Lecture 3 - Basic Theorems

Thursday, January 30, 2020 10:51 AM

Theorem: Finite Sample Theorem

$$S = \{ s_1, s_2, \ldots, s_n \}$$

so that $|S| = n < \infty$

We delineate some p_i's

$p_1, p_2, \ldots, p_n \leftarrow$ req: $p_i \geq 0$, $\sum_{i=1}^{n} p_i = 1$

Then define for $E \subset S$

$$P(E) = \sum_{i: s_i \in E} p_i .$$

Theorem says that P is a valid prob. fn.

Pf. (1) $\boxed{P(E) \geq 0}$

$P(E) = \sum_{i: s_i \in E} p_i =$ sum of positive p_i (non-neg) so it is non-neg.

(2) $\boxed{P(S) = 1}$

$$P(S) = \sum_{i: s_i \in S} p_i = \sum_{i=1}^{n} p_i = 1$$

by assumption

$i: s_i \in S \rightarrow 1, 2, 3, \ldots, n$

③ $\{E_i\}_{i=1}^{\infty}$ of disjoint events, then

$$\boxed{\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)}$$

<u>aside</u>: ③ basically continuity

$$\mathbb{P}\left(\lim_{i \to \infty} E_i\right) = \lim_{n \to \infty} \sum_{i=1}^{n} \mathbb{P}(E_i)$$

<u>proof</u>:

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i \,:\, \omega_i \in \bigcup_{j=1}^{\infty} E_j} p_i$$

$$\omega_i \in \bigcup_{j=1}^{\infty} E_j \iff \omega_i \in \text{at } \cancel{\text{least}} \text{ exactly one } E_j \quad (*)$$

if $\omega_i \in E_1$ and $\omega_i \in E_2$

then $\omega_i \in E_1 \cap E_2$

then intersection is non-empty

not possible $\leftarrow$

hence $(*)$

$\rightarrow$ example of sum

$$p_1 + p_3 + p_7 + p_6 + p_{11} + \cdots$$

$$\uparrow \quad \uparrow \quad \Updownarrow \quad \uparrow \quad \uparrow$$

index set of sum $\longrightarrow$ $\omega_1 \quad \omega_3 \quad \omega_7 \quad \omega_6 \quad \omega_{11}$

index ...
of sum $\longrightarrow$ $\omega_1 \quad \omega_3 \quad \omega_7 \quad \omega_6 \quad \omega_{11}$

$E_1 \quad E_2 \quad E_3$

re arrange the sum

$$\underbrace{(P_1 + P_7)}_{\sum_{i:\omega_i \in E_1}} + \underbrace{(P_3 + P_6)}_{\sum_{i:\omega_i \in E_2}} + \underbrace{P_{11}}_{\sum_{i:\omega_i \in E_3}} + \cdots$$

more generally we can write

$$\mathbb{P}\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{i:\omega_i \in \bigcup_j E_j} P_i$$

$$= \sum_{j=1}^{\infty} \left(\sum_{i:\omega_i \in E_j} P_i \right)$$

notice defn $\mathbb{P}(E_j)$

wanted.

$$= \sum_{j=1}^{\infty} \mathbb{P}(E_j)$$

Basic Theorems:

Ex. E = "its raining"

E^c = "not raining"

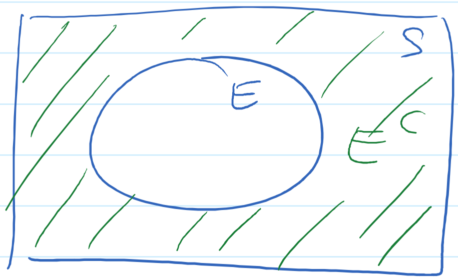
$\mathbb{P}(\text{"not raining"}) = \mathbb{P}(E^c)$

$= .25$

$= .75 = 1 - \mathbb{P}(E^c)$

Theorem: $\mathbb{P}(E^c) = 1 - \mathbb{P}(E)$

pf. $S = E \cup E^c$, $EE^c = \emptyset$



① $\mathbb{P}(S) = 1$

$\mathbb{P}(E \cup E^c) = 1$

② $\mathbb{P}(E \cup E^c) = \mathbb{P}(E) + \mathbb{P}(E^c)$

b/c $EE^c = \emptyset$

Combine: $\mathbb{P}(E) + \mathbb{P}(E^c) = 1$

$$\boxed{\mathbb{P}(E^c) = 1 - \mathbb{P}(E)}$$

Lemma: Finite Measure

$\mathbb{P}(E) \le 1$.

pf. $\mathbb{P}(E^c) = 1 - \mathbb{P}(E)$

$\mathbb{P}(E^c) \geq 0$ (Axiom 1)

So $1 - \mathbb{P}(E) \geq 0$

or $\boxed{\mathbb{P}(E) \le 1}$.

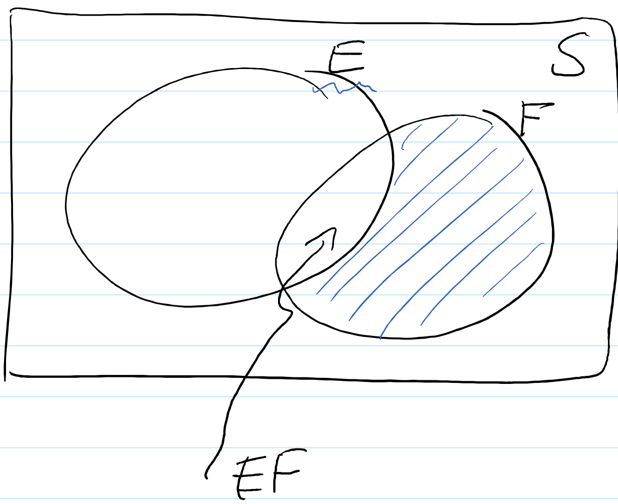
Theorem: Null Event Prob.

$\mathbb{P}(\emptyset) = 0$.

Pf. $P(S) = 1$. (Axiom 2)

$$S^c = \emptyset \qquad S^c = S \setminus S = \emptyset$$

$$P(\emptyset) = P(S^c) = 1 - P(S) = 1 - 1 = 0.$$

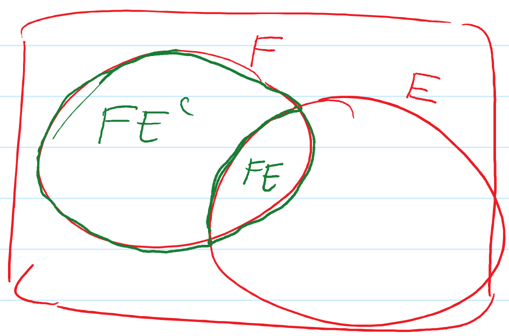


$$P(FE^c)$$
$$= P(F \setminus E)$$
$$= P(F) - P(EF)$$

Theorem: $P(FE^c) = P(F \setminus E) = P(F) - P(EF)$

$$F = FE \cup FE^c$$

disjoint



Axiom 3

$$P(F) = P(FE) + P(FE^c)$$

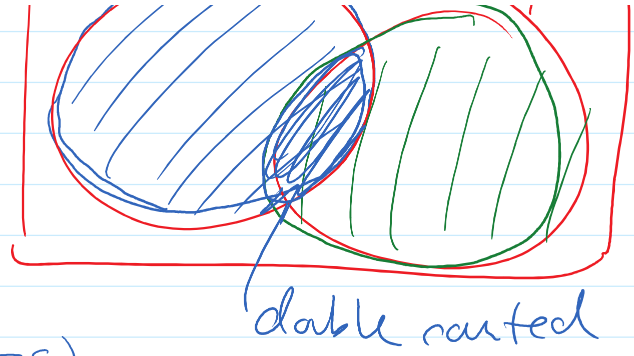
rearrange

$$\boxed{P(FE^c) = P(F) - P(FE)}$$

Theorem: $P(E \cup F)$

$$= P(E) + P(F)$$
$$- P(EF)$$

$= \mathbb{P}(E) + \mathbb{P}(F)$
$- \mathbb{P}(EF)$

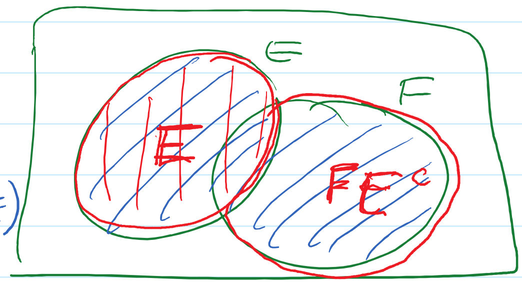


Pf. $E \cup F = E \cup (FE^c)$

Axiom 3

$\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(FE^c)$

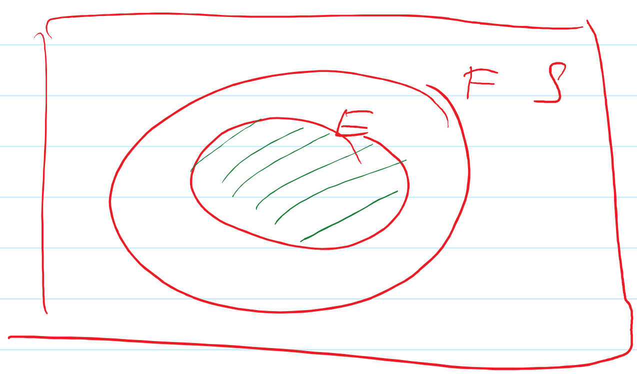
$= \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(FE)$



Theorem: Subset Prob.

If $E \subset F \subset S$

then $\mathbb{P}(E) \leq \mathbb{P}(F)$.



Pf. $\mathbb{P}(FE^c) \geq 0$ (Axiom 1)

$= \mathbb{P}(F) - \mathbb{P}(FE) \geq 0$

So $\mathbb{P}(F) \geq \mathbb{P}(FE)$

$E \subset F$ so $FE = E$

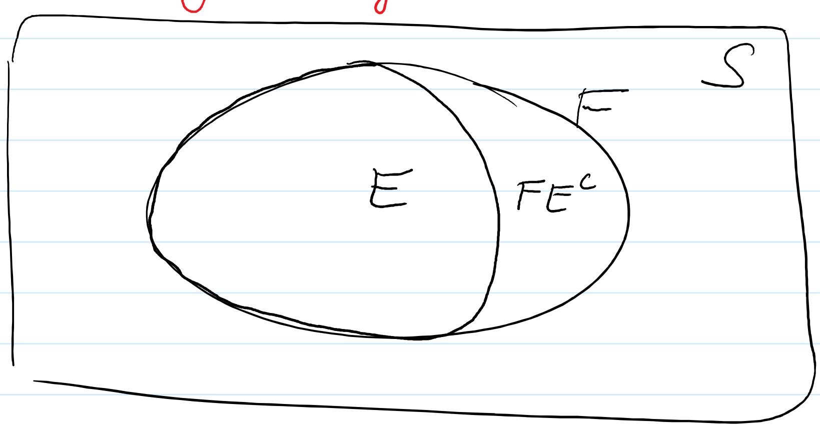
hence $\boxed{\mathbb{P}(F) \geq \mathbb{P}(E)}$.

hence $\boxed{\mathbb{P}(F) \geq \mathbb{P}(E).}$

Consider: $E \subset F$ but $E \neq F$.
(proper subset)

$$\mathbb{P}(E) \leq \mathbb{P}(F).$$

not true generally that $\mathbb{P}(E) < \mathbb{P}(F)$.



$FE^c \neq \emptyset$.
(proper subset)

If $\mathbb{P}(FE^c) = 0$ then $\mathbb{P}(E) = \mathbb{P}(F)$.

$$\left.\begin{aligned} 0 = \mathbb{P}(FE^c) &= \mathbb{P}(F) - \mathbb{P}(EF) \\ &= \mathbb{P}(F) - \mathbb{P}(E) \\ &\text{then } \mathbb{P}(E) = \mathbb{P}(F). \end{aligned}\right\}$$

We had a theorem: that

$$\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \underbrace{\mathbb{P}(EF)}_{\geq 0}$$

this shows: $\mathbb{P}(E \cup F) \leq \mathbb{P}(E) + \mathbb{P}(F)$.

Generalize?

Boole's Inequality

If $\{E_i\}_{i=1}^{\infty}$ then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} \mathbb{P}(E_i).$$

Proof. Want to use Axiom 3.

We will define a seq. of $\{B_i\}_{i=1}^{\infty}$ so that

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} B_i$$ but the B_i are disjoint.

define:

$B_1 = E_1$

$B_2 = E_2 \setminus E_1$

$B_3 = E_3 \setminus E_2 \setminus E_1$

$B_4 = E_4 \setminus E_3 \setminus E_3 \setminus E_2 \setminus E_1$

$\vdots$

Convince yourself: that B_i are disjoint and $\bigcup_i B_i = \bigcup_i E_i$

hint: $B_i = E_i \cap \left(\bigcup_{j<i} E_j\right)^c$

then

$$\mathbb{P}\left(\bigcup_i E_i\right) = \mathbb{P}\left(\bigcup_i B_i\right) = \sum_i \mathbb{P}(B_i)$$

notice: $\boxed{\mathbb{P}(B_i)} = \mathbb{P}(E_i \setminus \underbrace{\sim\sim}_{\text{something}}) \le \boxed{\mathbb{P}(E_i)}$

$\subset E_i$

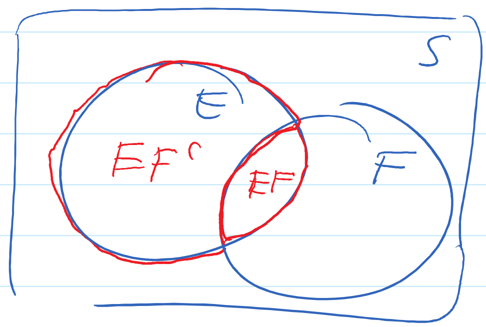
generally: $\mathbb{P}(A \setminus B) \le \mathbb{P}(A)$.

$$\le \sum_i \mathbb{P}(E_i)$$

result I want.

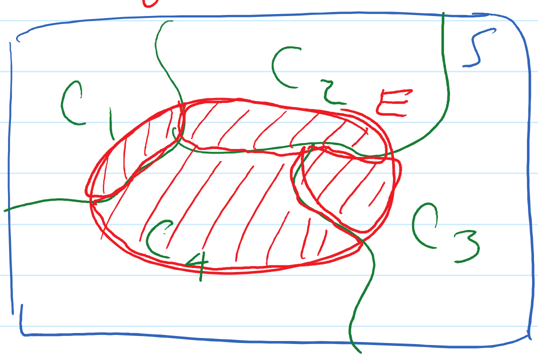
Theorem: Event Partitioning

Aside: done something like $E = \overbrace{EF \cup EF^c}^{\text{partition of } E}$



F and F^c partition S

Generally:



$$E = EC_1 \cup EC_2 \cup EC_3 \cup EC_4$$

disjoint

So

$$P(E) = \sum_{i=1}^{4} P(EC_i)$$

$$EC_1 \cap EC_2 = E(\underbrace{C_1 \cap C_2}_{\emptyset})$$

$$= E\emptyset = \emptyset$$

Generally: If $\{C_i\}$ is a partition of S, then for any $E \subset S$,

$$\boxed{P(E) = \sum_i P(EC_i)}$$

basically pf:

$(EC_i)(EC_j) = \emptyset$ (they are disjoint) hence

$$\mathbb{P}(E) = \mathbb{P}\left(\bigcup_i E C_i\right) = \sum_i \mathbb{P}(E C_i)$$