Introduction to Artificial Intelligence

Lecture 4: Representing uncertain knowledge

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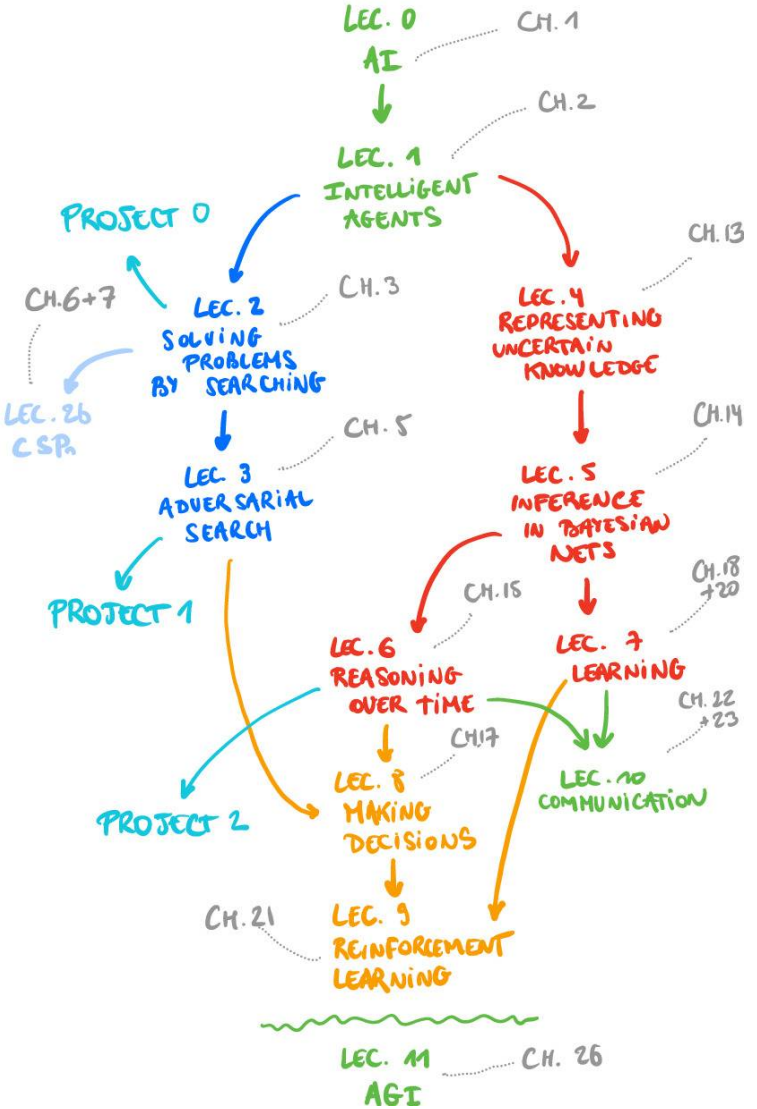
LIÈGE université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AI
CH. 1
LEC. 1
INTELLIGENT AGENTS
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PROJECT 0
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SOLVING PROBLEMS BY SEARCHING
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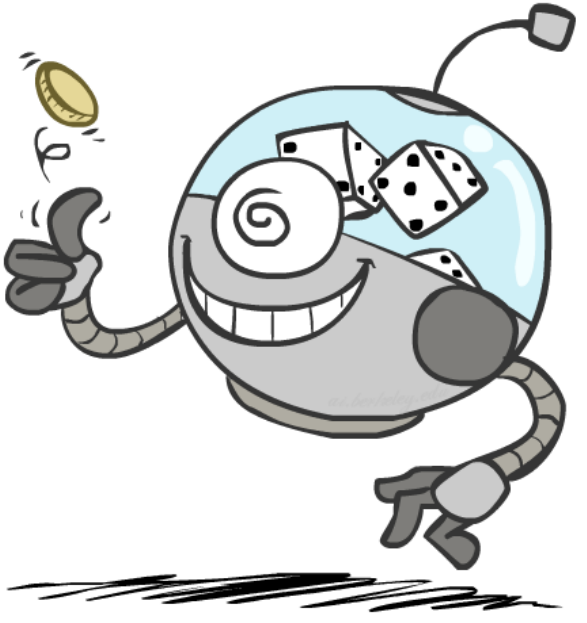
CH. 26

Today

- Probability:
  - Random variables
  - Joint and marginal distributions
  - Conditional distributions
  - Product rule, Chain rule, Bayes' rule
  - Inference
- Bayesian networks:
  - Representing uncertain knowledge
  - Semantics
  - Construction



Do not overlook this lecture!

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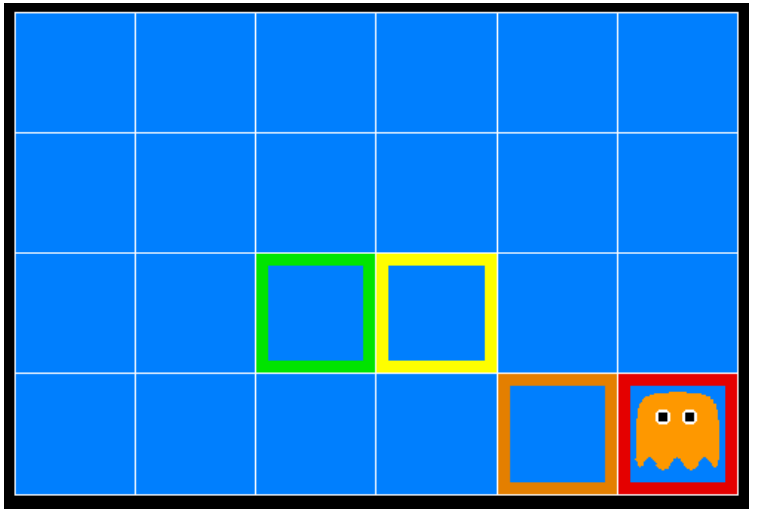
Image credits: CS188, UC Berkeley.

Quantifying uncertainty

A ghost is hidden in the grid somewhere.

Sensor readings tell how close a square is to the ghost:

- On the ghost: red
- 1 or 2 away: orange
- 3 away: yellow
- 4+ away green



Sensors are noisy, but we know the probability values $P(\text{color}|\text{distance})$, for all colors and all distances.

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Image credits: CS188, UC Berkeley.

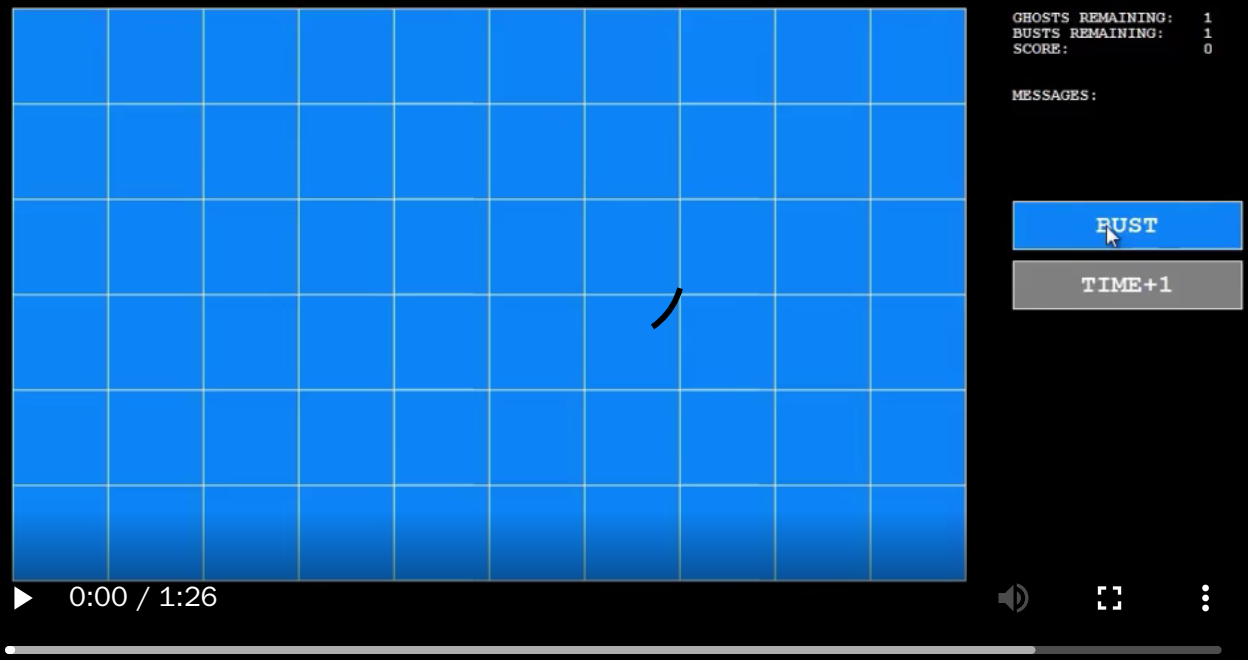

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Image credits: CS188, UC Berkeley.

Uncertainty

General setup:

- Observed variables or evidence: agent knows certain things about the state of the world (e.g., sensor readings).
- Unobserved variables: agent needs to reason about other aspects that are uncertain (e.g., where the ghost is).
- (Probabilistic) model: agent knows or believes something about how the observed variables relate to the unobserved variables.

Probabilistic reasoning provides a framework for managing our knowledge and beliefs.

Probabilistic assertions

Probabilistic assertions express the agent's inability to reach a definite decision regarding the truth of a proposition.

- Probability values summarize effects of
  - laziness (failure to enumerate all world states)
  - ignorance (lack of relevant facts, initial conditions, correct model, etc).
- Probabilities relate propositions to one's own state of knowledge (or lack thereof).
  - e.g., $P(\text{ghost in cell } [3, 2]) = 0.02$

Frequentism vs. Bayesianism

What do probability values represent?

- The objectivist frequentist view is that probabilities are real aspects of the universe.
  - i.e., propensities of objects to behave in certain ways.
  - e.g., the fact that a fair coin comes up heads with probability 0.5 is a propensity of the coin itself.
- The subjectivist Bayesian view is that probabilities are a way of characterizing an agent's beliefs or uncertainty.
  - i.e., probabilities do not have external physical significance.
  - This is the interpretation of probabilities that we will use!

Kolmogorov's probability theory

Begin with a set Ω, the sample space.

$\omega \in \Omega$ is a sample point or possible world.

A probability space is a sample space equipped with a probability function, i.e. an assignment $P : \mathcal{P}(\Omega) \to \mathbb{R}$ such that:

- 1st axiom: $P(\omega) \in \mathbb{R}, 0 \leq P(\omega)$ for all $\omega \in \Omega$
- 2nd axiom: $P(\Omega) = 1$
- 3rd axiom: $P(\{\omega_1, ..., \omega_n\}) = \sum_{i=1}^{n} P(\omega_i)$ for any set of samples

where $\mathcal{P}(\Omega)$ the power set of Ω.

Example

- Ω = the 6 possible rolls of a die.
- ω_i (for $i = 1, \ldots, 6$) are the sample points, each corresponding to an outcome of the die.
- Assignment P for a fair die:

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

Random variables

- A random variable is a function $X : \Omega \to D_X$ from the sample space to some domain defining its outcomes.
  - e.g., $\mathrm{Odd} : \Omega \to \{\mathrm{true}, \mathrm{false}\}$ such that $\mathrm{Odd}(\omega) = (\omega \bmod 2 = 1)$.
- P induces a probability distribution for any random variable X.
  - $P(X = x_i) = \sum_{\{\omega : X(\omega) = x_i\}} P(\omega)$
  - e.g., $P(\mathrm{Odd} = \mathrm{true}) = P(1) + P(3) + P(5) = \frac{1}{2}$.
- An event E is a set of outcomes $\{(x_1, ..., x_n)_i\}$ of the variables $X_1, ..., X_n$, such that

$$P(E) = \sum_{(x_1,...,x_n) \in E} P(X_1 = x_1, ..., X_n = x_n).$$

Notations

- Random variables are written in upper roman letters: X, Y, etc.
- Realizations of a random variable are written in corresponding lower case letters. E.g., x_1, x_2, ..., x_n could be of outcomes of the random variable X.
- The probability value of the realization x is written as $P(X = x)$.
- When clear from context, this will be abbreviated as $P(x)$.
- The probability distribution of the (discrete) random variable X is denoted as $\mathbf{P}(X)$. This corresponds e.g. to a vector of numbers, one for each of the probability values $P(X = x_i)$ (and not to a single scalar value!).

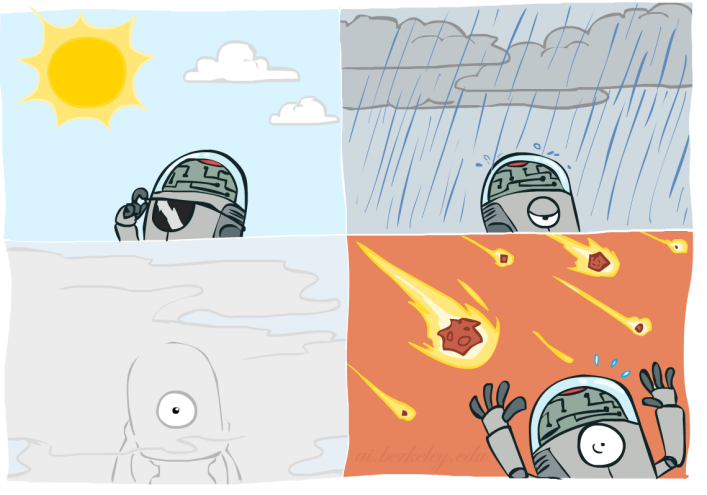
Probability distributions

For discrete variables, the probability distribution can be encoded by a discrete list of the probabilities of the outcomes, known as the probability mass function.

One can think of the probability distribution as a table that associates a probability value to each outcome of the variable.

$$\mathbf{P}(W)$$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0



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Joint distributions

A joint probability distribution over a set of random variables $X_1, \ldots, X_n$ specifies the probability of each (combined) outcome:

$$P(X_1 = x_1, \ldots, X_n = x_n) = \sum_{\{\omega : X_1(\omega) = x_1, \ldots, X_n(\omega) = x_n\}} P(\omega)$$

$$\mathbf{P}(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Marginal distributions

The marginal distribution of a subset of a collection of random variables is the joint probability distribution of the variables contained in the subset.

$\mathbf{P}(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$\mathbf{P}(T)$

T	P
hot	0.5
cold	0.5

$\mathbf{P}(W)$

W	P
sun	0.6
rain	0.4

$$P(t) = \sum_w P(t, w) \qquad P(w) = \sum_t P(t, w)$$

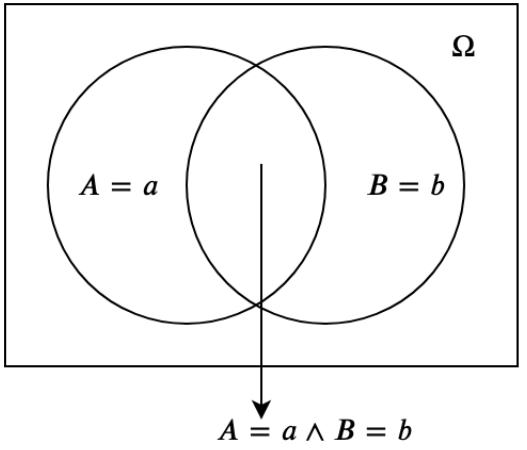
Intuitively, marginal distributions are sub-tables which eliminate variables.

Conditional distributions

The conditional probability of a realization a given the realization b is defined as the ratio of the probability of the joint realization a and b, and the probability of b:

$$P(a|b) = \frac{P(a,b)}{P(b)}.$$

Indeed, observing $B = b$ rules out all those possible worlds where $B \neq b$, leaving a set whose total probability is just $P(b)$. Within that set, the worlds for which $A = a$ satisfy $A = a \wedge B = b$ and constitute a fraction $P(a,b)/P(b)$.



Conditional distributions are probability distributions over some variables, given fixed values for others.

$\mathbf{P}(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$\mathbf{P}(W|T = \text{hot})$

T	P
sun	0.8
rain	0.2

$\mathbf{P}(W|T = \text{cold})$

W	P
sun	0.4
rain	0.6

Probabilistic inference

Probabilistic inference is the problem of computing a desired probability from other known probabilities (e.g., conditional from joint).

- We generally compute conditional probabilities.
  - e.g., $P(\text{on time}|\text{no reported accidents}) = 0.9$
  - These represent the agent's beliefs given the evidence.
- Probabilities change with new evidence:
  - e.g., $P(\text{on time}|\text{no reported accidents}, 5\text{AM}) = 0.95$
  - e.g., $P(\text{on time}|\text{no reported accidents}, \text{rain}) = 0.8$
  - e.g., $P(\text{ghost in } [3, 2]|\text{red in } [3, 2]) = 0.99$
  - Observing new evidence causes beliefs to be updated.

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Image credits: CS188, UC Berkeley.

General case

- Evidence variables: $E_1, ..., E_k = e_1, ..., e_k$
- Query variables: Q
- Hidden variables: $H_1, ..., H_r$
- $(Q \cup E_1, ..., E_k \cup H_1, ..., H_r)$ = all variables $X_1, ..., X_n$

Inference is the problem of computing $\mathbf{P}(Q|e_1, ..., e_k)$.

Inference by enumeration

Start from the joint distribution $\mathbf{P}(Q, E_1, ..., E_k, H_1, ..., H_r)$.

1. Select the entries consistent with the evidence $E_1, ..., E_k = e_1, ..., e_k$.
2. Marginalize out the hidden variables to obtain the joint of the query and the evidence variables:

$$\mathbf{P}(Q, e_1, ..., e_k) = \sum_{h_1, ..., h_r} \mathbf{P}(Q, h_1, ..., h_r, e_1, ..., e_k).$$

3. Normalize:

$$Z = \sum_q P(q, e_1, ..., e_k)$$

$$\mathbf{P}(Q|e_1, ..., e_k) = \frac{1}{Z}\mathbf{P}(Q, e_1, ..., e_k)$$

Example

- $\mathbf{P}(W)$?
- $\mathbf{P}(W|\text{winter})$?
- $\mathbf{P}(W|\text{winter}, \text{hot})$?

S	T	W	P
summer	hot	sun	0.3
summer	hot	rain	0.05
summer	cold	sun	0.1
summer	cold	rain	0.05
winter	hot	sun	0.1
winter	hot	rain	0.05
winter	cold	sun	0.15
winter	cold	rain	0.2

Complexity

- Inference by enumeration can be used to answer probabilistic queries for discrete variables (i.e., with a finite number of values).
- However, enumeration does not scale!
  - Assume a domain described by n variables taking at most d values.
  - Space complexity: $O(d^n)$
  - Time complexity: $O(d^n)$

Exercise
Can we reduce the size of the representation of the joint distribution?

Product rule

$$P(a, b) = P(b)P(a|b)$$

Example

$\mathbf{P}(W)$

W	P
sun	0.8
rain	0.2

$\mathbf{P}(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

$\mathbf{P}(D, W)$

D	W	P
wet	sun	?
dry	sun	?
wet	rain	?
dry	rain	?

Chain rule

More generally, any joint distribution can always be written as an incremental product of conditional distributions:

$$P(x_1, x_2, x_3) = P(x_1)P(x_2|x_1)P(x_3|x_1, x_2)$$

$$P(x_1, ..., x_n) = \prod_{i=1}^{n} P(x_i|x_1, ..., x_{i-1})$$

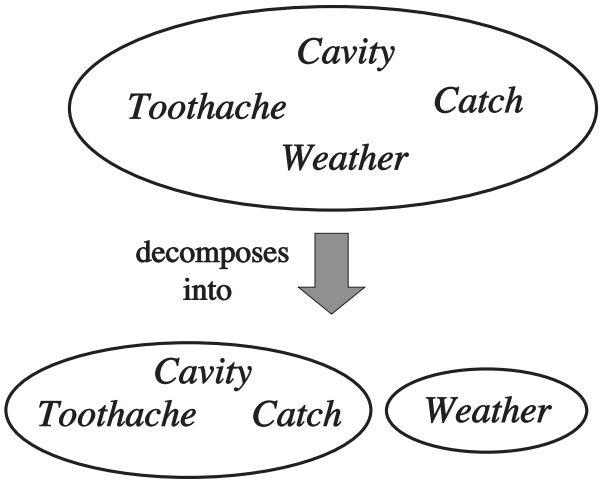
Independence

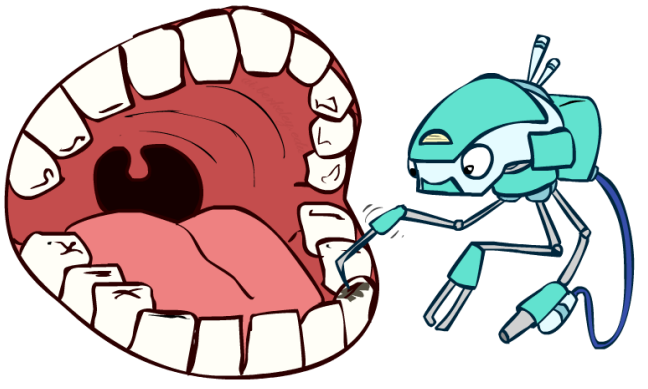
A and B are independent iff, for all $a \in D_A$ and $b \in D_B$,

- $P(a|b) = P(a)$, or
- $P(b|a) = P(b)$, or
- $P(a, b) = P(a)P(b)$

Independence is denoted as $A \perp B$.

Example 1





$$P(\text{toothache}, \text{catch}, \text{cavity}, \text{weather})$$
$$= P(\text{toothache}, \text{catch}, \text{cavity})P(\text{weather})$$

The original 32-entry table reduces to one 8-entry and one 4-entry table (assuming 4 values for Weather and boolean values otherwise).

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Image credits: CS188, UC Berkeley.

Example 2

For n independent coin flips, the joint distribution can be fully factored and represented as the product of n 1-entry tables.

- $2^n \rightarrow n$

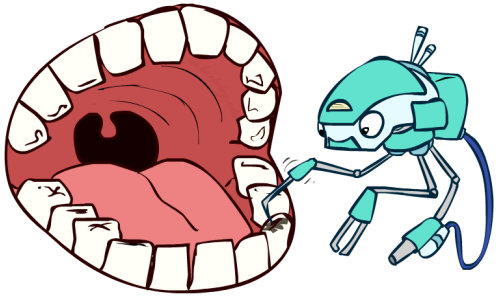
Conditional independence

A and B are conditionally independent given C iff, for all $a \in D_A, b \in D_B$ and $c \in D_C$,

- $P(a|b,c) = P(a|c)$, or
- $P(b|a,c) = P(b|c)$, or
- $P(a,b|c) = P(a|c)P(b|c)$

Conditional independence is denoted as $A \perp B|C$.

- Using the chain rule, the join distribution can be factored as a product of conditional distributions.
- Each conditional distribution may potentially be simplified by conditional independence.
- Conditional independence assertions allow probabilistic models to scale up.



Example 1

Assume three random variables $\mathrm{Toothache}$, Catch and Cavity.

Catch is conditionally independent of $\mathrm{Toothache}$, given Cavity. Therefore, we can write:

$$\begin{aligned}
&P(\mathrm{toothache}, \mathrm{catch}, \mathrm{cavity}) \\
&= P(\mathrm{toothache}|\mathrm{catch}, \mathrm{cavity})P(\mathrm{catch}|\mathrm{cavity})P(\mathrm{cavity}) \\
&= P(\mathrm{toothache}|\mathrm{cavity})P(\mathrm{catch}|\mathrm{cavity})P(\mathrm{cavity})
\end{aligned}$$

In this case, the representation of the joint distribution reduces to $2 + 2 + 1$ independent numbers (instead of $2^n - 1$).

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Image credits: CS188, UC Berkeley.

Example 2 (Naive Bayes)

More generally, from the product rule, we have

$$P(\text{cause}, \text{effect}_1, \ldots, \text{effect}_n) = P(\text{effect}_1, \ldots, \text{effect}_n | \text{cause}) P(\text{cause})$$

Assuming pairwise conditional independence between the effects given the cause, it comes:

$$P(\text{cause}, \text{effect}_1, \ldots, \text{effect}_n) = P(\text{cause}) \prod_i P(\text{effect}_i | \text{cause})$$

This probabilistic model is called a naive Bayes model.

- The complexity of this model is $O(n)$ instead of $O(2^n)$ without the conditional independence assumptions.
- Naive Bayes can work surprisingly well in practice, even when the assumptions are wrong.

Study the next slide. **Twice**.

The Bayes' rule

The product rule defines two ways to factor the joint distribution of two random variables.

$$P(a, b) = P(a|b)P(b) = P(b|a)P(a)$$

Therefore,

$$P(a|b) = \frac{P(b|a)P(a)}{P(b)}.$$

- $P(a)$ is the prior belief on a.
- $P(b)$ is the probability of the evidence b.
- $P(a|b)$ is the posterior belief on a, given the evidence b.
- $P(b|a)$ is the conditional probability of b given a. Depending on the context, this term is called the likelihood.

$$P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

The Bayes' rule is the foundation of many AI systems.

Example 1: diagnostic probability from causal probability.

$$P(\text{cause}|\text{effect}) = \frac{P(\text{effect}|\text{cause})P(\text{cause})}{P(\text{effect})}$$

where

- $P(\text{effect}|\text{cause})$ quantifies the relationship in the causal direction.
- $P(\text{cause}|\text{effect})$ describes the diagnostic direction.

Let S=stiff neck and M=meningitis. Given $P(s|m) = 0.7, P(m) = 1/50000$, $P(s) = 0.01$, it comes

$$P(m|s) = \frac{P(s|m)P(m)}{P(s)} = \frac{0.7 \times 1/50000}{0.01} = 0.0014.$$

Example 2: Ghostbusters, revisited

- Let us assume a random variable G for the ghost location and a set of random variables $R_{i,j}$ for the individual readings.
- We start with a uniform prior distribution $\mathbf{P}(G)$ over ghost locations.
- We assume a sensor reading model $\mathbf{P}(R_{i,j}|G)$.
  - That is, we know what the sensors do.
  - $R_{i,j}$ = reading color measured at $[i, j]$
    - e.g., $P(R_{1,1} = \text{yellow}|G = [1, 1]) = 0.1$
  - Two readings are conditionally independent, given the ghost position.

- We can calculate the posterior distribution $\mathbf{P}(G|R_{i,j})$ using Bayes' rule:

$$\mathbf{P}(G|R_{i,j}) = \frac{\mathbf{P}(R_{i,j}|G)\mathbf{P}(G)}{\mathbf{P}(R_{i,j})}.$$

- For the next reading $R_{i',j'}$, this posterior distribution becomes the prior distribution over ghost locations, which we update similarly.

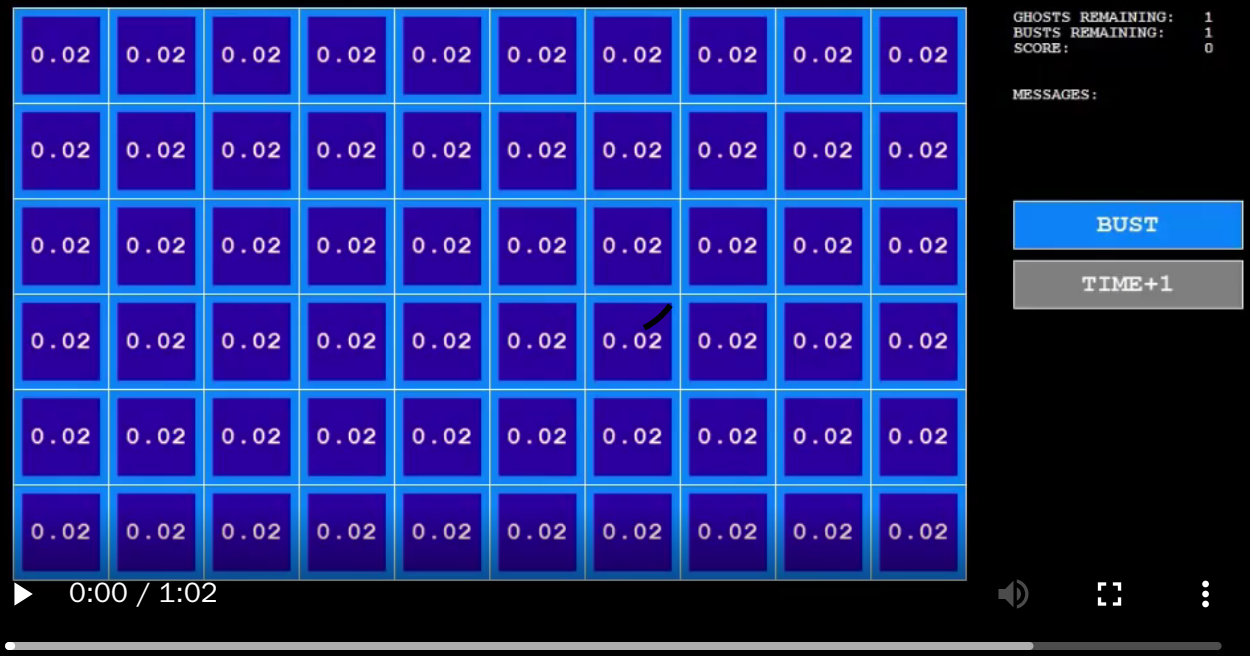
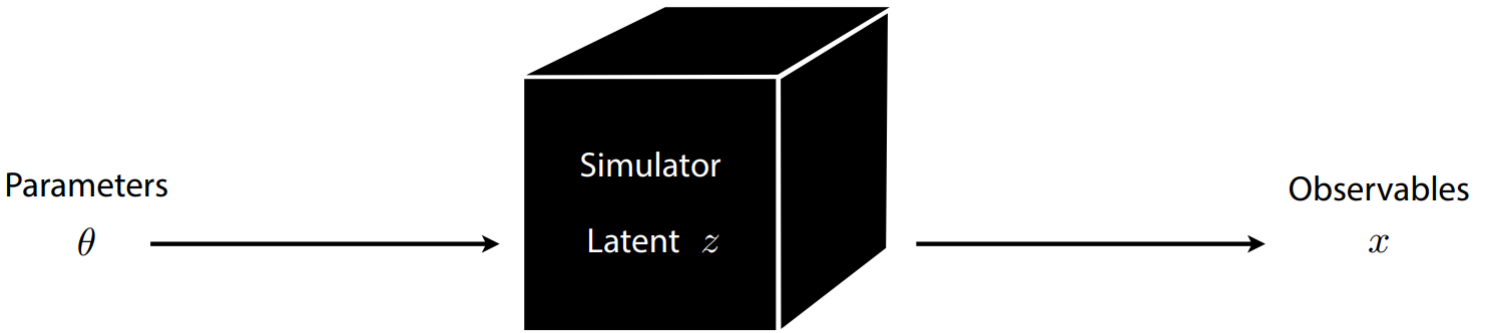
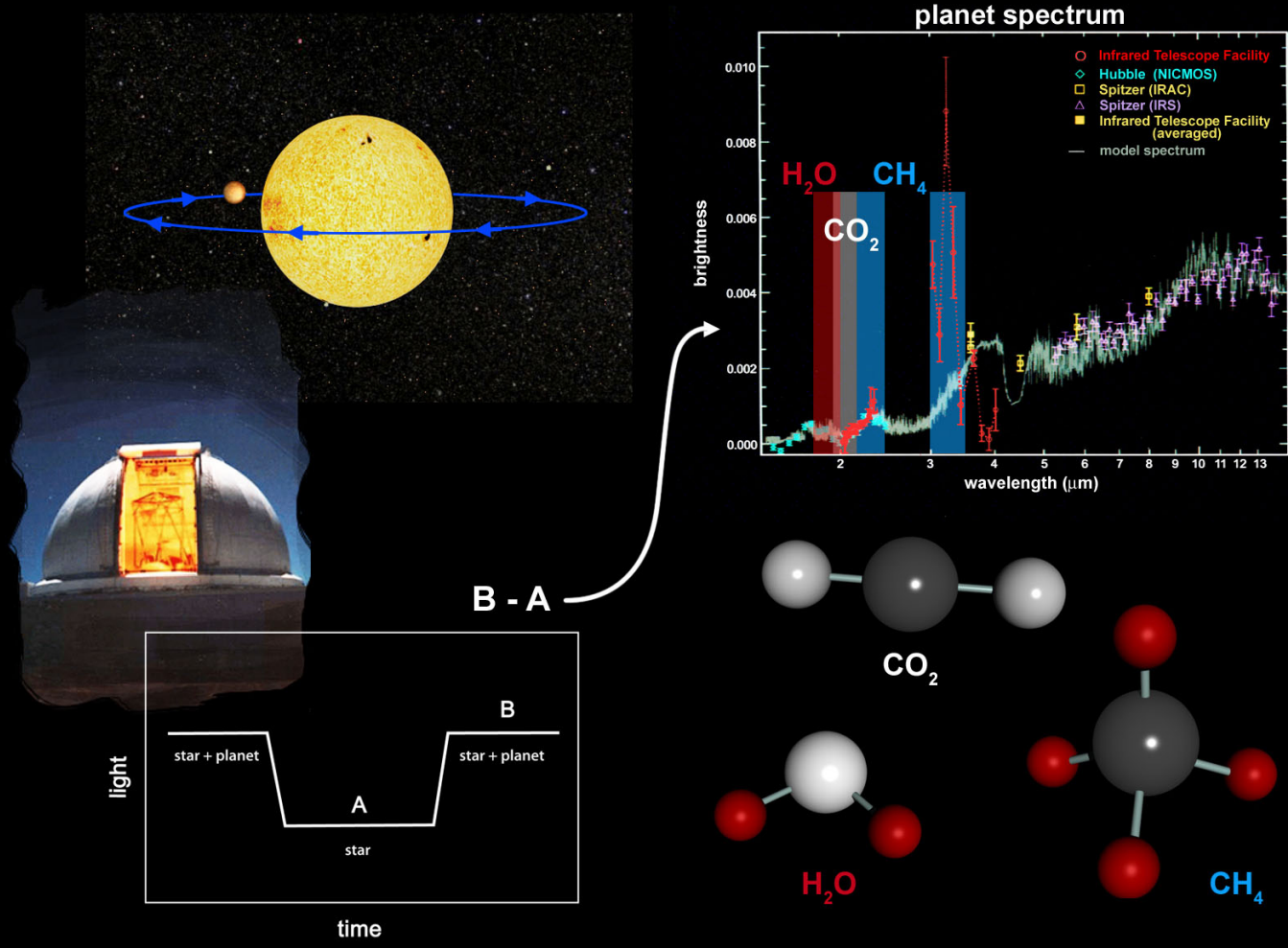


Image credits: CS188, UC Berkeley.

Example 3: AI for Science



Example 3: Exoplanet atmosphere characterization



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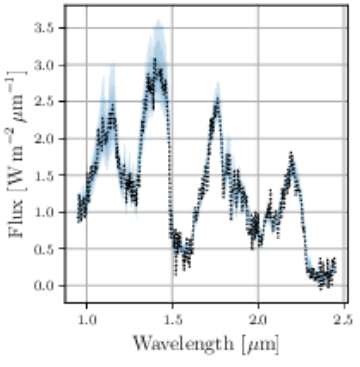
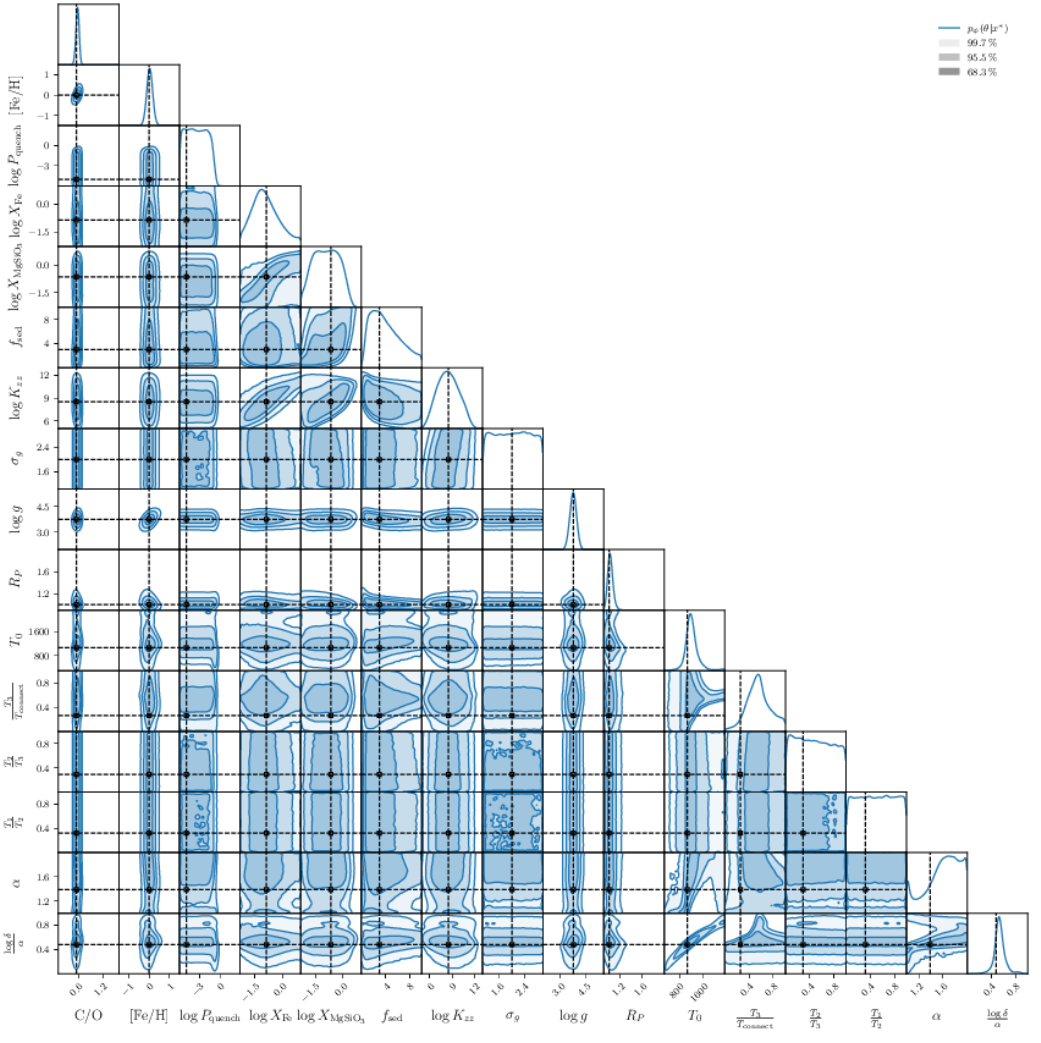
Credits: NSA/JPL-Caltech, 2010.


Flux [W m^{-2} μm^{-1}]
Wavelength [μm]


C/O
[Fe/H]
log $P_{\rm quench}$
log $X_{\rm Fe}$
log $X_{\rm MgSiO_3}$
$f_{\rm sed}$
log K_{zz}
σ_g
log g
R_P
T_0
$\frac{T_3}{T_{\rm connect}}$
$\frac{T_2}{T_3}$
$\frac{T_1}{T_2}$
α
$\frac{\log \delta}{\alpha}$

Probabilistic reasoning

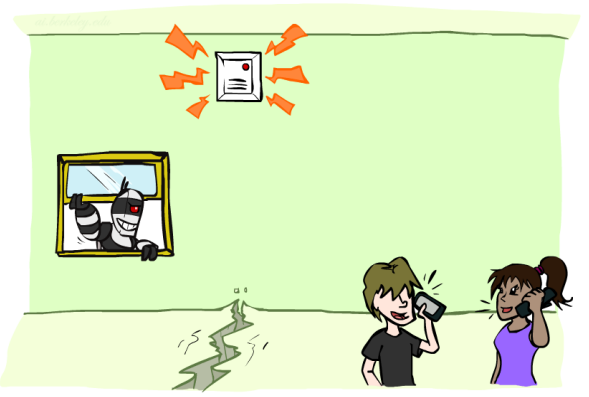
Representing knowledge

- The joint probability distribution can answer any question about the domain.
- However, its representation can become intractably large as the number of variable grows.
- Independence and conditional independence reduce the number of probabilities that need to be specified in order to define the full joint distribution.
- These relationships can be represented explicitly in the form of a Bayesian network.

Bayesian networks

A Bayesian network is a directed acyclic graph (DAG) in which:

- Each node corresponds to a random variable.
  - Can be observed or unobserved.
  - Can be discrete or continuous.
- Each edge indicates dependency relationships.
  - If there is an arrow from node X to node Y, X is said to be a parent of Y.
- Each node X_i is annotated with a conditional probability distribution $\mathbf{P}(X_i|\text{parents}(X_i))$ that quantifies the effect of the parents on the node.



Example 1

I am at work, neighbor John calls to say my alarm is ringing, but neighbor Mary does not call. Sometimes it's set off by minor earthquakes. Is there a burglar?

- Variables: $\mathrm{Burglar}$, $\mathrm{Earthquake}$, Alarm, $\mathrm{JohnCalls}$, $\mathrm{MaryCalls}$.
- Network topology from "causal" knowledge:
  - A burglar can set the alarm off
  - An earthquake can set the alaram off
  - The alarm can cause Mary to call
  - The alarm can cause John to call

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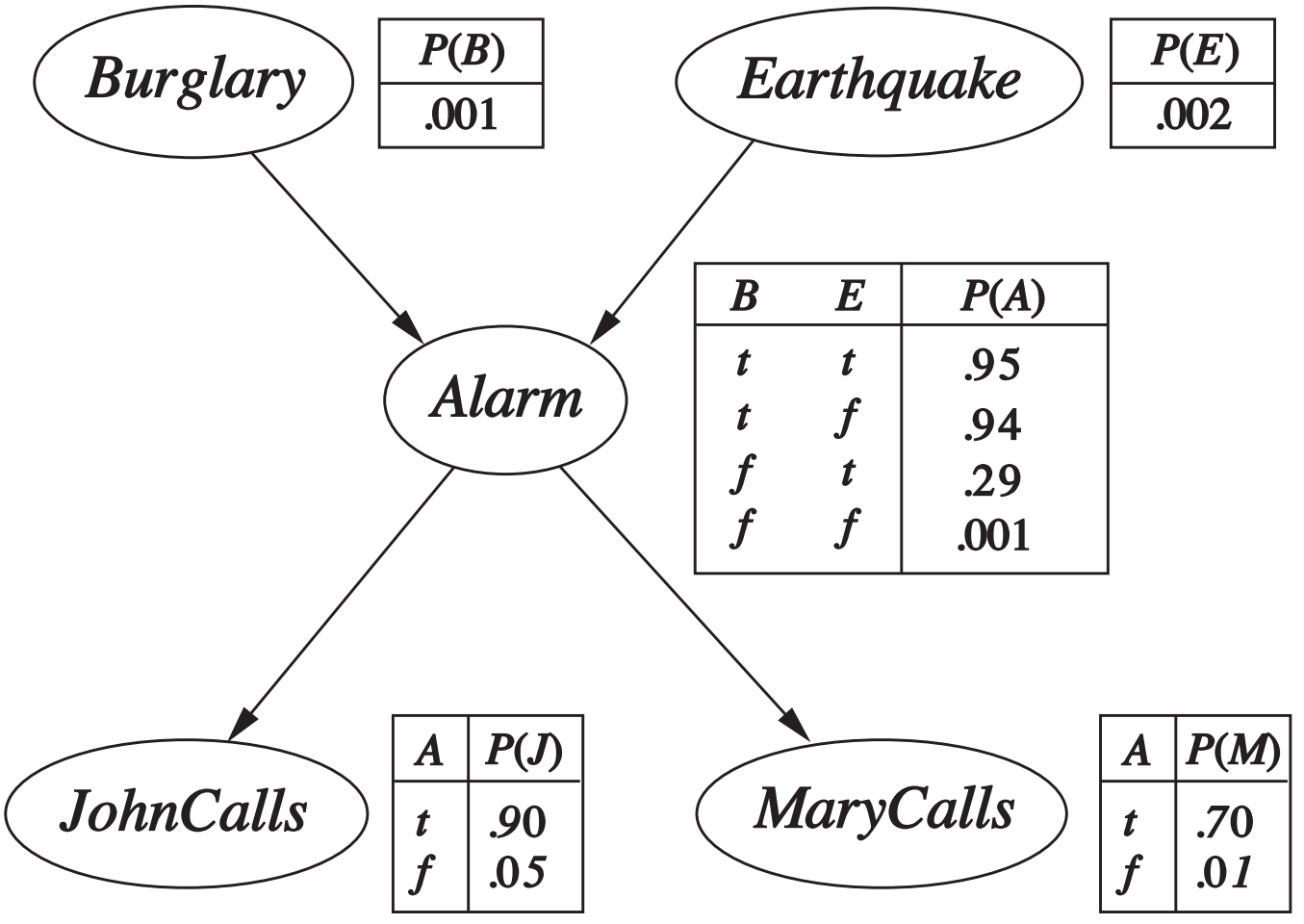
Image credits: CS188, UC Berkeley.


Burglary
P(B)
.001
Earthquake
P(E)
.002
Alarm
B E P(A)
t t .95
t f .94
f t .29
f f .001
JohnCalls
A P(J)
t .90
f .05
MaryCalls
A P(M)
t .70
f .01

Semantics

A Bayesian network implicitly encodes the full joint distribution as the product of the local distributions:

$$P(x_1, ..., x_n) = \prod_{i=1}^{n} P(x_i | \text{parents}(X_i))$$

Example

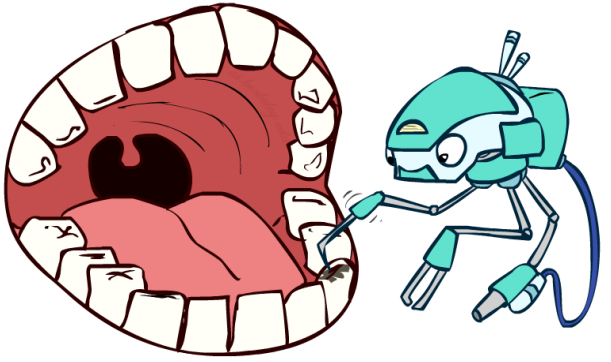
$$\begin{aligned} P(j, m, a, \neg b, \neg e) &= P(j|a)P(m|a)P(a|\neg b, \neg e)P(\neg b)P(\neg e) \\ &= 0.9 \times 0.7 \times 0.001 \times 0.999 \times 0.998 \\ &\approx 0.00063 \end{aligned}$$

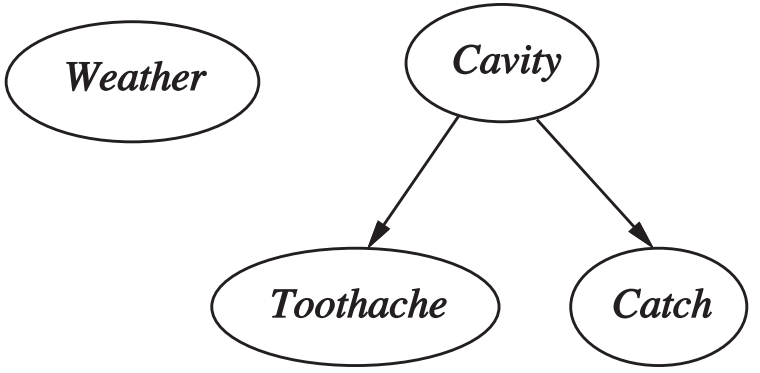
Why does $\prod_{i=1}^{n} P(x_i|\text{parents}(X_i))$ result in the proper joint probability?

- By the chain rule, $P(x_1, ..., x_n) = \prod_{i=1}^{n} P(x_i|x_1, ..., x_{i-1})$.
- Provided that we assume conditional independence of X_i with its predecessors in the ordering given the parents, and provided $\text{parents}(X_i) \subseteq \{X_1, ..., X_{i-1}\}$:

$$P(x_i|x_1, ..., x_{i-1}) = P(x_i|\text{parents}(X_i))$$

- Therefore $P(x_1, ..., x_n) = \prod_{i=1}^{n} P(x_i|\text{parents}(X_i))$.





Example 2

The topology of the network encodes conditional independence assertions:

- Weather is independent of the other variables.
- Toothache and Catch are conditionally independent given Cavity.

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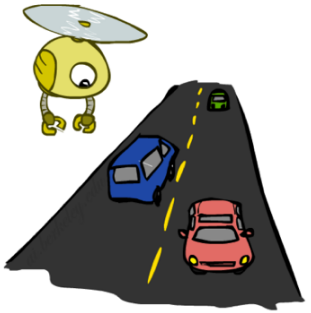
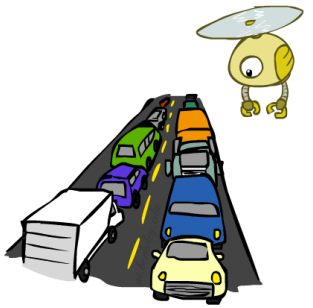
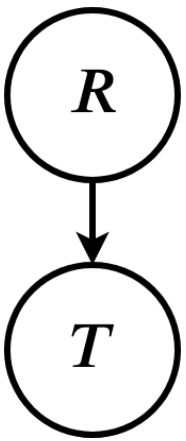
Image credits: CS188, UC Berkeley.





Example 3



$\mathbf{P}(R)$

R	P
r	0.25
¬r	0.75

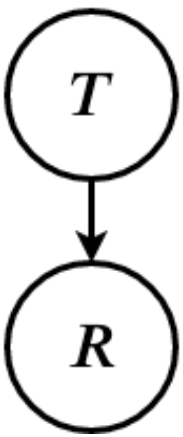
$\mathbf{P}(T|R)$

R	T	P
r	t	0.75
r	¬t	0.25
¬r	t	0.5
¬r	¬t	0.5

—

Image credits: CS188, UC Berkeley.

Example 3 (bis)



$\mathbf{P}(T)$

T	P
t	$9/16$
$\neg$t	$7/16$

$\mathbf{P}(R|T)$

T	R	P
t	r	$1/3$
t	$\neg$r	$2/3$
$\neg$t	r	$1/7$
$\neg$t	$\neg$r	$6/7$

—

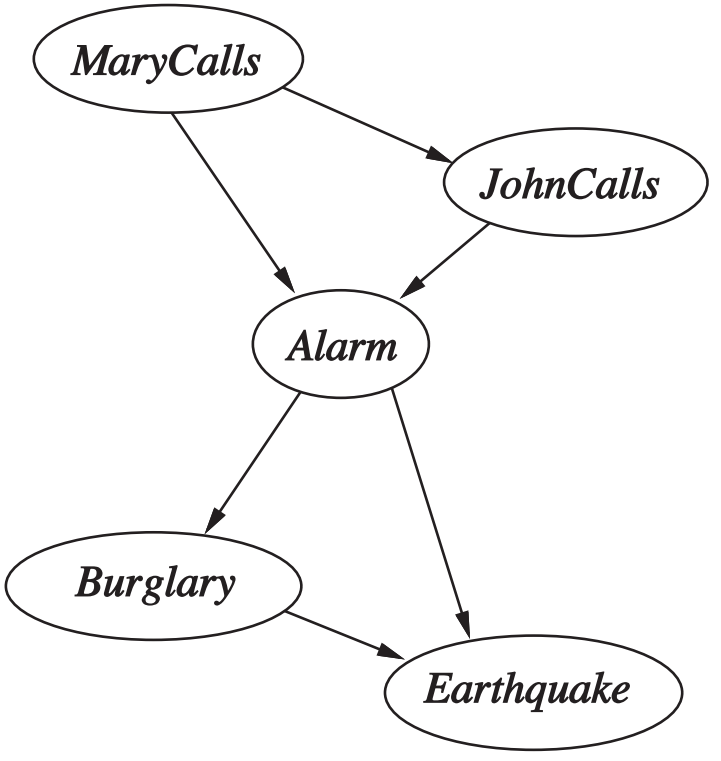
Image credits: CS188, UC Berkeley.

Construction

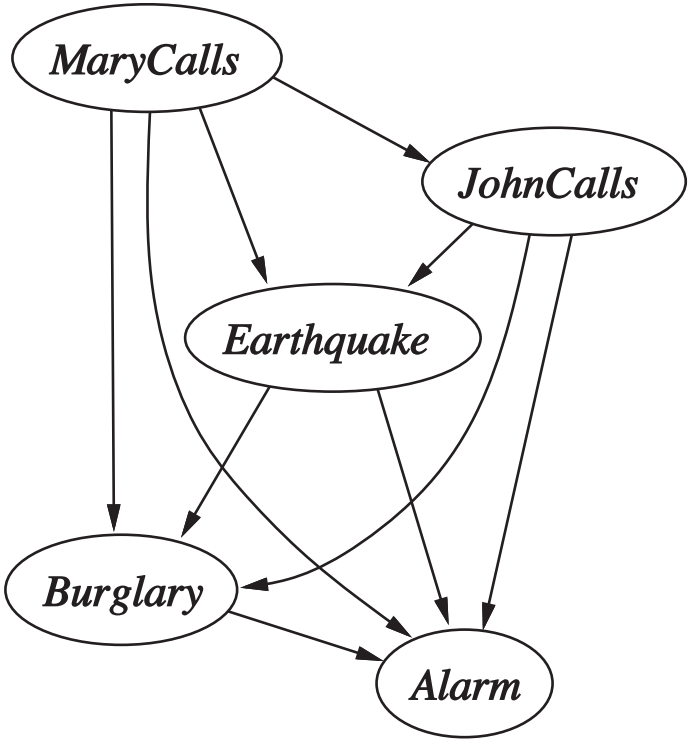
Bayesian networks are correct representations of the domain only if each node is conditionally independent of its other predecessors in the node ordering, given its parents.

Construction algorithm

1. Choose some ordering of the variables $X_1, ..., X_n$.
2. For $i = 1$ to n:
    1. Add X_i to the network.
    2. Select a minimal set of parents from $X_1, ..., X_{i-1}$ such that $P(x_i | x_1, ..., x_{i-1}) = P(x_i | \text{parents}(X_i))$.
    3. For each parent, insert a link from the parent to X_i.
    4. Write down the CPT.



(a)



(b)

Exercise
Do these networks represent the same distribution?

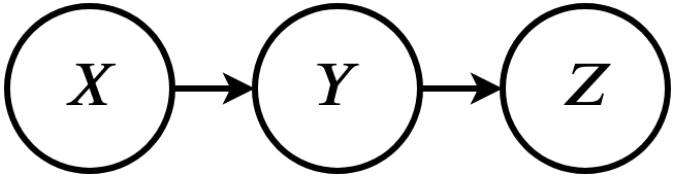
Compactness

- A CPT for boolean X_i with k boolean parents has 2^k rows for the combinations of parent values.
- Each row requires one number p for $X_i = \text{true}$.
  - The number for $X_i = \text{false}$ is just $1 - p$.
- If each variable has no more than k parents, the complete network requires $O(n \times 2^k)$ numbers.
  - i.e., grows linearly with n, vs. $O(2^n)$ for the full joint distribution.
- For the burglary net, we need $1 + 1 + 4 + 2 + 2 = 10$ numbers (vs. $2^5 - 1 = 31$).
- Compactness depends on the node ordering.

Independence

Important question: Are two nodes independent given certain evidence?

- If yes, this can be proved using algebra (tedious).
- If no, this can be proved with a counter example.



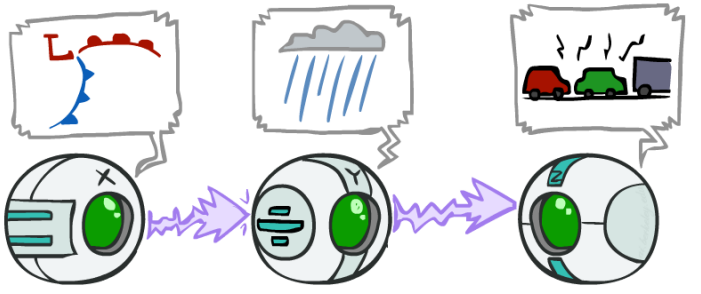
Example: Are X and Z necessarily independent?

Cascades

Is X independent of Z? No.

Counter-example:

- Low pressure causes rain causes traffic, high pressure causes no rain causes no traffic.
- In numbers:
  - $P(y|x) = 1$,
  - $P(z|y) = 1$,
  - $P(\neg y|\neg x) = 1$,
  - $P(\neg z|\neg y) = 1$



X: low pressure, Y: rain, Z: traffic.

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

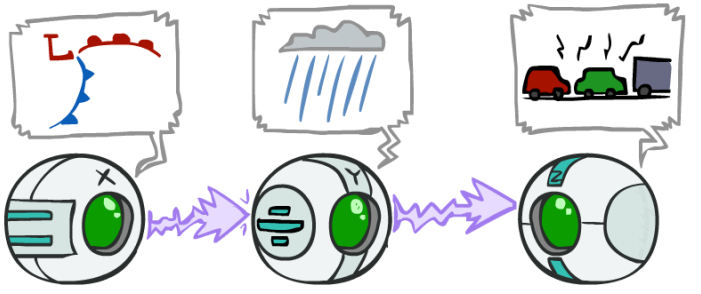
—

Image credits: CS188, UC Berkeley.

Is X independent of Z, given Y? Yes.

$$\begin{aligned} P(z|x,y) &= \frac{P(x,y,z)}{P(x,y)} \\ &= \frac{P(x)P(y|x)P(z|y)}{P(x)P(y|x)} \\ &= P(z|y) \end{aligned}$$

We say that the evidence along the cascade "blocks" the influence.



X: low pressure, Y: rain, Z: traffic.

$$P(x,y,z) = P(x)P(y|x)P(z|y)$$

—

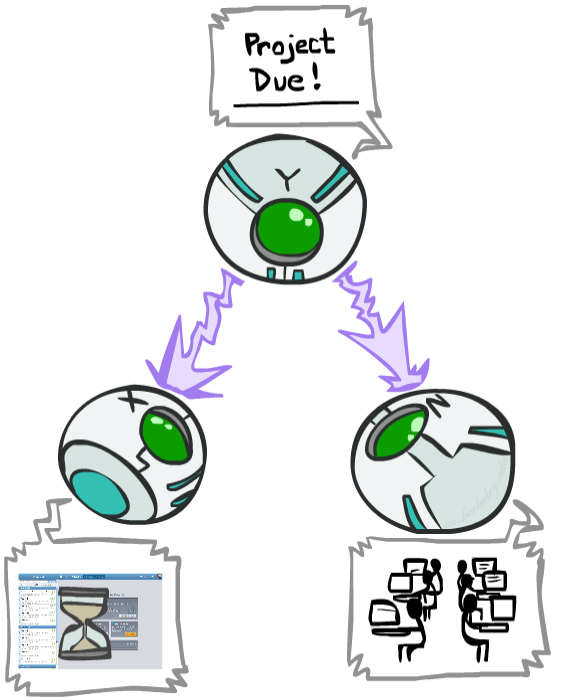
Image credits: CS188, UC Berkeley.

Common parent

Is X independent of Z? No.

Counter-example:

- Project due causes both forums busy and lab full.
- In numbers:
  - $P(x|y) = 1$,
  - $P(\neg x|\neg y) = 1$,
  - $P(z|y) = 1$,
  - $P(\neg z|\neg y) = 1$



X: forum busy, Y: project due, Z: lab full.

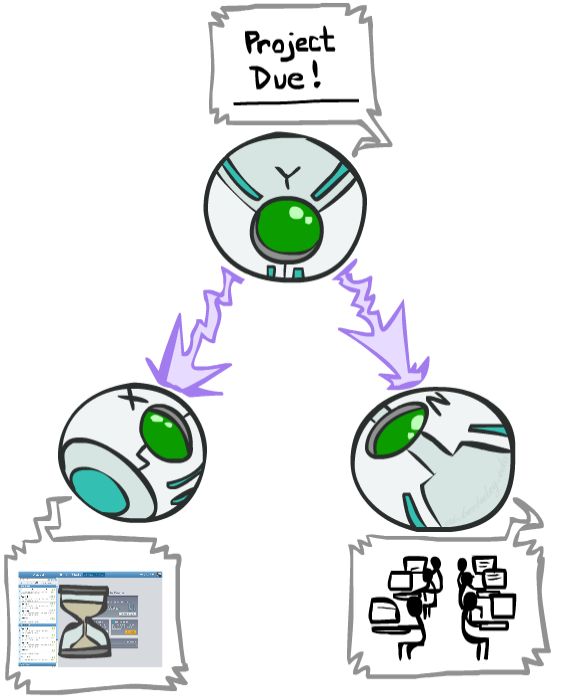
$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

—
Image credits: CS188, UC Berkeley.

Is X independent of Z, given Y? Yes

$$\begin{aligned} P(z|x,y) &= \frac{P(x,y,z)}{P(x,y)} \\ &= \frac{P(y)P(x|y)P(z|y)}{P(y)P(x|y)} \\ &= P(z|y) \end{aligned}$$

Observing the parent blocks the influence between the children.



X: forum busy, Y: project due, Z: lab full.

$$P(x,y,z) = P(y)P(x|y)P(z|y)$$

—
Image credits: CS188, UC Berkeley.

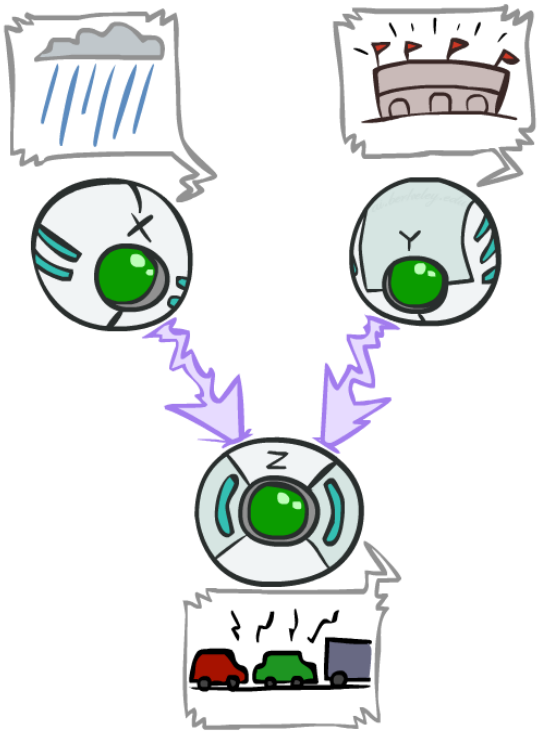
v-structures

Are X and Y independent? Yes.

- The ballgame and the rain cause traffic, but they are not correlated.
- (Prove it!)

Are X and Y independent given Z? No!

- Seeing traffic puts the rain and the ballgame in competition as explanation.
- This is backwards from the previous cases. Observing a child node activates influence between parents.



X: rain, Y: ballgame, Z: traffic.

$$P(x, y, z) = P(x)P(y)P(z|x, y)$$

—
Image credits: CS188, UC Berkeley.

d-separation

Let us assume a complete Bayesian network. Are X_i and X_j conditionally independent given evidence $Z_1 = z_1, ..., Z_m = z_m$?

Consider all (undirected) paths from X_i to X_j:

- If one or more active path, then independence is not guaranteed.
- Otherwise (i.e., all paths are inactive), then independence is guaranteed.

A path is active if each triple along the path is active:

- Cascade $A \rightarrow B \rightarrow C$ where B is unobserved (either direction).
- Common parent $A \leftarrow B \rightarrow C$ where B is unobserved.
- v-structure $A \rightarrow B \leftarrow C$ where B or one of its descendents is observed.

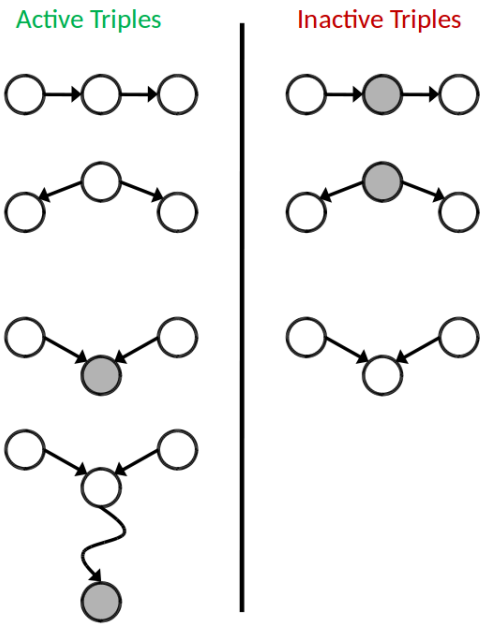


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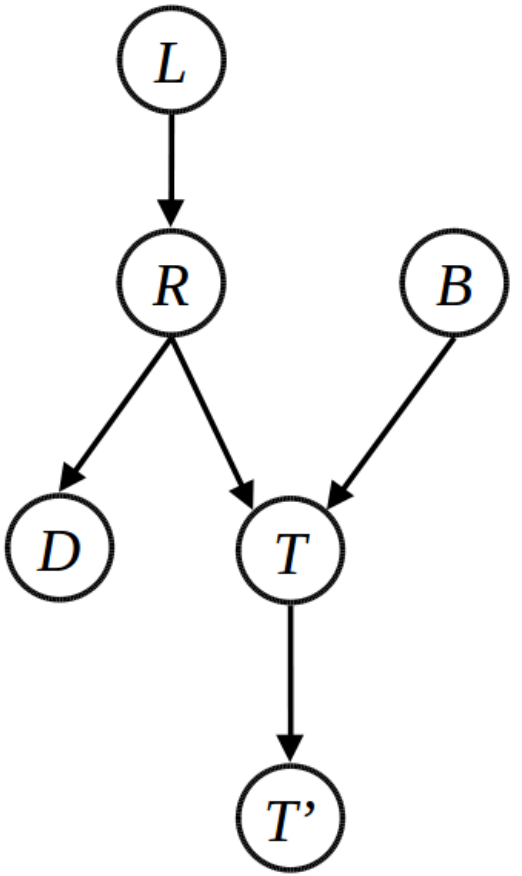
Image credits: CS188, UC Berkeley.

Example

- $L \perp T' | T$?
- $L \perp B$?
- $L \perp B | T$?
- $L \perp B | T'$?
- $L \perp B | T, R$?



Causality?

- When the network reflects the true causal patterns:
  - Often more compact (nodes have fewer parents).
  - Often easier to think about.
  - Often easier to elicit from experts.
- But, Bayesian networks need not be causal.
  - Sometimes no causal network exists over the domain (e.g., if variables are missing).
  - Edges reflect correlation, not causation.
- What do the edges really mean then?
  - Topology may happen to encode causal structure.
  - Topology really encodes conditional independence.

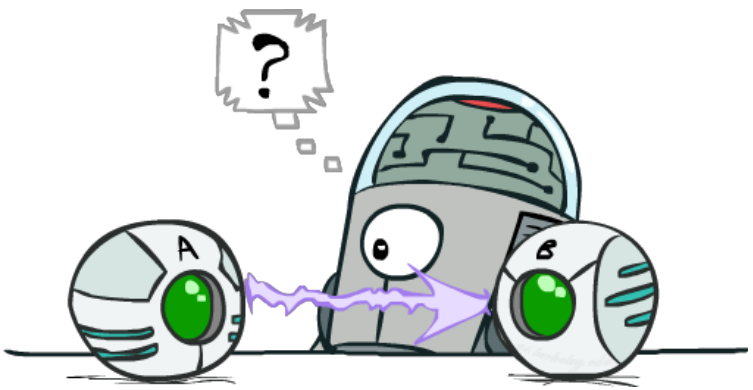


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Image credits: CS188, UC Berkeley.

- Correlation does not imply causation.
- Causes cannot be expressed in the language of probability theory.

Judea Pearl

Philosophers have tried to define causation in terms of probability: $X = x$ causes $Y = y$ if $X = x$ raises the probability of $Y = y$.

However, the inequality

$$P(y|x) > P(y)$$

fails to capture the intuition behind "probability raising", which is fundamentally a causal concept connoting a causal influence of $X = x$ over $Y = y$.

The correct formulation should read

$$P(y|\mathrm{do}(X = x)) > P(y),$$

where $\mathrm{do}(X = x)$ stands for an external intervention where X is set to the value x instead of being observed.

Observing vs. intervening

- The reading in barometer is useful to predict rain.

$$P(\text{rain}|\text{Barometer} = \text{high}) > P(\text{rain}|\text{Barometer} = \text{low})$$

- But hacking a barometer will not cause rain!

$$P(\text{rain}|\text{Barometer hacked to high}) = P(\text{rain}|\text{Barometer hacked to low})$$

Summary

- Uncertainty arises because of laziness and ignorance. It is inescapable in complex non-deterministic or partially observable environments.
- Probabilistic reasoning provides a framework for managing our knowledge and beliefs, with the Bayes' rule acting as the workhorse for inference.
- A Bayesian Network specifies a full joint distribution. They are often exponentially smaller than an explicitly enumerated joint distribution.
- The structure of a Bayesian network encodes conditional independence assumptions between random variables.

The end.