Monte Carlo Methods in Finance

Homework: Chapter 6

Please enter your answers in the homework unit at the end of the Chapter. The solutions to this homework will be posted in a separate unit after the due date.

1. In this exercise we will analyze the properties of two widely used stochastic models for interest rates: The Vasicek model and the Cox-Ingersoll-Ross (CIR) model. To formulate these models we will write stochastic differential equations for $r(t)$, the instantaneous spot rate or *short rate*. This is the rate at which interest is accrued at the instant t. Specifically, if $M(t)$ represents the value of a bank account that pays interest $r(t)$ at time t, then
$$dM(t) = r(t)M(t)dt.$$
Therefore, assuming that the initial value of the bank account at time t_0 is $M(t_0) = M_0$, the value at time t is
$$M(t) = M_0 \exp\left(\int_{t_0}^{t} r(s)ds\right)$$
The equations for the short rate in these models are
$$\begin{aligned} &\text{Vasicek model:} && dr(t) = a(b - r(t))dt + \sigma dW(t) \\ &\text{CIR model:} && dr(t) = a(b - r(t))dt + \sigma\sqrt{r(t)}dW(t), \end{aligned}$$
where $W(t)$ is a Wiener process. They are one-factor models, in the sense that they describe the evolution of interest rates as driven by only one source of risk ($W(t)$).

   Write code to generate $M = 1000$ trajectories for each of the models using the stochastic Euler method. In the simulation we will use $N = 200$ time steps from $t_0 = 0$ to $t = 4$. The parameters for the SDE's are $a = 2$, $b = 0.05$, $\sigma = 0.06$ and $r(0) = 0.1$.

   For each model, make a plot of the mean of $r(t)$, averaged over trajectories, for every value of t. Which of the following behaviors is observed?

   - The mean stays at $r(0)$ in the Vasicek model and tends to b in the CIR model.

- The mean tends to b in the Vasicek model and stays at $r(0)$ in the CIR model.
- In both models the mean stays at $r(0)$.
- In both models the mean tends to b.

Hint: Use the function `stochasticEulerIntegration`, which is provided in this chapter, to carry out the simulations.

2. For each model, display all the trajectories that you have simulated in the same plot. Which of the following is true?

   - $r(t)$ is always positive in the CIR model but not in the Vasicek model.
   - $r(t)$ is always positive in the Vasicek model but not in the CIR model.
   - $r(t)$ is always positive in both models.
   - $r(t)$ can take negative values in both models.

3. Obtain Monte Carlo estimates of $\mathtt{P}[r(t) > 0.055]$ with $t = 4$ in both cases. What are the closest values to these estimates?

   - 0.57 (Vasicek) and 0.32 (CIR).
   - 0.57 (Vasicek) and 0.78 (CIR).
   - 0.43 (Vasicek) and 0.22 (CIR).
   - 0.43 (Vasicek) and 0.68 (CIR).

4. The price of an asset, $S(t)$, is modeled using geometric Brownian motion with drift $\mu = 0.05$ and volatility $\sigma = 0.25$. What is the probability that the price in six years at least doubles the price in three years?

   - 0.19789
   - 0.10486
   - 0.07067
   - The probability depends on S_0, the price of the asset today.

   Hint: To compute the probability $\mathtt{P}[S(t_2) \geq 2S(t_1)]$ use the fact that the ratio $\frac{S(t_2)}{S(t_1)}$ is distributed as a lognormal.

5. Assume that the stochastic process $f(t)$ with $f(t_0) = f_0$ follows a geometric Brownian motion for $t > t_0$. Which of these statements is true?

   - The mean of $f(t)$ is greater than its median.
   - The median of $f(t)$ is greater than its mean.
   - The mean of $f(t)$ is equal to its median.
   - The mean of $\exp(f(t))$ is equal to its median.