Lecture 12- Variance, Moments, MGFs

Thursday, March 5, 2020 11:02 AM

Defn: Variance

$$Var(X) = E[(X - E(X))^2]$$
$$= E[X^2] - E[X]^2$$

Ex. $X \sim Bin(n,p)$

Recall: $E[X] = np$

Saw: $x\binom{n}{x} = n\binom{n-1}{x-1}$

$$E[X^2] = \sum_{x=0}^{n} x^2 f(x)$$
$$= \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x}$$

$y = x-1$

$x = y+1$

$$= \sum_{x=\cancel{0}1}^{n} x\, n\binom{n-1}{x-1} p^x (1-p)^{n-x}$$
$$= \sum_{y=0}^{n-1} (y+1)\, n\binom{n-1}{y} p^{y+1} (1-p)^{n-y-1}$$
$$= np\Bigg(\underbrace{\sum_{y=0}^{n-1} y\binom{n-1}{y} p^y (1-p)^{(n-1)-y}}_{\substack{\text{Sum of } y\cdot\text{pmf of } Bin(n-1,p) \\ = E[Bin(n-1,p)] = (n-1)p}} + \underbrace{\sum_{y=0}^{n-1}\binom{n-1}{y} p^y (1-p)^{(n-1)-y}}_{\substack{\text{Sum of pmf for } Bin(n-1,p) \\ = 1}}\Bigg)$$

$$\boxed{= np((n-1)p + 1)} = E[X^2]$$

So $Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

$$= np((n-1)p+1) - (np)^2$$

$$= np(\cancel{n}-1)p+1-\cancel{n}p)$$

$$\boxed{= np(1-p)}$$

or $S.d.(X) = \sqrt{Var(X)} = \sqrt{np(1-p)}$

Defn: Moment

If r is a pos. integer then the r^{th} moment of a r.v. X is

$$\mu_r \overset{def}{=} \mathbb{E}[X^r].$$

Can also defne the r^{th} central moment as

$$\mu_r' = \mathbb{E}[(X-\mu)^r]$$

$$\mu = \mu_1 = \mathbb{E}[X].$$

Note:

$$\mu_1 = \mathbb{E}[X], \mu_2 = \mathbb{E}[X^2], \mu_3 = \mathbb{E}[X^3], \ldots$$

$\mu_1' = \mathbb{E}[X-\mu] = 0$

$\mu_2' = \mathbb{E}[(X-\mu)^2] = Var(X)$

Defn: Moment Generating Function (MGF)

If X is a r.v. then the MGF is a function

$$M : \mathbb{R} \longrightarrow \mathbb{R}$$

defined for $t \in \mathbb{R}$ as

$$M(t) = \mathbb{E}\left[e^{tX}\right].$$

So long as this expectation exists in some neighborhood of 0.

For continuous r.v.s, $g(x) = e^{tx}$

$$M(t) = \mathbb{E}\left[e^{tX}\right] = \mathbb{E}[g(X)]$$

$$\boxed{= \int_{\mathbb{R}} e^{tx} f(x)\,dx}$$

discrete

$$\boxed{M(t) = \sum_x e^{tx} f(x)}$$

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x

Ex. $X \sim \mathrm{Exp}(\lambda)$

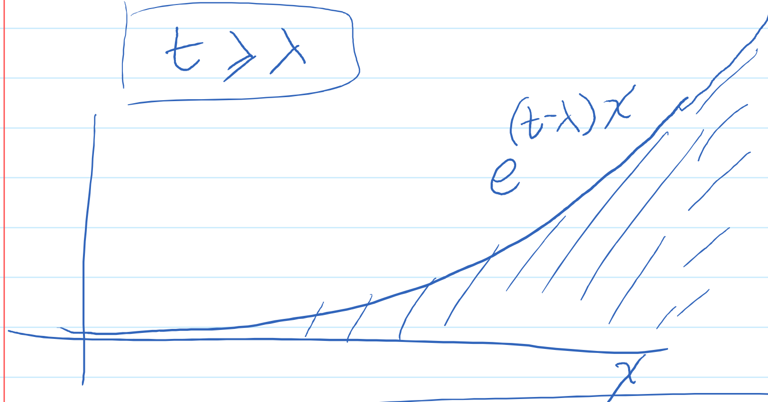
then

$$M(t) = E[e^{tX}] = \int_{\mathbb{R}} e^{tx} f(x)\,dx = \int_0^\infty e^{tx} \lambda e^{-\lambda x}\,dx$$

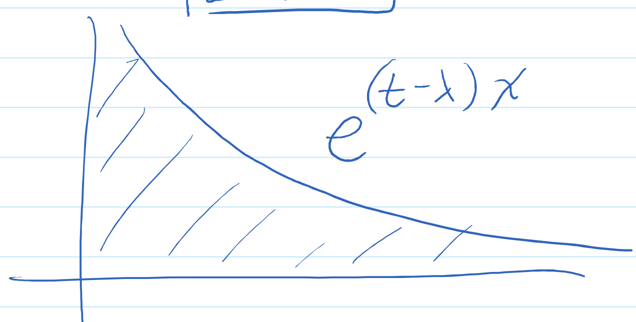
$$= \lambda \int_0^\infty e^{(t-\lambda)x}\,dx = \frac{\lambda}{t-\lambda}\Big[e^{(t-\lambda)x}\Big]_{x=0}^{\infty} = \frac{\lambda}{t-\lambda}[0-1]$$

if $t < \lambda$

$\boxed{t \geq \lambda}$



$\boxed{t < \lambda}$



So $\boxed{M(t) = \dfrac{\lambda}{\lambda - t} \quad \text{for } t < \lambda}$

Consider:

$$\frac{d}{dt} M(t) = \frac{\lambda}{(\lambda - t)^2} \quad \text{and} \quad \left.\frac{dM}{dt}\right|_{t=0} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$= E[X]$$

$$\frac{d^2 M}{dt^2} = \frac{2\lambda}{(\lambda - t)^3} \quad \text{and} \quad \left.\frac{d^2 M}{dt^2}\right|_{t=0} = \frac{2}{\lambda^2} = E[X^2]$$

Theorem:

$$\left.\frac{d^r M}{dt^r}\right|_{t=0} = E[X^r] = \mu_r$$

pf.

$$\frac{d^r M}{dt^r} = \frac{d^r}{dt^r} E[e^{tX}] = \frac{d^r}{dt^r}\int e^{tx} f(x)\,dx$$

$$= \int \frac{d^r}{dt^r} e^{tx} f(x)\,dx$$

$$\boxed{\frac{d^r}{dt^r} e^{tx} = x^r e^{tx}}$$

$$\frac{d}{dt} e^{tx} = x e^{tx}$$

$$\frac{d^2}{dt^2} e^{tx} = \frac{d}{d}(x e^{tx}) = x^2 e^{tx}$$

$$= \int x^r e^{tx} f(x)\,dx$$

$$\left.\frac{d^r M}{dt^r}\right|_{t=0} = \int x^r f(x)\,dx = E[X^r].$$

pf. $e^{tX} = 1 + tX + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \cdots$

Taylor series centered at zero.

So $\frac{d}{dt} e^{tX} = X + \frac{2tX^2}{2!} + \cdots$

$\left.\frac{d}{dt} e^{tX}\right|_{t=0} = X$ so $\left.\frac{d}{dt} E[e^{tX}]\right|_{t=0} = E[X].$

$$\frac{d}{dt} e^{tX}\Big|_{t=0} = X \quad \text{so} \quad \frac{d}{dt} E[e^{tX}]\Big|_{t=0} = E[X].$$

$$\frac{d^r}{dt^r} e^{tX} = X^r + t\,X^{r+1} + \dots \Rightarrow \frac{d^r}{dt^r} E[e^{tX}] = E[X^r].$$

$t = 0$

Ex. $X \sim \text{Bin}(n,p)$

$$M(t) = E[e^{tX}] = \sum_{x=0}^{n} e^{tx} f(x)$$

$$= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$\boxed{= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x}}$$

Binomial Theorem:

Ex. $(x+y)^2 = x^2 + 2xy + y^2 = \binom{2}{0} x^2 y^0 + \binom{2}{1} x^1 y^1 + \binom{2}{2} x^0 y^2$

$$= \sum_{i=0}^{2} \binom{2}{i} y^i x^{2-i}$$

$$\boxed{(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}}$$

Same form...

$$\boxed{= ((1-p) + pe^t)^n}$$

MGF for Bin(n,p)

$$\frac{dM}{dt} = n\left((1-p) + pe^t\right)^{n-1} pe^t$$

$$\frac{dM}{dt} = n\left((1-p)+pe^t\right)\ pe^t$$

$$\left.\frac{dM}{dt}\right|_{t=0} = n\left((1-p)+p(1)\right)^{n-1} p(1)$$
$$= np = E[X]$$

$$\frac{d^2M}{dt^2} = n(n-1)\left((1-p)+pe^t\right)^{n-2} pe^t pe^t$$
$$+ n\left((1-p)+pe^t\right)^{n-1} pe^t$$

$$\left.\frac{d^2M}{dt^2}\right|_{t=0} = n(n-1)p^2 + np$$

$$Var(X) = E[X^2] - E[X]^2 = n(n-1)p^2 + np - n^2p^2$$
$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$
$$= np - np^2 = \boxed{np(1-p).}$$

Theorem: If $a, b \in \mathbb{R}$ and

$$Y = aX + b$$

then
$$\boxed{M_Y(t) = e^{tb} M_X(at)}$$

pf. $M_Y(t) = E[e^{tY}] = E[e^{(aX+b)t}]$

$= e^{tb} E[e^{atX}]$

$$= E[e \quad e \quad]$$

$$= e^{tb} E[e^{atX}]$$

$$= e^{tb} M_X(at)$$

Theorem:

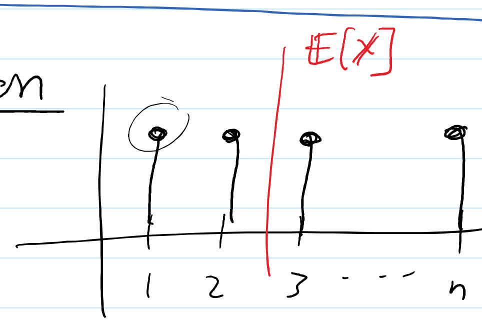
If X and Y are r.v.s. and

$$M_X(t) = M_Y(t) \quad (t \text{ in same neighborhood of zero})$$

then $X \overset{d}{=} Y$.

Discrete Uniform Distribution

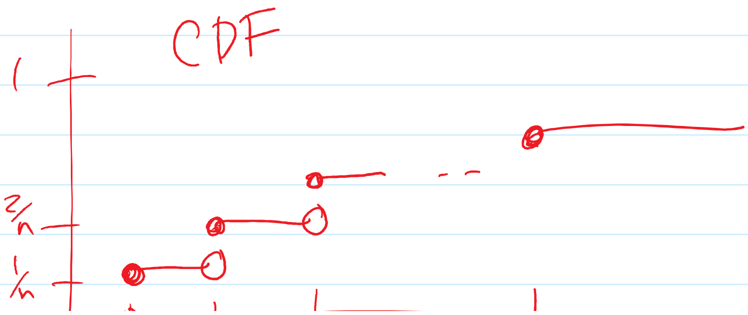
$X \sim U(\{1, .., n\})$



means

$$f(x) = 1/n \quad \text{for } x = 1, .., n \qquad \text{(PMF)}$$

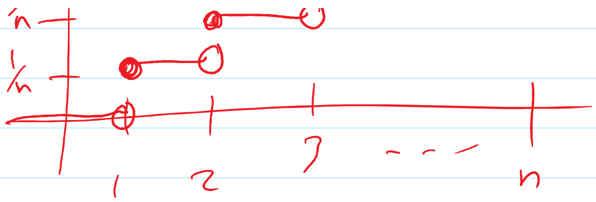
$$F(x) = \sum_{i \le x} f(i) = \begin{cases} 0 & x < 1 \\ 1/n & 1 \le x < 2 \\ 2/n & 2 \le x < 3 \\ \vdots & \\ 1 & x \ge n \end{cases} = \begin{cases} \frac{\lfloor x \rfloor}{n} & x \ge 0 \\ 0 & x < 0 \end{cases}$$



$\lfloor x \rfloor = \text{floor}(x)$
= round down to next integer

$\lfloor 1.5 \rfloor = 1$

$\lfloor .0271 \rfloor = 0$



$\lfloor 1.3 \rfloor = 1$

$\lfloor .027 \rfloor = 0$

$$E[X] = \sum_{x=1}^{n} x f(x) = \sum_{x=1}^{n} x \tfrac{1}{n} = \frac{1}{n}\,\frac{n(n+1)}{2} = \boxed{\frac{n+1}{2}}$$

$$E[X^2] = \cdots = \sum_{x=1}^{n} x^2 \tfrac{1}{n} = \boxed{\frac{1}{n}\,\frac{(n+1)(2n+1)}{6}}$$

$$\mathrm{Var}(X) = E[X^2] - E[X]^2 = \frac{(n+1)(2n+1)}{6n} - \frac{(n+1)^2}{4}$$

$$= \cdots = \boxed{\frac{(N+1)(N-1)}{12}}$$