Monte Carlo Methods in Finance

Homework: Chapter 5

Please enter your answers in the homework unit at the end of the Chapter. The solutions to this homework will be posted in a separate unit after the due date.

1. Consider the ordinary differential equation (ODE) for $y(t)$

$$\frac{dy(t)}{dt} = e^{-y(t)^2}$$

with $y(0) = 1$. What is the value of y(0.5) computed using a second order (quadratic) Taylor approximation around $t = 0$?

- 1.031
- 1.150
- 1.184
- 1.218
- 1.625

Hint: The second Taylor approximation of $y(t)$ in the neighborhood of $t = 0$ is

$$y(\Delta T) \approx y(0) + \left.\frac{dy(t)}{dt}\right|_{t=0} \Delta T + \frac{1}{2} \left.\frac{d^2y(t)}{dt^2}\right|_{t=0} (\Delta T)^2 + \mathcal{O}\left((\Delta T)^3\right).$$

The expression of the first derivative of $y(t)$ is directly given by the ODE. The expression of the second derivative of $y(t)$ is obtained by taking a derivative with respect to t of the expression of the first derivative given by the ODE.

2. Find the solution of the ODE for $z(t)$

$$\frac{dz(t)}{dt} = e^{at+z(t)}$$

with z(0)=0 using the method of separation of variables.

- $z(t) = \log\left(1 - \frac{1}{a}e^{at}\right)$
- $z(t) = \frac{e^{at}-1}{at+1}$

- $z(t) = t \log \left(e^{at} - 1 \right)$
- $z(t) = -\log \left(1 - \frac{1}{a}(e^{at} - 1) \right)$

3. Let $M(20)$ be the balance of a bank account in 20 years, with initial investment of \$50000 and an annual continuously compounded interest of 6% . What is the relative error, $(M_{\text{Euler}}(20) - M(20))/M(20)$, incurred when the balance is obtained by Euler integration using 1 time step per year?

   - 0%
   - +6.6%
   - -6.6%
   - +3.4%
   - -3.4%

   Hint: The approximate solution given by Euler integration is equivalent to the expression for discrete compounding.

4. Consider the Lorenz system of ordinary differential equations

$$
\begin{aligned}
\frac{dx(t)}{dt} &= \sigma(y(t) - x(t)), & x(t_0) &= x_0 \\
\frac{dy(t)}{dt} &= x(t)(r - z(t)) - y(t), & y(t_0) &= y_0 \\
\frac{dz(t)}{dt} &= x(t)y(t) - bz(t), & z(t_0) &= z_0,
\end{aligned}
$$

   with positive parameters σ, b and r. Write an Octave/MATLAB function to compute an approximation of the solution, $\mathbf{s}_N(t) = \{x_N(t), y_N(t), z_N(t)\}$, using the Euler integration scheme with N time steps. The header of this function is

```
function [t,x,y,z] = eulerIntegrationLorenz(t0,x0,y0,z0,sigma,r,b,T,N)
% eulerIntegrationLorenz:Solution of Lorenz attractor eqns. via Euler integration
%
% SYNTAX:
%       [t,x,y,z] = eulerIntegrationLorenz(t0,x0,y0,z0,sigma,r,b,T,N)
%
% INPUT:
%       t0 : Initial time
%  x0,y0,z0 : Initial position
% sigma,r,b : Parameters
%       T  : Length of integration interval [t0, t0+T]
%       N  : Number of time steps
%
% OUTPUT:
%       t  : Times at which the trajectory is monitored
%            t(n) = t0 + n Delta T
%    x,y,z  : Values of the position along the trajectory
```

```
%
% EXAMPLE:
%       t0 = 0; x0 = 1; y0 = 1; z0 = 1;
%       sigma = 10; b = 8/3; r = 28;
%       T = 50; N = 1e5;
%       [t,x,y,z] = eulerIntegrationLorenz(t0,x0,y0,z0,sigma,r,b,T,N);
```

Use this function to compute the solution at $T = 1$, with initial conditions $x_0 = y_0 = z_0 = 1$, parameters $\sigma = 10, b = \frac{8}{3}, r = 28$, and using $N = 10000$ steps. What is the closest value to this approximation?

- $\mathbf{s}_N(1) = [-9.351, -8.361, 29.294]$
- $\mathbf{s}_N(1) = [1.114, 1.982, 1.796]$
- $\mathbf{s}_N(1) = [12.810, 2.761, 1.616]$
- $\mathbf{s}_N(1) = [6.067, -12.004, 5.056]$
- $\mathbf{s}_N(1) = [-2.772, 17.277, -9.997]$

5. Consider the Lorenz system from the previous exercise. It is known that for the values of the parameters used in the previous exercise the system has chaotic behavior. Approximate the solution using the Euler integration scheme for $T = 25$, first with $N_1 = 10000$ time steps and then with $N_2 = 10001$ time steps. What is the relative difference

$$d = \frac{x_{N_2}(T) - x_{N_1}(T)}{x_{N_1}(T)}$$

between the x component of both solutions? Are $N_1 = 10000$ time steps sufficient for an accurate approximation of the solution?

- Diff $\leq 1\%$. Yes, N_1 is high enough.
- Diff $\approx 10\%$. Yes, N_1 is high enough.
- Diff $\geq 100\%$. Yes, N_1 is high enough.
- Diff $\leq 1\%$. No, N_1 is insufficient.
- Diff $\approx 10\%$. No, N_1 is insufficient.
- Diff $\geq 100\%$. No, N_1 is insufficient.