Lecture 9 - PMFs and PDFs

Tuesday, February 25, 2020   2:00 PM

Defn: Identically Distributed R.V.s

$$X \overset{d}{=} Y \text{ if } P(X \in A) = P(Y \in A) \text{ for all } A \subset S$$

Ex. $X = \#$ heads in 3 flips

$Y = \#$ tails in 3 flips

e.g. $P(X=1) = 3/8 = P(Y=1)$

however

HTT $X = 1$ and $Y = 2$.

Theorem: $X \overset{d}{=} Y$ iff $F_X = F_Y$

Ex. Toss coins (independently) until a H appears.

$$S = \{H, TH, TTH, TTTH, \dots\}$$

$$|S| = \infty$$

Let p be the prob. of getting a H on any flip.

and, $X = \#$ of flips to get a H.

S	X

S	X
H	1
TH	2
TTH	3
$\vdots$	$\vdots$

Q: What is the CDF of X?

$$F(x) = \mathbb{P}(X \le x)$$

To determine F, let's consider

$$\mathbb{P}(X = x)$$

prob. we make x flips to get first H

T_i = getting a T on i^{th} flip

$H_i = T_i^c$ = getting H " "

then "$X = x$" $= T_1 T_2 T_3 \cdots T_{x-1} H_x$

so

$$\mathbb{P}(X = x) = \mathbb{P}(T_1 T_2 \cdots T_{x-1} H_x)$$

$$= \mathbb{P}(T_1)\mathbb{P}(T_2)\cdots\mathbb{P}(T_{x-1})\mathbb{P}(H_x) \quad \text{by independence}$$

$$= (1-p)(1-p)\cdots(1-p)p$$

$$= (1-p)^{x-1}p$$

If W_i = make i flips to get first H
= "$X = i$"

claim:
disjoint union

$$\text{“}X \le x\text{”} = \omega_1 \cup \omega_2 \cup \cdots \cup \omega_x$$

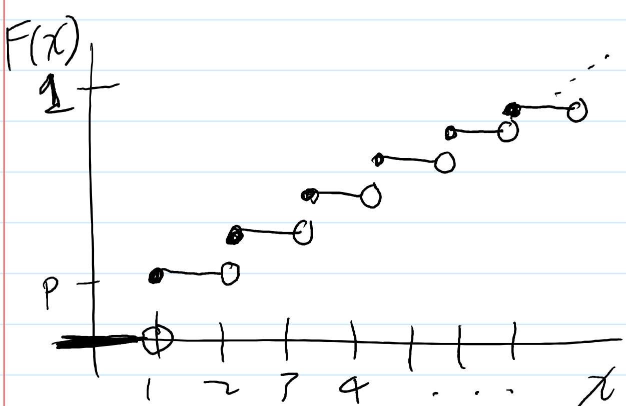
disjoint union

$$\begin{aligned}
F(x) = \mathbb{P}(X \le x) &= \mathbb{P}(\omega_1 \cup \omega_2 \cup \cdots \cup \omega_x) \\
&= \sum_{i=1}^{x} P(\omega_i) \\
&= \sum_{i=1}^{x} \mathbb{P}(X = i) \\
&= \sum_{i=1}^{x} (1-p)^{i-1} p \\
&= p \sum_{i=0}^{x-1} (1-p)^{i} \\
&= \cancel{p}\, \frac{1-(1-p)^x}{\cancel{1-(1-p)}} \\
&= 1-(1-p)^x = F(x)
\end{aligned}$$

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$$

$$\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$$

$r = 1-p$



We call this type of random variable a Geometric r.v.

We calculated the CDF by breaking it down into a sum of $\boxed{\mathbb{P}(X = x)}$ for each x

Defn: Probability Mass Function (PMF).

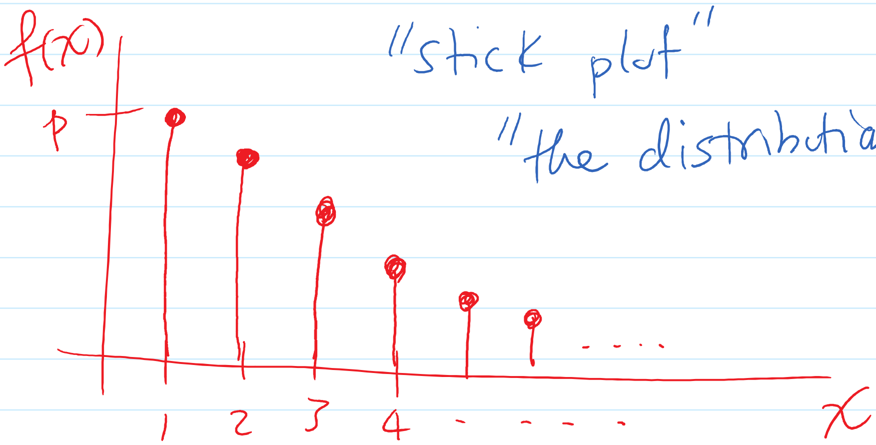
For a discrete random variable we call

$$f(x) = P(X=x)$$

the PMF. (Called the distribution of X)

Ex. For the geometric r.v.

$$f(x) = P(X=x) = (1-p)^{x-1}p$$



"stick plot"

"the distribution of X"

Theorem:

$$\boxed{F(x) = \sum_{i \le x} f(i)}$$

$P(X \le x)$ $\qquad\qquad$ $P(X=i)$

Pf. $\text{"}X \le x\text{"} = \bigcup_{i \le x} \text{"}X = i\text{"}$ disjoint union

$$F(x) = P(X \le x) = \sum_{i \le x} P(X=i)$$

$$= \sum_{i \le x} f(i)$$

Ex. Discrete Uniform

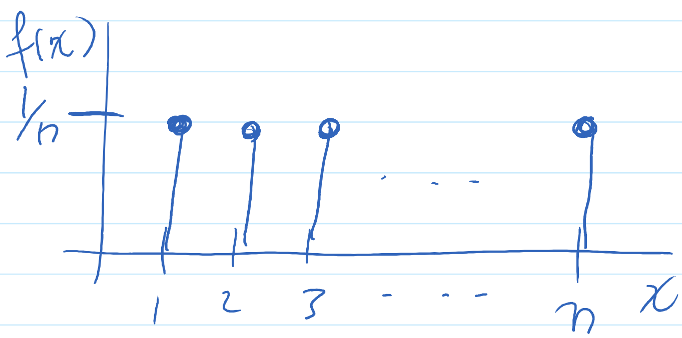
We say X has a discrete uniform distribution over $\{1,\dots,n\}$ denote

$$X \sim U(\underbrace{\{1,\dots,n\}}_{\text{set over which it is uniform}})$$

read: distributed as   uniform   set over which it is uniform

means

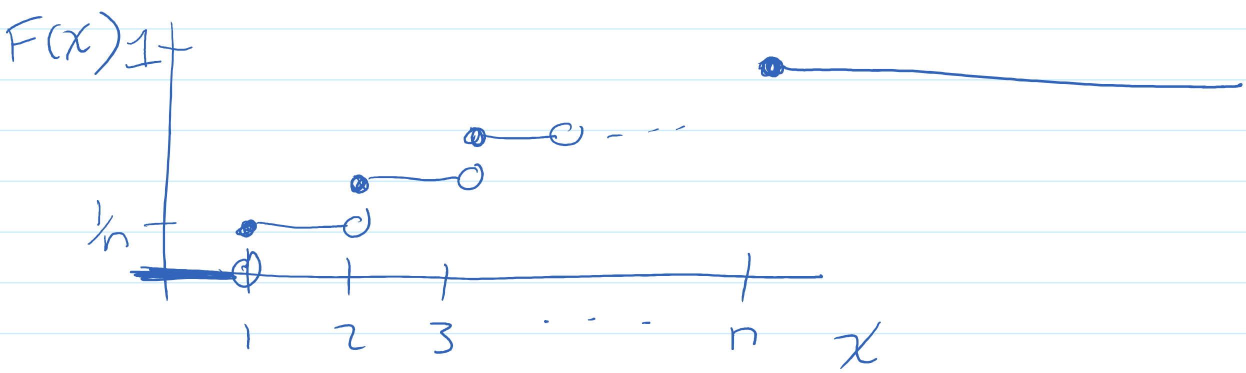
$$f(i) = \frac{1}{n} \quad \text{for } i=1,\dots,n$$



pmf view

Q: what is the CDF?

$$F(x) = \sum_{i=1}^{x} f(i) = \sum_{i=1}^{x} \frac{1}{n} = \frac{x}{n}$$



More generally $A \subset \mathbb{R}$

$$P(X \in A) = \sum_{x \in A} f(x)$$

sum of pmf over values in A

Ex. X has discrete uniform

$$P(2 \le X < 5)$$
$$= P(X \in \underbrace{\{2,3,4\}}_{A})$$
$$= \sum_{x=2,3,4} f(x) = \sum_{x=2,3,4} 1/n = 3/n$$

Ex. $P(X \in \{1,7,3\}) = \sum_{x=1,7,3} 1/n = 3/n$

Ex. Roll a die 60 times. (independently)

X = # of 6s I roll.

Lets derive the PMF.

$f(x) = P(X=x)$ = prob. I roll x 6s in 60 rolls

$$f(0) = P(X=0) = (5/6)(5/6)(5/6)\cdots(5/6)$$
$$= (5/6)^{60}$$

$$f(1) = \mathbb{P}(X=1) = 60\underbrace{(5/6)(5/6)\cdots(5/6)}_{59 \text{ rolls}}(1/6)$$

$$= 60\,(5/6)^{59}(1/6)$$

$$f(2) = \mathbb{P}(X=2) = \binom{60}{2}(1/6)(1/6)(5/6)^{58}$$

$$= \binom{60}{2}(1/6)^2(5/6)^{58}$$

$$\boxed{f(x) = \mathbb{P}(X=x) = \binom{60}{x}\left(1/6\right)^{x}\left(5/6\right)^{60-x}}$$

We call this a Binomial random variable.

I do n experiments each independent w/ a prob. p of success.

X = # of successes

then $X \sim \text{Bin}(n,p)$

above: $n = 60$, $p = 1/6$

What is prob. of getting an even # of 6s.

$\mathbb{P}(X \text{ is even})$

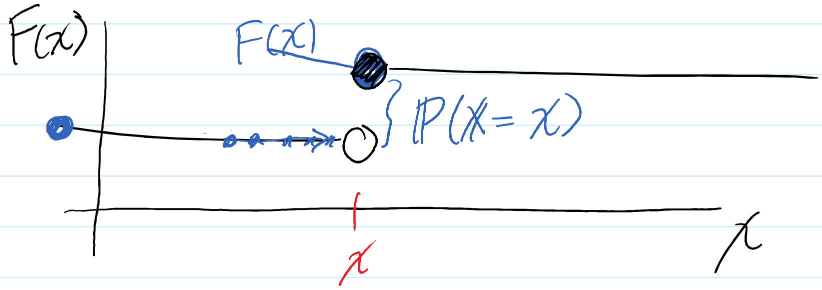
$= \mathbb{P}(X = 2, 4, 6, 8, \ldots, 58, 60)$

$$= \sum_{x=2,4,6,8,\dots,60} f(x)$$

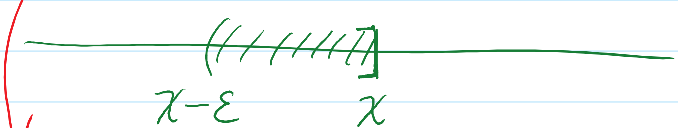
$$= \sum_{x \text{ even}} \binom{60}{x} \left(\tfrac{1}{6}\right)^x \left(\tfrac{5}{6}\right)^{60-x}.$$

Recall:

$$\boxed{\mathbb{P}(a < X \leq b) = F(b) - F(a)}$$



$$\mathbb{P}(X = x) = \lim_{\varepsilon \downarrow 0} \mathbb{P}(x - \varepsilon < X \leq x)$$



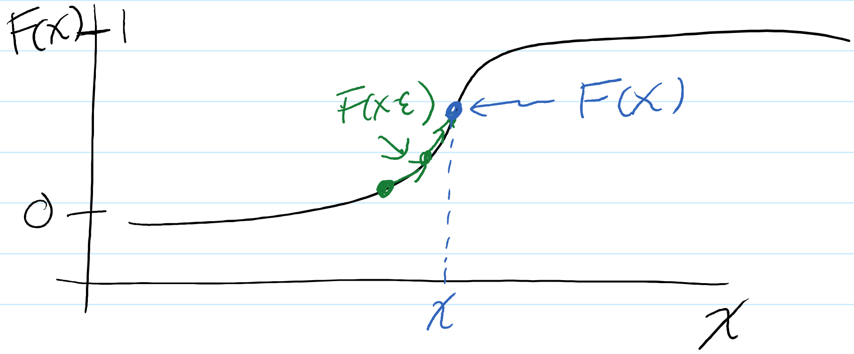
$$= \lim_{\varepsilon \downarrow 0} \left[F(x) - F(x-\varepsilon)\right]$$

$$= \text{gap} = \mathbb{P}(X = x)$$

Another way:

$$F(x) = \sum_{i \leq x} f(i)$$

Same argument for cts r.v.s.



$$P(X=x) = \lim_{\varepsilon \downarrow 0} P(x-\varepsilon < X \le x)$$
$$= \lim_{\varepsilon \downarrow 0} \left[F(x) - F(x-\varepsilon)\right]$$
$$= 0$$

So $\boxed{P(X=x) = 0 \quad \forall x}$

Can't do: define a pmf like the discrete case.

$$f(x) = P(X=x) = 0 \text{ always.}$$

Want: something like PMF for cts. r.v.

for discrete $\boxed{F(x) = \sum_{i \le x} f(i)}$

can we get something like this for cts.

Defn: Probability Density Function (PDF)

Analog of PMF for cts.

The pdf is a function f so that

$$F(x) = \int_{-\infty}^{x} f(t)\,dt.$$

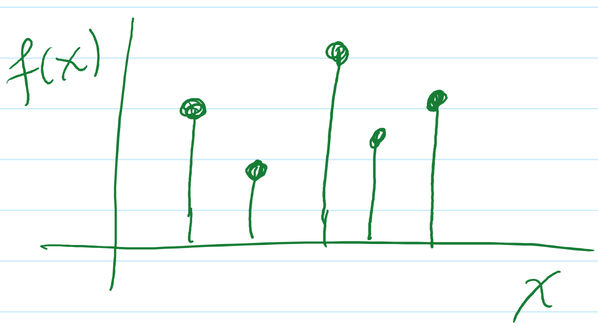
Notice the Fundamental theorem of Calc says

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_{-\infty}^{x} f(t)\,dt = f(x).$$

So $\quad f(x) = \frac{d}{dx}F(x).$

discrete

pmf



continuous

pdf