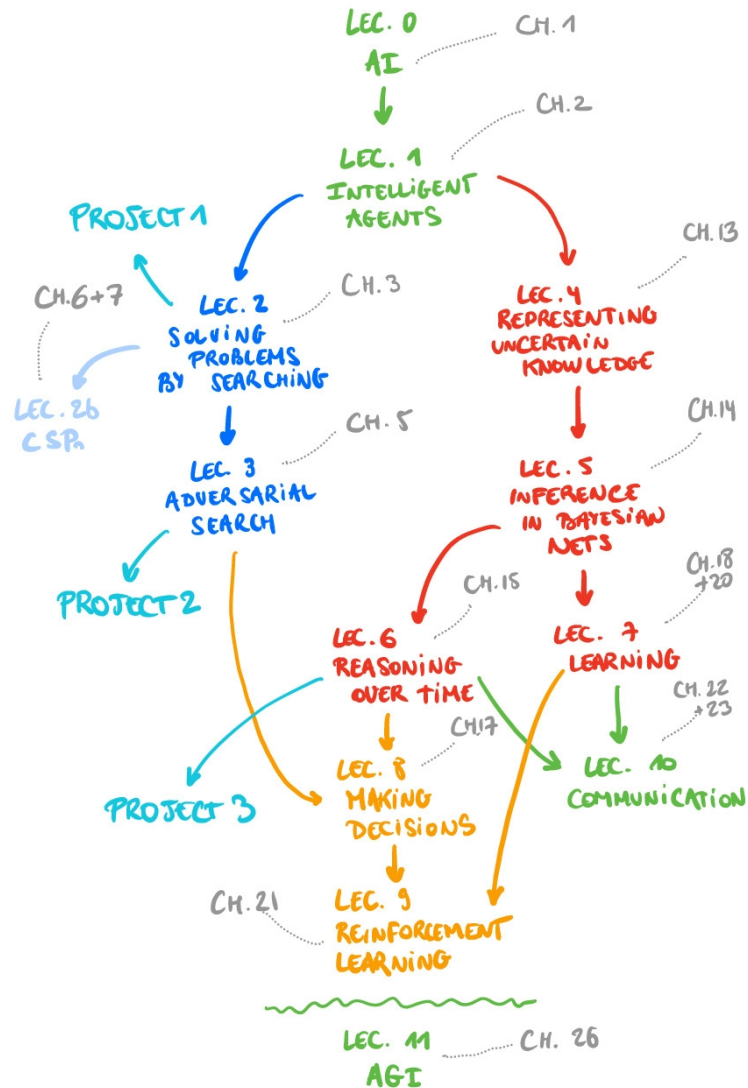


# Introduction to Artificial Intelligence

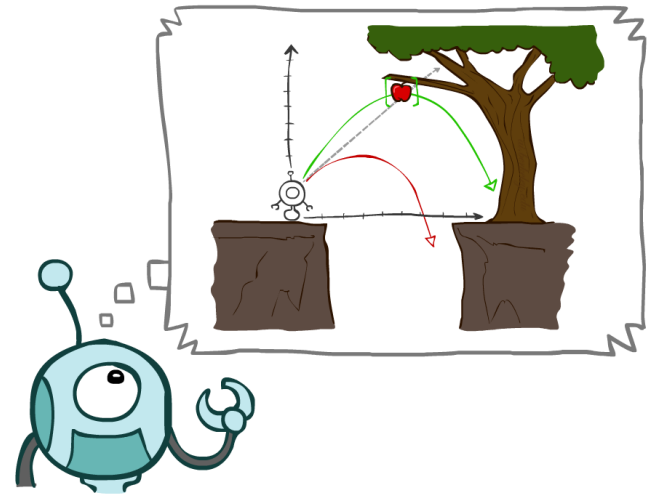
Lecture 2: Solving problems by searching

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# Today

- Planning agents
- Search problems
- Uninformed search methods
  - Depth-first search
  - Breadth-first search
  - Uniform-cost search
- Informed search methods
  - A\*
  - Heuristics

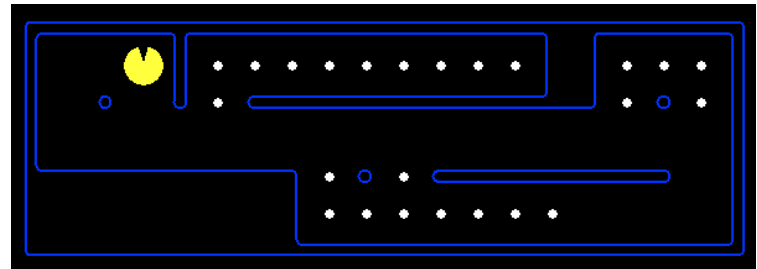
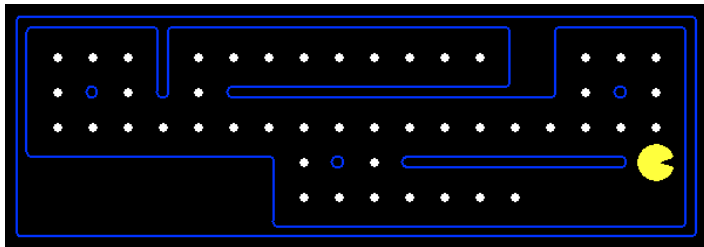


# Planning agents

# Reflex agents

## Reflex agents

- select actions on the basis of the current percept;
- may have a model of the world current state;
- do not consider the future consequences of their actions;
- consider only **how the world is now**.



*For example, a simple reflex agent based on condition-action rules could move to a dot if there is one in its neighborhood. No planning is involved to take this decision.*

[Q] Can a reflex agent be rational?

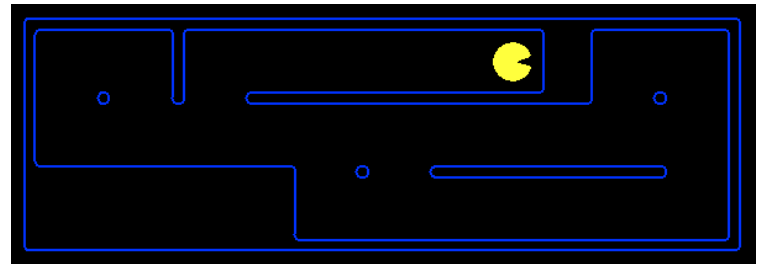
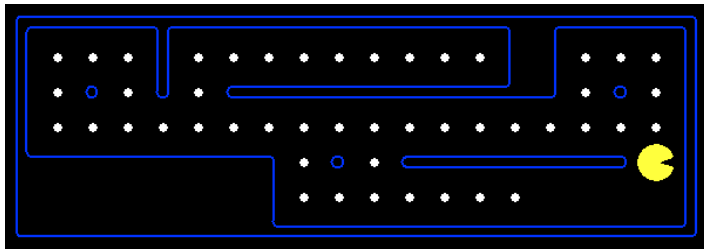
# Problem-solving agents

Assumptions:

- Observable, deterministic (and known) environment.

Problem-solving agents

- take decisions based on (hypothesized) consequences of actions;
- must have a model of how the world evolves in response to actions;
- formulate a goal, explicitly;
- consider how the world would be.



*A planning agent looks for sequences of actions to eat all the dots.*

**function** SIMPLE-PROBLEM-SOLVING-AGENT(*percept*) **returns** an action

**persistent:** *seq*, an action sequence, initially empty

*state*, some description of the current world state

*goal*, a goal, initially null

*problem*, a problem formulation

*state*  $\leftarrow$  UPDATE-STATE(*state*, *percept*)

**if** *seq* is empty **then**

*goal*  $\leftarrow$  FORMULATE-GOAL(*state*)

*problem*  $\leftarrow$  FORMULATE-PROBLEM(*state*, *goal*)

*seq*  $\leftarrow$  SEARCH(*problem*)

**if** *seq* = *failure* **then return** a null action

*action*  $\leftarrow$  FIRST(*seq*)

*seq*  $\leftarrow$  REST(*seq*)

**return** *action*

---

**Figure 3.1** A simple problem-solving agent. It first formulates a goal and a problem, searches for a sequence of actions that would solve the problem, and then executes the actions one at a time. When this is complete, it formulates another goal and starts over.

## Offline vs. Online solving

- Problem-solving agents are **offline**. The solution is executed "eyes closed", ignoring the percepts.
- **Online** problem solving involves acting without complete knowledge. In this case, the sequence of actions might be recomputed at each step.

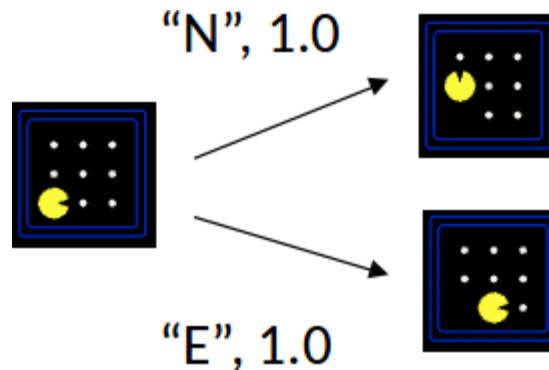


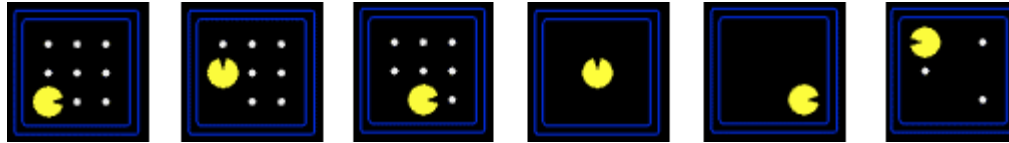
# Search problems

# Search problems

A **search problem** consists of the following components:

- The **initial state** of the agent.
- A description of the **actions** available to the agent given a state  $s$ , denoted  $\text{actions}(s)$ .
- A **transition model** that returns the state  $s' = \text{result}(s, a)$  that results from doing action  $a$  in state  $s$ .
  - We say that  $s'$  is a **successor** of  $s$  if there is an acceptable action from  $s$  to  $s'$ .





- Together, the initial state, the actions and the transition model define the **state space** of the problem, i.e. the set of all states reachable from the initial state by any sequence of action.
  - The state space forms a directed graph:
    - nodes = states
    - links = actions
  - A path is a sequence of states connected by actions.
- A **goal test** which determines whether the solution of the problem is achieved in state  $s$ .
- A **path cost** that assigns a numeric value to each path.
  - In this course, we will also assume that the path cost corresponds to a sum of positive **step costs**  $c(s, a, s')$  associated to the action  $a$  in  $s$  leading to  $s'$ .

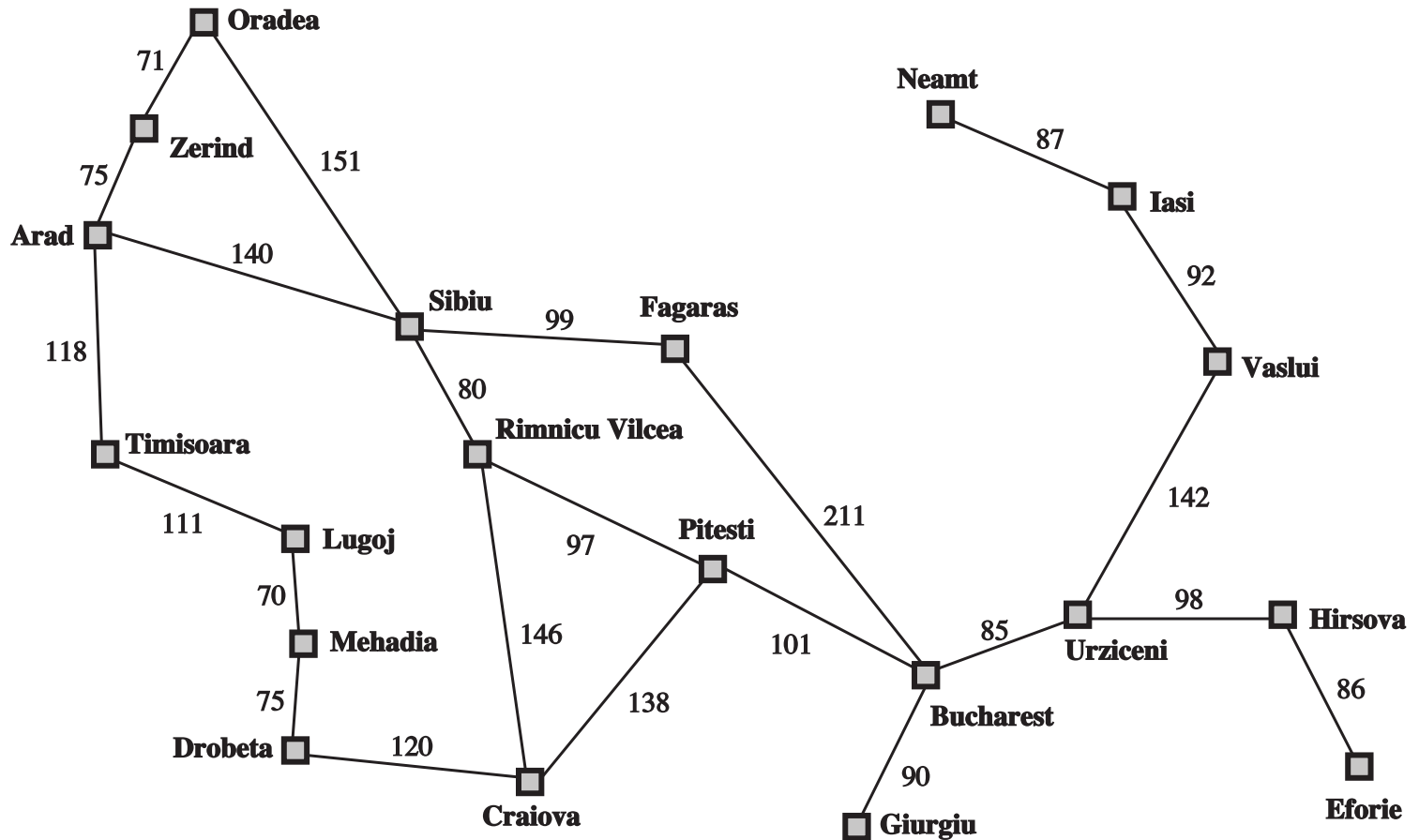
A **solution** to a problem is an action sequence that leads from the initial state to a goal state.

- A solution quality is measured by the path cost function.
- An **optimal solution** has the lowest path cost among all solutions.

### **Exercise**

What if the environment is partially observable? non-deterministic?

## Example: Traveling in Romania



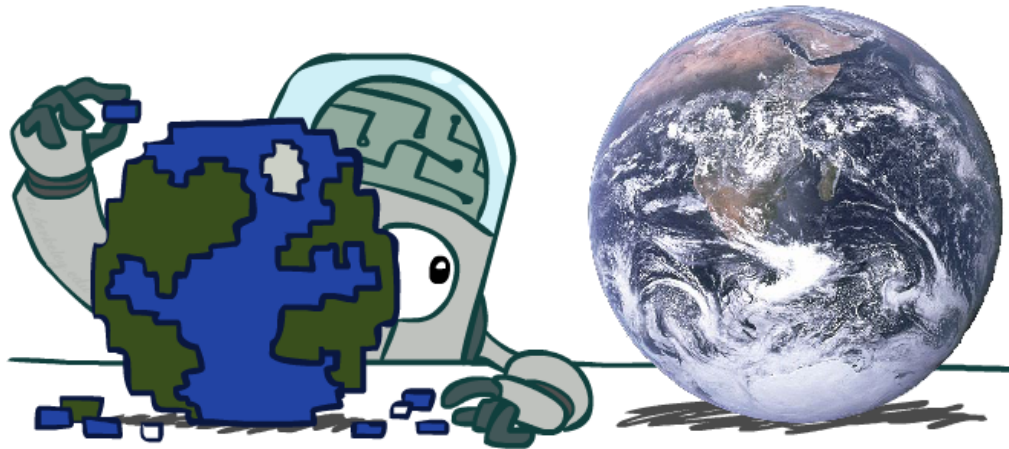
*How to go from Arad to Bucharest?*

- Initial state = the city we start in.
  - $s_0 = \text{in}(\text{Arad})$
- Actions = Going from the current city to the cities that are directly connected to it.
  - $\text{actions}(s_0) = \{\text{go}(\text{Sibiu}), \text{go}(\text{Timisoara}), \text{go}(\text{Zerind})\}$
- Transition model = The city we arrive in after driving to it.
  - $\text{result}(\text{in}(\text{Arad}), \text{go}(\text{Zerind})) = \text{in}(\text{Zerind})$
- Goal test: whether we are in Bucharest.
  - $s \in \{\text{in}(\text{Bucharest})\}$
- Step cost: distances between cities.

# Selecting a state space

The real world is absurdly **complex**.

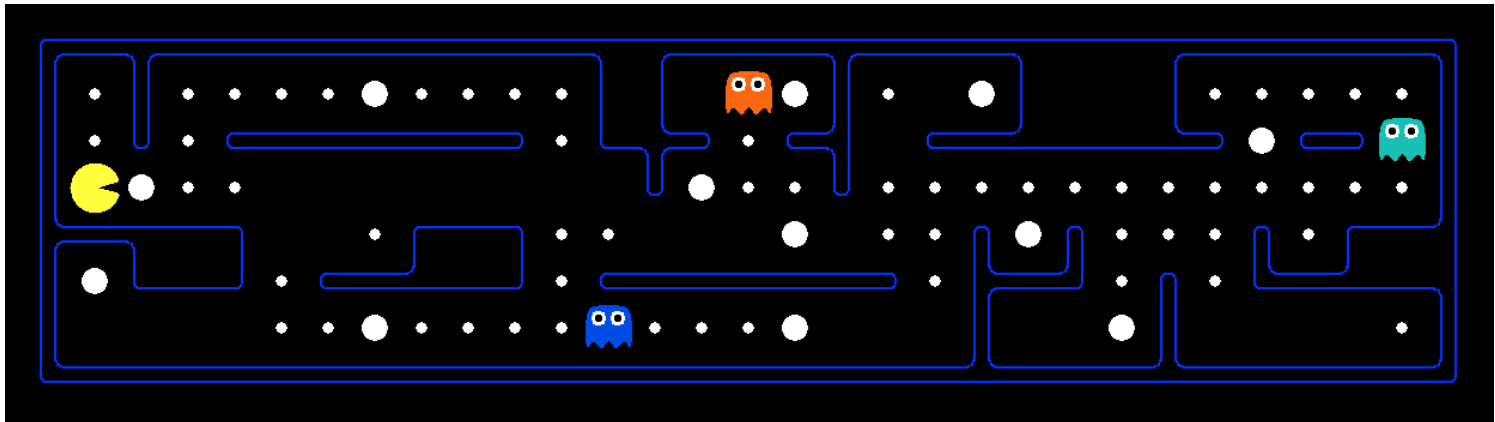
- The **world state** includes every last detail of the environment.
- A **search state** keeps only the details needed for planning.



Search problems are **models**.

## Example: eat-all-dots

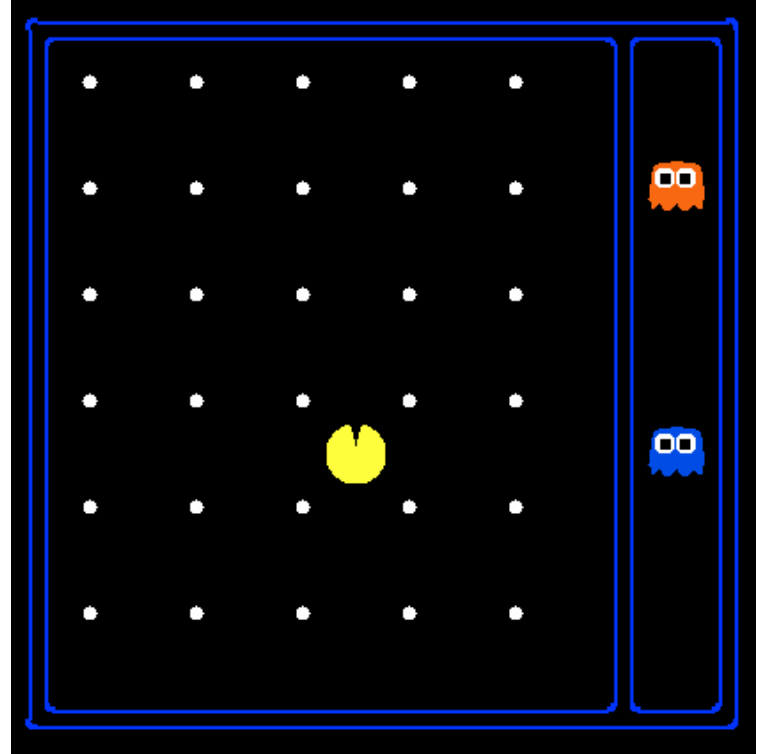
- States:  $\{(x, y), \text{dot booleans}\}$
- Actions: NSEW
- Transition: update location and possibly a dot boolean
- Goal test: dots all false





## State space size

- World state:
  - Agent positions: 120
  - Found count: 30
  - Ghost positions: 12
  - Agent facing: NSEW
- How many?
  - World states?
    - $120 \times 2^{30} \times 12^2 \times 4$
  - States for eat-all-dots?
    - $120 \times 2^{30}$

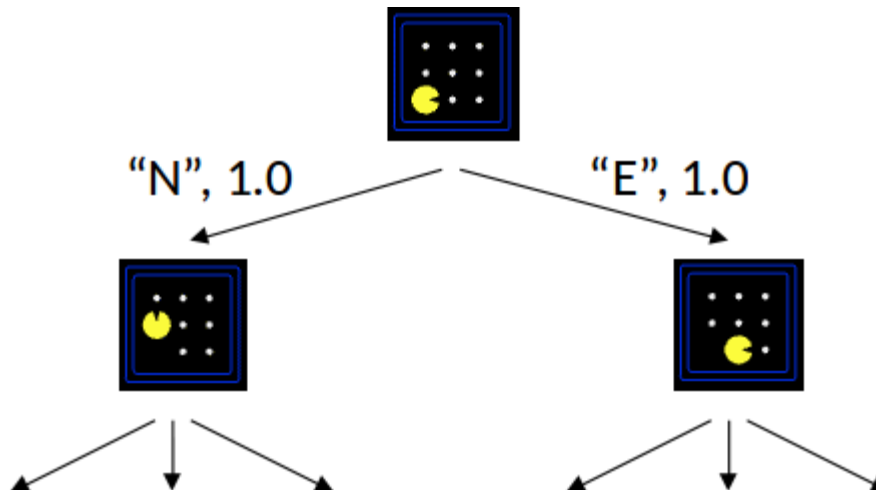


# Search trees

The set of acceptable sequences starting at the initial state form a **search tree**.

- Nodes correspond to states in the state space, where the initial state is the root node.
- Branches correspond to applicable actions, with child nodes corresponding to successors.

For most problems, we can never actually build the whole tree. Yet we want to find some optimal branch!



# Tree search algorithms

```
function TREE-SEARCH(problem, fringe) returns a solution, or failure
  fringe ← INSERT(MAKE-NODE(INITIAL-STATE[problem]), fringe)
  loop do
    if fringe is empty then return failure
    node ← REMOVE-FRONT(fringe)
    if GOAL-TEST(problem, STATE(node)) then return node
    fringe ← INSERTALL(EXPAND(node, problem), fringe)
```

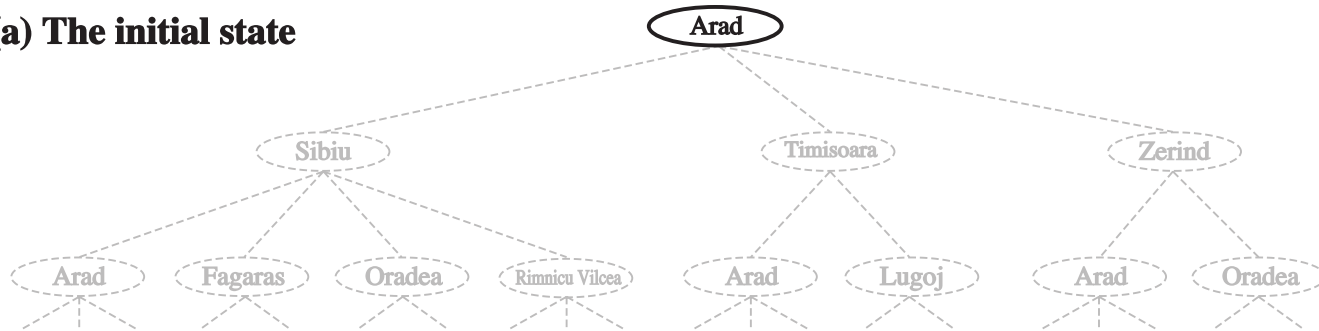
## Important ideas

- Fringe (or frontier) of partial plans under consideration
- Expansion
- Exploration

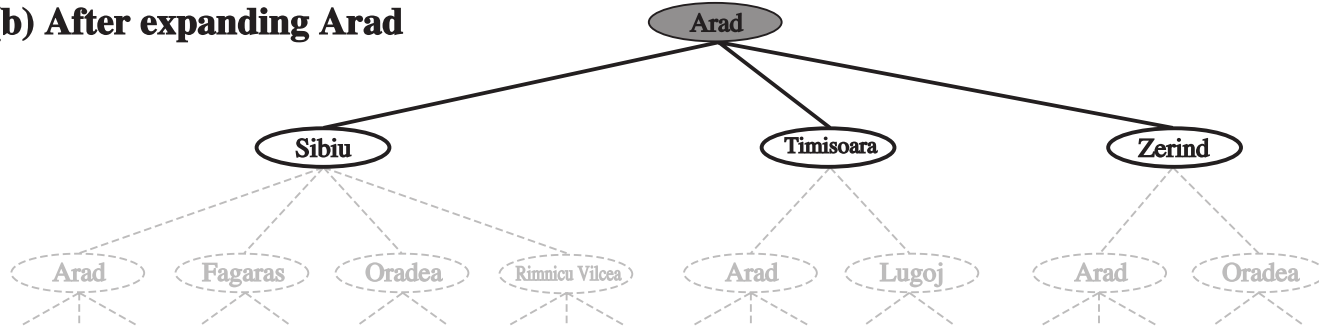
**Exercise**

Which fringe nodes to explore? How to expand as few nodes as possible, while achieving the goal?

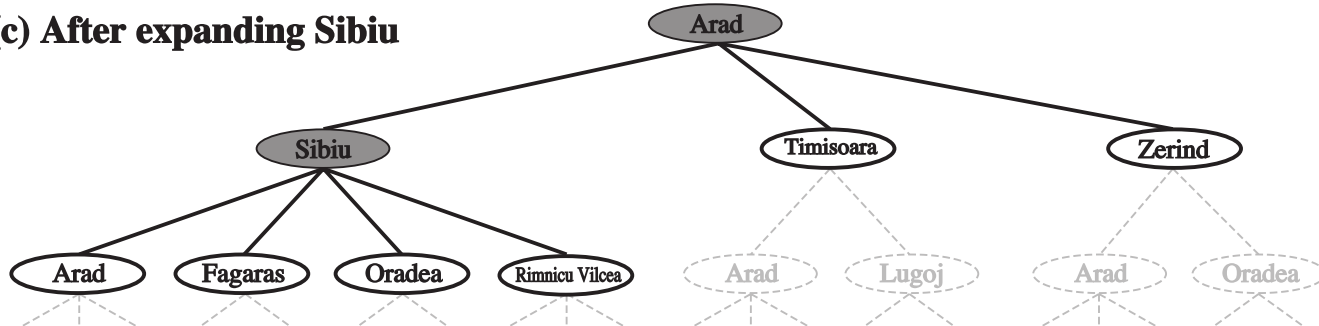
**(a) The initial state**



**(b) After expanding Arad**



**(c) After expanding Sibiu**



# Uninformed search strategies

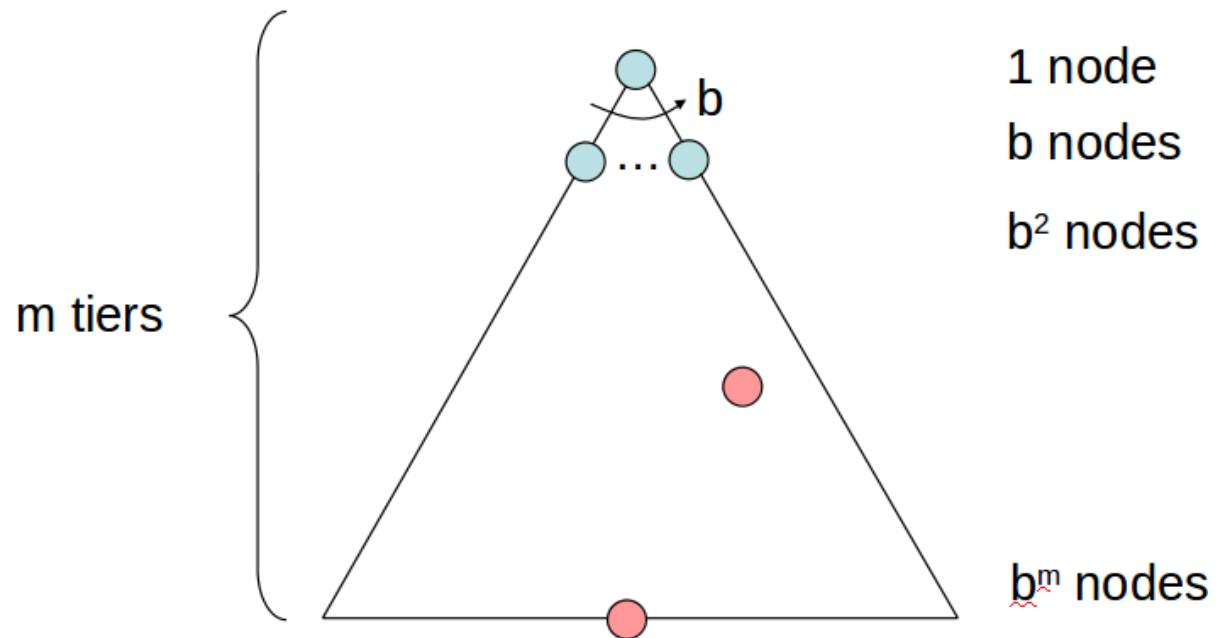
**Uninformed** search strategies use only the information available in the problem definition. They do not know whether a state looks more promising than some other.

## Strategies

- Depth-first search
- Breadth-first search
- Uniform-cost search
- Iterative deepening

# Properties of search strategies

- A strategy is defined by picking the **order of expansion**.
- Strategies are evaluated along the following dimensions:
  - **Completeness**: does it always find a solution if one exists?
  - **Optimality**: does it always find the least-cost solution?
  - **Time complexity**: how long does it take to find a solution?
  - **Space complexity**: how much memory is needed to perform the search?
- Time and complexity are measured in terms of
  - $b$ : maximum branching factor of the search tree
  - $d$ : depth of the least-cost solution
    - the depth of  $s$  is defined as the number of actions from the initial state to  $s$ .
  - $m$ : maximum length of any path in the state space (may be  $\infty$ )

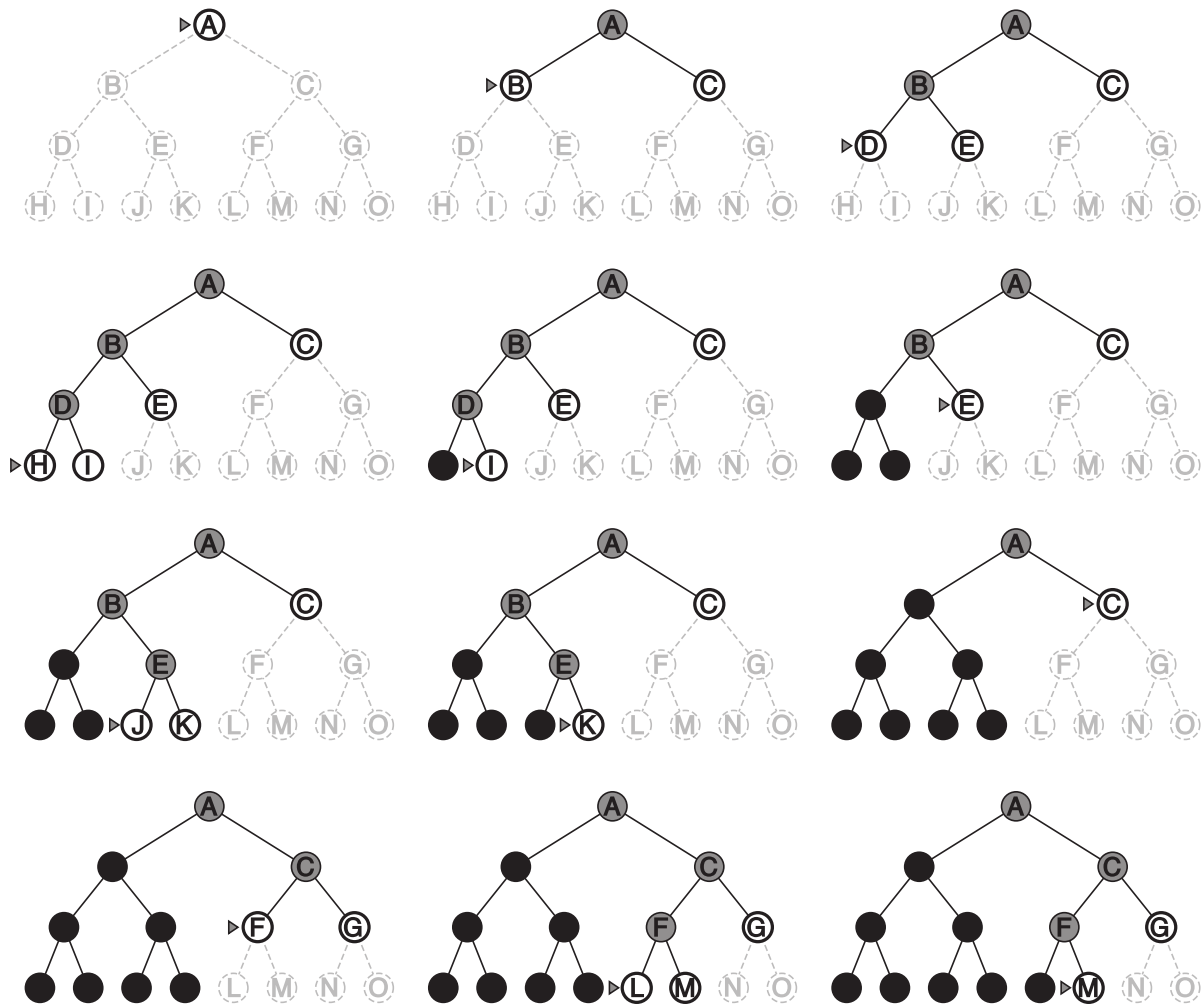


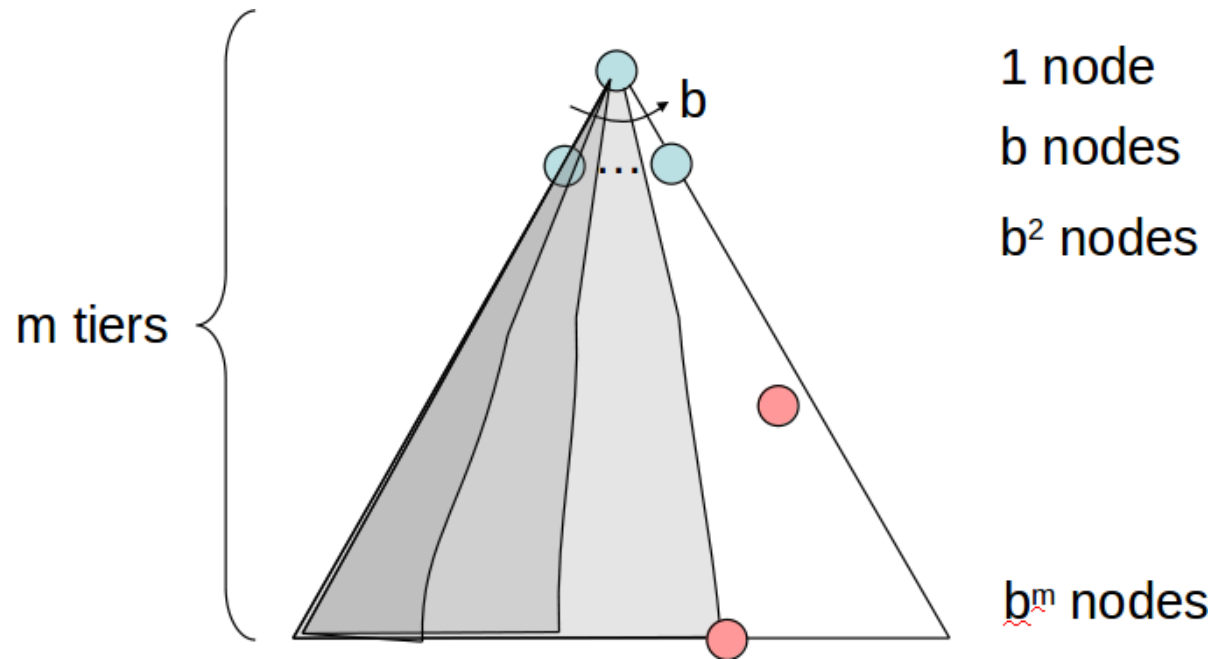


# Depth-first search



- **Strategy:** expand the deepest node in the fringe.
- **Implementation:** fringe is a **LIFO stack**.





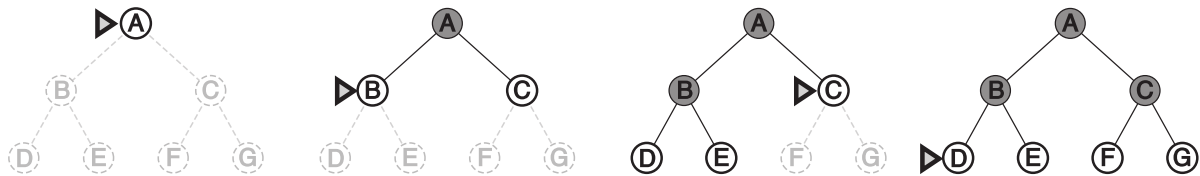
## Properties of DFS

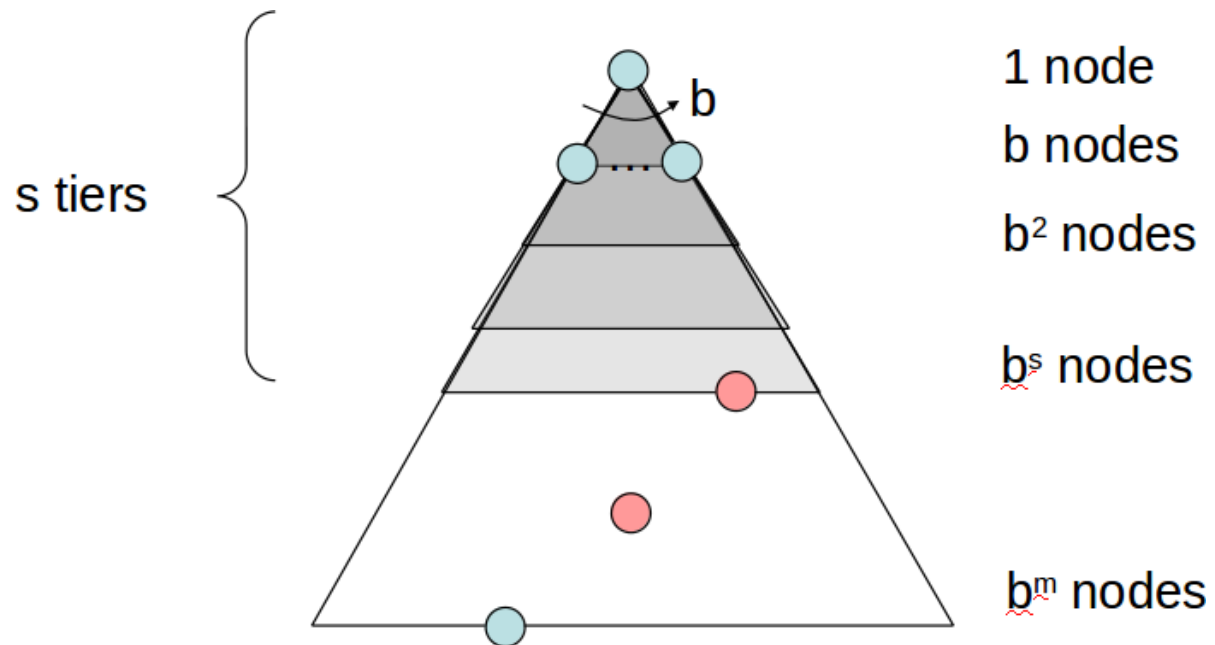
- **Completeness:**
  - $m$  could be infinite, so only if we prevent cycles (more on this later).
- **Optimality:**
  - No, DFS finds the leftmost solution, regardless of depth or cost.
- **Time complexity:**
  - May generate the whole tree (or a good part of it, regardless of  $d$ ). Therefore  $O(b^m)$ , which might be much greater than the size of the state space!
- **Space complexity:**
  - Only store siblings on path to root, therefore  $O(bm)$ .
  - When all the descendants of a node have been visited, the node can be removed from memory.

# Breadth-first search



- **Strategy:** expand the shallowest node in the fringe.
- **Implementation:** fringe is a **FIFO queue**.





## Properties of BFS

- **Completeness:**
  - If the shallowest goal node is at some finite depth  $d$ , BFS will eventually find it after generating all shallower nodes (provided  $b$  is finite).
- **Optimality:**
  - The shallowest goal is not necessarily the optimal one.
  - BFS is optimal only if the path cost is a non-decreasing function of the depth of the node.
- **Time complexity:**
  - If the solution is a depth  $d$ , then the total number of nodes generated before finding this node is  $b + b^2 + b^3 + \dots + b^d = O(b^d)$
- **Space complexity:**
  - The number of nodes to maintain in memory is the size of the fringe, which will be the largest at the last tier. That is  $O(b^d)$



Demo comparing DFS against BFS.

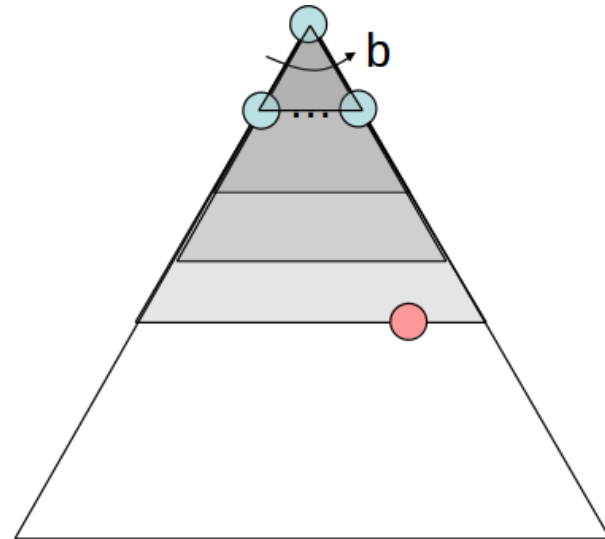
# Iterative deepening

Idea: get DFS's space advantages with BFS's time/shallow solution advantages.

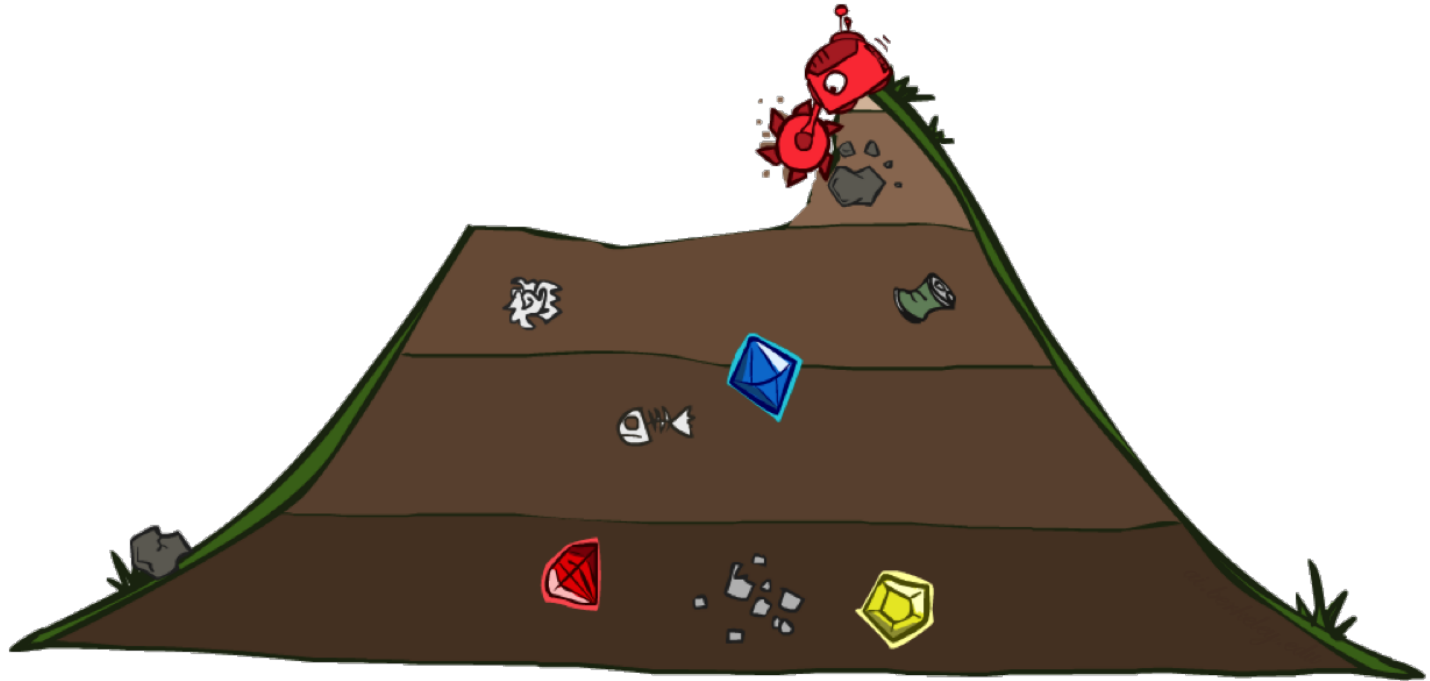
- Run DFS with depth limit 1.
- If no solution, run DFS with depth limit 2.
- If no solution, run DFS with depth limit 3.
- ...

## Exercise

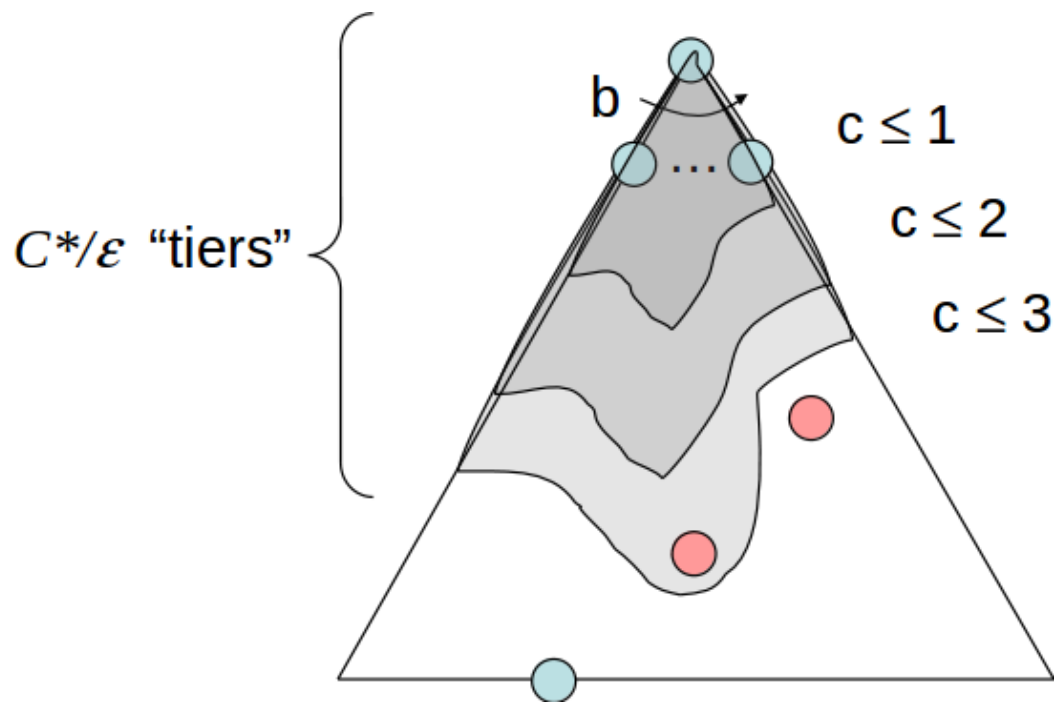
- What are the properties of iterative deepening?
- Isn't this process wastefully redundant?



# Uniform-cost search



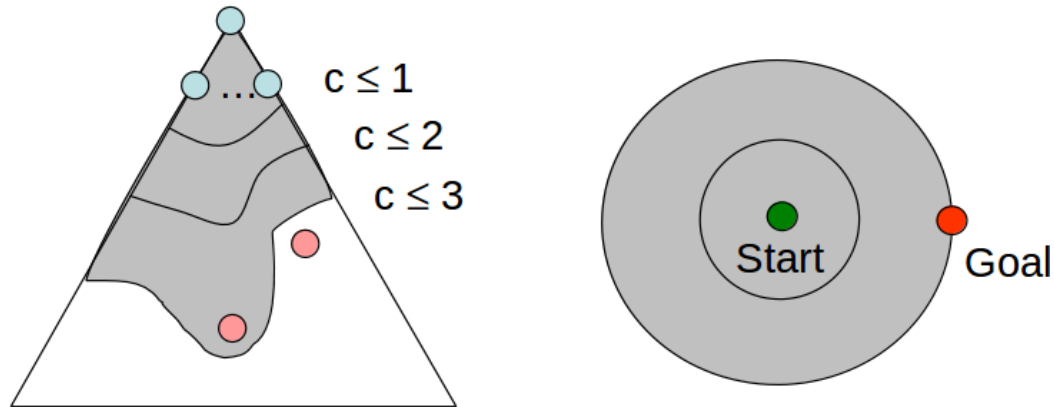
- **Strategy**: expand the cheapest node in the fringe.
- **Implementation**: fringe is a **priority queue**, using the cumulative cost  $g(n)$  from the initial state to node  $n$  as priority.



## Properties of UCS

- **Completeness:**
  - Yes, if step cost are all such that  $c(s, a, s') \geq \epsilon > 0$ . (Why?)
- **Optimality:**
  - Yes, since UCS expands nodes in order of their optimal path cost.
- **Time complexity:**
  - Assume  $C^*$  is the cost of the optimal solution and that step costs are all  $\geq \epsilon$ .
  - The "effective depth" is then roughly  $C^*/\epsilon$ .
  - The worst-case time complexity is  $O(b^{C^*/\epsilon})$ .
- **Space complexity:**
  - The number of nodes to maintain is the size of the fringe, so as many as in the last tier  $O(b^{C^*/\epsilon})$ .

# Informed search strategies



One of the **issues of UCS** is that it explores the state space in **every direction**, without exploiting information about the (plausible) location of the goal node.

**Informed** search strategies aim to solve this problem by expanding nodes in the fringe in decreasing order of **desirability**.

- Greedy search
- $A^*$

# Greedy search

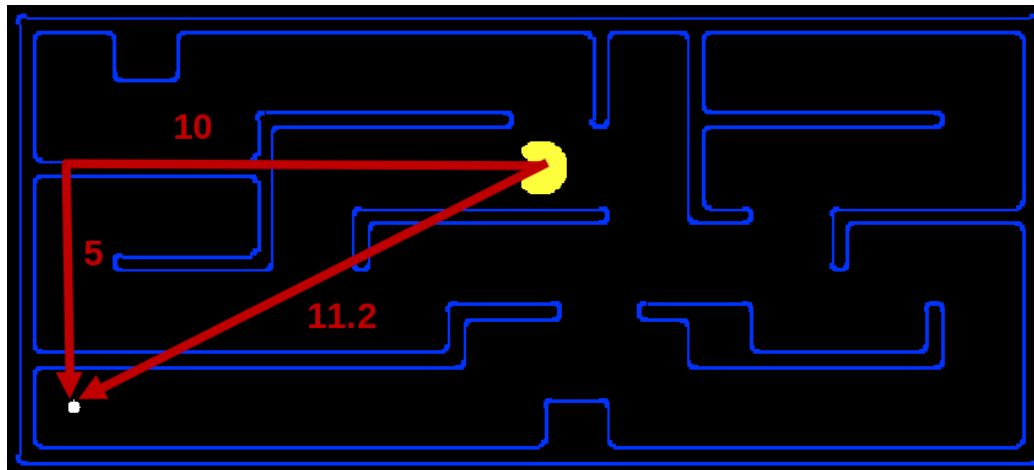




# Heuristics

A **heuristic** (or evaluation) function  $h(n)$  is:

- a function that **estimates** the cost of the cheapest path from node  $n$  to a goal state;
  - $h(n) \geq 0$  for all nodes  $n$
  - $h(n) = 0$  for a goal state.
- is designed for a **particular** search problem.



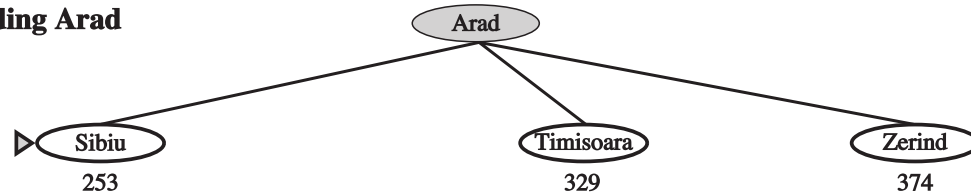
## Greedy search

- **Strategy**: expand the node  $n$  in the fringe for which  $h(n)$  is the lowest.
- **Implementation**: fringe is a **priority queue**, using  $h(n)$  as priority.

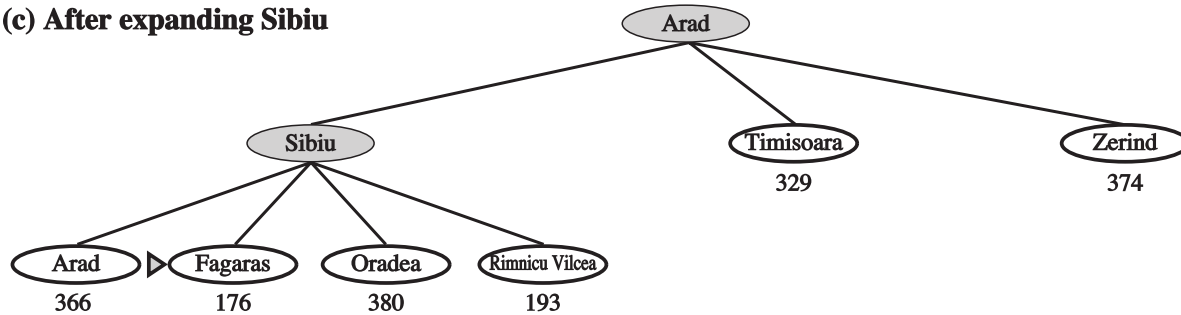
(a) The initial state



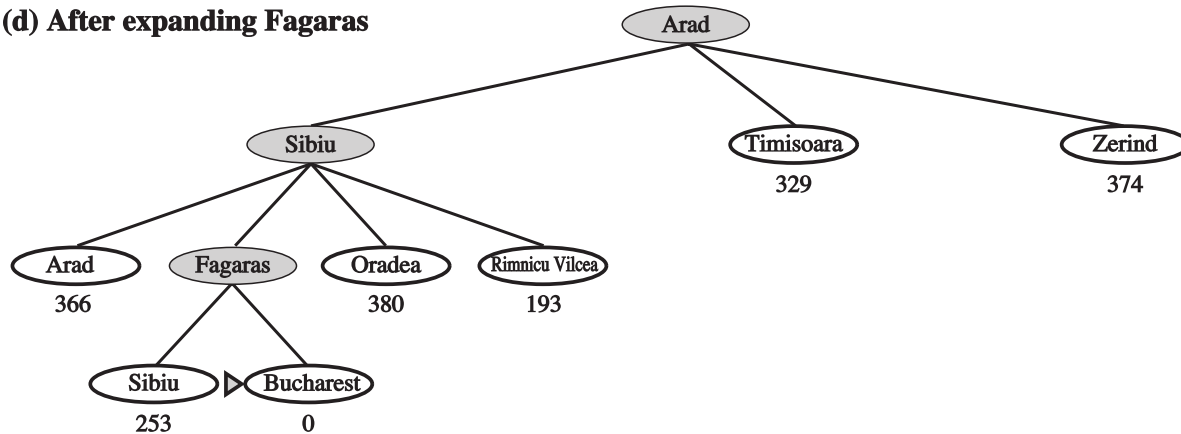
(b) After expanding Arad



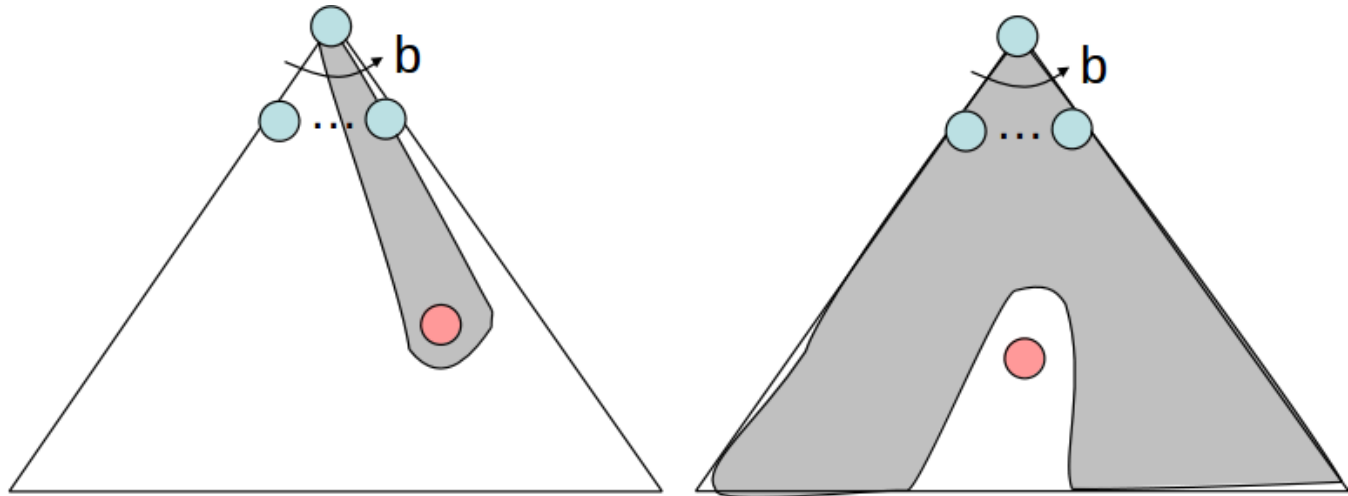
(c) After expanding Sibiu



(d) After expanding Fagaras



$h(n)$  = straight line distance to Bucharest.



At best, greedy search takes you straight to the goal.  
At worst, it is like a badly-guided BFS.

## Properties of greedy search

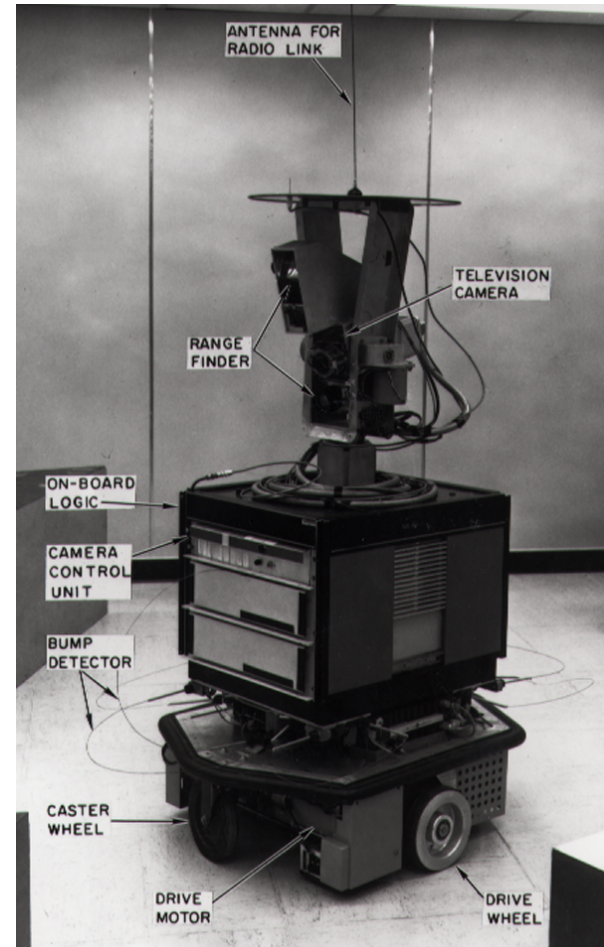
- **Completeness:**
  - No, unless we prevent cycles (more on this later).
- **Optimality:**
  - No, e.g. the path via Sibiu and Fagaras is 32km longer than the path through Rimnicu Vilcea and Pitesti.
- **Time complexity:**
  - $O(b^m)$ , unless we have a good heuristic function.
- **Space complexity:**
  - $O(b^m)$ , unless we have a good heuristic function.

A\*



## Shakey the Robot

- A\* was first proposed in 1968 to improve robot planning.
- Goal was to navigate through a room with obstacles.



## A\*

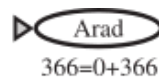
- Uniform-cost orders by path cost, or **backward cost**  $g(n)$
- Greedy orders by goal proximity, or **forward cost**  $h(n)$
- **A\*** combines the two algorithms and orders by the sum

$$f(n) = g(n) + h(n)$$

- $f(n)$  is the estimated cost of cheapest solution through  $n$ .



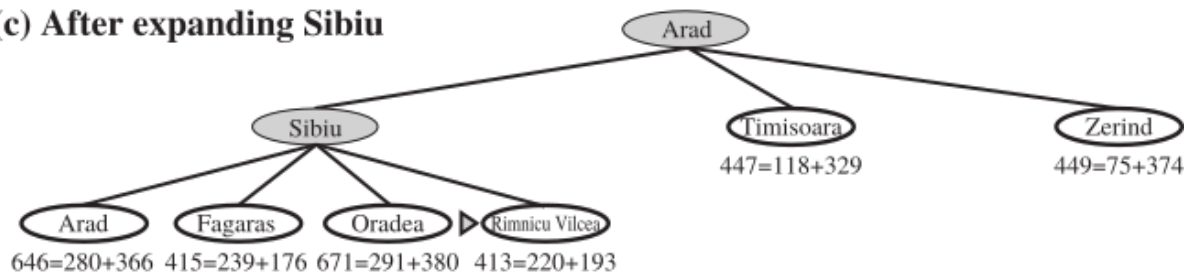
(a) The initial state



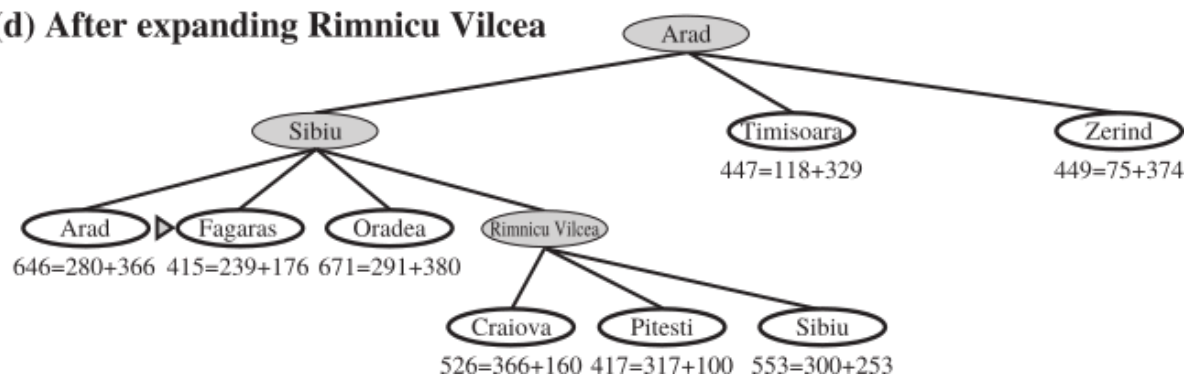
(b) After expanding Arad



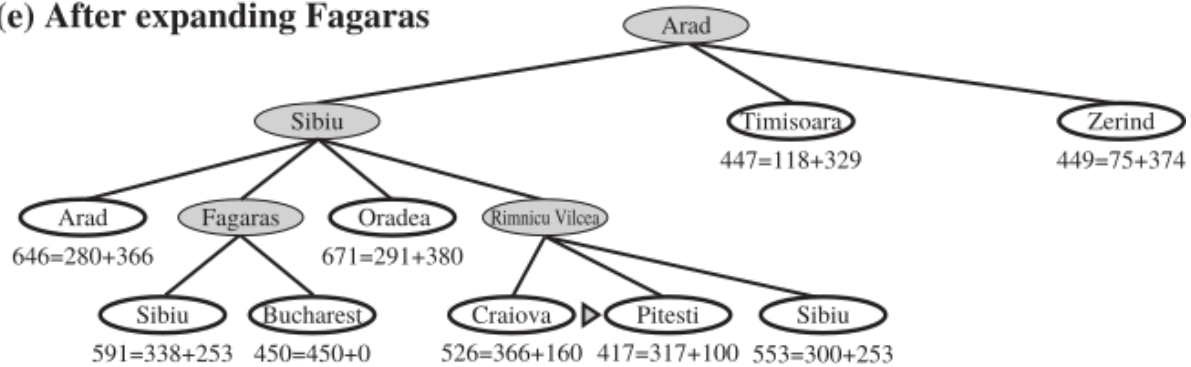
(c) After expanding Sibiu



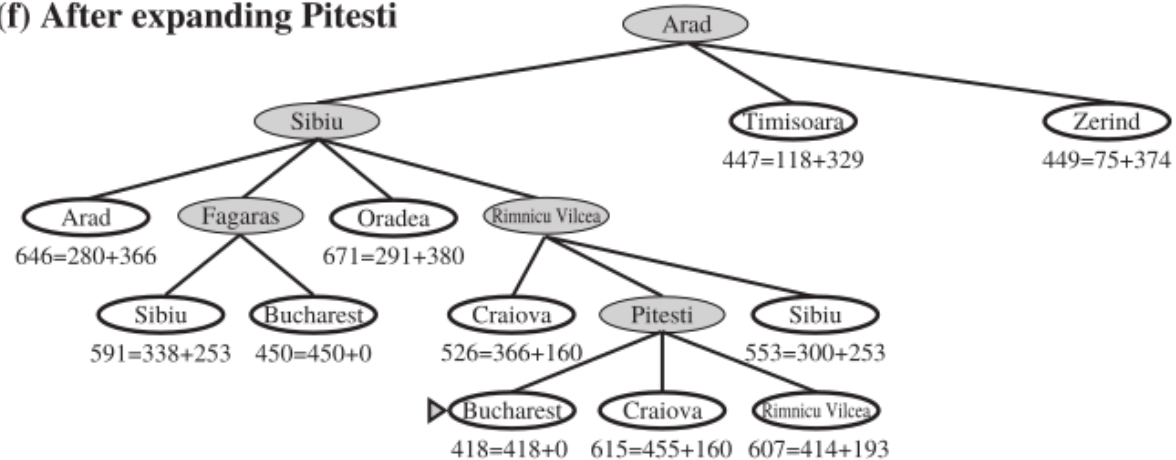
(d) After expanding Rimnicu Vilcea



(e) After expanding Fagaras



(f) After expanding Pitesti



## Exercise

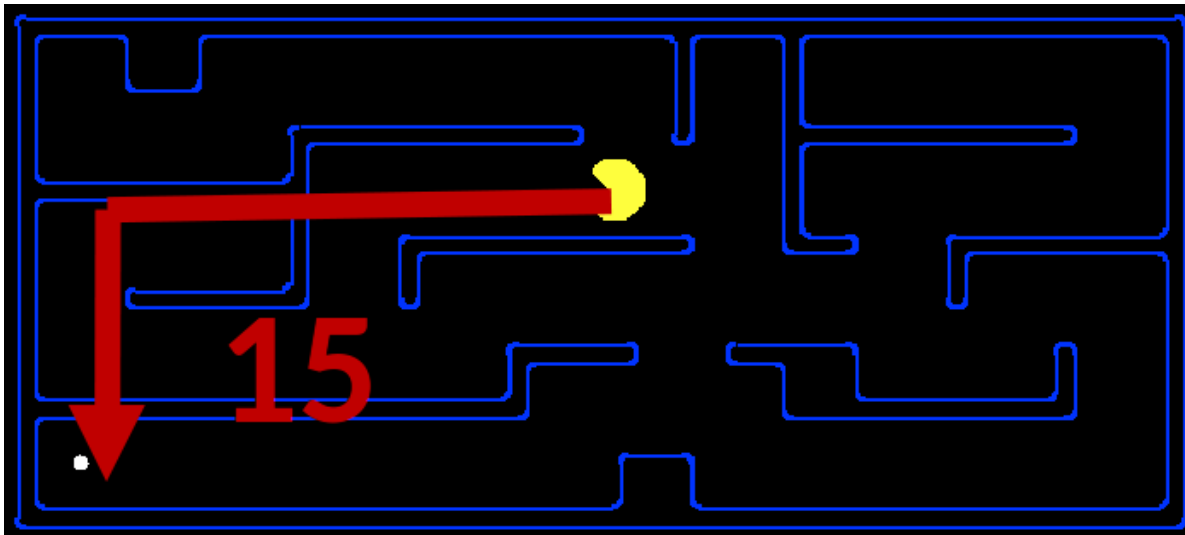
Why doesn't A\* stop at step (e), since Bucharest is in the fringe?

## Admissible heuristics

A heuristic  $h$  is **admissible** if

$$0 \leq h(n) \leq h^*(n)$$

where  $h^*(n)$  is the true cost to a nearest goal.



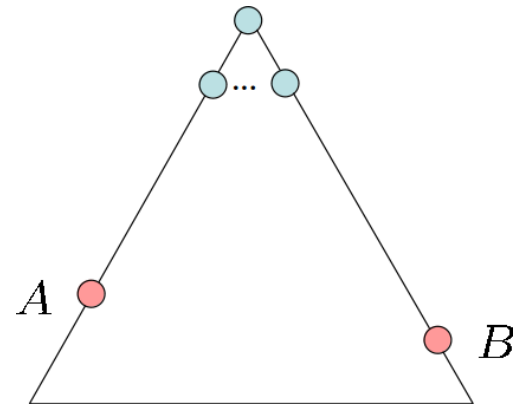
*The Manhattan distance is admissible*

## Optimality of A\*

Assumptions:

- $A$  is an optimal goal node
- $B$  is a suboptimal goal node
- $h$  is admissible

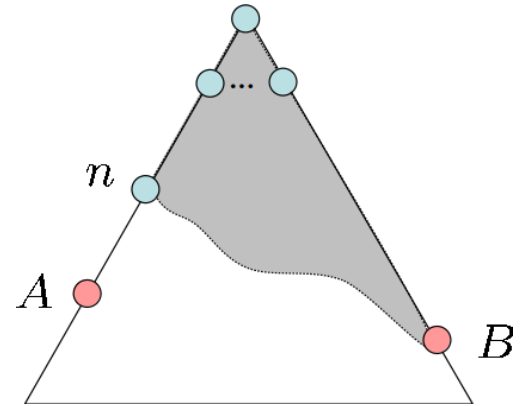
Claim:  $A$  will exit the fringe before  $B$ .



## Proof

Assume  $B$  is on the fringe. Some ancestor  $n$  of  $A$  is on the fringe too.

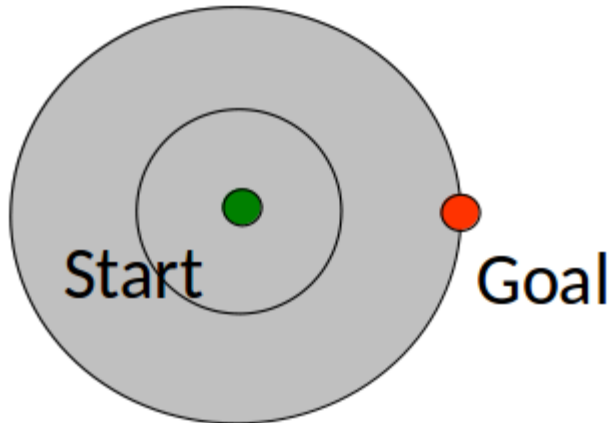
- $f(n) \leq f(A)$ 
  - $f(n) = g(n) + h(n)$  (by definition)
  - $f(n) \leq g(A)$  (admissibility of  $h$ )
  - $f(A) = g(A) + h(A) = g(A)$  ( $h = 0$  at a goal)
- $f(A) < f(B)$ 
  - $g(A) < g(B)$  ( $B$  is suboptimal)
  - $f(A) < f(B)$  ( $h = 0$  at a goal)
- Therefore,  $n$  expands before  $B$ .
  - since  $f(n) \leq f(A) < f(B)$



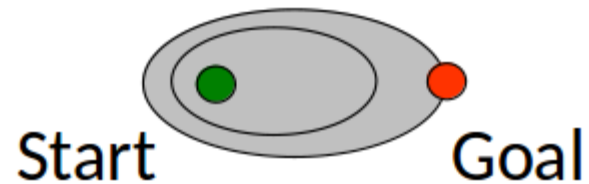
Similarly, all ancestors of  $A$  expand before  $B$ , including  $A$ . Therefore  $A^*$  is optimal.

## A\* contours

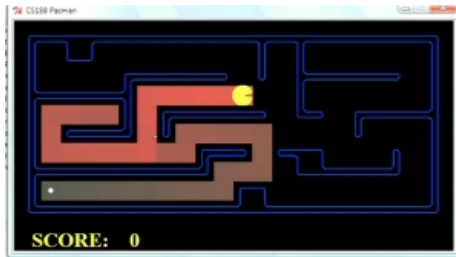
- Assume  $f$ -costs are non-decreasing along any path.
- We can define **contour levels**  $t$  in the state space, that include all nodes  $n$  for which  $f(n) \leq t$ .



For UCS ( $h(n) = 0$  for all  $n$ ), bands are circular around the start.



For A\* with accurate heuristics, bands stretch towards the goal.



Greedy search



UCS

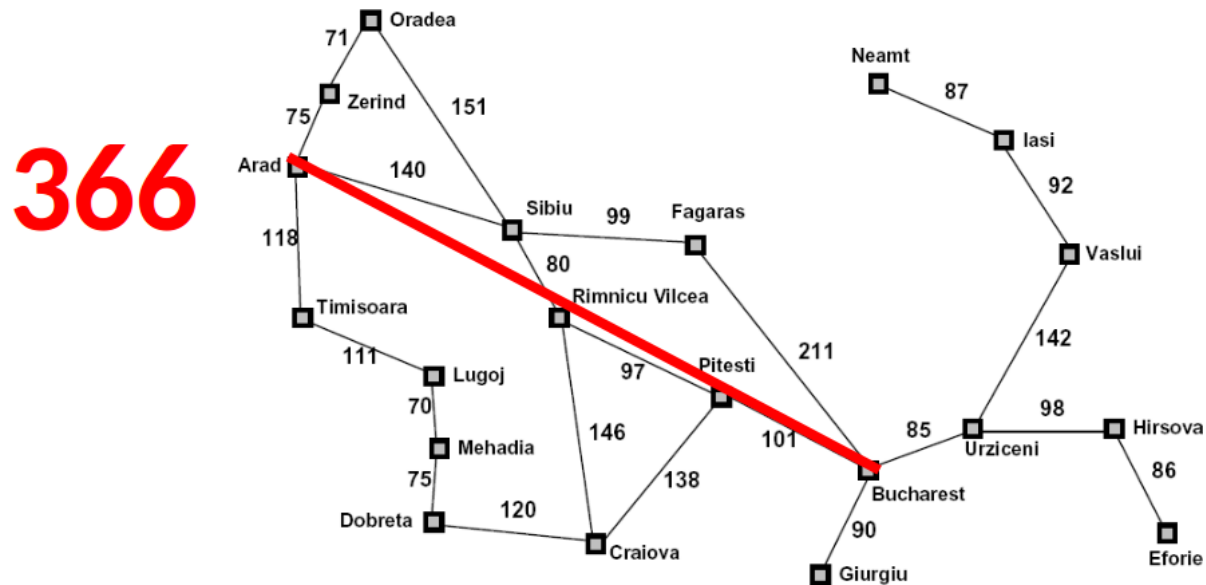


A\*

# Creating admissible heuristics

Most of the work in solving hard search problems optimally is in finding admissible heuristics.

Admissible heuristics can be derived from the exact solutions to [relaxed problems](#), where new actions are available.





## Dominance

- If  $h_1$  and  $h_2$  are both admissible and if  $h_2(n) \geq h_1(n)$  for all  $n$ , then  $h_2$  **dominates**  $h_1$  and is **better** for search.
- Given any admissible heuristics  $h_a$  and  $h_b$ ,

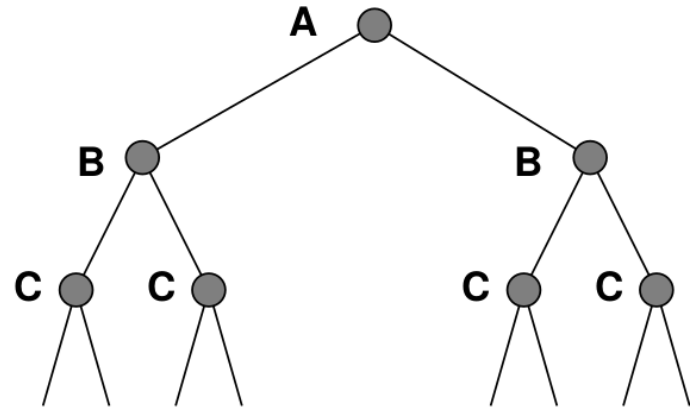
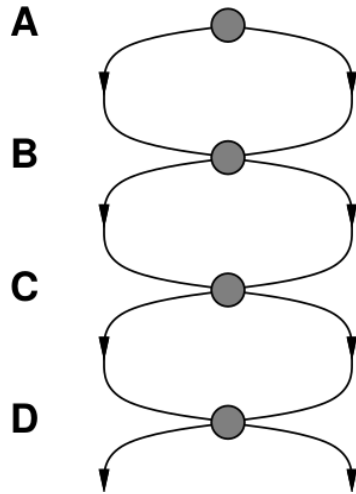
$$h(n) = \max(h_a(n), h_b(n))$$

is also admissible and dominates  $h_a$  and  $h_b$ .

## Learning heuristics from experience

- Assuming an **episodic** environment, an agent can **learn** good heuristics by playing the game many times.
- Each optimal solution  $s^*$  provides **training examples** from which  $h(n)$  can be learned.
- Each example consists of a state  $n$  from the solution path and the actual cost  $g(s^*)$  of the solution from that point.
- The mapping  $n \rightarrow g(s^*)$  can be learned with **supervised learning** algorithms.
  - Linear models, Neural networks, etc.

# Graph search



The failure to detect **repeated states** can turn a linear problem into an exponential one. It can also lead to non-terminating searches.

Redundant paths and cycles can be avoided by **keeping track** of the states that have been **explored**. This amounts to grow a tree directly on the state-space graph.

**function** GRAPH-SEARCH(*problem*, *fringe*) **returns** a solution, or failure

*closed*  $\leftarrow$  an empty set

*fringe*  $\leftarrow$  INSERT(MAKE-NODE(INITIAL-STATE[*problem*]), *fringe*)

**loop do**

**if** *fringe* is empty **then return** failure

*node*  $\leftarrow$  REMOVE-FRONT(*fringe*)

**if** GOAL-TEST(*problem*, STATE[*node*]) **then return** *node*

**if** STATE[*node*] is not in *closed* **then**

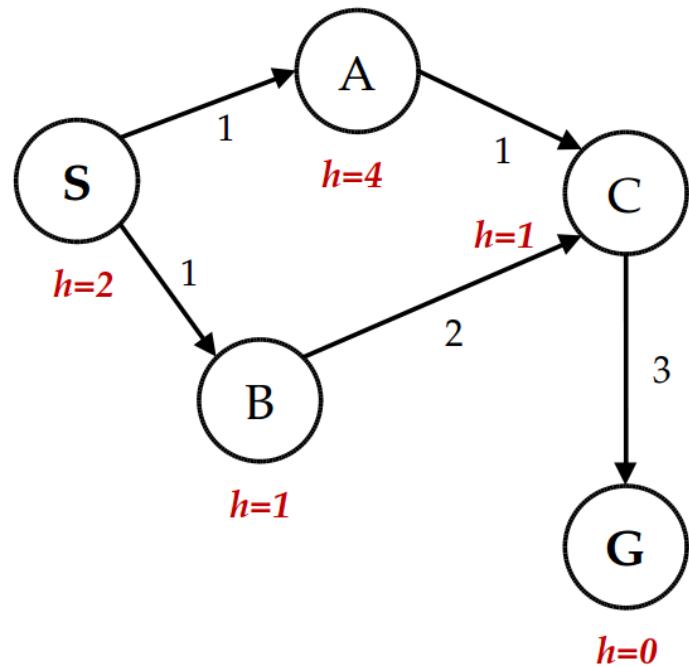
        add STATE[*node*] to *closed*

*fringe*  $\leftarrow$  INSERTALL(EXPAND(*node*, *problem*), *fringe*)

**end**

## A\* graph-search gone wrong?

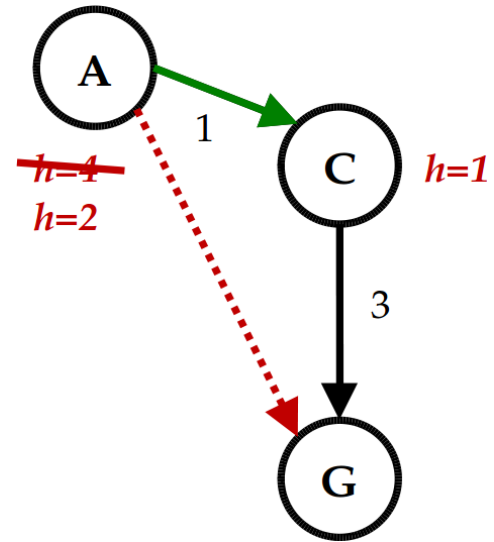
- We start at *S* and *G* is a goal state.
- Which path does graph search find?



## Consistent heuristics

A heuristic  $h$  is consistent if for every  $n$  and every successor  $n'$  generated by any action  $a$ ,

$$h(n) \leq c(n, a, n') + h(n').$$



Consequences of consistent heuristics:

- $f(n)$  is non-decreasing along any path.
- $h(n)$  is admissible.
- With a consistent heuristic, graph-search  $A^*$  is optimal.

# Recap example: Super Mario



- Task environment?
  - performance measure, environment, actuators, sensors?
- Type of environment?
- Search problem?
  - initial state, actions, transition model, goal test, path cost?
- Good heuristic?



A\* in action



# Summary

- Problem formulation usually requires abstracting away real-world details to define a state space that can feasibly be explored.
- Variety of uninformed search strategies (DFS, BFS, UCS, Iterative deepening).
- Heuristic functions estimate costs of shortest paths. Good heuristic can dramatically reduce search cost.
- Greedy best-first search expands lowest  $h$ , which shows to be incomplete and not always optimal.
- $A^*$  search expands lowest  $f = g + h$ . This strategy is complete and optimal.
- Graph search can be exponentially more efficient than tree search.

The end.

# References

- Hart, Peter E., Nils J. Nilsson, and Bertram Raphael. "A formal basis for the heuristic determination of minimum cost paths." IEEE transactions on Systems Science and Cybernetics 4.2 (1968): 100-107.