Time encoding and perfect recovery of non-bandlimited signals with an integrate-and-fire system

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***Abstract*—Time encoding represents an alternative method of sampling, based on mapping the amplitude information of a signal into a time sequence. In this paper, we investigate the problem of time encoding based on an integrate-and-fire model, consisting of an integrator and a threshold comparator. We focus on particular classes of non-bandlimited signals such as streams and bursts of Diracs, and prove we can recover these perfectly from their timing information.**

***Index Terms*—Integrate-and-fire, time encoding, non-uniform sampling, finite rate of innovation.**

I. Introduction

There has recently been a growing interest in sampling theory, which achieves the conversion of continuous signals into discrete sequences [1]. From the classical Shannon sampling theorem [2], to recent theories in compressed sensing [3], [4] and finite rate of innovation [5]–[8], sampling theory has provided precise answers on when a faithful conversion of a continuous waveform into a discrete sequence is possible. These methods are generally based on recording the amplitude of the signal at specified time instants, which lead to uniform sampling if evenly spaced, and non-uniform otherwise.

An alternative method to classical sampling is time encoding, which captures the amplitude information of a signal into a sequence of non-uniform time instants. Time encoding appears in nature, as a mechanism used by neurons to represent sensory information as a sequence of action potentials, allowing them to process information very efficiently [9]. Similarly, acquisition models inspired by this mechanism, such as analog to digital converters [10] or event-based vision sensors [11], lead to simple and efficient processing of information. Within the topic of time encoding, several authors have provided ways to sample and reconstruct bandlimited signals [12]–[15], typically connecting time-based sampling with the problem of non-uniform sampling in shift-invariant spaces [16]–[18].

In this paper, we focus on particular classes of continuous-time non-bandlimited signals such as streams or bursts of Diracs, and prove that it is possible to perfectly recover them from timing information. In Section II, we describe the Time Encoding Machine (TEM) based on an integrate-and-fire model. This acquisition model is based on first filtering the input with a sampling kernel which can locally reproduce exponentials, in the context of non-uniform sampling. Furthermore, in Section III we leverage the properties of the sampling kernel into an algorithm for perfect recovery of an input Dirac, and extend this method to streams and bursts of Diracs. Then, in Section IV we derive conditions for perfect retrieval of these signals. The results in Section V show that reconstruction of these signals from timing information is exact to numerical precision. Finally, conclusions are drawn in Section VI.

II. Time-based Sampling using an Integrate-and-Fire Mechanism

A. Acquisition Model

The operating principle of the time encoding strategy investigated in this paper is similar to the one in [12], and is depicted in Fig. 1. The signal is first filtered with the kernel $\varphi(-t)$, before being passed to an integrator. When the output of the integrator reaches the positive or negative trigger mark $\pm C_T$, the time encoding machine outputs a spike and the integrated signal $y(t)$ is reset to 0. The time instants when the integrator reaches the threshold $\pm C_T$ are recorded in the sequence $\{t_n\}$. Then, we can compute the output sample $y(t_n)$ at each spike t_n as:

$$y(t_n) = \pm C_T = \int_{t_{n-1}}^{t_n} f(\tau) d\tau, \tag{1}$$

where $f(t)$ is defined as:

$$f(t) = \int x(\tau)\varphi(\tau - t) d\tau, \text{ for } t \in [t_{n-1}, t_n]. \tag{2}$$

Hence, time encoding with an integrate-and-fire model is equivalent to a non-uniform sampling problem, where we aim to estimate the input $x(t)$ from non-uniform samples $y(t_n)$.

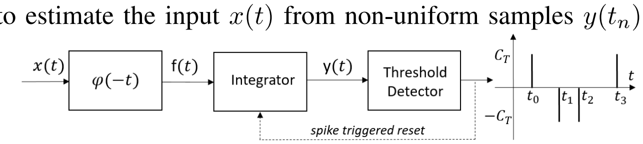

Fig. 1: Time Encoding Machine based on Integrate-and-fire.

B. Sampling Kernels

The authors in [5] have shown that many signals with a local rate of innovation can be perfectly retrieved from their *uniform* samples, using kernels that are able to reproduce polynomials or exponentials. This last family of kernels is also of particular interest in the context of time-based sampling using an integrate-and-fire mechanism.

An exponential reproducing kernel is a function $\varphi(t)$ that, together with its uniformly shifted versions, reproduces exponentials of the form $e^{\alpha_m t}$:

$$\sum_{n \in \mathbb{Z}} c_{m,n} \varphi(t - n) = e^{\alpha_m t}, \tag{3}$$

where $\alpha_m \in \mathbb{C}$ and for a proper choice of coefficients $c_{m,n}$.

In order to satisfy Eq. (3), $\varphi(t)$ has to verify the generalized Strang-Fix conditions [19], [20], and the E-splines [21] are an important family of functions that satisfy these conditions.

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The second-order E-spline of support L is defined as:

$$\varphi(t)=\begin{cases}\frac{e^{\alpha_1-\alpha_0}}{\alpha_1-\alpha_0}e^{-\alpha_0 t}+\frac{e^{-\alpha_1+\alpha_0}}{\alpha_0-\alpha_1}e^{-\alpha_1 t}, & -L\le t\le -\frac{L}{2},\\ \frac{1}{\alpha_0-\alpha_1}e^{-\alpha_0 t}+\frac{1}{\alpha_1-\alpha_0}e^{-\alpha_1 t}, & -\frac{L}{2}\le t\le 0,\\ 0, & \text{otherwise.}\end{cases} \tag{4}$$

In what follows, we show that the E-spline in Eq. (4) can be used to *locally* reproduce exponentials, in the context of non-uniform sampling. In other words, we prove that, within a time interval $[t_1, t_2]$ where there are no knots of the shifted splines $\varphi(t-t_n)$, the following equation holds:

$$\sum_{n=0}^{N-1} c_{m,n}\varphi(t-t_n)=e^{-\alpha_m t}, \tag{5}$$

where $N\ge 2$, $m\in\{0,1\}$, and $\{t_n\}$ are non-uniform.

We notice that within each of its knot-free regions, the second-order E-spline $\varphi(t-t_n)$ can be expressed as a linear combination of the exponentials $e^{-\alpha_0 t}$ and $e^{-\alpha_1 t}$:

$$\varphi(t-t_n)=\begin{cases}a_{1,n}e^{-\alpha_0 t}+b_{1,n}e^{-\alpha_1 t}, & t_n-L\le t\le t_n-\frac{L}{2},\\ a_{2,n}e^{-\alpha_0 t}+b_{2,n}e^{-\alpha_1 t}, & t_n-\frac{L}{2}\le t< t_n,\\ 0, & \text{otherwise,}\end{cases} \tag{6}$$

where $a_{1,n}, b_{1,n}, a_{2,n}$ and $b_{2,n}$ are chosen to satisfy Eq. (4).

Then, let I_1 be a knot-free interval with $I_1\subset[t_{n+1}-L, t_n-\frac{L}{2}]$. Moreover let $v_1(t)=\varphi(t-t_n)$ for $t\in I_1$ and $v_2(t)=\varphi(t-t_{n+1})$ for $t\in I_1$. In the vector space spanned by $e^{-\alpha_0 t}$ and $e^{-\alpha_1 t}$, the elements $v_1(t)$ and $v_2(t)$ are linearly independent, since $t_{n+1}\neq t_n$. Hence, using a linear combination of v_1 and v_2, we can uniquely represent any vector in this space, including $e^{-\alpha_0 t}$ and $e^{-\alpha_1 t}$. Therefore, in the interval I_1 where there are no knots, we can find unique $c_{m,n}$ and $c_{m,n+1}$ such that Eq. (5) holds for $m\in\{0,1\}$.

Similarly, reproduction of exponentials is possible on any time interval spanned by knot-free regions of at least two shifted E-splines. For different continuous intervals, the solution to Eq. (5) differs, as highlighted in Fig. 2, where reconstruction of exponentials in the continuous regions I_1 and I_2 is possible, but with different coefficients.

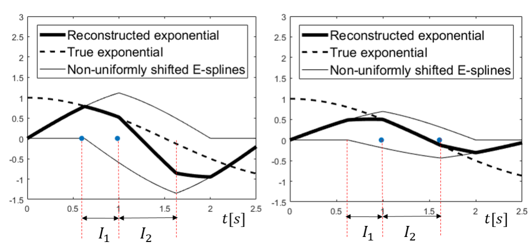

Fig. 2: Reproduction of $e^{j\frac{2\pi}{5}t}$ in two different intervals, $I_1=[0.625,1]s$ and $I_2=[1,1.625]s$, overlapped by continuous regions of two non-uniformly shifted second-order E-splines.

III. Perfect Recovery of Non-bandlimited Signals from Timing Information

In this section, we provide algorithms for perfect recovery from timing information, of the following classes of non-bandlimited signals: single Dirac, stream of Diracs, and bursts of Diracs. For all the signals studied, we assume sampling using the TEM in Fig. 1, where the filter is the second-order E-spline $\varphi(t)$ defined in Eq. (4), of support L, which can reproduce two different complex exponentials $e^{-\alpha_0 t}$ and $e^{-\alpha_1 t}$, with $\alpha_0=j\omega_0$, $\alpha_1=j\omega_1$ and $\omega_1=-\omega_0$. The choice of frequencies α_0 and α_1 ensures that $\varphi(t)$ is real-valued.

A. Estimation of a Single Dirac

Let us consider a single input Dirac given by:

$$x(t)=x_1\delta(t-\tau_1). \tag{7}$$

Suppose that in the interval $[\tau_1, \tau_1+\frac{L}{2}]$, the TEM outputs 3 spikes, at t_0, t_1 and t_2. Then, using the property in Eq. (1), we compute the following non-uniform output samples:

$$y(t_0)=\frac{x_1(e^{\alpha_0(t_0-\tau_1)}-1)}{\alpha_0(\alpha_0-\alpha_1)}+\frac{x_1(e^{\alpha_1(t_0-\tau_1)}-1)}{\alpha_1(\alpha_1-\alpha_0)},$$

$$y(t_1)=\frac{x_1(e^{\alpha_0 t_1}-e^{\alpha_0 t_0})}{\alpha_0(\alpha_0-\alpha_1)}e^{-\alpha_0\tau_1}+\frac{x_1(e^{\alpha_1 t_1}-e^{\alpha_1 t_0})}{\alpha_1(\alpha_1-\alpha_0)}e^{-\alpha_1\tau_1}, \tag{8}$$

$$y(t_2)=\frac{x_1(e^{\alpha_0 t_2}-e^{\alpha_0 t_1})}{\alpha_0(\alpha_0-\alpha_1)}e^{-\alpha_0\tau_1}+\frac{x_1(e^{\alpha_1 t_2}-e^{\alpha_1 t_1})}{\alpha_1(\alpha_1-\alpha_0)}e^{-\alpha_1\tau_1}. \tag{9}$$

The samples $y(t_1)$ and $y(t_2)$ are equivalent to those obtained by filtering the input signal $x(t)$ with a new kernel $\tilde{\varphi}(t)$:

$$y(t_1)=\langle x(t), \tilde{\varphi}(t-t_1)\rangle,$$

where the shifted kernel can be expressed as a linear combination of the exponentials $e^{-\alpha_0 t}$ and $e^{-\alpha_1 t}$:

$$\begin{aligned}\tilde{\varphi}(t-t_1)&=\frac{e^{\alpha_0 t_1}-e^{\alpha_0 t_0}}{\alpha_0(\alpha_0-\alpha_1)}e^{-\alpha_0 t}+\frac{e^{\alpha_1 t_1}-e^{\alpha_1 t_0}}{\alpha_1(\alpha_1-\alpha_0)}e^{-\alpha_1 t}\\ &=a_1e^{-\alpha_0 t}+b_1e^{-\alpha_1 t}.\end{aligned}$$

Similarly, we compute the sample $y(t_2)$ as:

$$y(t_2)=\langle x(t), \tilde{\varphi}(t-t_2)\rangle=\langle x(t), a_2e^{-\alpha_0 t}+b_2e^{-\alpha_1 t}\rangle.$$

Then, using the proof in Section II-B, we can find the unique coefficients $c_{m,1}$ and $c_{m,2}$ such that:

$$c_{m,1}\tilde{\varphi}(t-t_1)+c_{m,2}\tilde{\varphi}(t-t_2)=e^{-\alpha_m t}, \text{ for } m\in\{0,1\}. \tag{10}$$

Furthermore, we can define the signal moments as:

$$s_m=\sum_{n=1}^{2}c_{m,n}y(t_n)\overset{(a)}{=}x_1\sum_{n=1}^{2}c_{m,n}\tilde{\varphi}(\tau_1-t_n)\overset{(b)}{=}x_1e^{-\alpha_m\tau_1}, \tag{11}$$

where (a) follows from Eq. (7), and (b) from Eq. (10).

Finally, using Prony's method [22], we can uniquely estimate the input parameters x_1 and τ_1, from the two signal moments s_m given by Eq. (11), for $m\in\{0,1\}$ and $\alpha_1=-\alpha_0$.

B. Estimation of a Stream of Diracs

Let us now consider an input stream of Diracs:

$$x(t)=\sum_k x_k\delta(t-\tau_k).$$

Assuming that the Diracs are sufficiently separated, such that $\tau_k-\tau_{k-1}>L, \forall k$, we propose the following sequential retrieval algorithm. Let us suppose that the first Dirac $\delta_1=x_1\delta(t-\tau_1)$ is correctly estimated using the method in Section III-A. Moreover, let us denote the output spike locations in the interval $[\tau_1, \tau_1+L]$ with $t_0, t_1, ..., t_{n-1}$, and the times after τ_1+L with $t_n, t_{n+1}, ...$, which means that the location of the second Dirac must satisfy $\tau_2\in[\tau_1+L, t_n]$.

Using the model of Fig. 1, we compute the first 3 non-uniform output samples after τ_1+L as:

$$y(t_n)=\int_{t_{n-1}}^{\tau_1+L}x_1\varphi(\tau_1-\tau)d\tau+\int_{\tau_2}^{t_n}x_2\varphi(\tau_2-\tau)d\tau,$$

$$y(t_{n+1})=y_{n+1}=\int_{t_n}^{t_{n+1}}x_2\varphi(\tau_2-\tau)d\tau,$$

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$$y(t_{n+2}) = y_{n+2} = \int_{t_{n+1}}^{t_{n+2}} x_2 \varphi(\tau_2 - \tau) d\tau.$$

The sample $y(t_n)$ contains information of both δ_1 and δ_2, and hence cannot be used for estimation of the latter Dirac. On the other hand, Section IV describes sufficient conditions that guarantee that the output samples y_{n+1} and y_{n+2} occur in the time interval $[\tau_2, \tau_2 + \frac{L}{2}]$. Then, since these samples have contribution from δ_2 only, we can use the proof in Section III-A to compute the signal moments as:

$$s_m = c_{m,1} y_{n+1} + c_{m,2} y_{n+2} = x_2 e^{-\alpha_m \tau_2}.$$

Once δ_2 is estimated from the signal moments using Prony's method, we use the subsequent non-uniform output samples, after $\tau_2 + L$, in order to sequentially retrieve the next Diracs.

C. Estimation of a Dirac Burst with a Multichannel Approach

We now address the problem of estimation of a single input burst of K Diracs, given by:

$$x(t) = \sum_{k=1}^{K} x_k \delta(t - \tau_k),$$

where $\tau_1 < \tau_2 < ... < \tau_K$, and all x_k have the same sign.

Then, the first output samples can be computed as:

$$y(t_0) = \int_{\tau_1}^{t_0} \langle x(t), \varphi(t) \rangle dt$$

and

$$y(t_i) = y_i = \int_{t_{i-1}}^{t_i} \langle x(t), \varphi(t) \rangle dt, \text{ for } i \geq 1.$$

Assuming $\tau_K - \tau_1 < \frac{L}{2}$, Section IV describes conditions that guarantee $t_1, t_2, t_3 \in [\tau_K, \tau_1 + \frac{L}{2}]$. This means that whilst y_0 and y_1 may capture information of only some of the K Diracs, y_2 and y_3 will contain information from all the input Diracs. Then, using the definition of $\varphi(\tau_k - t)$ from Eq. (4) for $[\tau_k, \tau_k + \frac{L}{2}]$, we get:

$$y_2 = \sum_{k=1}^{K} \frac{x_k (e^{\alpha_0 t_2} - e^{\alpha_0 t_1})}{\alpha_0(\alpha_0 - \alpha_1)} e^{-\alpha_0 \tau_k} + \frac{x_k (e^{\alpha_1 t_2} - e^{\alpha_1 t_1})}{\alpha_1(\alpha_1 - \alpha_0)} e^{-\alpha_1 \tau_k},$$

$$y_3 = \sum_{k=1}^{K} \frac{x_k (e^{\alpha_0 t_3} - e^{\alpha_0 t_2})}{\alpha_0(\alpha_0 - \alpha_1)} e^{-\alpha_0 \tau_k} + \frac{x_k (e^{\alpha_1 t_3} - e^{\alpha_1 t_2})}{\alpha_1(\alpha_1 - \alpha_0)} e^{-\alpha_1 \tau_k}.$$

For $K = 1$, these samples are identical to the ones in Eq. (8) and (9). Hence, using the proof in Section III-A, we can find the unique coefficients $c_{m,1}$ and $c_{m,2}$ which give:

$$s_m = c_{m,1} y_2 + c_{m,2} y_3 = \sum_{k=1}^{K} x_k e^{-\alpha_m \tau_k}, \text{ for } m \in \{0, 1\}. \quad (12)$$

Since the input is a burst of K Diracs, there are $2K$ free parameters we need to retrieve, and Prony's method requires at least $2K$ signal moments in order to ensure correct estimation. Given that with one channel we obtain 2 signal moments as in Eq. (12), we need K channels to compute $2K$ different moments. Finally, the filter $\varphi(t)$ of the m^{th} channel is a second-order E-spline which can reproduce the exponentials $e^{j\omega_{m_0} t}$ and $e^{j\omega_{m_1} t}$, where $\omega_{m_0} = \omega_0 + \lambda m$, $\omega_{m_1} = \omega_0 + \lambda(2K - 1 - m)$ and $\lambda = \frac{-2\omega_0}{2K-1}$ (which ensures $\omega_{m_1} = -\omega_{m_0}$ such that $\varphi(t)$ is a real-valued function).

D. Estimation of Bursts of Diracs

We now consider the case of input bursts of K Diracs:

$$x(t) = \sum_{b} \sum_{k=1}^{K} x_{b,k} \delta(t - \tau_{b,k}), \quad (13)$$

where $\tau_{b,k}$ is the location of Dirac δ_k in burst b, and all amplitudes $x_{b,k}$ in a burst b have the same sign.

We assume that the Diracs in consecutive bursts are sufficiently separated, such that $\tau_{b,1} - \tau_{b-1,K} > L, \forall b$, and propose the following sequential reconstruction algorithm.

Suppose the first burst of Diracs is correctly estimated using the multichannel method presented in Section III-C, and that the retrieved locations are $\tau_{1,1}, \tau_{1,2}..., \tau_{1,K}$. Moreover, denoting the output spikes after $\tau_{1,K} + L$ with $t_n, t_{n+1}, t_{n+2}, ...$, we can impose sufficient conditions that guarantee $t_{n+1}, t_{n+2}, t_{n+3} \in [\tau_{2,K}, \tau_{2,1} + \frac{L}{2}]$, as detailed in Section IV. Therefore, using the proof in Section III-C, we can use the output samples $y(t_{n+2})$ and $y(t_{n+3})$ as in Eq. (12), to retrieve the subsequent input burst.

IV. Sufficient Conditions for Perfect Reconstruction

In this section, we prove sufficient conditions for loss-free recovery of a sequence of bursts of Diracs, and an input stream of Diracs. We impose conditions on the trigger mark of the comparator in Fig. 1, which ensure the output spike train captures sufficient information of the input signal, as well as on the minimum separation between consecutive bursts of Diracs, to ensure the input parameters are correctly retrieved using the sequential algorithms presented in Section III.

Proposition 1. *The timing information $t_{1,i}, t_{2,i}, ..., t_{M,i}$ for $i = 0, 1, ..., K - 1$ provided by K devices as in Fig. 1 is a sufficient representation of bursts of K Diracs as in Eq. (13) when the sampling kernel of the m^{th} time encoding machine is a second-order E-spline of support L, which can reproduce the exponentials $e^{j\omega_{m_0} t}$ and $e^{j\omega_{m_1} t}$ with $\omega_{m_0} = \omega_0 + \lambda m$, $\omega_{m_1} = \omega_0 + \lambda(2K - 1 - m)$, $\lambda = \frac{-2\omega_0}{2K-1}$ and $0 < \omega_0 \leq \frac{\pi}{L}$. Moreover, the spacing between bursts should be larger than L, and the separation between the last and first Diracs within any burst b must satisfy $\tau_{b,K} - \tau_{b,1} < \frac{L}{2}$. In addition, the comparator's trigger mark C_T must satisfy the following conditions for each device m and burst b:*

$$C_T > \frac{(K-1)A_{max}}{\omega_{m_0}^2} [1 - \cos(\omega_{m_0}(\tau_{b,K} - \tau_{b,1}))], \quad (14)$$

$$C_T < \frac{K A_{min}}{5\omega_{m_0}^2} [1 - \cos(\omega_{m_0}(\frac{L}{2} - (\tau_{b,K} - \tau_{b,1})))], \quad (15)$$

where A_{max} and A_{min} are the absolute maximum and minimum amplitudes of the input Diracs, and $\tau_{b,1}$ and $\tau_{b,K}$ are the locations of the first and last Diracs in burst b, respectively.

Proof. Suppose we want to estimate a burst of K Diracs with locations $\tau_1, \tau_2, ..., \tau_K$, and let us assume for simplicity that their amplitudes satisfy $x_1, x_2, ..., x_K > 0$. We denote with $t_n, t_{n+1}, ..., t_M$ the output spikes caused by this burst in a certain channel, such that $t_n > \tau_1 > t_{n-1}$. Then, for the derivations in Section III-D to hold, we need to ensure that $t_{n+1}, t_{n+2}, t_{n+3} \in [\tau_K, \tau_1 + \frac{L}{2}]$. The condition $t_{n+1} > \tau_K$ is equivalent to:

$$\int_{\tau_1}^{t_{n+1}} f(\tau) d\tau > \int_{\tau_1}^{\tau_K} f(\tau) d\tau. \quad (16)$$

The left-hand side of this inequality can be expressed as:

$$\int_{\tau_1}^{t_{n+1}} f(\tau) d\tau = \int_{t_{n-1}}^{t_{n+1}} f(\tau) d\tau - \int_{t_{n-1}}^{\tau_1} f(\tau) d\tau \overset{(a)}{>} C_T, \quad (17)$$

where (a) holds given Eq. (1) and $t_n > \tau_1 > t_{n-1}$.

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The right-hand side of Eq. (16) can be re-written as:

$$\int_{\tau_1}^{\tau_K} f(\tau)d\tau \overset{\text{(a)}}{<} \sum_{k=1}^{K-1} A_{max} \int_{\tau_k}^{\tau_K} \varphi(\tau_k - \tau)d\tau$$
$$\overset{\text{(b)}}{<} \frac{(K-1)A_{max}}{\omega_{m_0}^2}[1 - \cos(\omega_{m_0}(\tau_K - \tau_1))] \overset{\text{(c)}}{<} C_T \overset{\text{(d)}}{<} \int_{\tau_1}^{t_{n+1}} f(\tau)d\tau,$$

which shows that indeed $t_{n+1} > \tau_K$.

In the derivations above, (a) follows from Eq. (2), and the assumption $x_1, ..., x_K > 0$, (c) follows from Eq. (14) and (d) from Eq. (17). Finally, condition (b) follows from:

$$\int_{\tau_k}^{\tau_K} \varphi(\tau_k - \tau)d\tau \overset{\text{(a)}}{=} \frac{1}{\omega_{m_0}^2}[1 - \cos(\omega_{m_0}(\tau_K - \tau_k))]$$
$$\overset{\text{(b)}}{<} \frac{1}{\omega_{m_0}^2}[1 - \cos(\omega_{m_0}(\tau_K - \tau_1))].$$

where (a) follows from the definition of $\varphi(\tau_k - \tau)$ in Eq. (4) for $\tau \in [\tau_k, \tau_K]$ with $\tau_K < \tau_k + \frac{L}{2}$, and from the hypothesis that $\varphi(\tau)$ reproduces the exponentials $e^{\pm j\omega_{m_0}\tau}$. Moreover, (b) follows from the hypothesis that $0 < \omega_{m_0} \leq \frac{\pi}{L}$ which is equivalent to $0 < \frac{\omega_{m_0} L}{2} \leq \frac{\pi}{2}$, and from $\tau_K - \tau_k < \frac{L}{2}$, which means that $0 < \omega_{m_0}(\tau_K - \tau_k) < \frac{\pi}{2}$, and hence $1 - \cos(\omega_{m_0}(\tau_K - \tau_1)) > 1 - \cos(\omega_{m_0}(\tau_K - \tau_k))\ \forall k = 2, ..., K$.

Similarly, since $f(\tau) > 0$ for $x_1, ..., x_K > 0$, the condition $\tau_1 + \frac{L}{2} > t_{n+3}$ is equivalent to:

$$\int_{\tau_1}^{\tau_1 + \frac{L}{2}} f(\tau)d\tau > \int_{\tau_1}^{t_{n+3}} f(\tau)d\tau, \tag{18}$$

where the left-hand side can be expressed as:

$$\begin{aligned}
\int_{\tau_1}^{\tau_1 + \frac{L}{2}} f(\tau)d\tau &\overset{\text{(a)}}{=} \sum_{k=1}^{K} \int_{\tau_1}^{\tau_1 + \frac{L}{2}} x_k \varphi(\tau_k - \tau)d\tau \\
&\overset{\text{(b)}}{=} \frac{1}{\omega_{m_0}^2} \sum_{k=1}^{K} x_k [1 - \cos(\omega_{m_0}(\frac{L}{2} - (\tau_k - \tau_1)))] \\
&\overset{\text{(c)}}{>} \frac{1}{\omega_{m_0}^2} \sum_{k=1}^{K} x_k [1 - \cos(\omega_{m_0}(\frac{L}{2} - (\tau_K - \tau_1)))] \\
&\overset{\text{(d)}}{>} \frac{K A_{min}}{\omega_{m_0}^2}[1 - \cos(\omega_{m_0}(\frac{L}{2} - (\tau_K - \tau_1)))] \overset{\text{(e)}}{>} 5C_T,
\end{aligned} \tag{19}$$

where (a) follows from Eq. (2), (b) follows from the definition of $\varphi(\tau_k - \tau)$ in Eq. (4) for $\tau \in [\tau_k, \tau_1 + \frac{L}{2}]$, and (c) follows from the hypothesis that $0 < \omega_{m_0} \leq \frac{\pi}{L}$ which is equivalent to $0 < \frac{\omega_{m_0} L}{2} \leq \frac{\pi}{2}$, and since $\tau_k - \tau_1 < \frac{L}{2}\ \forall k = 2, ..., K$. Moreover, (d) holds since we assume $x_1, ..., x_K > 0$, and (e) follows from Eq. (15).

Finally, the right-hand side of Eq. (18) is equivalent to:

$$\int_{\tau_1}^{t_{n+3}} f(\tau)d\tau = \int_{t_{n-1}}^{t_{n+3}} f(\tau)d\tau - \int_{t_{n-1}}^{\tau_1} f(\tau)d\tau$$
$$\overset{\text{(a)}}{=} 4C_T - \int_{t_{n-1}}^{\tau_1} f(\tau)d\tau \overset{\text{(b)}}{<} 5C_T \overset{\text{(c)}}{<} \int_{\tau_1}^{\tau_1 + \frac{L}{2}} f(\tau)d\tau,$$

where (a) follows from Eq. (1), (b) holds since $t_n > \tau_1 > t_{n-1}$ and (c) follows from Eq. (19). □

In addition, we need to impose constraints on the separation $\tau_K - \tau_1$, such that the system of Eq. (14) and (15) is consistent.

Finally, the corresponding conditions for a stream of Diracs are: $0 < C_T < \frac{A_{min}}{4\omega_0^2}$. These are obtained by setting $K = 1$ in Eq. (14) and Eq. (15), and relaxing the latter condition to $\int_{\tau_1}^{\tau_1 + \frac{L}{2}} y(\tau)d\tau < 4C_T$. This is because for the retrieval of a Dirac δ_1 in a stream, we only need 3 output spikes in the interval $[\tau_1, \tau_1 + \frac{L}{2}]$ as detailed in Section III-B (rather than 4 samples as in the case of bursts of Diracs).

V. Simulations

The sampling and reconstruction of a stream of $K = 3$ Diracs are depicted in Fig. 3. Here, the filter is a second-order E-spline, of support $L = 2$, as seen in Fig. 3(b), the inter-Dirac separation is larger than the kernel support L, as seen in Fig. 3(a), and the threshold comparator's trigger mark is $C_T = 0.12$. The amplitudes and locations of the estimated Diracs are exact to numerical precision. The reconstruction of 3 bursts of 2 Diracs (with inter-burst separation larger than L) from non-uniform samples is depicted in Fig. 4. The reconstructed signal depicted in Fig. 4(d) is exact to numerical precision. Finally, in the plots in Fig. 3(c) and 4(c) we observe that there are no output spikes in a region where the input signal is zero, which leads to lower average density of samples.

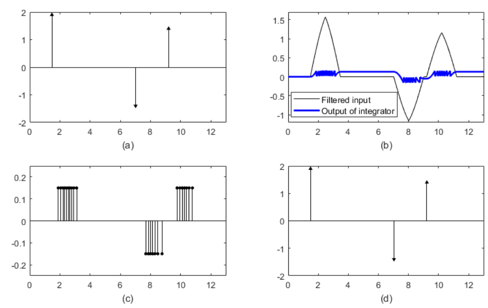

Fig. 3: Sampling of a stream of Diracs. The input signal is shown in (a), the integrator output in (b), the output non-uniform samples in (c), and the reconstructed signal in (d).

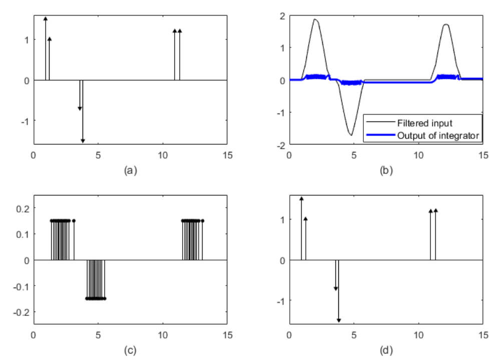

Fig. 4: Sampling of bursts of Diracs. The input signal is shown in (a), the first channel's integrator output in (b), the first channel's output non-uniform samples in (c), and the reconstructed signal in (d).

VI. Conclusions

This paper investigated the problem of time encoding using an integrate-and-fire mechanism, which consists of filtering the input with an exponential reproducing kernel, and obtaining the timing information with an integrator and a threshold comparator. We first showed that the sampling kernel can locally reproduce exponentials in the case of non-uniform sampling. Moreover, we designed algorithms for recovery of non-bandlimited signals from timing information. Simulations verified that reconstruction of these signals is exact.

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