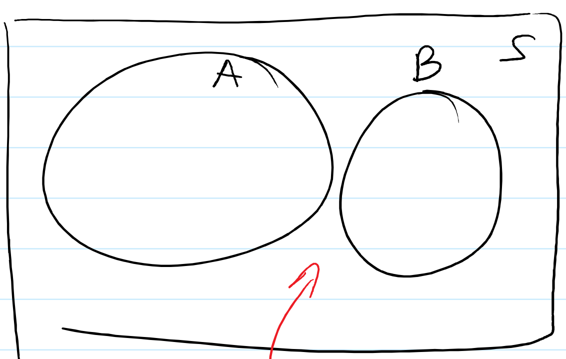
Defn: Independence (of two events)

If $A, B \subset S$ we say "A is independent of B"

denote $A \perp\!\!\!\perp B$, if

$$\mathbb{P}(AB) = \mathbb{P}(A)\mathbb{P}(B) \quad (*)$$

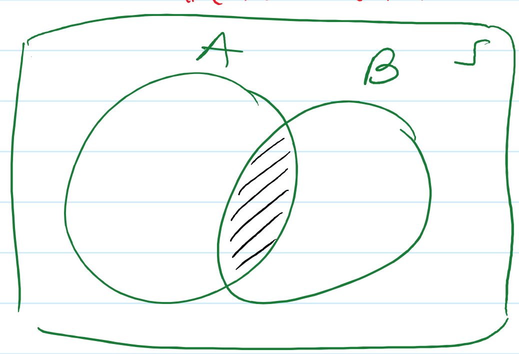
Mutually Exclusive $AB = \emptyset$



no overlap

Independence:

$\mathbb{P}(AB) = \mathbb{P}(A)\mathbb{P}(B)$



ratio of area of A to S
=
ratio of area of AB to B

Theorem: If $A \perp\!\!\!\perp B$ then

$$\mathbb{P}(A|B) = \mathbb{P}(A).$$

Pf: $$\mathbb{P}(A|B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)\cancel{\mathbb{P}(B)}}{\cancel{\mathbb{P}(B)}} = \mathbb{P}(A).$$

Ex. Roll two dice (independently).

What is the prob. we get at least one 6?

$P(\text{at least one } 6)$

$= 1 - P(\text{no } 6s)$ $\quad$ $\big(\text{"no 6 on roll 1"}\big) A_1 \cap \big(\text{"no 6 on roll 2"}\big) A_2$

$= 1 - P(A_1 A_2)$

$= 1 - P(A_1) P(A_2)$ $\quad$ Assume: $A_1 \perp\!\!\!\perp A_2$

$= 1 - \left(\frac{5}{6}\right)\left(\frac{5}{6}\right)$

$= \frac{11}{36}$

Solve from a counting perspective:

<u>Unordered</u>: $S = \{\text{all unordered pairs } \{i, j\} \text{ where } 1 \leq i, j \leq 6\}$

sampling w/o order and w/ replacement

$n = 6, \; r = 2$

$$|S| = \binom{n+r-1}{r} = \binom{6+2-1}{2} = \binom{7}{2}$$

$$= \frac{7!}{2!\,5!} = \frac{7 \cdot 6}{2} = 21$$

$S = \{\{1,1\}, \{1,2\}, \{2,2\}, \{6,1\}, \ldots\}$

$E = \text{"one or more 6"} = \{\{6,6\}, \{6,5\}, \{6,4\}, \{6,3\}, \{6,2\},$

$\{6,13\}$

So $|E| = 6$ hence $\mathbb{P}(E) = \frac{|E|}{|S|} = \frac{6}{21}$

Ordered: w/ ordering w/ replacement: n^r

$S = \{(1,1), (2,2), (1,2), (2,1), \ldots\}$

and so $|S| = 6^2 = 36$

$E = \{(6,6), (6,5), (5,6), (6,4), (4,6), (6,3), (3,6),$
$(6,2), (2,6), (6,1), (1,6)\}$

$|E| = 11$, hence $\mathbb{P}(E) = \frac{|E|}{|S|} = 11/36$.

Takeaway! Ordered counting gives same answer assuming independence.

Theorem: Complementary Independence

If $A \perp\!\!\!\perp B$ then

1. $A \perp\!\!\!\perp B^c$
2. $A^c \perp\!\!\!\perp B$
3. $A^c \perp\!\!\!\perp B^c$

pf. Case 1:

$\mathbb{P}(AB^c) = \mathbb{P}(A) - \mathbb{P}(AB)$
$= \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B)$
$= \mathbb{P}(A)(1 - \mathbb{P}(B))$
$= \mathbb{P}(A)\mathbb{P}(B^c)$.

Defn: Mutual Independence

(generalize independence to multiple events).

If $\{A_i\}_{i=1}^{n}$ are a seq. of events, we say they are mutually independent if for any subsequence of length $k \leq n$

$$A_{i_1}, A_{i_2}, A_{i_3}, \ldots, A_{i_k}$$

$$\mathbb{P}\left(\bigcap_{j=1}^{k} A_{i_j}\right) = \prod_{j=1}^{k} \mathbb{P}(A_{i_j})$$

Ex.

$$\mathbb{P}(A_1 A_3 A_4) = \mathbb{P}(A_1)\mathbb{P}(A_3)\mathbb{P}(A_4)$$

$$\mathbb{P}(A_2 A_7 A_{11} A_{12}) = \mathbb{P}(A_2)\mathbb{P}(A_7)\mathbb{P}(A_{11})\mathbb{P}(A_{12}).$$

$\vdots$ etc. for all subsequences.

Q: Is this the same as

$$\mathbb{P}(A_1 A_2 A_3 \cdots A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2)\cdots\mathbb{P}(A_n) \ ?$$

Ex. Roll two dice.

$|A| = 6$ $A =$ "roll doubles" $= \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

$|B| = 18$ $B =$ "sum is between 7 and 10"

$\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1),(2,6),(3,5),(4,4),(5,3),(6,2),(3,6),(4,5),(5,4),(6,3),(4,6),(5,5),(6,4)\}$

$C =$ "Sum is 2, 7 or 8"

$|C| = 12 \quad = \{(1,1), \ \ldots\}$

Since $|S| = 36$ then

$$\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = \left(\frac{6}{36}\right)\left(\frac{18}{36}\right)\left(\frac{12}{36}\right)$$
$$= \left(\tfrac{1}{6}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{3}\right) = \tfrac{1}{36}$$

$$\mathbb{P}(\underbrace{ABC}_{\{(4,4)\}}) = \tfrac{1}{36}$$

good

Consider $BC =$ "sum is 7 or 8"

$|BC| = 11$ hence

$\mathbb{P}(BC) = 11/36$

$\mathbb{P}(B)\mathbb{P}(C) = \left(\tfrac{1}{2}\right)\left(\tfrac{1}{3}\right) = \tfrac{1}{6} \neq \tfrac{11}{36}$

Fail.

Our condition isn't met for all subsequences.

Hence $\{A, B, C\}$ aren't mutually independent.

Defn: Pairwise Independence

$\{A_i\}_i^n$ are pairwise independent if

$\{A_i\}_{i=1}^{n}$ are pairwise independent if

$$\mathbb{P}(A_i A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j) \quad i \neq j$$

Aside: $A \perp\!\!\!\perp A$

$$\underbrace{\mathbb{P}(AA)}_{\mathbb{P}(A)} = \underbrace{\mathbb{P}(A)\mathbb{P}(A)}_{\mathbb{P}(A)^2}$$

$$\mathbb{P}(A) = \mathbb{P}(A)^2$$

$$\mathbb{P}(A) = 0 \text{ or } 1$$

Q: Pairwise = Mutual? No.

Ex. $S = \{aaa, bbb, ccc, abc, acb, bac, bca, cab, cba\}$

$|S| = 9$ Assume all are equally likely.

$A_i = \{i^{th}$ place in the triplet is an "a"$\}$

$A_1 = \{aaa, abc, acb\}$

$A_2 = \{aaa, bac, cab\}$

$A_3 = \{aaa, bca, cba\}$

$\mathbb{P}(A_1) = \mathbb{P}(A_2) = \mathbb{P}(A_3) = 3/9 = 1/3$

Mutual Independence? Pairwise?

To check pairwise independence:

$\{aaa\}$ $\quad \mathbb{P}(A_iA_j) = \mathbb{P}(A_i)\mathbb{P}(A_j)$ for $i \neq j$

$\quad\quad 1/9 \quad = (1/3)(1/3)$

So the A_i are pairwise independent.

To check mutual independence:

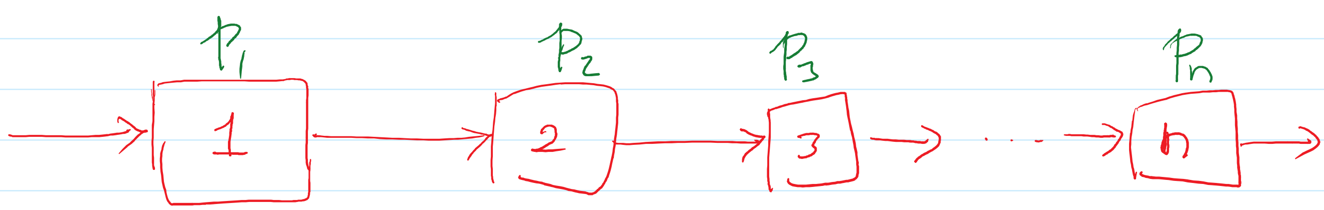
$\{aaa\}$ $\quad \mathbb{P}(A_1A_2A_3) = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3)$

$\quad\quad 1/9 \quad \neq (1/3)(1/3)(1/3) = \frac{1}{27}$

doesn't work

hence events aren't mutually independent.

Ex. System:



the prob. of failure at each subsystem is p_i

The process fails if any subsystem fails

If the failure/success of each subsystem is independent of the others.

What is the prob. the entire system works?

Let F_i = "i^{th} component works"

then $\mathbb{P}(F_i) = 1 - \mathbb{P}(F_i^c) = 1 - p_i$

$\mathbb{P}$(system works)

$$= \mathbb{P}(F_1 \cap F_2 \cap F_3 \cap \cdots \cap F_n)$$

$$= \mathbb{P}(F_1)\mathbb{P}(F_2)\cdots\mathbb{P}(F_n)$$

$$= (1-p_1)(1-p_2)(1-p_3)\cdots(1-p_n)$$

Exam 1 materials stop here

Random Variables

Often we want to summarize outcomes in S.

Ex, Flip a coin 3 times. X = # of heads.

s	$X(s)$
HHH	3
HHT	2
HTH	2
HTT	1
THH	2
THT	1
TTH	1
TTT	0

For each $s \in S$ I compute $X(s) \in \mathbb{R}$