Lecture 4 - Counting

Tuesday, February 4, 2020 10:57 AM

Equally likely outcomes

$$S = \{s_1, \dots, s_n\}$$

then we could assume that

$$\mathbb{P}(\{s_i\}) = \text{equal for all } s_i$$

Ex. Roll a six sided die

$$S = \{1, \dots, 6\}$$

equally likely

$$\mathbb{P}(\{i\}) = 1/6$$

In general we can say

$$\mathbb{P}(E) = \frac{|E|}{|S|} = \frac{\#\text{ of outcomes in } E}{\#\text{ of possible outcomes}}$$

Ex. E = "rolling a 2 or 6"

$= \{2, 6\}$

then $\mathbb{P}(E) = \frac{|E|}{|S|} = 2/6 = 1/3$

In this case counting $|E|$ is easy, but it could be difficult.

Or, counting $|S|$ could be complicated.

Counting

Fundamental Theorem of Counting

If a "job" or "task" consists of k sub-tasks each of which has n_i ways of being done $(i = 1, \ldots, k)$ then the # of ways to accomplish the over-all task is

$$N = n_1 \cdot n_2 \cdot n_3 \cdots n_k$$

$$= \prod_{i=1}^{k} n_i$$

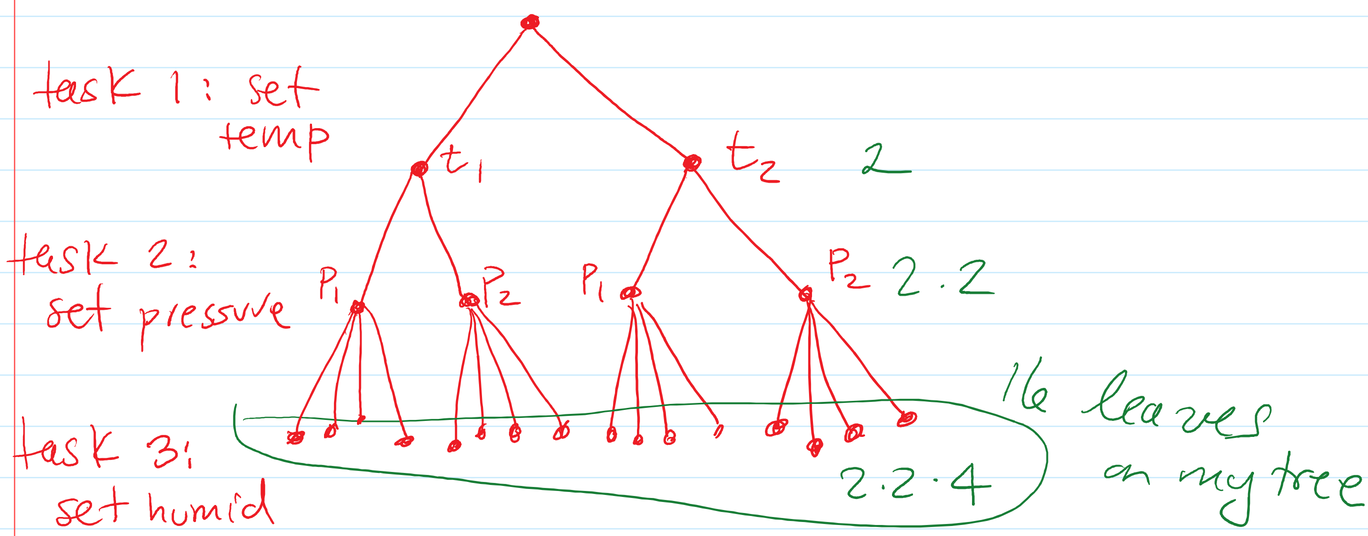
Ex. Experiment consists of 3 factors

1. 2 possible temperature settings
2. 2 " pressure "
3. 4 " humidity "

How many possible experiments can I do?

According to the FTC there are

$2 \cdot 2 \cdot 4 = 16$ ways of doing this experiment.



Ex. A man has 5 shirts, 2 pairs of pants, 2 pairs of shoes.

How many outfits can he create?

By FTC $k=3$; $\underbrace{n_1}_{\text{pants}} = 5$, $\underbrace{n_2}_{\text{shirts}} = 2$, $\underbrace{n_3}_{\text{shoes}} = 2$

Overall we have $(5)(2)(2) = 20$ ways.

Ex. Shuffle a deck of cards. (52 cards)

What is the prob. that after shuffling the cards are in order? (A–K, C, D, H, S)

$S = \{\text{all possible ordering of the 52 cards}\}$

Assume that shuffling is perfect and all orderings are equally likely.

E = "cards in order after shuffle"

$$P(E) = \frac{|E|}{|S|}$$ ← need to count numerators/denominator

① $|E| = 1$

② $|S| =$ by FTC $= (52)(51)(50)(49) \cdots (1)$

52 tasks i.e. picking card 1, 2, 3, ..., 52

$n_1 = 52$, $n_2 = 51$, $n_3 = 50$, ...

$n_{52} = 1$

So

$$P(E) = \frac{|E|}{|S|} = \frac{1}{52 \cdot 51 \cdot 50 \cdots 1} \approx 1.24 \cdot 10^{-68}$$

Defn: Factorial

For a non-negative integer n we define n-factorial, denoted $n!$, as

$$n! = n(n-1)(n-2)\cdots(3)(2)(1)$$
$$= \prod_{i=1}^{n} i$$

We define $0! = 1$.

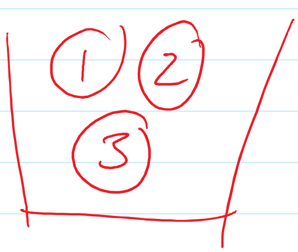
<u>Ex.</u> $\mathbb{P}(E) = 1/52!$ (from above example).

Ordering of Outcomes and Drawing w/ or w/o Repl.

We can think of many counting problems as drawing samples from a population and we may care about

1. Ordering — do we care about order in which sample is drawn
2. Replacement — do we replace items back into pop. after drawing a sample

<u>Ex.</u>



<u>Ordering:</u>

1 2 3 different than 3 2 1

Replacement: can I sample the same ball twice?

① ② ① ?

can do if I replace ① after drawing sample

can't get if sampling w/o replacement.

4 situations:

	w/ replacement	w/o replacement
ordered	①	②
un-ordered	④	③

Permutations

A permutation is an ordering of objects.

Ex. objects A_1, A_2, A_3

possible permutations:

$$\left.\begin{matrix} A_1A_2A_3, & A_2A_1A_3, & A_1A_3A_2 \\ A_3A_2A_1, & A_2A_3A_1, & A_3A_1A_2 \end{matrix}\right\}$$ 6 possible permutations

Theorem: Permutation Counting

The # of ways to permute n items is $n!$

Pf: Use FTC

n substasks
$$\left\{\begin{array}{lll} n & \text{ways to choose} & 1^{st} \text{ item in permutation} \\ n-1 & \text{"} & 2^{nd} \text{ "} \\ n-2 & \text{"} & 3^{rd} \text{ "} \\ & \vdots & \\ 1 & \text{"} & n^{th} \text{ "} \end{array}\right.$$

and the $n_i = n-i+1$

So by FTC I have

$$n! = n(n-1)(n-2)(n-3)\cdots(1)$$

ways of making a permutation of n items

Theorem: Ordered Sampling w/o Replacement

If I have n items and I draw a sample of r of them $(r \leq n)$ then the number of ways to draw this sample is

$$(n) \stackrel{\text{def}}{=} \boxed{\frac{n!}{}}$$

$$(n)_r \stackrel{\text{def}}{=} \boxed{\frac{n!}{(n-r)!}}$$

pf. My task has r sub-tasks

n ways to choose 1^{st} item

$n-1$ ways " 2^{nd} "

$n-2$ " 3^{rd} "

$\vdots$

$n-r+1$ " r^{th} "

So by FTC the # of ways to draw my sample is

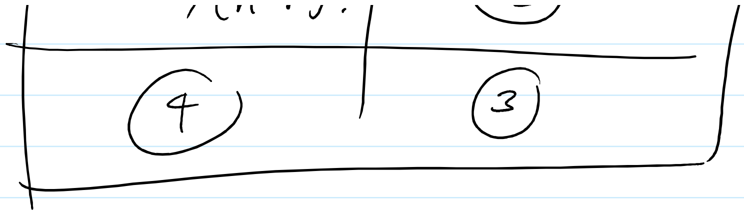
$$\boxed{n(n-1)(n-2)(n-3)\cdots(n-r+2)(n-r+1)}$$

$$= \frac{n!}{(n-r)!}$$

$$\frac{n(n-1)(n-2)\cdots(n-r+1)\cancel{(n-r)}\cancel{(n-r-1)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{(n-r)}\cancel{(n-r-1)}\cancel{(n-r-2)}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}$$

	w/o repl.	w/ repl.
ordered	$n!/(n-r)!$	2
ordered	[illegible]	[illegible]

unordered



draw r items from n

Ex. 10 students in total and want to choose 3 to be president, VP, treasurer

How many ways can we choose the 3?

you can only be one role

Sample 3 from 10

call the 1st the pres. → care about ordering b/c the order of draw determines the role

2nd the VP

3rd the T

→ no repl. b/c a person cannot occupy > 1 role.

the # of ways is

$$= \boxed{10!/(10-3)!} = \frac{n!}{(n-r)!} \quad n = 10 \quad r = 3$$

$$\frac{10 \cdot 9 \cdot 8 \cdot \cancel{7 \cdots 1}}{\cancel{7 \cdot 6 \cdot 5 \cdots 1}} = 10 \cdot 9 \cdot 8 = 720$$

Ex. Lotto: draw 4 numbered balls from a set of 25 balls

1 2 3 . . . 25

I pick lotto #s

2 5 22 13 = E

If each set of numbers is equally likely what is the prob. I win?

$$P(E) = \frac{|E|}{|S|} = \frac{1}{|S|}$$

S = { all ordered draws w/o repl. }

$$|S| = \frac{25!}{(25-4)!} = \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot \cancel{21 \cdot 20 \cdots 1}}{\cancel{21 \cdot 20 \cdot 19 \cdots}}$$

$$= 25 \cdot 24 \cdot 23 \cdot 22$$

hence

$$P(E) = \frac{1}{25 \cdot 24 \cdot 23 \cdot 22}$$

= bad idea

Theorem: Sampling Ordered w/ Replacement

If I draw a sample of r items from a population of n paying attention to order but replacing items after each draw.

I can get

$$n^r$$

Samples.

Pf: Again, use FTC

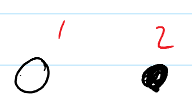
r – subtasks choosing each of r samples.

n ways to choose 1^{st} sampled item
n " 2^{nd} "
n " 3^{rd} "
⋮
n " r^{th} "

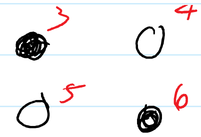
FTC says there are

$$\underbrace{n \cdot n \cdot n \cdot n \cdots n}_{r \text{ times}} = n^r.$$

Ex. Braille has 6 locations for dots:



• = raised bumps



How many braille "letters" can I represent?

Sample ○ or ● six times.

$\boxed{2^6}$