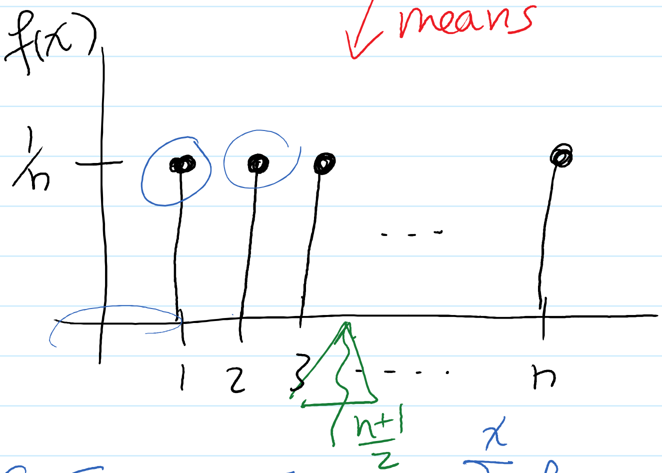
Discrete Uniform Distribution

$$X \sim \text{Unif}\{1, \dots, n\}$$

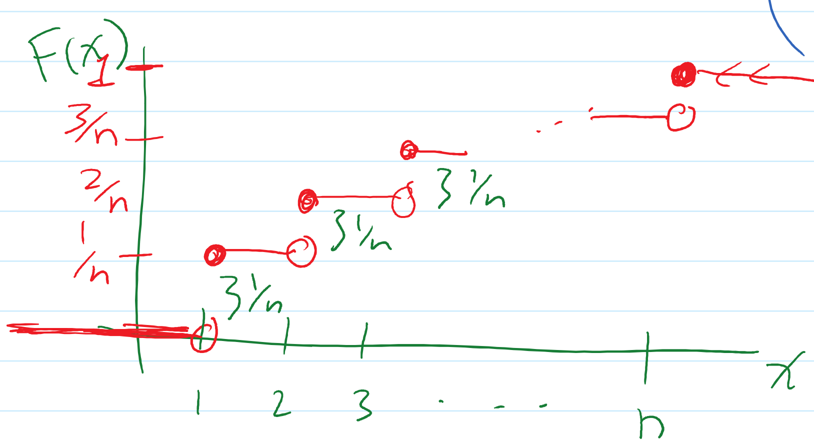
means



i.e. $$f(x) = \begin{cases} 1/n & x = 1, \dots, n \\ 0 & \text{else} \end{cases}$$

CDF:

$$F(x) = \sum_{i=1}^{x} f(i) = \begin{cases} 0 & i < 1 \\ 1/n & 1 \le i < 2 \\ 2/n & 2 \le i < 3 \\ 3/n & 3 \le i < 4 \\ \vdots & \vdots \\ 1 = n/n & n \le i \end{cases}$$



1. non-decreasing
2. right cts
3. $\lim_{x \to -\infty} F(x) = 0$

   $\lim_{x \to \infty} F(x) = 1$

Expected Value

$$E[X] = \sum_{i=1}^{n} i f(i) = \sum_{i=1}^{n} i \frac{1}{n} = \frac{1}{n}\left[\sum_{i=1}^{n} i\right]$$

$$= \frac{1}{n}\left[\frac{n(n+1)}{2}\right]$$

$$= \frac{n+1}{\;}$$

$$= \frac{n+1}{2}$$

$$E[X^2] = \sum_{x=1}^{n} x^2 f(x) = \sum_{x=1}^{n} x^2 \, 1/n = \frac{1}{n} \sum_{x=1}^{n} x^2$$

$$= \frac{1}{\cancel{n}} \frac{\cancel{n}(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(2n+1)}{6}$$

Variance: $Var(X) = E[(X - E[X])^2]$

Short-cut formula says:

$$Var(X) = E[X^2] - E[X]^2$$

$$= \frac{(n+1)(2n+1)}{6} - \left(\frac{(n+1)}{2}\right)^2$$

$$= \ldots \text{algebra}$$

$$\boxed{= \frac{(n+1)(n-1)}{12}}$$

MGF (moment generating function)

defn: $M(t) = E[e^{tX}] = \sum_{x=1}^{n} e^{tx} f(x) = \frac{1}{n} \sum_{x=1}^{n} e^{tx}$

$$= \frac{1}{n} \sum_{x=1}^{n} (e^t)^x$$

starts to look like a partial geometric series

$n-1 \qquad x+1 \qquad n-1 \quad i \qquad 1-r^n$

series

$$\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$$

$|r|<1$

$$= \frac{1}{n}\sum_{x=0}^{n-1}(e^t)^{x+1}$$

$r = e^t$

$$= \frac{e^t}{n}\sum_{x=0}^{n-1}(e^t)^{x}$$

$$= \frac{e^t}{n}\,\frac{1-(e^t)^n}{1-e^t} = \boxed{\frac{e^t - e^{t(n+1)}}{n(1-e^t)}}$$

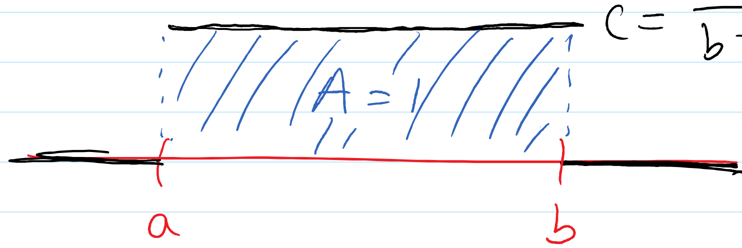
for t s.t. $e^t < 1$

Continuous Uniform Distribution

$X \sim U(a,b)$ "X is unif over (a,b)"

<u>means</u> uniform density over (a,b)

<u>pdf</u> $f(x)$



$c = \frac{1}{b-a}$

Know that $\int_{\mathbb{R}} f(x)\,dx = 1$

$$= \int_a^b f(x)\,dx = \int_a^b c\,dx = c(b-a) = 1$$

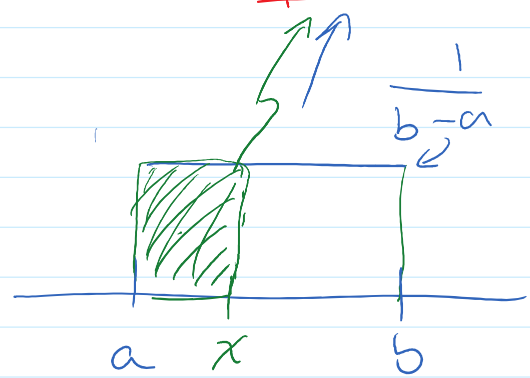
So $c = \frac{1}{b-a}$

So $c = \frac{1}{b-a}$

CDF:

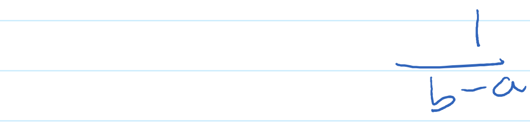
$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{a}^{x} f(t)\,dt = \int_{a}^{x} \frac{1}{b-a}\,dt$$

$$= \frac{1}{b-a}\int_{a}^{x} dt = \frac{1}{b-a}(x-a) = \boxed{\frac{x-a}{b-a}}$$

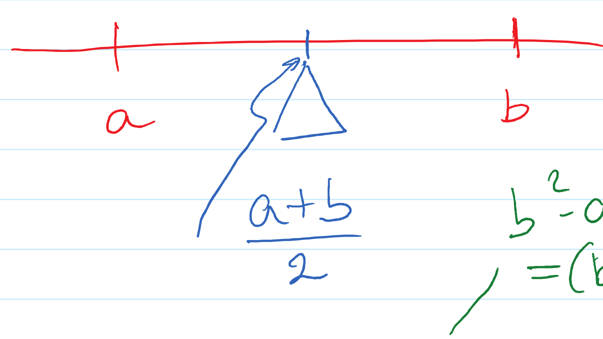


$$F(x) = \begin{cases} \frac{x-a}{b-a}, & a < x < b \\ 0 & x \leq a \\ 1 & x \geq b \end{cases}$$

Expected Value



$$E[X] = \int_{\mathbb{R}} x f(x)\,dx$$



$$= \int_{a}^{b} x \frac{1}{b-a}\,dx$$

$b^2 - a^2 = (b-a)(b+a)$

$$= \frac{1}{\ } \int_{a}^{b} x\,dx = \frac{1}{\ } \left[\frac{x^2}{\ }\right]_{a}^{b} = \frac{b^2 - a^3}{\ } = \frac{\cancel{(b-a)}(b+a)}{\ }$$

$$= \frac{1}{b-a}\int_a^b x\,dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}}$$

$$\boxed{= \frac{a+b}{2}}.$$

$$E[X^2] = \int_a^b x^2 f(x)\,dx = \int_a^b x^2 \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{b^3-a^3}{3(b-a)} = \frac{\cancel{(b-a)}(b^2+ba+a^2)}{3\cancel{(b-a)}}$$

$$\boxed{= \frac{b^2+ba+a^2}{3}}$$

Short-cut formula

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \frac{b^2+ba+a^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$= \ldots$ algebra

$$\boxed{= \frac{(b-a)^2}{12}}$$

MGF

$$M(t) = \mathbb{E}[e^{tX}] = \int_{\mathbb{R}} e^{tx} f(x)\,dx$$

$$M(t) = E[e^{tX}] = \int_{\mathbb{R}} e^{tx} f(x)\,dx$$

$$= \int_a^b e^{tx} \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{1}{t} e^{tx}\right]_{x=a}^{x=b}$$

$$\boxed{= \frac{1}{t(b-a)}\left(e^{bt} - e^{at}\right)}$$

for $t \neq 0$

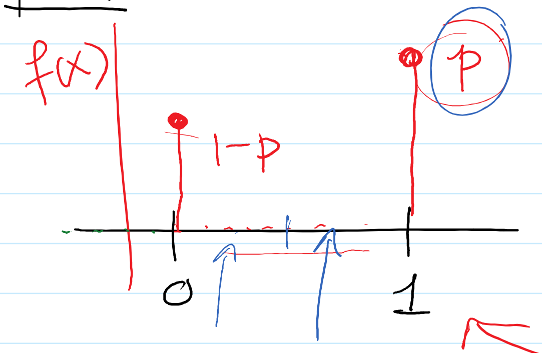
Bernoulli Distribution

$X \sim \text{Bern}(p)$ prob. of success, $p \in [0,1]$

- flip a coin w/ prop. p of H,
  $X = 1$ if H, $X = 0$ otherwise
  then $X \sim \text{Bern}(p)$

- any experiment w/ two outcomes 0 or 1
  w/ prob. p of getting a 1

pmf



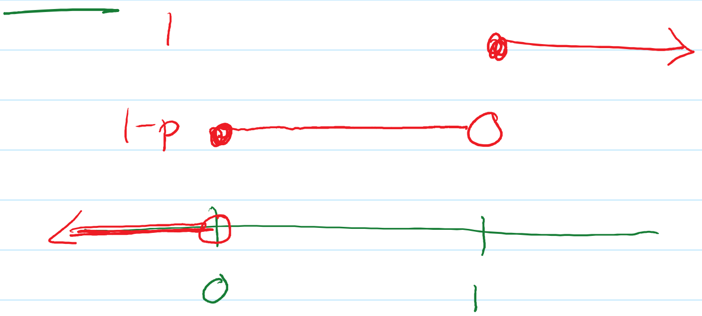
$$f(x) = \begin{cases} p^x (1-p)^{1-x} & \text{for } x = 0, 1 \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} 1-p & x = 0 \\ p & x = 1 \end{cases}$$

0		1

$$= \begin{cases} p & x = 1 \\ 0 & \text{else} \end{cases}$$

CDF



$$F(x) = \begin{cases} 0 & x < 0 \\ 1-p & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

Expectation:

$$\mathbb{E}[X] = \sum_{x=0}^{1} x f(x) = (0)f(0) + (1)f(1)$$
$$= f(1) = p$$

$$\mathbb{E}[X^2] = (0)^2 f(0) + (1)^2 f(1)$$
$$= f(1) = p$$

aside

$$\mathbb{E}[X^r] = f(1) = p$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = p - p^2 = \boxed{p(1-p).}$$

Note Bernoulli(p) same as Bin(n,p) where $n=1$

MGF: $\quad tx$

MGF: $M(t) = E[e^{tX}] = \sum_{x=0,1} e^{tx} f(x)$

$$= e^{t(0)} f(0) + e^{t(1)} f(1)$$

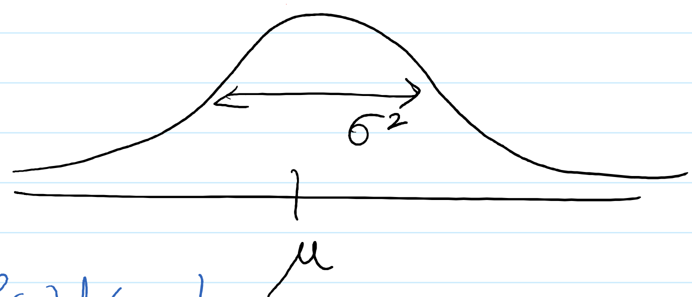
$$= f(0) + e^t f(1)$$

$$\boxed{= 1-p+pe^t}$$

Normal Distribution / Gaussian Distribution

$X \sim N(\mu, \sigma^2)$

$\mu \in \mathbb{R}, \sigma^2 > 0$



Shared: $\int f(x)dx = 1$

$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$, for all $x \in \mathbb{R}$

$\boxed{E[X] = \mu}$ and $\boxed{Var(X) = \sigma^2}$

CDF: No closed form.

Expected Value: $\boxed{E[X] = \mu}$

$$E[X] = \int_{\mathbb{R}} x f(x)\,dx$$

$$= \int x \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx$$

change of variables: $\underline{y = x-\mu}$ or $x = y+\mu$

$dy = dx$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int (y+\mu) \exp\left(-\frac{1}{2\sigma^2} y^2\right) dy$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \underbrace{y \exp\left(-\frac{1}{2\sigma^2} y^2\right)}_{g(y)} dy + \frac{1}{\sqrt{2\pi\sigma^2}} \int \mu \exp\left(-\frac{1}{2\sigma^2} y^2\right) dy$$

$g(y)$ is a odd function

$g(-y) = -g(y)$

If I integrate an odd function over a symmetric interval

$A - A = 0$

$$\mu \int_{\mathbb{R}} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2} y^2\right)}_{\text{pdf of a } N(0,\sigma^2)} dy$$

= integral is 1

$\mu(1)$

$$= 0 + \mu = \mu = E[X]$$

$$= 0 + \mu = \mu = \mathbb{E}[X]$$

$$\mathrm{Var}(X) = \mathbb{E}\left[(X-\mu)^2\right] \qquad \mu = \mathbb{E}[X]$$

$$= \int_{\mathbb{R}} (x-\mu)^2 f(x)\,dx$$

$$= \int_{\mathbb{R}} (x-\mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx$$

$y = \frac{x-\mu}{\sigma}$ then $dy = \frac{1}{\sigma}dx \iff dx = \sigma dy$

$$= \frac{\sigma}{\sqrt{2\pi\sigma^2}} \int \sigma^2 y^2 \exp\left(-\frac{1}{2}y^2\right) dy$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int y^2 \exp\left(-\frac{1}{2}y^2\right) dy = \frac{\sigma^2}{\sqrt{2\pi}} \int_{\mathbb{R}} u\,dv$$

Integration by parts: $u = y \quad dv = y\exp\left(-\frac{1}{2}y^2\right)dy$

Can show: $\int u\,dv = \sqrt{2\pi}$

$$\sigma^2 \ldots$$

$$\frac{\sigma^2}{\sqrt{2\pi}}\sqrt{2\pi} = \boxed{\sigma^2 = Var(X)}$$

MGF:

$$M(t) = E[e^{tX}] = \int_{\mathbb{R}} e^{tx} f(x)\,dx$$

$$(x-\mu)^2 = x^2 - 2x\mu + \mu^2$$

$$= \int_{\mathbb{R}} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\left(x^2 - 2x\mu + \mu^2 - 2tx\sigma^2\right)\right) dx$$

Complete the square

$$x^2 - 2x(\mu + \sigma^2 t) + \mu^2$$

$$= (x - (\mu + \sigma^2 t))^2 + \mu^2 - (\mu + \sigma^2 t)^2$$

bring out

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{\mu^2 - (\mu + \sigma^2 t)^2}{-2\sigma^2}\right) \exp\left(-\frac{1}{2\sigma^2}(x - (\mu + \sigma^2 t))^2\right) dx$$

pdf of a $N(\mu + \sigma^2 t, \sigma^2)$

pdf of a $N(\mu+\sigma^2 t, \sigma^2)$

$$= \underbrace{\boxed{\exp\left(-\frac{1}{2\sigma^2}\left(\mu^2-(\mu+\sigma^2 t)^2\right)\right)}}_{MGF} \underbrace{\int N(\mu+\sigma^2 t, \sigma^2)}_{1}$$

$= \dots =$ Simplify

$$\boxed{= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)}$$

test: $\left.\frac{dM}{dt}\right|_{t=0} = E[X] = \mu$

$$\frac{dM}{dt} = (\mu + \underbrace{\sigma^2 t}_{0}) \underbrace{\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)}_{1}$$

so $\left.\frac{dM}{dt}\right|_{t=0} = \mu$

Theorem: Linear Functions of a $N(\mu, \sigma^2)$

If $X \sim N(\mu, \sigma^2)$ then

$$Y = aX + b$$

then $Y \sim N(a\mu+b,\ a^2\sigma^2)$.

intuition: $\mathbb{E}[Y] = \mathbb{E}[aX+b] = a\mathbb{E}[X]+b$
$= a\mu + b$

$Var(Y) = Var(aX+b) = a^2 Var(X)$
$= a^2\sigma^2$.

pf. $M_Y(t) = M_{aX+b}(t) = e^{tb} M_X(at)$
$= e^{tb} \exp\left(\mu a t + \frac{\sigma^2}{2} a^2 t^2\right)$
$= \exp\left((a\mu+b)t + \frac{a^2\sigma^2 t^2}{2}\right)$

MGF of a $N(a\mu+b,\ a^2\sigma^2)$