

Introduction to Artificial Intelligence (INFO8006)

Exercises 4 – Reasoning over time

November 4, 2021

Learning outcomes

At the end of this session you should be able to

- formulate a Markov model for discrete-time reasoning problems.
- define the simplifying assumptions of Markov processes.
- define and apply prediction, filtering and smoothing in Markov processes.
- apply the simplified matrix algorithm(s) to hidden Markov models.
- define the Kalman filter assumptions and manipulate multivariate Gaussian distributions.

Exercise 1 Umbrella World (AIMA, Section 15.1.1)

You are a security guard stationed at a secret underground installation. You want to know whether it is raining today, but your only access to the outside world occurs each morning when you see the director coming in with, or without, an umbrella. For each day t , the evidence is a single variable $Umbrella_t \in \{1, 0\}$, *i.e.* whether the umbrella appears or not, and the (hidden) state is a single variable $Rain_t \in \{1, 0\}$, *i.e.* whether it is raining or not.

You believe that from one day $t - 1$ to the next t , the chances that the weather stays the same are 70 %. You also believe that the director brings his umbrella 90 % of the time when it is raining, and 20 % of the time otherwise.

1. You would like to represent your umbrella world as a Markov model. What formal assumptions correspond to your beliefs ?

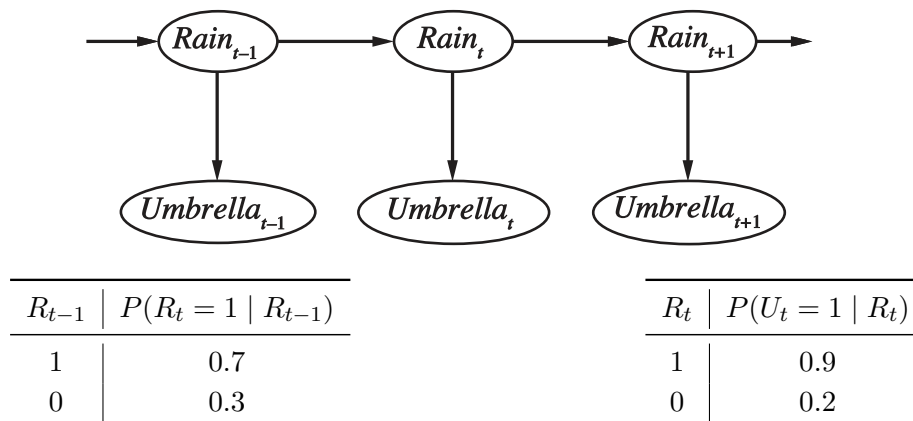
The first belief describes the umbrella world as a *first-order Markov process*, such that

$$P(R_t \mid R_{0:t-1}) = P(R_t \mid R_{t-1}),$$

i.e. R_t is independent from $R_{0:t-2}$ given R_{t-1} . The second belief is equivalent to a *first-order sensor Markov assumption*, implying that

$$P(U_t \mid R_{0:t}, U_{0:t-1}) = P(U_t \mid R_t).$$

2. Sketch a Bayesian network structure describing the umbrella world and provide the transition and sensor models.



3. Express the distributions $P(R_{t+1} | R_{t-1})$, $P(U_t | R_{t-1})$ and $P(R_t | R_{t-1}, U_t)$ in terms of the transition and sensor models.

$$\begin{aligned}
P(R_{t+1} | R_{t-1}) &= \sum_{r_t} P(R_{t+1} | r_t) P(r_t | R_{t-1}) \\
&= \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix} = \begin{pmatrix} 0.58 & 0.42 \\ 0.42 & 0.58 \end{pmatrix} \\
P(U_t | R_{t-1}) &= \sum_{r_t} P(U_t | R_{t-1}, r_t) P(r_t | R_{t-1}) \\
&= \sum_{r_t} P(U_t | r_t) P(r_t | R_{t-1}) \\
&= \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix} \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix} = \begin{pmatrix} 0.69 & 0.41 \\ 0.31 & 0.59 \end{pmatrix} \\
P(R_t | R_{t-1}, U_t) &= \frac{P(U_t | R_{t-1}, R_t) P(R_t | R_{t-1})}{P(U_t | R_{t-1})} \\
&= \frac{P(U_t | R_t) P(R_t | R_{t-1})}{P(U_t | R_{t-1})}
\end{aligned}$$

4. Suppose you observe an unending sequence of days on which the umbrella appears. Show that, as the days go by, the probability of rain on the current day tends monotonically towards a fixed point. Calculate this fixed point.

We are asked to prove that $x_t = P(R_t = 1 | U_{1:t} = 1)$ tends monotonically (with respect to t) towards a fixed point x^* . To do so, we need to find the value of x_t , which corresponds to apply *filtering* on the Markov model defined by the umbrella world. On this basis, we have

$$\begin{aligned}
P(R_t | U_{1:t} = 1) &= \alpha P(U_t = 1 | R_t) \sum_{r_{t-1}} P(R_t | r_{t-1}) P(r_{t-1} | U_{1:t-1} = 1) \\
&= \alpha \begin{pmatrix} 0.9 & 0 \\ 0 & 0.2 \end{pmatrix} \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix} P(R_{t-1} | U_{1:t-1} = 1) \\
\Leftrightarrow \begin{pmatrix} x_t \\ 1 - x_t \end{pmatrix} &= \alpha \begin{pmatrix} 0.63 & 0.27 \\ 0.06 & 0.14 \end{pmatrix} \begin{pmatrix} x_{t-1} \\ 1 - x_{t-1} \end{pmatrix} \\
&= \alpha \begin{pmatrix} 0.27 + (0.63 - 0.27) x_{t-1} \\ 0.14 + (0.06 - 0.14) x_{t-1} \end{pmatrix} \\
&= \frac{1}{0.41 + 0.28 x_{t-1}} \begin{pmatrix} 0.27 + 0.36 x_{t-1} \\ 0.14 - 0.08 x_{t-1} \end{pmatrix}.
\end{aligned}$$

Then, if there is a fixed point $x^* \in [0, 1]$, it satisfies

$$\begin{aligned}
x^* &= \frac{0.27 + 0.36 x^*}{0.41 + 0.28 x^*} \\
\Leftrightarrow 0.41 x^* + 0.28 x^{*2} &= 0.27 + 0.36 x^* \\
\Leftrightarrow 0 &= 0.28 x^{*2} + 0.05 x^* - 0.27 \\
\Rightarrow x^* &= \frac{-0.05 + \sqrt{0.05^2 + 4 \times 0.27 \times 0.28}}{2 \times 0.28} \approx 0.897.
\end{aligned}$$

To show that x_t tends monotonically towards x^* , it is sufficient to prove that x_t is always between x^* and x_{t-1} , *i.e.*

$$(x^* - x_t)(x_t - x_{t-1}) > 0,$$

when $x_{t-1} \neq x^*$, which is left as an exercise to the motivated reader.

5. Now consider forecasting further and further into the future, given t umbrella observations. Is there a fixed point ? If yes, compute its exact value.

We are asked to forecast the rain probability $x_k = P(R_{t+k} = 1 \mid U_{1:t} = 1)$, *i.e.* to perform *prediction*. We have

$$\begin{aligned}
 P(R_{t+k} \mid U_{1:t} = 1) &= \sum_{r_{t+k-1}} P(R_{t+k} \mid r_{t+k-1})P(r_{t+k-1} \mid U_{1:t} = 1) \\
 &= \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix} P(R_{t+k-1} \mid U_{1:t} = 1) \\
 \Leftrightarrow \begin{pmatrix} x_k \\ 1 - x_k \end{pmatrix} &= \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix} \begin{pmatrix} x_{k-1} \\ 1 - x_{k-1} \end{pmatrix}.
 \end{aligned}$$

By symmetry, we find that the fixed point $x^* = 0.5$, which highlights the fact that, without new evidences, uncertainty about the states accumulates.

Exercise 2 The coins

You are in a room containing 3 precious biased coins a , b and c . You inspect the coins and notice that the coins a , b and c have a head probability of 80 %, 50 % and 20 %, respectively.

Another person enters the room, takes the coins and put them into a bag. They draw a coin from the bag and tell you that they will repeat 4 times the same routine: hide their hand in the bag and either keep the current coin with probability $\frac{2}{3}$ or replace it by another, then toss it and show you the result. They proceed and the sequence of results are heads, heads, tail, heads. If you answer right to the following questions they will give you the coins.

1. Provide a hidden Markov model (HMM) that describes the process.

- The hidden state at time t is $X_t \in \{a, b, c\}$ and represents the tossed coin.
- The evidence at time t is $E_t \in \{0, 1\}$ and represents the result (heads or tail) of the toss. We have $e_1 = e_2 = e_4 = 0$ and $e_3 = 1$.
- The prior vector

$$f_0 = P(X_0) = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}^T$$

- The transition matrix

$$T = P(X_t | X_{t-1}) = \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{pmatrix}$$

- The sensor matrix

$$B = P(E_t | X_t) = \begin{pmatrix} 0.8 & 0.5 & 0.2 \\ 0.2 & 0.5 & 0.8 \end{pmatrix}$$

2. What are the probabilities of the last coin given the sequence of evidences?

We are asked to calculate the distribution $f_t = P(X_t | e_{1:t})$ for $t = 4$. Applying Bayes filter, we have

$$\begin{aligned} P(X_t | e_{1:t}) &= \alpha P(e_t | X_t, e_{1:t-1}) P(X_t | e_{1:t-1}) \\ &= \alpha P(e_t | X_t, e_{1:t-1}) \sum_{x_{t-1}} P(X_t | x_{t-1}, e_{1:t-1}) P(x_{t-1} | e_{1:t-1}) \\ &= \alpha P(e_t | X_t) \sum_{x_{t-1}} P(X_t | x_{t-1}) P(x_{t-1} | e_{1:t-1}) \\ \Leftrightarrow f_t &= \alpha O_t T f_{t-1} \end{aligned}$$

where we define the observation matrices

$$O_1 = O_2 = O_4 = \begin{pmatrix} 0.8 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.2 \end{pmatrix} \quad \text{and} \quad O_3 = \begin{pmatrix} 0.2 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.8 \end{pmatrix}.$$

Then, to calculate f_4 , we first need f_1 , f_2 and f_3 .

$$\begin{aligned}
f_1 &= \alpha_1 O_1 T f_0 = \alpha_1 \begin{pmatrix} 0.8 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.2 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \alpha_1 \begin{pmatrix} \frac{4}{15} \\ \frac{1}{6} \\ \frac{1}{15} \end{pmatrix} = \begin{pmatrix} \frac{8}{15} \\ \frac{1}{3} \\ \frac{2}{15} \end{pmatrix} \\
f_2 &= \alpha_2 O_2 T f_1 = \dots \\
f_3 &= \alpha_3 O_3 T f_2 = \dots \\
f_4 &= \alpha_4 O_4 T f_3 = \dots \approx \begin{pmatrix} 0.472 & 0.374 & 0.154 \end{pmatrix}^T
\end{aligned}$$

3. What are the probabilities of the first coin given the sequence of evidences? And of the first coin tossed?

We are asked to calculate the distribution $P(X_k | e_{1:t})$ for $t = 4$ and $k = 0$. As $k < t$ this corresponds to *smoothing* our belief of the past. We have

$$\begin{aligned}
P(X_k | e_{1:t}) &= \alpha P(X_k, e_{k+1:t} | e_{1:k}) \\
&= \alpha P(e_{k+1:t} | X_k, e_{1:k}) P(X_k | e_{1:k}) \\
&= \alpha P(e_{k+1:t} | X_k) P(X_k | e_{1:k})
\end{aligned}$$

As we already know $f_k = P(X_k | e_{1:k})$, we only need to compute $b_k = P(e_{k+1:t} | X_k)$. We have

$$\begin{aligned}
P(e_{k+1:t} | X_k) &= \sum_{x_{k+1}} P(x_{k+1} | X_k) P(e_{k+1:t} | X_k, x_{k+1}) \\
&= \sum_{x_{k+1}} P(x_{k+1} | X_k) P(e_{k+1} | x_{k+1}) P(e_{k+2:t} | x_{k+1}) \\
\Leftrightarrow \quad b_k &= T^T O_{k+1} b_{k+1}
\end{aligned}$$

where $b_t = b_4 = P(\text{anything} | X_4) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$. Therefore, to calculate b_0 , we first need b_3 , b_2 and b_1 .

$$\begin{aligned}
b_3 &= T^T O_4 b_4 = \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 0.8 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.65 \\ 0.5 \\ 0.35 \end{pmatrix} \\
b_2 &= T^T O_3 b_3 = \dots \\
b_1 &= T^T O_2 b_2 = \dots \\
b_0 &= T^T O_1 b_1 = \dots \approx \begin{pmatrix} 0.076 & 0.055 & 0.035 \end{pmatrix}^T
\end{aligned}$$

Eventually,

$$\begin{aligned}
P(X_0 | e_{1:4}) &= \alpha b_0 \times f_0 \approx \begin{pmatrix} 0.457 & 0.331 & 0.212 \end{pmatrix}^T \\
P(X_1 | e_{1:4}) &= \alpha b_1 \times f_1 \approx \begin{pmatrix} 0.580 & 0.329 & 0.091 \end{pmatrix}^T
\end{aligned}$$

4. What is the most likely sequence of tossed coins?

The most likely sequence $x_{1:t}^*$ given the sequence of evidences $e_{1:t}$ is the one that satisfies

$$x_{1:t}^* = \arg \max_{x_{1:t}} P(x_{1:t} | e_{1:t})$$

or, equivalently,

$$\begin{aligned}
x_k^* &= \arg \max_{x_k} \left[\max_{x_{1:k-1}} P(x_{1:k}, x_{k+1:t}^* \mid e_{1:t}) \right] \\
&= \arg \max_{x_k} \left[\max_{x_{1:k-1}} \alpha P(x_{1:k}, x_{k+1:t}^*, e_{k+1:t} \mid e_{1:k}) \right] \\
&= \arg \max_{x_k} \left[\max_{x_{1:k-1}} \alpha P(e_{k+1:t} \mid x_{1:k}, x_{k+1:t}^*, e_{1:k}) P(x_{k+1:t}^* \mid x_{1:k}, e_{1:k}) P(x_{1:k} \mid e_{1:k}) \right] \\
&= \arg \max_{x_k} \left[\max_{x_{1:k-1}} \alpha P(e_{k+1:t} \mid x_{k+1:t}^*) P(x_{k+2:t}^* \mid x_{k+1}^*) P(x_{k+1:t}^* \mid x_k) P(x_{1:k} \mid e_{1:k}) \right] \\
&= \arg \max_{x_k} P(x_{k+1}^* \mid x_k) \left[\max_{x_{1:k-1}} P(x_{1:k} \mid e_{1:k}) \right] \\
&= \arg \max_{x_k} P(x_{k+1}^* \mid x_k) m_k(x_k)
\end{aligned}$$

where

$$\begin{aligned}
m_k &= \max_{x_{1:k-1}} P(x_{1:k-1}, X_k \mid e_{1:k}) \\
&= \max_{x_{1:k-1}} \alpha P(x_{1:k-1}, X_k, e_k \mid e_{1:k-1}) \\
&= \max_{x_{1:k-1}} \alpha P(e_t \mid x_{1:k-1}, X_t, e_{1:k-1}) P(X_k \mid x_{1:k-1}, e_{1:k-1}) P(x_{1:k-1} \mid e_{1:k-1}) \\
&= \max_{x_{1:k-1}} \alpha P(e_k \mid X_k) P(X_k \mid x_{k-1}) P(x_{1:k-1} \mid e_{1:k-1}) \\
&= \alpha P(e_k \mid X_k) \max_{x_{k-1}} P(X_k \mid x_{k-1}) \max_{x_{1:k-2}} P(x_{1:k-1} \mid e_{1:k-1}) \\
&= \alpha P(e_k \mid X_k) \max_{x_{k-1}} P(X_k \mid x_{k-1}) m_{k-1}(x_{k-1})
\end{aligned}$$

and $m_1 = P(X_1 \mid e_1) = f_1$. Therefore, we can iteratively build the vectors m_1, m_2, \dots, m_t to find the most likely last state

$$x_t^* = \arg \max_{x_t} m_t(x_t)$$

and, then, the most likely sequence with

$$x_k^* = \arg \max_{x_k} P(x_{k+1}^* \mid x_k) m_k(x_k).$$

Exercise 3 September 2019 (AIMA, Ex 15.13 and 15.14)

A professor wants to know if students are getting enough sleep. Each day, the professor observes whether the students sleep in class, and whether they have red eyes. The professor has the following hypotheses:

- The prior probability of getting enough sleep, with no observations, is 0.7.
- The probability of getting enough sleep on night t is 0.8 given that the student got enough sleep the previous night, and 0.3 otherwise.
- The probability of having red eyes is 0.2 if the student got enough sleep, and 0.7 otherwise.
- The probability of sleeping in class is 0.1 if the student got enough sleep, and 0.3 otherwise.

The professor asks you to answer the following questions:

1. Formulate the environment and hypotheses as a dynamic Bayesian network that the professor could use to detect sleep deprived students, from a sequence of observations. Provide the associated probability tables.
2. Reformulate the dynamic Bayesian network as a hidden Markov model that has only a single observation variable. Give the complete probability tables for the model.
3. For the sequence $e_{1:3}$ of observations “no red eyes, not sleeping in class”, “red eyes, not sleeping in class” and “red eyes, sleeping in class”, calculate the distributions $P(\textit{EnoughSleep}_t \mid e_{1:t})$ and $P(\textit{EnoughSleep}_t \mid e_{1:3})$ for $t \in \{1, 2, 3\}$.

Exercise 4 Hyperloop

A few years ago, ULiège decided to get into SpaceX's Hyperloop competition. Briefly, what one should do to win this competition is to build the fastest and most reliable autonomous pod. One of the most important engineering problem is to be able to compute a robust estimation of the state of the pod, *i.e.* its position p (m), speed \dot{p} (m s^{-1}) and acceleration \ddot{p} (m s^{-2}), given noisy sensor measurements. You received an email from students asking you what would be your solution to this estimation problem. After some research you find out that you should use a Kalman filter to solve this task. Define the components of your Kalman filter in the context of the state estimation of the pod.

The email contains information about the competition and the sensors placed on the pod. The track is straight (one-dimensional) and the pod has $T = 60\text{ s}$ to go as far as possible. The initial position and speed are assumed to be centered around 0 with a standard deviation of 0.1 m and 0.01 m s^{-1} , respectively. The time between sensor measurements is $\Delta t = 1\text{ s}$. The sensors placed on the pod are 1) an unbiased GPS sensor providing the pod's position with 68 % chance to be less than 100 m away from the true position and 2) an unbiased accelerometer measuring the pod's acceleration with 68 % chance to be less than 1 m s^{-2} away from the true acceleration.

The acceleration \ddot{p} is the result of the combination of thrust and drag. The thrust is assumed to be normally distributed around μ_t with standard deviation σ_t . The drag is proportional but opposite in direction to the speed, *i.e.* it takes the form $-\eta \dot{p}$.

The Kalman filter is a special case of the continuous Bayes filter, which assumes

- a Gaussian prior

$$p(x_0) = \mathcal{N}(\mu_0, \Sigma_0),$$

- a linear Gaussian transition model

$$p(x_t | x_{t-1}) = \mathcal{N}(Ax_{t-1} + b, \Sigma_x)$$

- and a linear Gaussian sensor model

$$p(e_t | x_t) = \mathcal{N}(Cx_t + d, \Sigma_e).$$

In our case, the state x is a vector $\begin{pmatrix} p & \dot{p} & \ddot{p} \end{pmatrix}^T$ and according to the provided information, the prior is defined by

$$\mu_0 = \begin{pmatrix} 0 \\ 0 \\ \mu_a \end{pmatrix} \quad \text{and} \quad \Sigma_0 = \begin{pmatrix} 0.1^2 & 0 & 0 \\ 0 & 0.01^2 & 0 \\ 0 & 0 & \sigma_a^2 \end{pmatrix}.$$

Concerning the transition, we can use what we learned in Newtonian mechanics¹:

$$\begin{aligned} p_t &= p_{t-1} + \Delta t \dot{p}_{t-1} + \frac{1}{2} \Delta t^2 \ddot{p}_{t-1} \\ \dot{p}_t &= \dot{p}_{t-1} + \Delta t \ddot{p}_{t-1} \end{aligned}$$

As mentioned in the statement, the acceleration combines thrust and drag, which results in the distribution

$$\begin{aligned} \ddot{p}_t &\sim \mathcal{N}(\mu_t - \eta \dot{p}_t, \sigma_t) \\ &\sim \mathcal{N}(\mu_t - \eta (\dot{p}_{t-1} + \ddot{p}_{t-1} \Delta t), \sigma_t). \end{aligned}$$

¹Finally!

Then, the transition model² is defined by

$$A = \begin{pmatrix} 1 & \Delta t & \frac{1}{2}\Delta t^2 \\ 0 & 1 & \Delta t \\ 0 & -\eta & -\eta\Delta t \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ \mu_a \end{pmatrix} \quad \text{and} \quad \Sigma_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_a^2 \end{pmatrix}.$$

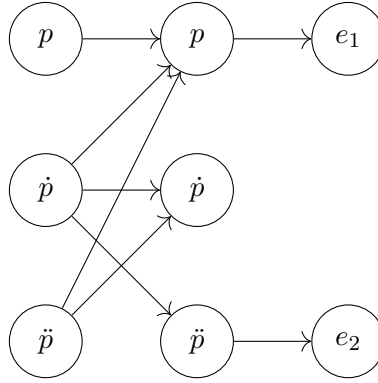
There are one position sensor (e_1) and one accelerometer (e_2), both unbiased, which translates to

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We also know that 68 % corresponds to the 1-sigma confidence interval of a one-dimensional Gaussian distribution and we can assume that sensors are independent. Therefore,

$$\Sigma_e = \begin{pmatrix} 100^2 & 0 \\ 0 & 1^2 \end{pmatrix}.$$

The discrete-time process can be represented as a *dynamic Bayesian network*.



²A model only approximates the reality. It could be inaccurate. For instance, at high speed, drag is usually proportional to the square of the speed, which our model does not take into account.

Supplementary materials

- Hidden Markov Models (UC Berkeley CS188, Spring 2014 Section 6).



- Hyperloop Pod Competition



- 68–95–99.7 rule



- Chapter 15 of the reference textbook.