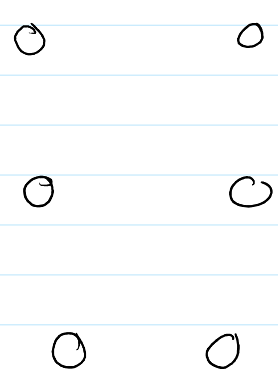
Lecture 5 - Unordered Counting

Thursday, February 6, 2020 10:37 AM

Last time: ordered counting

| | w/ replacement | w/o replacement. |
|---|---|---|
| ordered | $n^r$ | $\frac{n!}{(n-r)!}$ |
| unordered | | |

Ex.



Braille

each of 6 spots can be raised or not

(2 options)

The number of possible Braille letters:

$$\boxed{2^6} = n^r \qquad n=2$$

can think of sampling from {raised, not raised}

we take $6=r$ ordered sample (w/ repl.)

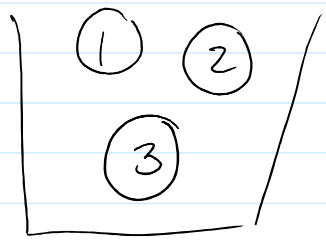
Theorem: Sampling Unordered w/o replacement.

The num. of ways to draw $r$ unordered samples from $n$ w/o replacement is

$$\boxed{\frac{n!}{r!(n-r)!} = \binom{n}{r}} = \text{"}n \text{ choose } r\text{"}$$

Binomial coefficient.

Ex.



draw $r=2$ from $n=3$
w/o repl. w/o ordering

If order mattered: we should have $\frac{n!}{(n-r)!} = \frac{3!}{1!} = 6$

2 each

| | | |
|---|---|---|
| $(1,2)$ | $(1,3)$ | $(2,3)$ |
| $(2,1)$ | $(3,1)$ | $(3,2)$ |

possible ordered samples.

If ordering doesn't matter:

$\{1,2\}$ $\{1,3\}$ $\{2,3\}$ 3 possible samples w/o ordering

In general: If I have an unordered sample

$\{a, b, c, \ldots\}$ of size $r$

an ordered sample is a permutation of the unordered

So I can make $r!$ ordered samples from my unordered

$$\left(\#\text{ of ordered samples}\right) = r!\left(\#\text{ unordered}\right)$$

$$\frac{n!}{(n-r)!} = r!\left(\#\text{ unordered}\right)$$

So $\boxed{\#\text{ unordered sample} = \frac{n!}{r!(n-r)!}} = \binom{n}{r}$

Ex. I have 10 professors, how many co-equal committees can I form w/ 4 members?

$$\binom{10}{4} = \boxed{\frac{10!}{4!(10-4)!}} = \frac{10!}{4!\,6!}$$

$$= \frac{10\cdot 9\cdot 8\cdot 7}{4!} = \frac{10\cdot \cancel{9}^{3}\cdot \cancel{8}\cdot 7}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}}$$

$$= 3\cdot 7\cdot 10$$

$= 3 \cdot 7 \cdot 10$

$= 210$

---

Ex. How many 5-card poker hands are there in a deck of 52?

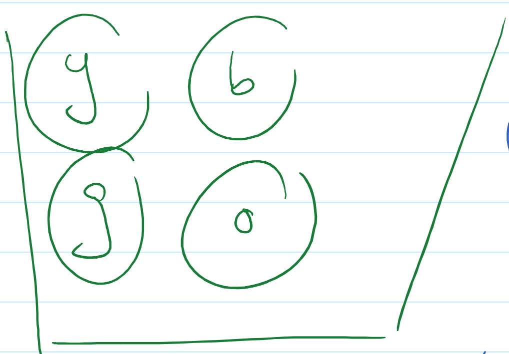
$r=5$, $n=52$

$$\binom{52}{5} \approx 2.5 \text{ mil}$$

$$\boxed{= \frac{52!}{5!(52-5)!}}$$

---

Ex. Jar w/ yellow, blue, orange, green marbles (1 of each)




Choose 3 "at random" (all ~~choices~~ samples are equally likely)

What is the prob. I choose yellow and blue in my sample?

$r=3$ $\quad$ $n=4$

$S = \{$ all samples of 3 from 4 w/o repl. and w/o ordering $\}$

$E = \{\{y, b, o\}, \{y, b, g\}\}$

$|E| = 2$

$$|S| = \binom{4}{3} = \frac{4!}{3!(4-3)!} = 4$$

So $\mathbb{P}(E) = 2/4 = 1/2$.

---

## Theorem: Unordered w/ Replacement.

The number of samples of size $r$ that may be drawn from $n$ w/o ordering mattering and w/ replacement is

$$\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!}$$

$$= \binom{n+r-1}{n-1}$$

---

pf.

Tempting to say:

true w/o replacement.

what about w/ repl.?

$$\cancel{\underbrace{(\#\text{ ordered})} = r!(\#\text{ unordered})}$$

(# ordered) ... (# unordered)

~~$n^r / r! = \#$ Unordered~~

Bad wrong

F

**Ex.** ① ② ③

Ordered samples: of size 2

(2,1) (1,2)   (2,3) (3,2)   (3,1) (1,3)   (1,1)   (2,2)   (3,3)

extra b/c we allow replacement

Unordered samples:

{1,2}, {2,3}, {1,3}, {1,1}, {2,2}, {3,3}

## Need a better argument:

## Game of Partioning:

**Ex.** $n=5$   $r=2$

Game: how many ways can I partition $r$ objects using $n-1$ walls

| O | |   O |

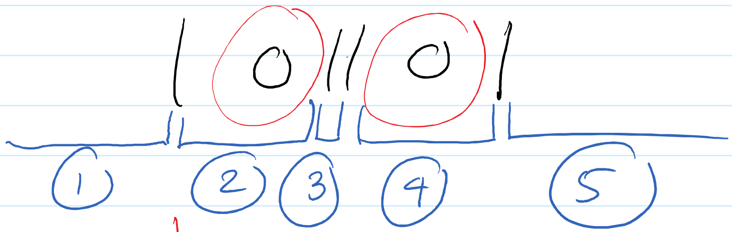
$r=2$ objects
$n-1=4$ walls

or

| O | | | O

or 

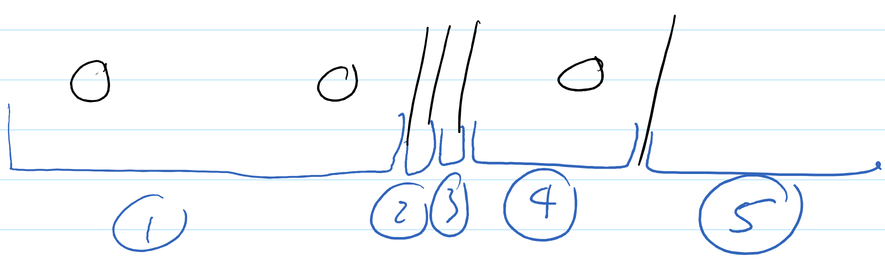
Can use each configuration to create unordered sample w/ repl.




| | |
|---|---|
| 1 | 0 |
| 2 | 1 |
| 3 | 0 |
| 4 | 1 |
| 5 | 0 |

$\Longrightarrow \{2, 4\}$

$r = 3 \quad n = 5$




| | # obj. |
|---|---|
| 1 | 2 |
| 2 | 0 |
| 3 | 0 |
| 4 | 1 |
| 5 | 0 |

$\Longrightarrow \{1, 1, 4\}$

# sample $\overset{1-1}{\Longleftrightarrow}$ # of ways to partition objects

"
want this

'
count this

Q: how may ways can I partion $r$ objects w/ $n-1$ walls?

things

I have: $\underbrace{o \ o \ o \ \cdots \ o}_{r}$ $\underbrace{/ \ / \ / \ / \cdots /}_{n-1}$

→ in total I have $r+n-1$ things

→ each partition I make is a certain permutation of these things

$|| \ o \ | \ o \ |$

the # of ways to permute the $r+n-1$ things is $(r+n-1)!$

swap

$|| \ o \ ||| \ o \ oo \ | \ o \ |$

→ objects are not distinct and same for walls

I can swap any objects o w/ each other or any walls | w/ each other and get the same partition (picture)

Need to divide through by # of such swaps to get

$$\frac{(r+n-1)!}{r!\,(n-1)!}$$

# of object swaps

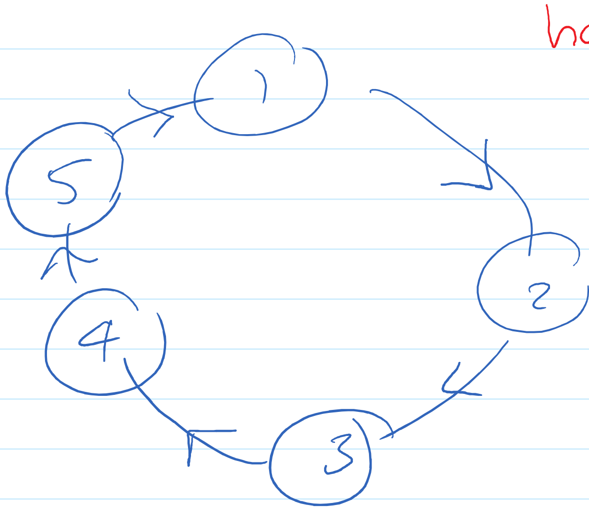
# wall swaps

= # of distinct partitions I can make.

---

Ex.

10 passengers on a bus on a route w/ 5 hotels.



Bus driver records how many passengers get off at each stop.

How many possible records are there?

Ex record

| hotel | # passengers |
|---|---|
| 1 | 0 |
| 2 | 3 |
| 3 | 1 |
| 4 | 2 |
| 5 | 4 |

$\Longleftrightarrow$ $\{2,2,2,3,4,4,5,5,5,5\}$

unordered sample (of size 10) from $\{1,\dots,5\}$

w/ replacement

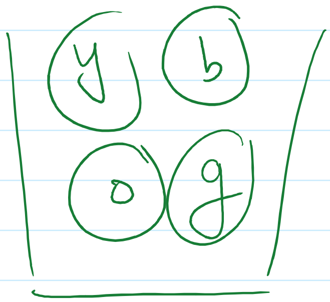
$n=5 \quad r=10$

according to formula: there are

$$\binom{r+n-1}{r} = \binom{14}{10} = 1001$$

---

Ex. Jar contains yellow, blue, orange, green marbles




Prob. of choosing y and b at least once in a sample of 3 where I sample w/ repl.

(all samples w/ repl. are equally likely)

$$E = \{\{y,b,y\}, \{y,b,b\}, \{y,b,o\}, \{y,b,g\}\}$$

$$S = \{\text{all unordered samples of size 3 from 4 w/ repl.}\}$$

$|E| = 4$ and $|S| = \binom{r+n-1}{r} = \binom{3+4-1}{3}$

$$= \binom{6}{3} = 20$$

hence $P(E) = \dfrac{|E|}{|S|} = \dfrac{4}{20} = \dfrac{1}{5}$.

| | w/ repl. | w/o repl. |
|---|---|---|
| ordered | $n^r$ | $\frac{n!}{(n-r)!}$ |
| un-ordered | $\binom{n+r-1}{r}$ | $\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$ |

## Conditional Probability

Ex.

political party A or B

| | A | B | Tot |
|---|---|---|---|
| men | 501 | 238 | 739 |
| women | 782 | 123 | 905 |
| Tot | 1283 | 361 | 1644 |

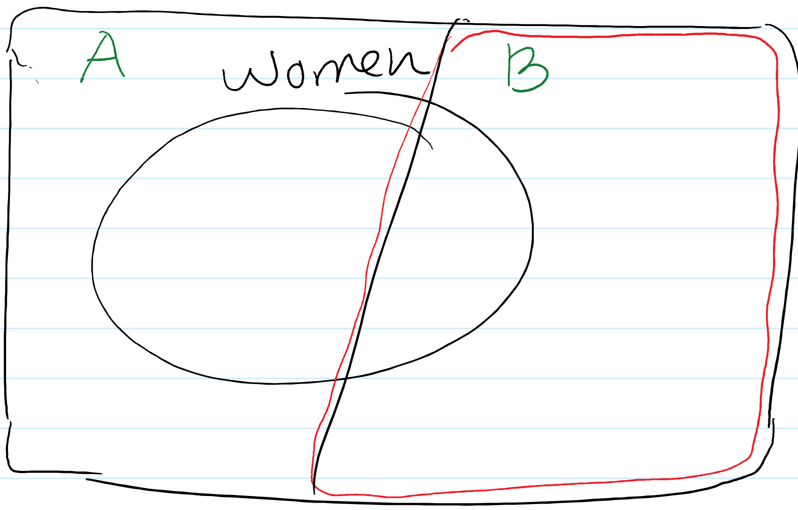
of W&M students

Q: If I randomly select a student,

$$P(\text{woman}) = \frac{\#\text{ women}}{\#\text{ students}} = \frac{905}{1644} \approx 55\%$$

Q: GIVEN the student is in B what is the prob. they are a woman?

$$\mathbb{P}(\text{woman given } B) = \frac{123}{361} \approx 39\%$$



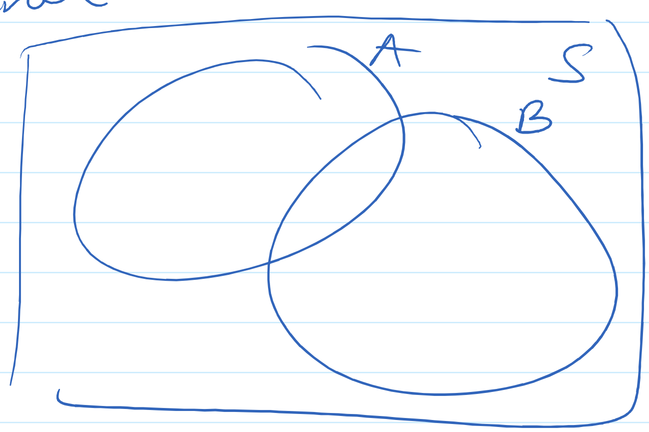

$\mathbb{P}(\text{women})$

$=$

$P(\text{woman given } B)$

$=$

In general:




$$\mathbb{P}(A \text{ given } B) = \frac{\text{area of } AB}{\text{area of } B}$$

---

Defn: Conditional Probability

If $A, B \subset S$ then

$P(A \mid B) \overset{\text{defined}}{=} \mathbb{P}(AB)$ (So long as $\mathbb{P}(B) > 0$)

$$\mathbb{P}(A \mid B) \overset{\text{defined}}{=} \frac{\mathbb{P}(AB)}{\mathbb{P}(B)} \quad \left(\text{so long as } \mathbb{P}(B) > 0\right)$$

read as "A given B"