Introduction to Artificial Intelligence (INFO8006)

Exercises 4 – Reasoning over time

November 19, 2021

Learning outcomes

At the end of this session you should be able to

- formulate a Markov model for discrete-time reasoning problems.
- define the simplifying assumptions of Markov processes.
- define and apply prediction, filtering and smoothing in Markov processes.
- apply the simplified matrix algorithm(s) to hidden Markov models.
- define the Kalman filter assumptions and manipulate multivariate Gaussian distributions.

Exercise 1 Umbrella World (AIMA, Section 15.1.1)

You are a security guard stationed at a secret underground installation. You want to know whether it is raining today, but your only access to the outside world occurs each morning when you see the director coming in with, or without, an umbrella. For each day t, the evidence is a single variable $Umbrella_t \in \{1, 0\}$, *i.e.* whether the umbrella appears or not, and the (hidden) state is a single variable $Rain_t \in \{1, 0\}$, *i.e.* whether it is raining or not.

You believe that from one day $t-1$ to the next t, the chances that the weather stays the same are 70 %. You also believe that the director brings his umbrella 90 % of the time when it is raining, and 20 % of the time otherwise.

1. You would like to represent your umbrella world as a Markov model. What formal assumptions correspond to your beliefs ?

2. Sketch a Bayesian network structure describing the umbrella world and provide the transition and sensor models.

3. Express the distributions $P(R_{t+1} \mid R_{t-1})$, $P(U_t \mid R_{t-1})$ and $P(R_t \mid R_{t-1}, U_t)$ in terms of the transition and sensor models.

4. Suppose you observe an unending sequence of days on which the umbrella appears. Show that, as the days go by, the probability of rain on the current day tends monotonically towards a fixed point. Calculate this fixed point.

5. Now consider forecasting further and further into the future, given t umbrella observations. Is there a fixed point ? If yes, compute its exact value.

Exercise 2 The coins

You are in a room containing 3 precious biased coins a, b and c. You inspect the coins and notice that the coins a, b and c have a head probability of 80 %, 50 % and 20 %, respectively.

Another person enters the room, takes the coins and put them into a bag. They draw a coin from the bag and tell you that they will repeat 4 times the same routine: hide their hand in the bag and either keep the current coin with probability $\frac{2}{3}$ or replace it by another, then toss it and show you the result. They proceed and the sequence of results are heads, heads, tail, heads. If you answer right to the following questions they will give you the coins.

1. Provide a hidden Markov model (HMM) that describes the process.
2. What are the probabilities of the last coin given the sequence of evidences?
3. What are the probabilities of the first coin given the sequence of evidences? And of the first coin tossed?
4. What is the most likely sequence of tossed coins?

Exercise 3 September 2019 (AIMA, Ex 15.13 and 15.14)

A professor wants to know if students are getting enough sleep. Each day, the professor observes whether the students sleep in class, and whether they have red eyes. The professor has the following hypotheses:

- The prior probability of getting enough sleep, with no observations, is 0.7.
- The probability of getting enough sleep on night t is 0.8 given that the student got enough sleep the previous night, and 0.3 otherwise.
- The probability of having red eyes is 0.2 if the student got enough sleep, and 0.7 otherwise.
- The probability of sleeping in class is 0.1 if the student got enough sleep, and 0.3 otherwise.

The professor asks you to answer the following questions:

1. Formulate the environment and hypotheses as a dynamic Bayesian network that the professor could use to detect sleep deprived students, from a sequence of observations. Provide the associated probability tables.
2. Reformulate the dynamic Bayesian network as a hidden Markov model that has only a single observation variable. Give the complete probability tables for the model.
3. For the sequence $e_{1:3}$ of observations "no red eyes, not sleeping in class", "red eyes, not sleeping in class" and "red eyes, sleeping in class", calculate the distributions $P(EnoughSleep_t \mid e_{1:t})$ and $P(EnoughSleep_t \mid e_{1:3})$ for $t \in \{1, 2, 3\}$.

Exercise 4 Hyperloop

A few years ago, ULiège decided to get into SpaceX's Hyperloop Pod competition. Briefly, what one should do to win this competition is to build the fastest and most reliable autonomous pod. One of the most important engineering problem is to be able to compute a robust estimation of the state of the pod, *i.e.* its position p (m), speed[1] $\dot{p}$ ($\mathrm{m\,s^{-1}}$) and acceleration $\ddot{p}$ ($\mathrm{m\,s^{-2}}$), given noisy sensor measurements. You received an email from students asking you what would be your solution to this estimation problem.

The email contains information about the competition and the sensors placed on the pod. The track is straight (one-dimensional) and the pod has $T = 30\,\mathrm{s}$ to go as far as possible. The initial position and speed are assumed to be centered around 0 with a standard deviation of 0.1 m and $0.01\,\mathrm{m\,s^{-1}}$, respectively. The time between sensor measurements is $\Delta t = 0.5\,\mathrm{s}$. The sensors placed on the pod are 1) an unbiased GPS sensor providing the pod's position with 68 % chance to be less than 100 m away from the true position and 2) an unbiased accelerometer measuring the pod's acceleration with 68 % chance to be less than $1\,\mathrm{m\,s^{-2}}$ away from the true acceleration. The acceleration $\ddot{p}$ is the result of the combination of thrust and drag. The thrust is assumed to be normally distributed around μ_a with standard deviation σ_a. The drag is proportional but opposite in direction to the speed, *i.e.* it takes the form $-\eta\,\dot{p}$.

After some research you find out that you should use a Kalman filter to solve this task.

1. Define the components of your Kalman filter in the context of the state estimation of the pod.
2. While the pod is on the track, we wish to estimate at each time step t its state x_t given the evidences $e_{1:t}$ provided by the sensors until t. Determine the distribution $p(x_t \mid e_{1:t})$, with respect to the components defined previously.
3. Represent the transition and sensor processes as a dynamic Bayesian network.

[1] In physics, the "dot" is a short-hand for the derivative with respect to time. This is called Newton's notation.

Supplementary materials

- Hidden Markov Models (UC Berkeley CS188, Spring 2014 Section 6).
- 68–95–99.7 rule
- Chapter 15 of the reference textbook.