Lecture 10 - PMFs/PDFs and Expected Value

Thursday, February 27, 2020 10:52 AM

Defn: PDF

Analog to PMF but for cts r.v.s. PDF f satisfies

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

Key connection: $\frac{d}{dx} F(x) = f(x)$

and, generally,

discrete:

$$\mathbb{P}(X \in A) = \sum_{x \in A} f(x)$$

continuous

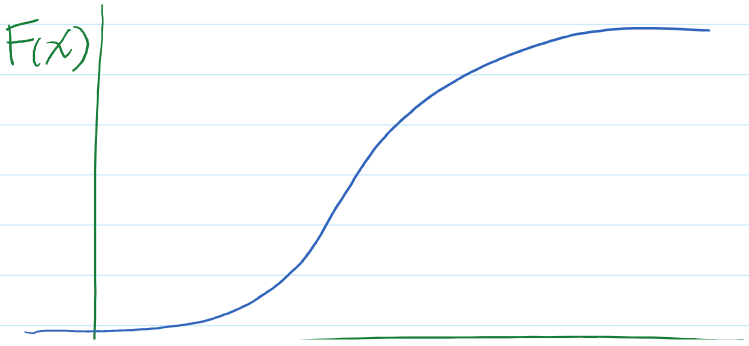
$$\mathbb{P}(X \in A) = \int_{x \in A} f(x)\,dx$$

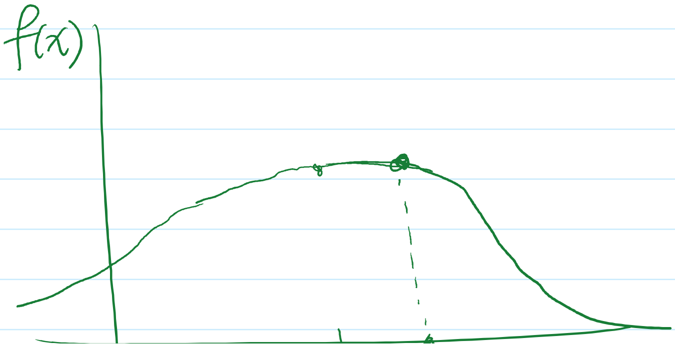
Ex.

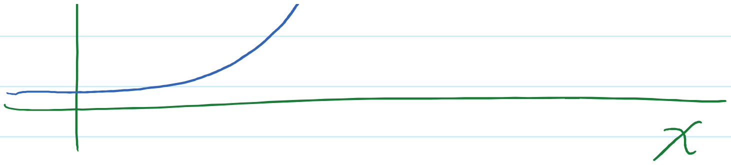
$$F(x) = \frac{1}{1+e^{-x}}$$

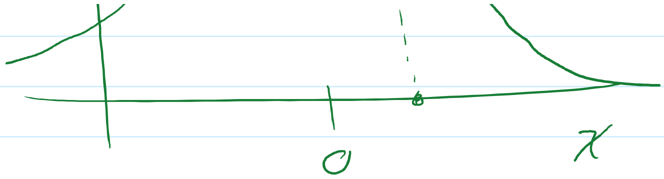
so the corresponding PDF

$$f(x) = F'(x) = \frac{dF}{dx} = \frac{e^{-x}}{(1+e^{-x})^2}.$$







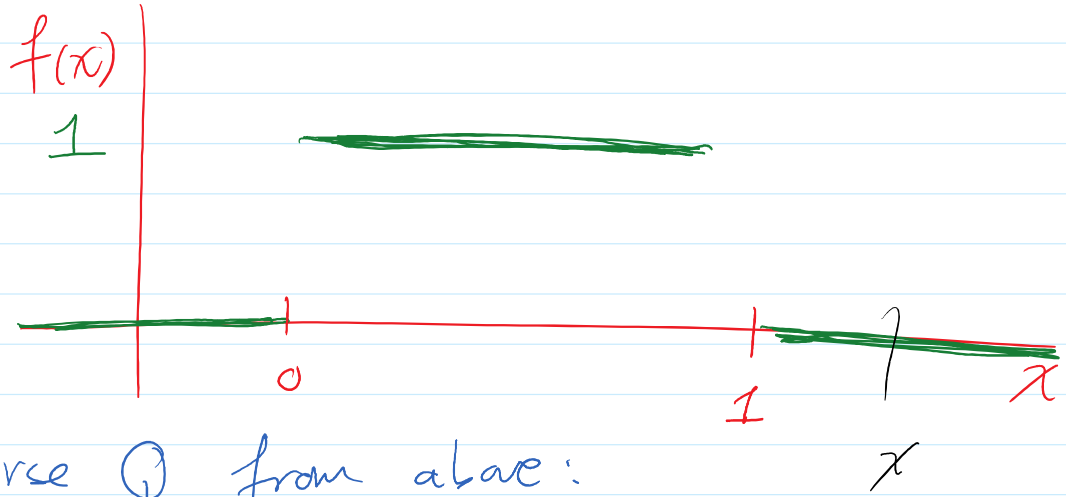


Ex Continuous Uniform Distribution

$$X \sim U(0,1) \quad \leftarrow \text{cts unif. dist. on } [0,1]$$

this means

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$



Reverse Q from above:

$$\text{CDF} \xrightarrow{\text{differentiation}} \text{PDF}$$

$$\text{PDF} \xrightarrow{\text{integrate}} \text{CDF}$$

What is the CDF?

$$F(x) = \int^{x} f(t)\,dt = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x \in [0,1] \end{cases}$$

$$F(x) = \int_{-\infty} f(t)\,dt = \begin{cases} x & \text{if } x\in[0,1] \\ 1 & \text{if } x \geq 1 \end{cases}$$

$x \leq 0$

Case 1: $\displaystyle\int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0$

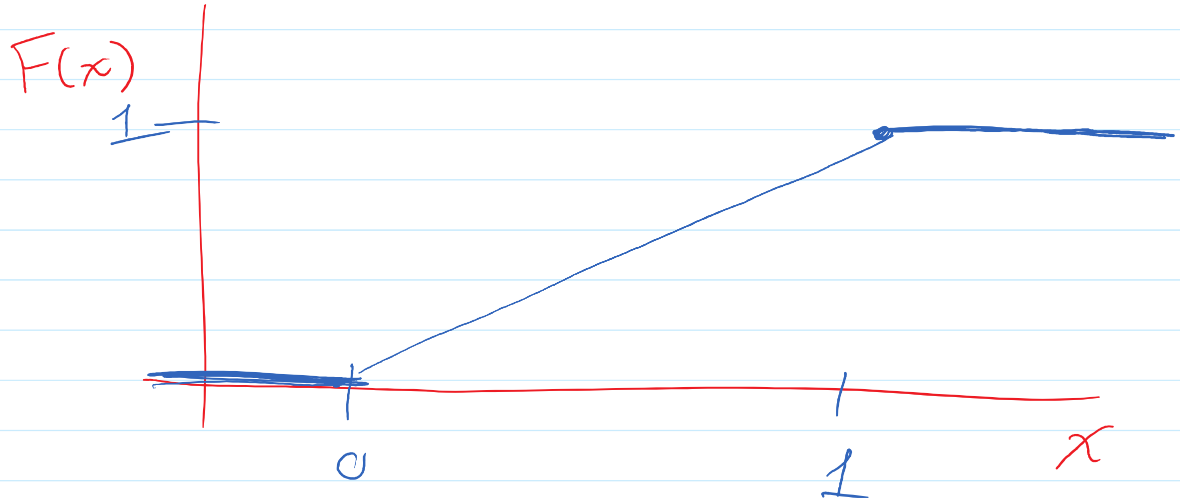
$x \in [0,1]$

Case 2: $\displaystyle\int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} f(t)\,dt = \int_{0}^{x} dt = [t]_{0}^{x} = x$

Case 3: $x \geq 1$

$$\int_{-\infty}^{x} f(t)\,dt = \int_{0}^{1} f(t)\,dt = \int_{0}^{1} dt = 1$$

plot.

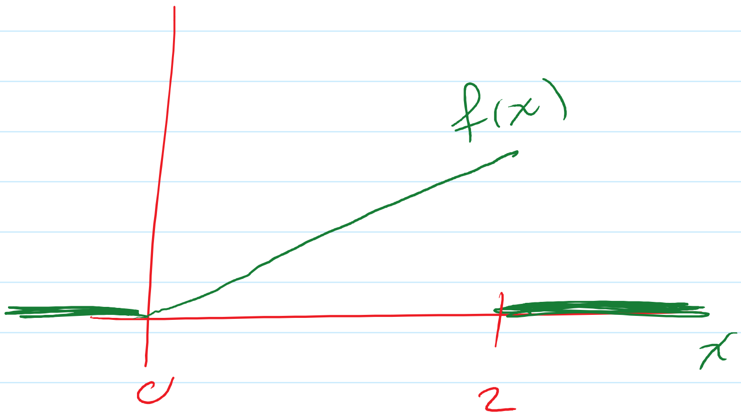


Ex, Recall $\displaystyle P(X \in A) = \int_A f(t)\,dt$

Let

$$f(x) = \begin{cases} x/2 & 0 < x < 2 \end{cases}$$

$$f(x) = \begin{cases} 2 & \\ 0 & \text{else} \end{cases}$$



Q: $\mathbb{P}(X > 1)$?

$$\mathbb{P}(X > 1) = \int_1^{\infty} f(t)\,dt = \int_1^{2} f(t)\,dt$$

$$= \int_1^2 \frac{t}{2}\,dt = \left[\frac{t^2}{4}\right]_1^2$$

$$= \frac{4}{4} - \frac{1}{4} = 3/4 .$$

Ex, Lets assme X has a CDF

$$F(x) = 1 - e^{-x}$$

$\mathbb{P}(1 < X < 2)$?

$$\mathbb{P}(1 < X < 2) = F(2) - F(1).$$

$$= (1 - e^{-2}) - (1 - e^{-1})$$

$$\boxed{= e^{-1} - e^{-2}}$$

Second solution: find PDF and integrate over 1 to 2

$$f(x) = \frac{dF}{dx} = \frac{d}{dx}(1 - e^{-x}) = e^{-x}$$

So

$$\mathbb{P}(1 < X < 2) = \int_1^2 f(t)\,dt$$

$$= \int_1^2 e^{-t}\,dt = \left[-e^{-t}\right]_1^2$$

Recall!

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$= -e^{-2} + e^{-1}$$

$$\boxed{= e^{-1} - e^{-2}}.$$

Theorem: PMF/PDF Characterization

A function f is a valid pmf or pdf iff

① $f(x) \geq 0 \quad \forall x \in \mathbb{R}$

② (discrete) $\sum_{x\in\mathbb{R}} f(x) = 1$

(continuous) $\int_{\mathbb{R}} f(x)\,dx = 1$

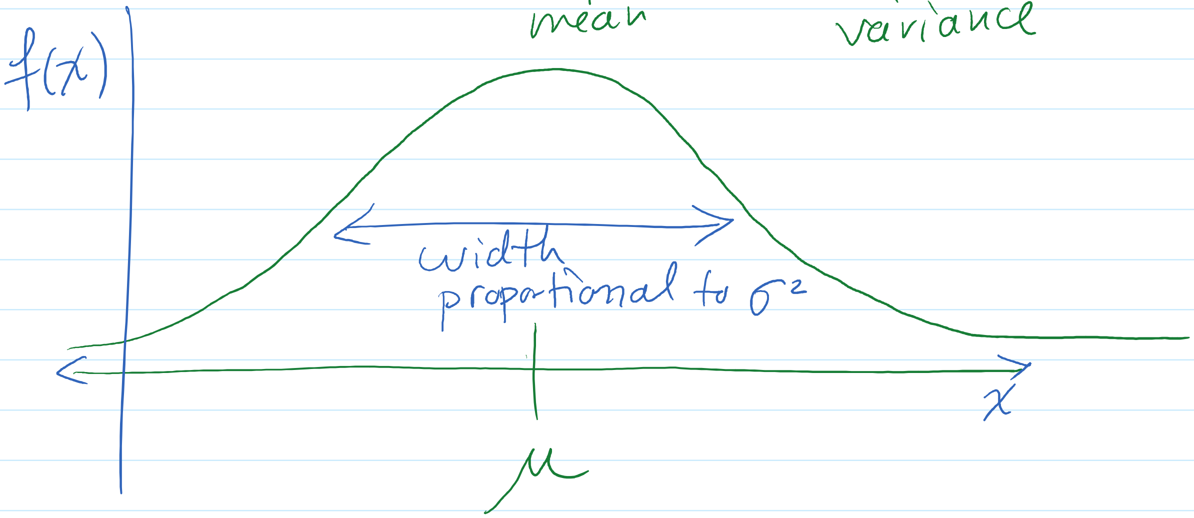
Ex. Normal Distribution

(Gaussian Distribution)

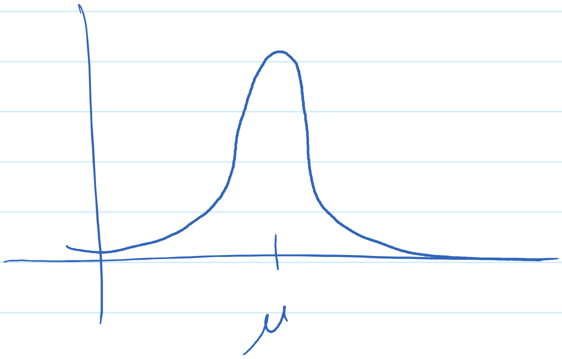
$X \sim N(\mu, \sigma^2)$ μ and σ^2 are parameters

$\mu \in \mathbb{R}$ and $\sigma^2 > 0$

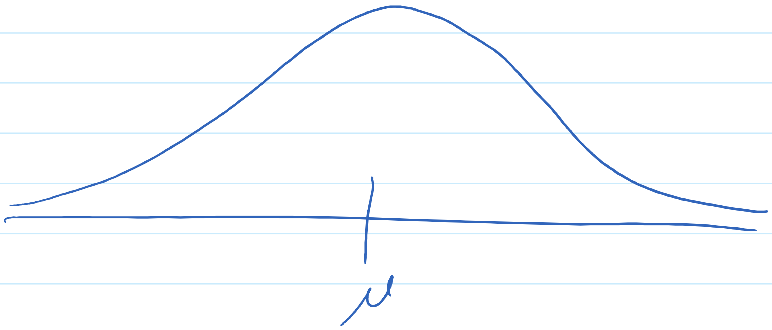
mean variance



Small σ^2



large σ^2



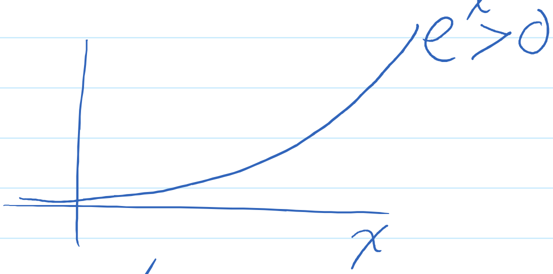
the PDF has the form

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right), \quad x \in \mathbb{R}$$

$$\exp(a) = e^{a}$$

Special case "standard" normal $X \sim N(0,1)$

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$$



Is this a valid PDF?

① $f(x) \geq 0 \quad \forall x$ ✓

② $\int_{\mathbb{R}} f(x)\,dx = 1$. $P(X \in \mathbb{R}) = 1$

$$\underbrace{\int \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)dx}_{I} = 1$$

same number

Equiv. since $I \geq 0$ I could show $I^2 = 1$

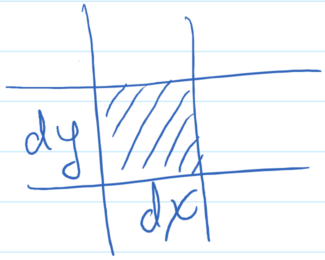
Equiv. Since $I \geq 0$ I could show $I = 1$

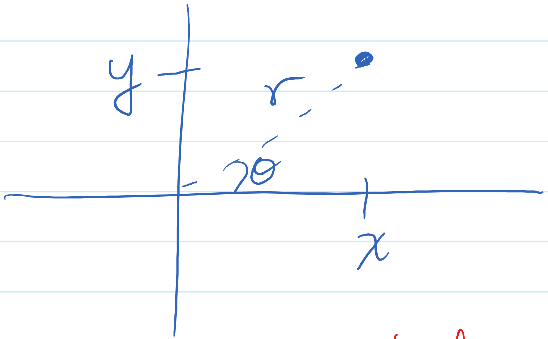
$$I^2 = I \cdot I = \left(\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}x^2)\, dx\right)\left(\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}y^2)\, dy\right)$$

$$= \iint_{\mathbb{R}\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}x^2) \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}y^2)\, dx\, dy$$

$$\boxed{= \frac{1}{2\pi} \int_{\mathbb{R}^2} \exp\left(-\tfrac{1}{2}(x^2+y^2)\right) dx\, dy}$$

in cartesian





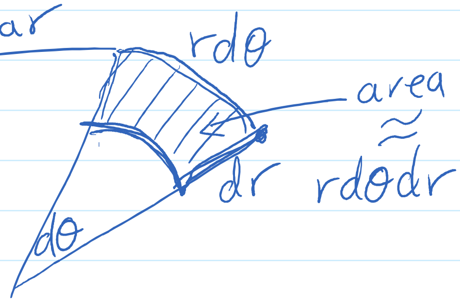
$dxdy \longrightarrow r dr d\theta$

$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

in polar



$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{-\frac{1}{2}r^2} \underline{r} dr d\theta$$

u-substitution

$u = \frac{1}{2}r^2 \quad du = r dr$

$u = \frac{1}{2} r \qquad du = r\,dr$

$$\int_0^\infty e^{-\frac{1}{2}r^2} r\,dr = \int e^{-u}\,du = -e^{-u} = \left[-e^{-\frac{1}{2}r^2}\right]_0^\infty$$

$$= \left[0 - (-1)\right]$$

$$= 1$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1 = I^2 \text{ so } I = 1.$$

Expected Value

If X is a r.v. then the expected value or mean of X is defined as

① discrete: $\mathbb{E}[X] = \sum_{x \in \mathbb{R}} x f(x)$

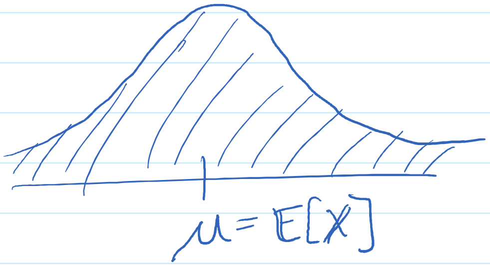
② Continuous: $\mathbb{E}[X] = \int_{\mathbb{R}} x f(x)\,dx$

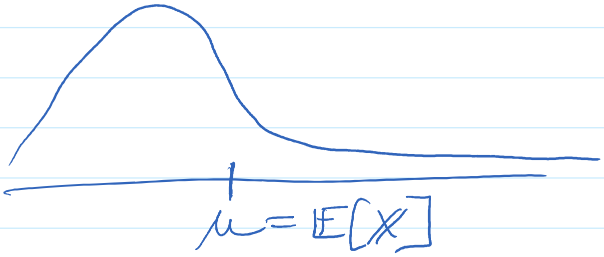
$\mathbb{E}$ is basically a weighted sum of the possible values of X, weighted by their likelihood (density or mass)

discrete case:

$$x_1 P(X=x_1) + x_2 \mathbb{P}(X=x_2) + x_3 P(X=x_3) + \cdots$$

balancing point of distribution





Ex,

let

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{else} \end{cases}$$

This is called the Exponential distribution w/ rate parameter $\lambda > 0$

write : $X \sim \text{Exp}(\lambda)$

Q: what is $E[X]$?

$$E[X] = \int_{\mathbb{R}} x f(x)\,dx = \int_{\cancel{-\infty}\ 0}^{\infty} x \lambda e^{-\lambda x}\,dx$$

$u = x \quad dv = \lambda e^{-\lambda x} dx$

$\mathbb{R}$ $\cancel{-\infty}$ 0

$u = x \qquad dv = \lambda e^{-\lambda x} dx$

$du = dx \qquad v = -e^{-\lambda x}$

Recall: Integration by Parts

$$\boxed{\int u\,dv = uv - \int v\,du}$$

$$= -xe^{-\lambda x} - \int (-e^{-\lambda x})\,dx$$

$$= xe^{-\lambda x} + \int e^{-\lambda x}\,dx$$

$$= \left[xe^{-\lambda x} - \frac{1}{\lambda} e^{-\lambda x} \right]_0^\infty = \left[xe^{-\lambda x} \right]_0^\infty - \frac{1}{\lambda}\left[e^{-\lambda x} \right]_0^\infty$$

$$= [0 - 0] - \frac{1}{\lambda}[0 - 1]$$

$$\boxed{\mathbb{E}[X] = 1/\lambda}$$

Theorem: Law of the Unconscious Statistician

$$\mathbb{E}[g(X)] = \int g(x) f(x)\,dx \qquad (\text{cts})$$

$$\mathbb{E}[g(X)] = \sum_x g(x) f(x) \qquad (\text{discrete})$$