4.21 Plot $E\left[(X-c)^2\right]$ and $E\left[|X-c|\right]$ for $X \sim$ binomial(3, 1/2) for $-1 < c < 4$.

4.22 Let X be the number of fives that appear in 100 rolls of a fair die.

(a) Write a mathematical expression for $P(10 \le X \le 20)$.

(b) Write an R statement to calculate $P(10 \le X \le 20)$.

4.23 Chet rolls a single die 600 times and gets only 81 ones. Based on these results, you suspect that his die is biased (as opposed to fair), yielding ones with some probability that is less than 1/6. One way to test your hypothesis is to calculate the probability of getting 81 or fewer ones in 600 rolls assuming that the die is fair. If this probability is small, then there is statistically significant evidence that the die is biased. Calculate the probability of getting 81 or fewer ones in 600 rolls of a fair die and write a sentence or two interpreting the result.

4.24 Ian rides across the country on his bicycle in 60 days. Every day on his trip, he purchases two bottles of "Purple Passion" grape soda. If the soda company claims that the probability that a consumer will win a prize on one of their bottle cap contests is 1/12, find the probability that Ian will win one or more prizes on his trip.

4.25 The World Series is being played by the Cubs and Sox. Let p be the probability that the Cubs win on a particular game, $0 < p < 1$. Assume that the outcome of each game is independent. If instead of the current "play until you win 4 games," the Commissioner is considering a marathon series consisting of "play until you win 20 games." Plot the probability that the Cubs win the series for both scenarios on $0 < p < 1$ on a single set of axes using R.

4.26 The 2004 United States Presidential election was predicted to be quite close. Assume that the incumbent, President George W. Bush, has 260 electoral votes secure. Assume that the challenger, Senator John Kerry, also has 260 electoral votes secure. The only "swing" states where the outcome of the election hangs in the balance are: Colorado (9 electoral votes), New Mexico (5 electoral votes), and New Hampshire (4 electoral votes). Assume that the outcome of each of these three swing states is a Bernoulli trial with parameter p, where p is the probability that President Bush wins the state. Use R to plot:

(a) the probability of a win for President Bush, a win for Senator Kerry, and a tie,

(b) the expected number of electoral votes for President Bush and Senator Kerry,

for $0 < p < 1$.

4.27 Consider an m-member jury that requires n or more votes to convict a defendant. Let p be the probability that a juror votes a guilty person innocent and let q be the probability that a juror votes an innocent person guilty, $0 < p < 1, 0 < q < 1$. Assuming that r is the fraction of guilty defendants and that jurors vote independently, what is

(a) the probability a defendant is convicted?

(b) the probability a defendant is convicted when $n = 9$, $m = 12$, $p = 1/4$, $q = 1/5$, and $r = 5/6$? Use R to calculate the result.

4.28 Let X be a binomial random variable with population mean $\mu = 3$ and population variance $\sigma^2 = 2$. Find $P(X \ge 2)$.

4.29 Ten fair coins are tossed. Each coin showing tails is tossed a second time. Finally, each coin still showing tails is tossed a third time. Let the random variable X be the number of heads showing after the third set of tosses.

(a) Find $f(x)$.

(b) Find $E[X]$.

(c) Write and execute a Monte Carlo simulation to give numerical support to your solution to part (b).

4.30 Some consider the digits after the decimal in $\pi = 3.14159265\ldots$ to be mutually independent random digits. If this conjecture is assumed to be true, write an R statement to calculate the probability that there will be 8 or fewer fours in the first 100 digits after the decimal point in π.

4.31 A coin has a probability p of turning up heads when tossed. The coin is tossed repeatedly until a head appears. Let X denote the number of tails that occur before the first head is tossed. Find the probability mass function of Y, the remainder when X is divided by 4.

4.32 Let X denote the number of tails prior to the first occurrence of a head in repeated tosses of a fair coin. Find $P(X \bmod 5 = 2)$.

4.33 There is a bear in the woods. Some think the bear is dangerous, others do not. A hunter can hit the bear on each shot with $p = 0.4$. Each shot is independent (slow bear). What is the minimum number of bullets that the hunter should carry to be at least 75% certain that he can hit the bear?

4.34 Katie has decided to sell her car, so she places an ad in the local newspaper. She will sell the car upon the first offer that exceeds some constant c. If the offer amounts are independent, each with cumulative distribution function $F(x)$, what is the expected number of offers that it takes for Katie to sell her car?

4.35 What is the minimum number of times that Pedro should play poker in order to be at least 95% certain that he will be dealt a four-of-a-kind?

4.36 Let the random variable X denote the number of independent rolls of a fair die required to obtain the *second* occurrence of a "five." Find $P(X \geq 4)$.

4.37 The probability of winning a game of craps is $244/495$. If Barbara plays craps repeatedly, find the probability that her third win will occur on the fifth game she plays.

4.38 Mr. Legonaton bets repeatedly on double-zero in roulette until he wins three times. If double-zero occurs with probability $1/38$, find the probability mass function, expected value, and population standard deviation of the number of bets that Mr. Legonaton will make.

4.39 Consider repeated independent Bernoulli trials with probability of success p. What is the probability mass function of X, the *number of trials before* the rth success, where r is a positive integer?

4.40 An automobile manufacturer is implementing a "zero-defects" drive in order to improve the quality of their automobiles. They have found that the number of manufacturing defects on each car follows a Poisson distribution. If their goal is to have at least 98 percent of their cars defect-free, find the largest population mean μ that achieves their goal.

4.48 Let X be a Poisson random variable with population mean λ. Find the probability that X is even. *Hint*: consider the expansion of $e^{-\lambda} + e^{\lambda}$.

4.49 Let X be a Poisson(λ) random variable. Find the value of λ that maximizes $P(X = x)$.

4.50 Show that a Poisson (λ) random variable X has a mode at $x = \lfloor \lambda \rfloor$ for noninteger λ and no mode when λ is an integer.

4.51 The number of customers arriving daily to TVs-R-Us is a Poisson(λ) random variable. If each customer's decision to buy a TV is an independent Bernoulli trial, where p is the probability of making a purchase, give an expression for the probability mass function of the number of TVs purchased daily.

4.52 The *probability generating function* of a discrete random variable X having support on the nonnegative integers is

$$G(t) = E\left[t^X\right].$$

Find the probability generating function of $X \sim$ Poisson(λ).

4.53 Let $X \sim$ Poisson(λ). Find $E[X(X-1)]$.

4.54 Let $X \sim$ Poisson(λ). Find λ such that $f_X(2) = f_X(4)$.

4.55 Let $X \sim$ Poisson(λ). If $f_X(2) = f_X(3)$, find $E\left[X^2\right]$.

4.56 Find the probability that a 13-card hand dealt from a well-shuffled deck contains 11 or more spades.

4.57 There are 1000 people who have each put a single ticket into a lottery for 50 identical prizes. The prizewinners will be selected without replacement from the entries. Cruella has possession of these lottery tickets and has hatched the following evil scheme. Rather than putting in a single additional ticket for herself as the rules dictate, she will put in k tickets. If a single ticket is drawn with her name on it, she will win a prize. But if two or more of her tickets are drawn, her cheating will be detected, she will be disqualified, and will receive no prizes. Cruella is not concerned about her reputation if she is caught cheating, for it is already bad. Find the value of k that maximizes the probability that Cruella will win a prize.

4.58 A random sample of size 12 is taken without replacement from a lot of 100 items that contains 8 defectives.

(a) Find the probability that there are exactly two defectives in the sample.

(b) Find the probability that there are two or fewer defectives in the sample.

4.59 Mrs. Sindy gives a spelling test to confirm that her students can spell basic words. Her test consists of n words, and a student passes if he or she makes 0, 1, or 2 errors.

(a) Let p be the true proportion of words that a student can spell correctly. Give a general expression for P(passing) for $0 < p < 1$ and $n = 2, 3, \ldots$ assuming that the population of basic words is infinite. Make a plot of p on the horizontal axis versus P(passing) on the vertical axis for $n = 5$, 10, and 20.

(b) Give a general expression for P(passing) for $n = 2, 3, \ldots$ assuming that there are 100 basic words and the students can spell r of these words correctly. Mrs. Sindy chooses, of course, to sample her words without replacement. Make a plot of the true proportion of words a student can spell correctly versus P(passing) for $n = 5$, 10, and 20.

5.6 Exercises

5.1 Let $X \sim U(-5, 5)$. Find $E\left[\left||X|-2\right|\right]$.

5.2 Let $X \sim U(2, 9)$. Find $E\left[X - \lfloor X \rfloor\right]$.

5.3 Let $X \sim U(0, 1)$. Find the cumulative distribution function of $Y = 1/X$.

5.4 Let $X \sim U(0, 3)$. Find the probability mass function of $Y = \lceil 2X \rceil - 2\lfloor X \rfloor$.

5.5 Let $X \sim U(0, 1)$. Find $V\left[\sqrt{X}\right]$.

5.6 Let $X \sim U(0, 10)$. Find $P(2X < 10 < 5X)$.

5.7 Let $X \sim U(0, \theta)$, for some positive parameter θ. Find θ such that $E[X] = V[X]$.

5.8 Let $X \sim U(0, 3)$. Let $Y = \lfloor 2X \rfloor - \lceil X \rceil$.

(a) Find $f_Y(y)$.

(b) Find $V[Y]$.

5.9 The radius of a circle is a $U(1, 5)$ random variable.

(a) Find the probability that the area exceeds the circumference of the circle.

(b) Find the expected area of the circle.

5.10 A random variable X assumes the value of a Bernoulli$(1/3)$ random variable with probability $1/4$ and assumes the value of a $U(0, 1)$ random variable with probability $3/4$. Find the moment generating function of X.

5.11 Locating pleasant facilities, such as parks, hospitals, or fire stations, is often done in a manner that minimizes the expected traveling distance for patrons. Likewise, locating unpleasant facilities, such as sewage treatment plants or high power cables, is often done in a manner that maximizes the expected traveling distance.

(a) Consider locating a pleasant facility on a road of length a where demand positions X that occur are located as $U(0, a)$ random variables. What is the position of the facility on the road x_0, such that $E\left[|X - x_0|\right]$ is minimized?

(b) Where should the facility be located for any arbitrary demand distribution with probability density function $f(x)$ defined on the length of the road?

(c) Where should the facility be located for any arbitrary demand distribution with probability mass function $f(x)$ defined on the length of the road?

5.12 A deck is made up of slats that are 6 inches wide with negligible distance between the slats. A frisbee with a 8 inch diameter lands at a random position on the deck. Find the probability that the frisbee covers a portion of three slats.

5.13 Consider the continuous random variable X with probability density function

$$f(x) = \frac{1}{b-a} \qquad a < x < b$$

for real constants $a < b$. Write this probability density function parameterized by its population mean μ and its population standard deviation σ.

5.43 The random variable $X \sim N(\mu, \sigma^2)$ has moment generating function

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2} \qquad -\infty < t < \infty.$$

What is the moment generating function and distribution of $Y = 3X + 4$?

5.44 Assume that men's heights, in inches, are $N(70, 16)$ random variables, and women's heights, in inches, are $N(67, 9)$ random variables. Let X_1 be the height of a man and X_2 be the height of a woman selected at random. Conduct a Monte Carlo simulation experiment in R to estimate the 1st and 99th percentile of the taller of the man and woman, that is, estimate $y_{0.01}$ and $y_{0.99}$ associated with $Y = \max\{X_1, X_2\}$. Use 1000 simulated pairs.

5.45 Let X be normally distributed with population mean $\mu = 2$ and population variance $\sigma^2 = 9$.

(a) What is $P(X < 8)$?

(b) What is $P(4 < X < 8)$?

(c) What is $P(X > 10)$?

(d) Find a constant a such that $P(X < a) = 0.95$.

(e) Find a constant b such that $P(-b < X < b) = 0.95$.

(f) If six random variables with this probability distribution are observed, what is the probability that exactly four of them are less than eight?

5.46 A fair die is tossed 1000 times.

(a) Find the expected number of threes.

(b) Find the probability of 200 or fewer threes exactly and by using the normal approximation to the binomial distribution.

(c) Find the probability of 200 or fewer threes exactly and by using the normal approximation to the binomial distribution given that there were exactly 500 even numbers that appeared in the 1000 tosses.

5.47 Adult men's heights (in inches) are normally distributed with $\mu = 70$ and $\sigma^2 = 16$. Adult women's heights (in inches) are normally distributed with $\mu = 67$ and $\sigma^2 = 9$. Members of the "Beanstalk Club" must be at least six feet tall.

(a) If a man and a woman are selected at random from their respective populations, what is the probability that both are eligible for the Beanstalk Club?

(b) If a woman who is eligible for the Beanstalk Club is selected at random, what is her expected height?

5.48 IQ scores are normally distributed with population mean 100 and standard deviation 10. If six independent IQ scores are collected, what is the probability that exactly three are less than 90, exactly two are between 90 and 120, and exactly one is greater than 120?

5.49 Show that the probability density function for $X \sim N(\mu, \sigma^2)$ has inflection points at the x-values $\mu \pm \sigma$.

5.50 Show that for $Z \sim N(0, 1)$ with probability density function $f_Z(z)$,

$$\int_z^\infty w f_Z(w)\,dw = f_Z(z).$$

5.66 The continuous random variable X has the *power distribution* with probability density function

$$f(x) = \frac{\beta x^{\beta-1}}{\alpha^{\beta}} \qquad 0 < x < \alpha,$$

where α is a positive scale parameter and β is a positive shape parameter. Find the population median of X.

5.67 The continuous random variable X has the *standard power distribution* with probability density function

$$f(x) = \beta x^{\beta-1} \qquad 0 < x < 1,$$

where β is a positive shape parameter.

(a) Find the population mean of X.

(b) Find the population median of X.

5.68 Let X be a continuous random variable with probability density function

$$f(x) = ax + b \qquad 0 < x < 1,$$

where a and b are parameters.

(a) What conditions are required for the parameters a and b?

(b) Name two special cases of this distribution associated with particular settings of the parameters a and b which coincide with common continuous parametric distributions.

5.69 Show that the *standard double-exponential* (Laplace) random variable X with probability density function

$$f(x) = \frac{1}{2}e^{-|x|} \qquad -\infty < x < \infty$$

has moment generating function $M(t) = 1/\left(1-t^2\right)$, for $|t| < 1$.

5.70 Consider the random variable X having the *standard double exponential distribution* (Laplace) with probability density function

$$f(x) = \frac{1}{2}e^{-|x|} \qquad -\infty < x < \infty.$$

Find $E\left[X^n\right]$ for any positive integer n.

5.71 For each probability below, give an expression and *carefully* plot (that is, use a computer or carefully plot by hand) on a single set of axes.

(a) $P(\mu - k\sigma < X < \mu + k\sigma)$ when $X \sim N(\mu, \sigma^2)$ for $0 \le k \le 4$.

(b) $P(\mu - k\sigma < X < \mu + k\sigma)$ when X is exponentially distributed with population mean 2 for $0 \le k \le 4$.

(c) $P(\mu - k\sigma < X < \mu + k\sigma)$ when X has a Poisson distribution with population mean 4 for $0 \le k \le 4$.

(d) The lower bound on $P(\mu - k\sigma < X < \mu + k\sigma)$ provided by Chebyshev's inequality for $1 \le k \le 4$.

5.14 Let $X \sim U(0, 5280)$. Find the probability mass function of $Y = \gcd(\lceil X \rceil, 10)$, where $\lceil \cdot \rceil$ is the ceiling function and $\gcd(\cdot, \cdot)$ is the greatest common divisor function (for example, $\gcd(9, 12) = 3$).

5.15 A stick of length a is broken at a random point that is equally likely along the length of the stick. What is the probability that the longer piece is more than twice as long as the shorter piece?

5.16 Matrix theory traditionally emphasizes matrices whose elements are real or complex constants. But what if the elements of a matrix are random variables? Such matrices are referred to as "stochastic" or "random" matrices. Although a myriad of questions can be asked concerning random matrices, our emphasis here will be limited to the following question: if the elements of a 3×3 matrix are independent $U(0, 1)$ random variables with positive diagonal elements and negative off-diagonal elements, what is the probability that the matrix has a positive determinant? That is, what is the probability that

$$\begin{vmatrix} +u_{11} & -u_{12} & -u_{13} \\ -u_{21} & +u_{22} & -u_{23} \\ -u_{31} & -u_{32} & +u_{33} \end{vmatrix} > 0,$$

where the u_{ij}'s are mutually independent $U(0, 1)$ random variables? This question is rather vexing using probability theory due to the appearance of some of the random numbers multiple times in the expression for the determinant. Use Monte Carlo simulation in R to estimate the probability of interest.

5.17 Consider a circle with a random radius $R \sim U(0, 5)$. What is the probability that the area of the circle is greater than twice the circumference? Support your solution by conducting a Monte Carlo simulation experiment.

5.18 What is the probability mass function of the value printed by the following R code?

```
count = 0
for (i in 1:10000)
  if (sum(runif(2)) > 1.5) count = count + 1
print(count)
```

5.19 Let $X \sim U(0, \theta)$. Find the probability density function of

$$Y = \frac{X - \mu}{\sigma},$$

where $\mu = E[X]$ and $\sigma^2 = V[X]$.

5.20 Show that if $X \sim$ exponential(λ), then $\lambda X \sim$ exponential(1).

5.21 Show that if $X \sim$ exponential(λ), then $Y = \ln X$ has an *extreme value distribution* with cumulative distribution function

$$F_Y(y) = 1 - e^{-\lambda e^y} \qquad -\infty < y < \infty.$$

5.22 In a *time-truncated life test*, there are n light bulbs simultaneously placed on test at time 0. The test is concluded at time $c > 0$. Assuming that the lifetimes of the light bulbs are drawn from an exponential population with population mean $1/\lambda$, where $\lambda > 0$, find the distribution of the *number* of failures that occur by time c.