Defn: Variance

$$\mathrm{Var}(X) = E\left[(X - E[X])^2\right]$$
$$= E[X^2] - E[X]^2$$

Ex. $X \sim \mathrm{Bin}(n, p)$

recall: $E[X] = np$

$$E[X^2] = \sum_{x=0}^{n} x^2 f(x)$$

$$= \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x}$$

$$x\binom{n}{x} = n\binom{n-1}{x-1}$$

$$= \sum_{x=\cancel{0}1}^{n} x\, n \binom{n-1}{x-1} p^x (1-p)^{n-x}$$

$$\boxed{\begin{aligned} y &= x-1 \\ x &= y+1 \end{aligned}}$$

$$= \sum_{y=0}^{n-1} (y+1)\, n \binom{n-1}{y} p^{y+1} (1-p)^{n-y-1}$$

$$= np\left(\underbrace{\sum_{y=0}^{n-1} y \binom{n-1}{y} p^y (1-p)^{(n-1)-y}}_{E[\mathrm{Bin}(n-1,p)] \,=\, (n-1)p} + \underbrace{\sum_{y=0}^{n-1} \binom{n-1}{y} p^y (1-p)^{(n-1)-y}}_{\text{Sum of a } \mathrm{Bin}(n-1,p) \text{ pmf} \,=\, 1}\right)$$

$$= np\left((n-1)p + 1\right) = E[X^2]$$

$$\mathrm{Var}(X) = E[X^2] - E[X]^2$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$
$$= np((n-1)p+1) - (np)^2$$
$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$
$$= np - np^2$$
$$\boxed{np(1-p)}$$

$$\boxed{\begin{aligned} \mathrm{Sd}(X) &= \sqrt{\mathrm{Var}(X)} \\ &= \sqrt{np(1-p)} \end{aligned}}$$

Defn:

If r is a pos. integer we define the r^{th} moment of a r.v. as

$$\mu_r \stackrel{\text{def}}{=} \mathbb{E}[X^r]$$

Ex. $\mu_1 = \mathbb{E}[X] = \mu$

$\mu_2 = \mathbb{E}[X^2]$ $\ldots$ $\mu_3 = \mathbb{E}[X^3]$, $\ldots$

We define the r^{th} central moment

$$\mu_r' \stackrel{\text{def}}{=} \mathbb{E}[(X-\mu)^r]$$

Ex. $\mu_1' = \mathbb{E}[X-\mu] = 0$

$\mu_1' = E[X-\mu] = 0$

$\mu_2' = E[(X-\mu)^2] = Var(X) = \mu_2 - \mu_1^2$

Defn: Moment Generating Function (MGF)

If X is a r.v. the MGF is a function

$$M : \mathbb{R} \longrightarrow \mathbb{R}$$

defined for $t \in \mathbb{R}$ as

$$M(t) = E[e^{tX}].$$

Its defined in some neighborhood of zero.
(Possible $M(t) = \infty$ for some t)

For continuous r.v.s

$$M(t) = E[e^{tX}] = \int_{\mathbb{R}} e^{tx} f(x)\, dx$$

discrete

$$M(t) = \sum_x e^{tx} f(x).$$

Ex. $X \sim Exp(\lambda) \rightsquigarrow f(x) = \lambda e^{-\lambda x}$ for $x > 0$

$M(t) = E[e^{tX}] = \int e^{tx} \lambda \ldots$

$f(x) = \lambda e^{-\lambda x}$ for $x > 0$

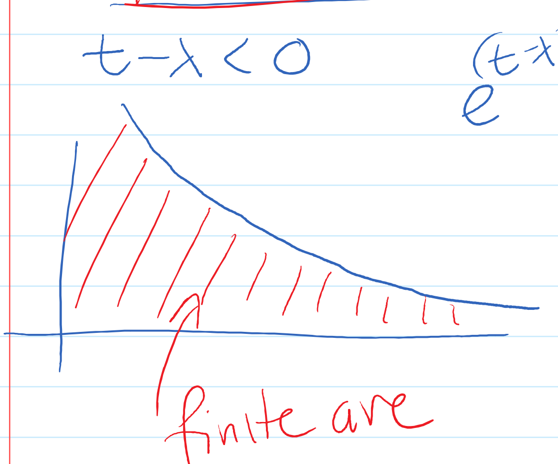
$$M(t) = \mathbb{E}[e^{tX}] = \int_{\mathbb{R}} e^{tx} f(x)\,dx$$

$$= \int_0^\infty e^{tx} \lambda e^{-\lambda x}\,dx$$

$$= \lambda \underbrace{\int_0^\infty e^{(t-\lambda)x}\,dx}$$

$\boxed{t < \lambda}$

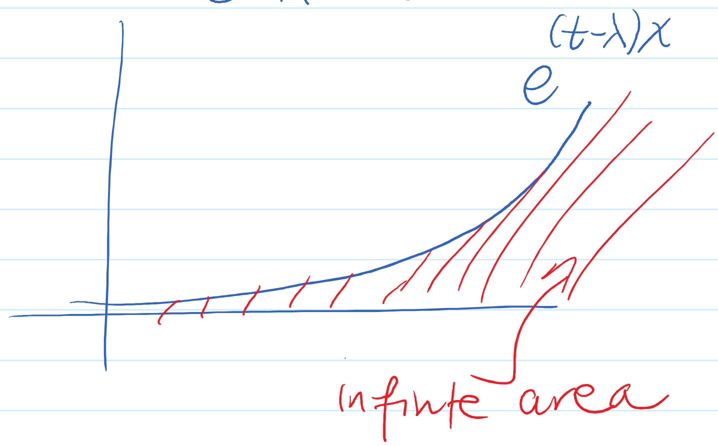
$t - \lambda < 0$



$t \in (0, \lambda)$

$\boxed{t \geq \lambda}$

$t - \lambda \geq 0$



if $t < \lambda$ then

$$\lambda \int_0^\infty e^{(t-\lambda)x}\,dx = \frac{\lambda}{t-\lambda}\left[e^{(t-\lambda)x}\right]_0^\infty$$

$$= \frac{\lambda}{t-\lambda}\left[0 - 1\right]$$

$$M(t) \quad \boxed{= \frac{\lambda}{\lambda - t} \quad \text{for } t < \lambda}$$

Consider:

$$\frac{dM}{dt} = \frac{\lambda}{(\lambda - t)^2}, \quad \left.\frac{dM}{dt}\right|_{t=0} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$= \mathbb{E}[X]$$

$$\frac{d^2M}{dt^2} = \frac{d}{dt}\left(\frac{\lambda}{(\lambda - t)^2}\right) = \frac{2\lambda}{(\lambda - t)^3}$$

$$\left.\frac{d^2M}{dt^2}\right|_{t=0} = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2} = \mathbb{E}[X^2]$$

Theorem:

$$\boxed{\left.\frac{d^rM}{dt^r}\right|_{t=0} = \mathbb{E}[X^r] = \mu_r}$$

Pf.

$$\frac{d^rM}{dt^r} = \frac{d^r}{dt^r}\mathbb{E}[e^{tX}] = \frac{d^r}{dt^r}\int_{\mathbb{R}} e^{tx} f(x)\,dx$$

$$= \int_{\mathbb{R}} \frac{d^r}{dt^r} e^{tx} f(x)\,dx$$

$\mathbb{R}$

$$\frac{d}{dt} e^{tx} = x e^{tx}, \quad \frac{d^2}{dt^2} e^{tx} = \frac{d}{dt}(x e^{tx}) = x^2 e^{tx}$$

generally: $\frac{d^r}{dt^r} e^{tx} = x^r e^{tx}$

$$= \int_{\mathbb{R}} x^r e^{tx} f(x)\,dx = \frac{d^r M}{dt^r}$$

So $\left.\frac{d^r M}{dt^r}\right|_{t=0} = \int_{\mathbb{R}} x^r f(x)\,dx = \mathbb{E}[X^r] = \mu_r$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

2

$$e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \cdots$$

$$\frac{d}{dt} e^{tX} = X + \frac{2tX^2}{2!} + \frac{3t^2X^3}{3!} + \cdots$$

$\mathbb{E}\left[\frac{d}{dt} e^{tX}\right]$ at $t=0 \to \mathbb{E}[X]$

Ex. $X \sim Bin(n, p)$

$$M(t) = \mathbb{E}[e^{tX}] = \sum_{x=0}^{n} e^{tx} f(x)$$

$$= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x}$$

Binomial Theorem

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$= \binom{2}{0} a^2 b^0 + \binom{2}{1} a^1 b^1 + \binom{2}{2} a^0 b^2$$

$$\sum_{i=0}^{2} \binom{2}{i} a^{2-i} b^i$$

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

$$\sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x} = \boxed{(pe^t + (1-p))^n = M(t)}$$

$$\left.\frac{dM}{dt}\right|_{t=0} = \left. n(pe^t + 1-p)^{n-1} pe^t \right|_{t=0}$$

$$= n(p(1) + 1-p)^{n-1} p(1)$$

$$= np = E[X]$$

$$\frac{d^2M}{dt^2} = \frac{d}{dt}\left(n(pe^t + 1-p)^{n-1} pe^t\right)$$

$$= n(n-1)(pe^t + 1-p)^{n-2} pe^t pe^t$$

$$= n(n-1)(pe^t+1-p)^{n-2}\, pe^t\, pe^t$$
$$+ n(pe^t+1-p)^{n-1} pe^t$$

So

$$\left.\frac{d^2M}{dt^2}\right|_{t=0} = n(n-1)p^2 + np$$
$$= n^2p^2 - np^2 + np = \mathbb{E}[X^2]$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$
$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$
$$= np - np^2$$
$$\boxed{= np(1-p)}$$

Theorem: If $a, b \in \mathbb{R}$, and $Y = aX + b$

$$M_Y(t) = e^{bt} M_X(at)$$

pf.

$$M_Y(t) = \mathbb{E}[e^{tY}] = \mathbb{E}[e^{t(aX+b)}]$$
$$= \mathbb{E}[e^{atX} e^{bt}]$$
$$= e^{bt}\mathbb{E}[e^{atX}]$$
$$= e^{bt} M_X(at)$$

Theorem:

Theorem:

If X and Y are r.v.s. and

$$M_X(t) = M_Y(t)$$

for t in some neighborhood of zero, then

$$X \stackrel{d}{=} Y.$$

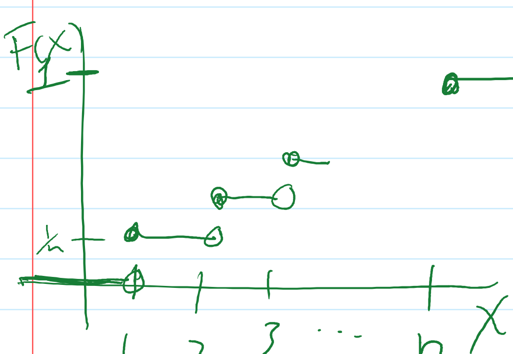
Discrete Uniform

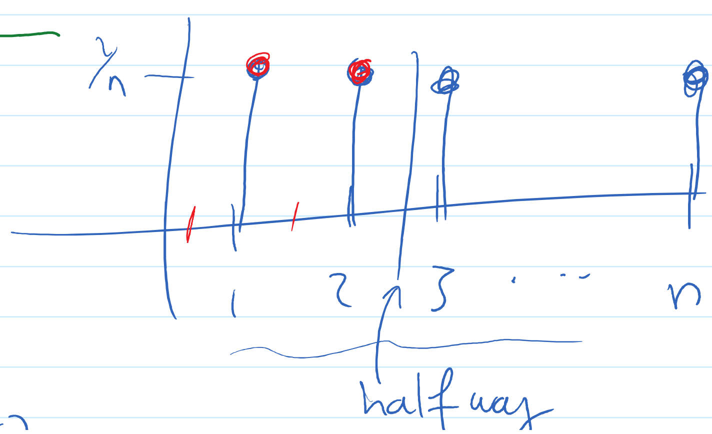
$$X \sim U(\{1, \dots, n\})$$

means $f(x) = 1/n$ for $x = 1, \dots, n$

CDF:

$$F(x) = \sum_{i \le x} f(i) = \begin{cases} 0 & i < 1 \\ 1/n & 1 \le i < 2 \\ 2/n & 2 \le i < 3 \\ \vdots & \vdots \\ 1 & i \ge n \end{cases}$$





$E[X] = \sum^{n} x f(x)$

halfway

$$\mathbb{E}[X] = \sum_{x=1}^{n} x f(x)$$

$$= \sum_{x=1}^{n} x \, \frac{1}{n} = \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}$$

$$\mathbb{E}[X^2] = \sum_{x=1}^{n} x^2 / n = \frac{1}{n} \frac{(n+1)(2n+1)}{6}$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \ldots = \boxed{\frac{(N+1)(N-1)}{12}}$$