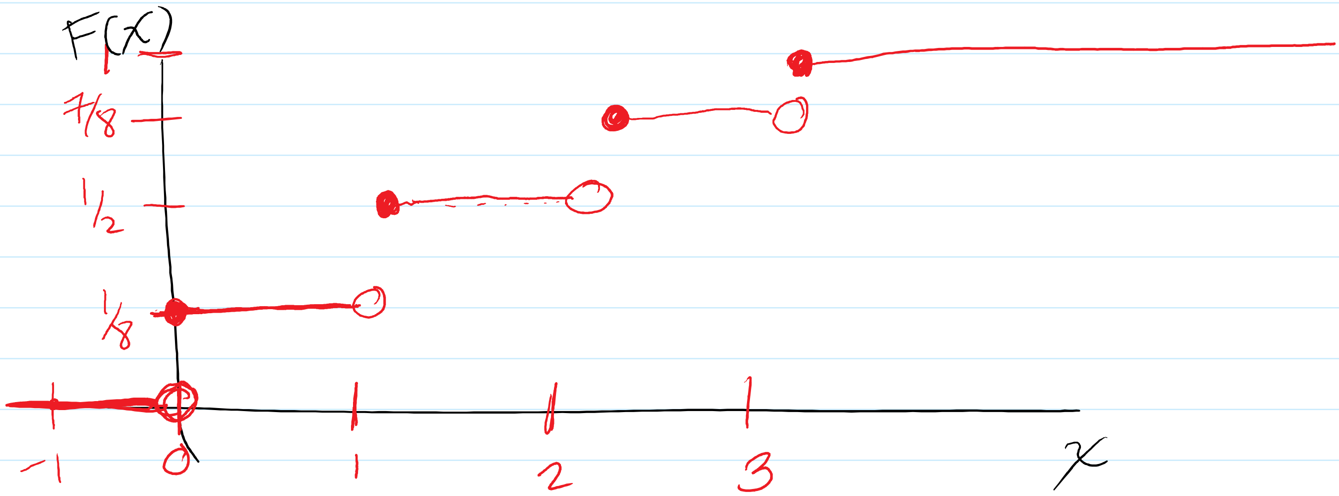
$$F(x) = \mathbb{P}(\mathbb{X} \le x) = \mathbb{P}(\mathbb{X} \in (-\infty, x])$$

$$= \mathbb{P}(\mathbb{X}^{-1}((-\infty, x]))$$

Ex. Toss a coin 3 times

$X = \#$ heads.

What does F look like?



$F(0) = \mathbb{P}(X \leq 0) = 1/8$

$F(1) = \mathbb{P}(X \leq 1) = 4/8 = 1/2$

$F(2) = \mathbb{P}(X \leq 2) = 7/8$

$F(3) = \mathbb{P}(X \leq 3) = 1$

$F(-1) = \mathbb{P}(X \leq -1) = 0$

$F(x) = 0$ for $x < 0$

Theorem: If F is a CDF then

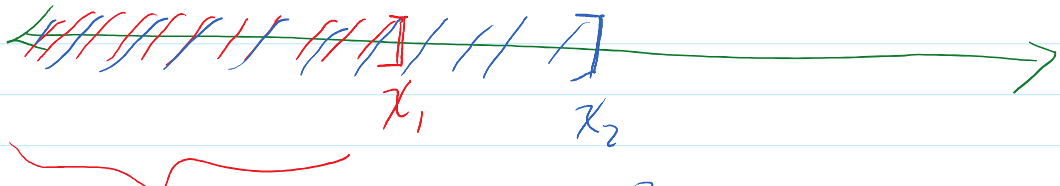
1. $0 \leq F(x) \leq 1$

   Note: $F(x) = \mathbb{P}(\ldots) \in [0,1]$

2. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$.

③ F is non-decreasing: if $x_1 < x_2$

$$F(x_1) \le F(x_2)$$



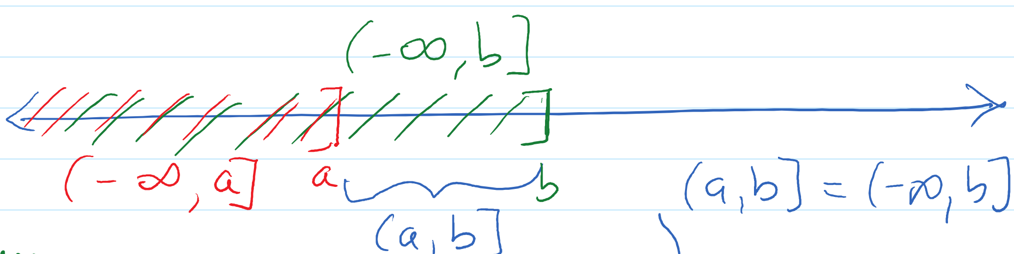
$$(-\infty, x_1] \subset (-\infty, x_2]$$

So

$$F(x_1) = \mathbb{P}(X \in (-\infty, x_1]) \le \mathbb{P}(X \in (-\infty, x_2]) = F(x_2).$$

④

$$\boxed{\mathbb{P}(a < X \le b) = F(b) - F(a)}$$



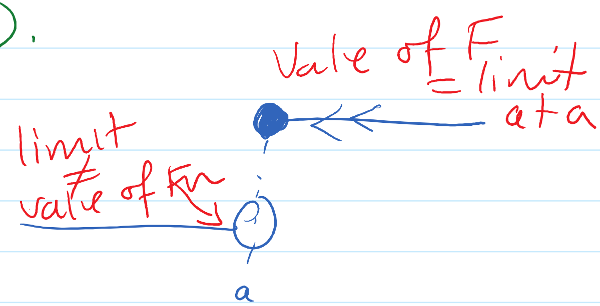
$$(a,b] = (-\infty, b] \setminus (-\infty, a]$$

then

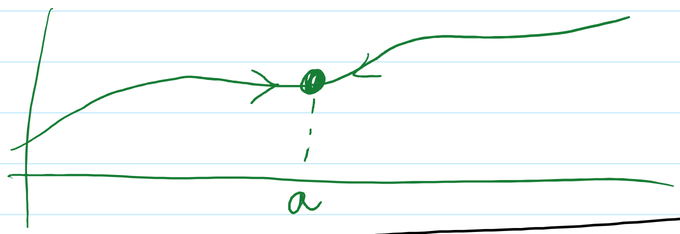
$$\mathbb{P}(a < X \le b) = \mathbb{P}(X \in (a,b])$$
$$= \mathbb{P}(X \in (-\infty, b]) - \mathbb{P}(X \in (-\infty, a])$$
$$= F(b) - F(a).$$

⑤ F is <u>right-continuous.</u>

$$\lim_{x \to a^+} F(x) = F(a)$$



recall a cts fn f: $\lim_{x \to a} f(x) = f(a)$



Theorem: If $F: \mathbb{R} \to \mathbb{R}$ then F is the CDF for some r.v. X if

1. $\lim_{x \to \infty} F(x) = 1 \quad \lim_{x \to -\infty} F(x) = 0$
2. F is non-decreasing
3. F is right continuous.

(Note if F is continuous, it is right continuous)

Ex, let $F(x) = \dfrac{1}{1+e^{-x}}$, $x \in \mathbb{R}$

Q: is this a valid CDF?

1. $\lim_{x \to \infty} F(x) = \lim_{x \to \infty} \dfrac{1}{1+e^{-x}} = 1$ ✓

$\lim F(x) = \lim \dfrac{1}{\quad} = 0$ ✓

$$x \to -\infty \quad 1+e^{-x}$$

(2) non-decreasing?

$$F'(x) = \frac{e^{-x}}{(1+e^{-x})^2} > 0 \quad ✓$$

(3) Right continuous?

yes, its continuous ✓

Defn: We say two random variables X and Y are identically distributed if for any $A \subset \mathbb{R}$,

$$\mathbb{P}(X \in A) = \mathbb{P}(Y \in A).$$

This is not the same as saying $X = Y$ as functions!

If $X = Y$ as a function then they are identically distributed.

Converse is false.

---

ex. Flip 3 coins.

let X = # heads, Y = # tails.

$$X \overset{d}{=} Y$$

equal in dist.

e.g. $P(X=1) = 3/8$ and $P(Y=1) = 3/8$

$\{HTT, THT, TTH\}$ $\{THH, HTH, HHT\}$

note:

$X(HTT)=1$ but $Y(HTT)=2$.

Theorem: $X \overset{d}{=} Y$ iff $F_X = F_Y$

CDFs of X and Y are equal as fns.
