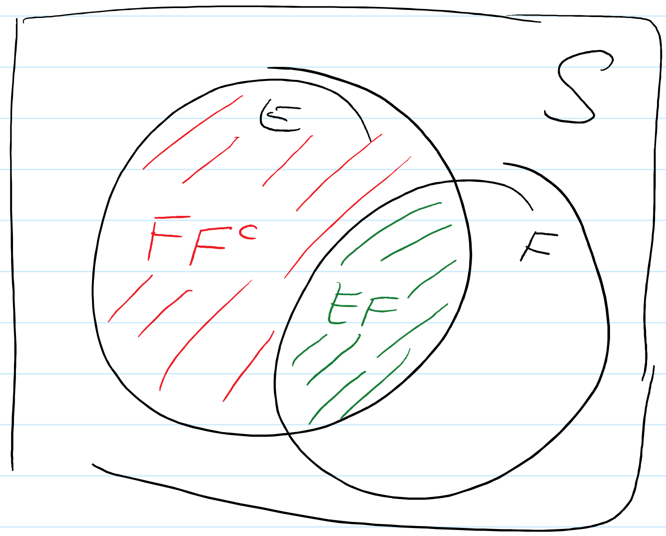
Always:

$$E = EF \cup EF^c$$

$\underbrace{EF \;\cup\; EF^c}_{\text{partition } E}$

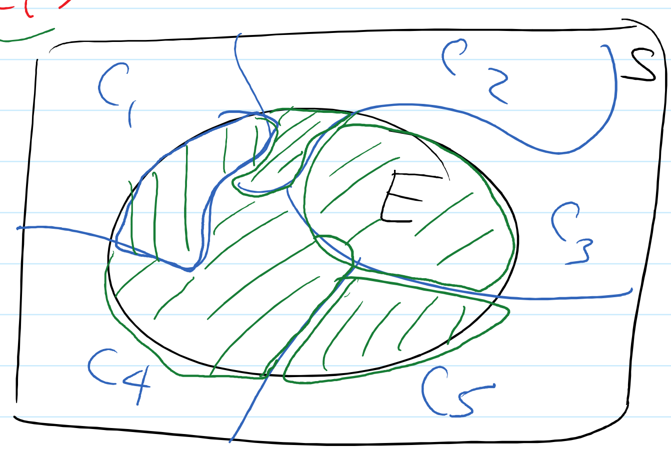


$$P(E) = P(EF) + P(EF^c)$$

Can do more generally:

If $\{C_i\}_{i=1}^{\infty}$ partition S

then $P(E) = \sum_{i=1}^{\infty} P(EC_i)$



Pf. ① $E = \bigcup_i (EC_i)$

② $\{EC_i\}_i$ are disjoint

b/c $(EC_i)(EC_j)$

$= EC_iC_j = E\emptyset = \emptyset$

from there $P(E) \overset{①}{=} P(\bigcup_i EC_i)$

$$\overset{(2)}{=} \sum_i P(EC_i).$$

Equally likely Sample Spaces

If I have a sample space

$$S = \{s_1, \dots, s_n\} \text{ so that } |S| < \infty$$

lets assume that

$$\mathbb{P}(\{s_i\}) \text{ are equal for all } i$$
$$= 1/n$$

(pf. $1 = \mathbb{P}(S) = \mathbb{P}(\{s_1\} \cup \{s_2\} \cup \cdots \cup \{s_n\})$
$$= \sum_{i=1}^{n} \mathbb{P}(\{s_i\}) = np$$
where $p = \mathbb{P}(\{s_i\})$

hence $p = 1/n$)

We say all outcomes are equally likely.

More generally: if $E \subset S$

$$\mathbb{P}(E) = \frac{\#\text{ of things/outcomes in } E}{\#\text{ of outcomes in } S}$$
$$= \frac{|E|}{|S|}$$

Ex. Roll a six-sided die

$$S = \{1, 2, \ldots, 6\}$$

let's assume all outcomes are equally likely
then if $E = \{2, 6\}$

$$P(E) = \frac{|E|}{|S|} = \frac{2}{6} = \frac{1}{3}$$

Notice:

$$P(E) = \frac{|E|}{|S|}$$

$|E|$ ← need to count # of outcomes in E

$|S|$ ← count # outcomes in S

Counting

Fundamental Theorem of Counting (FTC)

If a "job" or "task" consists of k sub-tasks each of which has n_i possible ways of being done, $(i = 1, 2, \ldots, k)$ then, the # of ways to complete the task is

$$N = n_1 \cdot n_2 \cdot n_3 \cdots n_k$$
$$= \prod_{i}^{n} n_i$$

$i=1$

Ex. An experiment consists of 3 factors:

1. 2 temperature settings
2. 2 pressure "
3. 4 humidity "

How many ways can I conduct the experiment?

$k = 3$ sub-tasks

$n_1 = 2$; $n_2 = 2$; $n_3 = 4$

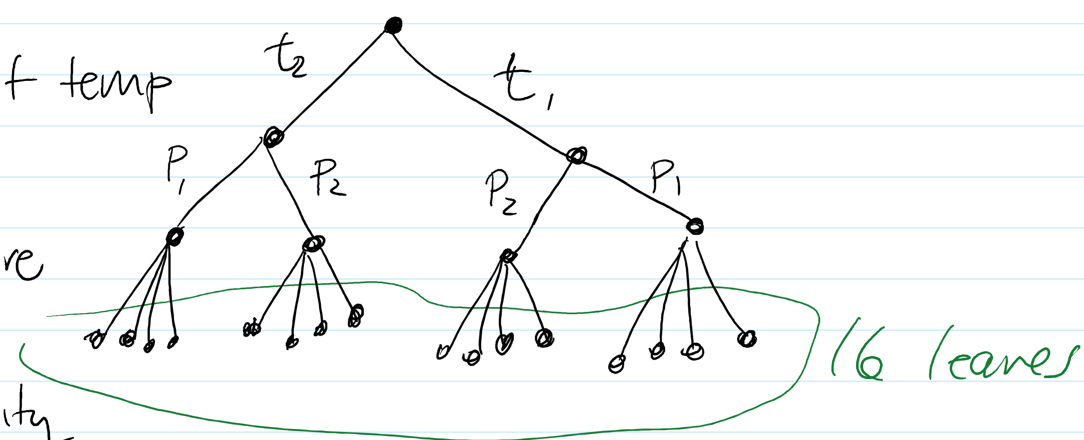
So I can conduct the experiment in

$2 \cdot 2 \cdot 4 = 16$ ways.

task 1: set temp

task 2: set pressure

task 3: set humidity



Ex. A man has 5 shirts, 2 pairs of pants, 2 pairs of shoes, How many outfits does he have?

$$5 \cdot 2 \cdot 2 = 20 \text{ outfits.}$$

Ex, Shuffle a deck of $\overbrace{\text{cards}}^{52}$.

What is the prob. they are in order?

(A–K, C, D, H, S)

Assume all outcomes are equally likely

$$S = \{\text{all possible orderings of 52 cards}\}$$

E = they are in order

$$\mathbb{P}(E) = |E| / |S|$$

$|E| = 1$ b/c only one of possible shuffles is in order.

$$|S| = ?$$

tasks are $k = 52$ cards we pick up

(52) ways/possibilities for card 1

52	ways/possibilities	for card	1
51	"		2
50	"		3
	⋮		
1	"		52

FTC says total # ways to choose cards this way is

$$52 \cdot 51 \cdot 50 \cdot 49 \cdots 3 \cdot 2 \cdot 1.$$

So $|S| =$

hence $$\mathbb{P}(E) = 1/(52 \cdot 51 \cdot 50 \cdots 3 \cdot 2 \cdot 1)$$

$$\approx 1.24 \cdot 10^{-68}.$$

Defn: Factorial

For any non-negative integer n we define $\underline{n\text{-factorial}}$, denoted $n!$, as

$$n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdots 3 \cdot 2 \cdot 1$$

$$= \prod_{i=1}^{n} i$$

note: $0! = 1$.

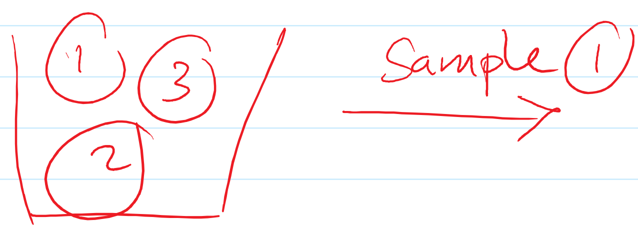
$i=1$

Ex. $\mathbb{P}(E) = 1/52!$ (from above).

Sampling w/ Ordering and Replacement.

Need to distinguish when drawing sample if

1. do we care about the order of the Sample
   E.g. is 1, 2, 3 different than 3, 2, 1
2. do we replace items back into population after drawing a sample?



(no replacement)

(w/ replacement)

Eg. can I get a sample 1, 1, 2 ?

4 different Senarios

	w/ replacement	w/o replacement

4 different scenarios

	w/ replacement	w/o replacement
ordered	2	1
unordered	3	4

Permutations:

A "permutation" is an ordering of objects.

Ex. A_1, A_2, A_3

permutations:

$$\left.\begin{array}{lll} A_1 A_2 A_3 & A_2 A_1 A_3 & A_1 A_3 A_2 \\ A_3 A_2 A_1 & A_3 A_1 A_2 & A_2 A_3 A_1 \end{array}\right\} 6 = 3!$$

Theorem: Permutation Counting

The # of ways to permute n objects is $n!$.

Pf. Use FTC

$k = n$ sub-tasks

n ways to choose first item

$n-1$ " 2nd "

$n-2$ " 3rd "

$\vdots$

1 " n^{th} "

multiply to get $n!$

Theorem: Ordered Sampling w/o Replacement.

If I have n items and I sample r (where $r \le n$) and I care about ordering and sample w/o replacement, then the number of ways to draw a sample is

$$(n)_r \stackrel{\text{def}}{=} \frac{n!}{(n-r)!}.$$

pf. Use FTC

$k = r$ sub-tasks of sampling the r items.

n ways to choose 1^{st} item in sample.

n ways to choose 1^{st} item in sample.

$n-1$ 〃 2^{nd} 〃

$n-2$ 〃 3^{rd} 〃

$\vdots$

$n-r+1$ 〃 r^{th} 〃

by FTC this means there are

$$\boxed{n(n-1)(n-2)\cdots(n-r+1)}$$

ways to sample.

equal.

$$\boxed{\frac{n!}{(n-r)!}} = \frac{\boxed{n(n-1)(n-2)\cdots(n-r+1)}\cancel{(n-r)(n-r-1)\cdots 1}}{\cancel{(n-r)(n-r-1)(n-r-2)\cdots 1}}$$

Ex. 10 students and I want to form a cabinet of a president, VP, treasurer from the 10 students. ← can only do one job

Q: how many possible cabinets can I form?

Think about it as:

Sampling 3 students from 10 w/o repl.

Ordering signifies position:

$\#1$ = pres.

$\#2$ = VP

$\#3$ = treasurer.

Our formula says I can do this in

$$\boxed{10!/(10-3)!}$$

$n = 10$

$r = 3$

$$\frac{10!}{7!} = \frac{10\cdot 9\cdot 8\cdot \cancel{7\cdot 6\cdots 1}}{\cancel{7\cdot 6\cdot 5\cdots 1}}$$

$$= 10\cdot 9\cdot 8 = 720.$$

Ex. Lotto.

I have a box of golf balls w/ the numbers $1, \dots, 25$ written on them

1 2 3 ... 24 25

Lotto: draw 4 balls. If you chose the 4 numbers in the correct order you win.

— my choice: 1 3 22 7

What is the prob. I win if all drawings are equally likely?

$S = \{$ all possible samples of size 4 from 25 w/o repl. and I care about order $\}$

$E = \{ 13\ 22\ 7 \}$

$$P(E) = \frac{|E|}{|S|} = \frac{1}{|S|} \leftarrow ???$$

$n = 25$, $r = 4$ hence

$$|S| = \frac{25!}{(25-4)!} = \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot \cancel{21 \cdots 1}}{\cancel{21 \cdot 20 \cdot 19 \cdots 1}}$$

$$= 25 \cdot 24 \cdot 23 \cdot 22$$

hence $P(E) = 1/(25 \cdot 24 \cdot 23 \cdot 22)$ = bad. !!

Theorem: Sampling w/ Ordering but w/ Replacement.

If I draw r samples from n where I care about the order but I replace items into population after each draw. I can draw

$$n^r$$

possible samples.

Note: we don't need $r \leq n$.

Pf. Use FTC

$k = r$ tasks

n ways to draw 1^{st} item

n " 2^{nd} "

n " 3^{rd} "

$\vdots$

n " r^{th} "

multiply by FTC

then there are $\underbrace{n \cdot n \cdot n \cdot n \cdots n}_{r\text{-times}} = n^r$,

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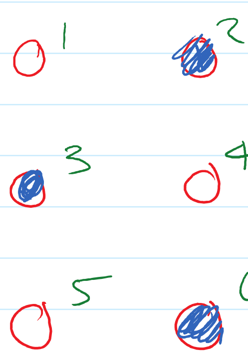
Ex, Braille has 6 locations for raised bumps



Q: how many possible Braille "letters" are

" there? "

Idea: drawing 6 samples from a population of either "raised" or "not raised"

sample w/ replacement of $r=6$ from $n=2$ care about the order.

Hence there are $2^6 = 64$ Braille letters