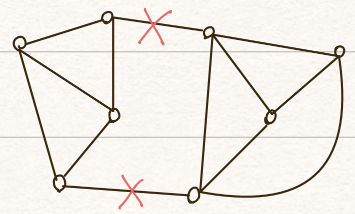
Randomized Minimum Cut

* Motivation

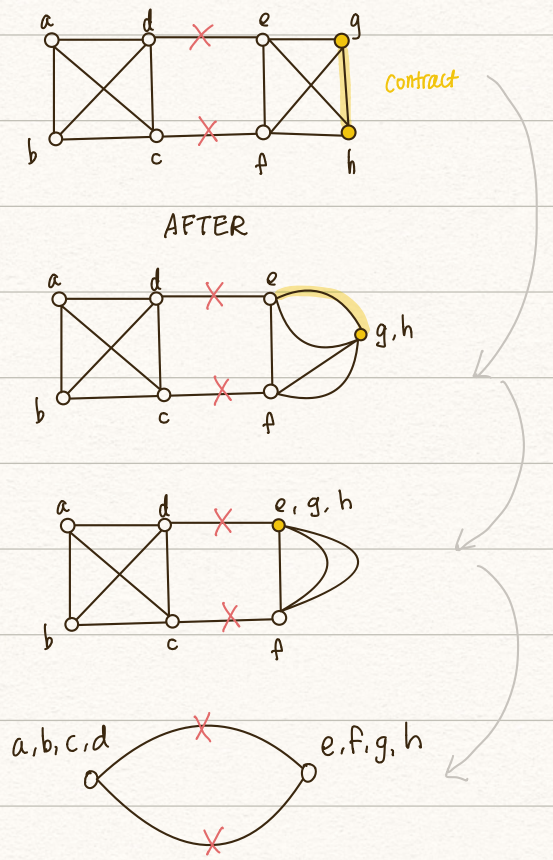
• Input : $G = (V, E)$



• Output : "Minimum Cut"

The minimum # of edges to remove to disconnect the original graph

BEFORE



↳ Observation

If the edge you picked is not in min.cut, min.cut of BEFORE = min.cut of AFTER

* Karger Algo:

• Randomly contract $(V-2)$ edges

• Return the cut between the two remaining vertices

In case there are multiple min cuts, stick with one of them.
there can be multiple, not unique

The probability that Karger's choose an edge from that min cut : $\frac{|\text{Min Cut}|}{E}$ ⇝ the size of min cut ⇝ the number of edges

↳ Observation

Let $\deg(v)$ be the number of vertices adjacent to v.

$\boxed{\deg(v) \geq |\text{min-cut}|}$ $\Rightarrow \sum \deg(v) \geq V \cdot |\text{min-cut}|$

$2E \geq V \cdot |\text{min-cut}|$

$\therefore \boxed{|\text{min-cut}| \leq \frac{2E}{V}}$

• Single Run → RT : $O(V^2)$

$\boxed{\text{The probability of success (not choosing min-cut) in one step : } \frac{V-2}{V}}$

$\Rightarrow$ Prob of success for Alg $\geq \frac{\cancel{V-2}}{V} \cdot \frac{\cancel{V-3}}{V-1} \cdot \frac{\cancel{V-4}}{\cancel{V-2}} \cdots$

$\Rightarrow \Pr[\text{success}] = 1 - \frac{2}{V} = \frac{V-2}{V}$

$\boxed{\Pr[\text{success of Karger}] \geq \frac{2}{V(V-1)}}$

• Multi-Run → RT : $V^2 \lg V \cdot O(V^2) = O(V^4 \lg V)$

Suppose we run Karger N times.

Probability that all N runs fail : $\left(1 - \frac{2}{V(V-1)}\right)^N$

Probability of success : $1 - \left(1 - \frac{2}{V(V-1)}\right)^N$

↳ How large N should be?

By using $\boxed{1 + x < e^x}$

$1 - \left(1 - \frac{2}{V(V-1)}\right)^N \rightsquigarrow 1 - \left(e^{-\frac{2}{V(V-1)}}\right)^N$

$\boxed{N = \frac{V(V-1)}{2} \cdot \lg V^c}$

$1 - e^{-\frac{\cancel{2}}{\cancel{V(V-1)}} \times \frac{\cancel{V(V-1)}}{\cancel{2}} \times \lg V^c}$

$1 - e^{-\lg V^c} = 1 - \frac{1}{V^c}$

with high probability