CS515: Algorithms and Data Structures, Winter 2021

Homework 3*

Due: Tue, Feb 23, 2021

Homework Policy:

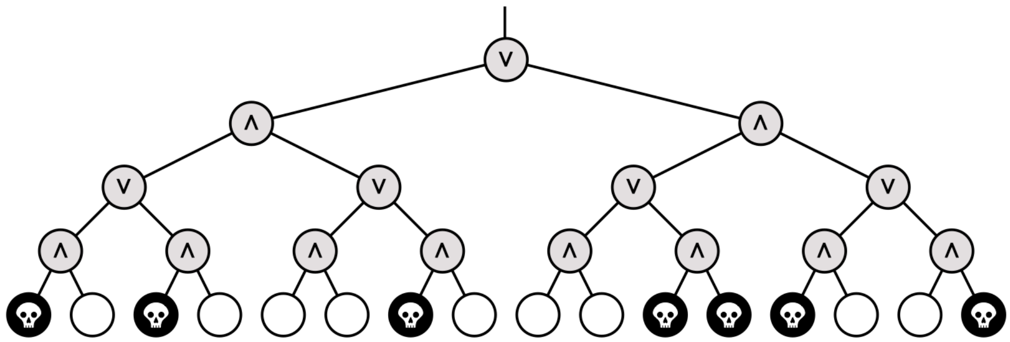
1. Students should work on group assignments in groups of preferably three people. Each group submits to CANVAS a *typeset* report in pdf format.
2. The goal of the homework assignments is for you to learn solving algorithmic problems. So, I recommend spending sufficient time thinking about problems individually before discussing them with your friends.
3. You are allowed to discuss the problems with other groups, and you are allowed to use other resources, but you *must* cite them. Also, you must write everything in your own words, copying verbatim is plagiarism.
4. *I don't know policy:* you may write "I don't know" *and nothing else* to answer a question and receive 25 percent of the total points for that problem whereas a completely wrong answer will receive zero.
5. Algorithms should be explained in plain english. Of course, you can use pseudocodes if it helps your explanation, but the grader will not try to understand a complicated pseudocode.
6. More items might be added to this list. ☺

Problem 1. Consider the following randomized algorithm for choosing the largest bolt. Draw a bolt uniformly at random from the set of n bolts, and draw a nut uniformly at random from the set of n nuts. If the bolt is smaller than the nut, discard the bolt, draw a new bolt uniformly at random from the unchosen bolts, and repeat. Otherwise, discard the nut, draw a new nut uniformly at random from the unchosen nuts, and repeat. Stop either when every nut has been discarded, or every bolt except the one in your hand has been discarded. What is the exact expected number of nut-bolt tests performed by this algorithm? Prove your answer is correct. [Hint: What is the expected number of unchosen nuts and bolts when the algorithm terminates?]

Problem 2. Death knocks on Dirk Gently's door one cold blustery morning and challenges him to a game. Emboldened by his experience with algorithms students, Death presents Dirk with a complete binary tree with 4^n leaves, each colored either black or white. There is a token at the root of the tree. To play the game, Dirk and Death will take turns moving the token from its current node to one of its children. The game will end after $2n$ moves, when the token lands on a leaf. If the final leaf is black, Dirk dies; if it's white, Dirk lives forever. Dirk moves first, so Death gets the last turn.

Unfortunately, Dirk slept through Death's explanation of the rules, so he decides to just play randomly. Whenever it's Dirk's turn, he flips a fair coin and moves left on heads, or right on tails,

*Some of the problems are from Erickson's lecture notes. Looking into similar problems about randomized algorithms from his notes is recommended.



confident that the Fundamental Interconnectedness of All Things will keep him alive, unless it doesn't. Death plays much more purposefully, of course, always choosing the move that maximizes the probability that Dirk loses the game.

(a) Describe an algorithm that computes the probability that Dirk wins the game against Death.

(b) Realizing that Dirk is not taking the game seriously, Death gives up in desperation and decides to play randomly as well! Describe an algorithm that computes the probability that Dirk wins the game again Death, assuming both players flip fair coins to decide their moves.

Problem 3. Prove the following basic facts about skip lists, where n is the number of keys.

(a) The expected number of nodes is $O(n)$.

(b) The number of nodes is $O(n)$ with high probability.

(c) A new key can be inserted in O(log n) time with high probability.

Problem 4. Answer the following questions about a treap T with n nodes.

(a) The left spine of a binary tree is a path starting at the root and following only left-child pointers. What is the expected number of nodes in the left spine of T?

(b) What is the expected number of leaves in T?

(c) What is the expected number of nodes in T with two children?