Lecture 10 - PMFs/PDFs and Expected Value

Thursday, February 27, 2020 9:14 AM

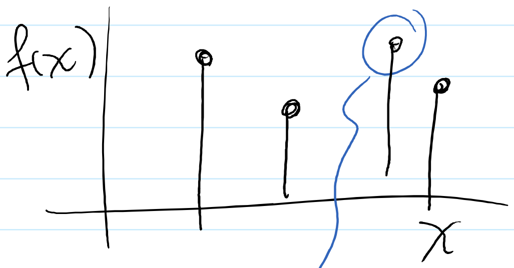
Defn: Probability Density Function (PDF)

A function f so that

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

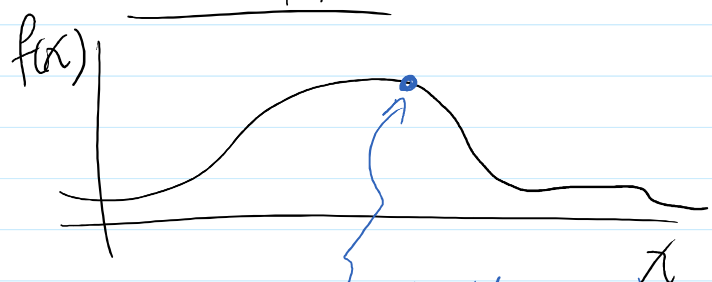
cts analog to the discrete pmf

discrete: PMF



$\mathbb{P}(X = x)$

cts: PDF



not the prob of $X = x$

Fact: $$f(x) = \frac{d}{dx} F(x)$$

Conversely: $$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

Properties:

$$\mathbb{P}(a < X \leq b) = F(b) - F(a)$$

$$= \int_{-\infty}^{b} f(t)\,dt - \int_{-\infty}^{a} f(t)\,dt$$

$$\int^{b}$$

$-\infty \qquad -\infty$

$$= \int_a^b f(t)\,dt$$

For cts. R.Vs.

$$P(a \le X \le b) = P(a < X \le b)$$
$$= P(a \le X < b)$$
$$= P(a < X < b)$$

$P(X=a) = 0$

$P(X=b) = 0$

General rule:

cts: $\mathbb{P}(X \in A) = \int_A f(t)\,dt$

discrete: $\mathbb{P}(X \in A) = \sum_{x \in A} f(x)$

Ex. Let

$$F(x) = \frac{1}{1+e^{-x}}$$

What is the PDF of this random var. w/ F as its CDF.

$$f = \frac{dF}{dx} = \frac{e^{-x}}{\ldots}$$

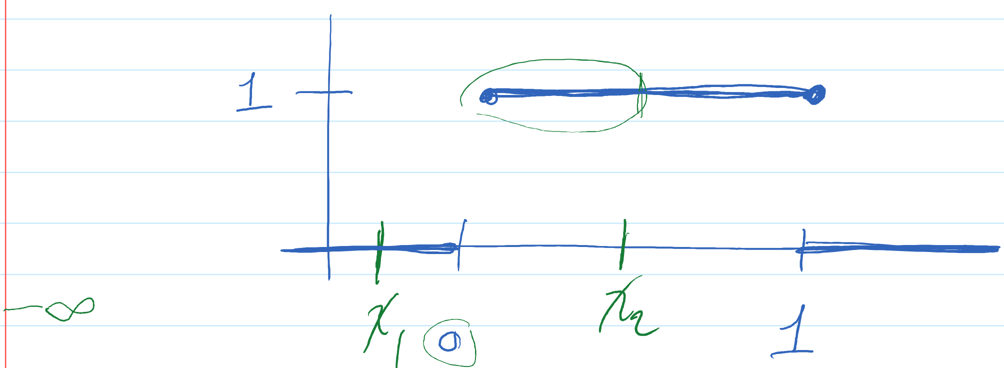
$$f(x) = \frac{dF}{dx} = \frac{e^{-x}}{(1+e^{-x})^2}$$

<u>Ex.</u> Continuous Uniform Distribution (on $[0, 1]$)

$$X \sim U(0,1)$$

<u>means</u>

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$



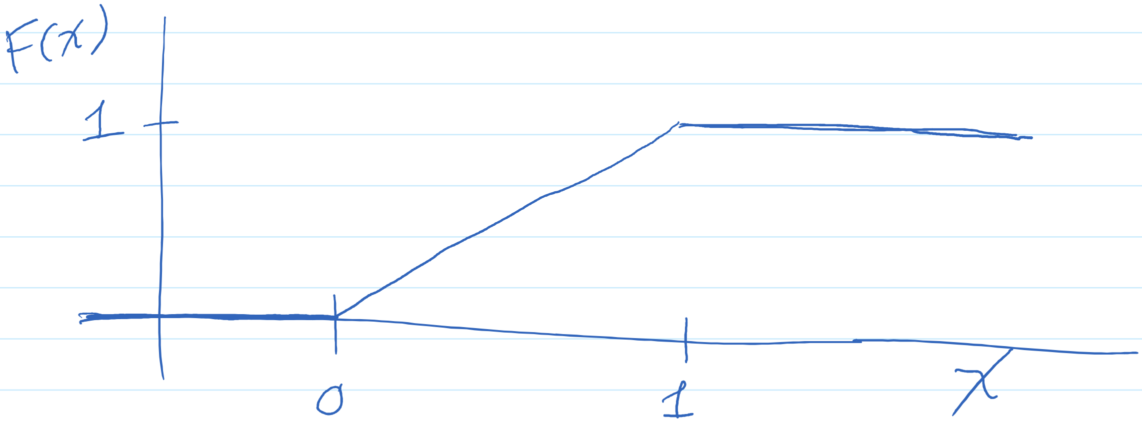
Q: What is the CDF?

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0 & x \leq 0 \\ x & x \in [0,1] \\ 1 & x \geq 0 \end{cases}$$

Case 1: if $x \le 0$ then $\int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0$

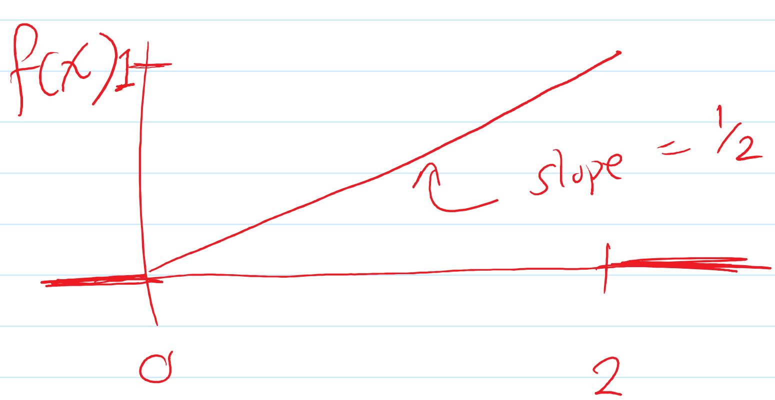
Case 2: $0 \le x \le 1$ $\int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} f(t)\,dt = \int_{0}^{x} 1\,dt = x$

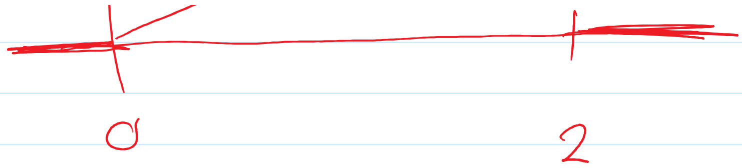
Case 3: $x \ge 1$ $\int_{-\infty}^{x} f(t)\,dt = \int_{0}^{1} f(t)\,dt = \int_{0}^{1} 1\,dt = 1$



Ex. The PDF of X is

$$f(x) = \begin{cases} x/2 & 0 < x < 2 \\ 0 & \text{else} \end{cases}$$





Recall: $P(X \in A) = \int_A f(t)\,dt$

$$P(X > 1) = \int_1^{\infty} f(t)\,dt = \int_1^{2} t/2\,dt = \left[\frac{t^2}{4}\right]_1^2$$

$$= \frac{4}{4} - \frac{1}{4} = 3/4$$

Ex. Cdf

$$F(x) = \begin{cases} 1 - e^{-x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

What is $P(1 < X < 2)$

Theorem that says: $P(a < X < b) = F(b) - F(a)$

So $P(1 < X < 2) = F(2) - F(1)$

$$= (1 - e^{-2}) - (1 - e^{-1})$$

$$= e^{-1} - e^{-2}$$

Calculate from the PDF

$$f(x) = \frac{dF}{dx} = \frac{d}{dx}(1 - e^{-x}) = e^{-x}$$

another way:

$$P(1 < X < 2) = \int_1^2 f(t)\,dt = \int_1^2 e^{-t}\,dt$$

$$= \left[-e^{-t}\right]_1^2$$

$$= -e^{-2} - (-e^{-1})$$

$$= e^{-1} - e^{-2}$$

Theorem: PMF/PDF characterization

A function f is a valid pmf/pdf iff

1. $f(x) \geq 0 \quad \forall x$

2. (discrete) $\sum_{x \in \mathbb{R}} f(x) = 1$

   (cts) $\int_{\mathbb{R}} f(x)\,dx = 1$

(aside $\mathbb{P}(X \in A) = \int_A f(t)\,dt \leftarrow$ better be a valid prob.)

Side: If $f(x) \geq 0$ and $\int_{\mathbb{R}} f(x)\,dx = c$

then $\frac{1}{c} f(x)$ is a valid PDF.

Normal Distribution

aka the Gaussian Distribution

notation

$$X \sim N(\mu, \sigma^2)$$

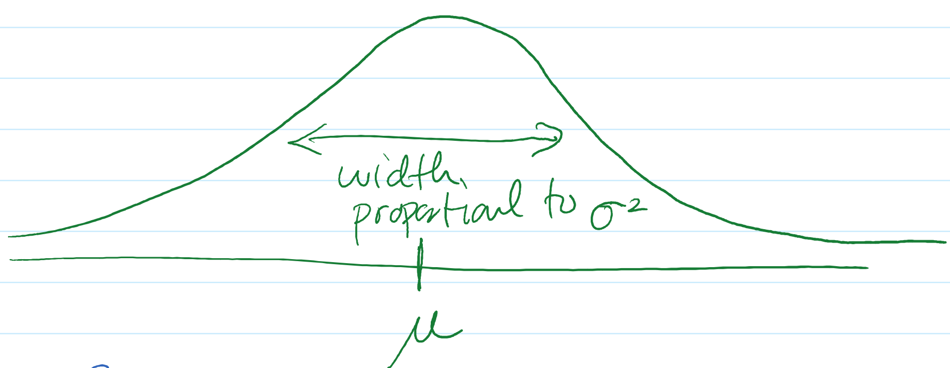
mean $\mu \in \mathbb{R}$

Variance $\sigma^2 > 0$

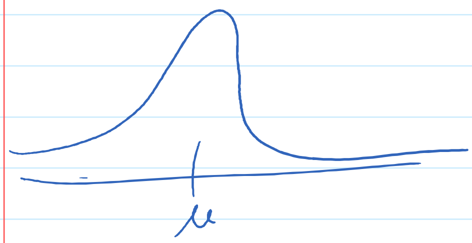
the pdf is

$\exp(a) = e^a$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}(x-\mu)^2\right), \quad x \in \mathbb{R}$$



small σ^2



large σ^2

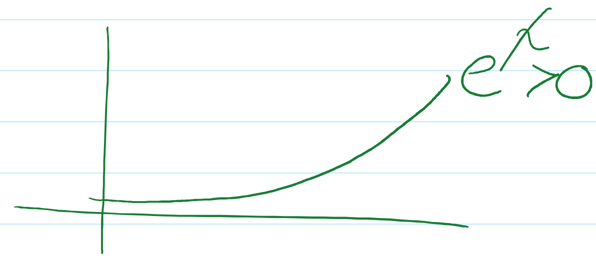
Special case: Standard Normal

$$X \sim N(0,1)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$$

Q:

(1) $f(x) \geq 0$ ✓



(2) $\int_{\mathbb{R}} f(x)\,dx = 1$

$$\underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) dx}_{I \,=\, \text{some number} \,\geq\, 0} = 1$$

Equiv. Show $I^2 = 1$. Then I must be 1 also.

$$I^2 = \left(\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) dx\right)\left(\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right) dy\right)$$

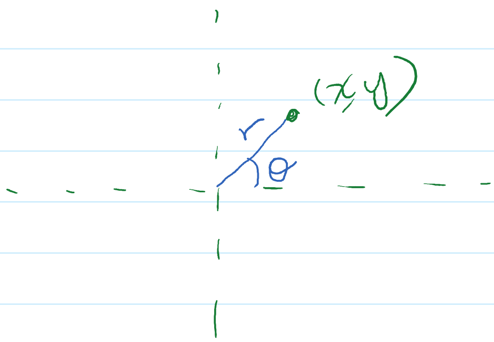
$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{2\pi} \exp\left(-\frac{1}{2}x^2\right) \exp\left(-\frac{1}{2}y^2\right) dx\,dy$$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{2\pi} \exp\left(-\tfrac{1}{2}x^2\right)\exp\left(-\tfrac{1}{2}y^2\right)dx\,dy$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}^2} \exp\left(-\frac{1}{2}(x^2+y^2)\right)dx\,dy$$

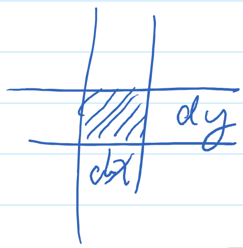
Recall: polar coordinates

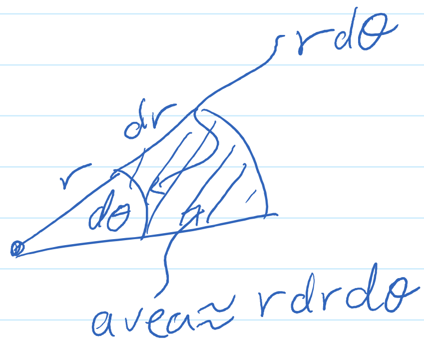
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$x^2+y^2 = r^2$$



$$dx\,dy = r\,dr\,d\theta$$

integrate x, y





$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} \exp\left(-\tfrac{1}{2}r^2\right) r\,dr\,d\theta$$

u-substitution

$$u = \frac{1}{2}r^2$$
$$du = r\,dr$$

$$\int_0^{\infty} e^{-u}\,du = \left[-e^{-u}\right]_0^{\infty} = [0-(-1)] = 1$$

$$= \frac{1}{2\pi}\int_0^{2\pi} 1\,d\theta = \frac{1}{2\pi}2\pi = 1 \;. = I^2.$$

Expected Value

Expected Value

If X is a r.v. then the mean or the expected value of X

denoted $\mathbb{E}[X]$

is defined as

(1) discrete

$$\mathbb{E}[X] = \sum_{x \in \mathbb{R}} x f(x)$$

(2) continuous

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x)\, dx$$

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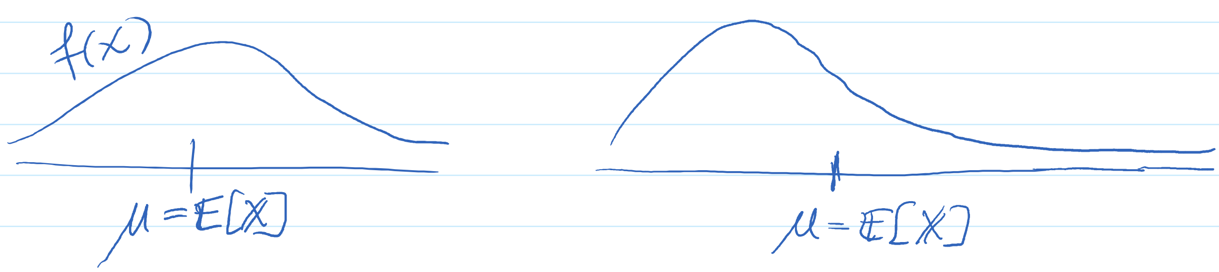
discrete case:

$$\mathbb{E}[X] = x_1 \mathbb{P}(X = x_1) + x_2 \mathbb{P}(X = x_2) + \cdots$$

balancing point of the distribution



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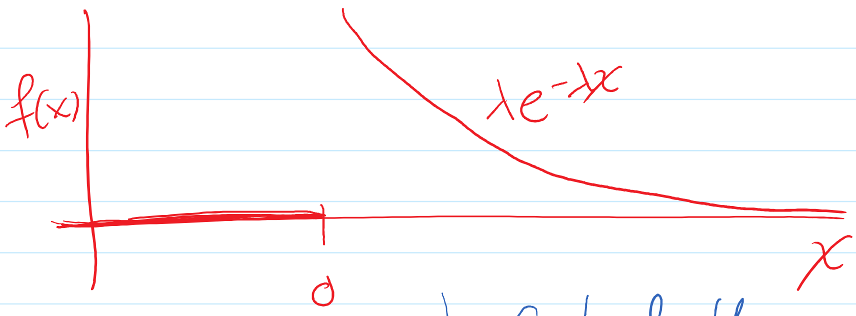
ex.

ex.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{else} \end{cases}$$

$\lambda > 0$ = rate parameter



Called the exponential distribution, denote $X \sim Exp(\lambda)$

Q: what is $E[X]$?

$$E[X] = \int_{\mathbb{R}} x f(x)\,dx = \int_0^\infty x \lambda e^{-\lambda x}\,dx$$

Integration By Parts

$$\int u\,dv = uv - \int v\,du$$

$u = x \qquad v = -e^{-\lambda x}$

$du = dx \qquad dv = \lambda e^{-\lambda x}\,dx$

$$\int_0^\infty u\,dv = uv - \int v\,du$$

$$= \left[-xe^{-\lambda x}\right]_0^\infty + \int_0^\infty e^{-\lambda x}\,dx$$

$$= \left[-xe^{-\lambda x} - \frac{1}{\lambda} e^{-\lambda x}\right]_0^\infty$$

$$= \left[-xe^{-\lambda x}\right]_0^\infty - \frac{1}{\lambda}\left[e^{-\lambda x}\right]^\infty$$

$$= \left[-xe^{-\lambda x}\right]_0^{\infty} - \frac{1}{\lambda}\left[e^{-\lambda x}\right]_0^{\infty}$$

$$= \left[0 - 0\right] - \frac{1}{\lambda}\left[0 - 1\right]$$

$$= \frac{1}{\lambda}$$