Poisson Distribution

- discrete
- support non-negative integers $\{0, 1, 2, \dots\}$
- counting the number of "events" in a time period

Ex. – radioactive decay
– # of pieces of equip that fails

# of event is a time period

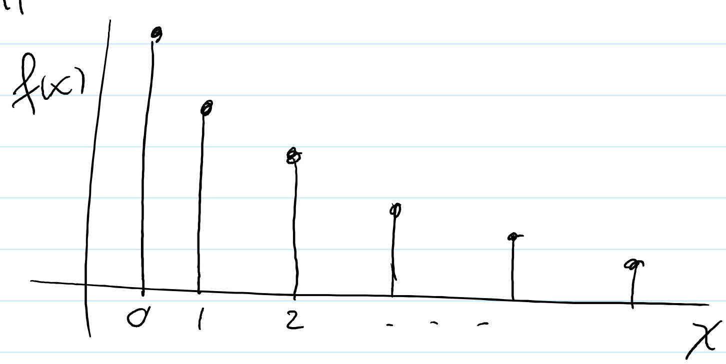
$X \sim \text{Pois}(\lambda)$

λ: rate parameter
= rate at which events occur (in some time period)

PMF:

$$f(x) = \frac{e^{-\lambda} \lambda^{x}}{x!} \quad \text{for } x = 0, 1, 2, 3, \dots$$

PMF



Expected Value

$$E[X] = \sum_{x=0}^{\infty} x f(x) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda} \lambda^{x}}{x!} = \sum_{x=1}^{\infty} x \frac{e^{-\lambda} \lambda^{x}}{x!}$$

0 when $x = 0$

$$= \sum^{\infty} \frac{e^{-\lambda} \lambda^{x}}{(\dots)} = \lambda e^{-\lambda} \sum^{\infty} \frac{\lambda^{x-1}}{(\dots)} = \lambda e^{-\lambda} \sum^{\infty} \frac{\lambda^{x}}{x!}$$

Aside!

$$e^{\lambda} = \sum_{x=0}^{\infty} \frac{\lambda^{x}}{x!}$$

Calc II

$$= \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{(x-1)!} = \lambda e^{-\lambda}\sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} = \lambda e^{-\lambda} \underbrace{\sum_{x=0}^{\infty} \frac{\lambda^{x}}{x!}}_{e^{\lambda}}$$

$$= \lambda e^{-\lambda} e^{\lambda} \boxed{= \lambda} = E[X]$$

$$E[X(X-1)] = \sum_{x=0}^{\infty} x(x-1) f(x) = \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^{x}}{x!}$$

when $x=0$ or $x=1$ summand is zer

$$= \sum_{x=2}^{\infty} \frac{x(x-1)e^{-\lambda}\lambda^{x}}{x!} = \sum_{x=2}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{(x-2)!} = e^{-\lambda}\lambda^{2} \underbrace{\sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!}}_{e^{\lambda}}$$

$= $ similar to above $= e^{-\lambda}\lambda^{2}e^{\lambda} = \lambda^{2}$

Showed: $E[X(X-1)] = \lambda^2$

$$E[X^2] - \overbrace{E[X]}^{\lambda} = \lambda^2$$

& $E[X^2] = \lambda^2 + \lambda$

$$Var(X) = E[X^2] - E[X]^2 = \lambda^2 + \lambda - (\lambda)^2 \boxed{= \lambda} = E[X]$$

$$\boxed{Var(X) = E[X] = \lambda}$$

MGF:

MGF:

$$M(t) = E[e^{tx}] = \sum_{x=0}^{\infty} e^{tx} f(x) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= e^{-\lambda} \underbrace{\sum_{x=0}^{\infty} \frac{(\overbrace{\lambda e^{t}}^{a = \lambda e^t})^{x}}{x!}}_{e^{(\lambda e^t)}} = e^{-\lambda} e^{\lambda e^{t}}$$

$$\boxed{= \exp(\lambda(e^{t}-1))}$$

$$\boxed{e^{a} = \sum_{x=0}^{\infty} a^{x}/x!}$$

Continuous distribution

Gamma Distribution: Generalization of Exponential

Recall: $X \sim Exp(\lambda)$ $f(x) = \lambda e^{-\lambda x}$ for $x > 0$

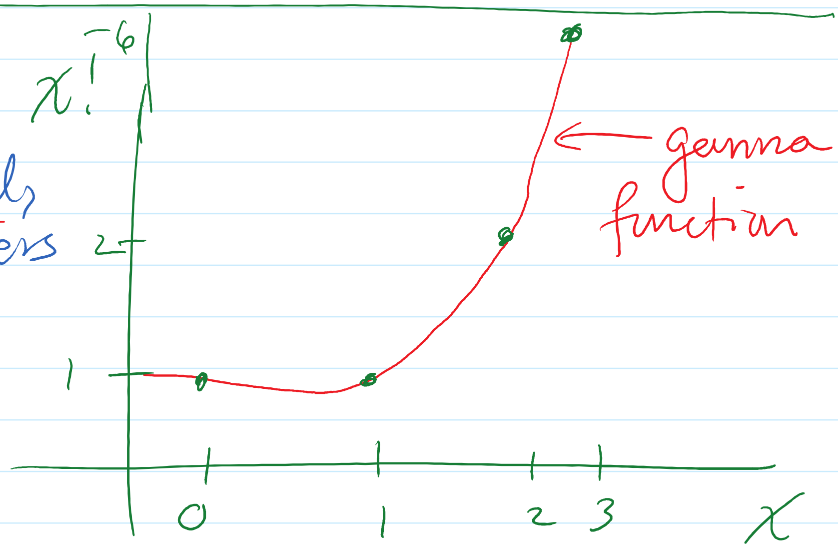
Gamma Function

Extension of factorials to positive numbers

$\Gamma : \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$

defined for $a \in \mathbb{R}^{+}$

$$\Gamma(a) = \int_{0}^{\infty} x^{a-1} e^{-x} \, dx$$



If a is an integer then

$$\Gamma(a) = (a-1)!$$

or

$$\Gamma(a+1) = a!$$

Notice that $x! = x(x-1)!$ for integers x

so $\boxed{\Gamma(x+1) = x\,\Gamma(x)}$

This holds for all $x \in \mathbb{R}^+$

Practical facts to know about Γ

1. $\Gamma(x+1) = x!$
2. $\Gamma(x+1) = x\,\Gamma(x)$

Gamma Distribution

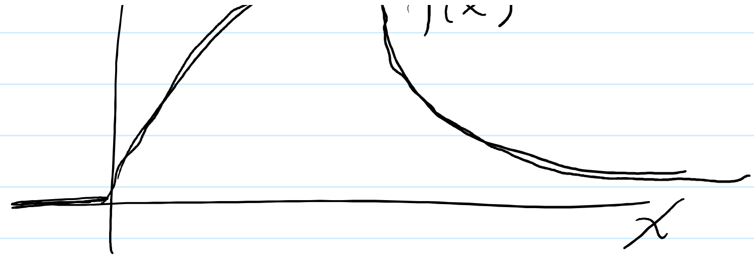
$$X \sim \text{Gamma}(a, \lambda)$$

PDF

$$\boxed{f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{a-1}}{\Gamma(a)}} \quad \text{for } x > 0$$

If $a = 1$ then $f(x) = \dfrac{\lambda e^{-\lambda x}(1)}{(1)}$ $\leftarrow \Gamma(1) = 0! = 1$

X has an $\text{Exp}(\lambda)$.



Expected Value:

$$E[X] = \int_0^\infty x f(x)\,dx = \int_0^\infty \frac{x \lambda e^{-\lambda x} (\lambda x)^{a-1}}{\Gamma(a)}\,dx$$

$$\frac{1}{\lambda}\int_0^\infty \frac{\lambda e^{-\lambda x} (\lambda x)^{a}}{\Gamma(a)}\,dx$$

$$\Gamma(a+1) = a\Gamma(a)$$

$$\Updownarrow$$

$$\Gamma(a) = \frac{\Gamma(a+1)}{a}$$

$$= \frac{a}{\lambda}\int_0^\infty \frac{\lambda e^{-\lambda x} (\lambda x)^{(a+1)-1}}{\Gamma(a+1)}\,dx$$

PDF of a Gamma$(\lambda, a+1)$

$$\frac{\lambda e^{-\lambda x} (\lambda x)^{(a+1)-1}}{\Gamma(a+1)}$$

$$= \frac{a}{\lambda}(1)$$

$$\boxed{= a/\lambda} = E[X]$$

$$E[X^r] = \int_0^\infty \frac{x^r \lambda e^{-\lambda x} (\lambda x)^{a-1}}{\Gamma(a)}\,dx$$

$$= \frac{1}{\lambda^r}\int_0^\infty \frac{(\lambda x)^r \lambda e^{-\lambda x} (\lambda x)^{a-1}}{\Gamma(a)}\,dx$$

$$\Gamma(a+r) \int^\infty (\lambda x)^r \lambda e^{-\lambda x} (\lambda x)^{a-1}$$

$$= \frac{\Gamma(a+r)}{\lambda^r} \int_0^\infty \frac{(\lambda x)^r \lambda e^{-\lambda x} (\lambda x)^{a-1}}{\Gamma(a+r)\Gamma(a)} dx$$

$$= \frac{\Gamma(a+r)}{\Gamma(a)} \frac{1}{\lambda^r} \underbrace{\int_0^\infty \frac{\lambda e^{-\lambda x} (\lambda x)^{(a+r)-1}}{\Gamma(a+r)} dx}_{\text{pdf of Gama}(\lambda,\, a+r) \;=\; 1}$$

$$\Rightarrow \boxed{E[X^r] = \frac{\Gamma(a+r)}{\Gamma(a)} \frac{1}{\lambda^r}}$$

$\checkmark\ E[X] = \frac{a}{\lambda}$ $\quad r=1; \quad \frac{\Gamma(a+1)}{\Gamma(a)} \frac{1}{\lambda} = \frac{a\Gamma(a)}{\Gamma(a)} \frac{1}{\lambda}$

$$= a/\lambda$$

$E[X^2] = \frac{(a+1)a}{\lambda^2}$ $\quad r=2; \quad \frac{\Gamma(a+2)}{\Gamma(a)} \frac{1}{\lambda^2} = \frac{(a+1)\Gamma(a+1)}{\Gamma(a)} \frac{1}{\lambda^2}$

$$= \frac{(a+1)a\cancel{\Gamma(a)}}{\cancel{\Gamma(a)}} \frac{1}{\lambda^2}$$

$$= \frac{(a+1)a}{\lambda^2}$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{(a+1)a}{\lambda^2} - \left(\frac{a}{\lambda}\right)^2$$

$$= \cdots = \boxed{a/\lambda^2}$$

Geometric Distribution

Geometric Distribution

If I have a sequence of trials

$Y_1, Y_2, Y_3, \ldots$

each independently w/ a prob. of success of p,

let W = waiting time until the first H appears

$$\boxed{W \sim \text{Geometric}(p)}$$

Discrete random variable w/ support:

$\{1, 2, 3, 4, \ldots\}$

pmf:

$$f(x) = p(1-p)^{x-1} \quad \text{for } x = 1, 2, 3, \ldots$$

CDF: $F(x) = P(W \leq x) = \sum_{i=1}^{x} f(i) = \sum_{i=1}^{x} p(1-p)^{i-1}$

$$= p \sum_{i=1}^{x} (1-p)^{i-1} = p \sum_{i=0}^{x-1} (1-p)^{i}$$

$\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$

$r = 1-p$

$$= \cancel{p} \frac{1-(1-p)^{x-1}}{\cancel{1-(1-p)}}$$

$$\boxed{= 1-(1-p)^{x-1}}$$

$$= 1-(1-p)^{x-1}$$

Expected Value:

$$E[W] = \sum_{i=1}^{\infty} i \underbrace{p(1-p)^{i-1}}_{f(i)} = p \sum_{i=1}^{\infty} i(1-p)^{i-1}$$

$$\boxed{x(1-p)^{x-1} = -\frac{d}{dp}(1-p)^{x}}$$

$$= p \sum_{i=1}^{\infty} \left(-\frac{d}{dp}(1-p)^{i}\right) = -p \frac{d}{dp} \underbrace{\sum_{i=1}^{\infty} (1-p)^{i}}_{\text{geometric series}}$$

$$= -p \frac{d}{dp}\left[(1-p) \sum_{i=1}^{\infty} (1-p)^{i-1}\right]$$

$$= -p \frac{d}{dp}\left[(1-p) \sum_{i=0}^{\infty} (1-p)^{i}\right]$$

$$= -p \frac{d}{dp}\left[(1-p) \frac{1}{p}\right]$$

$$= -p \frac{d}{dp}\left(\frac{1-p}{p}\right)$$

$$= -p\left(-\frac{1}{p^2}\right) = \boxed{\frac{1}{p}} = E[W]$$

$W \sim \text{Geometric}(p)$

$$= -p\left(-\frac{1}{p^2}\right) = \boxed{\frac{1}{p}} = \mathbb{E}[W]$$

$$\text{MGF: } M(t) = \mathbb{E}[e^{tX}] = \sum_{x=1}^{\infty} e^{tx} p(1-p)^{x-1}$$

$$= pe^t \sum_{x=1}^{\infty} e^{t(x-1)} (1-p)^{x-1}$$

$$e^t e^{t(x-1)} = \cancel{e^t} e^{tx} \cancel{e^{-t}}$$

$r = e^t(1-p)$

$$= pe^t \sum_{x=0}^{\infty} \left(e^t(1-p)\right)^x$$

$$= pe^t \frac{1}{1-e^t(1-p)} = \boxed{\frac{pe^t}{1-(1-p)e^t}}$$

$$\left.\frac{d^2M}{dt^2}\right|_{t=0} = \ldots \text{algebra + Calculus} = \frac{2-p}{p^2}$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2$$

$$\boxed{= \frac{1-p}{p^2}}$$

Beta Distribution

Beta Distribution

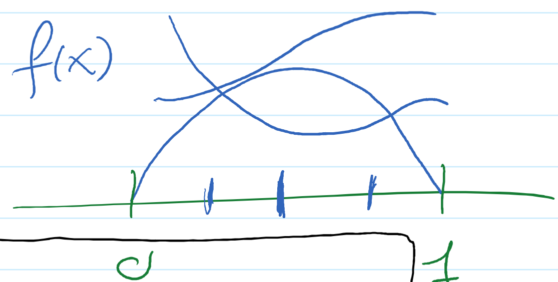
Beta Function:

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$$

$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Beta dist: cts r.v.

two parameters a, b



$$\boxed{f(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} \quad \text{for } x \in (0,1)}$$

Expected value:

$$E[X] = \int_0^1 x f(x)\,dx = \int_0^1 \frac{x\, x^{a-1}(1-x)^{b-1}}{B(a,b)}\,dx$$

$$= \frac{B(a+1,b)}{B(a,b)} \underbrace{\int_0^1 \frac{x^{(a+1)-1}(1-x)^{b-1}}{B(a+1,b)}\,dx}_{\text{pdf of Beta}(a+1,\,b)}$$

pdf of Beta$(a+1, b)$

$$= \frac{B(a+1,b)}{B(a,b)}$$

$$= \frac{\dfrac{\overset{a\,\cancel{\Gamma(a)}}{\Gamma(a+1)}\cancel{\Gamma(b)}}{\Gamma(a+b+1)}}{\dfrac{\cancel{\Gamma(a)}\cancel{\Gamma(b)}}{\Gamma(a+b)}} = \frac{a\,\Gamma(a+b)}{\Gamma(a+b+1)}$$

$$= \frac{a\,\cancel{\Gamma(a+b)}}{(a+b)\,\cancel{\Gamma(a+b)}}$$

$$\boxed{= \frac{a}{a+b}} = E[X]$$

$$\boxed{E[X^r]} = \int_0^1 x^r \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}\,dx$$

$$= \frac{B(a+r,b)}{B(a,b)} \int_0^1 \underbrace{\frac{x^{a+r-1}(1-x)^{b-1}}{B(a+r,b)}}_{\text{pdf of Beta}(a+r,\,b)}\,dx$$

$$\boxed{= \frac{B(a+r,b)}{B(a,b)}}$$

For $r=2$

$$\mathbb{E}[X^2] = \cdots = \frac{(a+1)\,a}{(a+b)(a+b+1)}$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \frac{(a+1)\,a}{(a+b)(a+b+1)} - \left(\frac{a}{a+b}\right)^2$$

$$= \text{algebra} = \boxed{\frac{ab}{(a+b+1)(a+b)^2}}$$