6.55 Let T_1 and T_2 be independent lifetimes of two light bulbs. Find the probability density function of time of the first failure $X = \min\{T_1, T_2\}$. Also, find the probability density function of the time between failures $Y = |T_1 - T_2|$. Assume the following pairs of distribution assumptions for T_1 and T_2.

(a) $T_1 \sim$ exponential(λ) and $T_2 \sim$ exponential(λ).

(b) $T_1 \sim$ exponential(λ_1) and $T_2 \sim$ exponential(λ_2).

(c) $T_1 \sim$ Weibull(λ, κ) and $T_2 \sim$ Weibull(λ, κ).

6.56 Let $X_1 \sim$ geometric(p_1) and $X_2 \sim$ geometric(p_2) using the parameterization of the geometric distribution beginning at 0. If X_1 and X_2 are independent, find the probability mass function of $Y = \max\{X_1, X_2\}$.

6.57 Consider the linear programming problem:

$$\begin{array}{ll} \text{maximize} & Ax_1 + Bx_2 \\ \text{subject to} & x_1 + x_2 \le 1 \\ & x_1 \ge 0 \\ & x_2 \ge 0 \end{array}$$

where the random coefficients $A \sim U(0, 1)$ and $B \sim U(0, 2)$ are independent random variables. Find the joint probability mass function of X_1 and X_2, the random values of x_1 and x_2 that solve the linear program.

6.58 Let the random variable X have probability density function

$$f_X(x) = 1 \qquad 0 < x < 1.$$

Let the random variable Y have probability density function

$$f_Y(y) = \frac{1}{3} \qquad -1 < y < 2.$$

Assume that X and Y are independent.

(a) Find the probability density function of $V = \min\{X, Y\}$.

(b) Find the probability mass function of $W = \min\{\lfloor 2X \rfloor, \lfloor Y \rfloor\}$.

6.59 Let X_1 and X_2 be independent and identically distributed random variables with support on the open interval $(0, 1)$ and common probability density function $f(x)$. Write an equation that $f(x)$ must satisfy so that $X_1 X_2 \sim U(0, 1)$.

6.60 Determine, by inspection, whether X and Y with distribution described by $f(x, y)$ given below, are independent or dependent random variables.

(a) $f(x, y) = \frac{1}{27}\left(x^3 + y\right)$ $\qquad x = 0, 1, 2;\ y = 1, 2$

(b) $f(x, y) = \frac{1}{40}x^2 y$ $\qquad (x, y) = (1, 1), (3, 1), (1, 3), (3, 3)$

(c) $f(x, y) = e^{-x}$ $\qquad x > 0;\ 0 < y < 1$

(d) $f(x, y) = 1/2$ $\qquad |x| + |y| < 1$

(e) $f(x, y) = 1/4$ $\qquad -1 < x < 1;\ -1 < y < 1$

6.81 In Buffon's needle problem, a needle of length l is tossed n times, and the number of times that it crosses one of the parallel lines that are a distance d apart is denoted by W. In order to estimate π, the fraction of crossings

$$\hat{p} = \frac{W}{n}$$

is equated to $\frac{2l}{\pi d}$ (in the case of $l \le d$) and π is solved for, yielding an estimate $\hat{\pi}$. The question here is: *how long should the needle be?* If one uses a long needle (l large), $\hat{p}$ will be close to one. On the other hand, if one uses a short needle (l small), $\hat{p}$ will be close to zero. A reasonable middle ground would be to choose l so as to maximize $V\left(\frac{W}{n}\right)$. Find the value of l that maximizes $V\left(\frac{W}{n}\right)$.

6.82 Let X and Y be random variables with $V[X] = 10$, $V[Y] = 16$, and $V[X+Y] = 24$. Find $\text{Cov}(X, Y)$.

6.83 The entries in the table below give the joint probability mass function of X and Y, where $p_1 + p_2 + p_3 + p_4 = 1$. Find $E[E[Y \mid X]]$.

x \ y	0	1
0	p_1	p_2
1	p_3	p_4

6.84 The entries in the table below give the joint probability mass function of X and Y, where $p_1 + p_2 + p_3 + p_4 = 1$. Find $E\left[X^5 \mid Y = 1\right]$.

x \ y	0	1
0	p_1	p_2
1	p_3	p_4

6.85 Let the random variable X be the number of successes in n mutually independent Bernoulli trials and the random variable $Y = n - X$ be the number of failures in these n mutually independent Bernoulli trials. Find ρ, the population correlation between X and Y.

6.86 For the joint probability density function defined by

$$f(x, y) = \frac{2}{k^2} \qquad 0 < x < y < k,$$

for some positive real constant k, find the conditional expected value of X given $Y = y$ and find the population correlation between X and Y.

6.87 Stefani draws three cards without replacement from a well-shuffled deck of cards. Let the random variable X be the number of spades and let the random variable Y be the number of diamonds. Find the population covariance between X and Y.

6.88 The random variables X and Y are uniformly distributed over the *interior* of a circle of radius 2 centered at (3, 4). Find $E[X \mid Y = 5]$.

6.89 Let X be an exponential random variable with population mean 2. Let $Y = 5X$. Find ρ, the population correlation between X and Y.

6.90 A die is weighted so that the number of spots that appear on the up face has probability mass function

$$f(x) = \frac{x}{21} \qquad x = 1, 2, \ldots, 6.$$

Bix rolls a pair of two such dice n times. Let the random variable X be the number of double sixes that are tossed and the random variable Y be the number of times that a three appears on one or both of the faces. Find the population correlation between X and Y.

6.91 Consider the square with vertices $(1, 1)$, $(-1, 1)$, $(1, -1)$, and $(-1, -1)$. Find the expected rectilinear (Manhattan) distance between a random point chosen in the interior of the square and the origin.

6.92 The random variables X and Y are uniformly distributed over the support

$$\mathcal{A} = \{(x, y) \,|\, 0 < x^2 < y < 4\}.$$

Find $E[Y \,|\, X = 1]$ and $V[Y \,|\, X = 1]$.

6.93 The first significant digit X_1 and the second significant digit X_2 have joint probability mass function

$$f(x_1, x_2) = \log_{10}\left(1 + \frac{1}{10x_1 + x_2}\right) \qquad x_1 = 1, 2, \ldots, 9;\; x_2 = 0, 1, 2, \ldots, 9.$$

(a) Are X_1 and X_2 defined on a product space?

(b) Are X_1 and X_2 independent?

(c) Find the marginal distribution of X_1.

(d) Find the marginal distribution of X_2.

(e) Find the population mean of X_2 to four digits.

6.94 Let X_1 and X_2 be independent standard normal random variables. Let $Y_1 = X_1 + X_2$ and $Y_2 = 2X_1 - X_2$.

(a) Find the distribution of Y_1.

(b) Find $\text{Cov}(Y_1, Y_2)$.

6.95 Let the random variables X and Y have joint probability density function

$$f(x, y) = 1 \qquad 0 < x < 1, 0 < y < 1.$$

Find $V[E[Y \,|\, X]]$.

6.96 Andrew draws two cards without replacement from a well-shuffled deck. Let X be the number of aces and Y be the number of face cards (that is, kings, queens, and jacks) drawn. Find $E[V[E[Y \,|\, X]]]$.

6.97 Consider the random variables X and Y defined on a product space with joint probability density function $f(x, y)$. Find $V[E[V[E[X \,|\, Y]]]]$.

7.18 Let X and Y be independent, continuous random variables with marginal probability density functions $f_X(x)$ and $f_Y(y)$, respectively. Let $Z = X + Y$. Use the cumulative distribution function technique to find the probability density function of Z.

7.19 Let $X \sim N(\mu, \sigma^2)$. The random variable $Y = e^X$ is known as a *lognormal* random variable. Find the probability density function of a lognormal random variable.

7.20 Let $X \sim \text{beta}(\alpha, \beta)$. The support of the beta distribution, $\mathcal{A} = \{x \mid 0 < x < 1\}$, limits its applicability. Some have suggested that by adding a positive location parameter c_1 and a positive scale parameter c_2 and transforming X by

$$Y = c_1 + c_2 X,$$

the resulting probability distribution has wider applicability. What is the probability density function of Y?

7.21 Let X have probability density function

$$f_X(x) = e^{-x} \qquad x > 0.$$

Find the probability density function of $Y = 1/X$ using

(a) the cumulative distribution function technique,

(b) the transformation technique.

7.22 Let Z_1 and Z_2 be independent standard normal random variables.

(a) Find the joint probability density function of $Y_1 = Z_1 + Z_2$ and $Y_2 = Z_1 - Z_2$.

(b) Are Y_1 and Y_2 independent?

(c) Find the marginal probability density function of Y_1.

7.23 Let $X \sim \text{Bernoulli}(p)$ and $Y \sim \text{Poisson}(\lambda)$ be independent random variables.

(a) Find the probability mass function of $X + Y$.

(b) Find the probability mass function of $X \cdot Y$.

7.24 Let $X \sim \text{Bernoulli}(p)$ and $Y \sim \text{Poisson}(\lambda)$ be independent random variables.

(a) Find the probability mass function of $\min\{X, Y\}$.

(b) Find the probability mass function of $\max\{X, Y\}$.

7.25 Let X_1, X_2, X_3 be mutually independent and identically distributed random variables, each with probability mass function

$$f(x) = \begin{cases} 1 - p & x = -1 \\ p & x = 1, \end{cases}$$

where p is a real-valued parameter satisfying $0 < p < 1$.

(a) Find the probability mass function of $X_1 + X_2 + X_3$.

(b) Find the probability mass function of X_1/X_2.

7.26 Let X be a continuous random variable with probability density function

$$f_X(x) = \frac{1}{x^2} \qquad x > 1.$$

Let Y be a continuous random variable with the same probability distribution as X. Assume that X and Y are independent. Find the probability density function of $Z = XY$.

7.27 Let X_1 and X_2 be continuous random variables with joint probability density function $f_{X_1,X_2}(x_1, x_2)$ defined on the support $\{(x_1, x_2) \mid x_1 > 0, x_2 > 0\}$. Give an expression for the probability density function of $X_1^{X_2}$.

7.28 Let X_1 and X_2 be independent $U(0, 1)$ random variables. Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1/(X_1 + X_2)$.

(a) Find the joint probability density function of Y_1 and Y_2.

(b) Find the probability density function of Y_1.

(c) Find the probability density function of Y_2.

7.29 Let the distribution of the continuous random variable X be described by the probability density function $f_X(x)$ on positive support $\{x \mid x > 0\}$. Likewise, let the distribution of the continuous random variable Y be described by the probability density function $f_Y(y)$ on positive support $\{y \mid y > 0\}$. Assume that X and Y are independent.

(a) Derive an expression for the probability density function of X/Y.

(b) Use the expression from part (a) to find the probability density function of the quotient of two independent and identically distributed $\chi^2(1)$ random variables.

7.30 Arno and Ted play the following game. Each independently selects a real number that is greater than 1. If the first digit of the product of the numbers is 1, 2, or 3, Arno wins. If the first digit of the product of the numbers is 4, 5, 6, 7, 8, or 9, Ted wins. Using his knowledge of probability, Arno generates his real number X from a probability distribution with probability density function

$$f_X(x) = \frac{1}{x \ln 10} \qquad 1 \le x < 10.$$

Find the probability that Arno wins if

(a) Ted generates Y via a probability distribution with probability density function

$$f_Y(y) = 1 \qquad 1 < y < 2.$$

(b) Ted generates Y via a probability distribution with probability density function

$$f_Y(y) = 2(y - 1) \qquad 1 < y < 2.$$

This game appeared in a chapter by Arno Berger and Ted Hill titled "A Short Introduction to the Mathematical Theory of Benford's Law" from the book titled "Benford's Law: Theory and Applications," edited by Steven J. Miller, published in 2015 by Princeton University Press. The game first appeared in "The Multiplication Game" by K.E. Morrison in *Mathematics Magazine*, Volume 83, 2010, pages 100–110.

7.31 Let the distribution of the continuous random variable X be described by the probability density function $f_X(x)$ on positive support $\{x \mid x > 0\}$. Likewise, let the distribution of the continuous random variable Y be described by the probability density function $f_Y(y)$ on positive support $\{y \mid y > 0\}$. Assume that X and Y are independent.

(a) Derive an expression for the probability density function of the product of X and Y.

(b) Use the expression from part (a) to find the probability density function of the product of the independent and identically distributed random variables X and Y with identical probability density functions

$$f_X(x) = \frac{1}{\sqrt{2\pi}x} e^{-(\ln x)^2/2} \qquad x > 0.$$

7.32 Euclidean Eunice walks along a line from the origin of a Cartesian coordinate system at $(0, 0)$ to a random point (X, Y), where X and Y have joint probability density function

$$f(x, y) = 1 \qquad 0 < x < 1, 0 < y < 1.$$

Find the probability density function of the distance that she will walk.

7.33 A stick of unit length is broken at two random points X_1 and X_2 with joint probability density function $f(x_1, x_2)$ defined on the unit square $\mathcal{A} = \{(x_1, x_2) \mid 0 < x_1 < 1, 0 < x_2 < 1\}$.

(a) Find an expression for the probability that the portion of the stick that was broken on both ends (that is, the *middle* stick) is the longest of the three sticks.

(b) Find an expression for the expected length of the middle stick.

(c) Find an expression for the expected product of the lengths of the end sticks.

7.34 Let $X_1, X_2, \ldots, X_5$ be mutually independent exponential(λ) random variables.

(a) Find the probability density function of the second order statistic $X_{(2)}$.

(b) Find $P\left(X_{(2)} \le 1/\lambda\right)$.

7.35 Let X_1 and X_2 be independent $U(0, 4)$ random variables. Let $X_{(1)}$ and $X_{(2)}$ be the associated order statistics.

(a) Find the joint probability density function of $X_{(1)}$ and $X_{(2)}$.

(b) Find $F_{X_{(1)}, X_{(2)}}(1, 3)$.

7.36 Let $X_1, X_2, \ldots, X_n$ be mutually independent and identically distributed $U(0, 1)$ random variables. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the associated order statistics. Find the population covariance between $X_{(1)}$ and $X_{(n)}$.

7.37 Cookie chooses n points randomly and independently on the circumference of a circle. What is the probability that the interior of the n-sided polygon obtained by connecting adjacent points contains the center of the circle?

7.38 Let X_1 and X_2 be independent random variables, each from a population probability distribution with common probability density function:

$$f(x) = \frac{2}{x^3} \qquad x > 1.$$

Find the joint probability density function of $Y_1 = X_1X_2$ and $Y_2 = X_2$.

7.47 Consider the linear programming problem:

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 \\ \text{subject to} & Ax_1 + Bx_2 \le 1 \\ & x_1 \ge 0 \\ & x_2 \ge 0 \end{array}$$

where the random coefficients A and B are independent $U(0, 1)$ random variables. Find the population median of the value of the objective function, $X_1 + X_2$, at its optimal value, where X_1 and X_2 are the random optimal solution values.

7.48 Let X_1 and X_2 be independent $U(0, \theta)$ random variables, where θ is a fixed, positive parameter.

(a) Find the probability density function of the sample mean

$$\bar{X} = \frac{X_1 + X_2}{2}.$$

(b) Find the probability density function of the sample standard deviation

$$S = \sqrt{(X_1 - \bar{X})^2 + (X_2 - \bar{X})^2}.$$

7.49 Let $X_1 \sim U(a, b)$, where $0 < a < b$, and $X_2 \sim U(-\pi, \pi)$ be independent random variables. Find the joint probability density function of

$$Y_1 = X_1 \cos X_2 \qquad \text{and} \qquad Y_2 = X_1 \sin X_2.$$

7.50 Let X_1 and X_2 denote a random sample from a $N(\mu, \sigma^2)$ population. Consider the random variables $(X_1 + X_2)/2$ and $(X_1 - X_2)^2/2$, which happen to be the sample mean and the unbiased version of the sample variance. Graphically and/or analytically examine the properties of this transformation.

7.51 Consider the stochastic project network precedence diagram shown in Figure 7.16. Let X_1, X_2, and X_3 be mutually independent $U(0, 1)$ random variables that denote the activity duration times. Find the probability density function of the project completion time, which is $\max\{X_1 + X_2, X_3\}$.

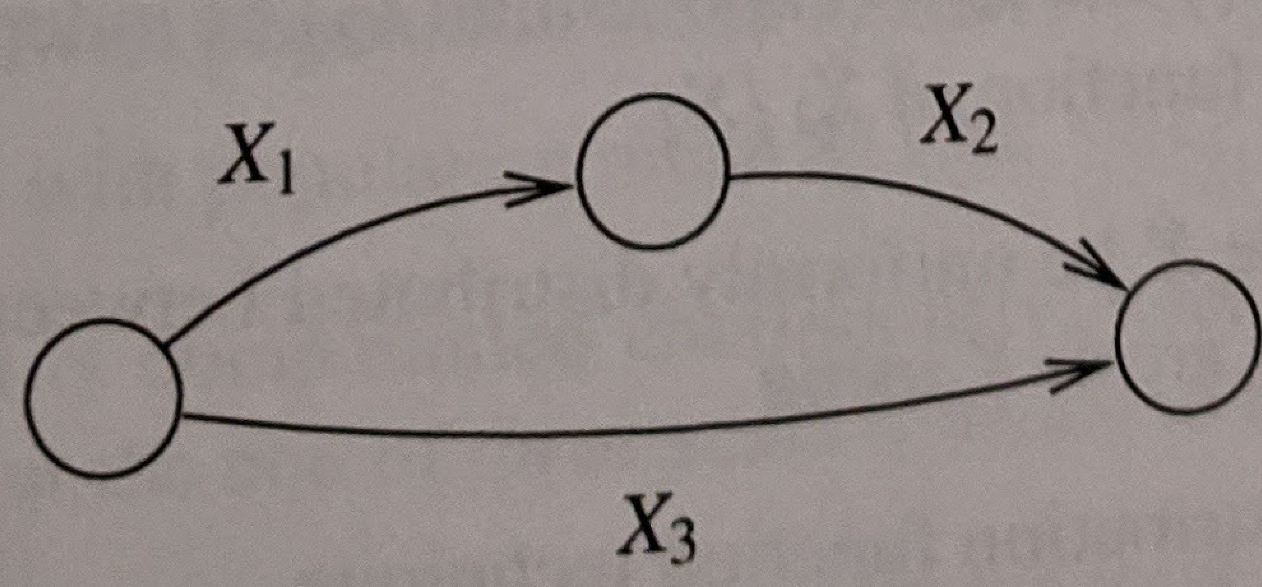

Figure 7.16: Stochastic project network precedence diagram.

7.52 Annette draws two cards without replacement from a well-shuffled deck of 52 cards. Let X_1 be the number of black cards (that is, clubs and spades) and let X_2 be the number of diamonds. Find the probability mass function of $2X_1^2 + X_2$.