Defn: Conditional PMF/PDF

Given X, Y

$$f(x|y) = \frac{f(x,y)}{f(y)}$$

$\uparrow$ cond. pmf/pdf of "$\overbrace{X \text{ given } Y = y}^{\text{a univariate r.v.}}$"

Defn: Conditional Expectation

If $g : \mathbb{R} \to \mathbb{R}$ then

$$\mathbb{E}[g(X) \mid Y = y] = \begin{cases} \int g(x) f(x|y)\, dx & \text{cts} \\ \sum_x g(x) f(x|y) & \text{discrete} \end{cases}$$

recall!

$$\boxed{\mathbb{E}[g(X)] = \int g(x) f(x)\, dx}$$

Ex. Shifted Exponential

$$\begin{cases} Y \mid X = x \sim \mathrm{Exp}(\lambda = 1, \text{shift} = x) \\ f(y|x) = e^{x-y} \text{ for } 0 < x < y \end{cases}$$

$$\boxed{\mathbb{E}[Y \mid X = x]} = \int y\, f(y|x)\, dy = \underbrace{\int_x^\infty y e^{x-y}\, dy}_{\text{by parts}}$$

by parts
$u = y \quad dv = e^{-y}$

$$\cdots = \boxed{1 + x} \quad \text{(an exercise)}$$

Defn: Cond. Variance

$$\mathrm{Var}(Y \mid X = x) = \mathbb{E}\left[\left(Y - \mathbb{E}[Y \mid X = x]\right)^2 \,\middle|\, X = x\right]$$

Short-cut formula:

$$\mathrm{Var}(Y \mid X = x) = \mathbb{E}[Y^2 \mid X = x] - \mathbb{E}[Y \mid X = x]^2$$

Ex. Continue example above

$$\mathrm{Var}(Y \mid X = x) = \mathbb{E}[Y^2 \mid X = x] - \underbrace{\mathbb{E}[Y \mid X = x]^2}_{(1+x)^2}$$

$$\int y^2 f(y|x)\, dy = \int_x^\infty y^2 e^{x-y}\, dy$$

$\hookrightarrow \int y^2 f(y|x)\,dy = \int_x y^2 e^{x-y}\,dy$

$= \text{integration by parts}$

$= x^2 + 2x + 2$

So $\mathrm{Var}(Y|X=x) = x^2+2x+2-(1+x)^2 = 1$

Independence

For events: $\quad P(AB) = P(A)P(B)$

For r.v.s.: For any A, B

$\longrightarrow \quad P(X\in A,\ Y\in B) = P(X\in A)P(Y\in B)$

then $X \perp\!\!\!\perp Y$.

Independence of RVs

bootstraps of independence for events using events $\underline{X^{-1}(A)}$ and $\underline{Y^{-1}(B)}$

r.v.s are independent if those events are independent $\forall A, B$.

Lemma: Factorization lemma

① $X \perp\!\!\!\perp Y \iff F(x,y) = F_X(x)\,F_Y(y)$

② $X \perp\!\!\!\perp Y \iff f(x,y) = f_X(x)\,f_Y(y)$

Fact: If $X \perp\!\!\!\perp Y$ then

$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{\cancel{f(x)}\,f(y)}{\cancel{f(x)}} = f(y)$$

Simly

$$f(x|y) = f(x).$$

Ex. $\longleftarrow$ joint pmf

$y \backslash x$	10	20
3	1/5	3/10
2	1/5	1/10
1	1/10	1/10

$\boxed{\text{Q! Is } X \perp\!\!\!\perp Y\text{?}}$

marginals:

x	10	20
$f(x)$	1/2	1/2

y	1	2	3
$f(y)$	1/5	3/10	1/2

$$f(x)f(y) = f(x,y)$$

$$f(10)f(3) = \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) \neq \tfrac{1}{5} = f(10,3)$$

So $X \not\perp\!\!\!\perp Y$.

<u>Corollary:</u>

$$X \perp\!\!\!\perp Y \iff f(x,y) = \underbrace{h(x)}_{\text{fn only of } x}\underbrace{g(y)}_{\text{fn only of } y}$$

<u>Ex.</u> $f(x,y) = \frac{1}{384} x^2 y^4 e^{-y-(x/2)}, \quad x>0, y>0$

<u>Joint pdf</u>

Q: $X \perp\!\!\!\perp Y$?

$$f(x,y) = \overbrace{\underbrace{\left(\frac{1}{384}x^2 e^{-x/2}\right)}_{\text{fn of } x \text{ only}}}^{h(x)}\overbrace{\underbrace{\left(y^4 e^{-y}\right)}_{\text{fn of } y \text{ only}}}^{g(y)}$$

So $X \perp\!\!\!\perp Y$.

<u>Theorem: $\mathbb{E}$ for independent RVs.</u>

If $X \perp\!\!\!\perp Y$ then for $g_1, g_2 : \mathbb{R} \longrightarrow \mathbb{R}$

$$\mathbb{E}\left[g_1(X)\, g_2(Y)\right] = \mathbb{E}[g_1(X)]\,\mathbb{E}[g_2(Y)].$$

pf. (cts case)

$$\mathbb{E}[g_1(X)g_2(Y)] = \iint g_1(x)g_2(y) f(x,y)\, dx\, dy$$

$$= \iint g_1(x)\, g_2(y)\, \underline{f(x)}\,\underline{f(y)}\, dx\, dy$$

$$= \left[\int g_1(x) f(x)\, dx\right]\left[\int g_2(y)\, f(y)\, dy\right]$$

$$= \mathbb{E}[g_1(X)]\,\mathbb{E}[g_2(Y)].$$

Ex. $X, Y \sim \text{Exp}(1)$ and $X \perp\!\!\!\perp Y$

then $\mathbb{E}[X^2 Y] = \mathbb{E}[X^2]\,\mathbb{E}[Y] = 2(1) = 2.$

recall for $Z \sim \text{Exp}(\lambda)$, $\mathbb{E}[Z] = 1/\lambda$

$\mathbb{E}[Z^2] = 2/\lambda^2$

Lemma: MGFs of Independent

$$\boxed{\text{If } X \perp\!\!\!\perp Y \text{ then} \quad M_{X+Y}(t) = M_X(t)\,M_Y(t)}$$

pf.

$$M_{X+Y}(t) = E\left[e^{t(X+Y)}\right] = E\left[e^{tX}e^{tY}\right]$$
$$= E\left[e^{tX}\right]E\left[e^{tY}\right]$$
$$= M_X(t)\,M_Y(t).$$

Ex. Let $X \sim N(\mu, \sigma^2)$ and $Y \sim N(\gamma, \tau^2)$ and $X \perp\!\!\!\perp Y$

then

$$M_{X+Y}(t) = M_X(t)\,M_Y(t)$$
$$= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)\exp\left(\gamma t + \frac{\tau^2 t^2}{2}\right)$$
$$= \exp\left(\underbrace{(\mu+\gamma)}_{a}t + \frac{\overbrace{(\sigma^2+\tau^2)}^{b}t^2}{2}\right)$$
$$= \exp\left(at + \frac{bt^2}{2}\right)$$

MGF of $N(a, b)$

So $X+Y \sim N(a,b)$ i.e. $X+Y \sim N(\mu+\delta, \sigma^2+\tau^2)$

$X \perp\!\!\!\perp Y$.

Theorem: Cov/Cor for Independent r.v.s

If $X \perp\!\!\!\perp Y$ then $\overset{\Longrightarrow}{}$ $\mathrm{Cov}(X,Y) = \mathrm{Cor}(X,Y) = 0$.

pf. $\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$

$= \mathbb{E}[X]\,\mathbb{E}[Y] - \mathbb{E}[Y]\,\mathbb{E}[X] = 0$

$\mathrm{Cor}(X,Y) \propto \mathrm{Cov}(X,Y)$

$\mathrm{Cor}(X,Y) = 0$ also.

Converse is generally not true.

If $\mathrm{cor}(X,Y)=0$ we don't know they are independent.

Ex. $X \sim N(0,1)$ and $Y = X^2$.

$\mathrm{Cov}(X,Y) = \mathrm{Cov}(X,X^2) = \mathbb{E}[XX^2] - \mathbb{E}[X^2]\,\mathbb{E}[X]$

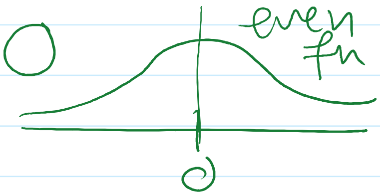
$= \underbrace{\mathbb{E}[X^3]}_{0} - \underbrace{\mathbb{E}[X]}_{0}\,\mathbb{E}[X^2]$

$= 0$

Fact: $\mathbb{E}[X^r] = 0$ for odd r

fact: $E[X^r] = 0$ for odd r $= 0$

$$E[X^r] = \int_{\mathbb{R}} \underbrace{\underbrace{x^r}_{\text{odd}} \underbrace{f(x)}_{\text{even}}}_{\text{odd}} dx = 0$$



integral of odd fn over sym. interval

Bayes' Theorem for RVs.

recall for events: $P(A|B) = \dfrac{P(B|A)P(A)}{P(B)}$

Now: $$\boxed{f(y|x) = \frac{f(x|y)f(y)}{f(x)}}$$ for $f(x) > 0$

$f(y|x) = \dfrac{f(x,y)}{f(y)} \Leftrightarrow$ $f(x,y) = f(y|x)f(x)$
$= f(x|y)f(y)$

$f(y|x)f(x) = f(x|y)f(y)$

Law of total prob:

recall: events

Partion $\{C_i\}_{i=1}^n$ then

$$P(A) = \sum_{i=1}^n P(A|C_i)P(C_i)$$

For RVs, Law of total Prob.

discrete: $f(y) = \sum_x f(y|x)f(x)$

continuous: $f(y) = \int f(y|x) f(x)\, dx$

pf ① $f(y|x) = \frac{f(x,y)}{f(x)} \Leftrightarrow \underline{f(x,y)} = \underline{f(y|x) f(x)}$

② $f(y) = \int f(x,y)\, dx$

Combine ①, ②: $\boxed{f(y) = \int f(y|x) f(x)\, dx}$

ex, Assume $Y | X = x \sim \text{Poisson}(x)$
and $X \sim \text{Exp}(\lambda)$

Q: Dist of Y?

$f(y) = \int f(y|x) f(x)\, dx$

$$= \int_0^\infty \underbrace{\frac{x^y e^{-x}}{y!}}_{\text{Poisson}(x)} \underbrace{\lambda e^{-\lambda x}}_{\text{Exp}(\lambda)} dx = \frac{\lambda}{y!} \int_0^\infty \underbrace{x^{\overset{\alpha-1}{y}} e^{-\overset{\beta}{(\lambda+1)} x} dx}$$

look like

Poisson(λ) Exp(λ)

look like Gamma PDF

$Z \sim \text{Gamma}(\alpha, \beta)$, $f(z) = \dfrac{\beta^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\beta z}$

Gamma$(y+1, \lambda+1)$

$$= \frac{\lambda}{y!} \frac{\Gamma(y+1)}{(\lambda+1)^{y+1}} \underbrace{\int_0^\infty \frac{(\lambda+1)^{y+1}}{\Gamma(y+1)} x^y e^{-(\lambda+1)x} \, dx}_{\text{Integrate to } 1}$$

$$= \frac{\lambda \cancel{y!}}{\cancel{y!}(\lambda+1)^{y+1}} = \boxed{\frac{\lambda}{(\lambda+1)^{y+1}}}$$

Same Neg. Bin.

marginal of Y.

Ex. $X \mid Y = y \sim \text{Bin}(y, p)$ $p \in [0,1]$

$Y \sim \text{Poisson}(\lambda)$ $0 \leq x \leq y$; $y \geq x$

Q: what dist of X?

Law of Tot. prob. (discrete)

$$f(x) = \mathbb{P}(X = x) = \sum^{\infty} f(y|x) f(y)$$

$$f(x) = \mathbb{P}(X=x) = \sum_{y=0}^{\infty} f(y|x) f(y)$$

$$= \sum_{y=x}^{\infty} \binom{y}{x} p^x (1-p)^{y-x} \frac{\lambda^y e^{-\lambda}}{y!} \qquad y < x$$

$$\binom{y}{x} \frac{1}{y!} = \frac{y!}{x!(y-x)!} \frac{1}{y!} = \frac{1}{x!(y-x)!}$$

$$= \frac{1}{x!} p^x e^{-\lambda} \lambda^x \sum_{y=x}^{\infty} \frac{1}{(y-x)!} (1-p)^{y-x} \lambda^{y-x}$$

$$= \frac{1}{x!} p^x e^{-\lambda} \lambda^x \underbrace{\sum_{y=0}^{\infty} \frac{1}{y!} \left((1-p)\lambda\right)^y}_{\exp(\underbrace{(1-p)\lambda}_{a})} \qquad \boxed{\sum_{y=0}^{\infty} \frac{a^y}{y!} = e^a}$$

$$= \frac{(p\lambda)^x \cancel{e^{-\lambda}} e^{(1-p)\lambda}}{x!}$$

$$= \frac{(p\lambda)^x e^{-p\lambda}}{x!} \qquad \text{Poisson}(\lambda p)$$

So $X \sim \text{Poisson}(\lambda p)$; $\mathbb{E}[X] = \lambda p$

Theorem: Iterated Expectation

If X, Y are r.v.s. then

$$E[X] = E_Y\big[E[X|Y]\big].$$

Seen $\boxed{E[X|Y=y]} = g(y)$

$g: \mathbb{R} \to \mathbb{R}$

Know: $g(Y)$ denoted $E[X|Y]$ is a r.v.

$$\boxed{E[X|Y] = g(Y)}$$

that is a trans of Y

Since this is a r.v. we can get its expectation:

$$E_Y\big[\underbrace{E[X|Y]}_{g(Y)}\big] = \int g(y)\, \underline{f(y)}\, dy$$

Theorem Says:

$$E_Y\big[E[X|Y]\big] = E[X].$$

$f(x)$

"

pf. in cts.

$$\boxed{E[X]} = \int x f(x)\,dx = \int x \left[\overbrace{\int f(x,y)\,dy}^{f(x)}\right] dx$$

$$= \int x \int \underbrace{f(x|y) f(y)}_{f(x,y)}\,dy\,dx$$

$$= \iint x f(x|y) f(y)\,dy\,dx$$

$$= \int \underbrace{\left[\int x f(x|y)\,dx\right]}_{E[X|Y=y]} f(y)\,dy$$

$$= \int E[X|Y=y]\, f(y)\,dy$$

$$\boxed{= E_Y\left[E[X|Y]\right].}$$

$E[X|Y=y]$ is a number

$E[X|Y]$ is a r.v.