Theorem: Finite Sample Space Theorem

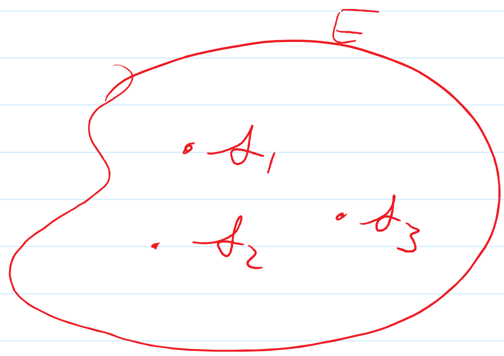
If $S = \{s_1, s_2, s_3, \ldots, s_n\}$

and $p_1, p_2, p_3, \ldots, p_n$

so that $\boxed{p_i \geq 0}$ and $\sum_{i=1}^{n} p_i = 1.$

Define: $E \subset S$,

$$\mathbb{P}(E) = \sum_{i : s_i \in E} p_i$$



This is a valid prob. fn.

Pf-

(Axiom 1) $\boxed{\mathbb{P}(E) \geq 0}$

$\mathbb{P}(E) = p_1 + p_2 + p_3.$

show: $\mathbb{P}(E) = \sum_{i : s_i \in E} p_i$ = Sum of non-neg. p_i ≥ 0

≥ 0

(Axiom 2) $\boxed{\mathbb{P}(S) = 1}$

$$\mathbb{P}(S) = \sum_{i : s_i \in S} p_i = \sum_{i=1}^{n} p_i = 1 \checkmark$$

$1, 2, 3, \ldots, n$

(Axiom 3) $\{E_i\}_{i=1}^{\infty}$ are disjoint

then ... ∞

Axiom 3) if $\{E_i\}_{i=1}^{\infty}$ are disjoint

then $$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

<u>aside</u>: $$\mathbb{P}\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{P}(E_i)$$

interchange limits and $\mathbb{P}$
(i.e. continuity of $\mathbb{P}$)

$$\mathbb{P}\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{i : s_i \in \bigcup_{j=1}^{\infty} E_j} p_i$$

$$\underbrace{i : s_i \in \bigcup_{j=1}^{\infty} E_j} \iff s_i \in \text{~~at least~~ exactly one of the } E_j.$$

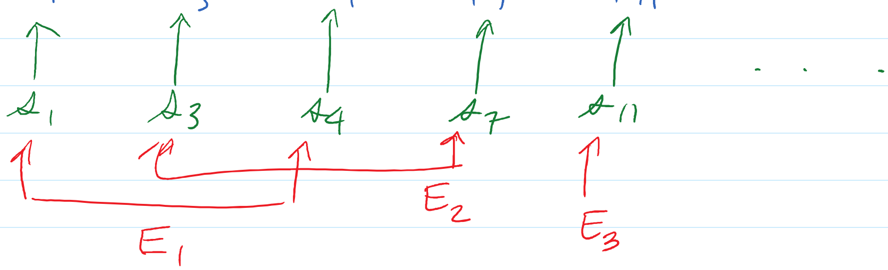
Q: can s_i be in two diffent E_j ?

<u>Ex.</u> If $s_i \in E_1$ and $s_i \in E_2$

then $s_i \in E_1 E_2$ so $E_1 E_2 \neq \emptyset$.
can't happen b/c the E_j are disjoint.

the sum of p_i might be e.g.

$$p_1 + p_3 + p_4 + p_7 + p_{11} + \cdots$$



re-arrange my sum as

$$= (p_1 + p_4) + (p_3 + p_7) + p_{11} + \cdots$$

$$= \left(\sum_{i:s_i \in E_1} p_i\right) + \left(\sum_{i:s_i \in E_2} p_i\right) + \left(\sum_{i:s_i \in E_3} p_i\right)$$

$$= \sum_{j=1}^{\infty} \sum_{i:s_i \in E_j} p_i \quad P(E_j)$$

$$= \sum_{j=1}^{\infty} P(E_j)$$

I have thus shown that $P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$.

This proof works if S is countable.

Basic Theorems about P.

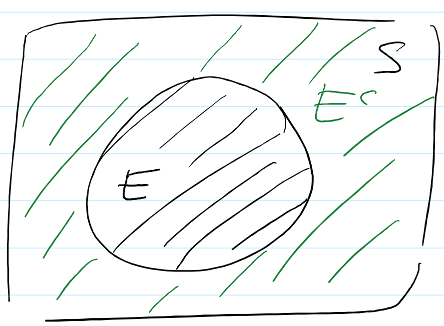
Ex, If E = "it is raining"

$P(E) = 1/4$

$P(\text{"not raining"}) = P(E^c) = 3/4 = 1 - P(E)$

Theorem: $P(E^c) = 1 - P(E)$

pf. $P(S) = 1$. (Axiom 2)



$S = E \cup E^c$

disjoint

(Axiom 3) $\underbrace{\mathbb{P}(E \cup E^c)}_{\mathbb{P}(S)} = \mathbb{P}(E) + \mathbb{P}(E^c)$

We have $1 = \mathbb{P}(S) = \mathbb{P}(E \cup E^c) = \mathbb{P}(E) + \mathbb{P}(E^c)$

$$1 = \mathbb{P}(E) + \mathbb{P}(E^c)$$

or $\mathbb{P}(E^c) = 1 - \mathbb{P}(E)$.

(Axiom 1) $\mathbb{P}(E) \geq 0$ and (Axiom 2) $\mathbb{P}(S) = 1$.

Theorem: $\mathbb{P}(E) \leq 1$

Pf. Know: $\mathbb{P}(E^c) = 1 - \mathbb{P}(E)$

and: $\mathbb{P}(E^c) \geq 0$ (Axiom 1)

combining: $1 - \mathbb{P}(E) \geq 0$

or $\boxed{\mathbb{P}(E) \leq 1}$

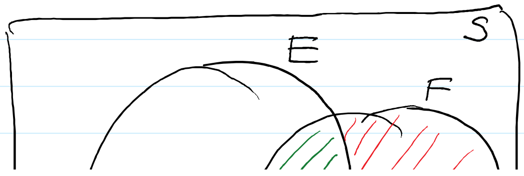
Theorem: $\mathbb{P}(\emptyset) = 0$.

$$\mathbb{P}(S^c) = 1 - \mathbb{P}(S) = 1 - 1 = 0$$

$S^c = \emptyset$ $\quad S^c = S \setminus S = \emptyset$

hence $\mathbb{P}(\emptyset) = 0$.

Theorem:



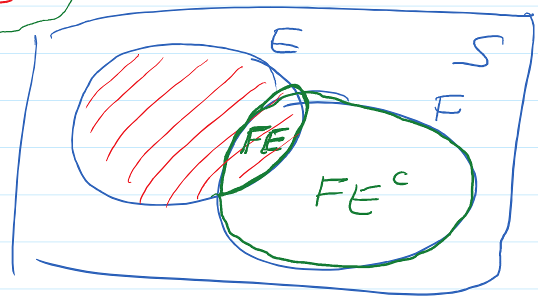
$\mathbb{P}(F \setminus E)$

$= \mathbb{P}(FE^c)$

$= \mathbb{P}(FE^c)$

$= \mathbb{P}(F) - \mathbb{P}(EF)$

proof. $\boxed{F = FE \cup FE^c}$



E and E^c partition S

(1) $EE^c = \emptyset$

(2) $E \cup E^c = S$

FE and FE^c partition F

(1) $FEFE^c = FEE^c = \emptyset$

(2) $FE \cup FE^c = F$

b/c they are disjoint : (Axiom 3)

$\mathbb{P}(F) = \mathbb{P}(FE \cup FE^c) = \mathbb{P}(FE) + \mathbb{P}(FE^c)$

rearrange

$$\boxed{\mathbb{P}(FE^c) = \mathbb{P}(F) - \mathbb{P}(FE)}$$

Axiom 3: If E, F disjoint then

$$\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F).$$

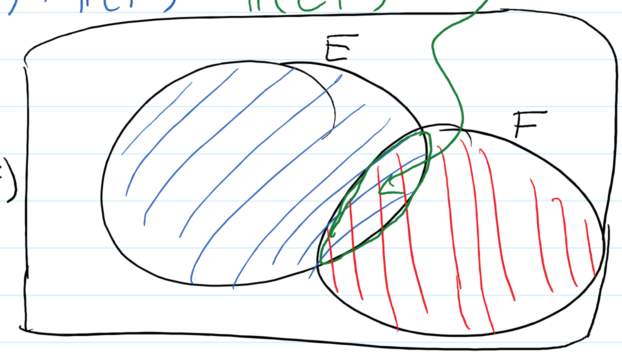
What if E and F aren't disjoint?

double counted

What if E and F aren't disjoint?

$$P(E\cup F) = P(E) + P(F) - P(EF)$$

double counted



Theorem:

$$P(E\cup F) = P(E) + P(F) - P(EF)$$

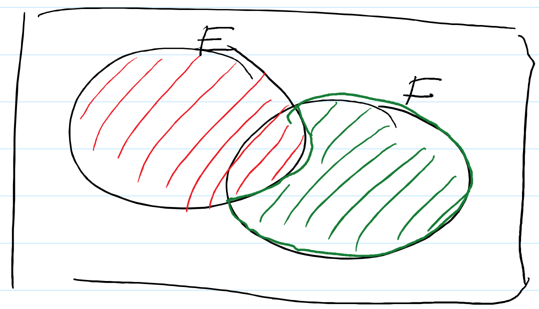
Pf.

$$E\cup F = E \cup FE^c$$

disjoint

So by Axiom 3

$$P(E\cup F) = P(E) + P(FE^c)$$

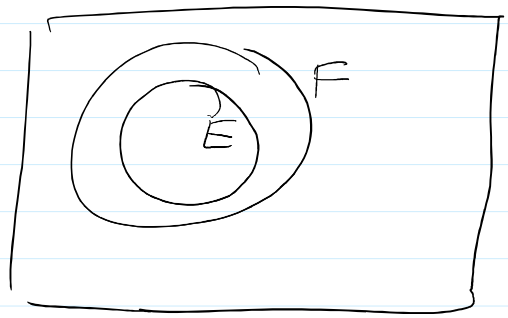


$$= P(E) + P(F) - P(EF),$$

by prev. theorem.

Theorem: Prob. of Subsets

If $E \subset F \subset S$ then $P(E) \le P(F)$.



Pf. $P(\text{any event}) \geq 0$

$$P(FE^c) \geq 0$$

$$P(F) - P(EF) \geq 0$$

$$EF = E$$

we get

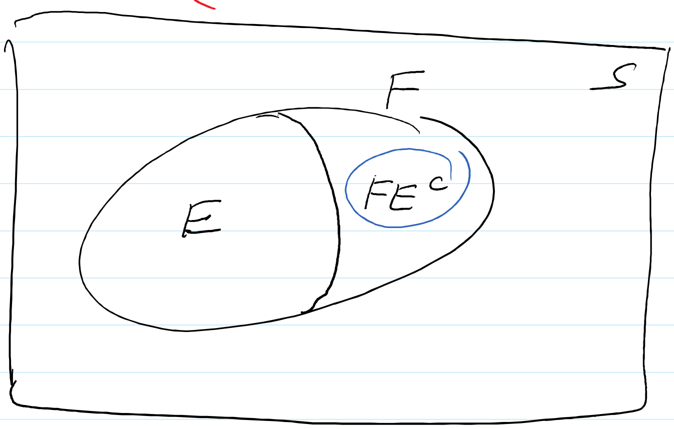
$$P(F) - P(E) \geq 0 \quad \text{or} \quad P(E) \leq P(F).$$

$$P(EF^c) = P(E) - P(EF) = P(E) - P(E) = 0.$$

$\emptyset$

$$\underbrace{\text{Let } E \subset F \text{ and } E \neq F.}_{E \text{ proper subset of } F}$$

$\cancel{P(E) < P(F)}$? Still: $P(E) \le P(F)$.



$FE^c \neq \emptyset$

$P(FE^c) = 0.$

Then $P(E) = P(F)$ but $E \neq F$.

$$P(F) = P(FE^c \cup E) = P(FE^c) + P(E)$$
$$= P(E)$$

<u>Useful Fact:</u>

$P(F \setminus E) \le P(F).$

Simply b/c $F \setminus E \subset F$.

For two events E and F, theorem:

$$P(E \cup F) = P(E) + P(F) - P(EF).$$

$$P(E \cup F) = P(E) + P(F) - \underbrace{P(EF)}_{\geq 0}.$$

alt.

$$P(E \cup F) \leq P(E) + P(F).$$

Generalizable?

<u>Boole's Inequality</u>

If $\{E_i\}_{i=1}^{\infty}$ of events,

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} P(E_i).$$

<u>Pf.</u> We'd like to say

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i) \text{ but}$$

the E_i aren't disjoint.

<u>Instead</u>: Come up w/ a seq. $\{B_i\}_{i=1}^{\infty}$

where (1) $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} B_i$

(2) $\{B_i\}$ disjoint.

$B_1 = E_1$

$B_2 = E_2 \setminus E_1$

$B_3 = E_3 \setminus E_2 \setminus E_1$

$B_4 = E_4 \setminus E_3 \setminus E_2 \setminus E_1$

$\vdots$

Convince yourself: the above two facts are true.

Axiom 3

⋮

hence

$$P\Big(\bigcup_i E_i\Big) = P\Big(\bigcup_i B_i\Big) \overset{\text{Axiom 3}}{=} \sum_{i=1}^{\infty} P(B_i)$$

$B_i = E_i \setminus \text{some stuff} \subset E_i$

hence $P(B_i) \le P(E_i)$

$$\le \sum_{i=1}^{\infty} P(E_i).$$

Previously:

$$E = EF \cup EF^c$$

more generally if $\{C_i\}$ is a partion of S

$E = EC_1 \cup EC_2 \cup EC_3 \cup EC_4$

notice they are disjoint!