Chapter 6: Stochastic differential equations

In this chapter we will learn how to solve stochastic differential equations (SDE's) using simulation. A stochastic differential equation of the Itô form is built by adding a stochastic term, which is expressed in terms of a Wiener process, to an ordinary differential equation [**Unit 6.1**].

The solution of a first order ordinary differential equation is a trajectory that starts at the initial condition. By contrast, the solution of a stochastic differential equation is a stochastic process. In **Unit 6.2** we will illustrate how the Euler method can be extended to take into account the stochastic term to approximate the solution of the SDE.

In practice, to solve the SDE we simulate a finite number of trajectories, which are possible realizations of the stochastic process [**Unit 6.3**].

In **Unit 6.4** we examine a stochastic model for the time-dependent volatility of a financial asset. The SDE consist of a deterministic (or drift) term that is mean-reverting and a stochastic (or diffusive) term that is proportional to the volatility.

One of the central assumptions made in the Black-Scholes model is that the time evolution of asset prices in financial markets follows a geometric Brownian motion (GBM). The stochastic differential equation for this stochastic process is derived in **Unit 6.5**.

As shown in **Unit 6.6**, the simulation of geometric Brownian motion can be carried out approximately using the stochastic Euler integration method.

Because of its central role in finance, the statistical properties of geometric Bronian motion (GBM) are analyzed in some detail in **Unit 6.7**. Specifically, we will show that a quantity that undergoes GBM is distributed as a lognormal with time dependent parameters. In **Unit 6.8** we simulate geometric Brownian motion using the exact expression for the time evolution instead of the approximation given by the Euler integration method.

Finally, in **Units 6.9** and **6.10** we analyze diffusive processes with jumps. In finance, abrupt changes in the price of stock price can occur because of new unexpected information about the asset (e.g. an acquisition or a merger) becomes available to the market agents. A common example is the jump-diffusion SDE introduced by the American economist and Nobel laureate Robert C. Merton, in which the jumps model idiosyncratic shocks in the price of the asset. As described in **Unit 6.10** the Merton model assumes that the number of jumps follows a homogeneous Poisson process and that the magnitude of the jumps, which are assumed to be multiplicative, is distributed as a lognormal random variable [Merton, 1976].

An advanced reference text on stochastic differential equations is [Øksendal]. Most textbooks on financial derivatives, such as [Hull, 2009, Glasserman, Wilmott, 1998] discuss geometric Brownian motion in some detail.

References

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