Introduction to Artificial Intelligence (INFO8006)

Exercises 4 – Reasoning over time

November 10, 2022

Learning outcomes

At the end of this session you should be able to

- formulate a Markov model for discrete-time reasoning problems;
- define the simplifying assumptions of Markov processes;
- define and apply prediction, filtering and smoothing in Markov processes;
- apply the simplified matrix algorithm(s) to hidden Markov models;
- define the Kalman filter assumptions and manipulate multivariate Gaussian distributions.

Exercise 1 Umbrella World (AIMA, Section 15.1.1)

You are a security guard stationed at a secret underground installation. You want to know whether it is raining today, but your only access to the outside world occurs each morning when you see the director coming in with, or without, an umbrella. For each day t, the evidence is a single variable $Umbrella_t \in \{1, 0\}$, *i.e.* whether the umbrella appears or not, and the (hidden) state is a single variable $Rain_t \in \{1, 0\}$, *i.e.* whether it is raining or not.

You believe that from one day $t-1$ to the next t, the chances that the weather stays the same are 70 %. You also believe that the director brings his umbrella 90 % of the time when it is raining, and 20 % of the time otherwise.

1. You would like to represent your umbrella world as a Markov model. What formal assumptions correspond to your beliefs ?

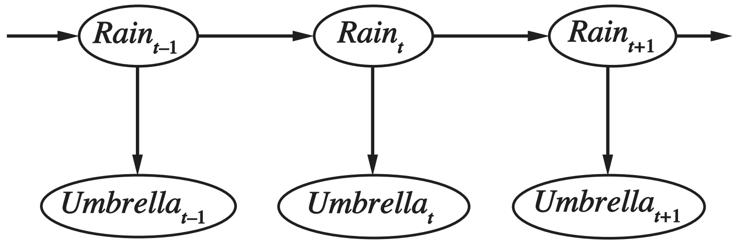
   The first belief describes the umbrella world as a *first-order Markov process*, such that

   $$P(R_t|R_{1:t-1}) = P(R_t|R_{t-1}),$$

   i.e. R_t is independent from $R_{1:t-2}$ given R_{t-1}. The second belief is equivalent to a *first-order sensor Markov assumption*, implying that

   $$P(U_t|R_{1:t}, U_{1:t-1}) = P(U_t|R_t).$$

2. Sketch a Bayesian network structure describing the umbrella world and provide the transition and sensor models.



R_{t-1}	$P(R_t = 1\|R_{t-1})$
1	0.7
0	0.3

R_t	$P(U_t = 1\|R_t)$
1	0.9
0	0.2

3. Express the distributions $P(R_{t+1}|R_{t-1})$, $P(U_t|R_{t-1})$ and $P(R_t|R_{t-1},U_t)$ in terms of the transition and sensor models.

$$\begin{aligned}
P(R_{t+1}|R_{t-1}) &= \sum_{r_t} P(R_{t+1}|r_t)P(r_t|R_{t-1}) \\
&= \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix} = \begin{pmatrix} 0.58 & 0.42 \\ 0.42 & 0.58 \end{pmatrix} \\
P(U_t|R_{t-1}) &= \sum_{r_t} P(U_t|R_{t-1},r_t)P(r_t|R_{t-1}) \\
&= \sum_{r_t} P(U_t|r_t)P(r_t|R_{t-1}) \\
&= \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix} \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix} = \begin{pmatrix} 0.69 & 0.41 \\ 0.31 & 0.59 \end{pmatrix} \\
P(R_t|R_{t-1},U_t) &= \frac{P(U_t|R_{t-1},R_t)P(R_t|R_{t-1})}{P(U_t|R_{t-1})} \\
&= \frac{P(U_t|R_t)P(R_t|R_{t-1})}{P(U_t|R_{t-1})}
\end{aligned}$$

4. Suppose you observe an unending sequence of days on which the umbrella appears. Show that, as the days go by, the probability of rain on the current day tends monotonically towards a fixed point. Calculate this fixed point.

We are asked to prove that $x_t = P(R_t = 1|U_{1:t} = 1)$ tends monotonically (with respect to t) towards a fixed point x^*. To do so, we need to find the value of x_t, which corresponds to apply *filtering* on the Markov model defined by the umbrella world. On this basis, we have

$$\begin{aligned}
P(R_t|U_{1:t} = 1) &= \alpha P(U_t = 1|R_t) \sum_{r_{t-1}} P(R_t|r_{t-1})P(r_{t-1}|U_{1:t-1} = 1) \\
&= \alpha \begin{pmatrix} 0.9 & 0 \\ 0 & 0.2 \end{pmatrix} \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix} P(R_{t-1}|U_{1:t-1} = 1) \\
\Leftrightarrow \qquad \begin{pmatrix} x_t \\ 1 - x_t \end{pmatrix} &= \alpha \begin{pmatrix} 0.63 & 0.27 \\ 0.06 & 0.14 \end{pmatrix} \begin{pmatrix} x_{t-1} \\ 1 - x_{t-1} \end{pmatrix} \\
&= \alpha \begin{pmatrix} 0.27 + (0.63 - 0.27)\, x_{t-1} \\ 0.14 + (0.06 - 0.14)\, x_{t-1} \end{pmatrix} \\
&= \frac{1}{0.41 + 0.28\, x_{t-1}} \begin{pmatrix} 0.27 + 0.36\, x_{t-1} \\ 0.14 - 0.08\, x_{t-1} \end{pmatrix}.
\end{aligned}$$

Then, if there is a fixed point $x^* \in [0, 1]$, it satisfies

$$\begin{aligned}
x^* &= \frac{0.27 + 0.36\, x^*}{0.41 + 0.28\, x^*} \\
\Leftrightarrow \quad 0.41\, x^* + 0.28\, {x^*}^2 &= 0.27 + 0.36\, x^* \\
\Leftrightarrow \quad 0 &= 0.28\, {x^*}^2 + 0.05\, x^* - 0.27 \\
\Rightarrow \quad x^* &= \frac{-0.05 + \sqrt{0.05^2 + 4 \times 0.27 \times 0.28}}{2 \times 0.28} \approx 0.897.
\end{aligned}$$

To show that x_t tends monotonically towards x^*, it is sufficient to prove that x_t is always between x^* and x_{t-1}, *i.e.*

$$(x^* - x_t)(x_t - x_{t-1}) > 0,$$

when $x_{t-1} \neq x^*$, which is left as an exercise to the motivated reader.

5. Now consider forecasting further and further into the future, given t umbrella observations. Is there a fixed point ? If yes, compute its exact value.

   We are asked to forecast the rain probability $x_k = P(R_{t+k} = 1 | U_{1:t} = 1)$, *i.e.* to perform *prediction*. We have

$$
\begin{aligned}
P(R_{t+k}|U_{1:t} = 1) &= \sum_{r_{t+k-1}} P(R_{t+k}|r_{t+k-1}) P(r_{t+k-1}|U_{1:t} = 1) \\
&= \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix} P(R_{t+k-1}|U_{1:t} = 1) \\
\Leftrightarrow \qquad \begin{pmatrix} x_k \\ 1 - x_k \end{pmatrix} &= \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} x_{k-1} \\ 1 - x_{k-1} \end{pmatrix}.
\end{aligned}
$$

   By symmetry, we find the fixed point $x^* = 0.5$, which highlights the fact that, without new evidences, uncertainty about the states accumulates.

Exercise 2 The coins

You are in a room containing 3 precious biased gold coins a, b and c. You inspect the coins and notice that the coins a, b and c have a head probability of 80 %, 50 % and 20 %, respectively.

Another person enters the room, takes the coins and put them into a bag. They draw a coin from the bag and tell you that they will repeat 4 times the same routine: hide their hand in the bag and either keep the current coin with probability $\frac{2}{3}$ or replace it by another, then toss it and show you the result. They proceed and the sequence of results are heads, heads, tail, heads. If you answer right to the following questions they will give you the coins.

1. Provide a hidden Markov model (HMM) that describes the process.
   - The hidden state at time t is $X_t \in \{a, b, c\}$ and represents the tossed coin.
   - The evidence at time t is $E_t \in \{0, 1\}$ and represents the result (heads or tail) of the toss. We have $e_1 = e_2 = e_4 = 0$ and $e_3 = 1$.
   - The prior vector
   $$f_0 = P(X_0) = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}^T$$
   - The transition matrix
   $$T = P(X_t|X_{t-1}) = \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{pmatrix}$$
   - The sensor matrix
   $$B = P(E_t|X_t) = \begin{pmatrix} 0.8 & 0.5 & 0.2 \\ 0.2 & 0.5 & 0.8 \end{pmatrix}$$

2. What are the probabilities of the last coin given the sequence of evidences?

   We are asked to calculate the distribution $f_t = P(X_t|e_{1:t})$ for $t = 4$. Applying Bayes filter, we have
   $$\begin{aligned} P(X_t|e_{1:t}) &= \alpha P(e_t|X_t, e_{1:t-1}) P(X_t|e_{1:t-1}) \\ &= \alpha P(e_t|X_t, e_{1:t-1}) \sum_{x_{t-1}} P(X_t|x_{t-1}, e_{1:t-1}) P(x_{t-1}|e_{1:t-1}) \\ &= \alpha P(e_t|X_t) \sum_{x_{t-1}} P(X_t|x_{t-1}) P(x_{t-1}|e_{1:t-1}) \\ \Leftrightarrow \qquad f_t &= \alpha\, O_t\, T\, f_{t-1} \end{aligned}$$
   where $O_t = \mathrm{diag}(P(e_t|X_t))$ is the observation matrix. By substitution,
   $$O_1 = O_2 = O_4 = \begin{pmatrix} 0.8 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.2 \end{pmatrix} \quad \text{and} \quad O_3 = \begin{pmatrix} 0.2 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.8 \end{pmatrix}.$$
   Then, to calculate f_4, we first need f_1, f_2 and f_3.
   $$\begin{aligned} f_1 &= \alpha_1\, O_1\, T\, f_0 = \alpha_1 \begin{pmatrix} 0.8 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.2 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \alpha_1 \begin{pmatrix} \frac{4}{15} \\ \frac{1}{6} \\ \frac{1}{15} \end{pmatrix} = \begin{pmatrix} \frac{8}{15} \\ \frac{1}{3} \\ \frac{2}{15} \end{pmatrix} \\ f_2 &= \alpha_2\, O_2\, T\, f_1 = \dots \\ f_3 &= \alpha_3\, O_3\, T\, f_2 = \dots \\ f_4 &= \alpha_4\, O_4\, T\, f_3 = \dots \approx \begin{pmatrix} 0.472 & 0.374 & 0.154 \end{pmatrix}^T \end{aligned}$$

3. What are the probabilities of the first coin given the sequence of evidences? And of the first coin tossed?

   We are asked to calculate the distribution $P(X_k|e_{1:t})$ for $t = 4$ and $k = 0$. As $k < t$ this corresponds to *smoothing* our belief of the past. We have

$$\begin{aligned} P(X_k|e_{1:t}) &= \alpha P(X_k, e_{k+1:t}|e_{1:k}) \\ &= \alpha P(e_{k+1:t}|X_k, e_{1:k})P(X_k|e_{1:k}) \\ &= \alpha P(e_{k+1:t}|X_k)P(X_k|e_{1:k}) \end{aligned}$$

   As we already know $f_k = P(X_k|e_{1:k})$, we only need to compute $b_k = P(e_{k+1:t}|X_k)$. We have

$$\begin{aligned} & P(e_{k+1:t}|X_k) = \sum_{x_{k+1}} P(x_{k+1}|X_k)P(e_{k+1:t}|X_k, x_{k+1}) \\ & \qquad = \sum_{x_{k+1}} P(x_{k+1}|X_k)P(e_{k+1}|x_{k+1})P(e_{k+2:t}|x_{k+1}) \\ \Leftrightarrow \qquad & b_k = T^T \, O_{k+1} \, b_{k+1} \end{aligned}$$

   where $b_t = b_4 = P(anything|X_4) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$. Therefore, to calculate b_0, we first need b_3, b_2 and b_1.

$$\begin{aligned} b_3 &= T^T \, O_4 \, b_4 = \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 0.8 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.65 \\ 0.5 \\ 0.35 \end{pmatrix} \\ b_2 &= T^T \, O_3 \, b_3 = \dots \\ b_1 &= T^T \, O_2 \, b_2 = \dots \\ b_0 &= T^T \, O_1 \, b_1 = \dots \approx \begin{pmatrix} 0.076 & 0.055 & 0.035 \end{pmatrix}^T \end{aligned}$$

   Eventually,

$$\begin{aligned} P(X_0|e_{1:4}) &= \alpha b_0 \times f_0 \approx \begin{pmatrix} 0.457 & 0.331 & 0.212 \end{pmatrix}^T \\ P(X_1|e_{1:4}) &= \alpha b_1 \times f_1 \approx \begin{pmatrix} 0.580 & 0.329 & 0.091 \end{pmatrix}^T \end{aligned}$$

4. What is the most likely sequence of tossed coins?

   The most likely sequence $x^*_{1:t}$ given the sequence of evidences $e_{1:t}$ is the one that satisfies

$$x^*_{1:t} = \arg\max_{x_{1:t}} P(x_{1:t}|e_{1:t})$$

   or, equivalently,

$$\begin{aligned} x^*_k &= \arg\max_{x_k} \left[\max_{x_{1:k-1}} P(x_{1:k}, x^*_{k+1:t}|e_{1:t}) \right] \\ &= \arg\max_{x_k} \left[\max_{x_{1:k-1}} \alpha P(x_{1:k}, x^*_{k+1:t}, e_{k+1:t}|e_{1:k}) \right] \\ &= \arg\max_{x_k} \left[\max_{x_{1:k-1}} \alpha P(e_{k+1:t}|x_{1:k}, x^*_{k+1:t}, e_{1:k}) P(x^*_{k+1:t}|x_{1:k}, e_{1:k}) P(x_{1:k}|e_{1:k}) \right] \\ &= \arg\max_{x_k} \left[\max_{x_{1:k-1}} \underbrace{\alpha P(e_{k+1:t}|x^*_{k+1:t}) P(x^*_{k+2:t}|x^*_{k+1})}_{\text{constant}} P(x^*_{k+1}|x_k) P(x_{1:k}|e_{1:k}) \right] \\ &= \arg\max_{x_k} P(x^*_{k+1}|x_k) \left[\max_{x_{1:k-1}} P(x_{1:k}|e_{1:k}) \right] \\ &= \arg\max_{x_k} P(x^*_{k+1}|x_k) \, m_k(x_k) \end{aligned}$$

where

$$
\begin{aligned}
m_k &= \max_{x_{1:k-1}} P(x_{1:k-1}, X_k | e_{1:k}) \\
&= \max_{x_{1:k-1}} \alpha P(x_{1:k-1}, X_k, e_k | e_{1:k-1}) \\
&= \max_{x_{1:k-1}} \alpha P(e_t | x_{1:k-1}, X_t, e_{1:k-1}) P(X_k | x_{1:k-1}, e_{1:k-1}) P(x_{1:k-1} | e_{1:k-1}) \\
&= \max_{x_{1:k-1}} \alpha P(e_k | X_k) P(X_k | x_{k-1}) P(x_{1:k-1} | e_{1:k-1}) \\
&= \alpha P(e_k | X_k) \max_{x_{k-1}} P(X_k | x_{k-1}) \max_{x_{1:k-2}} P(x_{1:k-1} | e_{1:k-1}) \\
&= \alpha P(e_k | X_k) \max_{x_{k-1}} P(X_k | x_{k-1})\, m_{k-1}(x_{k-1})
\end{aligned}
$$

and $m_1 = P(X_1|e_1) = f_1$. Therefore, we can iteratively build the vectors m_1, m_2, $\ldots$, m_t to find the most likely last state

$$x_t^* = \arg\max_{x_t} m_t(x_t)$$

and, then, the most likely sequence with

$$x_k^* = \arg\max_{x_k} P(x_{k+1}^* | x_k)\, m_k(x_k).$$

Exercise 3 September 2019 (AIMA, Ex 15.13 and 15.14)

A professor wants to know if students are getting enough sleep. Each day, the professor observes whether the students sleep in class, and whether they have red eyes. The professor has the following hypotheses:

- The prior probability of getting enough sleep, with no observations, is 0.7.
- The probability of getting enough sleep on night t is 0.8 given that the student got enough sleep the previous night, and 0.3 otherwise.
- The probability of having red eyes is 0.2 if the student got enough sleep, and 0.7 otherwise.
- The probability of sleeping in class is 0.1 if the student got enough sleep, and 0.3 otherwise.

The professor asks you to answer the following questions:

1. Formulate the environment and hypotheses as a dynamic Bayesian network that the professor could use to detect sleep deprived students, from a sequence of observations. Provide the associated probability tables.
2. Reformulate the dynamic Bayesian network as a hidden Markov model that has only a single observation variable. Give the complete probability tables for the model.
3. For the sequence $e_{1:3}$ of observations "no red eyes, not sleeping in class", "red eyes, not sleeping in class" and "red eyes, sleeping in class", calculate the distributions $P(EnoughSleep_t|e_{1:t})$ and $P(EnoughSleep_t|e_{1:3})$ for $t \in \{1, 2, 3\}$.

Exercise 4 Super Spring Ultra Pro Max XXL

The "Foire de Liège" has a new attraction called "Super Spring Ultra Pro Max XXL" which consists in a ball attached to a spring on a platform. The participants take seat in the ball locked at some position. The ball is then released and pulled by the spring which makes it oscillate back and forth. After a few seconds, the ball is stopped by magnetic brakes. You notice that the final position of the ball is different each time. As the ball is opaque, you wonder if it is possible for the participants to guess where the ball stopped, given their perception of acceleration.

You more or less remember your Newtonian mechanics class and model the movement of the ball as a series of transitions

$$
\begin{aligned}
p_t &= p_{t-1} + \Delta t\, \dot{p}_{t-1} + \frac{1}{2}\Delta t^2\, \ddot{p}_{t-1} \\
\dot{p}_t &= \dot{p}_{t-1} + \Delta t\, \ddot{p}_{t-1} \\
\ddot{p}_t &= \rho w_t - \kappa\, p_{t-1} - \eta\, \dot{p}_{t-1} \\
w_t &\sim \mathcal{N}(\alpha w_{t-1}, \sigma_w^2)
\end{aligned}
$$

where p_t, $\dot{p}_t$ and $\ddot{p}_t$ are respectively the position, velocity and acceleration of the ball at timestep t, w_t is the wind at timestep t, Δt is the time elapsed between $t-1$ and t, ρ is the thrust coefficient of the wind, κ is the stiffness of the spring, η is the friction coefficient of the platform and α is the persistence of the wind. You estimate that the ball starts $10.0 \pm 0.5\,\mathrm{m}$ at the left of the spring with a negligible speed $0.0 \pm 0.1\,\mathrm{m\,s^{-1}}$ and acceleration $0.0 \pm 0.1\,\mathrm{m\,s^{-2}}$. You assume that the wind has a stationary distribution. Finally, you assume that the human perception of acceleration follows an unbiased Gaussian distribution.

1. You wish to predict the state of the ball given the perceptions of a participant. Define the components of a Kalman filter in this context.

   The Kalman filter is a special case of the continuous Bayes filter, which assumes

   - a Gaussian prior
   $$p(X_0) = \mathcal{N}(\mu_0, \Sigma_0),$$
   - a linear Gaussian transition model
   $$p(X_t|X_{t-1}) = \mathcal{N}(FX_{t-1} + u, \Sigma_x)$$
   - and a linear Gaussian sensor model
   $$p(E_t|X_t) = \mathcal{N}(HX_t + v, \Sigma_e).$$

   In a multivariate Gaussian distribution $x = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix}^T \sim \mathcal{N}(\mu, \Sigma)$, the first argument is the *mean vector* and the second is the *covariance matrix*. The mean vector is the vector of the variable means, *i.e.* $\mu_i = \mathbb{E}[x_i]$. An element Σ_{ij} of the covariance matrix is the covariance between the variables x_i and x_j, *i.e.* $\Sigma_{ij} = \mathbb{E}[(x_i - \mu_i)(x_j - \mu_j)]$. If x_i is independent from x_j, their covariance is null by definition, *i.e.* $\Sigma_{ij} = 0$. Interestingly, the diagonal elements Σ_{ii} are the variable variances $\mathbb{V}[x_i] = \mathbb{E}[(x_i - \mu_i)^2]$.

   In our case, the state X_t is a 4-dimensional vector $\begin{pmatrix} p_t & \dot{p}_t & \ddot{p}_t & w_t \end{pmatrix}^T$ and according to the provided information, the prior is defined by

   $$\mu_0 = \begin{pmatrix} 10 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \Sigma_0 = \begin{pmatrix} 0.5^2 & 0 & 0 & 0 \\ 0 & 0.1^2 & 0 & 0 \\ 0 & 0 & 0.1^2 & 0 \\ 0 & 0 & 0.0 & \frac{\sigma_w^2}{1-\alpha^2} \end{pmatrix}.$$

Then, our transition model is defined by

$$F = \begin{pmatrix} 1 & \Delta t & \frac{1}{2}\Delta t^2 & 0 \\ 0 & 1 & \Delta t & 0 \\ -\kappa & -\eta & 0 & \rho \\ 0 & 0 & 0 & \alpha \end{pmatrix}, \quad u = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \Sigma_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_w^2 \end{pmatrix}.$$

It should be noted that only the wind has a non-null variance in Σ_x, since the transitions of the position, velocity and acceleration are *deterministic*.

Concerning the sensor model, the evidence is a perturbed, but unbiased, perception of the acceleration, that is

$$e_t \sim \mathcal{N}(\ddot{p}_t, \sigma_e^2),$$

which translates to

$$H = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}, \quad v = 0 \quad \text{and} \quad \Sigma_e = \sigma_e^2.$$

2. Express the distribution $p(X_t|e_{1:t})$ with respect to the components defined previously.

This task corresponds exactly to filtering, *i.e.* inferring the distribution

$$\begin{aligned} p(X_t|e_{1:t}) &= \alpha\, p(e_t|X_t, e_{1:t-1})\, p(X_t|e_{1:t-1}) \\ &= \alpha\, p(e_t|X_t) \int p(X_t, x_{t-1}|e_{1:t-1})\, \mathrm{d}x_{t-1}. \end{aligned}$$

In the latter expression, the integral corresponds to the marginalization of the joint distribution

$$p(X_t, X_{t-1}|e_{1:t-1}) = p(X_t|X_{t-1})\, p(X_{t-1}|e_{1:t-1}).$$

Because the transition model $p(X_t|X_{t-1})$ is linear Gaussian, if the previous belief $p(X_{t-1}|e_{1:t-1})$ is a (multivariate) Gaussian distribution $\mathcal{N}(\mu_{t-1}, \Sigma_{t-1})$, we have

$$p\left(\begin{pmatrix} X_{t-1} \\ X_t \end{pmatrix} \middle| e_{1:t-1}\right) = \mathcal{N}\left(\begin{pmatrix} \mu_{t-1} \\ F\mu_{t-1} + u \end{pmatrix}, \begin{pmatrix} \Sigma_{t-1} & \Sigma_{t-1}F^T \\ F\Sigma_{t-1} & F\Sigma_{t-1}F^T + \Sigma_x \end{pmatrix}\right)$$

and

$$p(X_t|e_{1:t-1}) = \mathcal{N}\Big(\underbrace{F\mu_{t-1} + u}_{\mu_*}, \underbrace{F\Sigma_{t-1}F^T + \Sigma_x}_{\Sigma_*}\Big).$$

Similarly, because the sensor model $p(E_t|X_t)$ is linear Gaussian, we have the joint distribution

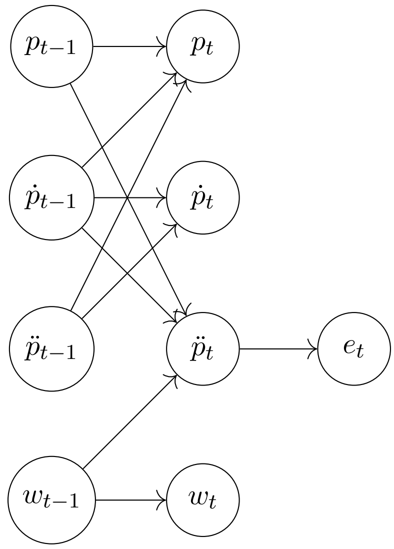
$$p\left(\begin{pmatrix} X_t \\ E_t \end{pmatrix} \middle| e_{1:t-1}\right) = \mathcal{N}\left(\begin{pmatrix} \mu_* \\ H\mu_* + v \end{pmatrix}, \begin{pmatrix} \Sigma_* & \Sigma_* H^T \\ H\Sigma_* & H\Sigma_* H^T + \Sigma_e \end{pmatrix}\right).$$

Then, from the cheat sheet for Gaussian models (slide 47, lecture 6) we have

$$p(X_t|e_{1:t}) = \mathcal{N}\Big(\underbrace{\mu_* + K(e_t - H\mu_* - v)}_{\mu_t}, \underbrace{\Sigma_* - KH\Sigma_*}_{\Sigma_t}\Big)$$

where $K = \Sigma_* H^T (H\Sigma_* H^T + \Sigma_e)^{-1}$ is the Kalman gain matrix. Since the prior $p(X_0)$ is Gaussian by assumption, all beliefs are Gaussian as well, by induction.

3. Represent the transition and sensor models as a dynamic Bayesian network.



Supplementary materials

- Hidden Markov Models (UC Berkeley CS188, Spring 2014 Section 6).
- 68–95–99.7 rule
- Chapter 15 of the reference textbook.