Lecture 20 - Bivariate Transformations

Friday, April 10, 2020 5:58 PM

Ex. $U = X + Y$
$V = X - Y$

$$\boxed{f_{U,V}(u,v) = f_{X,Y}\left(\frac{u+v}{2}, \frac{u-v}{2}\right) \cdot \frac{1}{2}}$$

Assume $X, Y \overset{iid}{\sim} N(0,1)$

iid = independent, identically distributed

$X \perp\!\!\!\perp Y$ and $X \sim N(0,1)$
$Y \sim N(0,1)$

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$

$$\boxed{f_{U,V}(u,v) = \frac{1}{2\pi} e^{-\frac{1}{2}\left(\frac{u+v}{2}\right)^2} e^{-\frac{1}{2}\left(\frac{u-v}{2}\right)^2} \cdot \frac{1}{2}}$$

$$\left(\frac{u+v}{2}\right)^2 = \frac{1}{4}(u^2 + 2uv + v^2)$$

$$+ \quad \left(\frac{u-v}{2}\right)^2 = \frac{1}{4}(u^2 - 2uv + v^2)$$

$$= \frac{1}{4}(2u^2 + 2v^2) = \frac{1}{2}(u^2 + v^2)$$

$$f \quad \frac{1}{\ } e^{-\frac{1}{2}\left(\frac{1}{2}(u^2+v^2)\right)}$$

+

$$= \underbrace{\frac{1}{\sqrt{2\cdot 2\pi}} e^{-\frac{1}{2}\frac{1}{2}u^2}}_{\substack{N(0,2) \\ f_u(u)}} \cdot \underbrace{\frac{1}{\sqrt{2\cdot 2\pi}} e^{-\frac{1}{2}\cdot\frac{1}{2}v^2}}_{\substack{N(0,2) \\ f_v(v)}}$$

So $U \perp\!\!\!\perp V$ see: $U \sim N(0,2)$, $V \sim N(0,2)$

Recap: $\boxed{X, Y \overset{iid}{\sim} N(0,1) \quad U = X+Y,\ V = X-Y \overset{iid}{\sim} N(0,2)}$

recall:

$X \sim N(0,1)$, $Y \sim N(0,1)$, $X \perp\!\!\!\perp Y$ the $X+Y \sim N(0,2)$

Theorem: If $X \perp\!\!\!\perp Y$, $X \sim N(\mu, \sigma^2)$

$Y \sim N(\lambda, \tau^2)$

then $X \pm Y$ are independent

and $X \pm Y \sim N(\mu \pm \lambda, \sigma^2 + \tau^2)$

Theorem: Independence and Transformations

If $X \perp\!\!\!\perp Y$ and g, h are two fns

$g: \mathbb{R} \to \mathbb{R}$, $h: \mathbb{R} \to \mathbb{R}$

then $U = g(X)$ and $V = h(Y)$

then $U \perp\!\!\!\perp V$.

<u>Idea</u>! Functions of independent r.vs are also independent.

<u>Ex.</u> $U = X^2$ and $V = -\log Y$

if $X \perp\!\!\!\perp Y$ then $U \perp\!\!\!\perp V$.

<u>pf.</u> $\boxed{F_{U,V}(u,v)} = P(U \le u, V \le v)$

$U = g(X)$ and $V = h(Y)$

$P(X \in g^{-1}((-\infty, u]), Y \in h^{-1}((-\infty, v]))$

b/c $X \perp\!\!\!\perp Y$ then

$= P(X \in g^{-1}((-\infty, u]))P(Y \in h^{-1}((\infty, v]))$

$= P(g(X) \le u)P(h(Y) \le v)$

$= P(U \le u)P(V \le v)$

$\boxed{= F_U(u) F_V(v)}$

So we say $U \perp\!\!\!\perp V$.

So we say $U \perp V$.

Ex. $X \sim Beta(\alpha, \beta)$, $Y \sim Beta(\alpha+\beta, \gamma)$

and $X \perp Y$.

$\boxed{\text{Q: What is the dist of } XY\text{?}}$

Consider: $\boxed{U = XY}$ and $\boxed{V = X}$

notice $0 < U < V < 1$

$\boxed{X = V}$ and $\boxed{U/V = Y}$

$$\boxed{x = g_1^{-1}(u,v) = v \quad \text{and} \quad y = g_2^{-1}(u,v) = u/v}$$

$$J = \left|\det\left(\frac{\partial g^{-1}}{\partial(u,v)}\right)\right|$$

$$\frac{\partial g^{-1}}{\partial(u,v)} = \begin{bmatrix} \frac{\partial g_1^{-1}}{\partial u} & \frac{\partial g_1^{-1}}{\partial v} \\ \frac{\partial g_2^{-1}}{\partial u} & \frac{\partial g_2^{-1}}{\partial v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{v} & -\frac{u}{v^2} \end{bmatrix}$$

$$J = \left|\det\left(\;\right)\right| = \left|0\left(-\frac{u}{v^2}\right) - \left(\frac{1}{v}\right)(1)\right| = 1/v$$

$$f_{U,V}(u,v) = \underline{f_{X,Y}}\left(g_1^{-1}(u,v), g_2^{-1}(u,v)\right) J$$

$X \perp\!\!\!\perp Y$ and $X \sim \text{Beta}(\alpha,\beta)$

$Y \sim \text{Beta}(\alpha+\beta, \gamma)$

$$f_{X,Y}(x,y) = \underline{f_X(x)} f_Y(y)$$

$$\boxed{= \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \cdot \frac{y^{\alpha+\beta-1}(1-y)^{\gamma-1}}{B(\alpha+\beta,\gamma)}}$$

$$\boxed{f_{U,V}(u,v) = f_{X,Y}(v, u/v)\,\frac{1}{v}}$$

$$= \frac{v^{\alpha-1}(1-v)^{\beta-1}}{B(\alpha,\beta)} \cdot \frac{\left(\frac{u}{v}\right)^{\alpha+\beta-1}\left(1-\frac{u}{v}\right)^{\gamma-1}}{B(\alpha+\beta,\gamma)}\,\frac{1}{v}$$

Simplify

Claim

$$\boxed{= \frac{1}{B(\alpha,\beta)B(\alpha+\beta,\gamma)} u^{\alpha-1}\left(\frac{u}{v}-u\right)^{\beta-1}\left(1-\frac{u}{v}\right)^{\gamma-1}\frac{u}{v^2}}$$

Joint PDF of U, V

Want marginal $(U = XY)$

Want marginal $\boxed{U = XY}$

$$\longrightarrow \underline{f_U(u)} = \int f_{U,V}(u,v)\,dv$$

<u>recall:</u> $0 < \boxed{U < V < 1}$

$$f_U(u) = \int_u^1 f_{U,V}(u,v)\,dv$$

$$= \frac{u^{\alpha-1}}{B(\alpha,\beta)B(\alpha+\beta,\gamma)} \int_u^1 \Big(\underbrace{\frac{u}{v} - u}_{y(1-u)}\Big)^{\beta-1} \Big(\underbrace{1 - u/v}_{(1-y)(1-u)}\Big)^{\gamma-1} \frac{u}{v^2}\,dv$$

Looks like a Beta

$x^{\alpha-1}(1-x)^{\beta-1}$

$$\boxed{y = \left(\frac{u}{v} - u\right)\frac{1}{1-u}} \Leftrightarrow \boxed{y(1-u) = \frac{u}{v} - u}$$

<u>also:</u> $\boxed{1 - u/v = (1-y)(1-u)}$

$$1 - y = 1 - \left(\frac{u}{v} - u\right)\frac{1}{1-u}$$

$$(1-u)(1-y) = (1 - \cancel{u}) - \left(\frac{u}{v} \cancel{- u}\right) = 1 - u/v$$

$$\boxed{dy = \frac{-u}{v^2(1-u)}\,dv} \qquad \Rightarrow \frac{u}{v^2}\,dv = -(1-u)\,dy$$

Plug into integral:

$$\frac{u^{\alpha-1}}{\ldots} \int_{\cancel{0}\,1} \big(y(1-u)\big)^{\beta-1} \big((1-y)(1-u)\big)^{\gamma-1} (\cancel{u}(1-u)\,dy)$$

$$\rightarrow \frac{u^{\alpha-1}}{B(\alpha,\beta)B(\alpha+\beta,\gamma)} \int (y(1-u)) \; ((1-y)(1-u)) \; (u(1-u)) dy$$

$\neq 0$

$$= \frac{u^{\alpha-1}(1-u)^{\beta+\gamma-1} B(\beta,\gamma)}{B(\alpha,\beta)B(\alpha+\beta,\gamma)} \int_0^1 \underbrace{\frac{y^{\beta-1}(1-y)^{\gamma-1} dy}{B(\beta,\gamma)}} \quad 1$$

basically is a $\text{Beta}(\beta,\gamma)$ PDF

$$= \boxed{\frac{u^{\alpha-1}(1-u)^{\beta+\gamma-1}}{B(\alpha,\beta+\gamma)}} \quad \text{Beta}(\alpha,\beta+\gamma)$$

So $\boxed{U \sim \text{Beta}(\alpha,\beta+\gamma)}$

$X Y$.

Theorem: Non-Invertible

As long as we can break g into chunks that are invertible, we're ok.

If $A = \text{Support}((X,Y)) \subset \mathbb{R}^2$

and A is partitioned into $A_1, \ldots, A_K$

and $(U, V) = g(X, Y)$ might not be invertible

so that $(U, V) = g^{(k)}(X, Y)$ on A_k, $k = 1, \ldots, K$

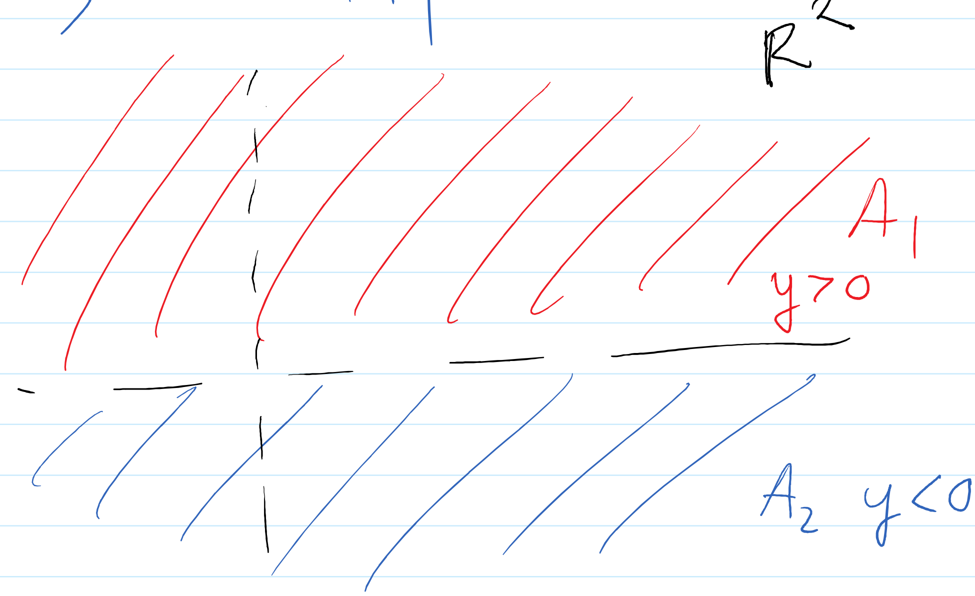
and $g^{(k)}$ are invertible

then

$$\boxed{f_{U,V}(u,v) = \sum_{k=1}^{K} f_{X,Y}\left({g^{(k)}}^{-1}(u,v)\right)\left|\det\left[\frac{\partial {g^{(k)}}^{-1}}{\partial(u,v)}\right]\right|}$$

Ex. $X, Y \overset{iid}{\sim} N(0,1)$

$U = X/Y, \quad V = |Y|$



On A_1 $\quad U = X/Y$ and $V = |Y|$

$A_1 = \{(x,y) / y > 0\}$

$u = x/y$ then $x = uy = \boxed{uv = x}$

for $y > 0$ $\boxed{v = |y| = y}$

$$\boxed{g_1^{(1)^{-1}}(u,v) = uv} \quad , \quad \boxed{g_2^{(1)^{-1}}(u,v) = v}$$

thus $$\frac{\partial g^{(1)-1}}{\partial(u,v)} = \begin{bmatrix} v & u \\ 0 & 1 \end{bmatrix}$$

$$\boxed{J_1 = |\det(\;)| = v}$$

On A_2: $\quad y < 0$ so $|y| = -y$

Can show: $\quad g_1^{(2)^{-1}}(u,v) = -uv$

$\qquad g_2^{(2)^{-1}}(u,v) = -v$

and $J_2 = v$

All together:

$$f\;\;(u,v)) = f\;\; (g^{(1)^{-1}}(u,v)\;\; g^{(1)^{-1}}(u,v)) \cdot J \;\; v$$

$$f_{U,V}(u,v) = \left[f_{X,Y}\left(g_1^{(1)^{-1}}(u,v), g_2^{(1)^{-1}}(u,v)\right) J_1 \right.$$

$$\left. + f_{X,Y}\left(\underbrace{g_1^{(2)^{-1}}(u,v)}_{-uv}, \underbrace{g_2^{(2)^{-1}}(u,v)}_{-v}\right) J_2 \right]$$

$J_1 = v$, $J_2 = v$

$X, Y \overset{iid}{\sim} N(0,1)$

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$$

$$= \underbrace{\frac{1}{2\pi} e^{-\frac{1}{2}((uv)^2+v^2)} \cdot v}_{} + \frac{1}{2\pi} e^{-\frac{1}{2}((-uv)^2+(-v)^2)}$$

$$\boxed{= \frac{v}{\pi} \exp\left(-\frac{1}{2} v^2 (1+u^2)\right)}$$

Joint of U, V

Marginal of $U = X/Y$. $(V = |Y|)$

$\beta = 1 + u^2$ so that

$$f_U(u) = \int_0^\infty f(u,v)\,dv = \int_0^\infty \frac{v}{\pi} e^{-\frac{1}{2}\beta v^2}\,dv$$

0 $\qquad 0$

$$z=\frac{1}{2}v^2 \quad ; \quad dz = v\,dv$$

$$= \frac{1}{\pi}\frac{1}{\beta}\underbrace{\int_0^\infty \beta e^{-\beta z}\,dz}_{\text{Integral of a Exp}(\beta)\ =\ 1}$$

$$= \frac{1}{\pi}\frac{1}{\beta} = \boxed{\frac{1}{\pi}\frac{1}{1+u^2} = f_U(u)}$$

$U = X/Y$

Cauchy r.v.

Ex. $\quad X \sim \text{Gamma}(\alpha, \lambda) \qquad X \perp\!\!\!\perp Y.$

$\qquad Y \sim \text{Gamma}(\beta, \lambda)$

look at: $\quad U = X+Y \qquad V = \dfrac{X}{X+Y}$

$$u = x+y \quad ; \quad v = x/u \Rightarrow \boxed{x = uv}$$

$$y = u - x = \boxed{u - uv = y}$$

$$g_1^{-1}(u,v) = uv \quad ; \quad g_2^{-1}(u,v) = u - uv$$

$$\frac{\partial g^{-1}}{\partial(u,v)} = \begin{bmatrix} v & u \\ 1-v & -u \end{bmatrix}$$

$$J = \left| \det\left(\frac{\partial g^{-1}}{\partial(u,v)}\right) \right| = \left| -uv - u(1-v) \right|$$
$$= u$$

$X \sim \underline{\mathrm{Gam}(\alpha, \lambda)}$ $\quad Y \sim \mathrm{Gamma}(\underline{\beta, \lambda})$, $X \perp Y$

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$
$$\boxed{= \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} \cdot \frac{\lambda^\beta y^{\beta-1} e^{-\lambda y}}{\Gamma(\beta)}}$$

---

all together

$$f_{U,V}(u,v) = f_{X,Y}(uv, u-uv)\, u$$
$$= \frac{\lambda^\alpha (uv)^{\alpha-1} e^{-\lambda uv}}{\Gamma(\alpha)} \frac{\lambda^\beta (u-uv)^{\beta-1} e^{-\lambda(u-uv)}}{\Gamma(\beta)} u$$

$\Gamma(\alpha) \qquad\qquad \Gamma(\beta)$

$= \text{algebra (check)}$

$$= \frac{\lambda^{\alpha+\beta}\, \underbrace{u^{\alpha+\beta-1} e^{-\lambda u}}\; \underbrace{v^{\alpha-1}(1-v)^{\beta-1}}}{\Gamma(\alpha)\Gamma(\beta)}$$

prop. to pdf of Gama$(\alpha+\beta, \lambda)$

Prop. to Beta(α, β) pdf

$$= h_1(u)\, h_2(v)$$

Hence $U = X+Y \sim \text{Gama}(\alpha+\beta, \lambda)$ $\quad U \perp V$

$V = X/X+Y \sim \text{Beta}(\alpha, \beta)$