Lecture 8 - Random Variables

Tuesday, February 18, 2020 10:54 AM

Ex. Flip a coin 3 times (independent)

$X = \#$ heads

8 outcomes in S

s	$X(s)$
H H H	3
H H T	2
H T H	2
H T T	1
T H H	2
T H T	1
T T H	1
T T T	0

Defn: Random Variable

a r.v. X is a function

$$X : S \longrightarrow \mathbb{R}.$$

Idea: Now I can ask questions like

"$\mathbb{P}(X=1)$"

recall that technically $\mathbb{P} : \mathcal{P}(S) \longrightarrow \mathbb{R}$

we really mean

$$\mathbb{P}(X=1) = \mathbb{P}(\{HTT, THT, TTH\})$$

$$\mathbb{P}(X=1) = \mathbb{P}(\underbrace{\{HTT, THT, TTH\}}_{\text{Event in } S \text{ where \# heads} = 1.})$$

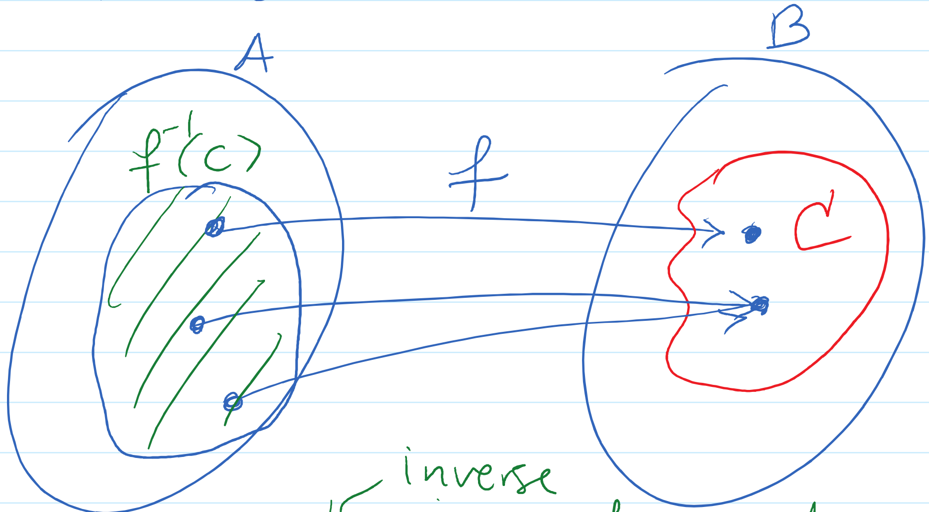
prob. that # heads is 1

"$X=1$" short hand for $\{s \in S \mid X(s) = 1\}$

Aside: Inverse Image (of 1 under X)

Inverse Image:

$$f : A \longrightarrow B$$



inverse image of C under f

$C \subset B$ then $f^{-1}(C)$

$$= \{a \in A \mid f(a) \in C\}$$

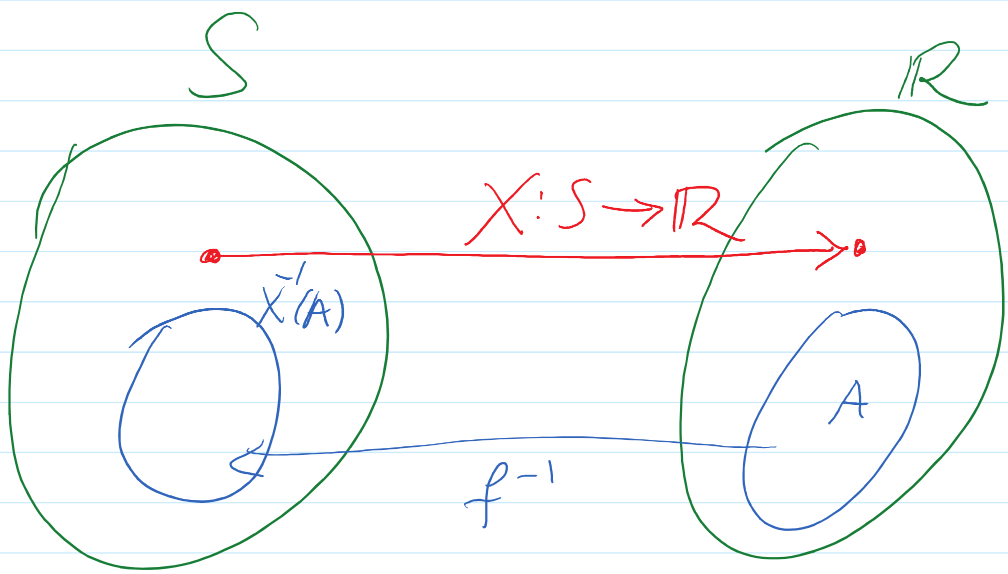
If f is bijective (invertible) then f^{-1} is our typical function inverse.

So for $b \in B$ then

$f^{-1}(\{b\})$ has exactly one element

$f(\{b\})$ has exactly one element
$f^{-1}(b)$ (using typical defn of inverse function)

For random variables



If $A \subset \mathbb{R}$ then $X^{-1}(A) \subset S$.

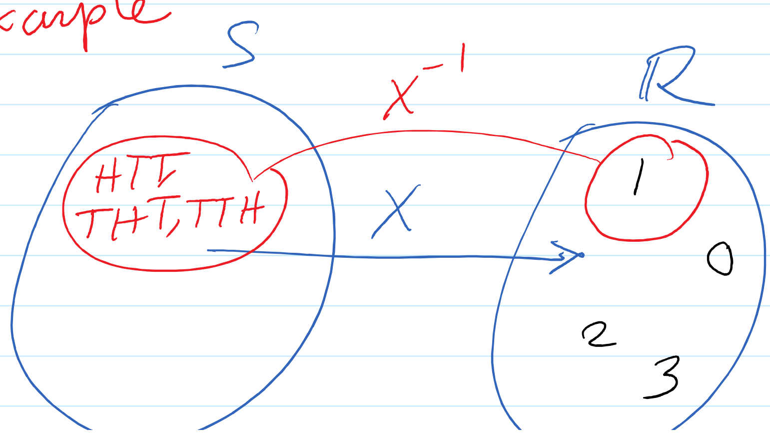
So I can write

$$\text{"}\mathbb{P}(X \in A)\text{"} \stackrel{\text{def}}{=} \mathbb{P}(X^{-1}(A)).$$

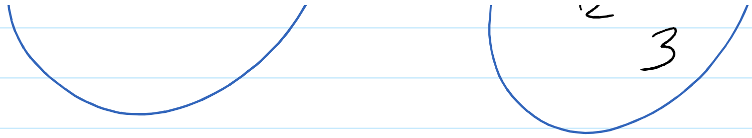
(technically $\mathbb{P} : \mathcal{P}(S) \to \mathbb{R}$)

Ex. From perivas example

$$\mathbb{P}(X=1)$$
$$= \mathbb{P}(X \in \{1\})$$
$$= \mathbb{P}(X^{-1}(\{1\}))$$
$$= \mathbb{P}(X^{-1}(1))$$



$$= \mathbb{P}(X^{-1}(1))$$



$$= \mathbb{P}(\{s \in S \mid X(s) = 1\})$$

$$= \mathbb{P}(\{HTT, THT, TTH\}) = 3/8$$

Ex.

$$\mathbb{P}(X = 1 \text{ or } 2)$$

$$= \mathbb{P}(X \in \{1, 2\})$$

$$= \mathbb{P}(X^{-1}(\{1, 2\}))$$

$$= \mathbb{P}(\{s \in S \mid X(s) = 1 \text{ or } 2\})$$

$$= \mathbb{P}(\{HTT, THT, TTH, HHT, HTH, THH\})$$

$$= 6/8$$

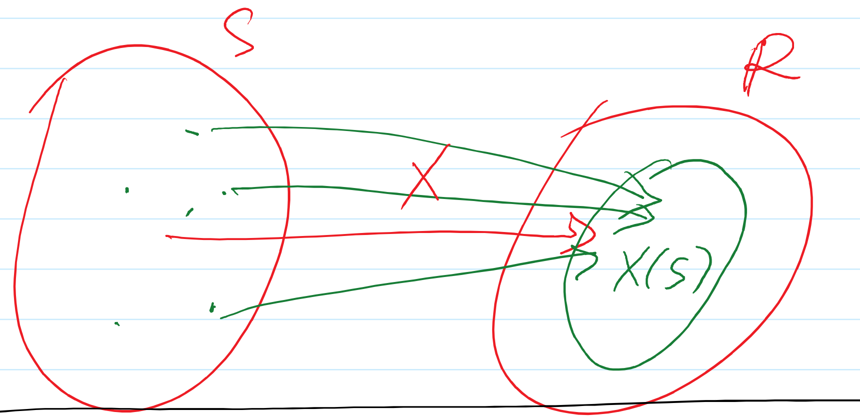
Defn: Support of a RV.

If X is a r.v. then the **Support** of X is the part of $\mathbb{R}$ where X has a pos. probability of being.

$$\text{Support}(X) = \underset{\text{image}}{\text{Im}}(X)$$

i.e. $X(S) = \{X(s) \mid s \in S\} \subset \mathbb{R}$

i.e. $X(S) = \{ X(s) \mid s \in S \} \subset \mathbb{R}$



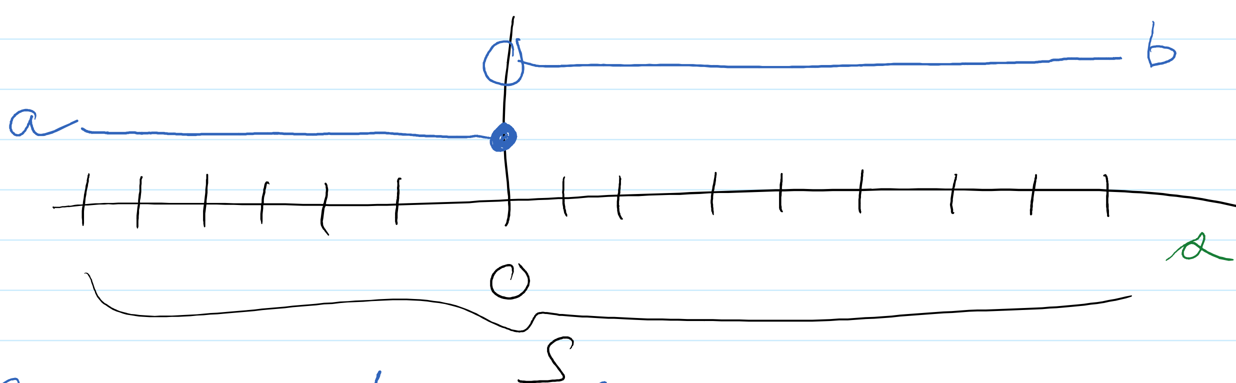
<u>Ex.</u> For coin flipping

$$\text{Support}(X) = \{0, 1, 2, 3\}.$$

Ex: Let $S = \{0, \pm 1, \pm 2, \pm 3, \dots\}$

and Y is a r.v. so that

$$Y(s) = \begin{cases} a & \text{if } s \le 0 \\ b & \text{if } s > 0 \end{cases}$$



Support of Y is $\{a, b\}$.

$P(Y = c)$ where $c \neq a$ and $c \neq b$

$\mathbb{P}(Y=c)$ where $c \neq a$ and $c \neq b$.
$= 0$?

$\rightarrow$ $\mathbb{P}(Y=c) = \mathbb{P}(Y^{-1}(c)) = \mathbb{P}(\emptyset) = 0$.

$\rightarrow$ $\mathbb{P}(Y=a) = \mathbb{P}(Y^{-1}(a)) = \mathbb{P}(\{s : s \leq 0\})$

$\rightarrow$ $\mathbb{P}(Y=b) = \mathbb{P}(Y^{-1}(b)) = \mathbb{P}(\{s : s > 0\})$

Defn: Cumulative Distribution Function (CDF).

If X is a r.v. then the CDF of X is a function

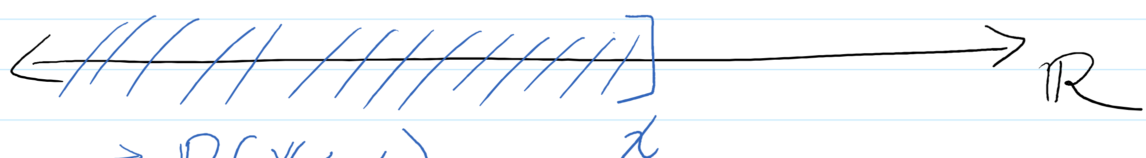
$$F : \mathbb{R} \rightarrow \mathbb{R}$$

defined for $x \in \mathbb{R}$

$$F(x) = \mathbb{P}(X \leq x)$$

capital/bold X is a r.v.

little x is a real number.



write $\rightarrow$ $\mathbb{P}(X \leq x)$

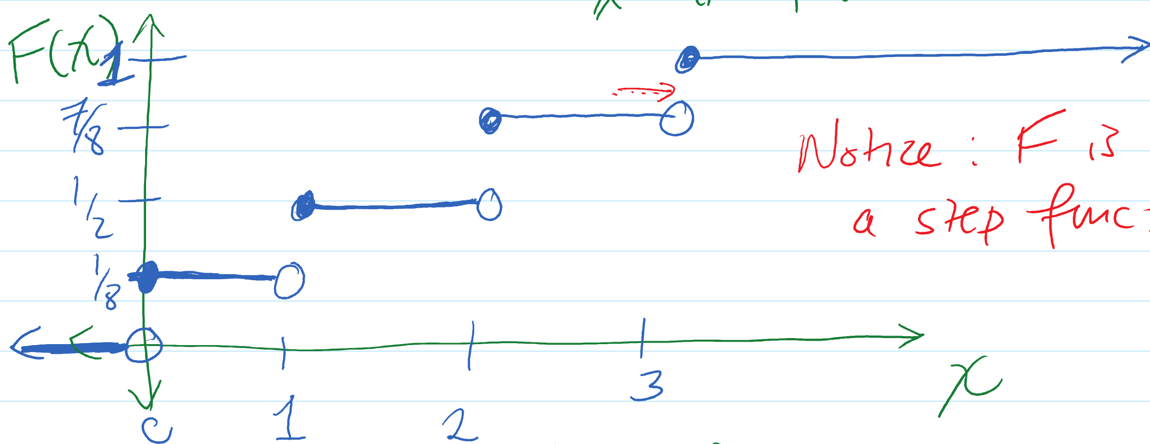
write $\longrightarrow$ $\mathbb{P}(X \le x)$

mean $\longrightarrow$ $= \mathbb{P}(X \in (-\infty, x])$

$= \mathbb{P}(X^{-1}((-\infty, x]))$

Ex. Toss a coin 3 times:

$X = \#$ of heads



Notice: F is a step function

recall: $F(x) = \mathbb{P}(X \le x)$

$\rightarrow$ If $x < 0$ then $F(x) = \mathbb{P}(X \le x) = 0$

$\rightarrow$ $F(0) = \mathbb{P}(X \le 0)$

$= \mathbb{P}(X^{-1}(0)) = \mathbb{P}(\{TTT\}) = 1/8$

$\rightarrow$ If $0 \le x < 1$, $F(x) = 1/8$

$\rightarrow$ $F(1) = \mathbb{P}(X \le 1) = \mathbb{P}(\{TTT, \ldots\}) = 1/2$

$\rightarrow$ $F(2) = \mathbb{P}(X \le 2) = 7/8$

$\rightarrow$ $F(3) = \mathbb{P}(X \le 3) = 1$

$F(x) = 1$ for $x \ge 3$

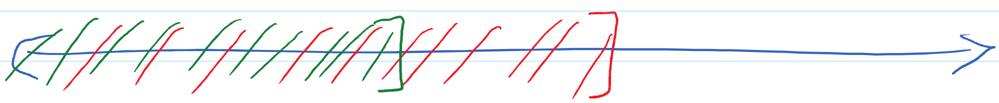
Theorem: Properties of CDFs

(1) $0 \le F(x) \le 1$ $\quad \left(F(x) = \mathbb{P}(\ldots) \in [0,1]\right)$

(2) $\lim_{x \to \infty} F(x) = 1$ and $\lim_{x \to -\infty} F(x) = 0$.

(3) F is non-decreasing

$x_1 < x_2$



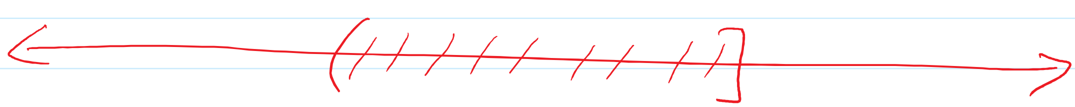
x_1 $\qquad$ x_2

$(-\infty, x_1] \subset (-\infty, x_2]$

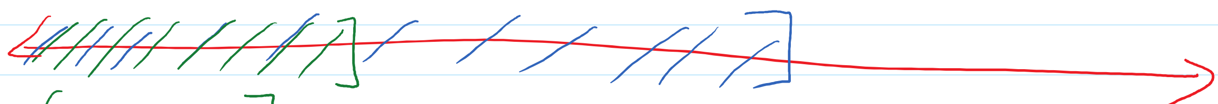
So

$F(x_1) = \mathbb{P}((-\infty, x_1]) \le \mathbb{P}((-\infty, x_2]) = F(x_2)$

(4) $\boxed{\mathbb{P}(a < X \le b) = F(b) - F(a)}$



a $\qquad$ b



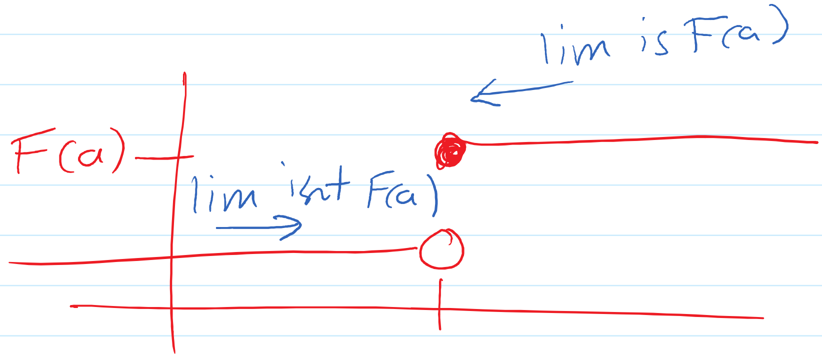
$(-\infty, a] \subset (-\infty, b]$

$(a, b] = (-\infty, b] \setminus (-\infty, a]$

$P(a < X \le b)$

(5) <u>F is right-continuous.</u>

$$\lim_{x \to a^+} F(x) = F(a)$$



a continuous function is a right continuous

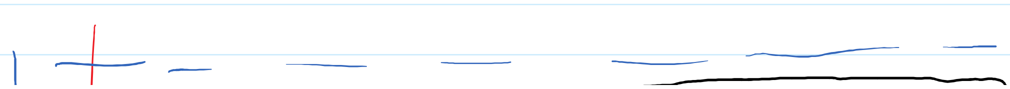
<u>Theorem:</u> If $F : \mathbb{R} \to \mathbb{R}$ then F is the CDF of some r.v. if

(1) $\lim_{x \to \infty} F(x) = 1$ $\quad \lim_{x \to -\infty} F(x) = 0$

(2) F is non-decreasing

(3) F is right continuous.

<u>Ex,</u> Let $F(x) = \dfrac{1}{1 + e^{-x}}$ for all $x \in \mathbb{R}$.



Q: is this a valid CDF?

1) $\lim_{x \to \infty} F(x) = \lim_{x \to \infty} \frac{1}{1+e^{-x}} = \frac{1}{1+0} = 1$

$\lim_{x \to -\infty} F(x) = \lim_{x \to -\infty} \frac{1}{1+e^{-x}} = \frac{1}{\infty} = 0$

2) F is non-decreasing:

$F'(x) = \frac{e^{-x}}{(1+e^{-x})^2} > 0 \quad \forall x$ ✓

3) F is continuous and hence right continuous.

Colloquial defn:

A <u>discrete</u> r.v. has a discrete support

e.g. X = # heads has a finite support

X = # of people arriving at bus stop

support(X) = $\mathbb{N}$

not finite but discrete.

A <u>continuous</u> r.v. has a non-discrete support.

e.g. X = waiting time for bus to arrive.

$$\text{Support}(X) = [0, \infty)$$

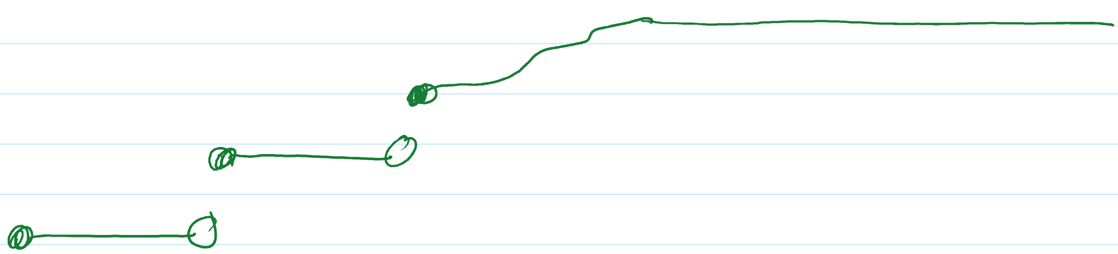
not discrete.

<u>Technical defn:</u>

<u>discrete</u> r.v.s have a CDF that is a step function

<u>Continuous</u> r.v.s have a continuous CDF

<u>mixed</u> r.v.s are somewhere between



<u>Defn: Identical Distribution</u>

We say two r.v.s. X and Y are <u>identically distributed</u> if for any $A \subset \mathbb{R}$

notation: $X \overset{d}{=} Y$

$$\boxed{\mathbb{P}(X \in A) = \mathbb{P}(Y \in A).}$$

$\boxed{X \overset{d}{=} Y}$ $\boxed{\mathbb{P}(X \in A) = \mathbb{P}(Y \in A).}$

DIFFERENT than saying $X = Y$. (as functions)

If $X = Y$ as functions, then $X \overset{d}{=} Y$.
Converse is false.

Ex. X = # heads in 3 coin flip
Y = # tails . . .

$X(HHT) = 2 \quad ; \quad Y(HHT) = 1$

yet $\mathbb{P}(X = 1) = \mathbb{P}(Y = 1)$ etc.