Lecture 2 - Axiomatic Probability`

Tuesday, January 28, 2020 1:59 PM

Defn: Sample Space

The "sample space" S is the set of possible outcomes for a random experiment.

Ex. Flip a coin

$$S = \{H, T\}$$

Ex. Rolling a six-sided die

$$S = \{1, 2, 3, 4, 5, 6\}.$$

Ex. Roll two six-sided dice.

$$S = \{(1,1), (1,2), (5,3), \ldots\}$$

$$= \{(i,j) \text{ where } 1 \le i,j \le 6\}$$

$1 \le j \le 6$ and $1 \le i \le 6$

Ex. Waiting time for a bus to arrive:

$$S = [0, \infty)$$

Ex. Number of customers arriving at my restaurant

$$S = \mathbb{N}_0 = \{0, 1, 2, 3, 4, \ldots\}$$

types of sample spaces:

1. finite
2. infinite
   - (i) countable (e.g. $\mathbb{N}_0$)
   - (ii) uncountable (e.g. $[0, \infty)$)

<u>Notation</u>: $|A|$ = cardinality of the set A
= # elements (for a finite set)

<u>Defn</u>: <u>Outcome</u>

We call elements of S "outcomes":

e.g. $s \in S$

outcome

sample space

<u>Ex.</u> $S = \{1, 2, 3, \ldots, 6\}$

then 1 is a possible outcome b/c $1 \in S$.

<u>Defn: Event</u>

An <u>event</u> is a subset of the sample space:

$$E \subset S.$$

event

sample space

Ex. Roll a die:

$$S = \{1, 2, \dots, 6\}$$

$$E = \{1, 2\} \subset S$$

rolling 1 or 2

We say an event E occurs if the observed outcome of an experiment is in E.

$F = \{3\}$ event that I roll a 3.

Ex. $S \subset S$, so S is an event.

the event that something happens

$\varnothing \subset S$, so $\varnothing$ is an event.

?? meaning ??

Axiomatic Probability

Given an experiment (a sample space S)

want: assign to each event a measure of its likelihood of occuring its likelihood of occuring → probability

mathematically, for each $E \subset S$ we want to assign a probability $\mathbb{P}(E)$.

What makes a valid prob. function $\mathbb{P}$. ← prob. fn.

Want to define $\mathbb{P}$:

1. to be mathematically consistant
2. to preserve (some) of our intuitions about probability

Defn: Probability Function $\mathbb{P}$

Given a sample space S a prob. fn $\mathbb{P}$ is a function

$$\mathbb{P} : \mathcal{P}(S) \longrightarrow \mathbb{R}$$

that satisfies the Kolmogorov Axioms

1. non-negativity

$$\mathbb{P}(E) \geq 0 \quad \forall E \subset S$$

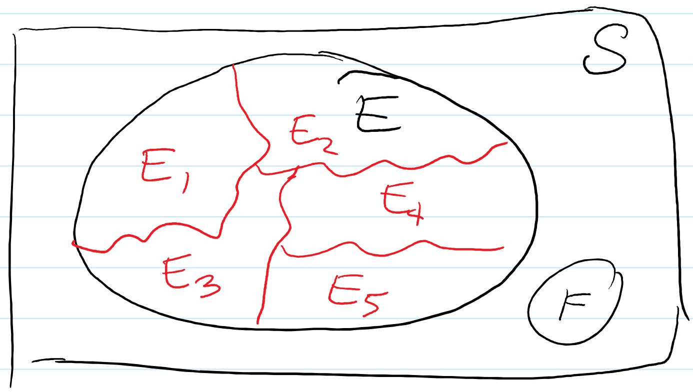
2. unit measure

$$\mathbb{P}(S) = 1$$

③ countable-additivity

$$\mathbb{P}(E) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

for a partition $\{E_i\}_{i=1}^{\infty}$ of E.



Alt. notice $E = \bigcup_{i=1}^{\infty} E_i$ so

$$\mathbb{P}(E) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

basically: countable additivity is a distributive law (for disjoint sets)

i.e. $\mathbb{P}$ is a valid prob. fn if it satisfies the Kolmogorov Axioms.

Ex. Flip a coin

$$S = \{H, T\} \leftarrow$$

What is a valid prob. fn on S?

$\mathbb{P}(\{H\}) = 1/2, \quad \mathbb{P}(\{T\}) = 1/2$

$\mathbb{P}(\underbrace{\{H, T\}}_{S}) = 1 \qquad \mathbb{P}(\emptyset) = 0$

Is this valid? Check Kolmogorov axioms:

(1) $\mathbb{P}(E) \geq 0$ ✓

(2) $\mathbb{P}(S) = 1$ ✓

(3) $\mathbb{P}(\bigcup_i E_i) = \sum_i \mathbb{P}(E_i)$ for disjoint $\{E_i\}$

Let $E_1 = \{H\}$, $E_2 = \{T\}$.

disjoint

$\mathbb{P}(\underbrace{E_1 \cup E_2}_{S}) = \mathbb{P}(\overset{\{H\}}{E_1}) + \mathbb{P}(\overset{\{T\}}{E_2})$

$1 \qquad = \boxed{1/2 + 1/2}$ ✓

think about other cases at home.

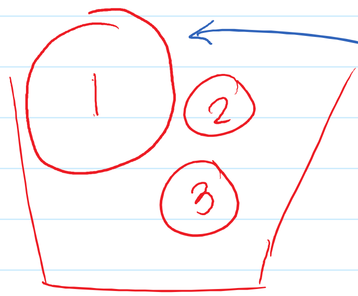
Since $\mathbb{P}$ satisfies the Kolmogorov axioms, it is a valid prob. fn.

Ex. Redefine $\mathbb{P}$

$\mathbb{P}(\{H\}) = .9$ and $\mathbb{P}(\{T\}) = .1$

Q: is this valid $\mathbb{P}$?

Ex. Basket containing 3 ball



ball 1 bigger than ball 2 and 3.

I randomly draw a ball from this basket,

Say: $S = \{1, 2, 3\}$.

defn $\mathbb{P}$: $\mathbb{P}(\{1\}) = 1/2$

$\mathbb{P}(\{2\}) = \mathbb{P}(\{3\}) = 1/4$

Claim: this will define a valid prob. fn.

Theorem: Finite Sample Space Theorem

Let $S = \{s_1, s_2, s_3, \dots, s_n\}$ i.e. $|S| = n$ finite

and choose a set of numbers

$p_1, p_2, p_3, \dots, p_n$

So that $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$.

Ex. $\mathbb{P}(\{s_1\}) = p_1$

$\mathbb{P}(\{s_2\}) = p_2$

$\vdots$

$\mathbb{P}(\{s_n\}) = p_n$

$\mathbb{P}(\{s_1\} \cup \{s_2\}) = \mathbb{P}(\{s_1\}) + \mathbb{P}(\{s_2\})$

$\mathbb{P}(\{s_1, s_2\}) = p_1 + p_2$

or $\mathbb{P}(\{s_1, s_3\}) = p_1 + p_3$

$\mathbb{P}(\{s_1, s_7, s_{11}, s_{15}\}) = p_1 + p_7 + p_{11} + p_{15}$

my defn of $\mathbb{P}$ is

$$\mathbb{P}(E) = \sum_{i : s_i \in E} p_i$$

$i : s_i \in E$

This is a valid prob. fn.

Pf. must show $\mathbb{P}$ satisfies the Kolmogorov axioms:

(1) $\boxed{\mathbb{P}(E) \geq 0}$

$\mathbb{P}(E) = \sum_{\cancel{i : s_i \in E}} p_i$ = sum of some stuff ≥ 0

≥ 0

(2) $\boxed{\mathbb{P}(S) = 1}$

$\{i \text{ s.t. } s_i \in S\}$

all i satisfy this

② $\boxed{\mathbb{P}(S) = 1}$

all i satisfy this

$$P(S) = \sum_{i : s_i \in S} p_i$$

$$= \sum_{i=1}^{n} p_i = 1.$$

③ $\boxed{\mathbb{P}\left(\bigcup_i E_i\right) = \sum_i \mathbb{P}(E_i)}$

$\{E_i\}$ disjoint