Monte Carlo Methods in Finance

Homework solutions: Chapter 6

1. In this exercise we will analyze the properties of two widely used stochastic models for interest rates: The Vasicek model and the Cox-Ingersoll-Ross (CIR) model. To formulate these models we will write stochastic differential equations for $r(t)$, the instantaneous spot rate or *short rate*. This is the rate at which interest is accrued at the instant t. Specifically, if $M(t)$ represents the value of a bank account that pays interest $r(t)$ at time t, then
$$dM(t) = r(t)M(t)dt.$$
Therefore, assuming that the initial value of the bank account at time t_0 is $M(t_0) = M_0$, the value at time t is
$$M(t) = M_0 \exp\left(\int_{t_0}^{t} r(s)ds\right)$$
The equations for the short rate in these models are
$$\begin{aligned} &\text{Vasicek model:} \qquad && dr(t) = a(b - r(t))dt + \sigma dW(t) \\ &\text{CIR model:} \qquad && dr(t) = a(b - r(t))dt + \sigma\sqrt{r(t)}dW(t), \end{aligned}$$
where $W(t)$ is a Wiener process. They are one-factor models, in the sense that they describe the evolution of interest rates as driven by only one source of risk ($W(t)$).

   Write code to generate $M = 1000$ trajectories for each of the models using the stochastic Euler method. In the simulation we will use $N = 200$ time steps from $t_0 = 0$ to $t = 4$. The parameters for the SDE's are $a = 2$, $b = 0.05$, $\sigma = 0.06$ and $r(0) = 0.1$.

   For each model, make a plot of the mean of $r(t)$, averaged over trajectories, for every value of t. Which of the following behaviors is observed?

   - The mean stays at $r(0)$ in the Vasicek model and tends to b in the CIR model.
   - The mean tends to b in the Vasicek model and stays at $r(0)$ in the CIR model.
   - In both models the mean stays at $r(0)$.
   - * In both models the mean tends to b.

Hint: Use the function `stochasticEulerIntegration`, which is provided in this chapter, to carry out the simulations.

Explanation: The trajectories can be simulated using the following code

```
a = 2; b = 0.05; sigma = 0.06; r0 = 0.1; t0 = 0;
T = 4; N = 200; M = 1000;
[t1,f1] = stochasticEulerIntegration(t0,r0,@(t,r)(a*(b-r)),...
                                    @(t,r)(sigma),T,N,M);
[t2,f2] = stochasticEulerIntegration(t0,r0,@(t,r)(a*(b-r)),...
                                    @(t,r)(sigma*sqrt(r)),T,N,M);
```

Then, the plots can be produced with

```
plot(t1,mean(f1),'-',t2,mean(f2),'-')
```

The answer should be clear in view of these plots: the mean tends to b in both models. Actually, it can be proved that, for every $a > 0$, both models present mean reversion to the value b, because of the form of the deterministic term.

2. For each model, display all the trajectories that you have simulated in the same plot. Which of the following is true?

   * $r(t)$ is always positive in the CIR model but not in the Vasicek model.
   - $r(t)$ is always positive in the Vasicek model but not in the CIR model.
   - $r(t)$ is always positive in both models.
   - $r(t)$ can take negative values in both models.

   Explanation: The plots can be obtained with

```
figure(1); plot(t1,f1);
figure(2); plot(t2,f2);
```

The correct answer should be clear from these plots: only the Vasicek model produces negative interest rates. This is one of the main drawbacks of the Vasicek model and the reason why the CIR model is usually preferred. Nevertheless, in the current economic situation, with very low reference interest rates, the possibility of negative values for the nominal interest rates cannot be ruled out.

Note that, even though solutions of the CIR equation are always non-negative (for positive a and b), this is not necessarily the case for the approximations obtained with the stochastic Euler method. For instance, it is possible that negative rates are generated in the simulation when `sigma` is increased in the previous code. A common fix for this numerical problem is to simulate from the model

$$dr(t) = a(b - r(t))dt + \sigma\sqrt{|r(t)|}dW(t).$$

3. Obtain Monte Carlo estimates of $\mathtt{P}[r(t) > 0.055]$ with $t = 4$ in both cases. What are the closest values to these estimates?

   - 0.57 (Vasicek) and 0.32 (CIR).
   - 0.57 (Vasicek) and 0.78 (CIR).
   * 0.43 (Vasicek) and 0.22 (CIR).
   - 0.43 (Vasicek) and 0.68 (CIR).

   Explanation: The MC estimates can be obtained with

```
mean(f1(:,end) > 0.055)
mean(f2(:,end) > 0.055)
```

   These estimates are close to 0.43 (Vasicek) and 0.22 (CIR). The exact values (without discretization error) can be calculated for both models and are, respectively, 0.4340 and 0.2225 (rounded to 4 significant figures).

4. The price of an asset, $S(t)$, is modeled using geometric Brownian motion with drift $\mu = 0.05$ and volatility $\sigma = 0.25$. What is the probability that the price in six years at least doubles the price in three years?

   - 0.19789
   - 0.10486
   * 0.07067
   - The probability depends on S_0, the price of the asset today.

   Hint: To compute the probability $\mathtt{P}[S(t_2) \geq 2S(t_1)]$ use the fact that the ratio $\frac{S(t_2)}{S(t_1)}$ is distributed as a lognormal.

   Explanation: Since $S(t)$ is a geometric Brownian motion, we have

$$S(t_2) = S(t_1) \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)(t_2 - t_1) + \sigma\sqrt{t_2 - t_1}X\right), \quad X \sim N(0,1).$$

   Therefore, $\frac{S(t_2)}{S(t_1)} \sim LN\left(\left(\mu - \frac{1}{2}\sigma^2\right)(t_2 - t_1), \sigma\sqrt{t_2 - t_1}\right)$.

$$\begin{aligned}\mathtt{P}\left[\frac{S(t_2)}{S(t_1)} \geq 2\right] &= 1 - \mathrm{logncdf}\left(2; \left(\mu - \frac{1}{2}\sigma^2\right)(t_2 - t_1), \sigma\sqrt{t_2 - t_1}\right) \\ &= 1 - \mathrm{logncdf}\left(2; 0.05625, 0.4330\right) = 0.07067\end{aligned}$$

5. Assume that the stochastic process $f(t)$ with $f(t_0) = f_0$ follows a geometric Brownian motion for $t > t_0$. Which of these statements is true?

   * The mean of $f(t)$ is greater than its median.
   - The median of $f(t)$ is greater than its mean.

- The mean of $f(t)$ is equal to its median.
- The mean of $\exp(f(t))$ is equal to its median.

Explanation: Since $f(t)$ is a geometric Brownian motion, $f(t)$ is distributed as a lognormal for every t, with parameters that depend on t. Since the mean of a Lognormal(μ, σ) is $\exp\left(\mu + \frac{1}{2}\sigma^2\right)$ and its median is $\exp(\mu)$, the former is always greater than the latter.