Chapter 5: Ordinary differential equations

In this chapter we will learn how to describe the evolution of quantities that change with time, be it the amount of money deposited in a bank account or the position of an oscillator. One way to describe change mathematically is to write down a differential equation that gives the relation between a time-varying quantity and its time derivatives [**Unit 5.1**].

In [**Unit 5.2**] we will describe how the Taylor series truncated to low orders can be used to approximate to the time evolution of a smoothly-varying quantity. This approximation, which is accurate for short times, can be used repeatedly as a basis for the design of an algorithm to approxiate this evolution at longer times.

In **Unit 5.3** we derive the a first order ordinary differential equation (ODE) that describes the evolution of the amount of money deposited in a bank account. The solution of this ODE using the method of separation of variables is given in **Unit 5.4**. The ODE can be approximately integrated (solved) using the Euler method [**Unit 5.5**]. In fact, the Euler method approximates the process of continuous compounding with discrete compounding. The relationship between these two types of compounding are analyzed in **Unit 5.6**.

The Euler integration method can be applied to solve any explicit first order ODE, as shown in **Unit 5.7**.

Some quantities, such as interest rates, tend to a constant in the long term. We say that these quantities revert to the mean. In **Unit 5.8** we will describe how to apply the Euler integration method to an ODE in which the reversion to the mean has an exponential form. This ODE is the basis for a stochastic model that describes the time-dependent volatility of an asset. This stochastic model will be analyzed in Chapter 6 of the course.

Up to this point we have used a constant step size for the Euler integration method. However, one can also use the method assuming steps of different sizes [**Unit 5.9**]. This can be useful to adapt the integration grid to the variability in the function: When the variations are abrupt, a finer integration grid should be used. Regions of smooth variation can be covered using a coarser integration grid.

The evolution of the position of the Van der Pol oscillator can be described by a second order differential equation. The ODE is second order because it relates the function that gives the position of the oscillator as a function of time with its first and second derivatives. Introducing a new variable, the momentum, which is proportional the first derivative of the position, it is possible to transform the second order ODE into two coupled first order ODE's, which can readily be solved using the Euler integration method. The details of this transformation are presented in **Unit 5.10**.

The ODE's considered are initial value problems. Euler's method is a simple but inaccurate procedure for the numerical integration of these these types of ODE's. Most books on numerical methods, such as [Press et al., 2007, Heath, 2002], discuss more sophisticated numerical solution methods. Nevertheless, in this course we restrict our attention to the Euler integration method because it can be readily extended to approximate the solution of Stochastic Differential Equations (SDE's). This extension will be introduced and analyzed in Chapter 6.

References

M.T. Heath. *Scientific computing: an introductory survey.* McGraw-Hill Higher Education. McGraw-Hill, 2002.

William H. Press, Saul A. Teukolsky, William T. Vetterling, and Brian P. Flannery. *Numerical Recipes 3rd Edition: The Art of Scientific Computing.* Cambridge University Press, New York, NY, USA, 3 edition, 2007. ISBN 0521880688, 9780521880688.