6.145 Wes is enrolled in four classes this semester. The final grades in each class are mutually independent random variables with an identical probability distribution described by:

$$P(A) = 0.5 \qquad P(B) = 0.3 \qquad P(C) = 0.2.$$

An A counts for 4 points, a B counts for 3 points, and a C counts for 2 points when calculating a grade point average (GPA).

(a) Find his expected GPA this semester.

(b) Find the population variance of his GPA this semester.

(c) Find the probability that Wes will get two As, one B, and one C this semester.

6.146 The weather on any day can take on one of two states: sunny and rainy. This Sunday has a 30% chance of rain. On the six days that follow Sunday,

- the probability that it is rainy is 0.6 if it is rainy on the previous day,
- the probability that it is rainy is 0.2 if it is sunny on the previous day.

What is the probability mass function of the number of rainy days during the next week?

6.147 Let X_1, X_2, and X_3 have joint moment generating function

$$M(t_1, t_2, t_3) = E\left[e^{t_1X_1 + t_2X_2 + t_3X_3}\right] = \left(\frac{1}{1-t_1}\right)\left(\frac{1}{1-t_2}\right)\left(\frac{1}{1-t_3}\right)$$

for $t_1 < 1$, $t_2 < 1$, and $t_3 < 1$. Are X_1, X_2, and X_3 mutually independent?

6.148 Let X_1, X_2, and X_3 have joint cumulative distribution function

$$F(x_1, x_2, x_3) = \begin{cases} 0 & x_1 < 4 \text{ or } x_2 < 5 \text{ or } x_3 < 6 \\ 1 & x_1 \geq 4 \text{ and } x_2 \geq 5 \text{ and } x_3 \geq 6. \end{cases}$$

Find $E[X_1X_2X_3]$.

6.149 Intelligence quotient (IQ) scores are normally distributed random variables with population mean 100 and population standard deviation 10. If nine children are tested, find the probability that two have scores that are less than 90, three have scores between 90 and 110, and four have scores higher than 110.

6.150 Let X_1, X_2, X_3 be random variables with variance–covariance matrix

$$\begin{bmatrix} 8 & 5 & 3 \\ 5 & 7 & 4 \\ 3 & 4 & 6 \end{bmatrix}.$$

Find the population variance of $X_1 + X_2 + X_3$.

6.151 Consider three consecutive sojourn times in a single-server queueing node: X_1, X_2, X_3. The population means of these three sojourn times are all equal to 8.1, and their variance–covariance matrix is

$$\Sigma = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 4 \end{bmatrix}.$$

If $\bar{X}$ is the sample mean of the three sojourn times, find $E\left[\bar{X}\right]$ and $V\left[\bar{X}\right]$.

6.152 Let $X_i \sim N(\mu_i, \sigma_i^2)$, $i = 1, 2, 3, 4$. Also, X_1, X_2, X_3, and X_4 are mutually independent random variables. If $F_{X_i}(x)$ is the cumulative distribution function of a normal random variable with population mean μ_i and population variance σ_i^2, for $i = 1, 2, 3, 4$, give an expression for the probability that *exactly one* of the four X_i values is less than k, where k is a real constant.

6.153 Let X_i have a Weibull distribution with $\lambda = 0.05$ and $\kappa = 2.5$ for $i = 1, 2, \ldots, 1000$, which denote the mutually independent lifetimes, in months, of light bulbs lighting the interior of a factory.

(a) What is the probability that one particular bulb survives six months?

(b) What is the expected number of bulbs that survive six months?

(c) What is the time when you would expect that 25% of the bulbs will be burned out?

6.154 Let X_1, X_2, and X_3 be mutually independent and identically distributed random variables each having moment generating function

$$M(t) = \frac{1}{2}e^t + \frac{1}{3}e^{2t} + \frac{1}{6}e^{5t}$$

for all real values of t.

(a) Find $V[X_1]$.

(b) Find $P(X_1 + X_2 + X_3 = 4)$.

6.155 Let X, Y, and Z be random variables with joint probability density function $f(x, y, z)$ with support on the interior of the unit sphere. Set up an expression (with appropriate integration limits) for $E[X + \cos Y \mid Z = z]$ in terms of the joint probability density function $f(x, y, z)$.

6.156 Let X_1, X_2, X_3, X_4 be mutually independent and identically distributed Bernoulli(p) random variables. Find $P(X_1 = X_2 = X_3 = X_4)$.

6.157 Let X_1, X_2, X_3, X_4 be mutually independent and identically distributed Bernoulli(p) random variables. Find $P(\bar{X} = 1)$, where $\bar{X}$ is the sample mean $\bar{X} = (X_1 + X_2 + X_3 + X_4)/4$.

6.158 Let X_1, X_2, X_3, X_4 be mutually independent and identically distributed Bernoulli(p) random variables. Find $E[X_1 X_2 X_3 X_4]$.

6.159 Let X_1, X_2, X_3, X_4 be mutually independent and identically distributed Bernoulli(p) random variables. Find $V\left[X_1 X_2^2 X_3^3 X_4^4\right]$.

6.160 Let X_1, X_2, X_3, X_4 be mutually independent and identically distributed Bernoulli(p) random variables. Find $P(X_1 \le X_2 \le X_3 \le X_4)$.

6.161 Let X_1, X_2, X_3, X_4 be mutually independent and identically distributed Bernoulli(p) random variables. Find the moment generating function of $X_1 X_2 X_3 X_4$.

6.162 Let X_1, X_2, X_3, X_4 be mutually independent and identically distributed Bernoulli(p) random variables. Find the moment generating function of $X_1 + X_2 + X_3 + X_4$.

6.163 Let X_1, X_2, X_3, X_4 be mutually independent and identically distributed Bernoulli(p) random variables. Find the moment generating function of the sample mean

$$\bar{X} = \frac{X_1 + X_2 + X_3 + X_4}{4}.$$

6.164 Mustafa draws four cards with replacement from a well-shuffled deck. Find the probability that all suits are represented in the sampling.

6.165 Let the joint probability density function for X_1, X_2, X_3 be

$$f(x_1, x_2, x_3) = \lambda_1 \lambda_2 \lambda_3 e^{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_3} \qquad x_1 > 0, x_2 > 0, x_3 > 0,$$

where $\lambda_1, \lambda_2, \lambda_3$ are positive real constants.

(a) Find $P(X_1 < X_2)$.

(b) Find $P(X_1 < X_2 < X_3)$.

6.166 Let T_1, T_2, T_3 be the mutually independent component lifetimes associated with the system of components arranged so that the system lifetime T is

$$T = \max\{T_1, \min\{T_2, T_3\}\}.$$

If $T_1 \sim$ exponential(λ_1), $T_2 \sim$ exponential(λ_2), and $T_3 \sim$ exponential(λ_3), where λ_1, λ_2, and λ_3 are positive failure rates, find an expression for the population mean time to system failure

(a) by hand,

(b) using APPL.

6.167 Elsayed bets a \$1 bill on each of 100 repeated games of craps. What are his expected earnings?

6.168 What is the name of the distribution of the sum of n mutually independent Bernoulli(p) random variables?

6.169 What is the name of the distribution of the sum of the following two independent random variables: $X \sim$ binomial(n, p) and $Y \sim$ binomial(m, p)?

6.170 What is the name of the distribution of the sum of n mutually independent and identically distributed geometric(p) random variables?

6.171 What is the name of the distribution of the sum of the squares of n mutually independent standard normal random variables?

6.172 Let $X_1, X_2, \ldots, X_n$ be mutually independent lifetimes of the components in a series system. Assume each has probability density function

$$f_X(x) = 1 \qquad 0 < x < 1.$$

Let Y be the system lifetime. Find the probability density function of Y.

6.173 Let X_1, X_2, X_3, X_4, X_5 be mutually independent lug nut lifetimes. Assume that each lug nut has probability density function

$$f_X(x) = 2x \qquad 0 < x < 1.$$

The lug nuts are arranged as a five-component parallel system. Find the probability density function of the system lifetime $Y = \max\{X_1, X_2, \ldots, X_5\}$.

6.174 Let X_1, X_2, and X_3 be mutually independent random variables such that $X_1 \sim N(1, 1)$, $X_2 \sim N(4, 4)$, and $X_3 \sim N(9, 9)$. Find the population variance of $X_1 + 5X_2 - 4X_3$.

6.175 Let X_1, X_2, X_3 have the trivariate normal distribution with population mean vector

$$\mu = (4, 5, 6)'$$

and variance–covariance matrix

$$\Sigma = \begin{bmatrix} 7 & 2 & 0 \\ 2 & 8 & 3 \\ 0 & 3 & 9 \end{bmatrix}.$$

Give the name of the distribution and the associated parameter values of the joint marginal distribution of X_2 and X_3.

6.176 Ted has a piece of licorice of unit length. Bill makes sequential mutually independent and identically distributed demands for a length of Ted's licorice X, each with probability density function

$$f(x) = e^{-x} \qquad x > 0.$$

Ted always satisfies Bill's demand if he is able to do so. Bill's demands stop once Ted is not able to satisfy a demand. Let the random variable N be the number of Bill's demands that Ted is able to successfully satisfy. Find $E[N]$.

6.177 Give the APPL statements and associated output to find the probability that the sum of the spots showing in a roll of five fair dice is 17. Check your solution by executing a Monte Carlo simulation.

6.178 Dr. Foote is a podiatrist. He has perused the podiatry literature and found the following population probabilities associated with patients having toe problems.

Toe	Probability
Big toe	0.56
Index toe	0.08
Middle toe	0.07
Ring (?!) toe	0.07
Pinky toe	0.22

If Dr. Foote sees six patients with complaints about one of their toes, find the probability that all of the five different toes are represented in his sample.

6.179 A bag contains 5 balls numbered 1, 2, 3, 4, 5. Arno selects three balls at random and with replacement from the bag. The numbers on the balls selected are X_1, X_2, and X_3. Find the population mean and population variance of

$$Y = X_1 + X_2 + X_3.$$

Support your results by executing a Monte Carlo simulation experiment.

6.180 Seth rolls a fair die repeatedly. Find the expected number of rolls until a six appears, conditioned on all rolls prior to the six being even-valued results.

6.181 A bag contains 5 balls numbered 1, 2, 3, 4, 5. Hillel selects three balls at random and without replacement from the bag. The numbers on the balls selected are X_1, X_2, and X_3. Find the population mean and population variance of

$$Y = X_1 + X_2 + X_3.$$

Support your results by executing a Monte Carlo simulation experiment.

6.182 Let (X_1, Y_1) and (X_2, Y_2) be two points chosen at random from the interior of a unit square with opposite vertices $(0, 0)$ and $(1, 1)$. Find

$$E\left[(X_1 - X_2)^2 + (Y_1 - Y_2)^2\right].$$

6.183 Professor Monte teaches a graduate course in probability at Riverwater College. He ignores a student's performance throughout the semester and assigns each student an A, B, or C with equal probability. Professor Monte's courses are not very popular. If six students are currently enrolled in his course, find the probability that he will assign two As, two Bs and two Cs.

6.184 Let X_1, X_2, X_3 be random variables with population mean vector and variance–covariance matrix

$$\mu = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \qquad \text{and} \qquad \Sigma = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 4 \end{bmatrix}.$$

(a) Find $E[X_1 - 8X_2 + 9X_3]$.

(b) Find $V[X_1 - 8X_2 + 9X_3]$.

6.185 Let $X_1 \sim U(0, 1)$, $X_2 \sim U(0, 2)$, $X_3 \sim U(0, 3)$ be mutually independent random variables.

(a) Find the joint probability density function of X_1, X_2, X_3.

(b) Find the probability density function of $\max\{X_1, X_2, X_3\}$.

6.186 Let the random variables X, Y and Z have a trivariate probability distribution with population mean vector $\boldsymbol{\mu} = (0, 1, 2)'$ and variance–covariance matrix

$$\Sigma = \begin{bmatrix} 7 & 3 & 4 \\ 3 & 8 & 5 \\ 4 & 5 & 9 \end{bmatrix}.$$

Let $W = 2X - Y + 3Z$.

(a) Find $E[W]$.

(b) Find $V[W]$.

6.187 The random variables X_1, X_2, and X_3 satisfy $P(X_1 < X_2) = 0.8$ and $P(X_2 < X_3) = 0.7$. Give the range of possible values of $P(X_1 < X_2 < X_3)$.

6.188 Let X_1, X_2, X_3 be mutually independent exponential(1) random variables.

(a) Find $P(X_1 + X_2 < X_3)$.

(b) Find $P(X_{(1)} + X_{(2)} < X_{(3)})$.

6.189 Let X_1, X_2, X_3, X_4 be mutually independent and identically distributed random variables drawn from a common continuous probability distribution which has finite population median m. Find $P(\min\{X_1, X_2, X_3, X_4\} < m < \max\{X_1, X_2, X_3, X_4\})$.

6.190 A bag contains four balls, numbered 1, 2, 3, and 4. Sharon draws three balls from the bag at random and without replacement. Let X_1 be the number on the first ball drawn, X_2 be the number on the second ball drawn, and X_3 be the number on the third ball drawn.

(a) Find $E[X_1]$, $E[X_2]$, and $E[X_3]$.

(b) Find the variance–covariance matrix of X_1, X_2, and X_3.

6.191 Let X_1, X_2, X_3 be mutually independent random variables from a $U(0, 10)$ distribution. Write R code that performs a Monte Carlo simulation experiment to estimate

$$E\left[X_1 \,\middle|\, X_{(3)} < 8\right]$$

to the nearest hundredth, where $X_{(3)} = \max\{X_1, X_2, X_3\}$.

6.192 Let X_1, X_2, X_3 be mutually independent and identically distributed random variables drawn from a continuous population distribution. Let Y_1, Y_2, Y_3 also be mutually independent and identically distributed random variables drawn from the same continuous population distribution as the first random sample. The second random sample Y_1, Y_2, Y_3 is independent of the first random sample X_1, X_2, X_3.

(a) Find the probability mass function of S, the number of pairs (X_i, Y_i) that satisfy $X_i < Y_i$ for $i = 1, 2, 3$.

(b) Consider the three random variables

$$Z_1 = |X_1 - Y_1| \qquad Z_2 = |X_2 - Y_2| \qquad Z_3 = |X_3 - Y_3|.$$

Assign rank 1 to the smallest Z_i value; assign rank 2 to the middle Z_i value; assign rank 3 to the largest Z_i value. Find the probability mass function of W, the sum of the ranks associated with pairs (X_i, Y_i) that satisfy $X_i < Y_i$ for $i = 1, 2, 3$.

6.193 A fair die has the numbers 1, 2, ..., 6 written on each face. A second fair die has the numbers 10, 20, ..., 60 written on each face. A third fair die has the numbers 100, 200, ..., 600 written on each face. Xi rolls the three dice. What is the probability mass function of the sum of the numbers appearing on the up faces?

6.194 The mutually independent continuous random variables X_1, X_2, and X_3 each have probability density function

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 1/4 & 1 \le x < 3. \end{cases}$$

Find the probability that

(a) all three random variables fall between 1/2 and 2,

(b) exactly two out of the three random variables fall between 1/2 and 2.

6.195 For an n-dimensional multivariate normal distribution having a population mean vector $(\mu_1, \mu_2, \ldots, \mu_n)'$ and variance–covariance matrix I, where I is the $n \times n$ identity matrix, find the largest value of the joint probability density function, that is, find $f(\mu_1, \mu_2, \ldots, \mu_n)$.

6.196 How many parameters are required to specify the n-dimensional multivariate normal distribution?