Evidence in favor of a scientific theory

With great complexity comes great honesty

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Why Bayesian inference?

Allows us to optimally update a priori beliefs given a model and data.

Where comes from?

	Not infected	Infected	
Not vaccinated	4	2	6
Vaccinated	76	18	94
	80	20	100

From conditional probability

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Bayes theorem:

$$P(A_1|B_1) = \frac{P(B_1 \cap A_1)}{P(B_1)} = \frac{P(B_1|A_1)P(A_1)}{P(B_1)} \tag{1}$$

Scientific test example

There is a test that correctly detects zombies 95% of the time.

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Someone receive a positive test:
She has **only 8.7% chance** to actually be a zombie!?

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In this example all frequencies were observables

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About our **belief distributions** of those hidden variables!

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A model is always there!

$$\underbrace{P(\text{Belief}|\text{Data},\text{Model})}_{\text{Posterior}} = \frac{\overbrace{P(\text{Data}|\text{Belief},\text{Model})}^{\text{Likelihood}}\,\overbrace{P(\text{Belief}|\text{Model})}^{\text{Prior}}}{\underbrace{P(\text{Data}|\text{Model})}_{\substack{\text{Evidence or}\\ \text{Average likelihood}}}}$$

- **Prior** belief (distribution):

$$P(B|M) = \frac{1}{\#\text{Beliefs}} \qquad \forall B \in \text{Beliefs}$$

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- **Posterior** belief (distribution):

$$P(B|D,M) = \frac{P(D|B,M)P(B|M)}{P(D|M)} \qquad \forall B \in \text{Beliefs}$$

The garden of forking paths

To update our beliefs (posterior), we need to consider every possible path in the model that could have lead us to the observed data (likelihood).

The garden of forking paths

Data (D): ● ○ ● Beliefs (B): ○○○○, ●○○○, ●●○○, ●●●○, ●●●●

Model (M): Data $\sim$ Binomial(n, p)

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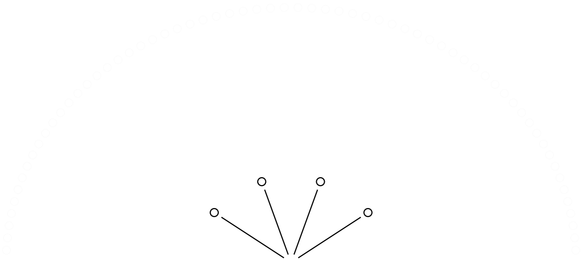

Ways given M and $B = \circ\circ\circ\circ$ (First marbel)

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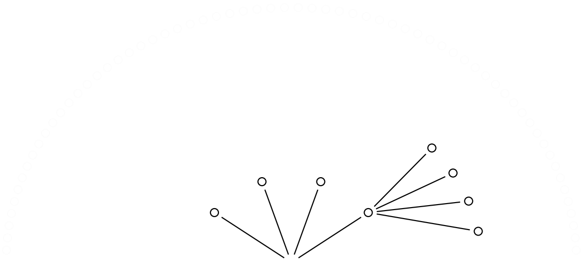

Ways given M and $B = \circ\circ\circ\circ$ (Second marbel)

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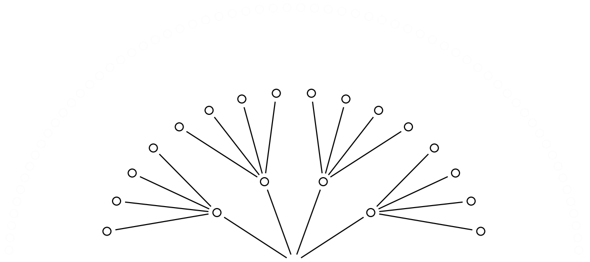

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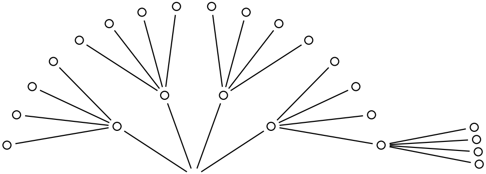

Ways given M and $B = \circ\circ\circ\circ$ (Third marbel)

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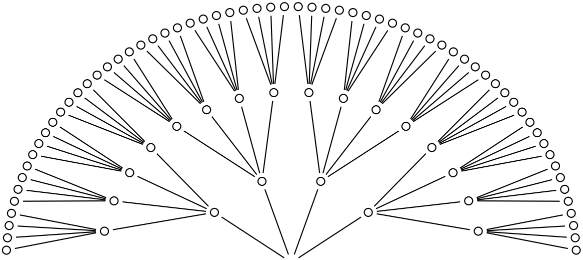

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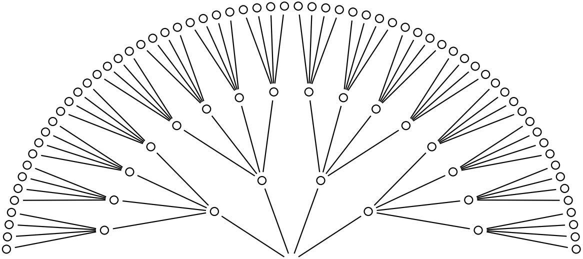

Ways given M and $B =$ ○○○○

Belief	Ways to produce ● ○ ●
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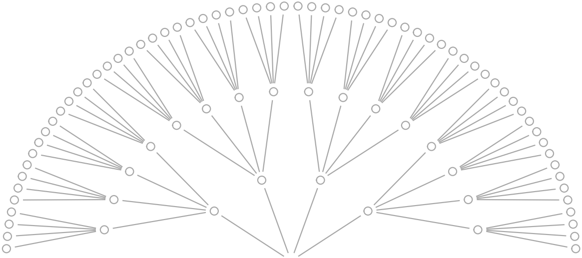

Ways given M and $B =$ ○○○○

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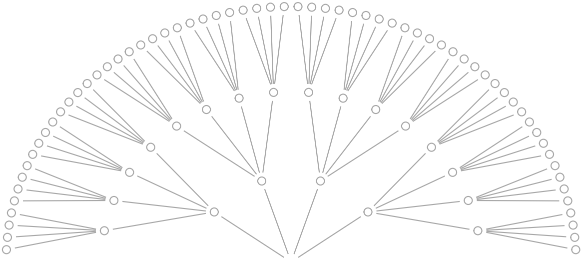

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Belief	Ways to produce ● ○ ●	Likelihood	Prior	Posterior $\propto$
○○○○	$0 \times 4 \times 0 = 0$	$\frac{0\times4\times0}{4\times4\times4} = \frac{0}{64}$	$1/5$	$\frac{0}{64}\,\frac{1}{5}$

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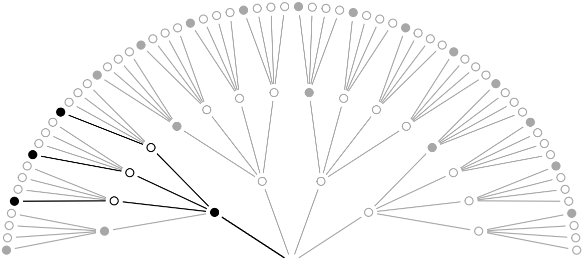

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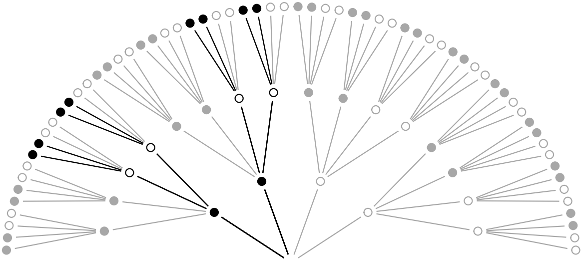

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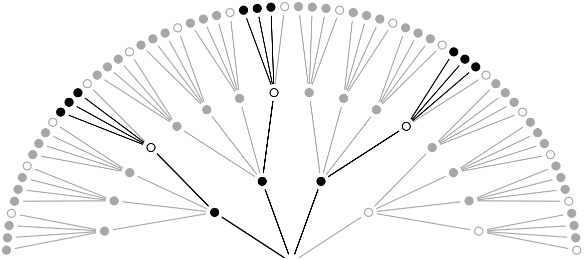

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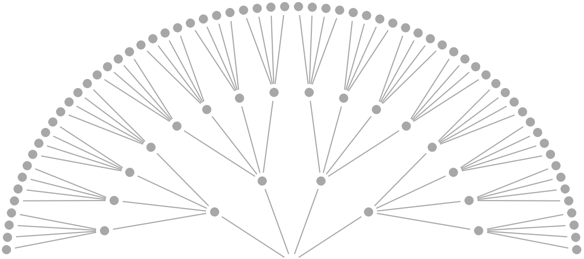

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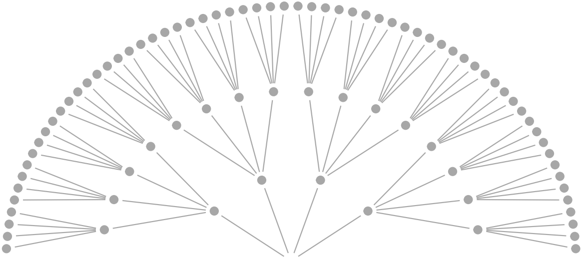

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				$P(D\|M)$

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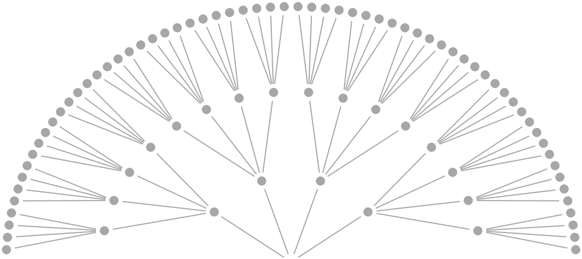

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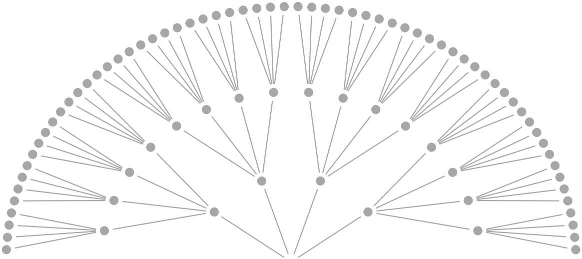

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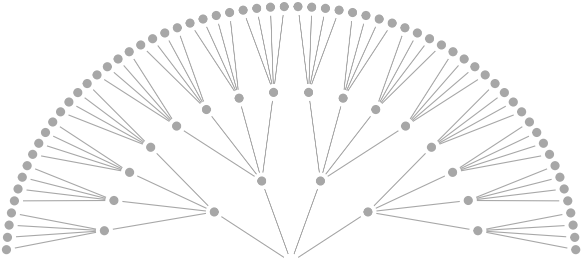

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○○○○	$0 \times 4 \times 0 = 0$	$\frac{0 \times 4 \times 0}{4 \times 4 \times 4} = \frac{0}{64}$	$1/5$	$\frac{0}{64}\,\frac{1}{5}$	$\frac{0}{3+8+9} = 0.00$
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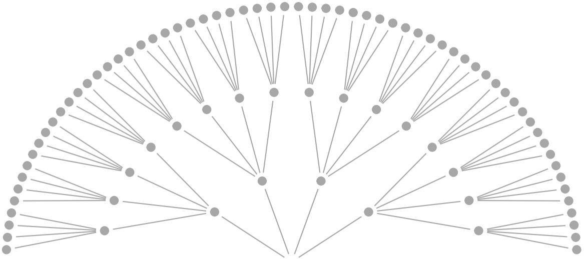
Model (M): Data $\sim$ Binomial(n, p)



Ways given M and $B =$ ●●●●

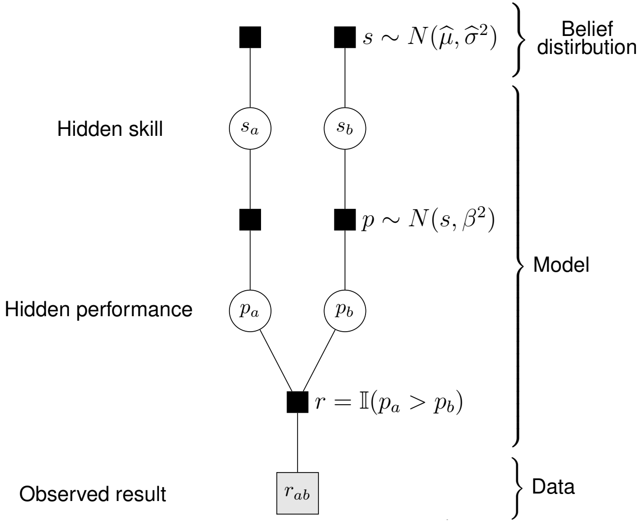
Belief	Ways to produce ●○●	Likelihood	Prior	Posterior $\propto$	Posterior
○○○○	$0 \times 4 \times 0 = 0$	$\frac{0\times4\times0}{4\times4\times4} = \frac{0}{64}$	$1/5$	$\frac{0}{64}\frac{1}{5}$	$\frac{0}{3+8+9} = 0.00$
●○○○	$1 \times 3 \times 1 = 3$	$3/64$	$1/5$	$\frac{3}{64}\frac{1}{5}$	$\frac{3}{3+8+9} = 0.15$
●●○○	$2 \times 2 \times 2 = 8$	$8/64$	$1/5$	$\frac{8}{64}\frac{1}{5}$	$\frac{8}{3+8+9} = 0.40$
●●●○	$3 \times 1 \times 3 = 9$	$9/64$	$1/5$	$\frac{9}{64}\frac{1}{5}$	$\frac{9}{3+8+9} = 0.45$
●●●●	$4 \times 0 \times 4 = 0$	$0/64$	$1/5$	$\frac{0}{64}\frac{1}{5}$	$\frac{0}{3+8+9} = 0.00$
				$\frac{3+8+9}{64\cdot5}$	

Bayesian skill estimator

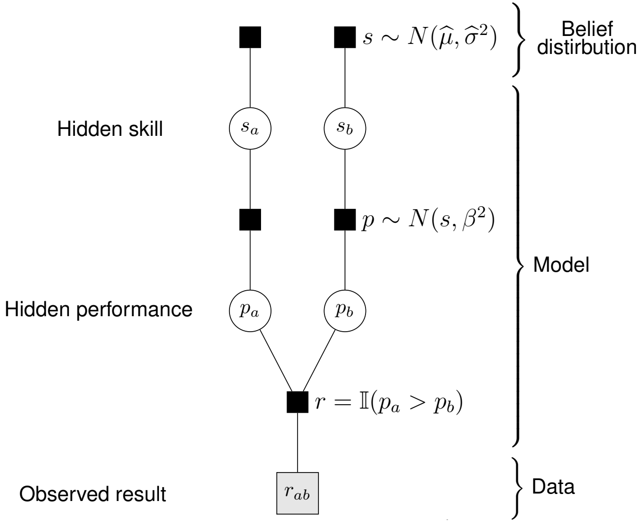
How to estimate skill of players?

Arpad Elo

Bayesian Elo factor graph



Bayesian Elo factor graph

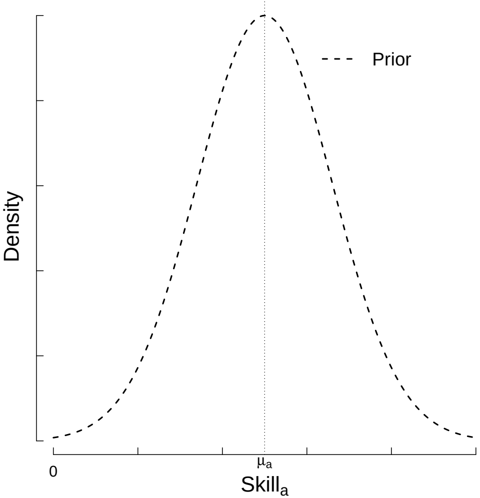


The factor graphs specifies the way to compute the posterior, likelihood, and evidence.

Kschischang FR, Frey BJ, Loeliger HA. Factor graphs and the sum-product algorithm. 2001

$$\overbrace{P(s_a \mid r_{ab}, \text{Elo model})}^{\text{Posterior}} \propto \overbrace{N(s_a \,|\, \widehat{\mu}_a, \widehat{\sigma}_a^2)}^{\text{Prior}} \; \overbrace{1 - \Phi(s_a \,|\widehat{\mu}_b, 2\beta^2 + \widehat{\sigma}_b^2)}^{\text{Likelihood}} \qquad \text{Win case}$$

$$\overbrace{P(s_a \mid r_{ab}, \text{Elo model})}^{\text{Posterior}} \propto \overbrace{N(s_a \mid \widehat{\mu}_a, \widehat{\sigma}_a^2)}^{\text{Prior}} \; \overbrace{1 - \Phi(s_a \mid \widehat{\mu}_b, 2\beta^2 + \widehat{\sigma}_b^2)}^{\text{Likelihood}} \qquad \text{Win case}$$



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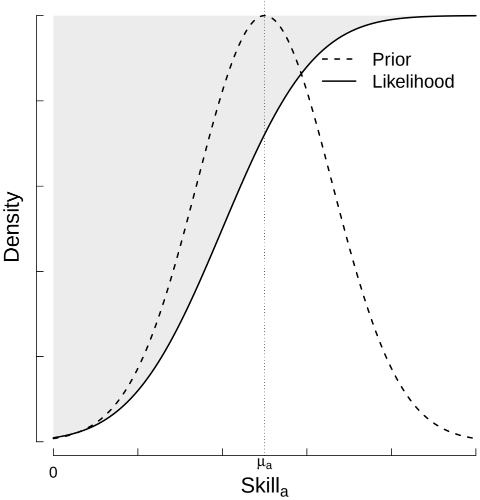

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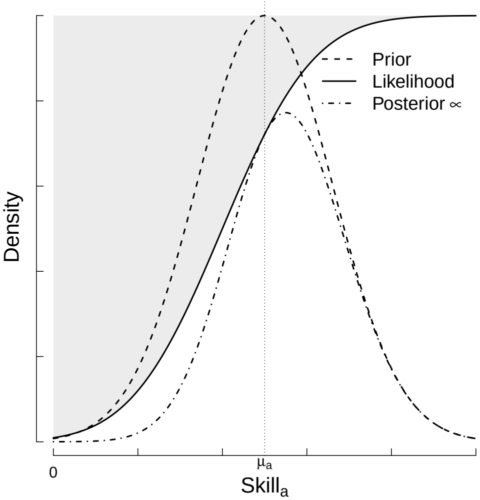

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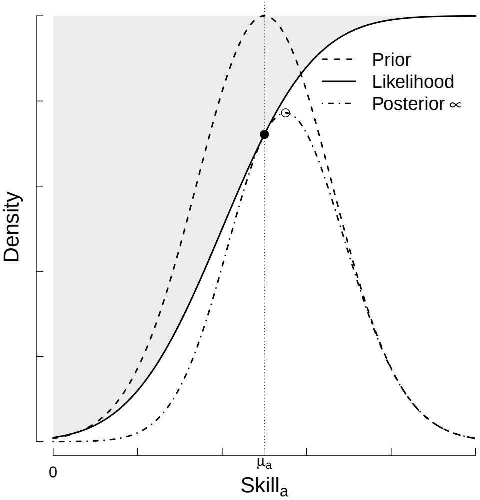

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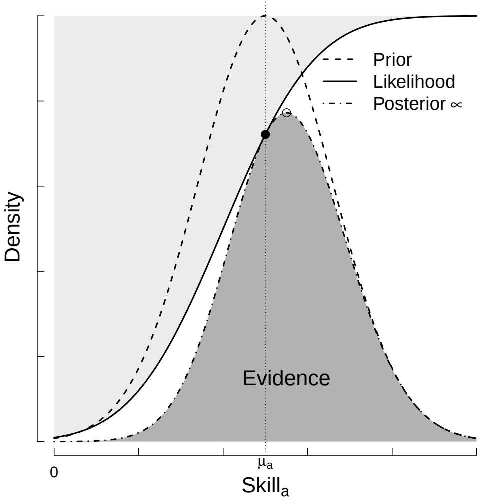

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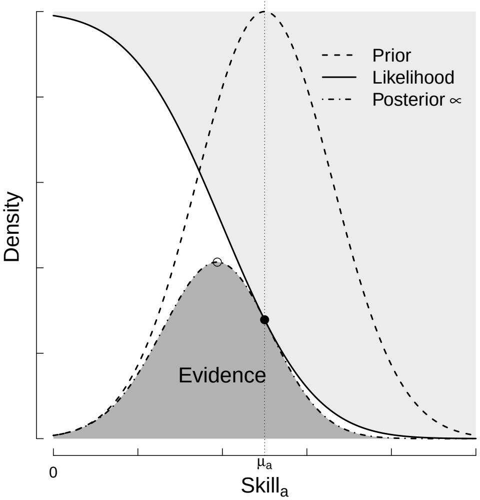

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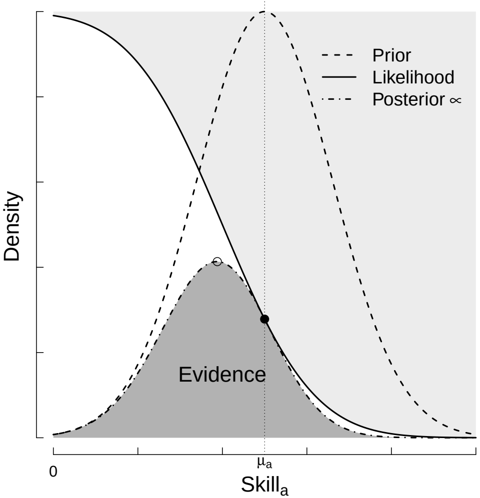
Loose case



For a detailed demostration, see Landfried. TrueSkill: Technical Report. 2019

Bayesian model inference

- Which are our beliefs about different hidden models M?

Bayesian model inference

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$$P(M_k|D) = \frac{P(D|M_k)P(M_k)}{\sum_{i=1}^{n} P(D|M_i)P(M_i)}$$

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$*$ With no prior preferences $P(M_q)=P(M_r)$

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All you need is evidence

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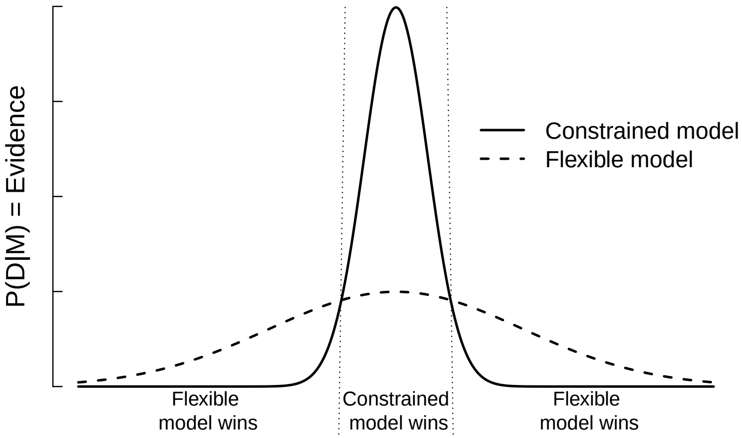
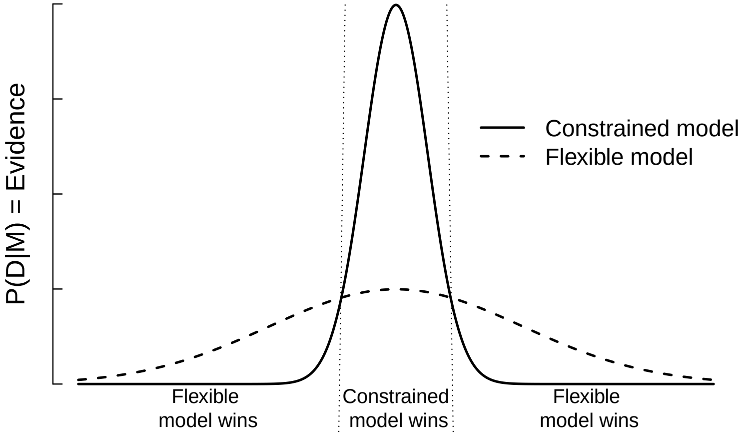
All you need is evidence

For a dicussion of bayes factor see Kass & Raftery. Bayes factors. 1995.

Evidence

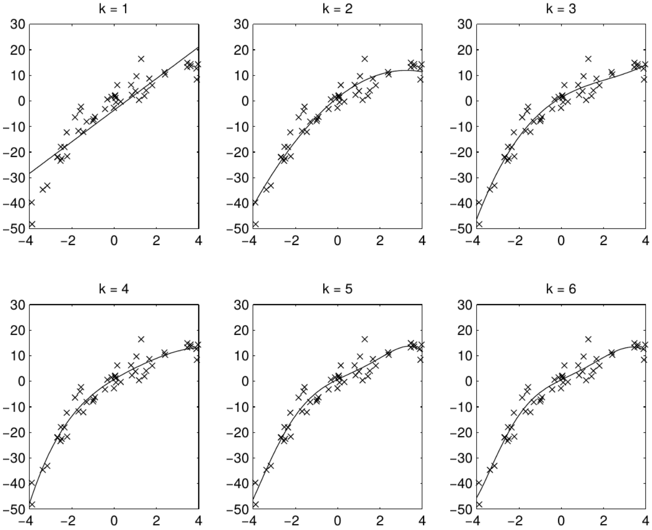

P(D|M) = Evidence
Constrained model
Flexible model
Flexible model wins
Constrained model wins
Flexible model wins

Evidence



Evience encode a trade-off between complexity and prediction.

Evidence vs maximum likelihood



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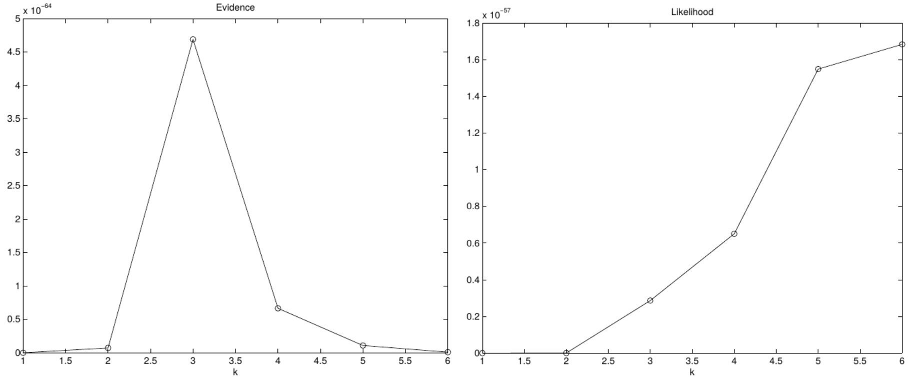

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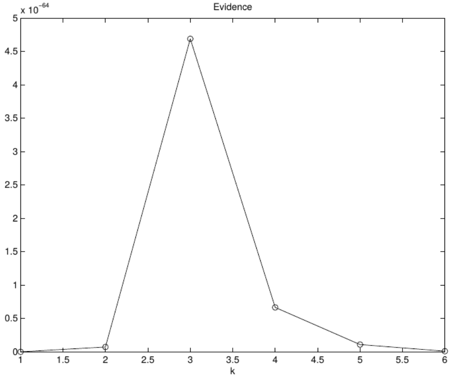
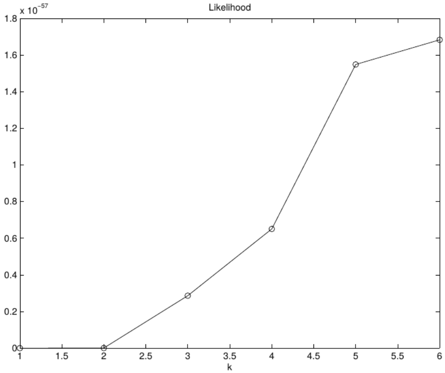
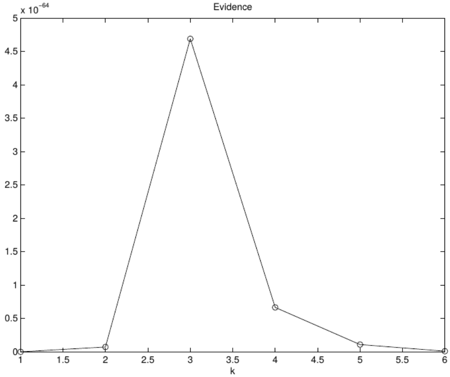
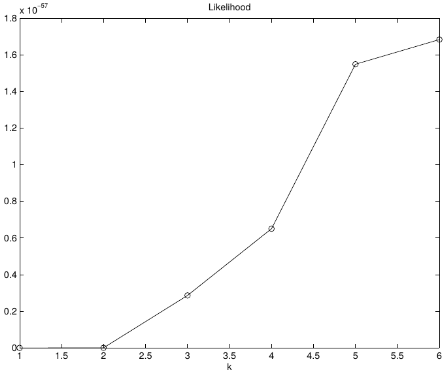




With evidence there is no need for regularization

Evidence vs maximum likelihood





With evidence there is no need for regularization

For more examples see Tom Minka

Why evidence is not widely used in machine learning?

Why evidence is not widely used in machine learning?

First let's take a look at Bayesian no-doubt case

Bayesian no-doubt case

Fixed beliefs, even with infinite new data

$$\#\text{Beliefs} = 1 \implies \underbrace{P(B)}_{\text{Prior}} = \underbrace{P(B|D)}_{\text{Posterior}} \qquad \begin{array}{c}\forall D \in \text{Data} \\ \forall B \in \text{Beliefs}\end{array}$$

Bayesian no-doubt case

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Likelihood is just the Evidence

$$\#\text{Beliefs} = 1 \iff \text{Likelihood} = \text{Evidence}$$

Who has no doubt? Who has only one belief?

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- God (if exists)

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- Mathematicians (and other non-empricial sciences)

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Who has no doubt? Who has only one belief?

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- **All non-bayesian machine learning** (the hacked-belief approach)

The hacked-belief approach

$$\overset{\text{The best belief after seeing the data}}{\cancel{\text{maximum likelihood estimator}}} = \underset{B}{\text{argmax}} P(D|B, M) = \widehat{B}$$

The hacked-belief approach

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Data appears back and forth!!

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$$\boxed{\text{Hacked evidence (with MLE)} = \text{Maximum likelihood}}$$

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$$\boxed{\text{Hacked evidence (with MLE)} = \text{Maximum likelihood}}$$

With great hacked-belief approach comes great overfitting!

With great overfitting comes great regularization!

$$\underbrace{\cancel{\text{maximum a posteriori (estimator)}}}^{\text{The best belief after seeing the data}} = \underset{B}{\text{argmax}}\, P(D|B, M)P(B|M) = \widehat{B}$$

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Again data appears back and forth!!

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Again data appears back and forth!!

$$\boxed{\text{Hacked evidence (with MAP)} = \text{L2 or L1 regularization}}$$

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Again data appears back and forth!!

> Hacked evidence (with MAP) = L2 or L1 regularization

For more regularization techniques for hacked-belief approach see
Zivik talk. With great complexity comes great regularization

Evidence and data science metrics

Is there any data science metrics equivalent to evidence?

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$$\overbrace{P(D|M)}^{\text{Evidence}}$$

Evidence and data science metrics

Is there any data science metrics equivalent to evidence?

$$\overbrace{P(D|M)}^{\text{Evidence}} \propto \log P(D|M)$$

Evidence and data science metrics

Is there any data science metrics equivalent to evidence?

$$\overbrace{P(D|M)}^{\text{Evidence}} \propto \log P(D|M) = \log\left(\prod_{i=1}^{|D|} P(D_i|M)\right)$$

Evidence and data science metrics

Is there any data science metrics equivalent to evidence?

$$\overbrace{P(D|M)}^{\text{Evidence}} \propto \log P(D|M) = \log \left(\prod_{i=1}^{|D|} P(D_i|M) \right) = \sum_{i=1}^{|D|} \log P(D_i|M)$$

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$$= E_p\left[\log q\right]$$

Evidence and data science metrics

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$$
\begin{aligned}
\overbrace{P(D|M)}^{\text{Evidence}} \propto \log P(D|M) &= \log\left(\prod_{i=1}^{|D|} P(D_i|M)\right) = \sum_{i=1}^{|D|} \log P(D_i|M) \\
&\propto \frac{1}{|D|}\sum_{i=1}^{|D|} \log P(D_i|M) = E_p\left[\log P(D|M)\right] \\
&= E_p\left[\log q\right] = -\underbrace{H(p,q)}_{\text{Cross entropy}}
\end{aligned}
$$

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$$= E_p\left[\log q\right] = -\underbrace{H(p,q)}_{\text{Cross entropy}}$$

$$\boxed{\text{Evidence} \propto \text{Cross entropy}}$$

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Evidence $\propto$ Cross entropy
(at validation data set, if hacked-belief approach)