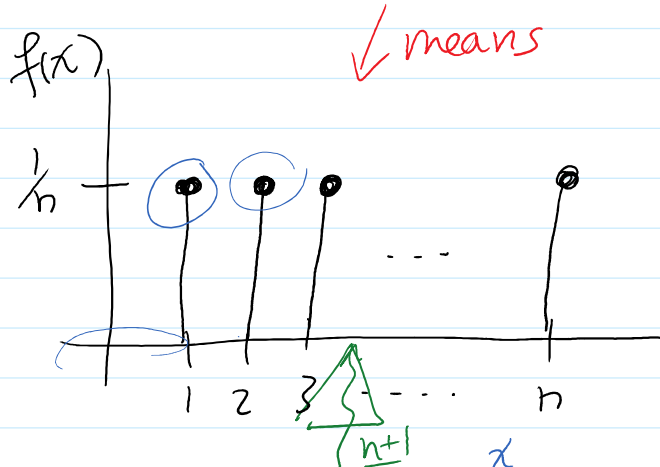


Discrete Uniform Distribution

$$X \sim \text{Unif} \{1, \dots, n\}$$

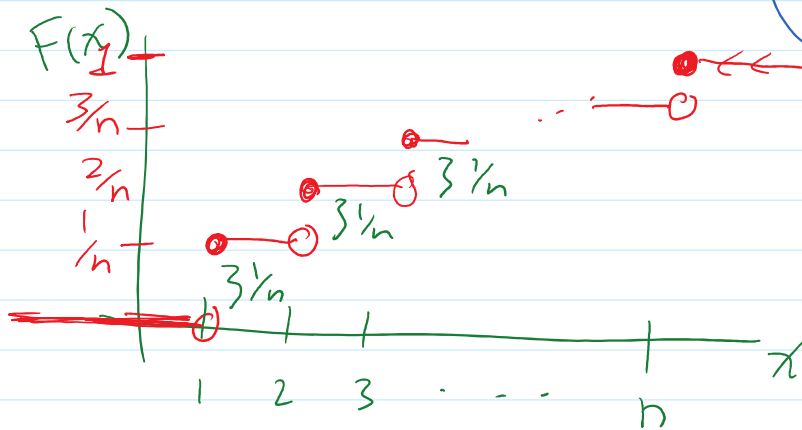


$$\text{i.e. } f(x) = \begin{cases} \frac{1}{n} & x=1, \dots, n \\ 0 & \text{else} \end{cases}$$

CDF:

$$F(x) = \sum_{i=1}^x f(i) =$$

$$\begin{cases} 0 & i < 1 \\ \frac{1}{n} & 1 \leq i < 2 \\ \frac{2}{n} & 2 \leq i < 3 \\ \frac{3}{n} & 3 \leq i < 4 \\ \vdots & \vdots \\ 1 = \frac{n}{n} & n \leq i \end{cases}$$



- ① non-decreasing
- ② right cts
- ③  $\lim_{x \rightarrow -\infty} F(x) = 0$   
 $\lim_{x \rightarrow \infty} F(x) = 1$

Expected Value

$$\begin{aligned} E[X] &= \sum_{i=1}^n i f(i) = \sum_{i=1}^n i \cdot \frac{1}{n} = \frac{1}{n} \left[ \sum_{i=1}^n i \right] \\ &= \frac{1}{n} \left[ \frac{n(n+1)}{2} \right] \\ &= \frac{n+1}{2} \end{aligned}$$

$$= \frac{n+1}{2}$$

$$\begin{aligned} E[X^2] &= \sum_{x=1}^n x^2 f(x) = \sum_{x=1}^n x^2 \frac{1}{n} = \frac{1}{n} \sum_{x=1}^n x^2 \\ &= \frac{1}{n} \frac{n(n+1)(2n+1)}{6} \\ &= \frac{(n+1)(2n+1)}{6} \end{aligned}$$

Variance:  $\text{Var}(X) = E[(X - E[X])^2]$

Short-cut formula says:

$$\begin{aligned} \text{Var}(X) &= E[X^2] - E[X]^2 \\ &= \frac{(n+1)(2n+1)}{6} - \left(\frac{(n+1)}{2}\right)^2 \end{aligned}$$

$$\begin{aligned} &= \dots \text{ algebra} \\ &= \boxed{\frac{(n+1)(n-1)}{12}} \end{aligned}$$

MGF (moment generating function)

defn:  $M(t) = E[e^{tX}] = \sum_{x=1}^n e^{tx} f(x) = \frac{1}{n} \sum_{x=1}^n e^{tx}$

$$\Rightarrow \frac{1}{n} \sum_{x=1}^n (e^t)^x$$

starts to look like a partial geometric series

$n-1$      $x+1$      $n-1$      $1-r^n$

$$\begin{aligned}
 &= \frac{1}{h} \sum_{x=0}^{n-1} (e^t)^{x+1} \\
 &= \frac{e^t}{h} \sum_{x=0}^{n-1} (e^t)^x \\
 &= \frac{e^t}{h} \frac{1 - (e^t)^n}{1 - e^t} = \boxed{\frac{e^t - e^{t(n+1)}}{n(1 - e^t)}}
 \end{aligned}$$

series  $\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$   
 $|r| < 1$

$r = e^t$

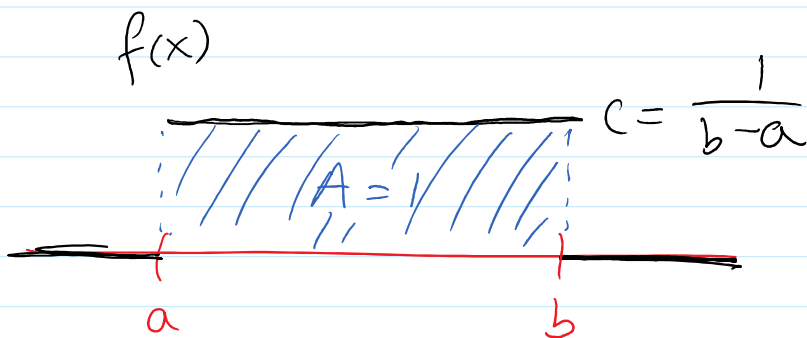
for  $t$  s.t.  $e^t < 1$

## Continuous Uniform Distribution

$X \sim U(a, b)$  "X is unif over  $(a, b)$ "

means uniform density over  $(a, b)$

pdf



Know that  $\int_{\mathbb{R}} f(x) dx = 1$

$$= \int_a^b f(x) dx = \int_a^b c dx = c(b-a) = 1$$

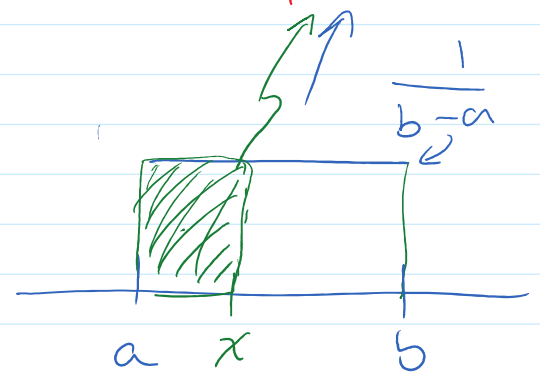
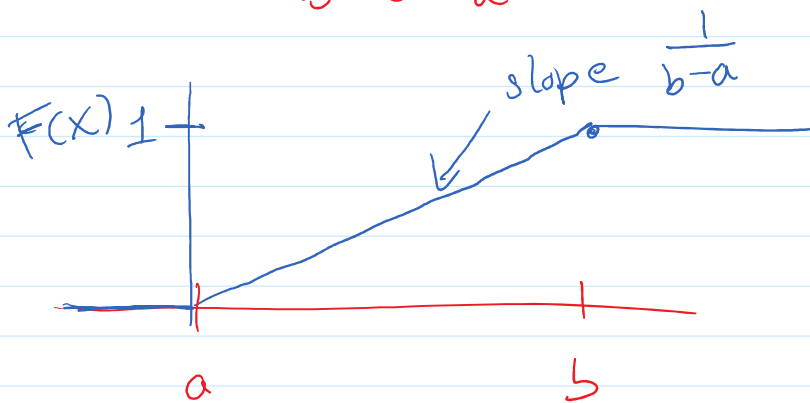
$$\text{so } c = \frac{1}{b-a}$$

$$\text{So } c = \frac{1}{b-a}$$

CDF:

$$F(x) = \int_{-\infty}^x f(t) dt = \int_a^x f(t) dt = \int_a^x \frac{1}{b-a} dt$$

$$= \frac{1}{b-a} \int_a^x dt = \frac{1}{b-a} (x-a) = \boxed{\frac{x-a}{b-a}}$$

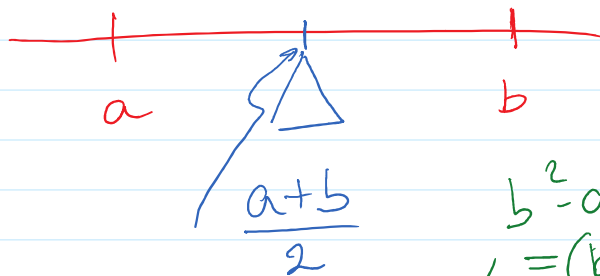


$$F(x) = \begin{cases} \frac{x-a}{b-a}, & a < x < b \\ 0, & x \leq a \\ 1, & x \geq b \end{cases}$$

$$\frac{1}{b-a}$$

Expected Value

$$\begin{aligned} E[X] &= \int_{\mathbb{R}} x f(x) dx \\ &= \int_a^b x \frac{1}{b-a} dx \end{aligned}$$



$$\frac{b^2 - a^2}{(b-a)(b+a)}$$

$$= \frac{1}{b-a} \int_a^b x dx = \frac{1}{b-a} \left[ \frac{x^2}{2} \right]_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{b+a}{2}$$

$$= \frac{1}{b-a} \int_a^b x dx = \frac{1}{b-a} \left[ \frac{x^2}{2} \right]_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{a+b}{2}$$

$$\begin{aligned} E[X^2] &= \int_a^b x^2 f(x) dx = \int_a^b x^2 \frac{1}{b-a} dx = \frac{1}{b-a} \left[ \frac{x^3}{3} \right]_a^b \\ &= \frac{b^3 - a^3}{3(b-a)} = \frac{(b-a)(b^2 + ba + a^2)}{3(b-a)} \\ &= \frac{b^2 + ba + a^2}{3} \end{aligned}$$

Short-cut formula

$$\begin{aligned} \text{Var}(X) &= E[X^2] - E[X]^2 \\ &= \frac{b^2 + ba + a^2}{3} - \left( \frac{a+b}{2} \right)^2 \end{aligned}$$

$$= \dots \text{ algebra}$$

$$= \frac{(b-a)^2}{12}$$

MGF

$$M(t) = E[e^{tX}] = \int_{\mathbb{R}} e^{tx} \underbrace{f(x)} dx$$

$$f(x) = \frac{1}{b-a} \quad x \in [a, b]$$

$$f(x) = \int_a^b e^{tx} \frac{1}{b-a} dx = \frac{1}{b-a} \left[ \frac{1}{t} e^{tx} \right]_{x=a}^{x=b}$$

$$= \frac{1}{t(b-a)} (e^{bt} - e^{at})$$

for  $t \neq 0$

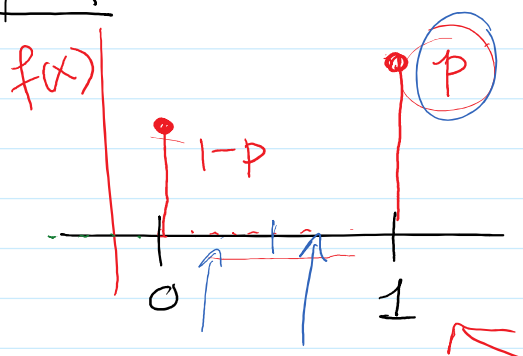
## Bernoulli Distribution

$X \sim \text{Bern}(p)$   $\swarrow$  prob. of success.  $p \in [0, 1]$

— flip a coin w/ prop.  $p$  of H,  
 $X=1$  if H,  $X=0$  otherwise  
 then  $X \sim \text{Bern}(p)$

— any experiment w/ two outcomes 0 or 1  
 w/ prob.  $p$  of getting a 1

pmf



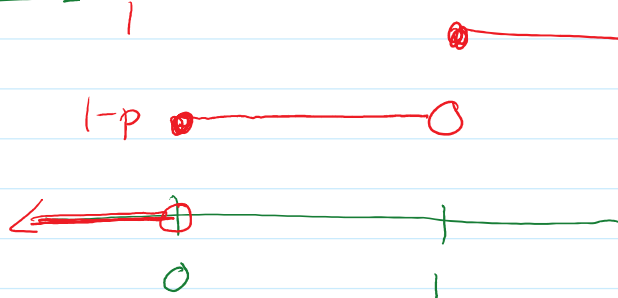
$$f(x) = \begin{cases} p^x (1-p)^{1-x} & \text{for } x=0, 1 \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} 1-p & x=0 \\ p & x=1 \end{cases}$$

$$P(X=0) = 1-p \quad \leftarrow \quad P(X=1) = p$$

$\hat{x} = 1$   
else

CDF



$$F(x) = \begin{cases} 0 & x < 0 \\ 1-p & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

Expectation:

$$E[X] = \sum_{x=0}^1 x f(x) = (0)f(0) + (1)f(1) = f(1) = p$$

$$E[X^2] = (0)^2 f(0) + (1)^2 f(1) = f(1) = p$$

aside

$$E[X'] = f(1) = p$$

$$\text{Var}(X) = E[X^2] - E[X]^2 = p - p^2 = \boxed{p(1-p)}$$

← Note Bernoulli(p) same as  
Bin(n, p) where  $n=1$

MGF.

✓✓

MGF:  $M(t) = \mathbb{E}[e^{tx}] = \sum_{x=0,1} e^{tx} f(x)$

$$= e^{t(0)} f(0) + e^{t(1)} f(1)$$

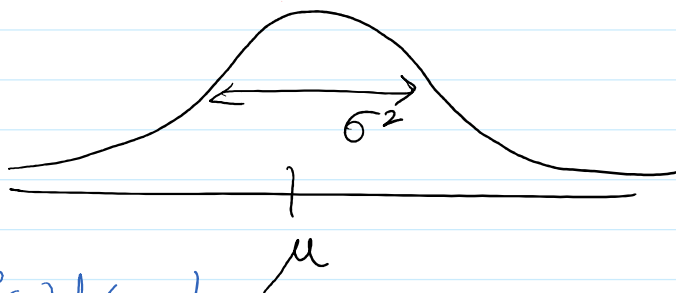
$$= f(0) + e^t f(1)$$

$1 - p + pe^t$

## Normal Distribution / Gaussian Distribution

$$X \sim N(\mu, \sigma^2)$$

$$\mu \in \mathbb{R}, \sigma^2 > 0$$



Shaped:  $\int f(x) dx = 1$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}(x-\mu)^2\right), \text{ for all } x \in \mathbb{R}$$

$\mathbb{E}[X] = \mu$

or

$\text{Var}(X) = \sigma^2$

CDF: No closed form.

Expected Value:  $\mathbb{E}[X] = \mu$



$$\underline{E[X]} = \int_{\mathbb{R}} x f(x) dx$$

$$= \int x \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx$$

change of variables:  $y = x - \mu$  or  $x = y + \mu$   
 $dy = dx$

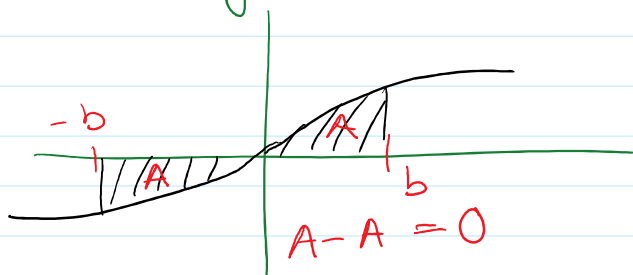
$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int (y + \mu) \exp\left(-\frac{1}{2\sigma^2}y^2\right) dy$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \underbrace{\int_{-\infty}^{\infty} y \exp\left(-\frac{1}{2\sigma^2}y^2\right) dy}_{g(y)} + \frac{1}{\sqrt{2\pi\sigma^2}} \int \mu \exp\left(-\frac{1}{2\sigma^2}y^2\right) dy$$

$g(y)$  is a odd function

$$g(-y) = -g(y)$$

If I integrate an odd function  
over a symmetric interval



$$\mu \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}y^2\right) dy}_{\text{pdf of a } N(0, \sigma^2) = \text{integral is } 1}$$

$\mu (1)$

$$= 0 + \mu = \mu = E[X]$$

$$= 0 + \mu = \mu = E[X]$$


---

$$\text{Var}(X) = E[(X - \mu)^2] \quad \mu = E[X]$$

$$= \int_{\mathbb{R}} (x - \mu)^2 f(x) dx$$

$$= \int_{\mathbb{R}} (x - \mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x - \mu)^2\right) dx$$

$$y = \frac{x - \mu}{\sigma} \quad \text{then} \quad dy = \frac{1}{\sigma} dx \Leftrightarrow dx = \sigma dy$$

$$= \frac{\sigma}{\sqrt{2\pi\sigma^2}} \int \sigma^2 y^2 \exp\left(-\frac{1}{2} y^2\right) dy$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int y^2 \exp\left(-\frac{1}{2} y^2\right) dy = \frac{\sigma^2}{\sqrt{2\pi}} \int_{\mathbb{R}} u dv$$

Integration by parts:  $u = y \quad dv = y \exp\left(-\frac{1}{2} y^2\right) dy$

$$\text{Can show: } \int u dv = \sqrt{2\pi}$$

$$\sigma^2 \quad \left[ \begin{array}{c} \text{---}^2 \text{---} \end{array} \right]$$

$$\frac{\sigma^2}{\sqrt{2\pi}} \sqrt{2\pi} = \boxed{\sigma^2 = \text{Var}(X)}$$

MGF:

$$M(t) = \mathbb{E}[e^{tx}] = \int_{\mathbb{R}} e^{tx} f(x) dx$$

$$(x-\mu)^2 = x^2 - 2x\mu + \mu^2$$

$$= \int_{\mathbb{R}} \underbrace{e^{tx}}_{\text{MGF of } N(\mu, \sigma^2)} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)}_{\text{pdf of } N(\mu, \sigma^2)} dx$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(\underbrace{x^2 - 2x\mu + \mu^2}_{\text{complete the square}} - 2tx\sigma^2)\right) dx$$

complete the square

$$x^2 - 2x(\mu + \sigma^2 t) + \mu^2$$

$$= (x - (\mu + \sigma^2 t))^2 + \mu^2 - (\mu + \sigma^2 t)^2$$

bring out

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{\mu^2 - (\mu + \sigma^2 t)^2}{-2\sigma^2}\right) \exp\left(-\frac{1}{2\sigma^2}(x - (\mu + \sigma^2 t))^2\right) dx$$

pdf of a  $N(\mu + \sigma^2 t, \sigma^2)$

pdf of a  $N(\mu + \sigma^2 t, \sigma^2)$

$$= \underbrace{\exp\left(-\frac{1}{2\sigma^2}(\mu^2 - (\mu + \sigma^2 t)^2)\right)}_{\text{MGF}} \underbrace{\int N(\mu + \sigma^2 t, \sigma^2)}_{1}$$

= ... = simplify

$$= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

test:  $\frac{dM}{dt} \Big|_{t=0} = E[X] = \mu$

$$\frac{dM}{dt} = (\underbrace{\mu + \sigma^2 t}_0) \exp\left(\underbrace{\mu t + \frac{\sigma^2 t^2}{2}}_1\right)$$

so  $\frac{dM}{dt} \Big|_{t=0} = \mu$

---

Theorem: Linear Functions of a  $N(\mu, \sigma^2)$

If  $X \sim N(\mu, \sigma^2)$  then

$$Y = aX + b$$

then  $Y \sim N(\underline{a\mu + b}, a^2\sigma^2)$ .

intuition:  $E[Y] = E[aX + b] = aE[X] + b$   
 $= a\mu + b$

$$\text{Var}(Y) = \text{Var}(aX + b) = a^2 \text{Var}(X) \\ = \underline{a^2\sigma^2}.$$

pf.  $M_Y(t) = M_{aX+b}(t) = e^{tb} M_X(at)$

$$= e^{tb} \exp\left(\mu at + \frac{\sigma^2}{2} a^2 t^2\right)$$
$$= \underline{\exp\left((a\mu + b)t + \frac{a^2\sigma^2}{2} t^2\right)}$$

MGF of a  $N(a\mu + b, a^2\sigma^2)$

---