- $X_3 \sim$ exponential(μ) because X_3 is the service time for customer 3.

Theorem 6.19 can now be invoked to determine the joint probability density function of T_1 and T_2 in Scenario 2. Substituting $\lambda_1 = \mu$, $\lambda_2 = \lambda + \mu$, and $\lambda_3 = \mu$ into the mixture of Scenarios 1 and 2 yields the joint probability density function of T_1 and T_2 as

$$f_{T_1,T_2}(t_1,t_2) = \begin{cases} \dfrac{\mu^2\left(\lambda e^{-\mu t_2} + \mu e^{-\lambda t_1 - \mu t_1 - \mu t_2}\right)}{\lambda + \mu} & 0 < t_1 \le t_2 \\[2ex] \dfrac{\mu^2\left(\lambda e^{-\lambda t_1 - \mu t_1 + \lambda t_2} + \mu e^{-\lambda t_1 - \mu t_1 - \mu t_2}\right)}{\lambda + \mu} & 0 < t_2 < t_1. \end{cases}$$

Using this joint probability density function, the population covariance between the sojourn times of customers 1 and 2 is

$$\text{Cov}(T_1, T_2) = \frac{\lambda(\lambda + 2\mu)}{(\lambda + \mu)^2 \mu^2}.$$

6.6 Exercises

6.1 A bag contains three balls numbered 1, 2, and 3. Two balls are sampled *with replacement.* Let X_1 be the number on the first ball sampled and let X_2 be the number on the second ball sampled. Find the probability mass function of the sample mean

$$\frac{X_1 + X_2}{2}.$$

6.2 A bag contains three balls numbered 1, 2, and 3. Two balls are sampled *without replacement.* Let X_1 be the number on the first ball sampled and let X_2 be the number on the second ball sampled. Find the probability mass function of the sample mean

$$\frac{X_1 + X_2}{2}.$$

6.3 The continuous random variables X and Y have joint probability density function

$$f(x,y) = \frac{1}{100} \qquad 0 < x < 10, 0 < y < 10.$$

Find $P(2X + Y < 12)$.

6.4 The continuous random variables X and Y have joint probability density function

$$f(x,y) = \frac{1}{2} \qquad |x| + |y| < 1.$$

Find $P(X > 1/2)$.

6.5 The random variables X and Y are uniformly distributed over the *interior* of a circle of radius 1 centered at the origin. Find the marginal probability density function $f_X(x)$.

6.17 Let X_1 be uniformly distributed between zero and one. Let x_1 be the realization of X_1 (that is, x_1 is a sample value of X_1). Let X_2 be uniformly distributed between zero and x_1.

(a) Find the marginal probability density function of X_2.

(b) Find the population mean of X_2 and execute a Monte Carlo simulation that supports the value that you derive analytically.

6.18 What is the probability that there are more spades than hearts in a five-card poker hand?

6.19 Kent leaves for work every day between 5:00 AM and 5:10 AM. He needs to be at work by 6:00 AM. He can take the highway or cut through town. The travel time to work via the highway is between 45 and 55 minutes. The travel time to work by cutting through town is between 40 and 60 minutes. Assuming that his departure and travel times are uniform on the given ranges, find the probability he will be on time by the two routes.

6.20 Let X be a Poisson random variable with a random parameter Λ, which has probability density function

$$f_\Lambda(\lambda) = e^{-\lambda} \qquad \lambda > 0.$$

Find the probability mass function of X. *Hint*: The conditional probability mass function of X given λ is $f_{X\,|\,\Lambda=\lambda}(x\,|\,\Lambda=\lambda) = \lambda^x e^{-\lambda}/x!$ for $x = 0, 1, 2, \ldots,$ and $\lambda > 0$.

6.21 Let X_1 and X_2 be the numbers on two balls drawn randomly from a bag of billiard balls numbered $1, 2, \ldots, 15$. Find the joint probability mass function of $Y_1 = \min\{X_1, X_2\}$ and $Y_2 = \max\{X_1, X_2\}$ and the marginal probability mass function of Y_1 when

(a) sampling is without replacement,

(b) sampling is with replacement.

6.22 For the random variables X and Y with joint probability density function

$$f(x, y) = 2 \qquad 0 < x < y < 1,$$

find the joint cumulative distribution function for all real values of x and y.

6.23 Let X and Y be uniformly distributed on the unit disk with joint probability density function

$$f(x, y) = \frac{1}{\pi} \qquad x^2 + y^2 < 1.$$

(a) Find $P(|X| + |Y| > 1)$ without using calculus.

(b) How many separate regions is the joint cumulative distribution function $F(x, y)$ defined on? *Hint* for part (b): Do not find the joint cumulative distribution function here—just find the number of regions it is defined on. Here is an example of the number of regions where the cumulative distribution function is defined: the joint cumulative distribution function of X and Y is

$$F(x, y) = \begin{cases} 0 & x \le 0 \text{ or } y \le 0 \\ 1 - e^{-(x+y)} & x > 0, y > 0 \end{cases}$$

is defined on *two* separate regions.

6.24 Let the random variables X and Y have joint probability density function

$$f(x, y) = \frac{1}{\pi} e^{-\frac{1}{2}(x^2+y^2)} \qquad x > 0, y < 0 \text{ or } x < 0, y > 0.$$

Find the marginal probability density functions. *Hint*: consider $\mathcal{A}$ carefully.

6.25 Consider the random variables X and Y with joint probability density function

$$f(x, y) = \frac{1}{\pi} \qquad x^2 + y^2 < 1.$$

(a) Find $P\left(X^2 + Y^2 \leq r^2\right)$ for any real $r \geq 0$.

(b) Set up, but do not evaluate, expressions for $P(X + Y \leq r)$ for any real r.

6.26 Let X_1 and X_2 be continuous random variables with joint probability density function

$$f(x_1, x_2) = \frac{1}{9} \qquad 0 < x_2 < x_1^2 < 9, x_1 > 0.$$

Find the marginal distribution for X_1.

6.27 Consider the Cartesian coordinate system. Li-Hsing selects a point that is uniformly distributed between the origin and (1, 0). Huarng selects a point that is uniformly distributed between the origin and (0, 1), independently of Li-Hsing's choice. Find the population median of the distribution of the distance between the two points.

6.28 Consider the random variables X and Y with joint probability density function

$$f(x, y) = cxy \qquad 0 < y < x < 1.$$

(a) What is the value of the constant c?

(b) What is $P(X > 1/2)$?

(c) What is $P\left(Y > X^2\right)$?

6.29 There are five balls in a bag numbered 1, 2, 3, 4, and 5. Two balls are drawn at random without replacement. Let X be the smaller number chosen and let Y be the larger number chosen.

(a) What is the joint probability mass function of X and Y?

(b) What is the probability mass function of $Z = Y - X$?

6.30 Let (x_i, y_i), for $i = 1, 2, 3$, denote the vertices of a triangle on the Cartesian coordinate system. Let the random ordered pair (X, Y) be uniformly distributed over the triangle (that is, the joint probability density function of X and Y is the reciprocal of the area of the triangle). Find the marginal distribution of X, $f_X(x)$. *Hint 1*: You may assume, without loss of generality, that $x_1 \leq x_2 \leq x_3$, although you must explicitly consider the cases $x_1 = x_2$ and $x_2 = x_3$. *Hint 2*: For a triangle with side lengths a, b, and c, and semi-perimeter $s = (a+b+c)/2$, Heron's formula gives the area as $\sqrt{s(s-a)(s-b)(s-c)}$.

6.31 Let X and Y be uniformly distributed on the support $|x| + |y| < 3$. Find $F(1, 1)$. *Hint:* Solve this problem geometrically rather than using integration.

6.32 Let X and Y be continuous random variables with joint probability density function

$$f(x, y) = \frac{1}{3\pi} \qquad 1 < x^2 + y^2 < 4.$$

Find the marginal probability density function of X.

6.33 The random variables X and Y have joint probability density function

$$f(x, y) = 2 \qquad x \ge 0, y \ge 0, x + y \le 1.$$

Find the point in the support of X and Y that maximizes $F(x, y)$.

6.34 The random variables X and Y have a joint probability distribution. Let

$$W = 2(\lceil X \rceil - \lfloor X \rfloor) + 3(\lceil Y \rceil - \lfloor Y \rfloor).$$

(a) If X and Y are continuous random variables, find the probability mass function of W.

(b) If X and Y are discrete random variables defined only on the integers, find the probability mass function of W.

6.35 Let X and Y have joint probability density function

$$f(x, y) = 2e^{-2x} \qquad x > 0, -x < y < x.$$

Find $P(2 < X < 3)$.

6.36 Let X and Y have joint probability density function

$$f(x, y) = 2e^{-2x} \qquad x > 0, -x < y < x.$$

Find $P(-3 < Y < 3 \mid X = 4)$.

6.37 Let the random variables X and Y have the joint probability density function

$$f(x, y) = \frac{3}{2}xy \qquad 0 < x < 2, 0 < y < 2 - x.$$

(a) Find $f_X(x)$.

(b) Find $P(2Y < X \mid X = 1)$.

6.38 The independent random variables T_1 and T_2 denote the lifetimes of two components arranged in parallel. The probability density function of T_1 is

$$f_{T_1}(t_1) = e^{-t_1} \qquad t_1 > 0.$$

The probability density function of T_2 is

$$f_{T_2}(t_2) = 2e^{-2t_2} \qquad t_2 > 0.$$

The cost of power to operate the system is $X = 3T_1 + 4T_2$. Find the cumulative distribution function of X.

6.39 Let $X_1 \sim U(0, 1)$ and $X_2 \sim U(0, 1)$ be independent random variables. Find the exact value of $P\left(X_1^2 + X_2^2 > 1\right)$ using a purely geometric argument (that is, use no calculus).

6.46 A fair die is tossed repeatedly. Let X be the toss number of the first even number to appear. Let Y be the toss number of the first time that a 3 appears. Without doing any mathematics, are X and Y independent or dependent random variables?

6.47 Let X and Y have joint probability density function

$$f(x, y) = c(2x + y) \qquad 0 < x < 1, 0 < y < 1,$$

where c is a positive constant.

(a) By inspection, are X and Y independent or dependent random variables?
(b) Find c.
(c) Find the marginal probability density function of X.
(d) Find the conditional probability density function of Y given $X = x$.

6.48 Let $X \sim$ Bernoulli(p) and $Y \sim$ Poisson(λ) be independent random variables. Find $P(X \neq Y)$.

6.49 Let the random variable X be the number of spots on the up side of a biased die, with probability mass function

$$f(x) = x/21 \qquad x = 1, 2, \ldots, 6.$$

The biased die is rolled once. The observed number of spots on the up side is x. A fair coin is then tossed x times, resulting in Y heads. Find $P(Y = 3)$.

6.50 By inspection, are the random variables X and Y with joint probability mass function

$$f(x, y) = \frac{1}{2^{x+y}} \qquad x = 1, 2, \ldots ; y = 1, 2, \ldots$$

independent?

6.51 A single possession in basketball results in 0, 1, 2, 3, or 4 points scored with probabilities p_0, p_1, p_2, p_3, and p_4, where $p_0 + p_1 + p_2 + p_3 + p_4 = 1$. What is the probability mass function of the total points scored in two independent possessions?

6.52 Let the independent random variables X_1 and X_2 be drawn from a population with probability mass function

$$f(x) = \begin{cases} 1/3 & x = 2 \\ 2/3 & x = 8. \end{cases}$$

Find the probability mass function of the sample mean

$$\frac{X_1 + X_2}{2}.$$

6.53 Let $X \sim$ binomial$(5, 7/10)$ and $Y \sim$ binomial$(4, 4/5)$ be independent random variables. Find $P(\min\{X, Y\} = 3)$.

6.54 Let the independent random variables X and Y have marginal probability mass functions

$$f_X(x) = \frac{1}{2} \qquad x = 1, 2$$

and

$$f_Y(y) = \frac{1}{3} \qquad y = 1, 2, 3.$$

Find $P(X = Y)$.

6.70 Let X and Y be random variables with joint probability density function

$$f(x, y) = \frac{8xy}{k^4} \qquad 0 < x < y < k,$$

where k is a positive constant.

(a) Find the population covariance between X and Y.

(b) Find $E[Y \mid X = x]$.

(c) Find $V[Y \mid X = x]$.

6.71 The Internal Revenue Service could classify couples using the following random variables

$$X = \begin{cases} 0 & \text{filing jointly} \\ 1 & \text{filing separately} \end{cases}$$

and

$$Y = \begin{cases} 0 & \text{no dependents} \\ 1 & \text{one or more dependents} \end{cases}$$

The joint probability mass function of X and Y is given below. Find the population covariance between X and Y.

		Y	
		0	1
X	0	0.34	0.46
	1	0.06	0.14

6.72 Ray and Jane are bridge partners. They each receive 13 cards dealt from a well-shuffled deck. Let X be the number of spades in Ray's hand and Y be the number of spades in Jane's hand.

(a) Find the joint probability mass function of X and Y.

(b) Find the population covariance between X and Y.

6.73 Consider the R code below which prints the sample mean of two random variates.

```
x = runif(2, 4, 8)
print((x[1] + x[2]) / 2)
```

(a) What is the population mean of the value printed?

(b) What is the population variance of the value printed?

6.74 For random variables X, Y, and Z, show that

$$\text{Cov}(X, Y + Z) = \text{Cov}(X, Y) + \text{Cov}(X, Z).$$

6.75 Without doing any mathematics, what is the population correlation between X, the number of red cards (that is, hearts and diamonds), and Y, the number of black cards (that is, spades and clubs), in a 5-card poker hand dealt from a standard 52-card deck?

6.76 Brennan tosses a biased coin n times. The probability of tossing a head on a single toss is p, where $0 < p < 1$.

(a) Find the expected value of the product of the number of heads tossed and the number of tails tossed.

(b) Find the population correlation between the number of heads tossed and the number of tails minus the number of heads. Note that this can be done with little or no mathematics.

(c) Write R statements to verify your solutions to parts (a) and (b) using a Monte Carlo simulation when $n = 17$ and $p = 1/3$ with 1000 replications.

6.77 Let X and Y have a bivariate distribution with variance–covariance matrix

$$\Sigma = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}.$$

Find $V[X+Y]$.

6.78 Let the continuous random variables X and Y be uniformly distributed over the support

$$\mathcal{A} = \{(x, y) \,|\, 0 < y < |x| < 1\}.$$

Find

(a) $P(X > 2/3)$,

(b) $P(Y > 2/3)$,

(c) $E\left[XY^2\right]$,

(d) $E[\arccos X]$,

(e) $f_Y(y)$.

6.79 Davina deals a five-card hand from a well-shuffled deck of cards. Let

- X_1 be the number of spades,
- X_2 be the number of diamonds,
- X_3 be the number of jacks,
- X_4 be the number of queens.

The population correlation between X_1 and X_2 is denoted by $\rho_{X_1X_2}$. The population correlation between X_3 and X_4 is denoted by $\rho_{X_3X_4}$. Without computing the correlations, choose one of the following statements that you believe to be true and write *two sentences* describing why you think it is true.

(a) $\rho_{X_1X_2} < \rho_{X_3X_4}$,

(b) $\rho_{X_1X_2} = \rho_{X_3X_4}$,

(c) $\rho_{X_1X_2} > \rho_{X_3X_4}$.

6.80 Solve Buffon's needle problem for $l > d$.

6.81 In Buffon's needle problem, a needle of length l is tossed n times, and the number of times that it crosses one of the parallel lines that are a distance d apart is denoted by W. In order to estimate π, the fraction of crossings

$$\hat{p} = \frac{W}{n}$$

is equated to $\frac{2l}{\pi d}$ (in the case of $l \le d$) and π is solved for, yielding an estimate $\hat{\pi}$. The question here is: *how long should the needle be?* If one uses a long needle (l large), $\hat{p}$ will be close to one. On the other hand, if one uses a short needle (l small), $\hat{p}$ will be close to zero. A reasonable middle ground would be to choose l so as to maximize $V\left(\frac{W}{n}\right)$. Find the value of l that maximizes $V\left(\frac{W}{n}\right)$.

6.82 Let X and Y be random variables with $V[X] = 10$, $V[Y] = 16$, and $V[X+Y] = 24$. Find $\text{Cov}(X, Y)$.

6.83 The entries in the table below give the joint probability mass function of X and Y, where $p_1 + p_2 + p_3 + p_4 = 1$. Find $E[E[Y \mid X]]$.

x \ y	0	1
0	p_1	p_2
1	p_3	p_4

6.84 The entries in the table below give the joint probability mass function of X and Y, where $p_1 + p_2 + p_3 + p_4 = 1$. Find $E\left[X^5 \mid Y = 1\right]$.

x \ y	0	1
0	p_1	p_2
1	p_3	p_4

6.85 Let the random variable X be the number of successes in n mutually independent Bernoulli trials and the random variable $Y = n - X$ be the number of failures in these n mutually independent Bernoulli trials. Find ρ, the population correlation between X and Y.

6.86 For the joint probability density function defined by

$$f(x, y) = \frac{2}{k^2} \qquad 0 < x < y < k,$$

for some positive real constant k, find the conditional expected value of X given $Y = y$ and find the population correlation between X and Y.

6.87 Stefani draws three cards without replacement from a well-shuffled deck of cards. Let the random variable X be the number of spades and let the random variable Y be the number of diamonds. Find the population covariance between X and Y.

6.88 The random variables X and Y are uniformly distributed over the *interior* of a circle of radius 2 centered at (3, 4). Find $E[X \mid Y = 5]$.