Introduction to Artificial Intelligence (INFO8006)

Exercises 4 – Reasoning over time

September 16, 2022

Learning outcomes

At the end of this session you should be able to

- formulate a Markov model for discrete-time reasoning problems;
- define the simplifying assumptions of Markov processes;
- define and apply prediction, filtering and smoothing in Markov processes;
- apply the simplified matrix algorithm(s) to hidden Markov models;
- define the Kalman filter assumptions and manipulate multivariate Gaussian distributions.

Exercise 1 Umbrella World (AIMA, Section 15.1.1)

You are a security guard stationed at a secret underground installation. You want to know whether it is raining today, but your only access to the outside world occurs each morning when you see the director coming in with, or without, an umbrella. For each day t, the evidence is a single variable $Umbrella_t \in \{1, 0\}$, *i.e.* whether the umbrella appears or not, and the (hidden) state is a single variable $Rain_t \in \{1, 0\}$, *i.e.* whether it is raining or not.

You believe that from one day $t - 1$ to the next t, the chances that the weather stays the same are 70 %. You also believe that the director brings his umbrella 90 % of the time when it is raining, and 20 % of the time otherwise.

1. You would like to represent your umbrella world as a Markov model. What formal assumptions correspond to your beliefs ?

2. Sketch a Bayesian network structure describing the umbrella world and provide the transition and sensor models.

3. Express the distributions $P(R_{t+1}|R_{t-1})$, $P(U_t|R_{t-1})$ and $P(R_t|R_{t-1}, U_t)$ in terms of the transition and sensor models.

4. Suppose you observe an unending sequence of days on which the umbrella appears. Show that, as the days go by, the probability of rain on the current day tends monotonically towards a fixed point. Calculate this fixed point.

5. Now consider forecasting further and further into the future, given t umbrella observations. Is there a fixed point ? If yes, compute its exact value.

Exercise 2 The coins

You are in a room containing 3 precious biased gold coins a, b and c. You inspect the coins and notice that the coins a, b and c have a head probability of 80 %, 50 % and 20 %, respectively.

Another person enters the room, takes the coins and put them into a bag. They draw a coin from the bag and tell you that they will repeat 4 times the same routine: hide their hand in the bag and either keep the current coin with probability $\frac{2}{3}$ or replace it by another, then toss it and show you the result. They proceed and the sequence of results are heads, heads, tail, heads. If you answer right to the following questions they will give you the coins.

1. Provide a hidden Markov model (HMM) that describes the process.
2. What are the probabilities of the last coin given the sequence of evidences?
3. What are the probabilities of the first coin given the sequence of evidences? And of the first coin tossed?
4. What is the most likely sequence of tossed coins?

Exercise 3 September 2019 (AIMA, Ex 15.13 and 15.14)

A professor wants to know if students are getting enough sleep. Each day, the professor observes whether the students sleep in class, and whether they have red eyes. The professor has the following hypotheses:

- The prior probability of getting enough sleep, with no observations, is 0.7.
- The probability of getting enough sleep on night t is 0.8 given that the student got enough sleep the previous night, and 0.3 otherwise.
- The probability of having red eyes is 0.2 if the student got enough sleep, and 0.7 otherwise.
- The probability of sleeping in class is 0.1 if the student got enough sleep, and 0.3 otherwise.

The professor asks you to answer the following questions:

1. Formulate the environment and hypotheses as a dynamic Bayesian network that the professor could use to detect sleep deprived students, from a sequence of observations. Provide the associated probability tables.
2. Reformulate the dynamic Bayesian network as a hidden Markov model that has only a single observation variable. Give the complete probability tables for the model.
3. For the sequence $e_{1:3}$ of observations "no red eyes, not sleeping in class", "red eyes, not sleeping in class" and "red eyes, sleeping in class", calculate the distributions $P(EnoughSleep_t|e_{1:t})$ and $P(EnoughSleep_t|e_{1:3})$ for $t \in \{1, 2, 3\}$.

Exercise 4 Super Spring Ultra Pro Max XXL

The “Foire de Liège” has a new attraction called “Super Spring Ultra Pro Max XXL” which consists in a ball hanging from a spring. The participants take seat in the ball locked at some position. The ball is then released and pulled by the spring which makes it oscillate back and forth. After a few seconds, the ball is stopped by magnetic brakes. You notice that the final position of the ball is different each time. As the ball is opaque, you wonder if it is possible for the participants to guess where the ball stopped, given their perception of acceleration.

You more or less remember your Newtonian mechanics class and model the movement of the ball as a series of transitions

$$\begin{aligned}
p_t &= p_{t-1} + \Delta t\, \dot{p}_{t-1} + \frac{1}{2}\Delta t^2\, \ddot{p}_{t-1} \\
\dot{p}_t &= \dot{p}_{t-1} + \Delta t\, \ddot{p}_{t-1} \\
\ddot{p}_t &= g - \kappa\, p_{t-1} - \eta\, \dot{p}_{t-1} + \tau
\end{aligned}$$

where p_t, $\dot{p}_t$ and $\ddot{p}_t$ are respectively the position, velocity and acceleration of the ball at timestep t, Δt is the time elapsed between $t-1$ and t, g is earth's surface gravity, κ is the stiffness of the spring, η is the linear drag coefficient of the air and $\tau \sim \mathcal{N}(0, \sigma_\tau^2)$ is the random thrust caused by the wind. You estimate that the ball starts $10.0 \pm 0.5\,\mathrm{m}$ below the top of the spring with a negligible speed $0.0 \pm 0.1\,\mathrm{m\,s^{-1}}$ and an acceleration $g - \kappa 10 \pm \sigma_\tau$. You also assume that the human perception of acceleration follows an unbiased Gaussian distribution.

1. You wish to predict the state of the ball given the perceptions of a participant. Define the components of a Kalman filter in this context.
2. Express the distribution $p(x_t|e_{1:t})$ with respect to the components defined previously.
3. Represent the transition and sensor models as a dynamic Bayesian network.

Supplementary materials

- Hidden Markov Models (UC Berkeley CS188, Spring 2014 Section 6).
- 68–95–99.7 rule
- Chapter 15 of the reference textbook.