Lecture 19 - Iterated Expectation and Bivariate Transformations

Thursday, April 9, 2020 4:57 PM

Theorem: Iterated Expectation

If X and Y then

$$\mathbb{E}[X] = \mathbb{E}\left[\mathbb{E}[X|Y]\right]$$

Recall: $\mathbb{E}[X|Y=y]$ here $y \in \mathbb{R}$

$$= \underbrace{\int_{\mathbb{R}} x f(x|y)\,dx}_{\mathbb{R}}$$

$g: \mathbb{R} \to \mathbb{R}$ defined as $\boxed{g(y) = \mathbb{E}[X|Y=y]}\ y \in \mathbb{R}$

Consider: $\boxed{g(Y)} = \cancel{\mathbb{E}[X|Y=Y]}$

$= \mathbb{E}[X|Y]$ same notation

Recap: $\mathbb{E}[X|Y=y]$ a number

$\mathbb{E}[X|Y]$ a random variable

Theorem: $\mathbb{E}[X] = \mathbb{E}_Y\left[\boxed{\mathbb{E}[X|Y]}\right]$

a r.v. that is a fn of Y

$\mathbb{E}(g(Y))$

a r.v. that is a fn of Y

outer expectation w/ resp. to Y

$$E(g(Y)) = \int_{\mathbb{R}} g(y) f(y)\, dy$$

pf. (cts case)

$$\mathbb{E}[X] = \int x f(x)\, dx$$

Fact: $f(x) = \int f(x,y)\, dy$

$$= \int x \int f(x,y)\, dy\, dx$$

Fact: $f(x|y) = \dfrac{f(x,y)}{f(y)}$

$$\Updownarrow$$

$$f(x,y) = f(x|y) f(y)$$

$$= \int x \int f(x|y) f(y)\, dy\, dx$$

$$= \iint x f(x|y) f(y)\, dx\, dy$$

$$= \int \underbrace{\left[\int x f(x|y)\, dx\right]}_{\mathbb{E}[X|Y=y] = g(y)} f(y)\, dy$$

$$= \int g(y) f(y)\, dy$$

$g(Y) = \mathbb{E}[X|Y]$

$$= \mathbb{E}[g(Y)]$$

$$= \mathbb{E}_Y[\mathbb{E}[X|Y]]$$

$E_Y[E[X|Y]]$

$$\boxed{E[X] = E_Y[E[X|Y]]}$$

Iterated Expectation.

Ex. $X|Y=y \sim Bin(y, p)$

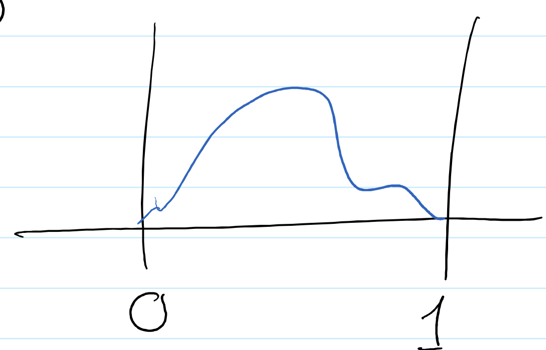
$Y \sim Pois(\lambda)$

$$E[X] = E[E[X|Y]]$$

1. $E[X|Y=y] = yp = g(y)$
2. $E[X|Y] = g(Y) = Yp$
3. $E[X] = E_Y[Yp] = p\,E[Y] = \boxed{p\lambda}$

Ex. $X|P=p \sim Bin(n, p)$

$P \sim Beta(\alpha, \beta)$



1. $E[X|P=p] = np$
2. $E[X|P] = nP = g(p)$

③ $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|P]] = \mathbb{E}[nP] = n\mathbb{E}[P]$

$$\boxed{= n\frac{\alpha}{\alpha+\beta}}$$

Theorem: Law of Total Variance

$$\boxed{Var(X) = \mathbb{E}[Var(X|P)] + Var(\mathbb{E}[X|P])}$$

pf. Similar to other theorem.

Ex. $X|P=p \sim Bin(n,p)$

$P \sim Beta(\alpha,\beta)$

$\boxed{Var(X)?}$ ① $\mathbb{E}[X|P=p] = np$

$Var(X|P=p) = np(1-p)$

② $\mathbb{E}[X|P] = nP \leftarrow$

$Var(X|P) = nP(1-P) \leftarrow$

(*) $Var(\mathbb{E}[X|P]) = Var(nP) = n^2 Var(P)$

$$= \boxed{n^2 \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}$$

(*) $\mathbb{E}[Var(X|P)] = \mathbb{E}[nP(1-P)]$

$= n\mathbb{E}[P - P^2]$

$$= n\,\mathbb{E}[P - P^2]$$

$$= n\,\mathbb{E}[P] - n\,\mathbb{E}[P^2]$$

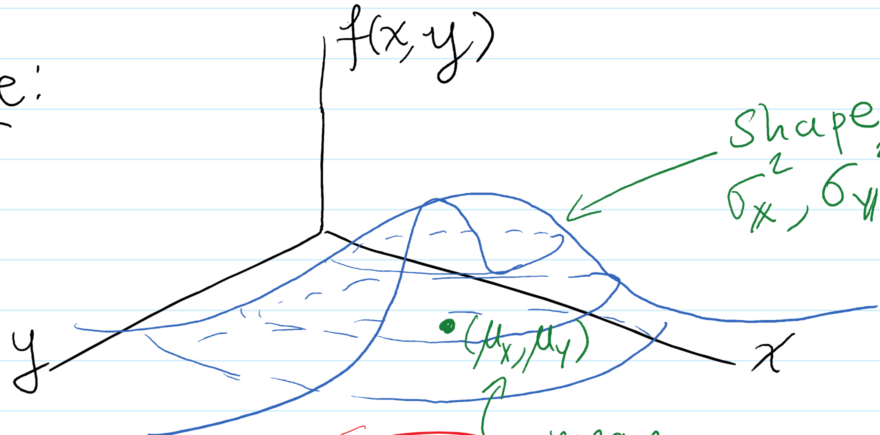
algebra ... $= \boxed{n \dfrac{\alpha\beta}{(\alpha+\beta)(\alpha+\beta+1)}}$

add to get $Var(X)$

Bivariate Normal

Univariate: $N(\mu, \sigma^2)$

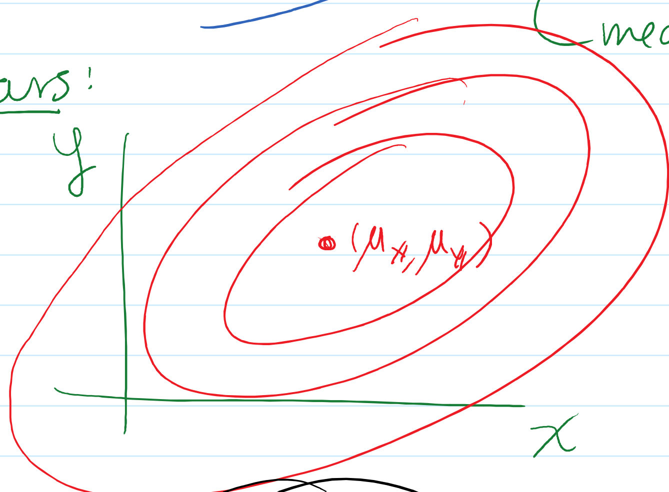
Bivariate:



Shape $\sigma_X^2, \sigma_Y^2, \rho$ — correlation btwn X and Y.

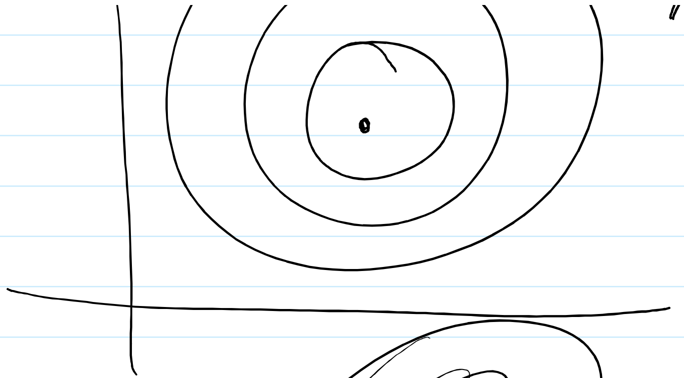
mean

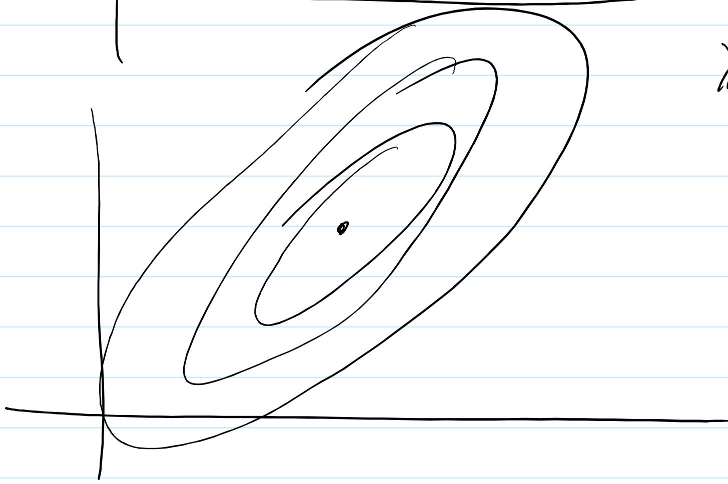
Contours:



Skewness/eccentricity of contours rel. to correlation.

X, Y uncorrelated





X, Y highly correlated

Bivariate normal:

means: μ_X, μ_Y ; $\mu = (\mu_X, \mu_Y)$

variance/correlation:

$$\sigma_X^2, \sigma_Y^2, \rho \; ; \; \Sigma = \begin{bmatrix} \sigma_X^2 & \sigma_X \sigma_Y \rho \\ \sigma_X \sigma_Y \rho & \sigma_Y^2 \end{bmatrix}$$

Var X

corr.

covariances

Var of Y

Covariance matrix

Joint PDF

$$(X, Y) \sim \text{BivNormal}(\mu_X, \mu_Y, \rho, \sigma_X^2, \sigma_Y^2)$$

Then

$$f(x,y) = \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2} \frac{1}{1-\rho^2} \left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 \right.\right.$$

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2}\frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)\right]\right\}$$

Simpler: μ mean vector, Σ covariance matrix

$$f(x,y) = \frac{1}{2\pi}\frac{1}{\sqrt{\det\Sigma}}\exp\left(-\frac{1}{2}z'\Sigma^{-1}z\right)$$

$z = (x,y)$

Facts:

1) $X \sim N(\mu_X, \sigma_X^2)$

$Y \sim N(\mu_Y, \sigma_Y^2)$

2) $Cor(X,Y) = \rho$

3) $aX + bY \sim N(a\mu_X + b\mu_Y,\ a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_X\sigma_Y\rho)$

4) Characterization of Biv. Normal

$(X,Y) \sim \text{BivNormal} \iff aX + bY \sim N(\ldots,\ldots)\ \forall a,b$

5) Recall: If $X \perp Y$ then $cor(X,Y) = 0$
$\Longrightarrow$

Theorem: If $(X,Y) \sim BivN$ and $Cor(X,Y) = 0$

then $X \perp Y$.

For biv. Normal : Independence = Uncorrelated

Pf. $Cor(X,Y) = 0 = \rho$

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\cancel{\sqrt{1-\rho^2}}^{\,1}}$$

$$\exp\left\{-\frac{1}{2}\underbrace{\cancel{\frac{1}{1-\rho^2}}}_{1}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2+\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2+\cancel{2\overset{0}{\rho}\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)}\right]\right\}$$

$$= \underbrace{\frac{1}{2\pi\sigma_X\sigma_Y}\exp\left(-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right)}_{h(x)}\underbrace{\exp\left(-\frac{1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right)}_{g(y)}$$

Shown: $f(x,y) = h(x)g(y)$

So $X \perp Y$.

Bivariate Transformation

Univariate: $g: \mathbb{R} \to \mathbb{R}$, $g(X)$?

Bivariate: $g: \mathbb{R}^2 \to \mathbb{R}^2$ $g(X,Y)$?

Bivariate: $g: \mathbb{R} \to \mathbb{R}$ $\quad g(X, Y)$?

Notation:

$$(X, Y) \xrightarrow{g} (U, V)$$

Ex. $(U, V) = (\underbrace{X^2 Y}_{g_1(X,Y)}, \underbrace{\log(Y)}_{g_2(X,Y)})$

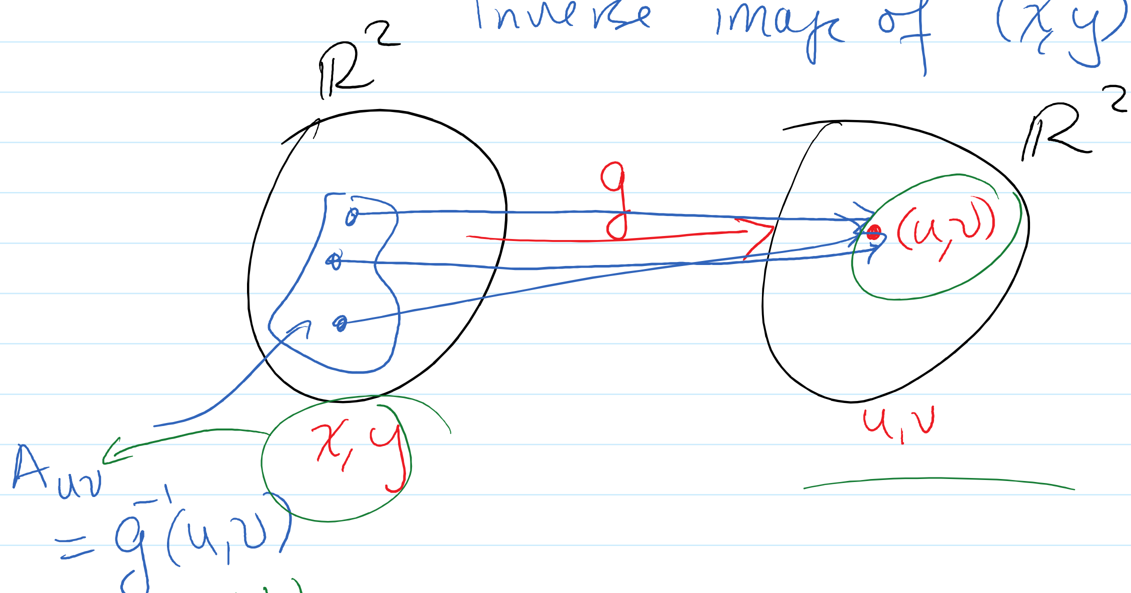
$(U, V) = (-X/Y, XY)$ $\quad g: \mathbb{R}^2 \to \mathbb{R}^2$

Discrete Case: $\quad (U, V) = (g_1(X, Y), g_2(X, Y))$

X, Y discrete.

Define: $A_{uv} = \{(x,y) \mid g_1(x,y) = u, g_2(x,y) = v\}$

Inverse image of (x,y) under g



Want (joint) pmf of (U, V) from (X, Y)

$$f_{U,V}(u,v) = \mathbb{P}(U=u, V=v)$$
$$= \mathbb{P}((X,Y) \in A_{uv})$$
$$= \sum_{(x,y)\in A_{uv}} \mathbb{P}(X=x, Y=y)$$
$$= \sum_{(x,y)\in A_{uv}} f_{X,Y}(x,y)$$

Ex. Let $X \perp\!\!\!\perp Y$ and

$$\left.\begin{aligned} X &\sim \text{Pois}(\theta) \\ Y &\sim \text{Pois}(\lambda) \end{aligned}\right\} \text{discrete}$$

$$f(x,y) = f(x)f(y) = \frac{\theta^x e^{-\theta}}{x!} \frac{\lambda^y e^{-\lambda}}{y!}$$

Let $U = X+Y$ and $V = Y$

$$(U,V) = (X+Y, Y)$$
$$g_1(x,y) = x+y \quad ; \quad g_2(x,y) = y$$

A_{uv} : Notice $u = x+y$ and $\boxed{v = y}$

$$\boxed{u - v = x}$$

So g invertible.

$$\underline{A_{u,v}} = g^{-1}(u,v) = \underline{\{(u-v, v)\}}$$

$$\boxed{f_{U,V}(u,v)} = \sum_{\boxed{(x,y)\in A_{u,v}}} f_{X,Y}(x,y)$$

$$= f_{X,Y}(u-v, v) = \boxed{\frac{\theta^{u-v} e^{-\theta}}{(u-v)!} \frac{\lambda^{v} e^{-\lambda}}{v!}}$$

$\boxed{U = X+Y}$ and $\boxed{V = Y}$ Poisson

So what is marginal of U?

$$f_U(u) = \sum_{v=0}^{u} f_{U,V}(u,v) = \sum_{v=0}^{u}$$

aside: $\boxed{\frac{1}{(u-v)!} \frac{1}{v!}} = \boxed{\frac{1}{u!}\binom{u}{v}}$

$$= \frac{e^{-(\theta+\lambda)}}{u!} \underbrace{\sum_{v=0}^{u} \binom{u}{v} \theta^{u-v} \lambda^{v}}_{\text{Binomial Theorem: } (\lambda+\theta)^u}$$

$$= \underbrace{\frac{e^{-(\theta+\lambda)}(\theta+\lambda)^u}{u!}}_{\text{Recognize: Poisson}(\theta+\lambda)}$$

Reccognize: Poisson$(\theta+\lambda)$

$U = X+Y \sim \text{Poisson}(\theta+\lambda)$

Theorem: $X \perp Y$, $X \sim \text{Pois}(\theta)$
$Y \sim \text{Pois}(\lambda)$

then $X+Y \sim \text{Pois}(\theta+\lambda)$.

Continuous Case: X, Y continuous r.v.s.

$(U,V) = g(X,Y)$

$U = g_1(X,Y)$, $V = g_2(X,Y)$

- g invertible $\quad g: \mathbb{R}^2 \to \mathbb{R}^2$
- g continuously differentable (in both coordinates)

Then

$$\boxed{f_{U,V}(u,v) = f_{X,Y}(g^{-1}(u,v)) \left|\det\left(\frac{\partial g^{-1}}{\partial(u,v)}\right)\right|}$$

Jacobian

$g: \mathbb{R}^2 \to \mathbb{R}^2$; $g^{-1}: \mathbb{R}^2 \to \mathbb{R}^2$

$$\boxed{g^{-1}(u,v) = (g_1^{-1}(u,v), g_2^{-1}(u,v)) \in \mathbb{R}^2}$$

Univariate: $f_{g(X)}(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$

Generally: $h: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$h(x,y) = (h_1(x,y), h_2(x,y))$

the Jacobian of h is

$$\frac{\partial h}{\partial(x,y)} = \begin{bmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \end{bmatrix}$$

$g^{-1}: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad g^{-1}(u,v) = (g_1^{-1}(u,v), g_2^{-1}(u,v))$

$$\frac{\partial g^{-1}}{\partial(u,v)} = \begin{bmatrix} \frac{\partial g_1^{-1}}{\partial u} & \frac{\partial g_1^{-1}}{\partial v} \\ \frac{\partial g_2^{-1}}{\partial u} & \frac{\partial g_2^{-1}}{\partial v} \end{bmatrix}$$

Deteminants: 2×2 matrix ; $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\det(A) = ad - cb$$

$(U,V) = g(X,Y)$ then

$$f_{u,v}(u,v) = f_{X,Y}\left(g_1^{-1}(u,v), g_2^{-1}(u,v)\right)\left|\det\left(\frac{\partial g^{-1}}{\partial(u,v)}\right)\right|$$

1. find g^{-1}
2. find Jacobian and take determinant
3. plug in above

<u>Ex.</u> $(u,v) = (X+Y, X-Y)$

$$u = g_1(X,Y) = X+Y$$
$$v = g_2(X,Y) = X-Y$$

1. Get g_1^{-1}, g_2^{-1}

$$\frac{u+v}{2} = X \quad \text{and} \quad \frac{u-v}{2} = Y$$

$$g_1^{-1}(u,v) = \frac{u+v}{2}, \quad g_2^{-1}(u,v) = \frac{u-v}{2}$$

2. Jacobian Determinant

$$J = \frac{\partial g^{-1}}{\partial(u,v)} = \begin{bmatrix} \frac{\partial g_1^{-1}}{\partial u} & \frac{\partial g_1^{-1}}{\partial v} \\ \frac{\partial g_2^{-1}}{\partial u} & \frac{\partial g_2^{-1}}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$|\det J| = \left|\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\right| = \left|-\frac{1}{2}\right| = \frac{1}{2}$$

3 Plug in

$$f_{U,V}(u,v) = f_{X,Y}(g_1^{-1}(u,v), g_2^{-1}(u,v))\,|\det J|$$

$$= f_{X,Y}\left(\frac{u+v}{2}, \frac{u-v}{2}\right)\frac{1}{2}$$