ORIE 633 Network Flows	October 11, 2007
Lecture 11	
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1 Minimum-Cost Circulations

We now turn to flow problems that include costs.

Minimum-Cost Circulation Problem

- **Input:**
  - A directed graph $G = (V, A)$.
  - Integer costs c_{ij}, $\forall (i,j) \in A$.
  - Integer capacities $u_{ij} \geq 0$, $\forall (i,j) \in A$.
  - Integer demands $0 \leq l_{ij} \leq u_{ij}$, $\forall (i,j) \in A$.
- **Goal:** Find a circulation f that minimizes $\sum_{(i,j) \in A} c_{ij} f_{ij}$.

Definition 1 *A circulation $f : A \to \Re^{\geq 0}$ such that*

$$\begin{aligned} l_{ij} \leq f_{ij} \leq u_{ij}, &\qquad \forall (i,j) \in A \\ \textstyle\sum_{k:(i,k)\in A} f_{ik} - \sum_{k:(k,i)\in A} f_{ki} = 0, &\qquad \forall i \in V. \end{aligned}$$

A related problem is the *minimum-cost flow problem.*

Minimum-Cost Flow Problem

- **Input:**
  - A directed graph $G = (V, A)$.
  - Integer costs c_{ij}, $\forall (i,j) \in A$.
  - Integer capacities $u_{ij} \geq 0$, $\forall (i,j) \in A$.
  - Integer demands b_i $\forall i \in V$, s.t. $\sum_{i \in V} b_i = 0$.
- **Goal:** Find a flow f that minimizes $\sum_{(i,j) \in A} c_{ij} f_{ij}$ s.t.

$$\begin{aligned} 0 \leq f_{ij} \leq u_{ij}, &\qquad \forall (i,j) \in A, \\ \textstyle\sum_{k:(k,i)\in A} f_{ki} - \sum_{k:(i,k)\in A} f_{ik} = b_i, &\qquad \forall i \in V. \end{aligned}$$

1.1 Equivalence with min-cost flows

We will show below that the minimum-cost circulation problem and the minimum-cost flow problem are equivalent to each other. Notice that the input for the minimum-cost flow problem is the same as the for the minimum-cost circulation problem, except that there are no demands l_{ij}, but instead, there are integer demands b_i $\forall i \in V$, such that the sum of demands over all the vertices is zero: $\sum_{i \in V} b_i = 0$. The goal is now to find a minimum-cost flow that satisfies demand at each of the vertices.

Theorem 1 *The minimum-cost flow problem and the minimum-cost circulation problem are equivalent.*

Proof: (*flow* $\Rightarrow$ *circulation*) Given an instance of the minimum-cost flow problem, we will show how to convert it to an instance of the minimum-cost circulation problem. Then, given an algorithm for minimum-cost circulation, we would be able to solve the minimum-cost flow problem. To convert, add a node s to the graph. For $i \in V$ such that $b_i > 0$, create an arc (i, s) with cost 0, and $l_{is} = u_{is} = b_i$. For $i \in V$ such that $b_i < 0$, we create an arc (s, i) of cost 0 such that $l_{si} = u_{si} = |b_i|$ (See Figure 1). Note that given a feasible flow in the original problem we can get a circulation of the same cost in the modified instance since the flow coming into each node is equal to the flow going out of each node (including the node s, since $\sum_{i:b_i>0} b_i = \sum_{i:b_i<0} |b_i|$). The reverse is also true – given a circulation in the modified instance, the flow on the arcs of the original problem is a feasible flow of the same cost. So by finding a minimum-cost circulation in the modified instance we can find a minimum-cost flow in the original instance.

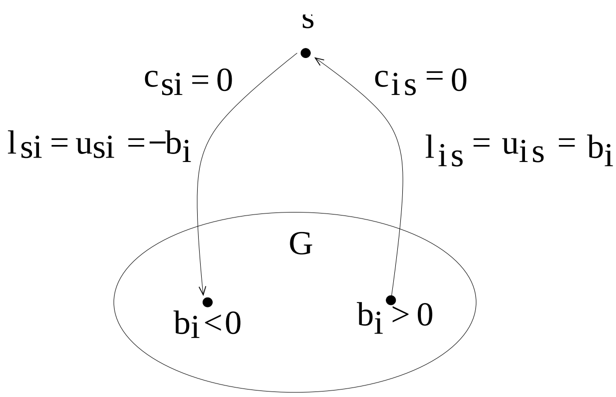

Figure 1: Transformation of minimum-cost flow instance to minimum-cost circulation instance.

(*circulation* $\Rightarrow$ *flow*) For this part, we change variables. Set $f'_{ij} = f_{ij} - l_{ij}$, and $u'_{ij} = u_{ij} - l_{ij}$. Set $b_i = \sum_{k:(i,k)\in A} l_{ik} - \sum_{k:(k,i)\in A} l_{ki}$. This provides a direct transformation between the two problems. Given a feasible circulation f in the original problem, we have a feasible flow f' in the modified problem of the same cost, and vice versa. Thus by finding a minimum-cost flow in the modified instance, we can find a minimum-cost circulation in the original instance. □

From here on, we will consider only the minimum-cost circulation problem and algorithms to solve it.

We will now change our notation slightly for the problem, as we did for the maximum flow problem, since it will make our algorithms and proofs simpler. Replace each arc by two arcs of opposite orientations. If f_{ij} is the flow in (i, j), then force $f_{ji} = -f_{ij}$. This is called antisymmetry. Also set $u_{ji} = -l_{ij}$. This removes the lower bound constraints, since $f_{ji} \leq u_{ji} \Rightarrow -f_{ij} \leq -l_{ij} \Rightarrow f_{ij} \geq l_{ij}$. We make the costs antisymmetric, too: $c_{ji} = -c_{ij}$. Thus the total cost for the two edges with flow f is $c_{ji}f_{ji} + c_{ij}f_{ij} = 2c_{ij}f_{ij}$. Hence optimizing for the total cost for this new graph is the same as optimizing for the total cost for the original graph. Thus our definition of a feasible circulation becomes the following.

Definition 2 *A circulation $f : A \to R$ such that*

$$\begin{aligned} f_{ij} &\leq u_{ij}, && \forall (i,j) \in A \\ f_{ij} &= -f_{ji}, && \forall (i,j) \in A \\ \textstyle\sum_{k:(i,k)\in A} f_{ik} &= 0, && \forall i \in V \end{aligned}$$

Claim 2 *Via one max flow computation, we can tell if the circulation problem is feasible and find a feasible circulation if one exists.*

Proof: See Problem Set 1 solutions. □

1.2 Optimality conditions

In the case of the maximum flow problem, we had conditions that told us when a flow was optimal; i.e. we knew a flow was maximum if and only if there was no augmenting path. We would like to give similar conditions for the minimum-cost circulation problem, but we need a few definitions first.

Definition 3 *The residual graph for a circulation f is $G_f = (V, A_f)$ where $A_f = \{(i,j) \in A : f_{ij} < u_{ij}\}$ with residual capacity $u_{ij}^f = u_{ij} - f_{ij}$.*

Definition 4 *Let $p : V \to \Re$. Then p are called* node potentials *(or sometimes* node prices*). The* reduced cost *of $(i,j) \in A$ with respect to potentials p is $c_{ij}^p = c_{ij} + p_i - p_j$. If Γ is a cycle, let $c(\Gamma) = \sum_{(i,j)\in\Gamma} c_{ij}$.*

Observe that the cost of a cycle Γ and the reduced cost of a cycle Γ is the same for any set of potentials p; that is, $c(\Gamma) = c^p(\Gamma)$, since the potentials cancel out (see Figure 2).

Definition 5 $c \cdot f = \sum_{(i,j)\in A} c_{ij} f_{ij}$

Theorem 3 $c \cdot f = c^p \cdot f$.

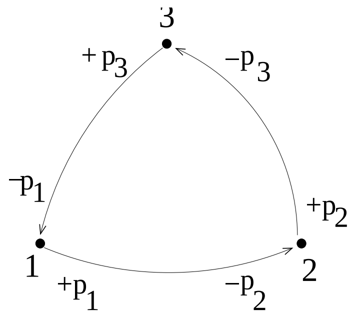

Figure 2: Example showing cost of cycle is same as the reduced cost of the cycle.

Proof:

$$\begin{aligned} c^p \cdot f &= \sum_{(i,j)\in A} (c_{ij} + p_i - p_j) f_{ij} \\ &= \sum_{(i,j)\in A} c_{ij} f_{ij} + \sum_{(i,j)\in A} (p_i - p_j) f_{ij} \\ &= c \cdot f + \sum_{(i,j)\in A} (p_i - p_j) f_{ij} \\ &= c \cdot f + \sum_{i \in V} p_i \Big(\sum_{k:(i,k)\in A} f_{ik} - \sum_{k:(k,i)\in A} f_{ki} \Big) \\ &= c \cdot f. \end{aligned}$$

This follows since the term in parentheses is zero because of flow conservation. □

We can now state the theorem giving us conditions under which a circulation is optimal.

Theorem 4 *The following are equivalent:*

1. *f is a minimal cost circulation,*
2. *there are no negative cost cycles in G_f, and,*
3. *there exist potentials p such that $c^p_{ij} \geq 0$ for all $(i,j) \in A_f$.*

Proof:

$[\neg(2) \Rightarrow \neg(1)]$ Let Γ be a negative cost cycle in A_f. Define

$$\delta = \min_{(i,j)\in\Gamma} u^f_{ij}.$$

Then $\delta > 0$. Let

$$f'_{ij} = \begin{cases} f_{ij} + \delta, & (i,j) \in \Gamma, \\ f_{ij} - \delta, & (j,i) \in \Gamma, \\ f_{ij}, & \text{otherwise.} \end{cases}$$

Thus, $f'_{ij} = -f'_{ji}$ and f' is a feasible circulation if f is. Also, $f'_{ij} \leq u_{ij}$. Furthermore,

$$c \cdot f' = c \cdot f + 2\delta c(\Gamma) < c \cdot f,$$

since Γ is a negative cost cycle. Therefore, f is not of minimum cost.
Note: In $G_{f'}$, Γ does not exist. This is so because $f'_{ij} = u_{ij}$ for some $(i,j) \in \Gamma$. Then $(i,j) \notin A_{f'}$, and so $\Gamma \nsubseteq A_{f'}$. We say that Γ has been *cancelled*.

[(2) $\Rightarrow$ (3)] Add a node s to G_f, and add arcs of cost 0 from s to each $i \in V$. Then let p_i be the length of the shortest path in the residual graph from s to i using costs c_{ij} as the edge lengths. These paths are well defined since there are no negative-cost cycles, by assumption. Moreover, by properties of shortest paths, for any $(i,j) \in A_f$, $p_j \leq p_i + c_{ij}$, so that $c^p_{ij} = c_{ij} + p_i - p_j \geq 0$.

[(3) $\Rightarrow$ (1)] Suppose f^* is any other valid circulation. We want to show that $c \cdot f \leq c \cdot f^*$. Consider the circulation f', where $f'_{ij} = f^*_{ij} - f_{ij}$. f' is a feasible circulation. Let p be a potential such that $c^p_{ij} \geq 0$ for all $(i,j) \in A_f$. Note that if $f'_{ij} > 0$ then $f_{ij} < f^*_{ij} \leq u_{ij}$. This implies $(i,j) \in A_f$ and $c^p_{ij} \geq 0$. Consider the following.

$$\begin{aligned} c \cdot f' = c^p \cdot f' = \sum_{(i,j)\in A} c^p_{ij} f'_{ij} &= \sum_{(i,j)\in A, f'_{ij}>0} c^p_{ij} f'_{ij} + \sum_{(i,j)\in A, f'_{ij}<0} (-c^p_{ji})(-f'_{ji}) \\ &= 2\left(\sum_{(i,j)\in A, f'_{ij}>0} c^p_{ij} f'_{ij} \right) \geq 0. \end{aligned}$$

Therefore, $c \cdot f' \geq 0 \Rightarrow \sum (i,j) \in A c_{ij}(f^*_{ij} - f_{ij}) \geq 0 \Rightarrow c \cdot f^* - c \cdot f \geq 0$. Therefore f is a min-cost circulation. □

1.3 A cycle-cancelling algorithm

This theorem yields a natural algorithm for computing a min-cost circulation:

Cycle-Cancelling Algorithm (Klein '67)

Let f be a feasible circulation.
While A_f contains a negative cycle Γ
    Cancel Γ, update f.

The correctness of the algorithm follows immediately from the above theorem. Note that cancelling a cycle Γ means to send enough flow so that the residual capacity of some arc goes to 0. Note that we can always find a feasible circulation, if one exists, by running one max flow computation (see Problem Set 1, # 3). Furthermore, we can find a negative cycle, if one exists, in $O(mn)$ time (Problem Set 3).

Also, notice that the algorithm implies that min-cost circulations, like max-flows, satisfies an **integrality property**: If u_{ij} and c_{ij} are integer for all $(i,j) \in A$, then if a feasible circulation exists, there is always integer-valued minimum-cost circulation. This is true, since we can always cancel a cycle with integer flow during each iteration of the cycle-cancelling algorithm.

To get a bound on the running time of the algorithm, define

$$U = \max_{(i,j)\in A} u_{ij} \qquad C = \max_{(i,j)\in A} |c_{ij}|.$$

Then any feasible circulation can cost at most mCU and must cost at least $-mCU$. Therefore, since a cycle cancellation improves the cost of a circulation by at least 1, at most $O(mCU)$ cancellations are needed in order to find an optimal circulation.