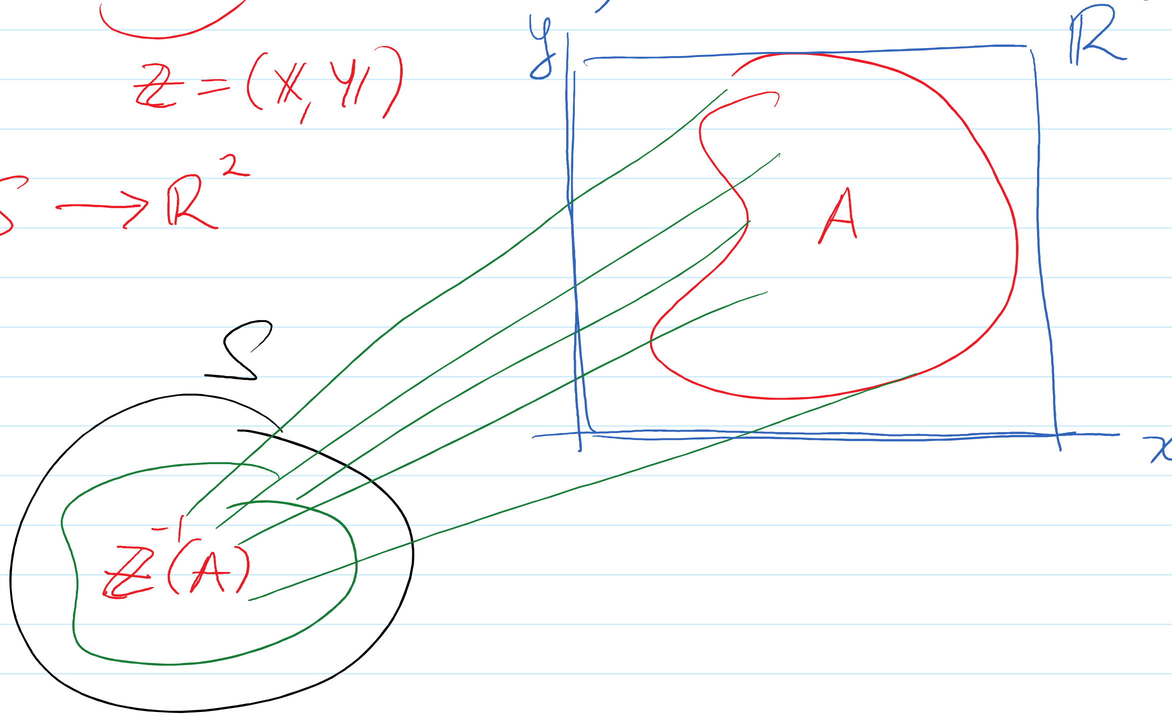
$X : S \to \mathbb{R}$ and $Y : S \to \mathbb{R}$
then $(\underline{X}, \underline{Y})$ a bivariate r.v.

Say: $\mathbb{P}(\underbrace{(X, Y)} \in A)$ here, $A \subset \mathbb{R}^2$

$Z = (X, Y)$

$Z : S \to \mathbb{R}^2$



Often $\mathbb{P}(X \in A, Y \in B)$ here $A, B \subset \mathbb{R}$

read: "and"

$= \mathbb{P}(X^{-1}(A) Y^{-1}(B))$

<u>Ex.</u> Coin flip 3 times

$$X = \begin{cases} 0 & \text{last flip is T} \\ 1 & \text{last flip is H} \end{cases}$$

Y = # heads among the 3 flips

$Z = (X, Y)$ (recall $Z : S \to \mathbb{R}^2$)

$s \in S$	$Z(s) = (X(s), Y(s))$
HHH	(1, 3)
HHT	(0, 2)
HTH	(1, 2)
HTT	(0, 1)
THH	(1, 2)
THT	(0, 1)
TTH	(1, 1)
TTT	(0, 0)

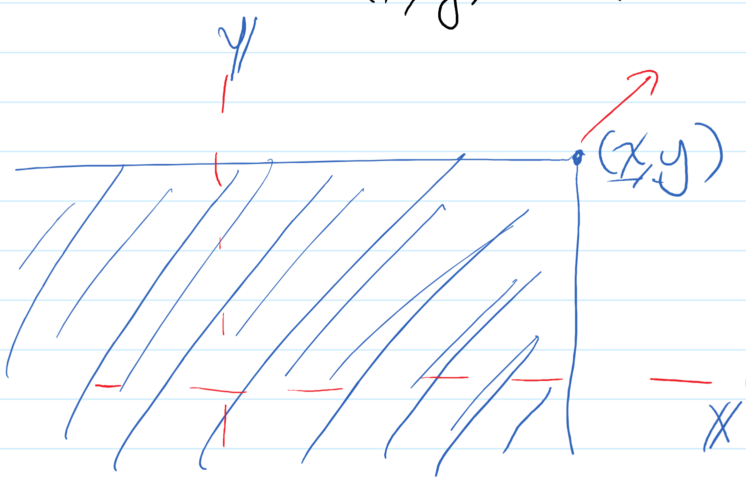
Defn: Bivariate CDF

$F : \mathbb{R}^2 \to \mathbb{R}$ defined for $(x,y) \in \mathbb{R}^2$

as

$$F(x,y) = \mathbb{P}(X \le x, Y \le y)$$

"and"



Theorem:

1. $F(x,y) \geq 0$
2. $\lim\limits_{x,y \to \infty} F(x,y) = 1$
3. $\lim\limits_{x \to -\infty} F(x,y) = 0$
4. F is non-decreasing and right-continuous

$\lim_{y \to -\infty} F(x,y) = 0$ (in each argument)

Defn: Marginal Distributions ⊛

If (X, Y) is a bivariate r.v. then the dists. of X and Y individually are called the marginal distributions of X and Y (resp.)

Theorem: ① $\lim_{x \to \infty} F(x,y) = F_Y(y)$

univariate marginal CDF of Y

② $\lim_{y \to \infty} F(x,y) = F_X(x)$

marginal CDF of X

idea: $F_X(x) = P(X \le x) = P(X \le x, Y = \text{anything})$

$= P(X \le x, Y \le \infty)$

$= \lim_{y \to \infty} F(x,y)$

Important note:

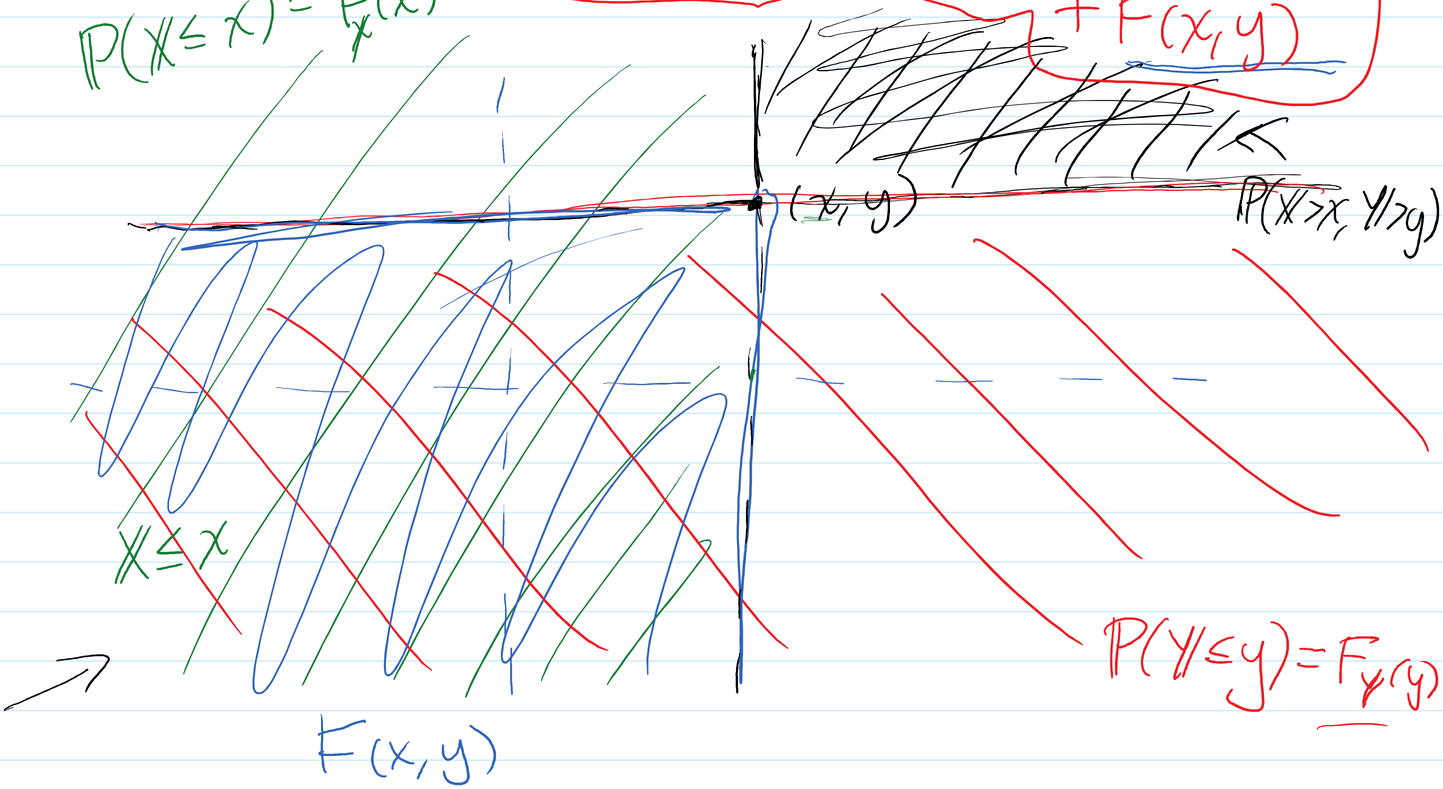
Knowing Joint CDF $\longrightarrow$ Marginals

Knowing Joint CDF $\longrightarrow$ Marginals

$\not\longleftarrow$ other direction doesn't work

Lemma: Univariate: $P(X > x) = 1 - F(x)$

Bivariate: $$\boxed{P(X > x, Y > y) = 1 - F_X(x) - F_Y(y) + F(x,y)}$$



Extension: $$\boxed{\begin{aligned} &P(x_1 \le X \le x_2,\ y_1 \le Y \le y_2) \\ &= F(x_2, y_2) + F(x_1, y_1) - F(x_1, y_2) - F(x_2, y_1) \end{aligned}}$$

Similar picture proof.

Defn: Joint PMF

If X and Y are discrete then we define the joint PMF

$$f(x,y) = \mathbb{P}(X=x, Y=y)$$

$$\boxed{\text{facts: } \sum_{x,y} f(x,y) = 1 \text{ and } f(x,y) \geq 0}$$

Theorem: Joint/Marginal PMF rel.

① $f_X(x) = \sum_{y \in \text{Support}(Y)} f(x,y)$

marginal pmf of X

② $f_Y(y) = \sum_{x} f(x,y)$

Pf

$f_X(x) = \mathbb{P}(X=x)$

$= \mathbb{P}(X=x, Y=\text{anything})$

$= \sum_{y} \mathbb{P}(X=x, Y=y)$

$= \sum_{y} f(x,y)$

Ex, Cointoss Coin flip example.

Flip a coin 3 times,

$$X = \begin{cases} 0 & \text{if } 3^{rd}\ T \\ 1 & \text{if } 3^{rd}\ H \end{cases}$$

$Y = \#$ heads

check: ① $f(x,y) \geq 0$

$Y = \#$ heads

check: ① $f(x,y) \geq 0$

② $\sum_{x,y} f(x,y) = 1$

$f(x,y)$

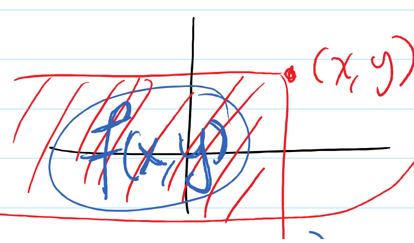
$X \backslash Y$	0	1	2	3	
0	1/8	2/8	1/8	0	$1/2 \leftarrow f_X(0)$
1	0	1/8	2/8	1/8	$1/2 \leftarrow f_X(1)$
	1/8	3/8	3/8	1/8	
	$f_Y(0)$	$f_Y(1)$	$f_Y(2)$	$f_Y(3)$	

Defn: Joint PDF

If X and Y are continuous we call the function $f : \mathbb{R}^2 \to \mathbb{R}$ the joint PDF if for all $A \subset \mathbb{R}^2$

$$\boxed{P((X, Y) \in A) = \int_A f(x,y)\,dx\,dy}$$

Facts:



(1)

1. $$\boxed{F(x,y) = \mathbb{P}(X \le x,\ Y \le y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(x,y)\,dx\,dy}$$

(recall in univariate case: $F(x) = \int_{-\infty}^{x} f(x)\,dx$)

2. (in univariate case: $f(x) = \frac{d}{dx}F(x)$)

$$\boxed{f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)}$$

3. Other similar properties:

   1. $f(x,y) \ge 0$
   2. $\int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)\,dx\,dy = 1$

(univariate: $f(x) \ge 0$ and $\int_{\mathbb{R}} f(x)\,dx = 1$)

Theorem:

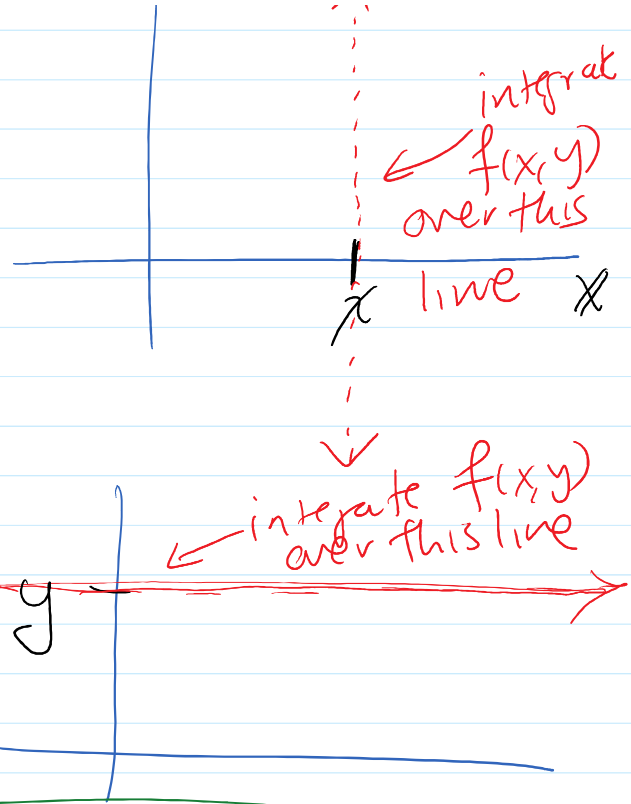
integrat

Theorem:

(1) $f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$

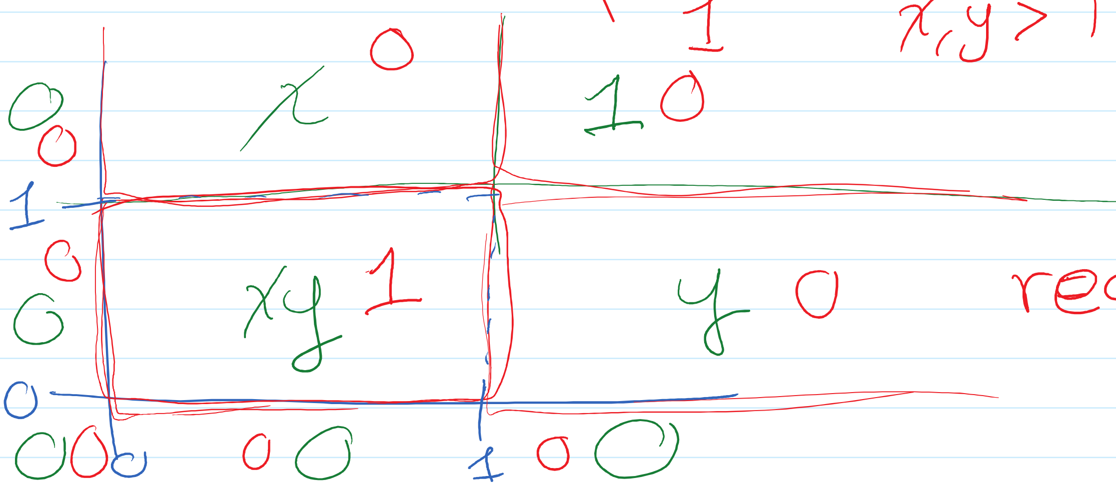
marginal pdf of X

(2) $f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx$



Ex.

$$F(x,y) = \begin{cases} 0 & x \le 0 \text{ or } y \le 0 \\ xy & (x,y) \in [0,1]^2 \\ x & y > 1,\ x \in [0,1] \\ y & x > 1,\ y \in [0,1] \\ 1 & x, y > 1 \end{cases}$$



red = pdf

Q: Joint PDF? $f(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y} F(x,y)$

$$\boxed{f(x,y) = 1 \text{ if } x, y \text{ are in } [0,1]}$$

basically a bivariate uniform over $[0,1]^2$

Marginals?

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy = \int_0^1 f(x,y)\,dy$$

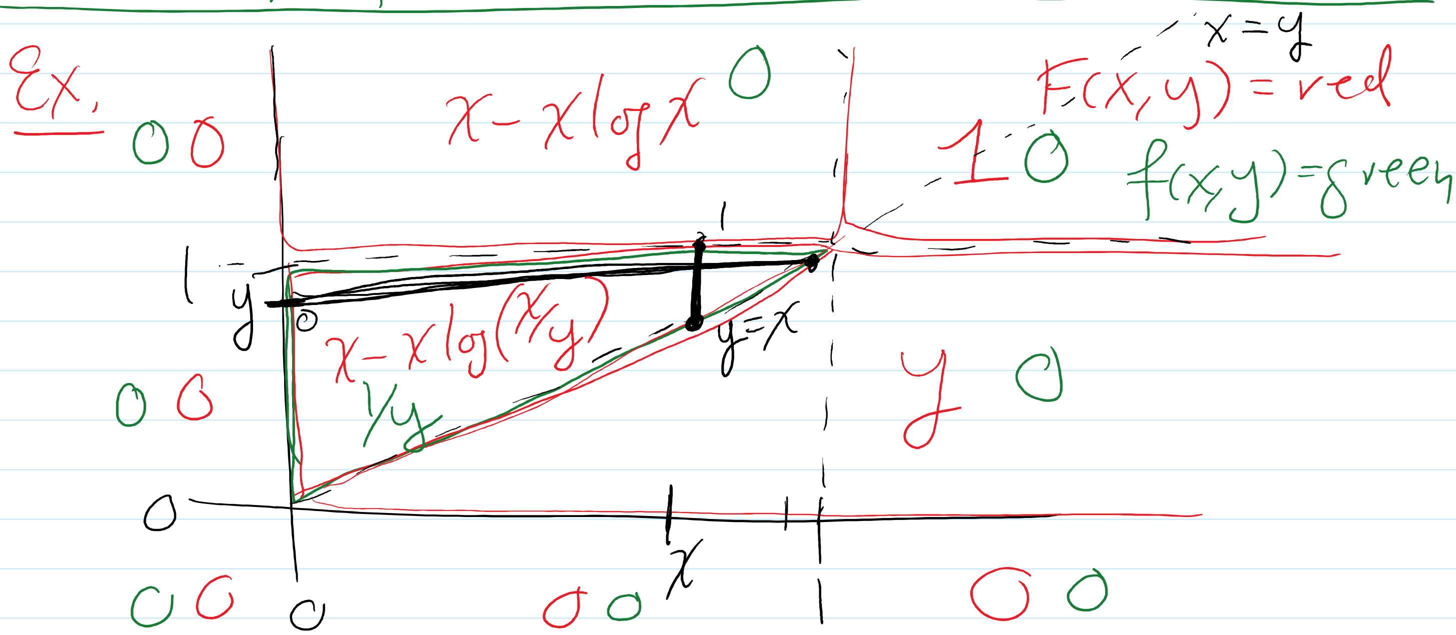
$$= \int_0^1 1\,dy = 1$$

$(\text{for } X \in [0,1])$

So $X \sim U(0,1)$

Similarly for $Y \sim U(0,1)$.

Ex.



__Recall:__ $f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$

For $0 \le x \le y \le 1$, $F(x,y) = x - x\log(x/y)$

$= \underline{x} - \underline{x\log(x)} + \underline{x\log(y)}$

$\rightarrow \frac{\partial F}{\partial y} = x/y$

$\rightarrow \frac{\partial^2 F}{\partial x \partial y} = 1/y$

$$\boxed{f(x,y) = \begin{cases} 1/y & \underline{0 \le x \le y \le 1} \\ 0 & \text{else} \end{cases}}$$

__Marginals:__

① $\underline{f_X}(x) = \int_{\mathbb{R}} \underline{f(x,y)}\, dy = \int_x^1 1/y\, dy = \log(y)\Big|_x^1$

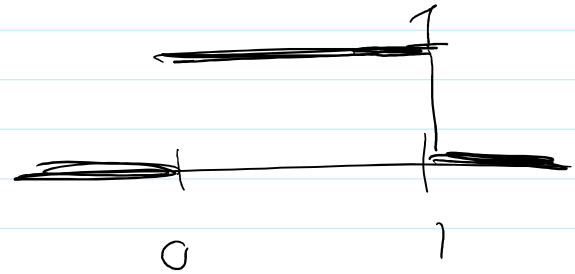
for $x \in [0,1]$

$= \log(1) - \log(x)$

$= \underline{-\log(x)}$

② $f_Y(y) = \int_0^y 1/y\, dx = \frac{1}{y}\int_0^y dx = \frac{1}{y}\, y = 1$

for $y \in [0,1]$



So $Y \sim U(0,1)$

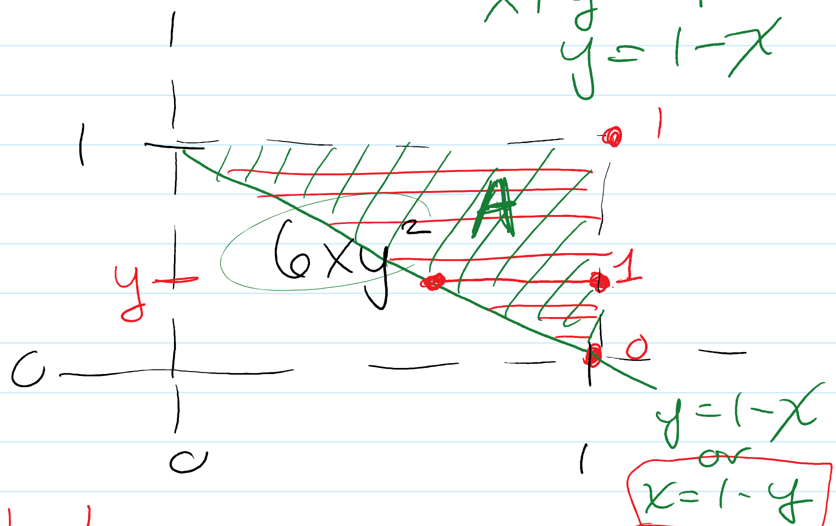
Ex. (X,Y) distributed with

$$f(x,y) = 6xy^2 \quad \text{for } 0 \le x, y \le 1$$

$P(X+Y \geq 1)$

recall:

$$P((X,Y) \in A) = \int_A f(x,y)\,dx\,dy$$

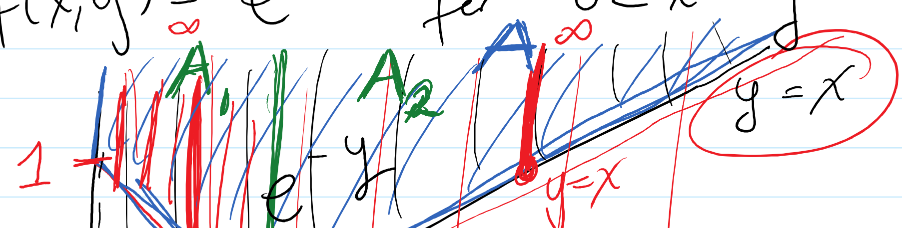


$$P(X+Y \geq 1) = \int_0^1 \int_{1-y}^1 6xy^2\,dx\,dy = 9/10$$

Ex. (X,Y) have a joint pdf of

$$f(x,y) = e^{-y} \quad \text{for } 0 < x < y$$

$P(X+Y \geq 1)$

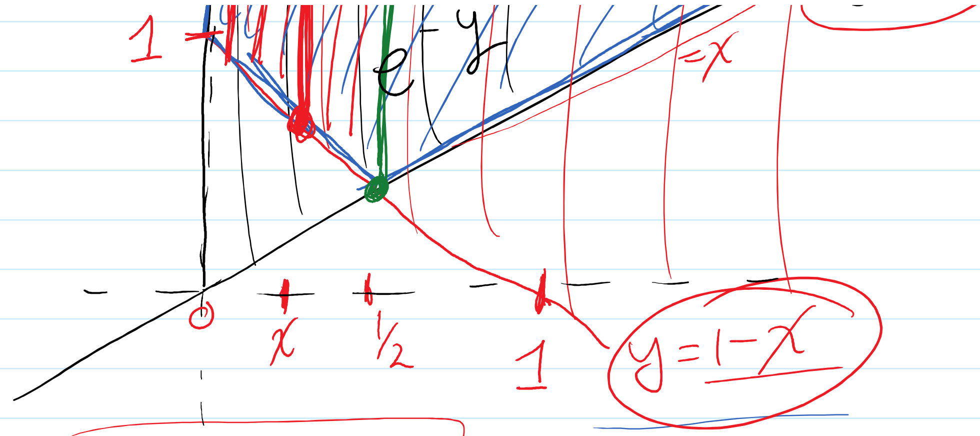


$P(X+Y \geq 1)$

$x+y=1$

$x=1-y$

$y=1-x$



$$P(X+Y \geq 1) = \boxed{\int_A f(x,y)\,dx\,dy} = \int_{A_1} f(x,y)\,dx\,dy + \int_{A_2} f(x,y)\,dx\,dy$$

A_1 $\qquad$ A_2

$$\int_0^{1/2}\int_{1-x}^{\infty} e^{-y}\,dy\,dx + \int_{1/2}^{\infty}\int_{x}^{\infty} e^{-y}\,dy\,dx$$

$$= \dots = \boxed{2e^{-1/2} - e^{-1}}$$