Introduction to Artificial Intelligence

Lecture 7: Reasoning over time

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Today

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  - Markov processes
  - Inference tasks
    - Prediction
    - Filtering
    - Smoothing
    - Most likely explanation
  - Hidden Markov models
  - Dynamic Bayesian networks
- Filters
  - Kalman filter
  - Particle filter

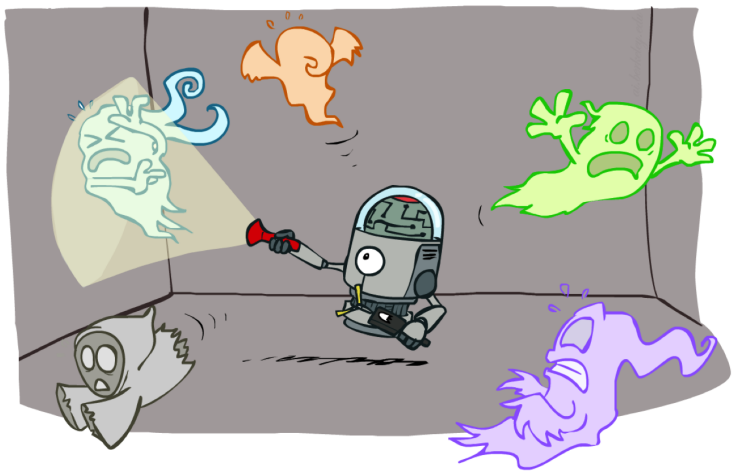
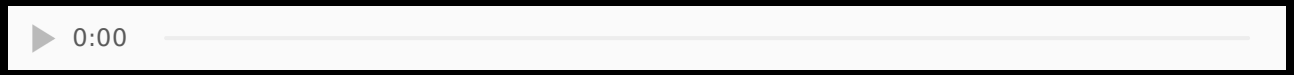


Image credits: CS188, UC Berkeley.



Pacman revenge: How to make good use of the sonar readings?

Image credits: CS188, UC Berkeley.

Markov models

Reasoning over time

Often, we want to reason about a sequence of observations.

- Speech recognition
- Robot localization
- User attention
- Medical monitoring.

For this reason, we need to introduce time (or space) in our model.

Modelling time

Consider the world as a discrete series of time slices, each of which contains a set of random variables:

- $\mathbf{X}_t$ denotes the set of unobservable state variables at time t.
- $\mathbf{E}_t$ denotes the set of observable evidence variables at time t.

We specify:

- a transition model $P(\mathbf{X}_t | \mathbf{X}_{0:t-1})$ that defines the probability distribution over the latest state variables, given the previous (unobserved) values.
- a sensor model $P(\mathbf{E}_t | \mathbf{X}_{0:t}, \mathbf{E}_{0:t-1})$ that defines the probability distribution over the latest evidence variables, given all previous (observed and unobserved) values.

Markov processes

Markov assumption

- The current state of the world depends only on its immediate previous state(s).
- i.e., $\mathbf{X}_t$ depends on only a bounded subset of $\mathbf{X}_{0:t-1}$.

Random processes that satisfy this assumption are called Markov processes.

First-order Markov processes

- Markov processes such that

$$P(\mathbf{X}_t|\mathbf{X}_{0:t-1}) = P(\mathbf{X}_t|\mathbf{X}_{t-1}).$$

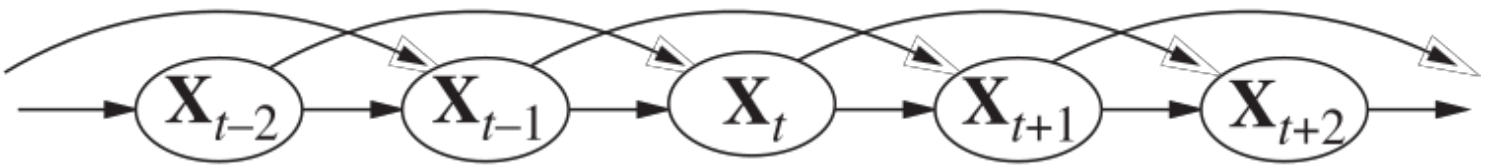
- i.e., $\mathbf{X}_t$ and $\mathbf{X}_{0:t-2}$ are conditionally independent given $\mathbf{X}_{t-1}$.

Second-order Markov processes

- Markov processes such that

$$P(\mathbf{X}_t|\mathbf{X}_{0:t-1}) = P(\mathbf{X}_t|\mathbf{X}_{t-2}, \mathbf{X}_{t-1}).$$

- i.e., $\mathbf{X}_t$ and $\mathbf{X}_{0:t-3}$ are conditionally independent given $\mathbf{X}_{t-1}$ and $\mathbf{X}_{t-2}$.



Sensor Markov assumption

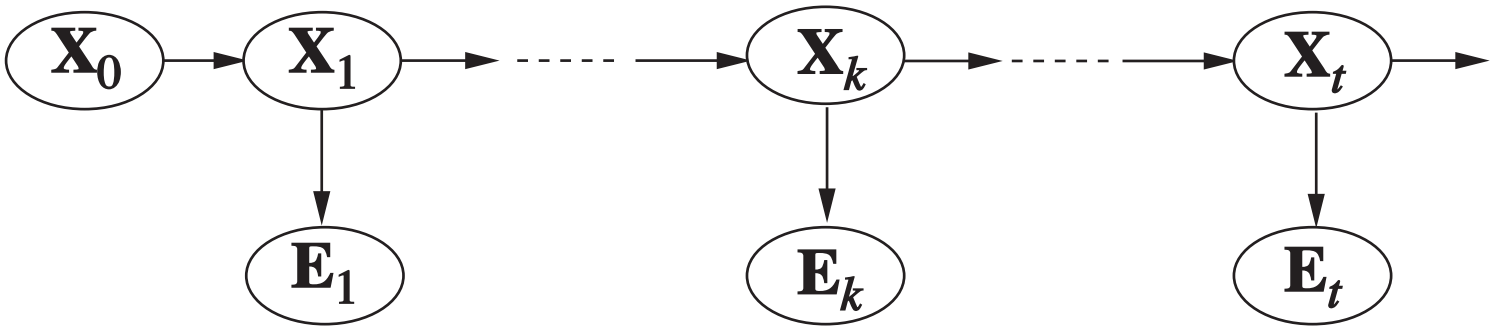
- Additionally, we make a (first-order) sensor Markov assumption:

$$P(\mathbf{E}_t | \mathbf{X}_{0:t}, \mathbf{E}_{0:t-1}) = P(\mathbf{E}_t | \mathbf{X}_t)$$

Stationarity assumption

- The transition and the sensor models are the same for all t (i.e., the laws of physics do not change with time).

Joint distribution

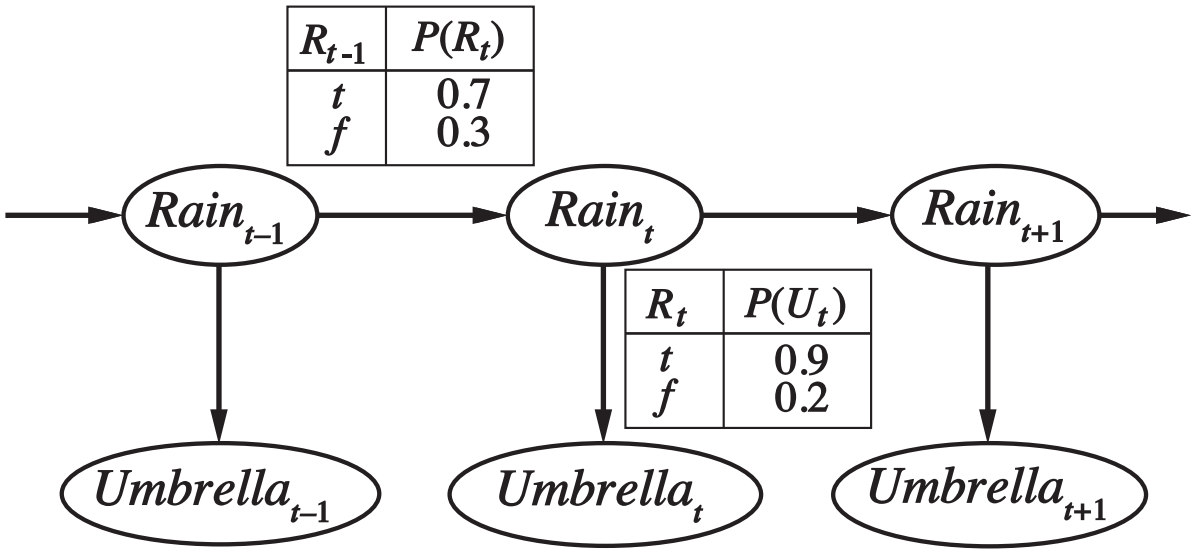


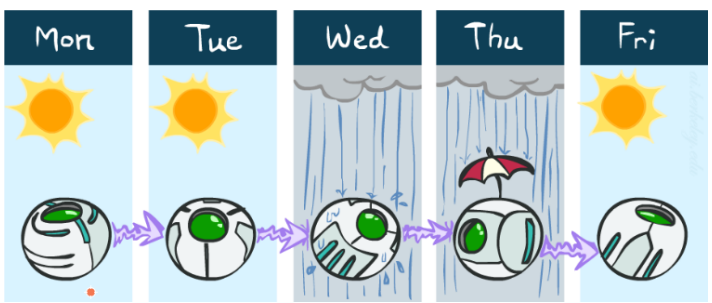
A Markov process can be described as a growable Bayesian network, unrolled infinitely through time, with a specified restricted structure between time steps.

Therefore, the joint distribution of all variables up to t in a (first-order) Markov process is

$$P(\mathbf{X}_{0:t}, \mathbf{E}_{1:t}) = P(\mathbf{X}_0) \prod_{i=1}^{t} P(\mathbf{X}_i|\mathbf{X}_{i-1}) P(\mathbf{E}_i|\mathbf{X}_i).$$

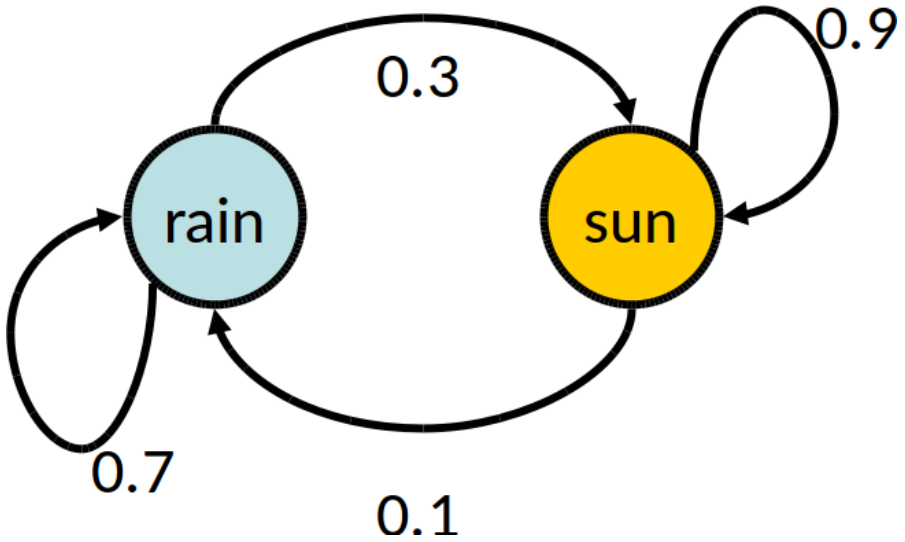
Example: Will you take your umbrella today?





- $P(\text{Umbrella}_t|\text{Rain}_t)$?
- $P(\text{Rain}_t|\text{Umbrella}_{0:t-1})$?
- $P(\text{Rain}_{t+2}|\text{Rain}_t)$?

Image credits: CS188, UC Berkeley.

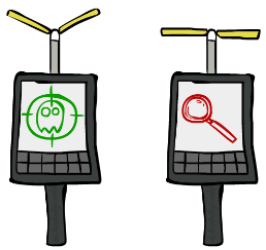


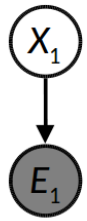
The transition model $P(\text{Rain}_t|\text{Rain}_{t-1})$ can equivalently be represented by a state transition diagram.

Inference tasks

- Prediction: $P(\mathbf{X}_{t+k}|\mathbf{e}_{1:t})$ for $k > 0$
  - Computing the posterior distribution over future states.
  - Used for evaluation of possible action sequences.
- Filtering: $P(\mathbf{X}_t|\mathbf{e}_{1:t})$
  - Filtering is what a rational agent does to keep track of the current hidden state $\mathbf{X}_t$, its belief state, so that rational decisions can be made.
- Smoothing: $P(\mathbf{X}_k|\mathbf{e}_{1:t})$ for $0 \leq k < t$
  - Computing the posterior distribution over past states.
  - Used for building better estimates, since it incorporates more evidence.
  - Essential for learning.
- Most likely explanation: $\arg\max_{\mathbf{x}_{1:t}} P(\mathbf{x}_{1:t}|\mathbf{e}_{1:t})$
  - Decoding with a noisy channel, speech recognition, etc.

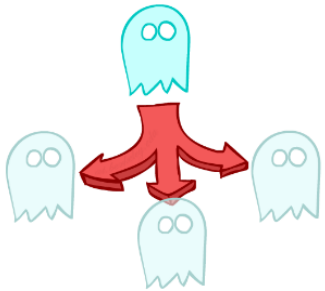
Base cases

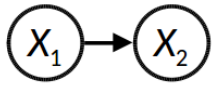




$$P(\mathbf{X}_1|\mathbf{e}_1) = \frac{P(\mathbf{X}_1, \mathbf{e}_1)}{P(\mathbf{e}_1)}$$
$$\propto P(\mathbf{X}_1, \mathbf{e}_1)$$
$$= P(\mathbf{X}_1)P(\mathbf{e}_1|\mathbf{X}_1)$$

Update $P(\mathbf{X}_1)$ with the evidence $\mathbf{e}_1$, given the sensor model.



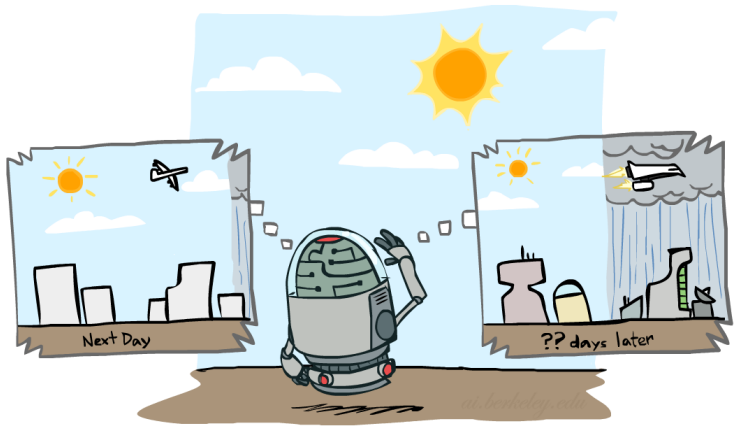


$$P(\mathbf{X}_2) = \sum_{\mathbf{x}_1} P(\mathbf{X}_2, \mathbf{x}_1)$$
$$= \sum_{\mathbf{x}_1} P(\mathbf{x}_1)P(\mathbf{X}_2|\mathbf{x}_1)$$

Push $P(\mathbf{X}_1)$ forward through the transition model.

Image credits: CS188, UC Berkeley.

Prediction



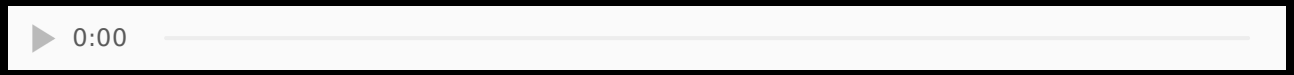
To predict the future $P(\mathbf{X}_{t+k}|\mathbf{e}_{1:t})$:

- Push the prior belief state $P(\mathbf{X}_t|\mathbf{e}_{1:t})$ through the transition model:

$$P(\mathbf{X}_{t+1}|\mathbf{e}_{1:t}) = \sum_{\mathbf{x}_t} P(\mathbf{X}_{t+1}|\mathbf{x}_t)P(\mathbf{x}_t|\mathbf{e}_{1:t})$$

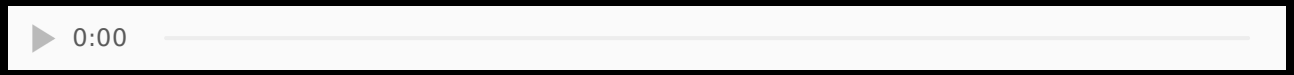
- Repeat up to $t+k$, using $P(\mathbf{X}_{t+k-1}|\mathbf{e}_{1:t})$ to compute $P(\mathbf{X}_{t+k}|\mathbf{e}_{1:t})$.

Image credits: CS188, UC Berkeley.



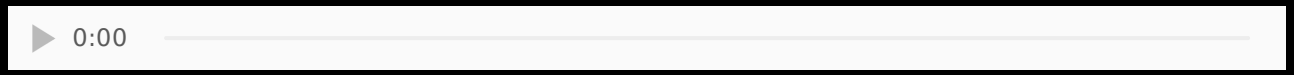
Basic dynamics (Ghostbusters)

Image credits: CS188, UC Berkeley.



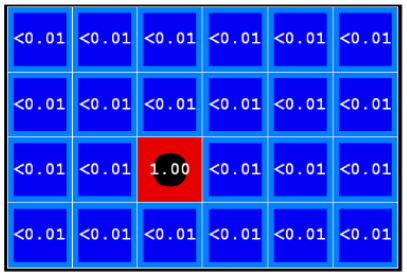
Circular dynamics (Ghostbusters)

Image credits: CS188, UC Berkeley.

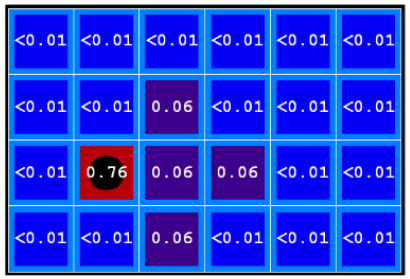


Whirlpool dynamics (Ghostbusters)

Image credits: CS188, UC Berkeley.

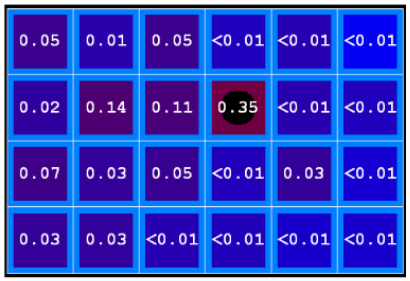


T = 1

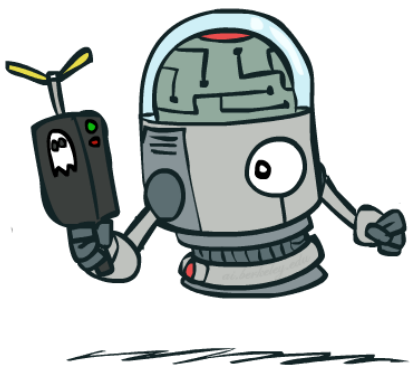


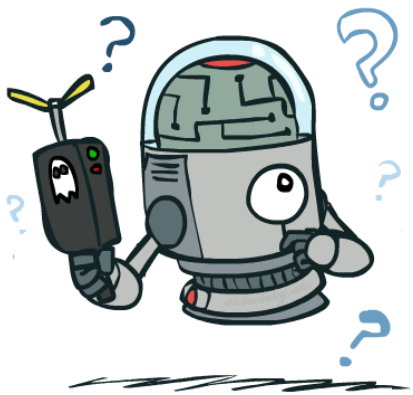
T = 2

...



T = 5





As time passes, uncertainty "accumulates" if we do not accumulate new evidence.

Image credits: CS188, UC Berkeley.

Stationary distributions

What if $t \to \infty$?

- For most chains, the influence of the initial distribution gets lesser and lesser over time.
- Eventually, the distribution converges to a fixed point, called the stationary distribution.
- This distribution is such that

$$P(\mathbf{X}_\infty) = P(\mathbf{X}_{\infty+1}) = \sum_{\mathbf{x}_\infty} P(\mathbf{X}_{\infty+1}|\mathbf{x}_\infty)P(\mathbf{x}_\infty)$$

Example

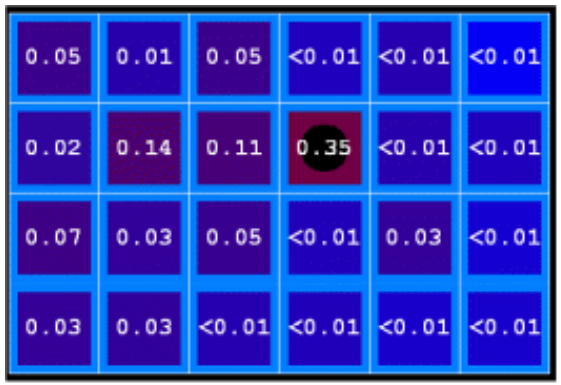
$$
\begin{aligned}
P(\mathbf{X}_\infty = \text{sun}) &= P(\mathbf{X}_{\infty+1} = \text{sun}) \\
&= P(\mathbf{X}_{\infty+1} = \text{sun} | \mathbf{X}_\infty = \text{sun}) P(\mathbf{X}_\infty = \text{sun}) \\
&\quad + P(\mathbf{X}_{\infty+1} = \text{sun} | \mathbf{X}_\infty = \text{rain}) P(\mathbf{X}_\infty = \text{rain}) \\
&= 0.9 P(\mathbf{X}_\infty = \text{sun}) + 0.3 P(\mathbf{X}_\infty = \text{rain})
\end{aligned}
$$

Therefore, $P(\mathbf{X}_\infty = \text{sun}) = 3P(\mathbf{X}_\infty = \text{rain})$.

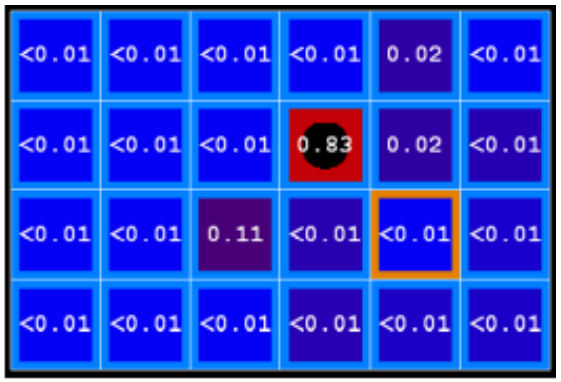
Which implies that $P(\mathbf{X}_\infty = \text{sun}) = \frac{3}{4}$ and $P(\mathbf{X}_\infty = \text{rain}) = \frac{1}{4}$.

Image credits: CS188, UC Berkeley.

Filtering



Before observation



After observation

What if we collect new observations? Beliefs get reweighted, and uncertainty "decreases":

$$P(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) \propto P(\mathbf{e}_{t+1}|\mathbf{X}_{t+1})P(\mathbf{X}_{t+1}|\mathbf{e}_{1:t})$$

Bayes filter

An agent maintains a belief state estimate $P(\mathbf{X}_t|\mathbf{e}_{1:t})$ (its prior) and updates it as new evidences $\mathbf{e}_{t+1}$ are collected (to obtain its posterior).

Recursive Bayesian estimation:

- (Predict step): Project the current state belief forward from t to $t+1$ through the transition model.
- (Update step): Update this new state using the evidence $\mathbf{e}_{t+1}$.

$$
\begin{aligned}
P(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) &= P(\mathbf{X}_{t+1}|\mathbf{e}_{1:t}, \mathbf{e}_{t+1}) \\
&= \alpha P(\mathbf{e}_{t+1}|\mathbf{X}_{t+1}, \mathbf{e}_{1:t}) P(\mathbf{X}_{t+1}|\mathbf{e}_{1:t}) \\
&= \alpha P(\mathbf{e}_{t+1}|\mathbf{X}_{t+1}) P(\mathbf{X}_{t+1}|\mathbf{e}_{1:t}) \\
&= \alpha P(\mathbf{e}_{t+1}|\mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} P(\mathbf{X}_{t+1}|\mathbf{x}_t, \mathbf{e}_{1:t}) P(\mathbf{x}_t|\mathbf{e}_{1:t}) \\
&= \alpha P(\mathbf{e}_{t+1}|\mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} P(\mathbf{X}_{t+1}|\mathbf{x}_t) P(\mathbf{x}_t|\mathbf{e}_{1:t})
\end{aligned}
$$

where

- the normalization constant

  $$\alpha = \frac{1}{P(\mathbf{e}_{t+1}|\mathbf{e}_{1:t})} = 1/\sum_{\mathbf{x}_{t+1}} P(\mathbf{e}_{t+1}|\mathbf{x}_{t+1}) P(\mathbf{x}_{t+1}|\mathbf{e}_{1:t})$$

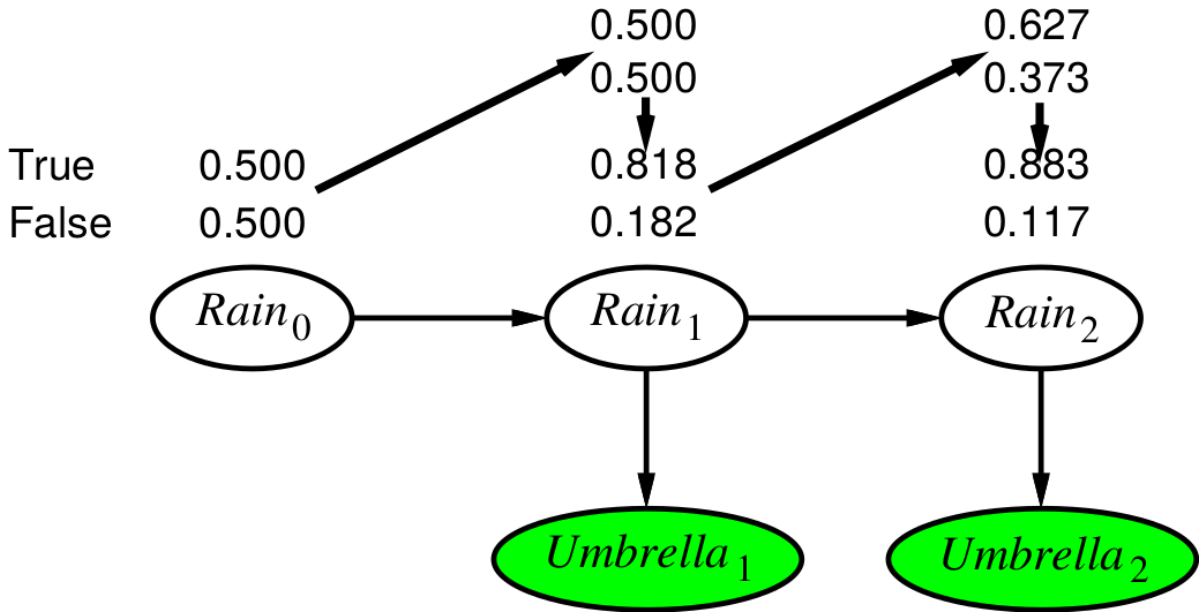
  is used to make probabilities sum to 1;

- in the last expression, the first and second terms are given by the model while the third is obtained recursively.

We can think of $P(\mathbf{X}_t | \mathbf{e}_{1:t})$ as a message $\mathbf{f}_{1:t}$ that is propagated forward along the sequence, modified by each transition and updated by each new observation.

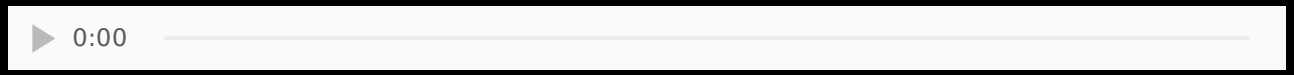
- Thus, the process can be implemented as $\mathbf{f}_{1:t+1} = \alpha \, \text{forward}(\mathbf{f}_{1:t}, \mathbf{e}_{t+1})$.
- The complexity of a forward update is constant (in time and space) with t.

Example



R_{t-1}	$P(R_t)$
$true$	0.7
$false$	0.3

R_t	$P(U_t)$
$true$	0.9
$false$	0.2



Ghostbusters with a Bayes filter

Image credits: CS188, UC Berkeley.

Smoothing

We want to compute $P(\mathbf{X}_k|\mathbf{e}_{1:t})$ for $0 \leq k < t$.

Divide evidence $\mathbf{e}_{1:t}$ into $\mathbf{e}_{1:k}$ and $\mathbf{e}_{k+1:t}$. Then,

$$\begin{aligned} P(\mathbf{X}_k|\mathbf{e}_{1:t}) &= P(\mathbf{X}_k|\mathbf{e}_{1:k}, \mathbf{e}_{k+1:t}) \\ &= \alpha P(\mathbf{X}_k|\mathbf{e}_{1:k}) P(\mathbf{e}_{k+1:t}|\mathbf{X}_k, \mathbf{e}_{1:k}) \\ &= \alpha P(\mathbf{X}_k|\mathbf{e}_{1:k}) P(\mathbf{e}_{k+1:t}|\mathbf{X}_k). \end{aligned}$$

Let the backward message $\mathbf{b}_{k+1:t}$ correspond to $P(\mathbf{e}_{k+1:t}|\mathbf{X}_k)$. Then,

$$P(\mathbf{X}_k|\mathbf{e}_{1:t}) = \alpha\, \mathbf{f}_{1:k}\mathbf{b}_{k+1:t}$$

This backward message can be computed using backwards recursion:

$$\begin{aligned} P(\mathbf{e}_{k+1:t}|\mathbf{X}_k) &= \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1:t}|\mathbf{X}_k, \mathbf{x}_{k+1}) P(\mathbf{x}_{k+1}|\mathbf{X}_k) \\ &= \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1:t}|\mathbf{x}_{k+1}) P(\mathbf{x}_{k+1}|\mathbf{X}_k) \\ &= \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1}|\mathbf{x}_{k+1}) P(\mathbf{e}_{k+2:t}|\mathbf{x}_{k+1}) P(\mathbf{x}_{k+1}|\mathbf{X}_k) \end{aligned}$$

The first and last factors are given by the model. The second factor is obtained recursively. Therefore,

$$\mathbf{b}_{k+1:t} = \text{backward}(\mathbf{b}_{k+2:t}, \mathbf{e}_{k+1}).$$

Forward-backward algorithm

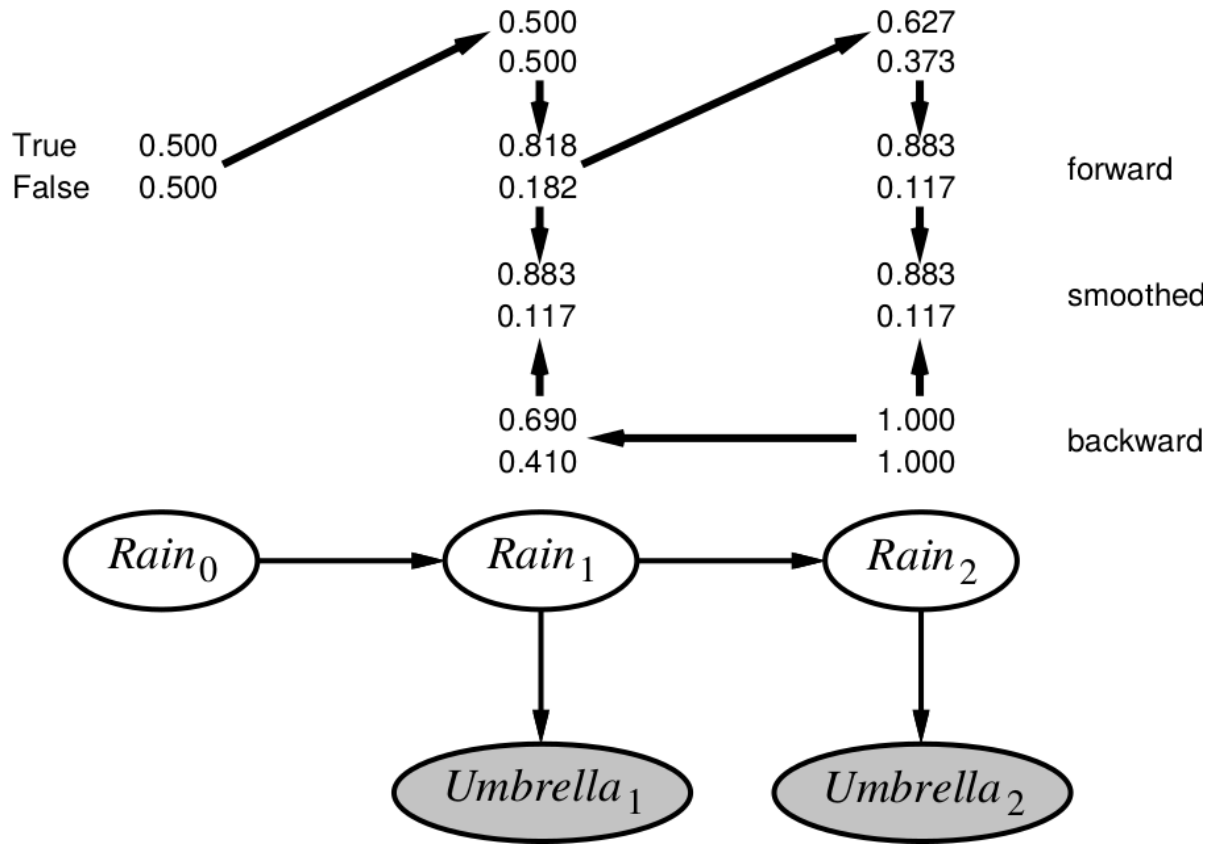
function FORWARD-BACKWARD(**ev**, $prior$) **returns** a vector of probability distributions
  inputs: **ev**, a vector of evidence values for steps $1, \ldots, t$
    $prior$, the prior distribution on the initial state, $\mathbf{P}(\mathbf{X}_0)$
  local variables: **fv**, a vector of forward messages for steps $0, \ldots, t$
    b, a representation of the backward message, initially all 1s
    sv, a vector of smoothed estimates for steps $1, \ldots, t$

  fv$[0] \leftarrow prior$
  for $i = 1$ **to** t **do**
    fv$[i] \leftarrow$ FORWARD(**fv**$[i-1]$, **ev**$[i]$)
  for $i = t$ **downto** 1 **do**
    sv$[i] \leftarrow$ NORMALIZE(**fv**$[i] \times$ **b**)
    b $\leftarrow$ BACKWARD(**b**, **ev**$[i]$)
  return sv

Complexity:

- Smoothing for a particular time step k takes: $O(t)$
- Smoothing a whole sequence (because of caching): $O(t)$

Example


0.500
0.500
0.627
0.373
True 0.500
False 0.500
0.818
0.182
0.883
0.117
forward
0.883
0.117
0.883
0.117
smoothed
0.690
0.410
1.000
1.000
backward
$Rain_0$
$Rain_1$
$Rain_2$
$Umbrella_1$
$Umbrella_2$

Most likely explanation

Suppose that $[true, true, false, true, true]$ is the umbrella sequence.

What is the weather sequence that is the most likely to explain this?

- Does the absence of umbrella at day 3 means it wasn't raining?
- Or did the director forget to bring it?
- If it didn't rain on day 3, perhaps it didn't rain on day 4 either, but the director brought the umbrella just in case?

Among all 2^5 sequences, is there an (efficient) way to find the most likely one?

Image credits: CS188, UC Berkeley.

- The most likely sequence is not the sequence of the most likely states!
- The most likely path to each $\mathbf{x}_{t+1}$, is the most likely path to some $\mathbf{x}_t$ plus one more step. Therefore,

$$\begin{aligned}&\max_{\mathbf{x}_{1:t}} P(\mathbf{x}_{1:t}, \mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) \\ &= \alpha P(\mathbf{e}_{t+1}|\mathbf{X}_{t+1}) \max_{\mathbf{x}_t}(P(\mathbf{X}_{t+1}|\mathbf{x}_t) \max_{\mathbf{x}_{1:t-1}} P(\mathbf{x}_{1:t-1}, \mathbf{x}_t|\mathbf{e}_{1:t}))\end{aligned}$$

- Identical to filtering, except that the forward message $\mathbf{f}_{1:t} = P(\mathbf{X}_t|\mathbf{e}_{1:t})$ is replaced with

$$\mathbf{m}_{1:t} = \max_{\mathbf{x}_{1:t-1}} P(\mathbf{x}_{1:t-1}, \mathbf{X}_t|\mathbf{e}_{1:t}),$$

  where $\mathbf{m}_{1:t}(i)$ gives the probability of the most likely path to state i.
- The update has its sum replaced by max, resulting in the Viterbi algorithm:

$$\mathbf{m}_{1:t+1} = \alpha P(\mathbf{e}_{t+1}|\mathbf{X}_{t+1}) \max_{\mathbf{x}_{1:t}} P(\mathbf{X}_{t+1}|\mathbf{x}_t)\mathbf{m}_{1:t}$$

Example

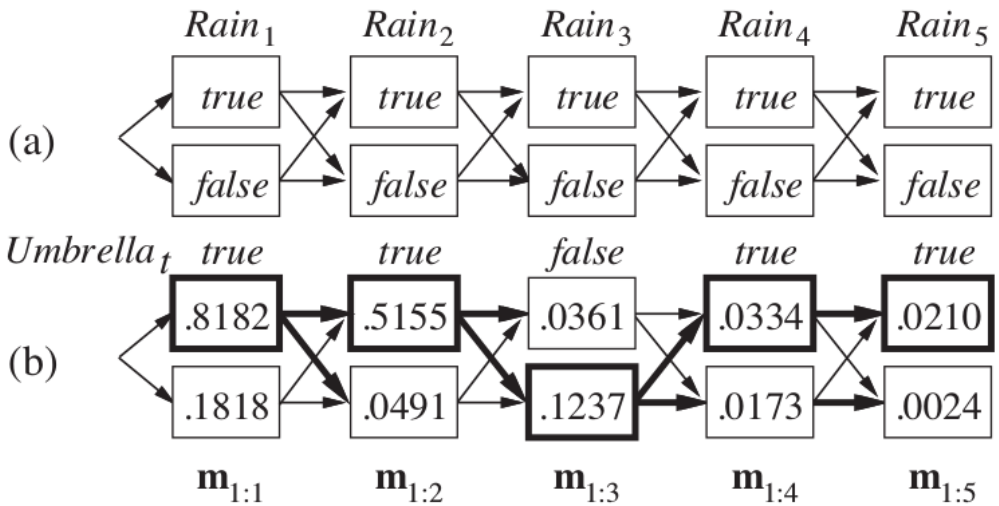


Figure 15.5 (a) Possible state sequences for $Rain_t$ can be viewed as paths through a graph of the possible states at each time step. (States are shown as rectangles to avoid confusion with nodes in a Bayes net.) (b) Operation of the Viterbi algorithm for the umbrella observation sequence $[true, true, false, true, true]$. For each t, we have shown the values of the message $\mathbf{m}_{1:t}$, which gives the probability of the best sequence reaching each state at time t. Also, for each state, the bold arrow leading into it indicates its best predecessor as measured by the product of the preceding sequence probability and the transition probability. Following the bold arrows back from the most likely state in $\mathbf{m}_{1:5}$ gives the most likely sequence.

Hidden Markov models

So far, we described Markov processes over arbitrary sets of state variables $\mathbf{X}_t$ and evidence variables $\mathbf{E}_t$.

- A hidden Markov model (HMM) is a Markov process in which the state $\mathbf{X}_t$ and the evidence $\mathbf{E}_t$ are both single discrete random variables.
  - e.g., $\mathbf{X}_t = X_t$, with domain $D_{X_t} = \{1, ..., S\}$.
- This restricted structure allows for a reformulation of the forward-backward algorithm in terms of matrix-vector operations.

Note on terminology

Some authors instead divide Markov models into two classes, depending on the observability of the system state:

- Observable system state: Markov chains
- Partially-observable system state: Hidden Markov models.

We follow here the terminology of the textbook.

Simplified matrix algorithms

- The transition model $P(X_t|X_{t-1})$ becomes an $S \times S$ transition matrix $\mathbf{T}$, such that

$$\mathbf{T}_{ij} = P(X_t = j|X_{t-1} = i).$$

- The sensor model $P(E_t|X_t)$ is defined as an $S \times S$ sensor matrix $\mathbf{O}_t$ whose i-th diagonal element is $P(e_t|X_t = i)$ and whose other entries are 0.
- If we use column vectors to represent forward and backward messages, then we have:

$$\mathbf{f}_{1:t+1} = \alpha \mathbf{O}_{t+1} \mathbf{T}^T \mathbf{f}_{1:t}$$

$$\mathbf{b}_{k+1:t} = \mathbf{T} \mathbf{O}_{k+1} \mathbf{b}_{k+2:t}$$

- Therefore the forward-backward algorithm needs time $O(S^2 t)$ and space $O(St)$.

Example

Suppose that $[true, true, false, true, true]$ is the umbrella sequence.

$$\mathbf{T} = \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix}$$

$$\mathbf{O}_1 = \mathbf{O}_2 = \mathbf{O}_4 = \mathbf{O}_5 = \begin{pmatrix} 0.9 & 0.0 \\ 0.0 & 0.2 \end{pmatrix}$$

$$\mathbf{O}_2 = \begin{pmatrix} 0.1 & 0.0 \\ 0.0 & 0.8 \end{pmatrix}$$

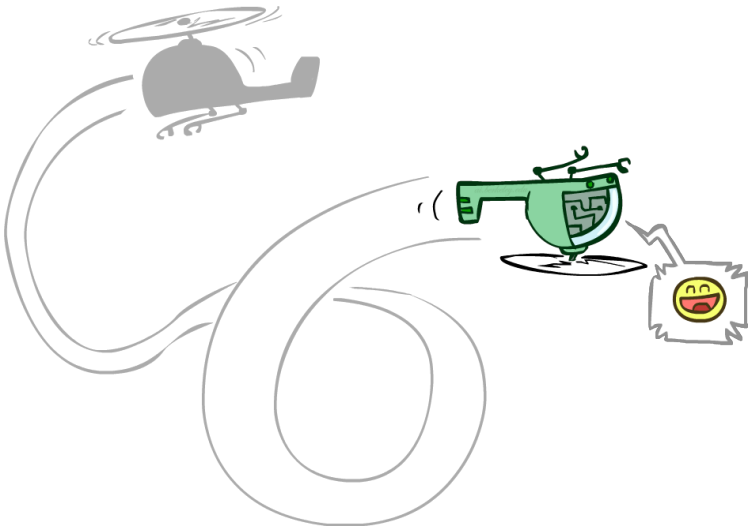
See `code/lecture7-forward-backward.ipynb` for the execution.

Applications

HMMs are used in many fields where the goal is to recover a data sequence that is not immediately observable, but other data that depend on the sequence are.

- Speech recognition (see Lecture 10)
- Speech synthesis
- Part-of-speech tagging
- Machine translation
- Robotics
- Computational finance
- Handwriting recognition
- Time series analysis
- Activity recognition

Filters



Suppose we want track the position and velocity of a robot from noisy observations collected over time.

Formally, we want to estimate continuous state variables such as

- the position $\mathbf{X}_t$ of the robot at time t,
- the velocity $\dot{\mathbf{X}}_t$ of the robot at time t.

We assume discrete time steps.

Image credits: CS188, UC Berkeley.

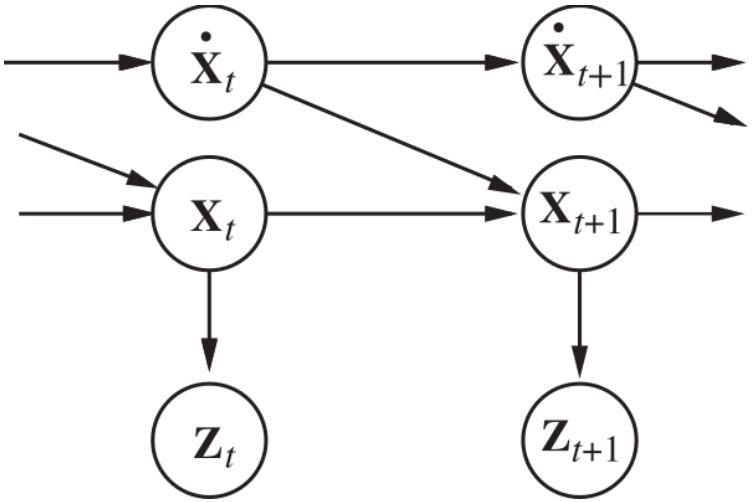
The Bayes filter similarly applies to continuous state and evidence variables $\mathbf{X}_t$ and $\mathbf{E}_t$, in which case summations are replaced with integrals:

$$p(\mathbf{x}_{t+1}|\mathbf{e}_{1:t+1}) = \alpha\, p(\mathbf{e}_{t+1}|\mathbf{x}_{t+1}) \int p(\mathbf{x}_{t+1}|\mathbf{x}_t) p(\mathbf{x}_t|\mathbf{e}_{1:t}) d\mathbf{x}_t$$

where the normalization constant is

$$\alpha = 1 \,/ \int p(\mathbf{e}_{t+1}|\mathbf{x}_{t+1}) p(\mathbf{x}_{t+1}|\mathbf{e}_{1:t}) d\mathbf{x}_{t+1}\,.$$

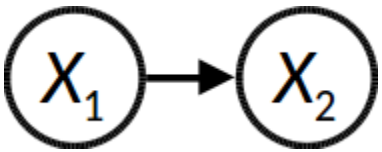
Kalman filter



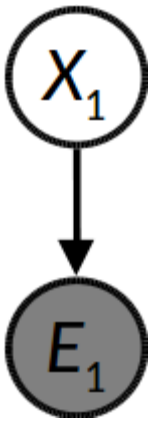
The Kalman filter is a special case of the Bayes filter, which assumes:

- Gaussian prior
- Linear Gaussian transition model
- Linear Gaussian sensor model

Linear Gaussian models



$$p(\mathbf{X}_{t+1}|\mathbf{x}_t) = \mathcal{N}(\mathbf{F}\mathbf{x}_t, \mathbf{\Sigma}_\mathbf{x})$$



$$p(\mathbf{E}_t|\mathbf{x}_t) = \mathcal{N}(\mathbf{H}\mathbf{x}_t, \mathbf{\Sigma}_\mathbf{e})$$

Filtering Gaussian distributions

- *Prediction step:*

  If the distribution $p(\mathbf{X}_t|\mathbf{e}_{1:t})$ is Gaussian and the transition model $p(\mathbf{X}_{t+1}|\mathbf{x}_t)$ is linear Gaussian, then the one-step predicted distribution given by

  $$p(\mathbf{X}_{t+1}|\mathbf{e}_{1:t}) = \int p(\mathbf{X}_{t+1}|\mathbf{x}_t)p(\mathbf{x}_t|\mathbf{e}_{1:t})d\mathbf{x}_t$$

  is also a Gaussian distribution.

- *Update step:*

  If the prediction $p(\mathbf{X}_{t+1}|\mathbf{e}_{1:t})$ is Gaussian and the sensor model $p(\mathbf{e}_{t+1}|\mathbf{X}_{t+1})$ is linear Gaussian, then after conditioning on new evidence, the updated distribution

  $$p(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) = \alpha p(\mathbf{e}_{t+1}|\mathbf{X}_{t+1})p(\mathbf{X}_{t+1}|\mathbf{e}_{1:t})$$

  is also a Gaussian distribution.

Therefore, for the Kalman filter, $p(\mathbf{X}_t|\mathbf{e}_{1:t})$ is a multivariate Gaussian distribution $\mathcal{N}(\mu_t, \mathbf{\Sigma}_t)$ for all t.

- Filtering reduces to the computation of the parameters μ_t and $\mathbf{\Sigma}_t$.
- By contrast, for general (nonlinear, non-Gaussian) processes, the description of the posterior grows unboundedly as $t \to \infty$.

1D example

Gaussian random walk:

- Gaussian prior:

$$p(x_0) = \alpha \exp\left(-\frac{1}{2}\frac{(x_0 - \mu_0)^2}{\sigma_0^2}\right)$$

- The transition model adds random perturbations of constant variance:

$$p(x_{t+1}|x_t) = \alpha \exp\left(-\frac{1}{2}\frac{(x_{t+1} - x_t)^2}{\sigma_x^2}\right)$$

- The sensor model yields measurements with Gaussian noise of constant variance:

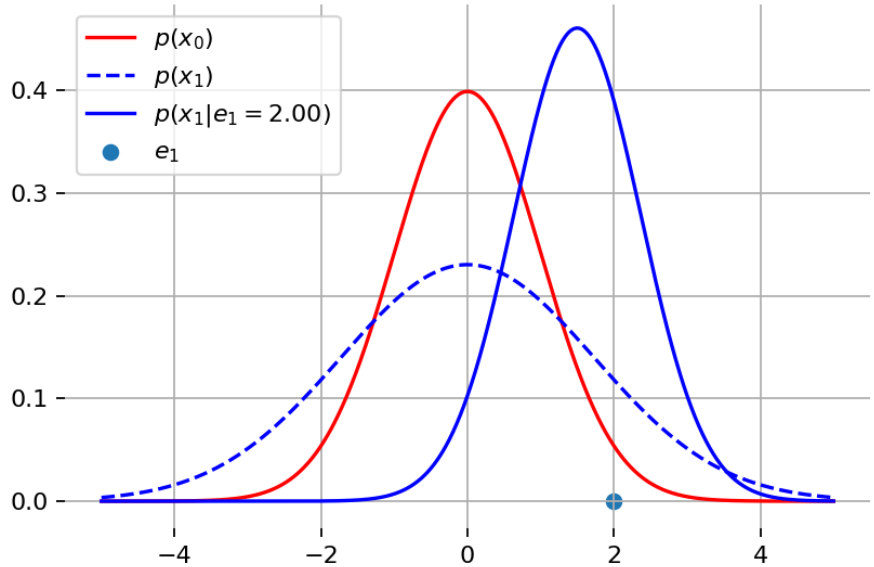
$$p(e_t|x_t) = \alpha \exp\left(-\frac{1}{2}\frac{(e_t - x_t)^2}{\sigma_e^2}\right)$$

The one-step predicted distribution is given by

$$
\begin{aligned}
p(x_1) &= \int p(x_1|x_0)p(x_0)dx_0 \\
&= \alpha \int \exp\left(-\frac{1}{2}\frac{(x_1-x_0)^2}{\sigma_x^2}\right)\exp\left(-\frac{1}{2}\frac{(x_0-\mu_0)^2}{\sigma_0^2}\right)dx_0 \\
&= \alpha \int \exp\left(-\frac{1}{2}\frac{\sigma_0^2(x_1-x_0)^2+\sigma_x^2(x_0-\mu_0)^2}{\sigma_0^2\sigma_x^2}\right)dx_0 \\
&\ldots \text{ (simplify by completing the square)} \\
&= \alpha \exp\left(-\frac{1}{2}\frac{(x_1-\mu_0)^2}{\sigma_0^2+\sigma_x^2}\right) \\
&= \mathcal{N}(\mu_0, \sigma_0^2+\sigma_x^2)
\end{aligned}
$$

For the update step, we need to condition on the observation at the first time step:

$$
\begin{aligned}
p(x_1|e_1) &= \alpha p(e_1|x_1)p(x_1) \\
&= \alpha \exp\left(-\frac{1}{2}\frac{(e_1 - x_1)^2}{\sigma_e^2}\right) \exp\left(-\frac{1}{2}\frac{(x_1 - \mu_0)^2}{\sigma_0^2 + \sigma_x^2}\right) \\
&= \alpha \exp\left(-\frac{1}{2}\frac{\left(x_1 - \frac{(\sigma_0^2+\sigma_x^2)e_1 + \sigma_e^2\mu_0}{\sigma_0^2+\sigma_x^2+\sigma_e^2}\right)^2}{\frac{(\sigma_0^2+\sigma_x^2)\sigma_e^2}{\sigma_0^2+\sigma_x^2+\sigma_e^2}}\right) \\
&= \mathcal{N}\left(\frac{(\sigma_0^2 + \sigma_x^2)e_1 + \sigma_e^2\mu_0}{\sigma_0^2 + \sigma_x^2 + \sigma_e^2}, \frac{(\sigma_0^2 + \sigma_x^2)\sigma_e^2}{\sigma_0^2 + \sigma_x^2 + \sigma_e^2}\right)
\end{aligned}
$$



In summary, the update equations given a new evidence e_{t+1} are:

$$\mu_{t+1} = \frac{(\sigma_t^2 + \sigma_x^2)e_{t+1} + \sigma_e^2 \mu_t}{\sigma_t^2 + \sigma_x^2 + \sigma_e^2}$$

$$\sigma_{t+1}^2 = \frac{(\sigma_t^2 + \sigma_x^2)\sigma_e^2}{\sigma_t^2 + \sigma_x^2 + \sigma_e^2}$$

General Kalman update

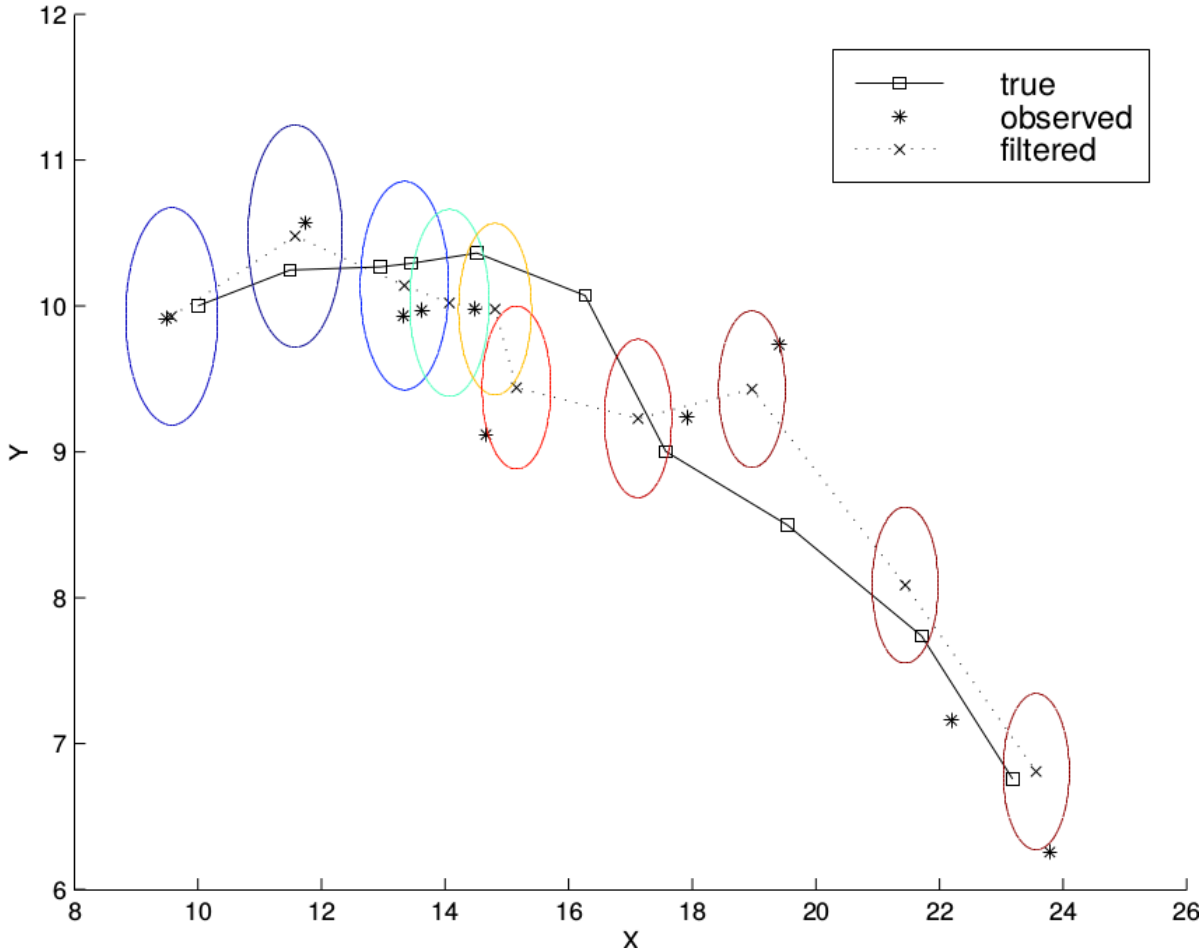
The same derivations generalize to multivariate normal distributions. In this case, we arrive at the following general update equations:

$$\begin{aligned}\mu_{t+1} &= \mathbf{F}\mu_t + \mathbf{K}_{t+1}(\mathbf{e}_{t+1} - \mathbf{H}\mathbf{F}\mu_t) \\ \mathbf{\Sigma}_{t+1} &= (\mathbf{I} - \mathbf{K}_{t+1}\mathbf{H})(\mathbf{F}\mathbf{\Sigma}_t\mathbf{F}^T + \mathbf{\Sigma}_x) \\ \mathbf{K}_{t+1} &= (\mathbf{F}\mathbf{\Sigma}_t\mathbf{F}^T + \mathbf{\Sigma}_x)\mathbf{H}^T(\mathbf{H}(\mathbf{F}\mathbf{\Sigma}_t\mathbf{F}^T + \mathbf{\Sigma}_x)\mathbf{H}^T + \mathbf{\Sigma}_e)^{-1}\end{aligned}$$

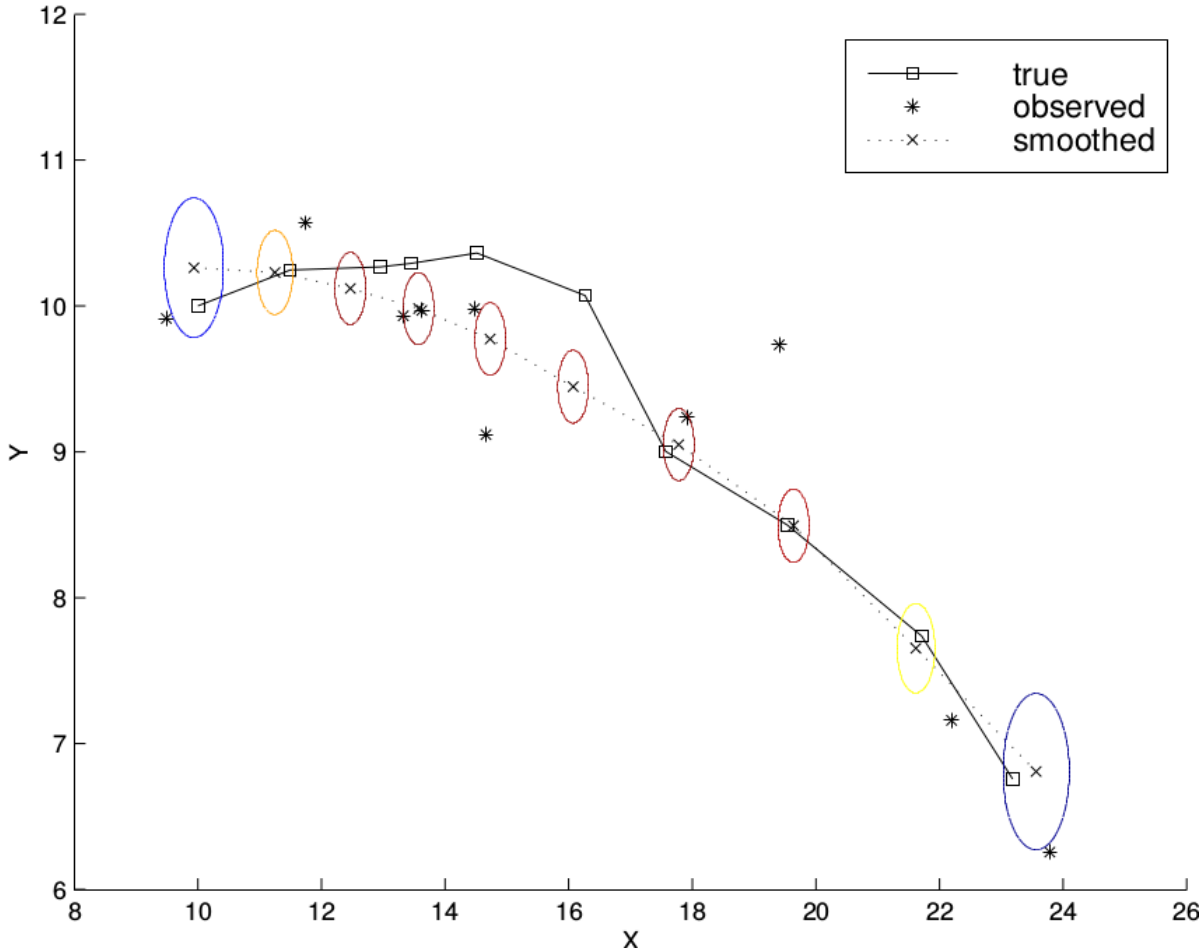
where $\mathbf{K}_{t+1}$ is the Kalman gain matrix.

Note that $\mathbf{\Sigma}_{t+1}$ and $\mathbf{K}_{t+1}$ are independent of the evidence. Therefore, they can be computed offline.

2D tracking: filtering



2D tracking: smoothing



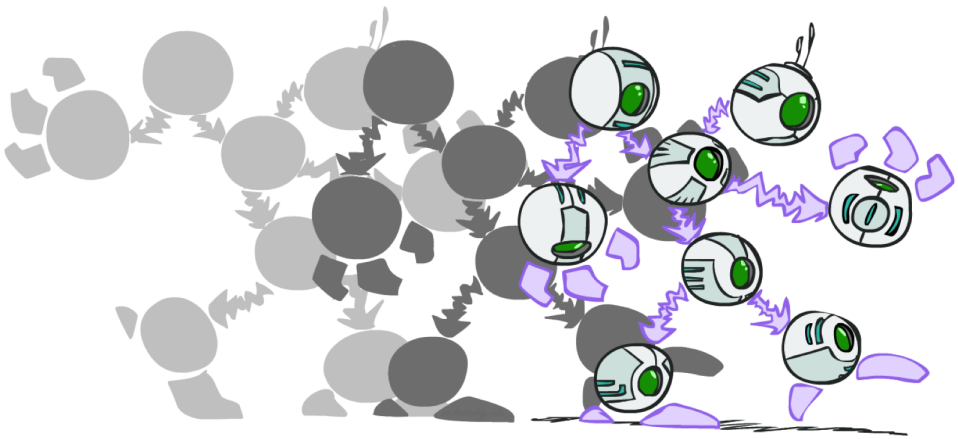
Apollo guidance computer

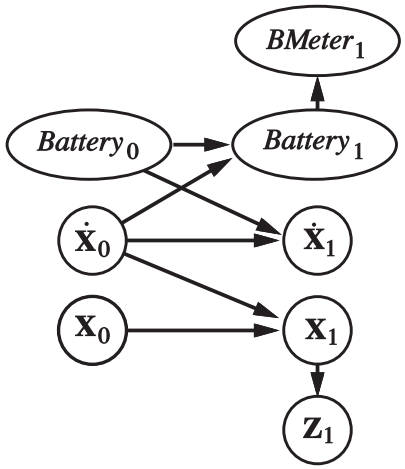
- The Kalman filter put man on the Moon, literally!
- The onboard guidance software of Saturn-V used a Kalman filter to merge new data with past position measurements to produce an optimal position estimate of the spacecraft.


```
# INCORP2 -- INCORPORATES THE COMPUTED STATE VECTOR DEVIATIONS INTO THE
# ESTIMATED STATE VECTOR.  THE STATE VECTOR UPDATED MAY BE FOR EITHER THE
# LEM OR THE CSM.  DETERMINED BY FLAG VEHUPFLG.  (ZERO = LEM) (1 = CSM)
#
# INPUT
#       PERMANENT STATE VECTOR FOR EITHER THE LEM OR CSM
#       VEHUPFLG = UPDATE VEHICLE C=LEM 1=CSM
#       W = ERROR TRANSITION MATRIX
#       DELTAX = COMPUTED STATE VECTOR DEVIATIONS
#       DMENFLG = SIZE OF W MATRIX (ZERO=6X6) (1=9X9)
#       GAMMA = SCALAR FOR INCORPORATION
#       ZI = VECTOR USED IN INCORPORATION
#       OMEGA = WEIGHTING VECTOR
#
# OUTPUT
#       UPDATED PERMANENT STATE VECTOR
```

Credits: Apollo-11 source code

Dynamic Bayesian networks



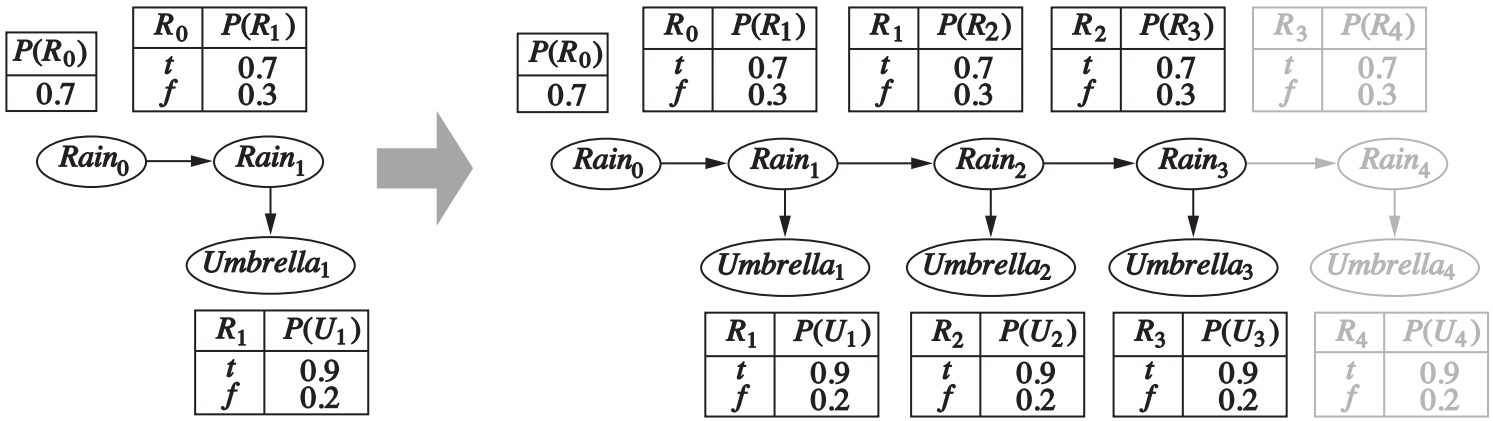


Dynamics Bayesian networks (DBNs) can be used for tracking multiple variables over time, using multiple sources of evidence.

- Idea: Repeat a fixed Bayes net structure at each time t.
- Variables from time t condition on those from $t - 1$.
- DBNs are a generalization of HMMs and of the Kalman filter.

Image credits: CS188, UC Berkeley.

Exact inference



Unroll the network through time and run any exact inference algorithm (e.g., variable elimination)

- Problem: inference cost for each update grows with t.
- Rollup filtering: add slice $t + 1$, sum out slice t using variable elimination.
  - Largest factor is $O(d^{n+k})$ and the total update cost per step is $O(nd^{n+k})$.
  - Better than HMMs, which is $O(d^{2n})$, but still infeasible for large numbers of variables.

Likelihood weighting for DBNs

If exact inference is intractable, then let's use instead approximate inference.

What about likelihood weighting? Generated LW samples pay no attention to the evidence!

- The fraction of samples that remain close to the actual series of events drops exponentially with t.
- Therefore, the number of required samples for inference grows exponentially with t.

We need a better solution!

Particle filtering

- Basic idea:
  - Maintain a finite population of samples, called particles.
  - Ensure the particles track the high-likelihood regions of the state space.
  - Throw away samples that have very low weight, according to the evidence.
  - Replicate those that have high weight.
- Scale to high-dimensional state spaces ($n > 10^5$).
- Can be shown to be consistent.

function PARTICLE-FILTERING($\mathbf{e}$, N, dbn) **returns** a set of samples for the next time step

inputs: $\mathbf{e}$, the new incoming evidence
N, the number of samples to be maintained
dbn, a DBN with prior $\mathbf{P}(\mathbf{X}_0)$, transition model $\mathbf{P}(\mathbf{X}_1|\mathbf{X}_0)$, sensor model $\mathbf{P}(\mathbf{E}_1|\mathbf{X}_1)$

persistent: S, a vector of samples of size N, initially generated from $\mathbf{P}(\mathbf{X}_0)$

local variables: W, a vector of weights of size N

for i = 1 to N **do**
$S[i] \leftarrow$ sample from $\mathbf{P}(\mathbf{X}_1 \mid \mathbf{X}_0 = S[i])$ /* step 1 */
$W[i] \leftarrow \mathbf{P}(\mathbf{e} \mid \mathbf{X}_1 = S[i])$ /* step 2 */
$S \leftarrow$ WEIGHTED-SAMPLE-WITH-REPLACEMENT(N, S, W) /* step 3 */
return S

Update cycle

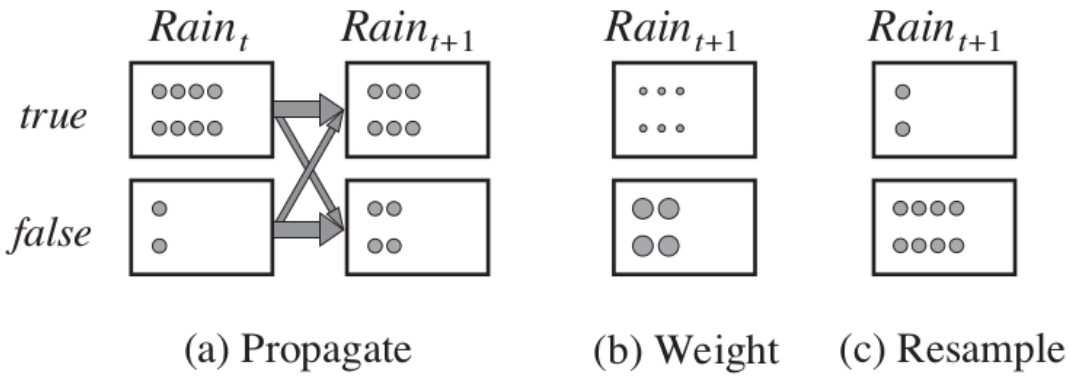
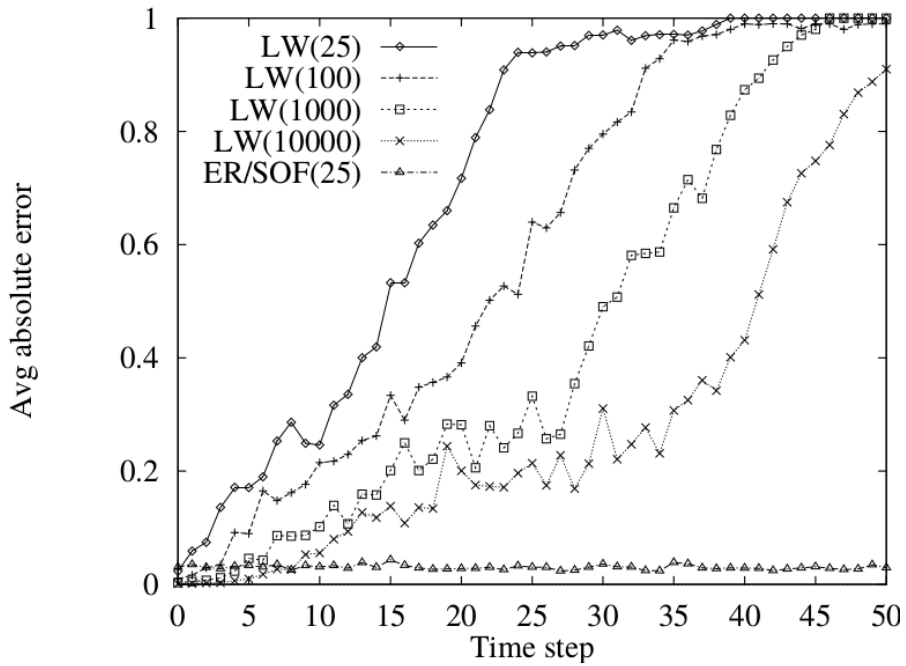


Figure 15.18 The particle filtering update cycle for the umbrella DBN with $N = 10$, showing the sample populations of each state. (a) At time t, 8 samples indicate *rain* and 2 indicate $\neg rain$. Each is propagated forward by sampling the next state through the transition model. At time $t + 1$, 6 samples indicate *rain* and 4 indicate $\neg rain$. (b) $\neg umbrella$ is observed at $t + 1$. Each sample is weighted by its likelihood for the observation, as indicated by the size of the circles. (c) A new set of 10 samples is generated by weighted random selection from the current set, resulting in 2 samples that indicate *rain* and 8 that indicate $\neg rain$.

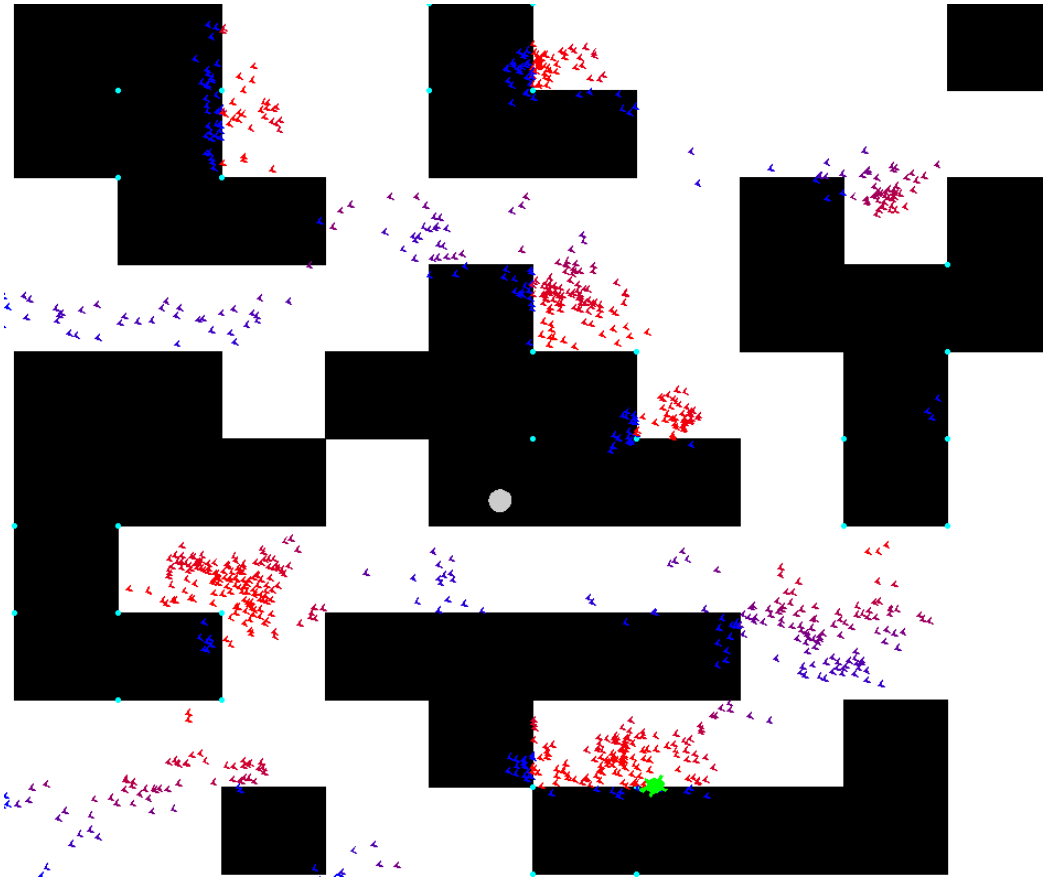
Performance

Approximation error of particle filtering remains bounded over time.

- At least empirically.
- Theoretical analysis is difficult.



Robot localization



(See demo.)

Summary

- Temporal models use state and sensor variables replicated over time.
- Markov assumptions and stationarity assumption. So we only need:
  - transition model $P(\mathbf{X}_{t+1}|\mathbf{X}_t)$
  - sensor model $P(\mathbf{E}_t|\mathbf{X}_t)$
- Inference tasks include filtering, prediction, smoothing and most likely sequence.
- HMMs have a signal discrete state variable.
- Kalman filters allow n continuous state variables, assume a linear Gaussian model.
- DBNs generalize HMMs and Kalman filters.
  - Exact inference is usually intractable.
  - Particle filtering is a good approximate filtering algorithm for DBNs.

The end.

References

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