Monte Carlo Methods in Finance

Homework solutions: Chapter 5

1. Consider the ordinary differential equation (ODE) for $y(t)$

$$\frac{dy(t)}{dt} = e^{-y(t)^2}$$

with $y(0) = 1$. What is the value of y(0.5) computed using a second order (quadratic) Taylor approximation around $t = 0$?

- 1.031
* 1.150
- 1.184
- 1.218
- 1.625

Hint: The second Taylor approximation of $y(t)$ in the neighborhood of $t = 0$ is

$$y(\Delta T) \approx y(0) + \left.\frac{dy(t)}{dt}\right|_{t=0} \Delta T + \frac{1}{2} \left.\frac{d^2y(t)}{dt^2}\right|_{t=0} (\Delta T)^2 + \mathcal{O}\left((\Delta T)^3\right).$$

The expression of the first derivative of $y(t)$ is directly given by the ODE. The expression of the second derivative of $y(t)$ is obtained by taking a derivative with respect to t of the expression of the first derivative given by the ODE.

Explanation: The second order Taylor approximation at $t = 0$ of the solution is

$$y(\Delta T) \approx y(0) + y'(0)\Delta T + \frac{1}{2}y''(0)\Delta T^2$$

The value of the first derivative of $y(t)$ at $t = 0$ is given by the ODE itself

$$y'(0) = \left.\frac{dy(t)}{dt}\right|_{t=0} = e^{-y(0)^2} = \frac{1}{e}.$$

The second derivative of $y(t)$ is obtained by differentiating both sides of the ODE

$$\frac{d^2y(t)}{dt^2} = -2y(t)\frac{dy(t)}{dt}e^{-y(t)^2} = -2y(t)e^{-2y(t)^2}.$$

The value of the second derivative of $y(t)$ at $t = 0$

$$y''(0) = \left.\frac{d^2y(t)}{dt^2}\right|_{t=0} = -\frac{2}{e^2}.$$

Substituting these values in the approximation we get

$$y(\Delta T) \approx 1 + \frac{\Delta T}{e} - \frac{\Delta T^2}{e^2},$$

that, for $\Delta T = 0.5$ yields $y(0.5) \approx 1.150$.

2. Find the solution of the ODE for $z(t)$

$$\frac{dz(t)}{dt} = e^{at+z(t)}$$

with z(0)=0 using the method of separation of variables.

- $z(t) = \log\left(1 - \frac{1}{a}e^{at}\right)$
- $z(t) = \frac{e^{at}-1}{at+1}$
- $z(t) = t\log\left(e^{at} - 1\right)$

* $z(t) = -\log\left(1 - \frac{1}{a}(e^{at} - 1)\right)$

Explanation: Since the exponential term in the left hand side can be factored in $e^{at+z(t)} = e^{at} \cdot e^{z(t)}$, the ODE can be written as

$$e^{-z}dz = e^{at}dt,$$

which, after integration,

$$\int_{z(t_0)=0}^{z(t)} e^{-z'}dz' = \int_{t_0=0}^{t} e^{at'}dt'$$

yields

$$-\left(e^{-z(t)} - 1\right) = \frac{1}{a}\left(e^{at} - 1\right).$$

Finally, solving for $z(t)$, the explicit solution is

$$z(t) = -\log\left(1 - \frac{1}{a}(e^{at} - 1)\right).$$

3. Let $M(20)$ be the balance of a bank account in 20 years, with initial investment of \$50000 and an annual continuously compounded interest of 6% . What is the relative error, $(M_{\text{Euler}}(20) - M(20))/M(20)$, incurred when the balance is obtained by Euler integration using 1 time step per year?

- 0%
- +6.6%
- -6.6%
- +3.4%
* -3.4%

Hint: The approximate solution given by Euler integration is equivalent to the expression for discrete compounding.

Explanation: The ODE for the bank account $M(t)$ is

$$\frac{dM(t)}{dt} = rM(t) \quad , \quad M(0) = M_0$$

. In this case, $r = 0.06$ and $M_0 = 50000$. The exact solution of the ODE is

$$M(t) = M_0 e^{r(t-t_0)},$$

so we get $M(20) = \$166005.85$. We can use the provided function `eulerIntegration.m` to get the Euler approximation, `MEuler20`, using the instructions

```
r  = 0.06;  % Interest rate
M0 = 50000; % Initial amount
derivM = @(t,M)(r*M); % Handle to the derivative function
t0 = 0; T = 20; % Initial time and time interval
N  = 20;        % Number of timesteps (1/year)
[t,M] = eulerIntegration(t0,M0,derivM,T,N);
MEuler20 = M(end) % Account value at t=20
```

which yields a value $M_{\text{Euler}}(20) = \$160356.77$. The error incurred is

$$\text{Error} = \frac{M_{\text{Euler}}(20) - M(20)}{M(20)} = -3.4\%.$$

Alternatively, recalling that the Euler integration for the bank account has the particularly simple expression of discrete compounding

$$M(t + \Delta T) = (1 + r\Delta T)\, M(t),$$

the approximate value can be obtained directly with the instruction

```
deltaT = T/N;
Meuler20 = M0*(1+r*deltaT)^N
```

4. Consider the Lorenz system of ordinary differential equations

$$\begin{aligned}
\frac{dx(t)}{dt} &= \sigma(y(t) - x(t)), & x(t_0) &= x_0 \\
\frac{dy(t)}{dt} &= x(t)(r - z(t)) - y(t), & y(t_0) &= y_0 \\
\frac{dz(t)}{dt} &= x(t)y(t) - bz(t), & z(t_0) &= z_0,
\end{aligned}$$

with positive parameters σ, b and r. Write an Octave/MATLAB function to compute an approximation of the solution, $\mathbf{s}_N(t) = \{x_N(t), y_N(t), z_N(t)\}$, using the Euler integration scheme with N time steps. The header of this function is

```
function [t,x,y,z] = eulerIntegrationLorenz(t0,x0,y0,z0,sigma,r,b,T,N)
% eulerIntegrationLorenz:Solution of Lorenz attractor eqns. via Euler integration
%
% SYNTAX:
%        [t,x,y,z] = eulerIntegrationLorenz(t0,x0,y0,z0,sigma,r,b,T,N)
%
% INPUT:
%        t0 : Initial time
%  x0,y0,z0 : Initial position
% sigma,r,b : Parameters
%        T  : Length of integration interval [t0, t0+T]
%        N  : Number of time steps
%
% OUTPUT:
%        t  : Times at which the trajectory is monitored
%             t(n) = t0 + n Delta T
%    x,y,z  : Values of the position along the trajectory
%
% EXAMPLE:
%        t0 = 0; x0 = 1; y0 = 1; z0 = 1;
%        sigma = 10; b = 8/3; r = 28;
%        T = 50; N = 1e5;
%        [t,x,y,z] = eulerIntegrationLorenz(t0,x0,y0,z0,sigma,r,b,T,N);
```

Use this function to compute the solution at $T = 1$, with initial conditions $x_0 = y_0 = z_0 = 1$, parameters $\sigma = 10, b = \frac{8}{3}, r = 28$, and using $N = 10000$ steps. What is the closest value to this approximation?

* $\mathbf{s}_N(1) = [-9.351, -8.361, 29.294]$
- $\mathbf{s}_N(1) = [1.114, 1.982, 1.796]$
- $\mathbf{s}_N(1) = [12.810, 2.761, 1.616]$
- $\mathbf{s}_N(1) = [6.067, -12.004, 5.056]$
- $\mathbf{s}_N(1) = [-2.772, 17.277, -9.997]$

Explanation: The Octave/MATLAB function is provided in the file `eulerIntegrationLorenz.m`. The set of instructions

```
t0 = 0; x0 = 1; y0 = 1; z0 = 1;
sigma = 10; b = 8/3; r = 28;
T = 1; N = 10000;
[t,x,y,z] = eulerIntegrationLorenz(t0,x0,y0,z0,sigma,r,b,T,N);
solution = [x(end), y(end), z(end)]
```

yields the result $\mathbf{s}_N(1) = [-9.351, -8.361, 29.294]$.

5. Consider the Lorenz system from the previous exercise. It is known that for the values of the parameters used in the previous exercise the system has chaotic behavior. Approximate the solution using the Euler integration scheme for $T = 25$, first with $N_1 = 10000$ time steps and then with $N_2 = 10001$ time steps. What is the relative difference

$$d = \frac{x_{N_2}(T) - x_{N_1}(T)}{x_{N_1}(T)}$$

between the x component of both solutions? Are $N_1 = 10000$ time steps sufficient for an accurate approximation of the solution?

- Diff $\leq 1\%$. Yes, N_1 is high enough.
- Diff $\approx 10\%$. Yes, N_1 is high enough.
- Diff $\geq 100\%$. Yes, N_1 is high enough.
* Diff $\leq 1\%$. No, N_1 is insufficient.
- Diff $\approx 10\%$. No, N_1 is insufficient.
- Diff $\geq 100\%$. No, N_1 is insufficient.

Explanation: To obtain the difference in the x component, we simply compute the approximations with the function coded in the previous exercise for both values of N

```
t0 = 0; x0 = 1; y0 = 1; z0 = 1;
sigma = 10; b = 8/3; r = 28;
T = 25;
N1 = 10000;
[t1,x1,y1,z1] = eulerIntegrationLorenz(t0,x0,y0,z0,sigma,r,b,T,N1);
N2 = 10001;
[t2,x2,y2,z2] = eulerIntegrationLorenz(t0,x0,y0,z0,sigma,r,b,T,N2);
x1T = x1(end)
x2T = x2(end)
```

which produces the values

```
x1T =
   -3.4534
x2T =
   -3.4726
```

for a relative difference of Diff $\approx 0.5\% < 1\%$. However, let us see what happens with all the components $[x, y, z]$ of both approximations

```
>> [x1(end),y1(end),z1(end)]
ans =
   -3.4534   -4.2110   19.3059
```

```
>> [x2(end),y2(end),z2(end)]
ans =
   -3.4726   -5.5354   14.8845
```

We can see that, just increasing 1 time step, the y and the z components have changed significantly. In fact, you can experiment increasing N some units: the approximations, far from stabilizing, exhibit large changes.

Therefore, although the variation in the x component was very small (in this case just accidentally), 10000 time steps are clearly insufficient.

The intuitive explanation of this fact is the following: in this parameter regime the Lorenz system exhibits chaotic behavior. In chaotic systems small changes in the initial conditions are amplified to produce very different trajectories. Due to this fact, the small errors introduced by the Euler integration scheme are amplified as well, leading to divergences in the solution. In some cases, as illustrated by this particular example, the number of time steps needed to obtain a sufficiently accurate approximation is too large for the Euler scheme to be of practical use.