Lecture 15 - Transformations

Friday, March 27, 2020 3:16 PM

$$\boxed{Y = g(X)}$$

$g: \mathbb{R} \to \mathbb{R}$

$X: S \to \mathbb{R}$

really mean: $Y = g \circ X$

$Y: S' \to \mathbb{R}$

Formalisms:

$A \subset \mathbb{R}$

$$\to P(Y \in A) = P((g \circ X) \in A) = P((g \circ X)^{-1}(A))$$
$$= P(X^{-1}(g^{-1}(A)))$$

$$\to \boxed{P(Y \in A) = P(g(X) \in A) = P(X \in g^{-1}(A))}$$

This lecture: Know something about X

what can I infer about $\underbrace{g(X)}_{Y}$?

How can I get the pmf of Y from the pmf of X?

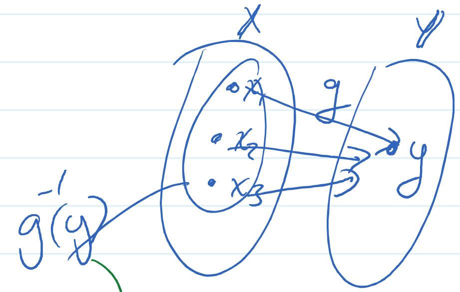
pmf $\to$ discrete

① discrete X

② Know $f_X \leftarrow$ pmf of X

(2) Know $f_X \leftarrow$ pmf of X

(3) Want: $f_Y \leftarrow$ pmf of Y



notice:

$$f_Y(y) = P(Y=y) = P(g(X)=y)$$

$$= P(X \in \underbrace{g^{-1}(y)}_{A})$$

g^{-1} is the inverse image

aside:

$$\boxed{P(X \in A) = \sum_{x \in A} f_X(x)}$$

$$= \sum_{x \in g^{-1}(y)} f_X(x)$$

Theorem:

$$\boxed{f_Y(y) = \sum_{x \in g^{-1}(y)} f_X(x)} \leftarrow$$

$$\boxed{P(Y=y) = \sum_{\substack{\text{all } x \text{ where} \\ g(x)=y}} P(X=x)}$$

intuition.

Ex. Let $X \sim Bin(n, p)$

let $Y = n - X$ alt. $Y = g(X)$

$= \#$ failures where $g(x) = n - x$

$y = g(x) = n - x$

$x = n - y$ so $g^{-1}(y) = n - y$ } aside: get g^{-1}

$$f_Y(y) = \sum_{x \in g^{-1}(y)} f_X(x) = \sum_{x = n-y} f_X(x)$$

a bit overkill

Bin. pmf

$$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$= f_X(n-y)$$

$$= \binom{n}{n-y} p^{n-y} (1-p)^{n-(n-y)}$$

$$= \binom{n}{y} p^{n-y} (1-p)^{y}$$

$$\binom{n}{n-y} = \frac{n!}{(n-y)!(n-(n-y))!} = \frac{n!}{(n-y)!\,y!} = \binom{n}{y}$$

$q = 1 - p$

$$= \boxed{\binom{n}{y} q^{y} (1-q)^{n-y}}$$

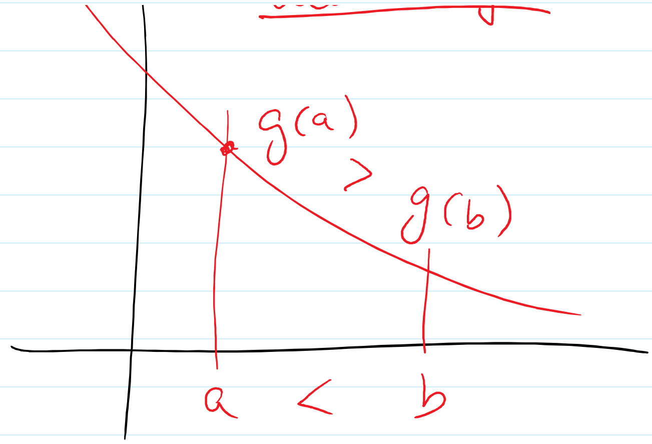
pmf of a Bin(n, q), $q = 1-p$

$f_Y(y) =$

So $\boxed{Y \sim \text{Bin}(n, 1-p)}$

Defn: a function is monotone if it is either increasing or decreasing

increasing — decreasing



Theorem: If $Y = g(X)$ then the CDF of Y is
X is continuous then

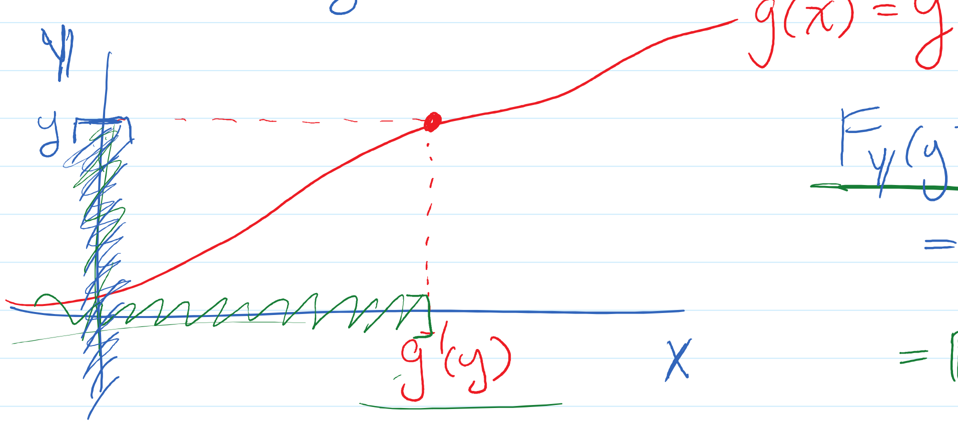
① If g is increasing:

$$F_Y(y) = F_X(g^{-1}(y))$$

② If g is decreasing:

$$F_Y(y) = 1 - F_X(g^{-1}(y)).$$

Case 1 pf. g increasing



$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \le y) \\ &= \mathbb{P}(Y \in (-\infty, y]) \\ &= \mathbb{P}(X \in [-\infty, g^{-1}(y)]) \\ &= F_X(g^{-1}(y)) \end{aligned}$$

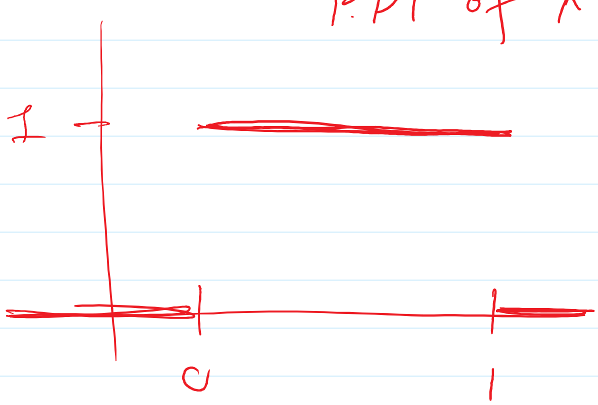
$= f_X(g^{-1}(y))$

Case 2: decreasing

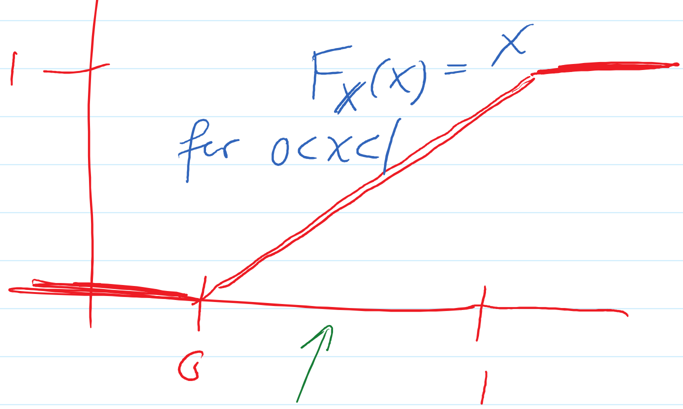
$$F_Y(y) = P(Y \le y)$$
$$= P(X \ge g^{-1}(y))$$
$$= 1 - P(X \le g^{-1}(y))$$
$$= 1 - F_X(g^{-1}(y))$$

Ex. $X \sim U(0,1)$

PDF of X



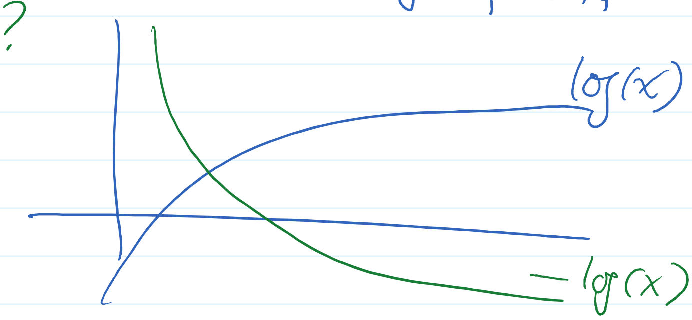
CDF of X



Let $Y = -\log(X)$. $g(x) = -\log(x)$.

decreasing function

What's the CDF of Y?



$$F_Y(y) = 1 - F_X(g^{-1}(y))$$
$$= 1 - F_X(e^{-y})$$

$0 < e^{-y} < 1$

for $y > 0$

aside: $y = -\log(x)$

$-y = \log(x)$

$0 < e^{-y} < 1$

for $y > 0$

$\to 1 - e^{-y}$

$-y = \log(x)$

$x = \boxed{e^{-y} = g^{-1}(y)}$

Showed $\boxed{F_Y(y) = 1 - e^{-y}}$ Notice: CDF of an Exp(1)

then $\boxed{Y \sim \text{Exp}(1).}$

What what pdf?

$Y = g(X)$

1. X continuous
2. g continuous, monotone
3. g^{-1} is continuously differentiable

$$\boxed{f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|} \quad \text{for } y \in \mathcal{Y}$$

support of Y

pf: Case 1: g increasing

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left[F_X(g^{-1}(y))\right]$$

$$= f_X(g^{-1}(y)) \underbrace{\left| \frac{dg^{-1}}{dy} \right|}_{>0}$$

$$\underbrace{1 - \cdots}_{>0}$$

Case 2: $\underline{g \text{ decreasing}}$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left[1 - F_X(g^{-1}(y))\right]$$

$$= -f_X(g^{-1}(y)) \underbrace{\frac{dg^{-1}}{dy}}_{<0}$$

chain rule.

$$= f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$

<u>Ex.</u> $X \sim \text{Gamma}(n, \beta)$

n an integer

$\Gamma(n) = (n-1)!$

So $f_X(x) = \dfrac{1}{(n-1)!\,\beta^n} x^{n-1} e^{-x/\beta}$ for $x > 0$

Consider $Y = 1/X$ equiv. $g(x) = 1/x$

- g monotone ✓
- g^{-1} cts. differentiable

$y = 1/x$

then $x = 1/y = g^{-1}(y)$

So $\dfrac{dg^{-1}}{dy} = -1/y^2$

<u>Theorem:</u>

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}}{dy} \right|$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dy}{dy} \right|$$

$$= f_X\left(\frac{1}{y}\right) \left| -\frac{1}{y^2} \right|$$

$$= \frac{1}{(n-1)!\,\beta^n} \left(\frac{1}{y}\right)^{n-1} e^{-\frac{1}{y\beta}} \frac{1}{y^2}$$

$\left(\frac{1}{y}\right)^2$

$$= \boxed{\frac{1}{(n-1)!\,\beta^n} \left(\frac{1}{y}\right)^{n+1} e^{-\frac{1}{y\beta}} \text{ for } y > 0}$$

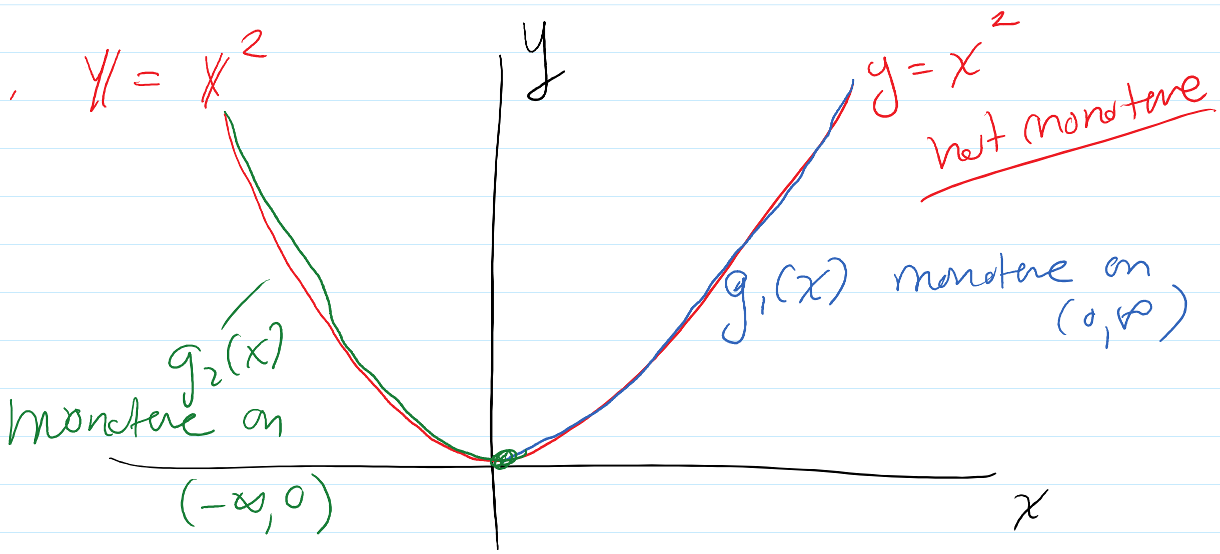
$x > 0$ then $\underbrace{\frac{1}{x}}_{y} > 0$

$Y \sim$ InverseGama(n, β).

What if g isn't monotone?

OK, as long as we can segment g into monotone chunks.

Ex. $Y = X^2$



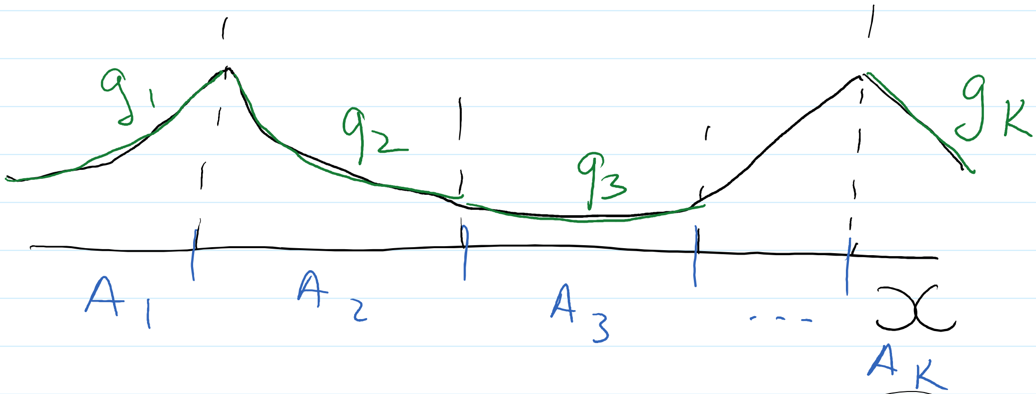
Theorem: Piecewise Monotone Transformation

X continuous has support $\mathcal{X}$

Let $A_1, A_2, \dots, A_K$ partition $\mathcal{X}$

let g_i be the function g on A_i for $i = 1, \dots, K$

(1) If each of these g_i satisfy the conditions of the previous theorem on A_i



(2) Image of A_i under g_i is $\mathcal{Y}$

then, if $Y = g(X)$

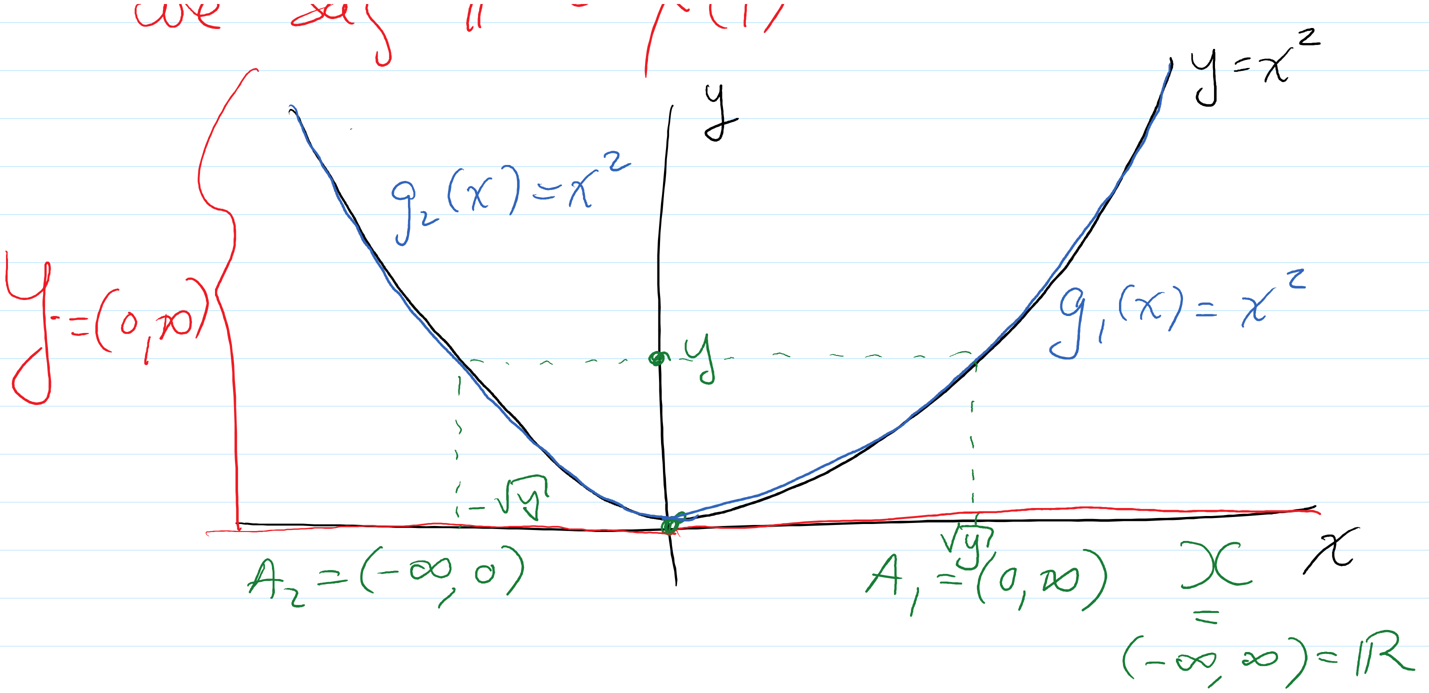
$$\boxed{f_Y(y) = \sum_{i=1}^{K} f_X(g_i^{-1}(y)) \left| \frac{d g_i^{-1}}{dy} \right|} \text{ for } y \in \mathcal{Y}$$

Ex. Chi-Sq.

let $X \sim N(0,1)$ and $Y = X^2$ $(g(x) = x^2)$

we say $Y \sim \chi^2(1)$

$Y = x^2$



$\underline{A_1 = (0, \infty)}$, $y = g_1(x) = x^2$; $g_1^{-1}(y) = \sqrt{y}$

$A_2 = (-\infty, 0)$, $g_2(x) = x^2$; $g_2^{-1}(y) = -\sqrt{y}$

Need: $\dfrac{dg_1^{-1}}{dy} = \dfrac{1}{2\sqrt{y}}$ and $\dfrac{dg_2^{-1}}{dy} = -\dfrac{1}{2\sqrt{y}}$

Recall: $f_X(x) = \dfrac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$ for $x \in \mathbb{R}$

$$\overbrace{f_Y(y)}^{y>0} = \sum_{i=1}^{2} f_X(g_i^{-1}(y)) \left| \frac{dg_i^{-1}}{dy} \right|$$

$$= f_X(g_1^{-1}(y)) \left| \frac{dg_1^{-1}}{dy} \right| + f_X(\underline{g_2^{-1}(y)}) \left| \frac{dg_2^{-1}}{dy} \right|$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(\sqrt{y})^2\right) \left| \frac{1}{2\sqrt{y}} \right| + \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(-\sqrt{y})^2\right) \left| \frac{-1}{2\sqrt{y}} \right|$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(\sqrt{y})^2\right) \left|\frac{1}{2\sqrt{y}}\right| + \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(-\sqrt{y})^2\right) \left|\frac{-1}{2\sqrt{y}}\right|$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y\right) \frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y\right) \frac{1}{2\sqrt{y}}$$

$$\boxed{= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y\right) \frac{1}{\sqrt{y}}}$$

pdf of a $\chi^2(1)$

$Y = X^2$ when $X \sim N(0,1)$

Theorem: Probability Integral Transformation

If X has CDF F_X and X is continuous (so is F_X)

then

$$Y = F_X(X) \sim U(0,1)$$

Pf. Assume F_X is (strictly) increasing

the F_X is invertible.

$Y = F_X(X) = g(X)$ where $g = F_X$

then

$$F_Y(y) = F_X(g^{-1}(y)) = F_X(F_X^{-1}(y)) = y.$$

Recall the CDF for $U \sim U(0,1)$

Recall the CDF for $U \sim \mathcal{U}(0,1)$

CDF of a $\mathcal{U}(0,1)$

$$F_U(u) = \begin{cases} 0 & u < 0 \\ \boxed{u \quad 0 < u < 1} \\ 1 & u > 1 \end{cases}$$

So $Y = F_X(X) \sim \mathcal{U}(0,1)$.

Generating RVs.

Want X, a r.v. w/ a CDF F_X

Know $\boxed{Y = F_X(X) \sim \mathcal{U}(0,1)}$

$F_X^{-1}(Y) = F_X^{-1}(F_X(X)) = X \leftarrow$ have CDF F_X

Algo: (1) generate $Y \sim \mathcal{U}(0,1)$

(2) plug Y into desired CDF inverse

(3) $X = F_X^{-1}(Y) \sim$ as I want.