Lecture 2 - Axiomatic Probability

Tuesday, January 28, 2020 11:05 AM

Pairwise disjoint:

For $\{A_i\}_{i=1}^{\infty}$ they are (pairwise) disjoint if

$$A_iA_j = \emptyset \text{ for } i \neq j$$

For a finite sequence

$\{B_i\}_{i=1}^{N}$ if $B_iB_j = \emptyset$ for $i \neq j$

can always extend this sequence so that

$$B_i = \emptyset \text{ for } i > N$$

Then this extended seq. is pairwise disjoint.

Empty set $\emptyset$ is disjoint from all other sets A

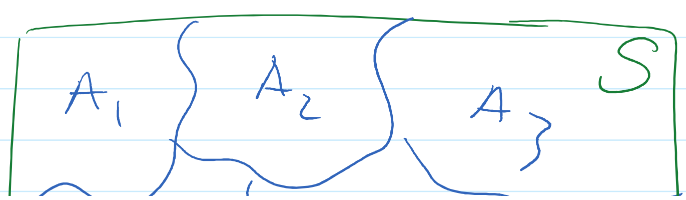
since $A\emptyset = \emptyset$.

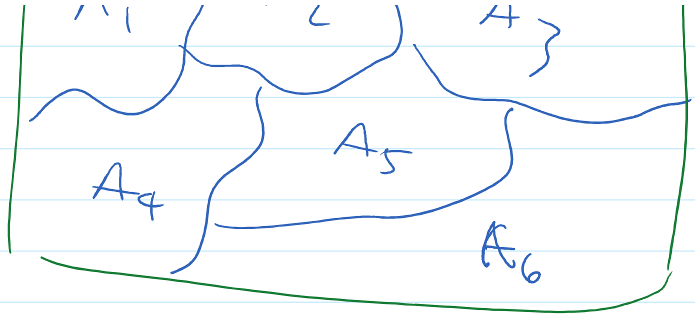
Defn: Partition

If $\{A_i\}_{i=1}^{\infty}$ where $A_i \subset S$ we say that my seq. are a partition of S if

① $\{A_i\}_{i=1}^{\infty}$ are mutually disjoint

② $\bigcup_{i=1}^{\infty} A_i = S$





Defn: Power Set

The power set of A, denoted $\mathcal{P}(A)$, is the collection (set) of all subsets of A.

$$\mathcal{P}(A) = \{B \mid B \subset A\}$$

Ex. $A = \{1, 2\}$ then

$$\mathcal{P}(A) = \{\underbrace{\emptyset, \{1\}, \{2\}, A}_{\text{\# elements (cardinality)} = 4 = 2^{|A|}}\}$$

$|A| = 2$ ← card. of A = # of elements in A

In general, $|2^A| = 2^{|A|}$.

Probability

Defn: Sample Space

The "sample space" S of an experiment is the set of all possible outcomes.

Ex. Flip a coin.

$$S = \{H, T\}$$

Ex. Roll a six-sided die

$$S = \{1, 2, 3, 4, 5, 6\}.$$

Ex. Roll two six-sided dice.

$$S = \{(1,1), (1,2), (3,4), \ldots\}$$
$$= \{(i,j) \text{ where } 1 \leq i,j \leq 6\}.$$

Ex. Waiting time for a bus to arrive:

$$S = [0, \infty)$$

Ex. Number of customers arriving at restaurant:

$$S = \mathbb{N}_0 = \{0, 1, 2, 3, 4, \ldots\}$$

Types of Sample spaces:

1. finite sample spaces
2. infinite
   1. Countably infinite (e.g. $\mathbb{N}_0$)
   2. uncountably infinite (e.g. $[0, \infty)$)

Defn: Outcome

An outcome is an element of the sample space:

$$\underbrace{s}_{\text{outcome}} \in \underbrace{S}_{\text{sample space}}$$

Ex. Rolling 6-sided die

$1 \in S$ so 1 is an outcome.

Define: Event

An event is a subset of S.

i.e. $\underbrace{E}_{\text{event}} \subset \underbrace{S}_{\text{sample space}}$.

Ex. Roll two dice

$$S = \{(i,j) \text{ where } 1 \le i,j \le 6\}$$

$$E = \text{"doubles"} = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$$

an event.

$\subset S$

Ex. $S \subset S$ so S is an event.

the event that something happens.

$\emptyset \subset S$ so $\emptyset$ is an event

the event that nothing happens (?)

Axiomatic Probability

Given an experiment (and hence a sample space S) we want to assign a measure of the likelihood of any event $E \subset S$.

probability

So for each $E \subset S$ we will assign a probability $\mathbb{P}(E)$.

Want to build $\mathbb{P}$ in a way that

1. makes mathematical sense
2. makes (some) intuitive sense.

Defn: Probability Function $\mathbb{P}$

Given a sample space S a prob. fn $\mathbb{P}$ is a fn

$$\mathbb{P} : \mathcal{P}(S) \longrightarrow \mathbb{R}$$

all possible $E \subset S$

following the Kolmogorov Axioms:

1. non-negativity

$$\mathbb{P}(E) \geq 0 \quad \forall E \subset S$$

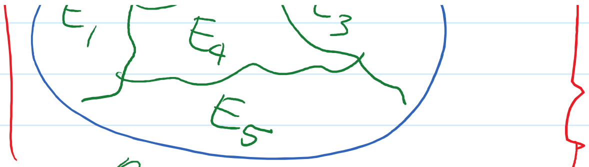
2. unit-measure

$$\mathbb{P}(S) = 1.$$

3. countable-additivity

$$\boxed{\mathbb{P}(E) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)}$$

$P(E) = \sum_{i=1} P(E_i)$

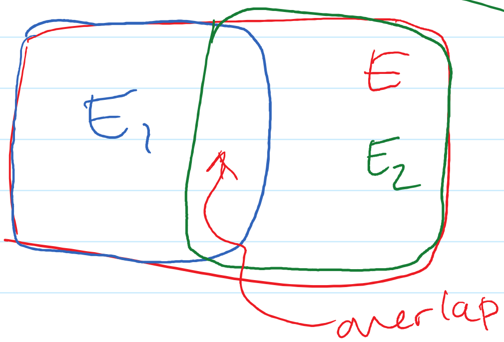


E is partitioned in $\{E_i\}_{i=1}^{\infty}$

① $E = \bigcup_{i=1}^{\infty} E_i$

② $\{E_i\}$ mutually disjoint.

$E = \bigcup_i E_i$



another way:

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i).$$

Ex. Flip a coin

$$S = \{H, T\}$$

what is a valid prob. fn on S?

Consider: $P(\{H\}) = 1/2$ $\quad P(S) = 1$

$P(\{T\}) = 1/2$ $\quad P(\emptyset) = 0$

Is this a valid prob. fn?

does this satisfy the Kolmogorov Axioms?

Axiom 1: all probs ≥ 0 ✓

Axiom 2: $\mathbb{P}(S) = 1$ ✓

Axiom 3: $E_1 = \{H\}$, $E_2 = \{T\}$

$$E = E_1 \cup E_2 = \{H, T\} = S$$

need:

$$\left.\begin{aligned} 1 = \mathbb{P}(S) = \mathbb{P}(E) &= \mathbb{P}(E_1 \cup E_2) \\ &\overset{?}{=} \mathbb{P}(E_1) + \mathbb{P}(E_2) \\ &= 1/2 + 1/2 \end{aligned}\right\} \checkmark$$

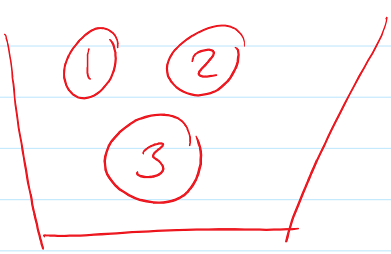
Since $\mathbb{P}$ satisfies the Kolmogorov Axioms it is a valid way of defining $\mathbb{P}$.

Ex. what if I define

$$\mathbb{P}(\{H\}) = \underbrace{.9 \text{ and } \mathbb{P}(\{T\}) = .1}_{\text{un-fair coin.}} \;?$$

Check yourself this works.

Ex. Basket w/ 3 balls



I will draw one ball randomly from the basket.

$$S = \{1, 2, 3\}$$

Maybe my balls in the basket are different sizes.

So I don't have an equal prob. of choosing each.

"Define my probs. (i.e. define $\mathbb{P}$) so that

$$\mathbb{P}(\{1\}) = 1/2, \quad \mathbb{P}(\{2\}) = 1/4, \quad \mathbb{P}(\{3\}) = 1/4."$$

Q: is this a valid way of specifying $\mathbb{P}$?
i.e. we list outcomes and probs that sum to 1.

A: Yes.

Theorem: Finite Sample Space Theorem

If $S = \{s_1, s_2, s_3, \dots, s_n\}$ is a finite sample space w/ n outcomes.

Let $p_1, p_2, \dots, p_n$ be a seq. of numbers so that (i) $p_i \geq 0$ and (ii) $\sum_{i=1}^{n} p_i = 1$.

Define $\mathbb{P}$ so that for any $E \subset S$

$$\mathbb{P}(E) = \sum_{i: s_i \in E} p_i$$

$i: s_i \in E$

$\mathbb{P}(\{s_1\}) = p_1, \quad \mathbb{P}(\{s_2\}) = p_2$

$\mathbb{P}(\{s_1, s_2\}) = p_1 + p_2$

$= \mathbb{P}(\{s_1\}) + \mathbb{P}(\{s_2\}) = p_1 + p_2$

('true by additivity of $\mathbb{P}$

$$\mathbb{P}(\{s_1, s_2, s_3\}) = p_1 + p_2 + p_3$$

$$\mathbb{P}(\{s_1, s_2, s_5\}) = p_1 + p_2 + p_5$$

Theorem says $\mathbb{P}$ is a valid prob. fn.