Lecture 9 - PMFs and PDFs

Tuesday, February 25, 2020 10:52 AM

<u>Defn:</u> Identically distributed r.v.s.

$X \overset{d}{=} Y$ if $\mathbb{P}(X \in A) = \mathbb{P}(Y \in A)$ for all $A \subset S$

<u>Ex.</u> $X = \#$ heads in 3 flips

$Y = \#$ tails ″

e.g. $\mathbb{P}(X=1) = \mathbb{P}(Y=1)$

$\frac{3}{8} = \frac{3}{8}$

however HTT we get $X = 1$ and $Y = 2$.

<u>Theorem:</u> $X \overset{d}{=} Y$ iff $F_X = F_Y$.

<u>Ex.</u> Toss coins independently until a H appears.

$S = \{H, TH, TTH, TTTH, \ldots\}$

$|S| = \infty$

Let p is the prob. of getting a H on any flip.

Let $X = \#$ of flips we make.

ρ | X

S	X
H	1
TH	2
TTH	3
$\vdots$	$\vdots$

CDF of X

$$F(x) = \mathbb{P}(X \le x)$$

First lets determine

$$\mathbb{P}(X = x)$$

T_i = event of getting tails on i^{th} flip

H_i = " heads "

then $"X = x" = T_1 T_2 T_3 \cdots T_{x-1} H_x$

independent

So $\underline{\mathbb{P}(X = x)} = \mathbb{P}(T_1 T_2 \cdots T_{x-1} H_x)$

$$= \mathbb{P}(T_1)\mathbb{P}(T_2) \cdots \mathbb{P}(T_{x-1})\mathbb{P}(H_x)$$

$$= (1-p)(1-p) \cdots (1-p)\,p$$

$$= (1-p)^{x-1} p$$

back to the CDF

disjoint

$$"X \le x" = \omega_1 \cup \omega_2 \cup \omega_3 \cup \cdots \cup \omega_x$$

ω_i = "make i flips to get 1^{st} heads"

$= "X = i"$

$\mathbb{P}(X \le x)$

$$= \mathbb{P}\left(\bigcup^{x} \omega_i\right)$$

ω_i = make i flips to get 1 heads

$$= \mathbb{P}\left(\bigcup_{i=1}^{x} \omega_i\right) \qquad = \text{"}X = i\text{"}$$

$$= \sum_{i=1}^{x} \mathbb{P}(\omega_i)$$

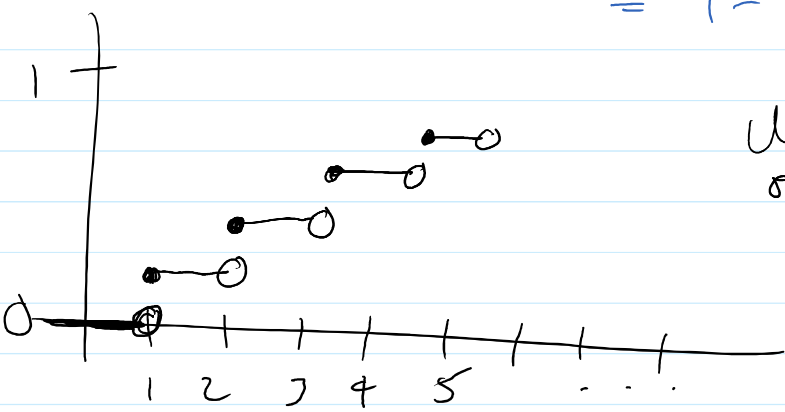
$$= \sum_{i=1}^{x} \mathbb{P}(X = i)$$

$$= \sum_{i=1}^{x} (1-p)^{i-1} p$$

$$= p \sum_{i=1}^{x} (1-p)^{i-1} = \cancel{p} \frac{1-(1-p)^x}{\cancel{1-(1-p)}}$$

Formula for finite Geometric series
$r = 1-p$

$$= 1-(1-p)^x = F(x)$$



We call this type of r.v. a Geometric R.V.

<u>Useful</u>: break down $F(x)$ into a sum of $\mathbb{P}(X=x)$

<u>Defn: Probability Mass Function (PMF)</u>

For a discrete r.v. X we call

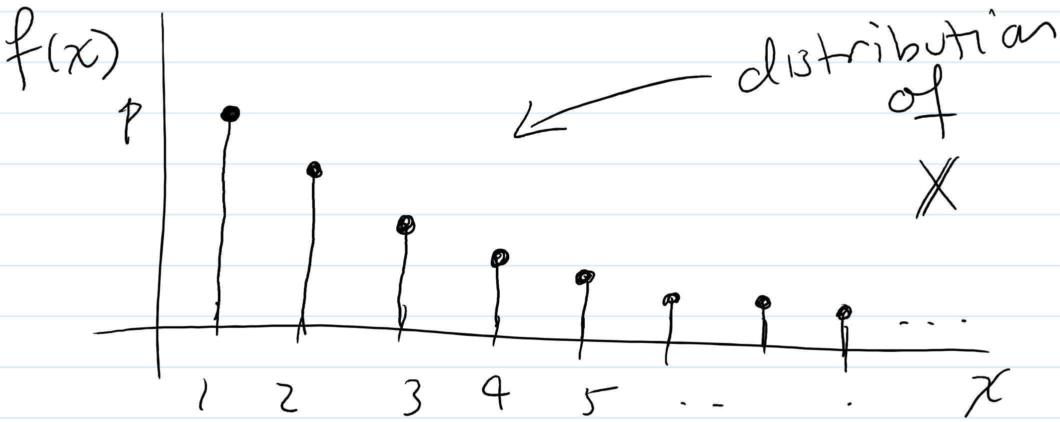
$$f(x) = \mathbb{P}(X=x)$$

the probability mass function.

the probability mass function.

Ex. For our geometric r.v.

$$f(x) = P(X=x) = p(1-p)^{x-1}.$$



Theorem:

$$\boxed{F(x) = \sum_{i \le x} f(i)}$$

pf. $$\text{"}X \le x\text{"} = \underbrace{\bigcup_{i \le x} \text{"}X = i\text{"}}_{\text{disjoint union}}$$

here

$$F(x) = P(X \le x) = P\Big(\bigcup_{i \le x} \text{"}X=i\text{"}\Big)$$

$$= \sum_{i \le x} P(X=i)$$

$$= \sum_{i \le x} f(i).$$

Ex. Discrete Uniform Distribution

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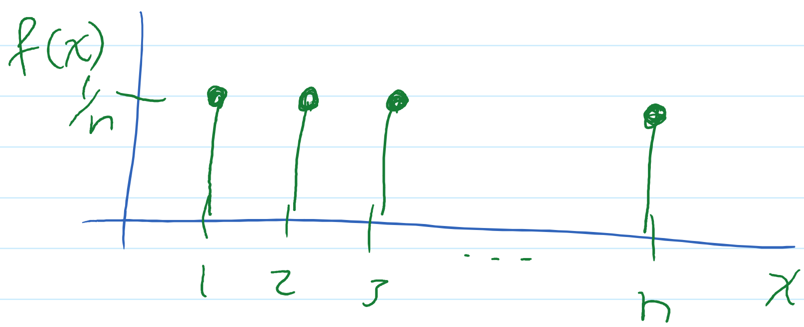
$$X \sim \underbrace{U(\{1, \dots, n\})}$$

Notation for discrete uniform on $\{1, \dots, n\}$

"$\sim$" read as "distributed as"

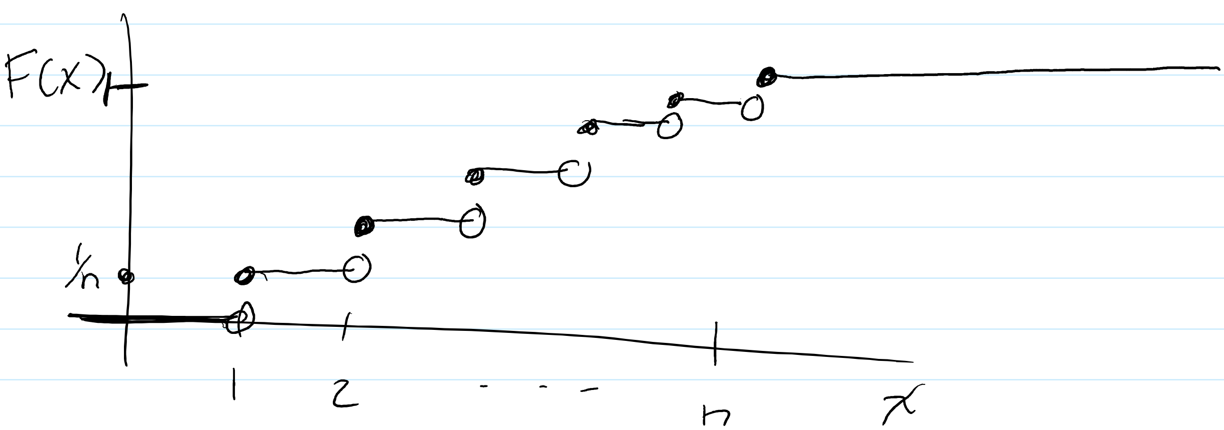
means

$$f(i) = \mathbb{P}(X = i) = \frac{1}{n} \qquad i = 1, \dots, n$$



what is the CDF?

$$F(x) = \sum_{i=1}^{x} f(i) = \sum_{i=1}^{x} \frac{1}{n} = \frac{x}{n}$$



We said

$$\boxed{F(x) = \mathbb{P}(X \le x) = \sum \mathbb{P}(X = i) = \sum f(i)}$$

$$\boxed{F(x) = \mathbb{P}(X \le x) = \sum_{i \le x} \mathbb{P}(X=i) = \sum_{i \le x} f(i)}$$

Generally:

$$\boxed{\mathbb{P}(X \in A) = \sum_{i \in A} f(i)}$$

Ex X has discrete uniform dist

$$\mathbb{P}(2 \le X < 5) = \sum_{\substack{i=2,3,4 \\ 2 \le i < 5}} f(i) = \sum_{i=2,3,4} 1/n$$

$$= 3/n.$$

$$\mathbb{P}(X \in \{1,7,3\}) = \sum_{i=1,7,3} f(i) = 3/n$$

Ex. Roll a die 60 times (independently)

X = # of 6s I roll.

$f(x) = \mathbb{P}(X = x)$ = prob. I roll x 6s in a total of 60 rolls.

$$f(0) = \mathbb{P}(X=0) = (5/6)(5/6)(5/6) \cdots (5/6)$$

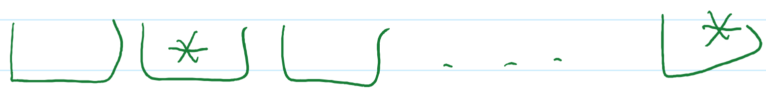
$$f(0) = P(X=0) = \underbrace{(5/6)(5/6)(5/6)\cdots(5/6)}_{60 \text{ times}}$$

$$= (5/6)^{60}$$

$$f(1) = P(X=1) = (5/6)^{59}(1/6)\cdot 60$$

$$f(2) = P(X=2) = (5/6)^{58}(1/6)^{2}\binom{60}{2}$$

$$\boxed{f(x) = P(X=x) = (5/6)^{60-x}(1/6)^{x}\binom{60}{x}}$$



60 buckets

We call this type of r.v. a Binomial distributed r.v.

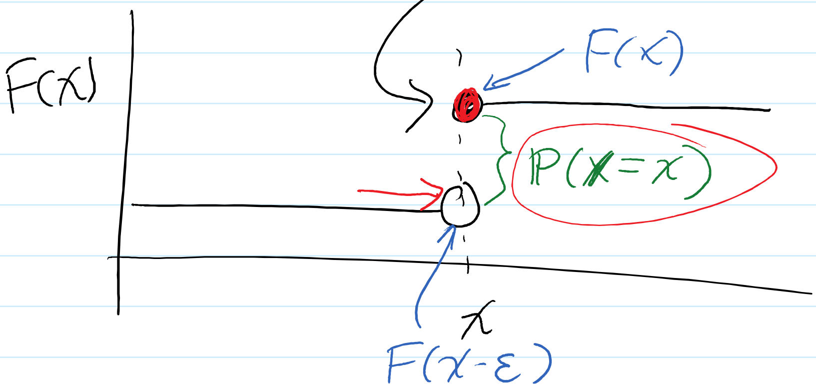
What is $P(X \text{ is even})$

$$= \sum_{x=2,4,6,8,\ldots,60} f(x)$$

$$= \sum_{x \text{ even}} \binom{60}{x}(5/6)^{60-x}(1/6)^{x}.$$

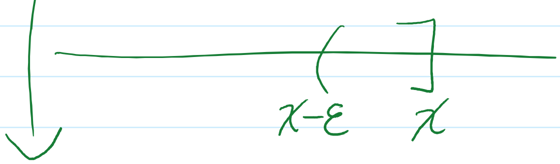
For discrete

$f(x) = \mathbb{P}(X = x)$ = jump of CDF of x



$\mathbb{P}(X = x) = \mathbb{P}(x-\varepsilon < X \le x)$ as $\varepsilon \to 0$

small ε



$\mathbb{P}(a < X \le b) = F(b) - F(a)$

$= \underbrace{F(x) - F(x-\varepsilon)}_{\text{jump}}$

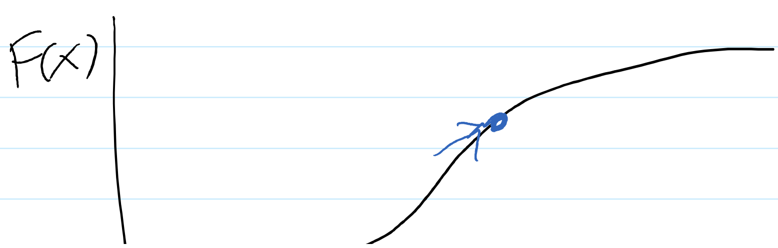
$\to \mathbb{P}(X = x)$

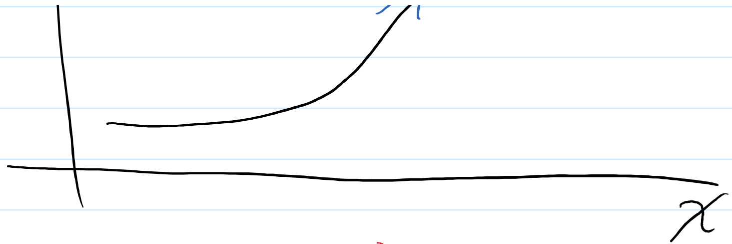
Another way:

$$F(x) = \sum_{i \le x} f(i)$$

when the CDF jumps it is adding on some $f(i)$

Problem, for continuous r.v.s the CDF is continuous. No jumps.





However $\mathbb{P}(x-\varepsilon < X \le x) = F(x) - F(x-\varepsilon)$

So

$$\mathbb{P}(X=x) = \lim_{\varepsilon \downarrow 0} \mathbb{P}(x-\varepsilon < X \le x)$$
$$= \lim_{\varepsilon \downarrow 0} \Big(F(x) - F(x-\varepsilon)\Big)$$
$$= F(x) - F(x)$$
$$= 0$$

So I can't define $f(x) = \mathbb{P}(X=x)$ for cts. r.v. b/c this is alway zero.

Want:

for discrete $F(x) = \sum_{i \le x} f(i)$

CDF = sum of PMF of values

Defn: Probability Density Function (PDF)

Analog of PMF for discrete.

The PDF is the function f_x so that

$$F(x) = \int_{-\infty}^{x} f(t)\,dt.$$

CDF = Integral of PDF.

Notice: Fundamental Theorem of Calc

says

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_{-\infty}^{x} f(t)\,dt = f(x)$$

so $\underset{\text{PDF}}{f(x)} = \underset{\text{deriv of CDF.}}{\frac{d}{dx}F(x)}$

Properties:

$$\mathbb{P}(a < X \leq b) = F(b) - F(a)$$

$$= \int_{-\infty}^{b} f(t)\,dt - \int_{-\infty}^{a} f(t)\,dt$$

$$= \int_{a}^{b} f(t)\,dt$$

notice: because $\mathbb{P}(X=a) = \mathbb{P}(X=b) = 0$

I don't care about differences btwn $<$ $\le$ or $>$ $\ge$

Ex.
$$\mathbb{P}(a < X \le b) = \mathbb{P}(a \le X \le b) - \underbrace{\cancel{\mathbb{P}(X=a)}}_{0}$$
$$\vdots$$
$$\left.\begin{aligned} &= \mathbb{P}(a \le X < b) \\ &= \mathbb{P}(a < X < b) \end{aligned}\right\} \text{all equal.}$$

In general I can ignore any finite number of points

More general fact:

discrete:
$$\mathbb{P}(X \in A) = \sum_{x \in A} f(x)$$

continuous:
$$\mathbb{P}(X \in A) = \int_A f(x)\,dx$$

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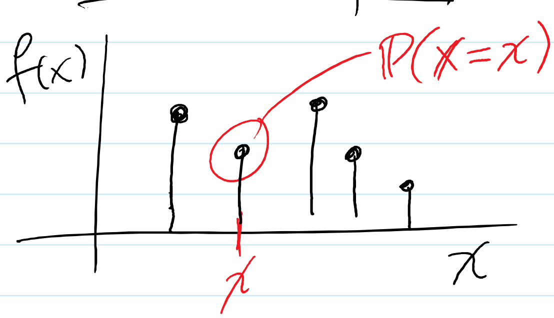
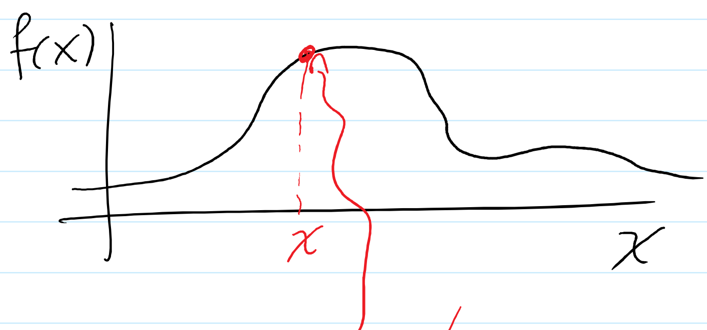
discrete pmf



continuous pdf



$f(x) \neq \mathbb{P}(X=x)$

$$f(x) \neq \mathbb{P}(X = x)$$