Deep Learning

Lecture 10: Auto-encoders and variational auto-encoders

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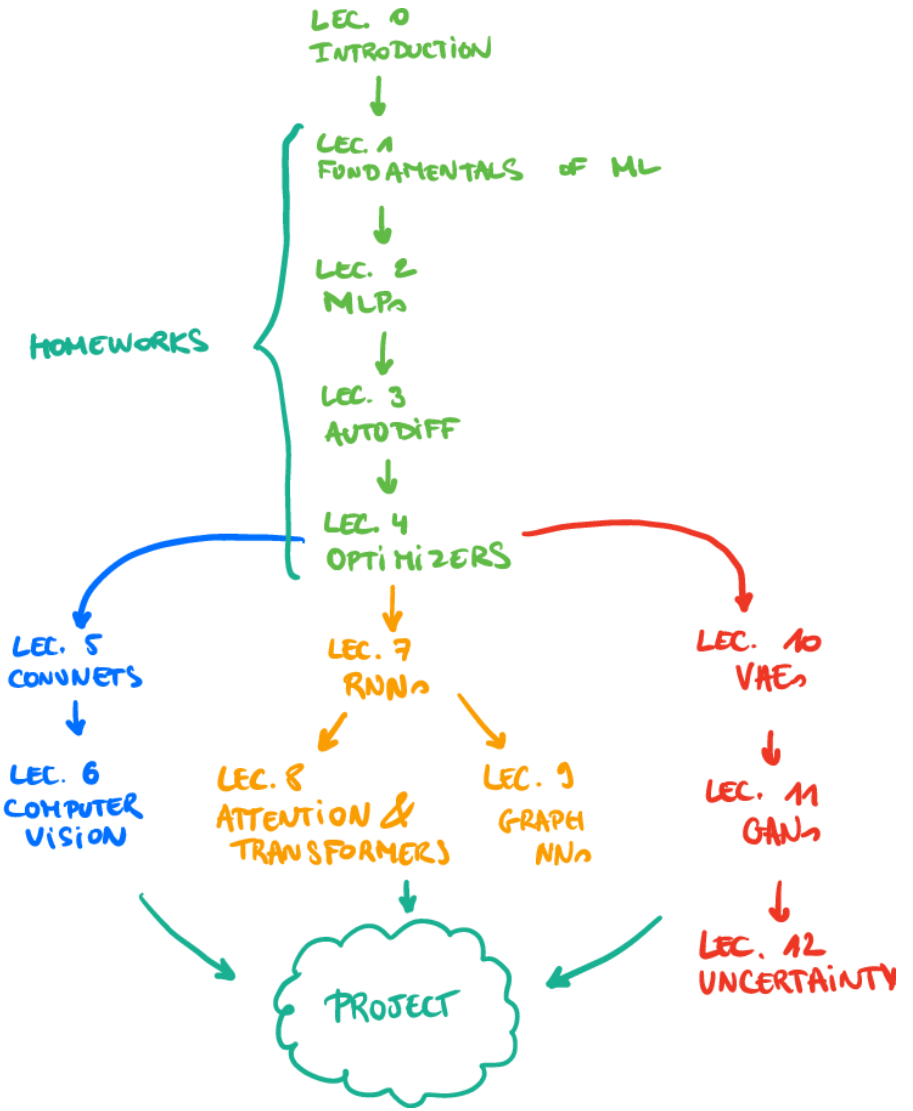
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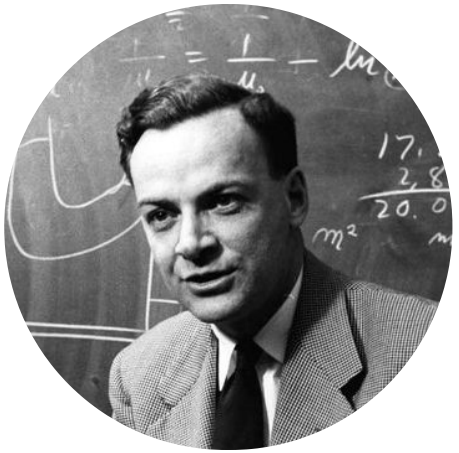
PROJECT

Today

Learn a model of the data.

- Auto-encoders
- Variational inference
- Variational auto-encoders



"What I cannot create, I do not understand."

Richard Feynman

"The brain has about 10^{14} synapses and we only live for about 10^{9} seconds. So we have a lot more parameters than data. This motivates the idea that we must do a lot of unsupervised learning since the perceptual input (including proprioception) is the only place we can get 10^{5} dimensions of constraint per second."

Geoffrey Hinton, 2014.

- "Pure" Reinforcement Learning (cherry)
  - The machine predicts a scalar reward given once in a while.
  - **A few bits for some samples**
- Supervised Learning (icing)
  - The machine predicts a category or a few numbers for each input
  - Predicting human-supplied data
  - **10→10,000 bits per sample**
- Unsupervised/Predictive Learning (cake)
  - The machine predicts any part of its input for any observed part.
  - Predicts future frames in videos
  - **Millions of bits per sample**
- (Yes, I know, this picture is slightly offensive to RL folks. But I'll make it up)

"We need tremendous amount of information to build machines that have common sense and generalize."

Yann LeCun, 2016.

Deep unsupervised learning

Deep unsupervised learning is about capturing rich patterns in raw data with deep networks in a label-free way.

- Generative models: recreate the raw data distribution (e.g., the distribution of natural images).
- Self-supervised learning: solve puzzle tasks that require semantic understanding (e.g., predict a missing word in a sequence).

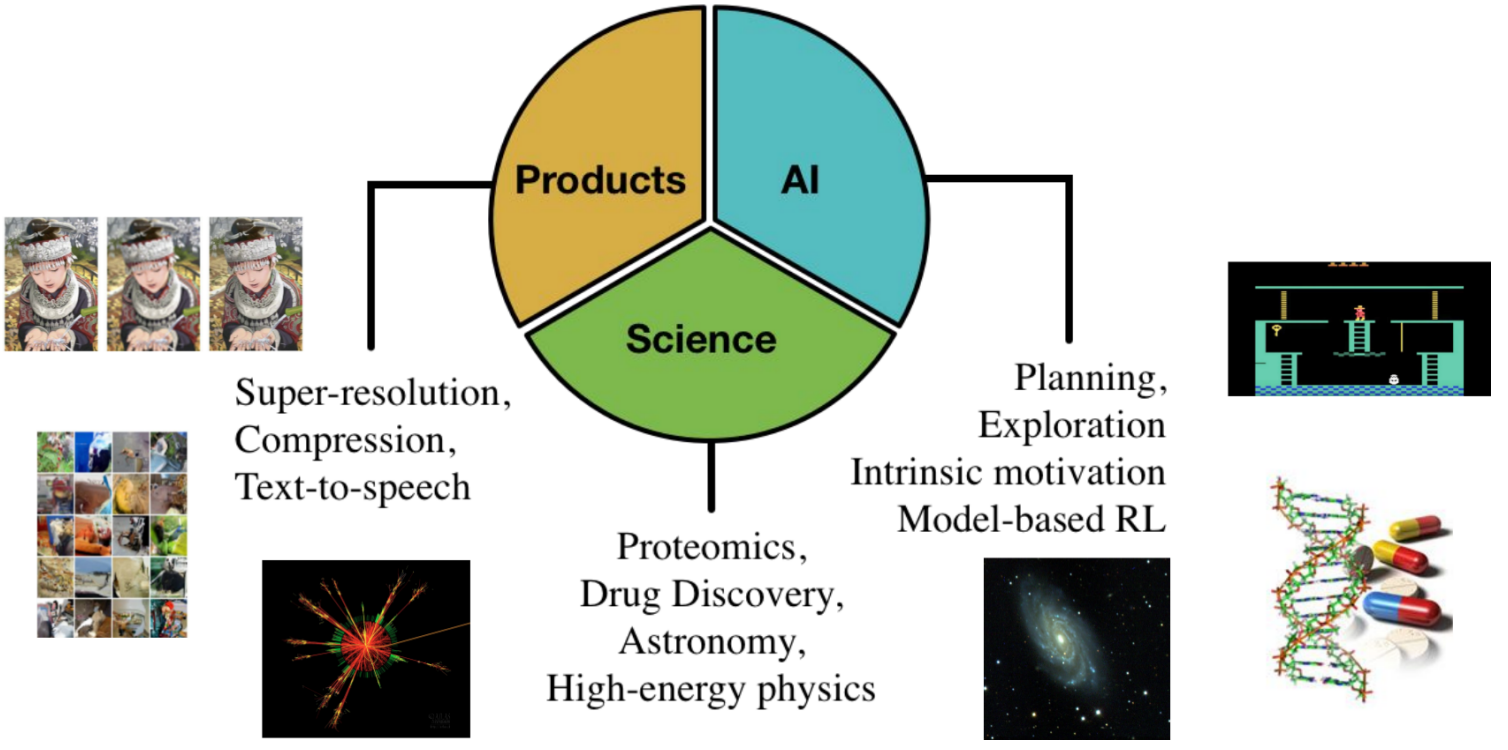
Generative models

A generative model is a probabilistic model p that can be used as a simulator of the data. Its purpose is to generate synthetic but realistic high-dimensional data

$$\mathbf{x} \sim p(\mathbf{x}; \theta),$$

that is as close as possible from the unknown data distribution $p(\mathbf{x})$, but for which we have empirical samples.

Immediate applications

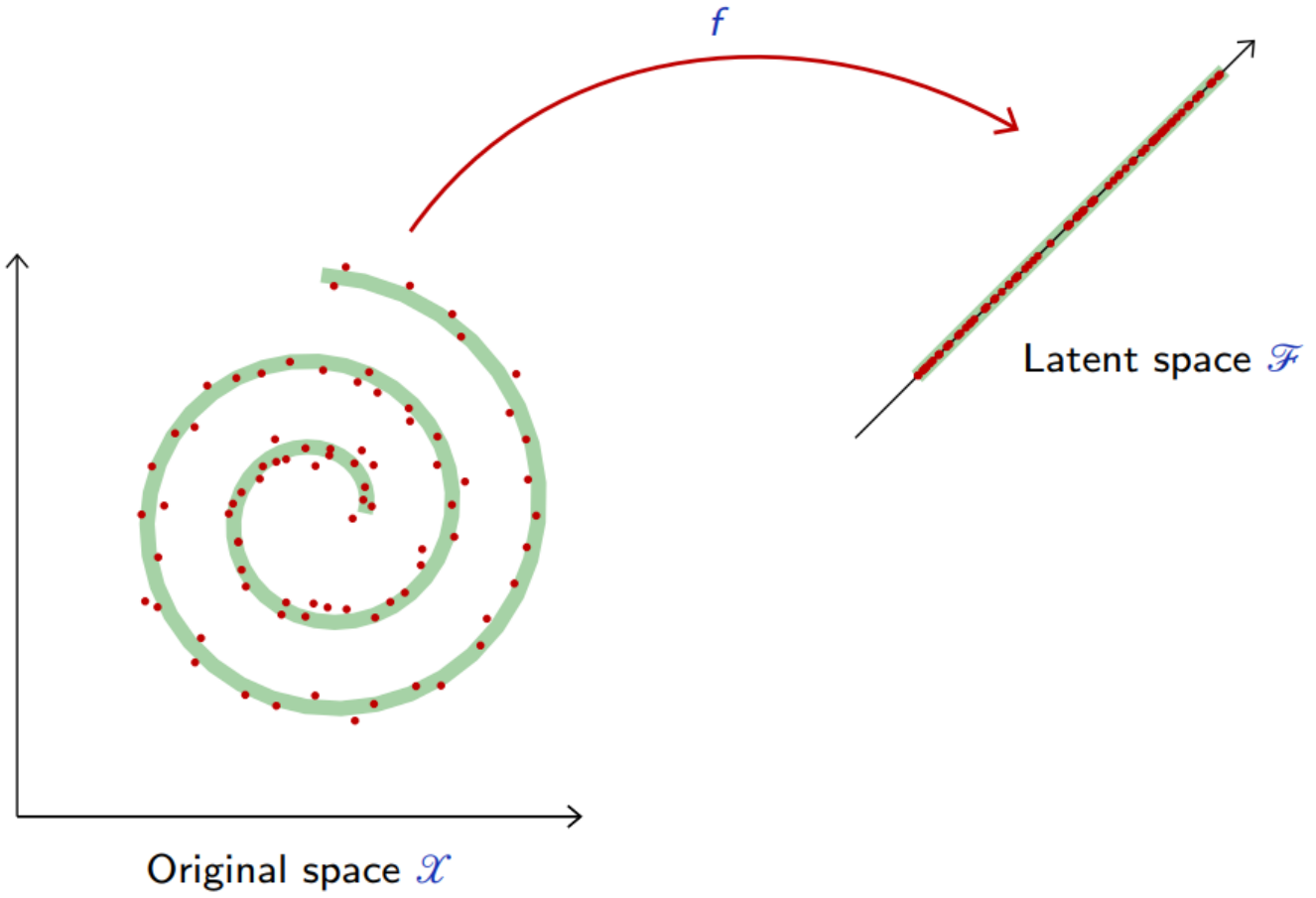


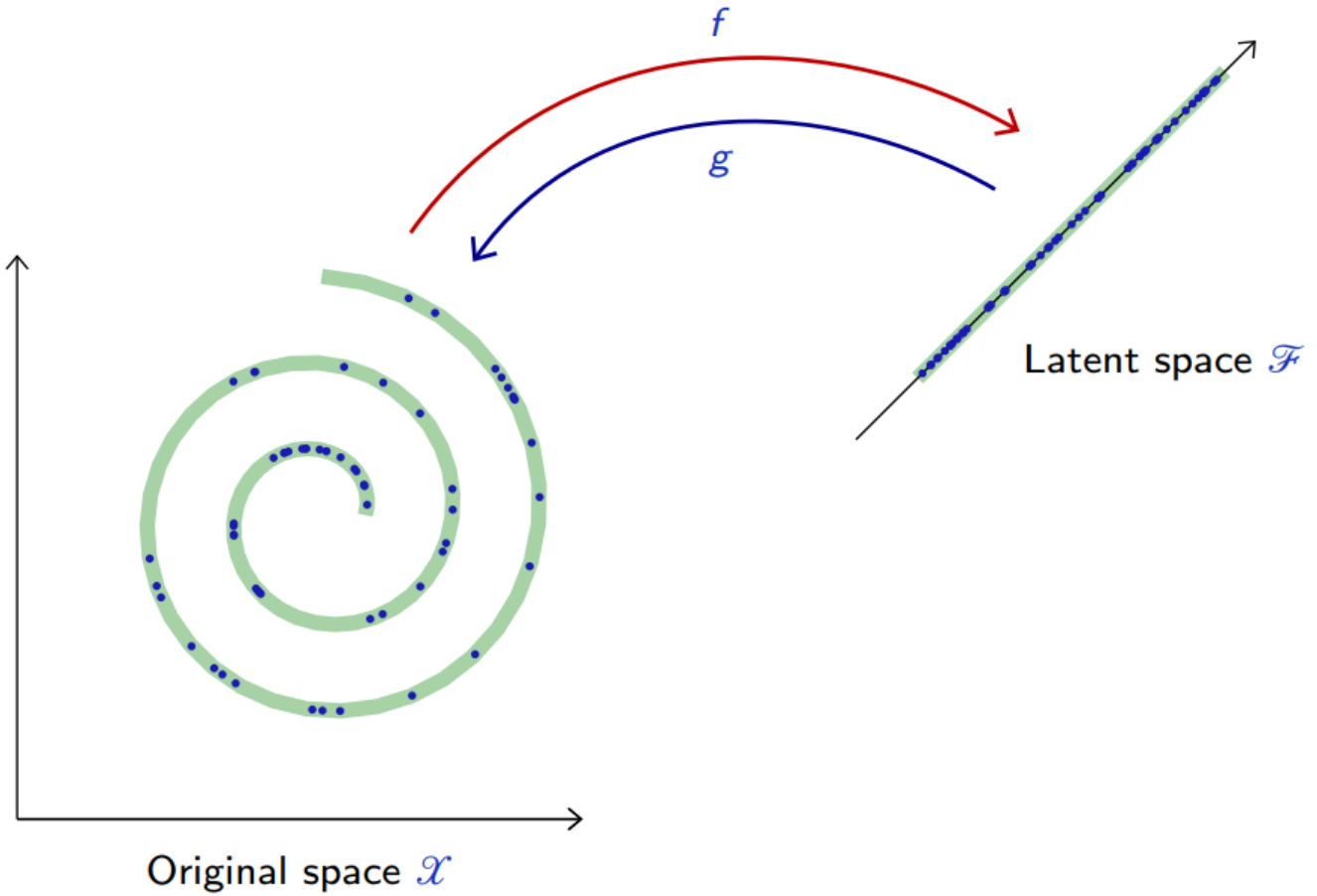
Generative models have a role in many important problems

Auto-encoders

Many applications such as image synthesis, denoising, super-resolution, speech synthesis or compression, require to go beyond classification and regression and model explicitly a high-dimensional signal.

This modeling consists of finding *"meaningful degrees of freedom"*, or *"factors of variations"*, that describe the signal and are of lesser dimension.


f
Latent space $\mathcal{F}$
Original space $\mathcal{X}$

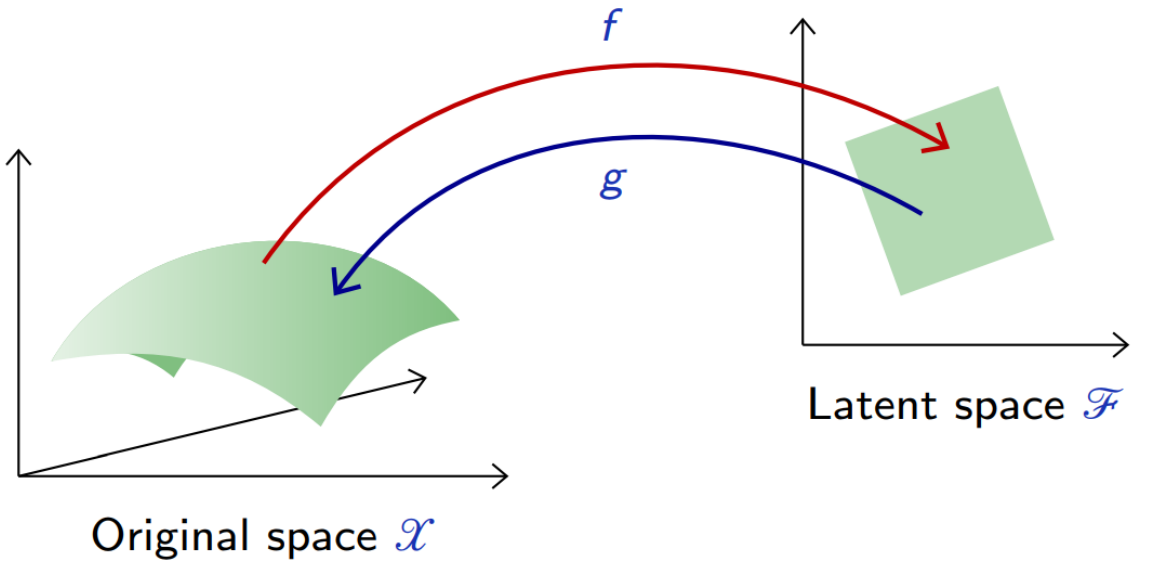

f
g
Latent space $\mathcal{F}$
Original space $\mathcal{X}$

Auto-encoders

An auto-encoder is a composite function made of

- an encoder f from the original space $\mathcal{X}$ to a latent space $\mathcal{Z}$,
- a decoder g to map back to $\mathcal{X}$,

such that $g \circ f$ is close to the identity on the data.



Let $p(\mathbf{x})$ be the data distribution over $\mathcal{X}$. A good auto-encoder could be characterized with the reconstruction loss

$$\mathbb{E}_{\mathbf{x} \sim p(\mathbf{x})} \left[||\mathbf{x} - g \circ f(\mathbf{x})||^2 \right] \approx 0.$$

Given two parameterized mappings $f(\cdot; \theta_f)$ and $g(\cdot; \theta_g)$, training consists of minimizing an empirical estimate of that loss,

$$\theta_f, \theta_g = \arg\min_{\theta_f, \theta_g} \frac{1}{N} \sum_{i=1}^{N} ||\mathbf{x}_i - g(f(\mathbf{x}_i, \theta_f), \theta_g)||^2.$$

For example, when the auto-encoder is linear,

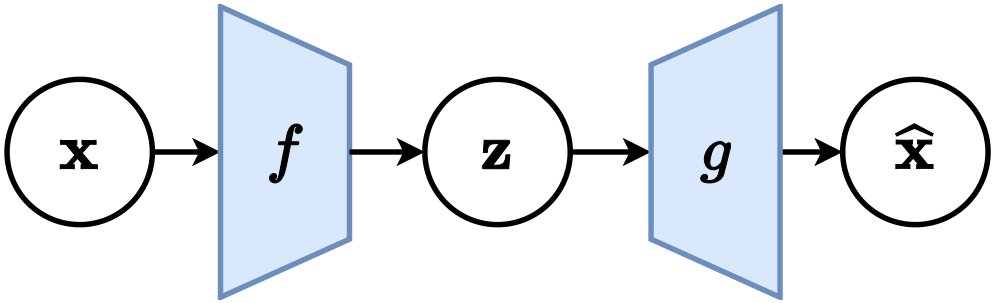
$$f : \mathbf{z} = \mathbf{U}^T \mathbf{x}$$
$$g : \hat{\mathbf{x}} = \mathbf{U}\mathbf{z},$$

with $\mathbf{U} \in \mathbb{R}^{p \times d}$, the reconstruction error reduces to

$$\mathbb{E}_{\mathbf{x} \sim p(\mathbf{x})} \left[||\mathbf{x} - \mathbf{U}\mathbf{U}^T \mathbf{x}||^2 \right].$$

In this case, an optimal solution is given by PCA.

Deep auto-encoders

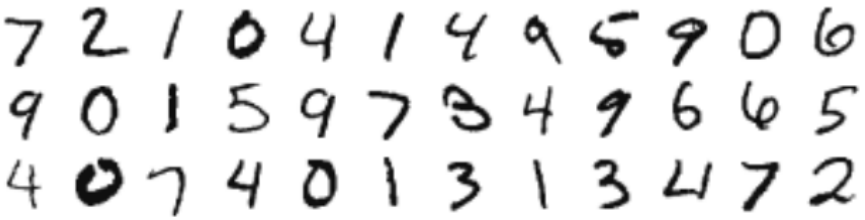


Better results can be achieved with more sophisticated classes of mappings than linear projections: use deep neural networks for f and g.

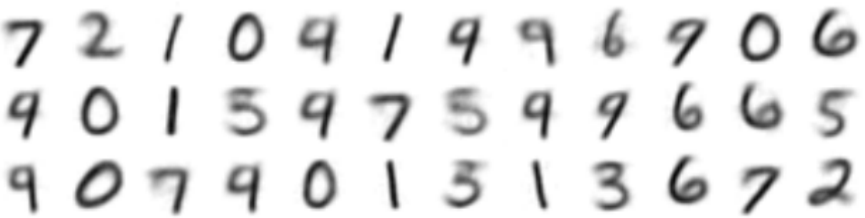
For instance,

- by combining a multi-layer perceptron encoder $f : \mathbb{R}^p \to \mathbb{R}^d$ with a multi-layer perceptron decoder $g : \mathbb{R}^d \to \mathbb{R}^p$.
- by combining a convolutional network encoder $f : \mathbb{R}^{w \times h \times c} \to \mathbb{R}^d$ with a decoder $g : \mathbb{R}^d \to \mathbb{R}^{w \times h \times c}$ composed of the reciprocal transposed convolutional layers.

X (original samples)

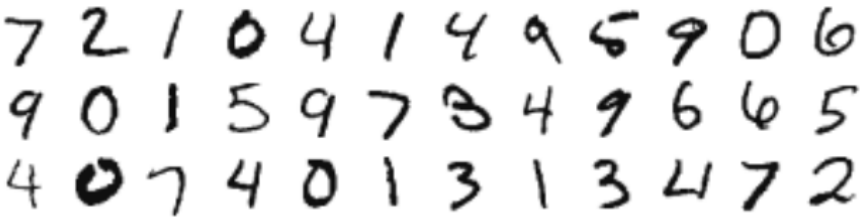


$g \circ f(X)$ (CNN, $d = 2$)

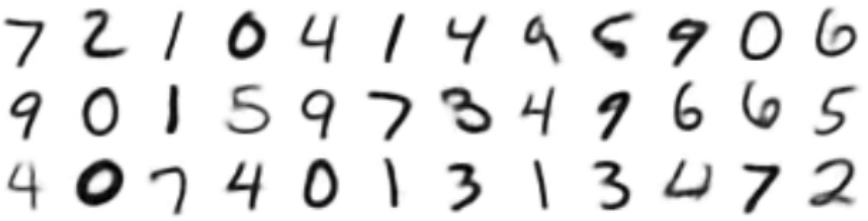


$g \circ f(X)$ (PCA, $d = 2$)

X (original samples)

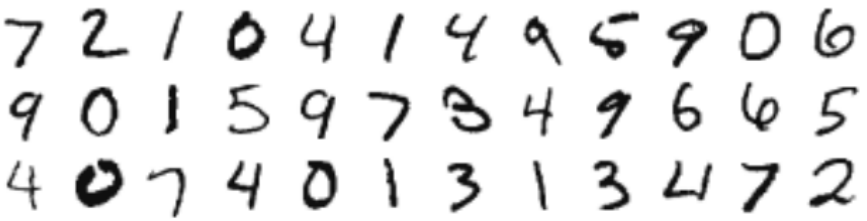


$g \circ f(X)$ (CNN, $d = 8$)

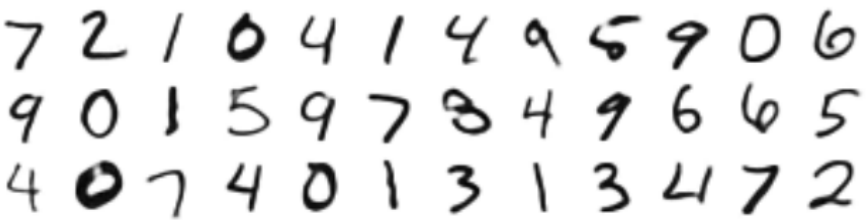


$g \circ f(X)$ (PCA, $d = 8$)

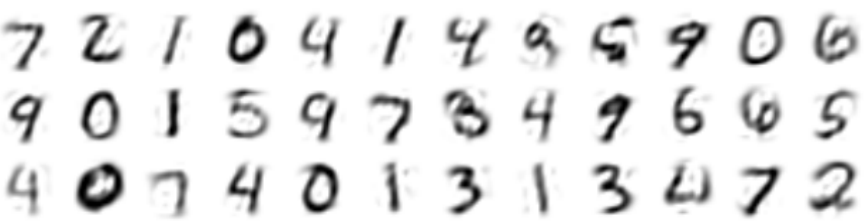
X (original samples)



$g \circ f(X)$ (CNN, $d = 32$)

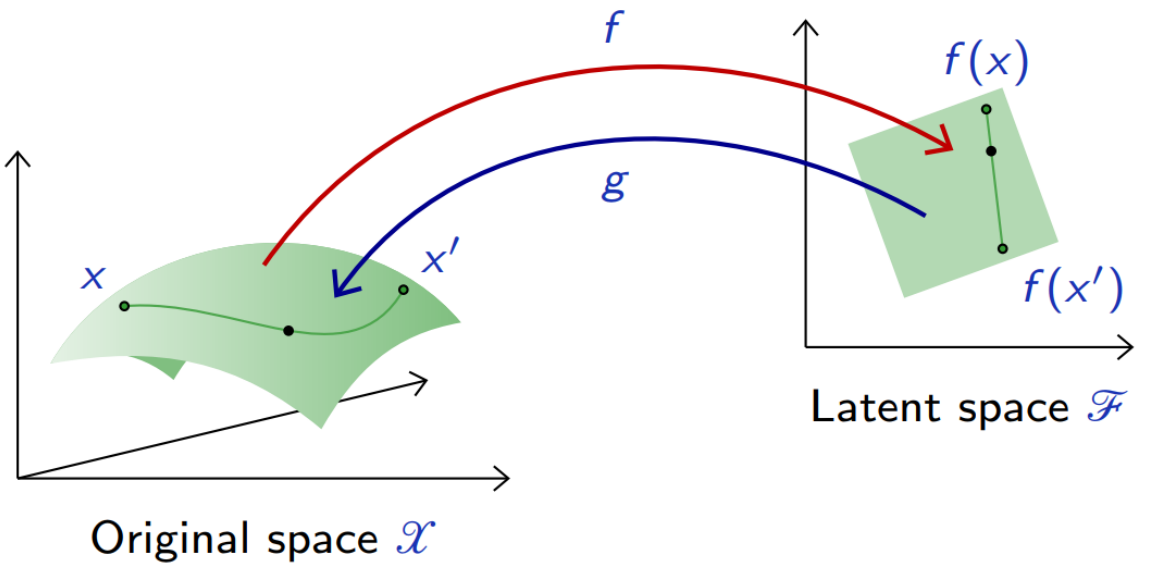


$g \circ f(X)$ (PCA, $d = 32$)

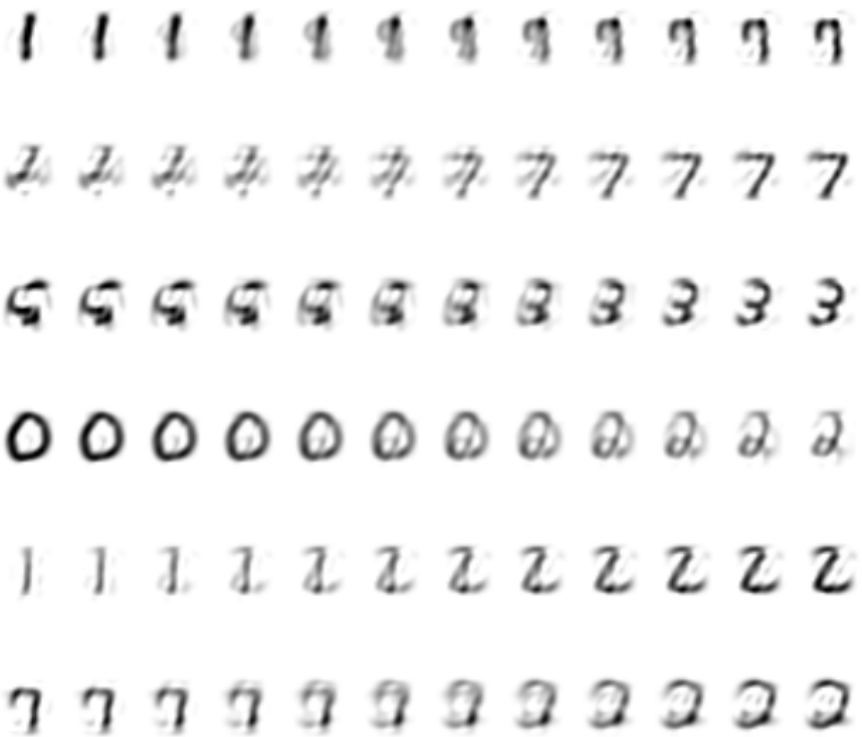


Interpolation

To get an intuition of the learned latent representation, we can pick two samples $\mathbf{x}$ and $\mathbf{x}'$ at random and interpolate samples along the line in the latent space.



PCA interpolation ($d = 32$)



Autoencoder interpolation ($d = 32$)

Denoising auto-encoders

Besides dimension reduction, auto-encoders can capture dependencies between signal components to restore degraded or noisy signals.

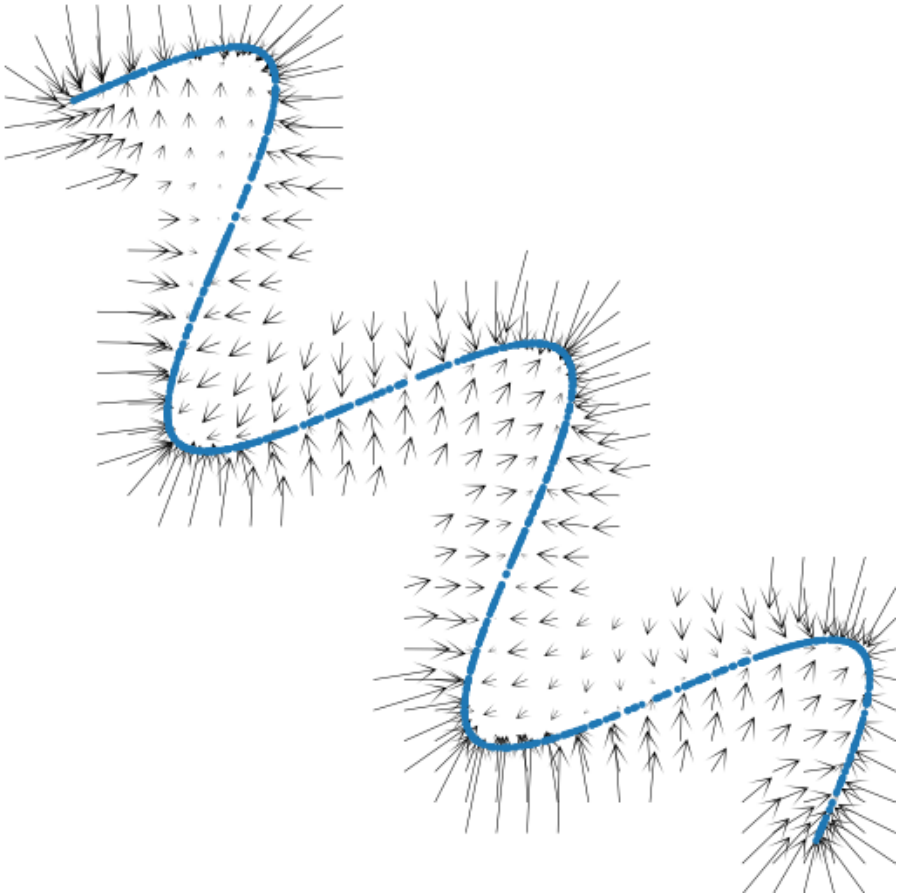
In this case, the composition

$$h = g \circ f : \mathcal{X} \to \mathcal{X}$$

is referred to as a denoising auto-encoder.

The goal is to optimize h such that a perturbation $\tilde{\mathbf{x}}$ of the signal $\mathbf{x}$ is restored to $\mathbf{x}$, hence

$$h(\tilde{\mathbf{x}}) \approx \mathbf{x}.$$



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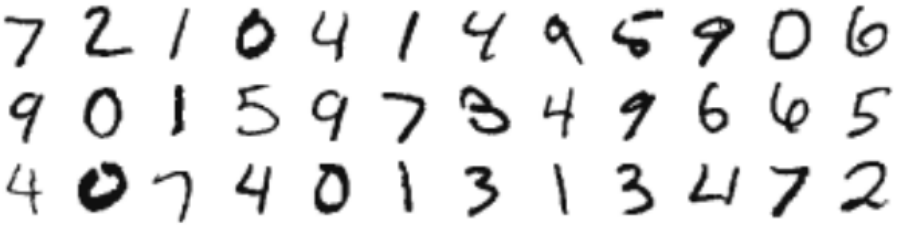
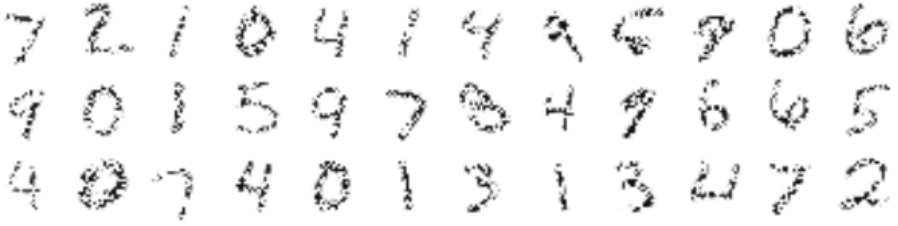
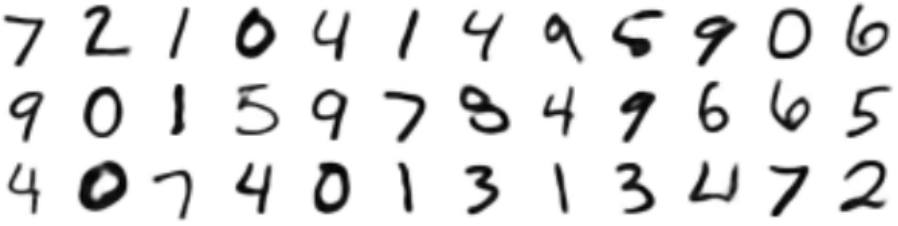
Original



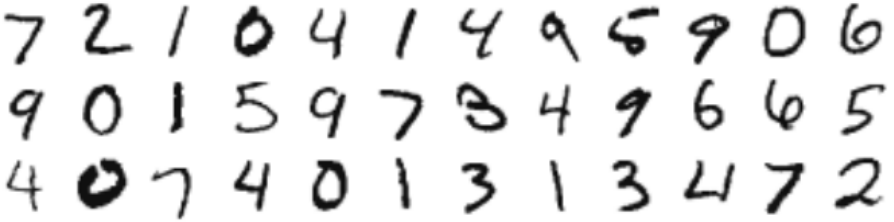
Corrupted ($p = 0.5$)



Reconstructed

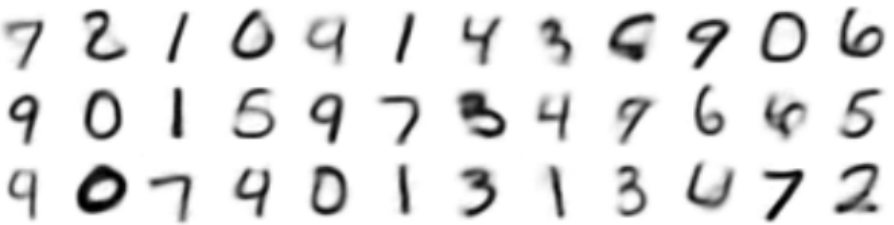


Original

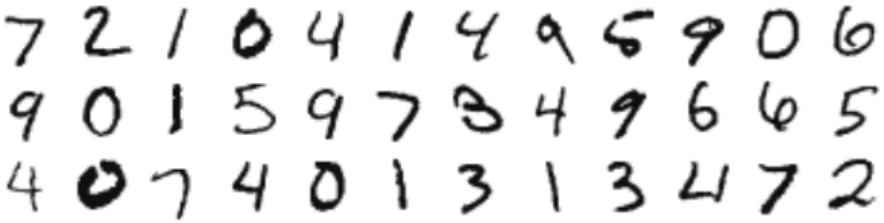


Corrupted ($p = 0.9$)

Reconstructed

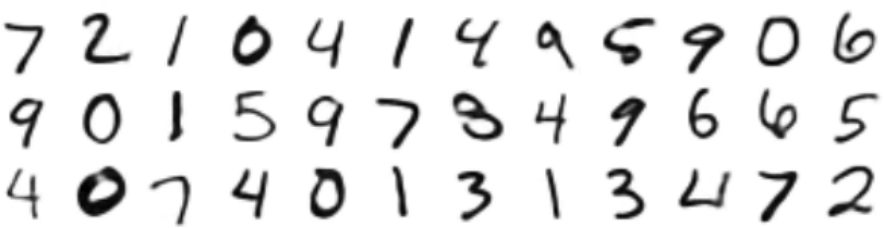


Original



Corrupted ($\sigma = 4$)

Reconstructed

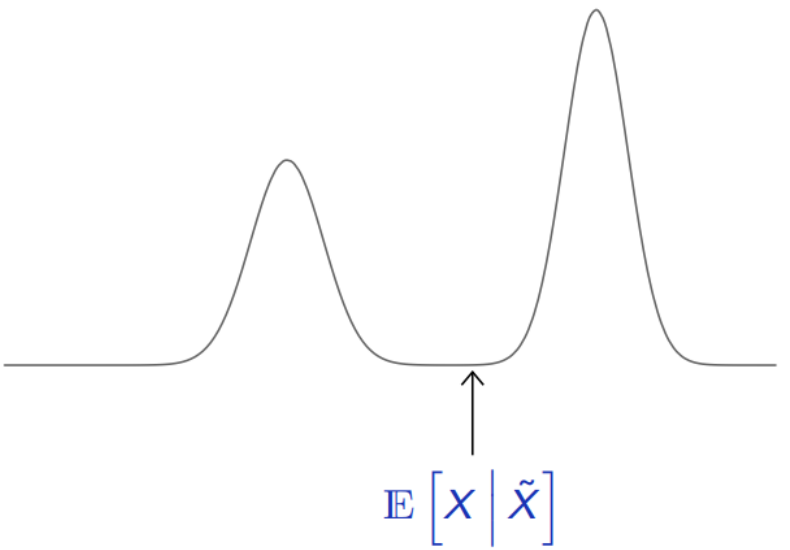


A fundamental weakness of denoising auto-encoders is that the posterior $p(\mathbf{x}|\tilde{\mathbf{x}})$ is possibly multi-modal.

If we train an auto-encoder with the quadratic loss (i.e., implicitly assuming a Gaussian posterior), then the best reconstruction is

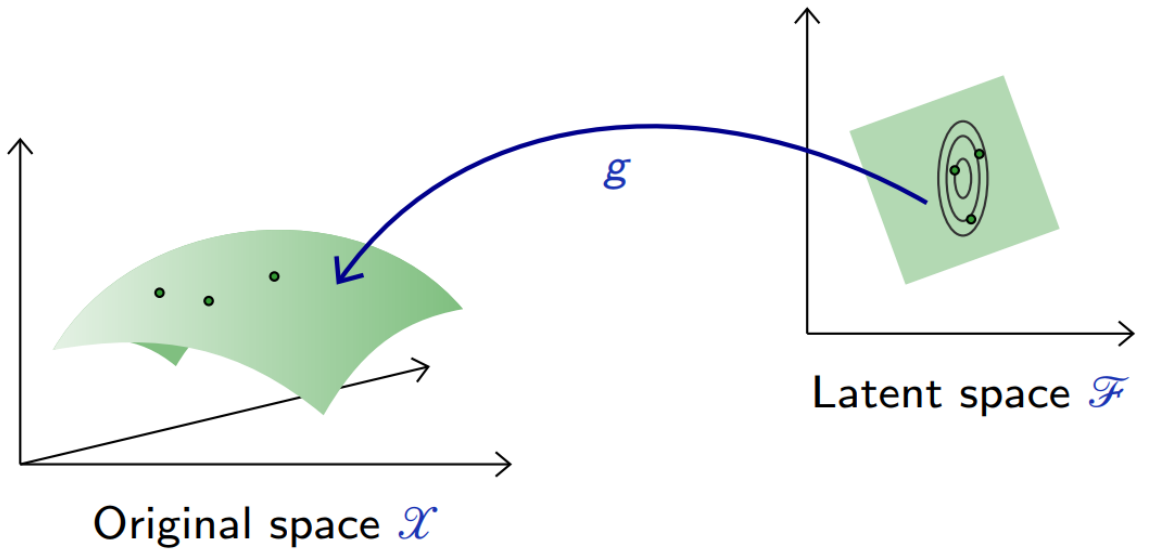
$$h(\tilde{\mathbf{x}}) = \mathbb{E}[\mathbf{x}|\tilde{\mathbf{x}}],$$

which may be very unlikely under $p(\mathbf{x}|\tilde{\mathbf{x}})$.



Sampling from an AE's latent space

The generative capability of the decoder g in an auto-encoder can be assessed by introducing a (simple) density model q over the latent space $\mathcal{Z}$, sample there, and map the samples into the data space $\mathcal{X}$ with g.

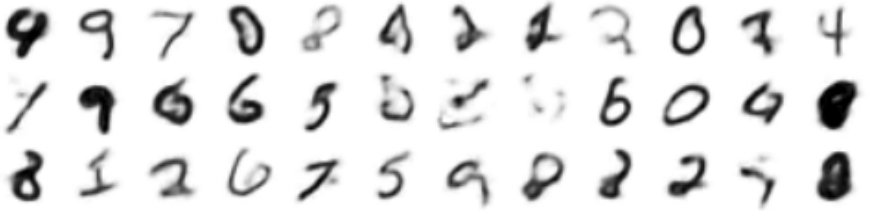


For instance, a factored Gaussian model with diagonal covariance matrix,

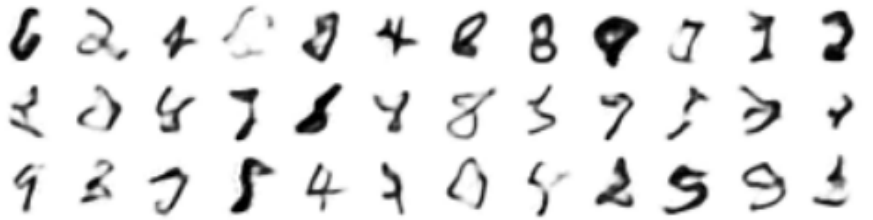
$$q(\mathbf{z}) = \mathcal{N}(\hat{\mu}, \hat{\Sigma}),$$

where both $\hat{\mu}$ and $\hat{\Sigma}$ are estimated on training data.

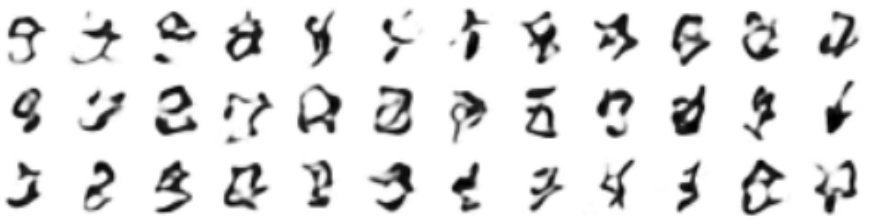
Autoencoder sampling ($d = 8$)



Autoencoder sampling ($d = 16$)



Autoencoder sampling ($d = 32$)

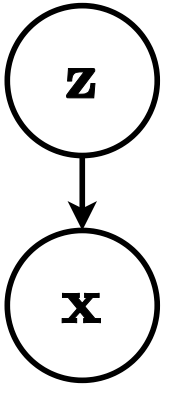


These results are not satisfactory because the density model on the latent space is too simple and inadequate.

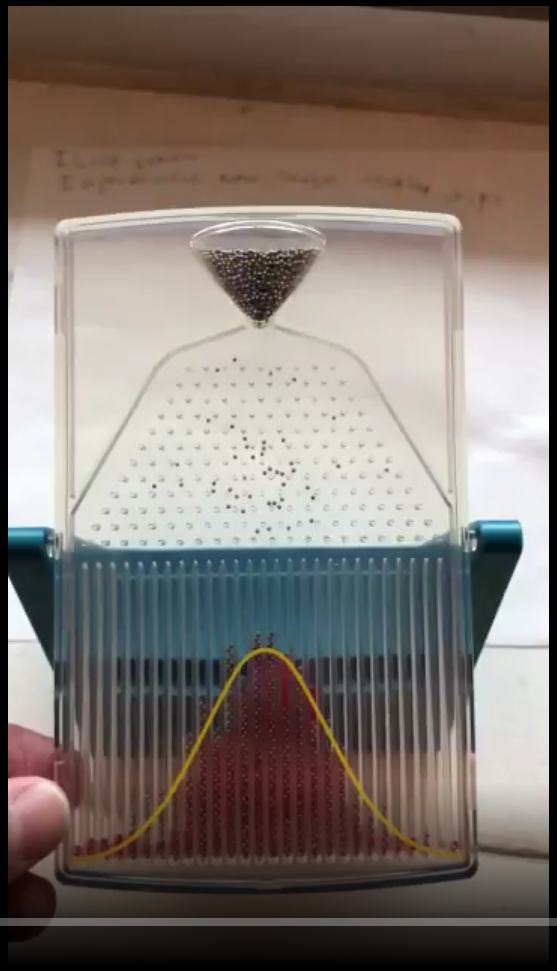
Building a good model in latent space amounts to our original problem of modeling an empirical distribution, although it may now be in a lower dimension space.

Variational inference

Latent variable model



Consider for now a prescribed latent variable model that relates a set of observable variables $\mathbf{x} \in \mathcal{X}$ to a set of unobserved variables $\mathbf{z} \in \mathcal{Z}$.



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The probabilistic model defines a joint probability distribution $p(\mathbf{x}, \mathbf{z})$, which decomposes as

$$p(\mathbf{x}, \mathbf{z}) = p(\mathbf{x}|\mathbf{z})p(\mathbf{z}).$$

If we interpret $\mathbf{z}$ as causal factors for the high-dimension representations $\mathbf{x}$, then sampling from $p(\mathbf{x}|\mathbf{z})$ can be interpreted as a stochastic generating process from $\mathcal{Z}$ to $\mathcal{X}$.

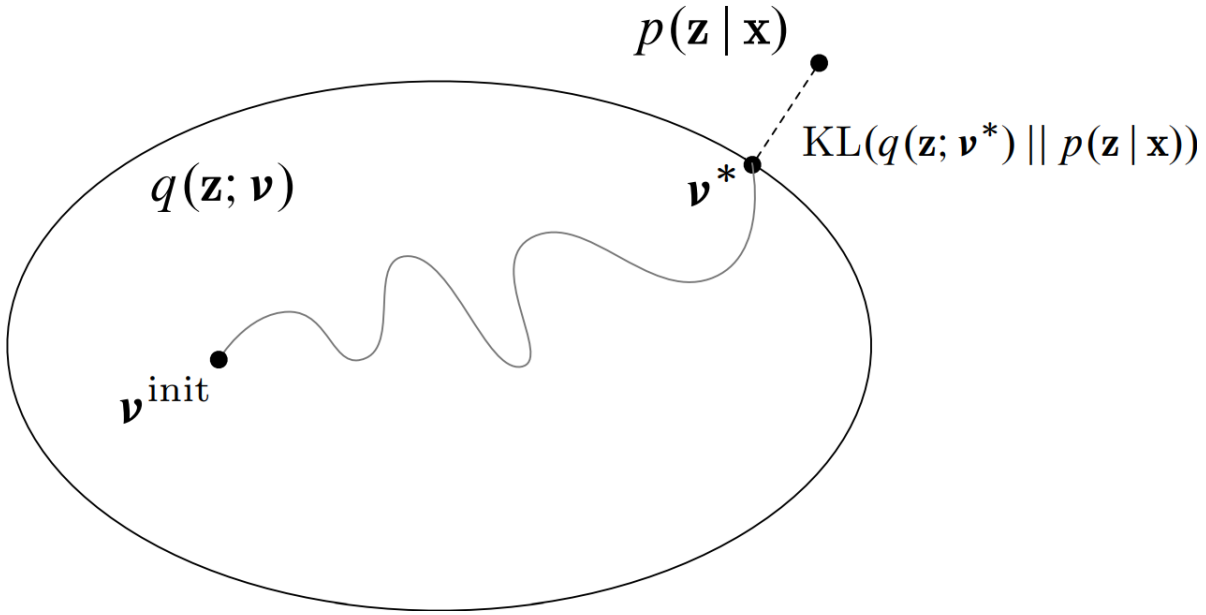
For a given model $p(\mathbf{x}, \mathbf{z})$, inference consists in computing the posterior

$$p(\mathbf{z}|\mathbf{x}) = \frac{p(\mathbf{x}|\mathbf{z})p(\mathbf{z})}{p(\mathbf{x})}.$$

For most interesting cases, this is usually intractable since it requires evaluating the evidence

$$p(\mathbf{x}) = \int p(\mathbf{x}|\mathbf{z})p(\mathbf{z})d\mathbf{z}.$$

Variational inference



Variational inference turns posterior inference into an optimization problem.

- Consider a family of distributions $q(\mathbf{z}|\mathbf{x}; \nu)$ that approximate the posterior $p(\mathbf{z}|\mathbf{x})$, where the variational parameters ν index the family of distributions.
- The parameters ν are fit to minimize the KL divergence between the approximation $q(\mathbf{z}|\mathbf{x}; \nu)$ and the posterior $p(\mathbf{z}|\mathbf{x})$.

Formally, we want to solve

$$
\begin{aligned}
&\arg\min_{\nu} \mathrm{KL}(q(\mathbf{z}|\mathbf{x};\nu)||p(\mathbf{z}|\mathbf{x})) \\
=&\arg\min_{\nu} \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\nu)}\left[\log \frac{q(\mathbf{z}|\mathbf{x};\nu)}{p(\mathbf{z}|\mathbf{x})}\right] \\
=&\arg\min_{\nu} \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\nu)}\left[\log q(\mathbf{z}|\mathbf{x};\nu) - \log p(\mathbf{x},\mathbf{z})\right] + \log p(\mathbf{x}).
\end{aligned}
$$

For the same reason as before, the KL divergence cannot be directly minimized because of the $\log p(\mathbf{x})$ term.

However, we can write

$$
\begin{aligned}
&\arg\min_{\nu} \mathrm{KL}(q(\mathbf{z}|\mathbf{x};\nu)||p(\mathbf{z}|\mathbf{x})) \\
=&\arg\min_{\nu} \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\nu)}\left[\log q(\mathbf{z}|\mathbf{x};\nu) - \log p(\mathbf{x},\mathbf{z})\right] + \log p(\mathbf{x}) \\
=&\arg\max_{\nu} \underbrace{\mathbb{E}_{q(\mathbf{z}|\mathbf{x};\nu)}\left[\log p(\mathbf{x},\mathbf{z}) - \log q(\mathbf{z}|\mathbf{x};\nu)\right]}_{\mathrm{ELBO}(\mathbf{x};\nu)}
\end{aligned}
$$

where $\mathrm{ELBO}(\mathbf{x};\nu)$ is called the evidence lower bound objective.

- Since $\log p(\mathbf{x})$ does not depend on ν, it can be considered as a constant, and minimizing the KL divergence is equivalent to maximizing the evidence lower bound, while being computationally tractable.
- Given a dataset $\mathbf{d} = \{\mathbf{x}_i | i = 1, ..., N\}$, the final objective is the sum $\sum_{\{\mathbf{x}_i \in \mathbf{d}\}} \mathrm{ELBO}(\mathbf{x}_i;\nu)$.

Remark that

$$\begin{aligned}\mathrm{ELBO}(\mathbf{x};\nu) &= \mathbb{E}_{q(\mathbf{z};|\mathbf{x}\nu)}\left[\log p(\mathbf{x},\mathbf{z}) - \log q(\mathbf{z}|\mathbf{x};\nu)\right] \\ &= \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\nu)}\left[\log p(\mathbf{x}|\mathbf{z})p(\mathbf{z}) - \log q(\mathbf{z}|\mathbf{x};\nu)\right] \\ &= \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\nu)}\left[\log p(\mathbf{x}|\mathbf{z})\right] - \mathrm{KL}(q(\mathbf{z}|\mathbf{x};\nu)||p(\mathbf{z}))\end{aligned}$$

Therefore, maximizing the ELBO:

- encourages distributions to place their mass on configurations of latent variables that explain the observed data (first term);
- encourages distributions close to the prior (second term).

Optimization

We want

$$\begin{aligned}\nu^* &= \arg\max_{\nu} \mathrm{ELBO}(\mathbf{x};\nu) \\ &= \arg\max_{\nu} \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\nu)}\left[\log p(\mathbf{x},\mathbf{z}) - \log q(\mathbf{z}|\mathbf{x};\nu)\right].\end{aligned}$$

We can proceed by gradient ascent, provided we can evaluate $\nabla_\nu \mathrm{ELBO}(\mathbf{x};\nu)$.

In general, this gradient is difficult to compute because the expectation is unknown and the parameters ν are parameters of the distribution $q(\mathbf{z}|\mathbf{x};\nu)$ we integrate over.

Variational auto-encoders

So far we assumed a prescribed probabilistic model motivated by domain knowledge. We will now directly learn a stochastic generating process with a neural network.

Variational auto-encoders

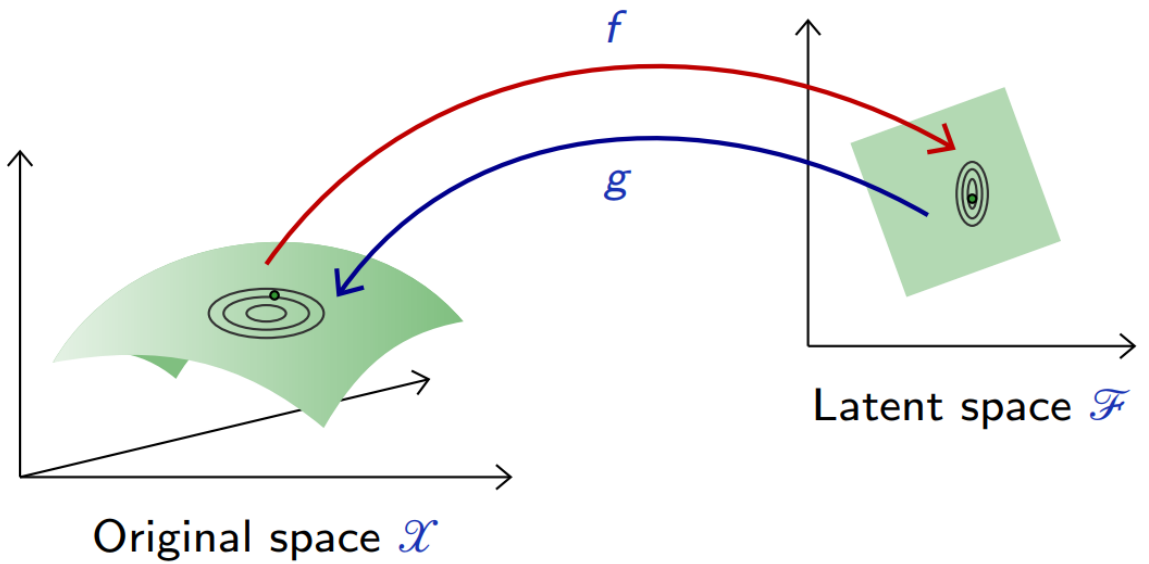
A variational auto-encoder is a deep latent variable model where:

- The prior $p(\mathbf{z})$ is prescribed, and usually chosen to be Gaussian.
- The likelihood $p(\mathbf{x}|\mathbf{z};\theta)$ is parameterized with a generative network NN_θ (or decoder) that takes as input $\mathbf{z}$ and outputs parameters $\phi = \text{NN}_\theta(\mathbf{z})$ to the data distribution. E.g.,

$$\mu, \sigma = \text{NN}_\theta(\mathbf{z})$$
$$p(\mathbf{x}|\mathbf{z};\theta) = \mathcal{N}(\mathbf{x};\mu,\sigma^2\mathbf{I})$$

- The approximate posterior $q(\mathbf{z}|\mathbf{x};\varphi)$ is parameterized with an inference network NN_φ (or encoder) that takes as input $\mathbf{x}$ and outputs parameters $\nu = \text{NN}_\varphi(\mathbf{x})$ to the approximate posterior. E.g.,

$$\mu, \sigma = \text{NN}_\varphi(\mathbf{x})$$
$$q(\mathbf{z}|\mathbf{x};\varphi) = \mathcal{N}(\mathbf{z};\mu,\sigma^2\mathbf{I})$$


f
g
Latent space $\mathcal{F}$
Original space $\mathcal{X}$

As before, we can use variational inference, but to jointly optimize the generative and the inference networks parameters θ and φ.

We want

$$\begin{aligned}\theta^*, \varphi^* &= \arg\max_{\theta,\varphi} \mathrm{ELBO}(\mathbf{x}; \theta, \varphi) \\ &= \arg\max_{\theta,\varphi} \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\varphi)} \left[\log p(\mathbf{x}|\mathbf{z};\theta)\right] - \mathrm{KL}(q(\mathbf{z}|\mathbf{x};\varphi)||p(\mathbf{z})).\end{aligned}$$

- Given some generative network θ, we want to put the mass of the latent variables, by adjusting φ, such that they explain the observed data, while remaining close to the prior.
- Given some inference network φ, we want to put the mass of the observed variables, by adjusting θ, such that they are well explained by the latent variables.

Unbiased gradients of the ELBO with respect to the generative model parameters θ are simple to obtain:

$$\begin{aligned}\nabla_\theta \mathrm{ELBO}(\mathbf{x};\theta,\varphi) &= \nabla_\theta \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\varphi)}\left[\log p(\mathbf{x},\mathbf{z};\theta) - \log q(\mathbf{z}|\mathbf{x};\varphi)\right] \\ &= \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\varphi)}\left[\nabla_\theta(\log p(\mathbf{x},\mathbf{z};\theta) - \log q(\mathbf{z}|\mathbf{x};\varphi))\right] \\ &= \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\varphi)}\left[\nabla_\theta \log p(\mathbf{x},\mathbf{z};\theta)\right],\end{aligned}$$

which can be estimated with Monte Carlo integration.

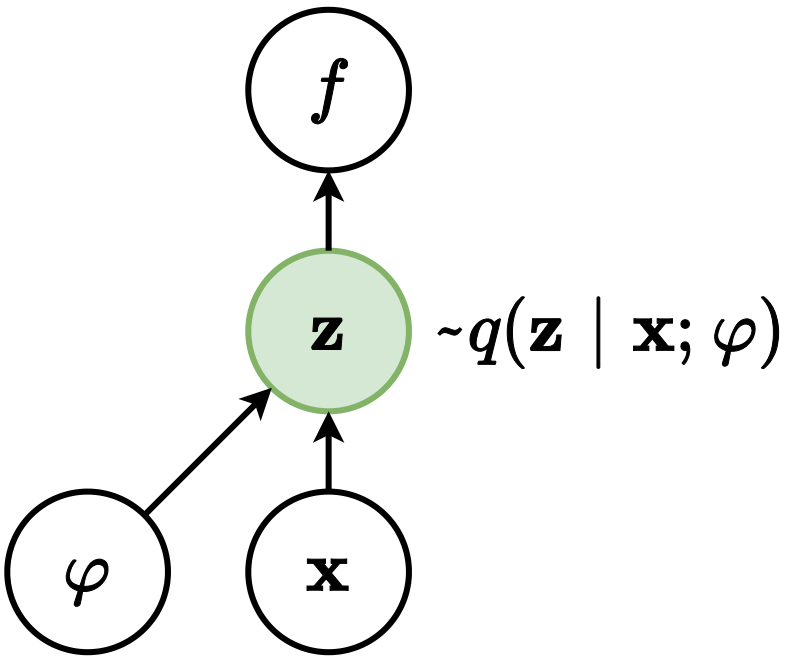
However, gradients with respect to the inference model parameters φ are more difficult to obtain:

$$\begin{aligned}\nabla_\varphi \mathrm{ELBO}(\mathbf{x};\theta,\varphi) &= \nabla_\varphi \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\varphi)}\left[\log p(\mathbf{x},\mathbf{z};\theta) - \log q(\mathbf{z}|\mathbf{x};\varphi)\right] \\ &\neq \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\varphi)}\left[\nabla_\varphi(\log p(\mathbf{x},\mathbf{z};\theta) - \log q(\mathbf{z}|\mathbf{x};\varphi))\right]\end{aligned}$$

Let us abbreviate

$$\begin{aligned}\mathrm{ELBO}(\mathbf{x};\theta,\varphi) &= \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\varphi)}\left[\log p(\mathbf{x},\mathbf{z};\theta) - \log q(\mathbf{z}|\mathbf{x};\varphi)\right] \\ &= \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\varphi)}\left[f(\mathbf{x},\mathbf{z};\varphi)\right].\end{aligned}$$

We have



We cannot backpropagate through the stochastic node $\mathbf{z}$ to compute $\nabla_\varphi f$!

Reparameterization trick

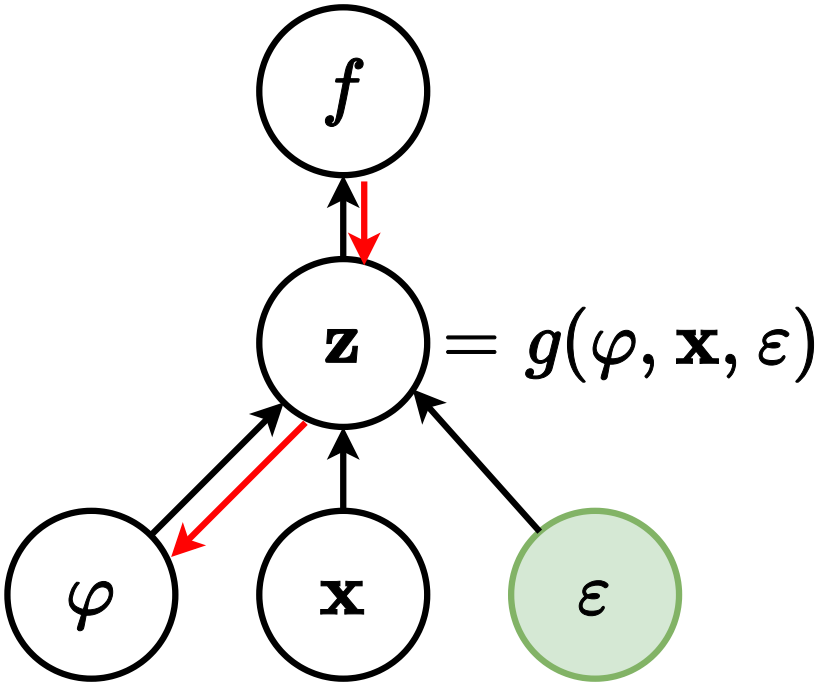
The **reparameterization trick** consists in re-expressing the variable

$$\mathbf{z} \sim q(\mathbf{z}|\mathbf{x}; \varphi)$$

as some differentiable and invertible transformation of another random variable ϵ given $\mathbf{x}$ and φ,

$$\mathbf{z} = g(\varphi, \mathbf{x}, \epsilon),$$

such that the distribution of ϵ is independent of $\mathbf{x}$ or φ.



For example, if $q(\mathbf{z}|\mathbf{x};\varphi) = \mathcal{N}(\mathbf{z};\mu(\mathbf{x};\varphi),\sigma^2(\mathbf{x};\varphi))$, where $\mu(\mathbf{x};\varphi)$ and $\sigma^2(\mathbf{x};\varphi)$ are the outputs of the inference network NN_φ, then a common reparameterization is:

$$p(\epsilon) = \mathcal{N}(\epsilon;\mathbf{0},\mathbf{I})$$
$$\mathbf{z} = \mu(\mathbf{x};\varphi) + \sigma(\mathbf{x};\varphi) \odot \epsilon$$

Given such a change of variable, the ELBO can be rewritten as:

$$\begin{aligned}\mathrm{ELBO}(\mathbf{x};\theta,\varphi) &= \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\varphi)}\left[f(\mathbf{x},\mathbf{z};\varphi)\right] \\ &= \mathbb{E}_{p(\epsilon)}\left[f(\mathbf{x},g(\varphi,\mathbf{x},\epsilon);\varphi)\right]\end{aligned}$$

Therefore,

$$\begin{aligned}\nabla_\varphi \mathrm{ELBO}(\mathbf{x};\theta,\varphi) &= \nabla_\varphi \mathbb{E}_{p(\epsilon)}\left[f(\mathbf{x},g(\varphi,\mathbf{x},\epsilon);\varphi)\right] \\ &= \mathbb{E}_{p(\epsilon)}\left[\nabla_\varphi f(\mathbf{x},g(\varphi,\mathbf{x},\epsilon);\varphi)\right],\end{aligned}$$

which we can now estimate with Monte Carlo integration.

The last required ingredient is the evaluation of the likelihood $q(\mathbf{z}|\mathbf{x};\varphi)$ given the change of variable g. As long as g is invertible, we have:

$$\log q(\mathbf{z}|\mathbf{x};\varphi) = \log p(\epsilon) - \log\left|\det\left(\frac{\partial \mathbf{z}}{\partial \epsilon}\right)\right|.$$

Example

Consider the following setup:

- Generative model:

$$
\begin{aligned}
\mathbf{z} &\in \mathbb{R}^d \\
p(\mathbf{z}) &= \mathcal{N}(\mathbf{z}; \mathbf{0}, \mathbf{I}) \\
p(\mathbf{x}|\mathbf{z}; \theta) &= \mathcal{N}(\mathbf{x}; \mu(\mathbf{z}; \theta), \sigma^2(\mathbf{z}; \theta)\mathbf{I}) \\
\mu(\mathbf{z}; \theta) &= \mathbf{W}_2^T \mathbf{h} + \mathbf{b}_2 \\
\log \sigma^2(\mathbf{z}; \theta) &= \mathbf{W}_3^T \mathbf{h} + \mathbf{b}_3 \\
\mathbf{h} &= \mathrm{ReLU}(\mathbf{W}_1^T \mathbf{z} + \mathbf{b}_1) \\
\theta &= \{\mathbf{W}_1, \mathbf{b}_1, \mathbf{W}_2, \mathbf{b}_2, \mathbf{W}_3, \mathbf{b}_3\}
\end{aligned}
$$

- Inference model:

$$
\begin{aligned}
q(\mathbf{z}|\mathbf{x};\varphi) &= \mathcal{N}(\mathbf{z};\mu(\mathbf{x};\varphi),\sigma^2(\mathbf{x};\varphi)\mathbf{I}) \\
p(\epsilon) &= \mathcal{N}(\epsilon;\mathbf{0},\mathbf{I}) \\
\mathbf{z} &= \mu(\mathbf{x};\varphi) + \sigma(\mathbf{x};\varphi) \odot \epsilon \\
\mu(\mathbf{x};\varphi) &= \mathbf{W}_5^T\mathbf{h} + \mathbf{b}_5 \\
\log\sigma^2(\mathbf{x};\varphi) &= \mathbf{W}_6^T\mathbf{h} + \mathbf{b}_6 \\
\mathbf{h} &= \mathrm{ReLU}(\mathbf{W}_4^T\mathbf{x} + \mathbf{b}_4) \\
\varphi &= \{\mathbf{W}_4, \mathbf{b}_4, \mathbf{W}_5, \mathbf{b}_5, \mathbf{W}_6, \mathbf{b}_6\}
\end{aligned}
$$

Note that there is no restriction on the generative and inference network architectures. They could as well be arbitrarily complex convolutional networks.

Plugging everything together, the objective can be expressed as:

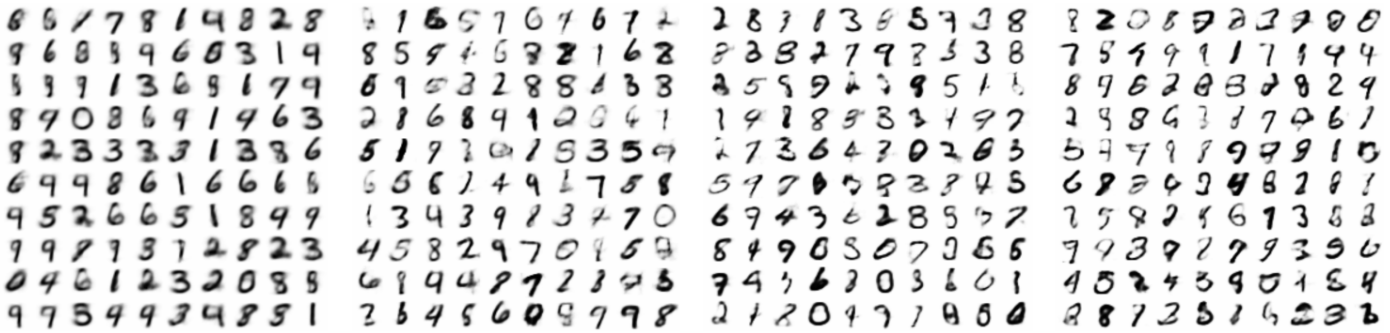
$$\begin{aligned}\mathrm{ELBO}(\mathbf{x};\theta,\varphi) &= \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\varphi)}\left[\log p(\mathbf{x}|\mathbf{z};\theta)\right] - \mathrm{KL}(q(\mathbf{z}|\mathbf{x};\varphi)||p(\mathbf{z})) \\ &= \mathbb{E}_{p(\epsilon)}\left[\log p(\mathbf{x}|\mathbf{z}=g(\varphi,\mathbf{x},\epsilon);\theta)\right] - \mathrm{KL}(q(\mathbf{z}|\mathbf{x};\varphi)||p(\mathbf{z}))\end{aligned}$$

where the KL divergence can be expressed analytically as

$$\mathrm{KL}(q(\mathbf{z}|\mathbf{x};\varphi)||p(\mathbf{z})) = \frac{1}{2}\sum_{j=1}^{d}\left(1+\log(\sigma_j^2(\mathbf{x};\varphi)) - \mu_j^2(\mathbf{x};\varphi) - \sigma_j^2(\mathbf{x};\varphi)\right),$$

which allows to evaluate its derivative without approximation.

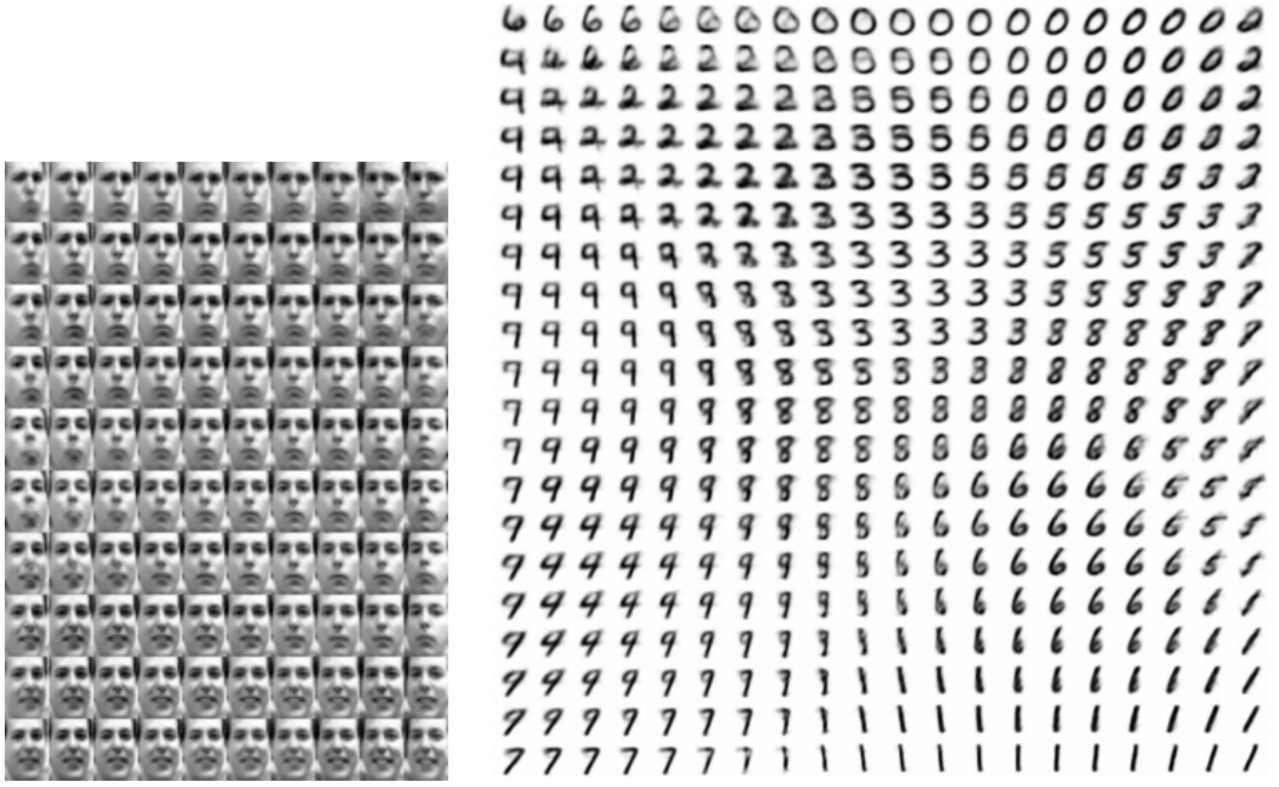
Consider as data $\mathbf{d}$ the MNIST digit dataset:



(a) 2-D latent space (b) 5-D latent space (c) 10-D latent space (d) 20-D latent space

Figure 5: Random samples from learned generative models of MNIST for different dimensionalities of latent space.

(Kingma and Welling, 2013)



(a) Learned Frey Face manifold

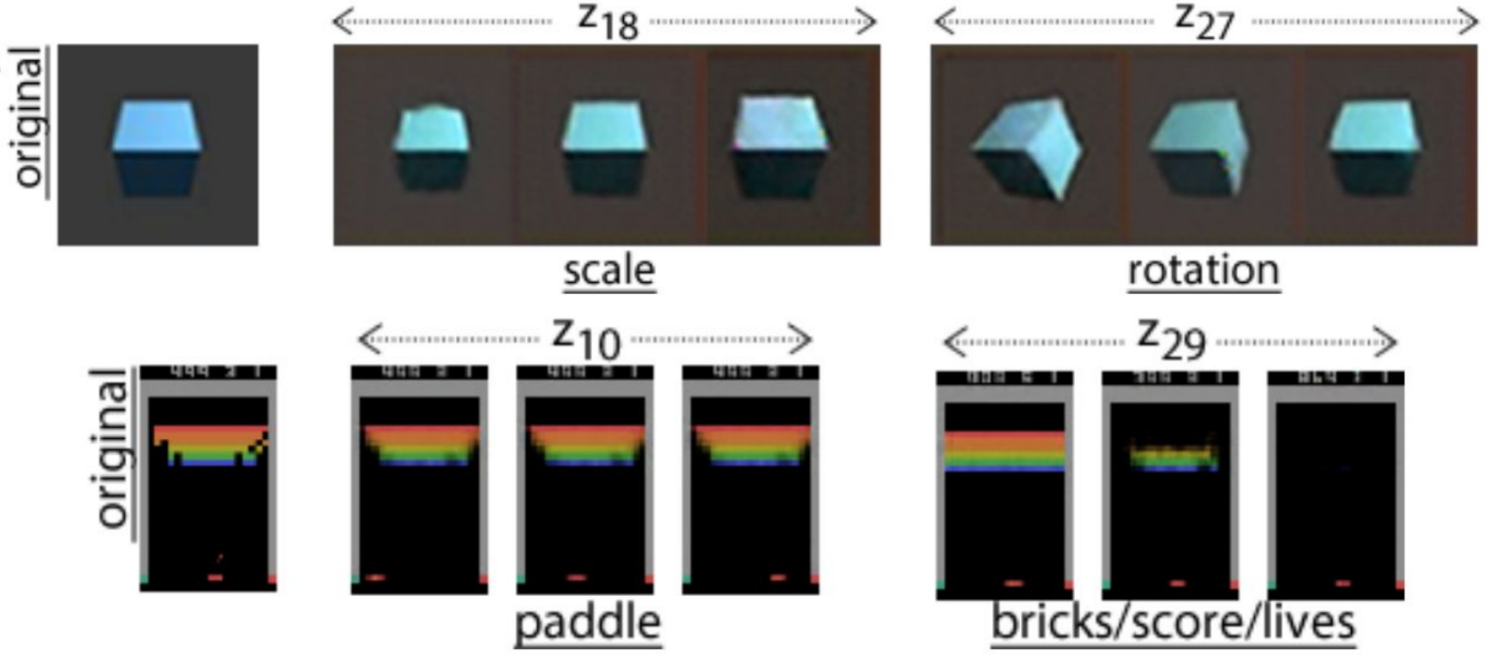
(b) Learned MNIST manifold

Figure 4: Visualisations of learned data manifold for generative models with two-dimensional latent space, learned with AEVB. Since the prior of the latent space is Gaussian, linearly spaced coordinates on the unit square were transformed through the inverse CDF of the Gaussian to produce values of the latent variables $\mathbf{z}$. For each of these values $\mathbf{z}$, we plotted the corresponding generative $p_{\boldsymbol{\theta}}(\mathbf{x}|\mathbf{z})$ with the learned parameters $\boldsymbol{\theta}$.

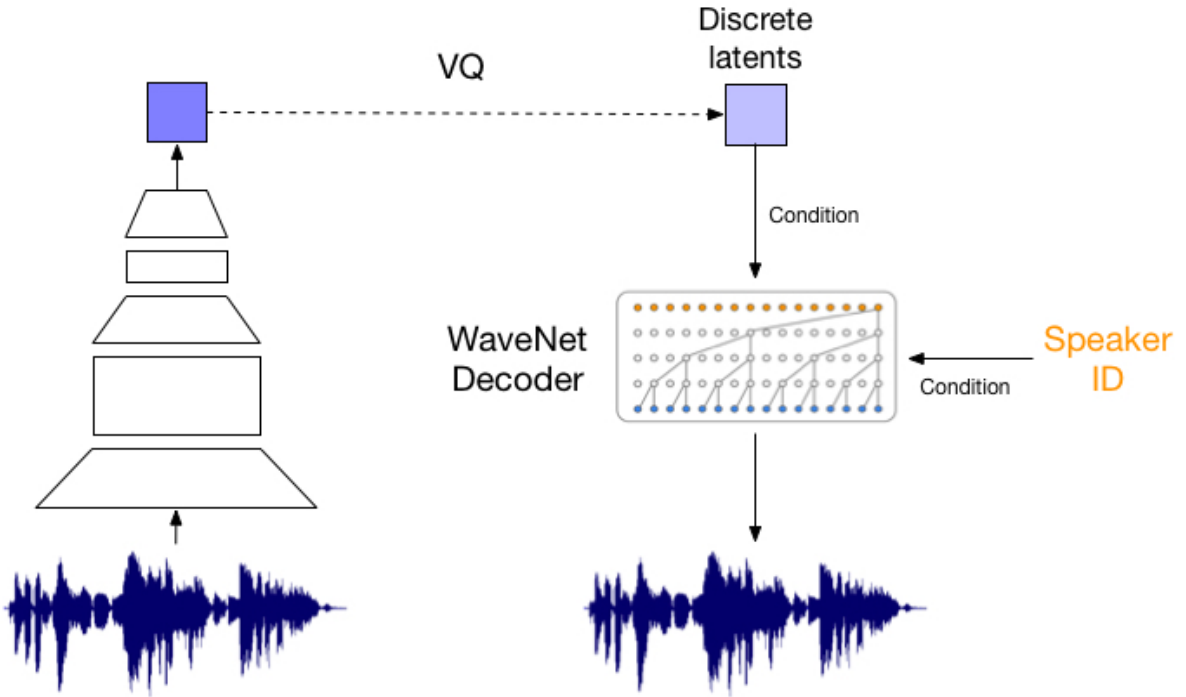
(Kingma and Welling, 2013)

Applications

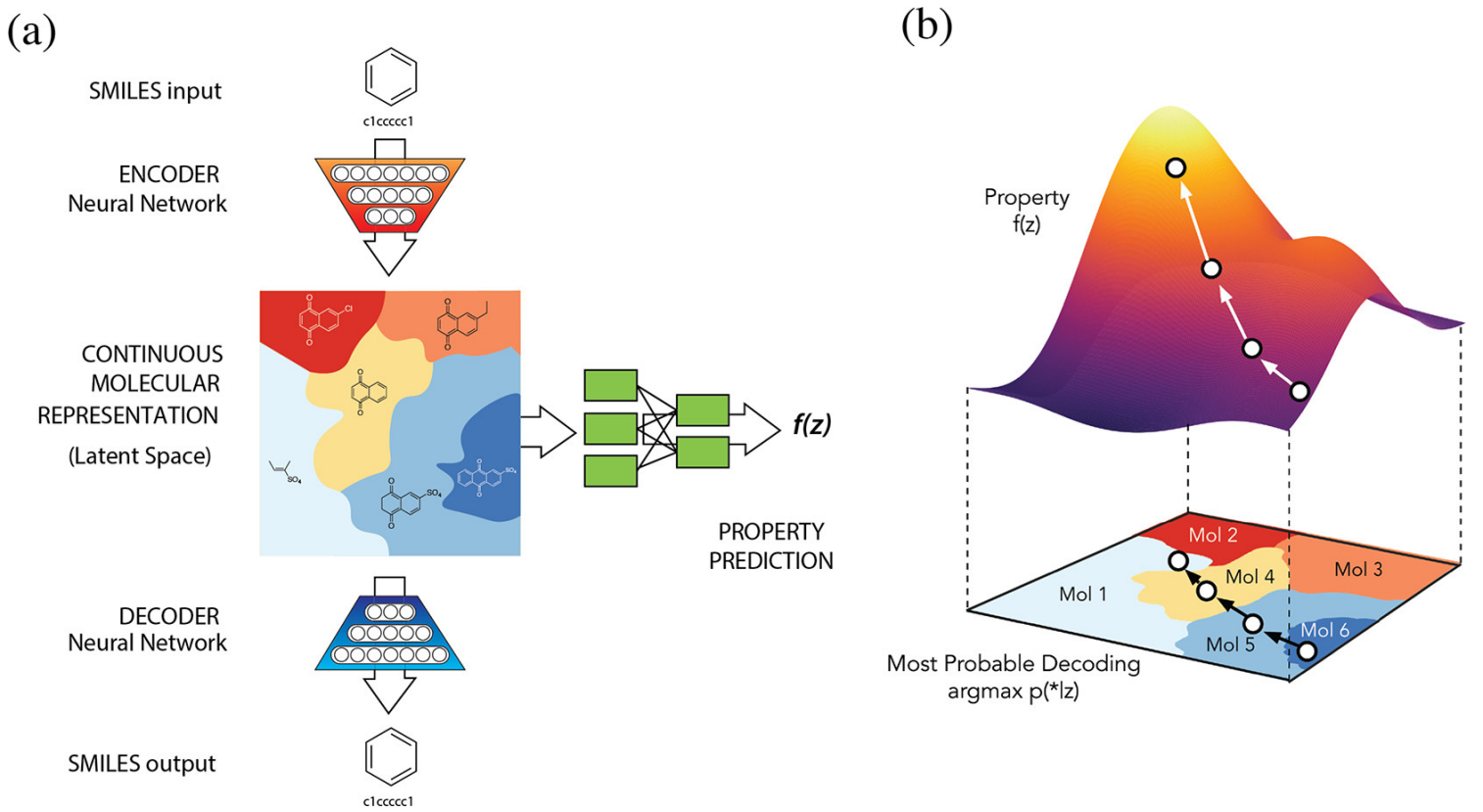
Hierarchical **compression of images and other data**,
e.g., in video conferencing systems (Gregor et al, 2016).



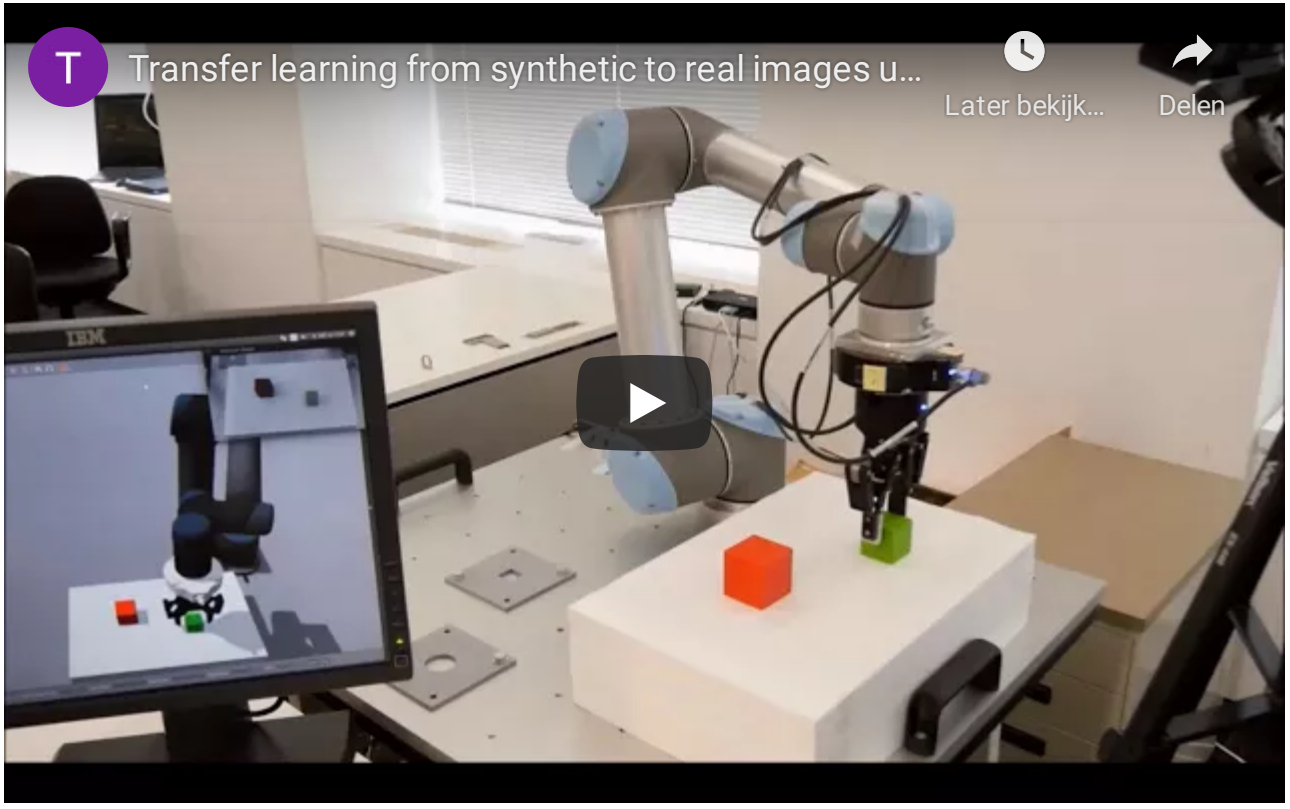
Understanding the factors of variation and invariances (Higgins et al, 2017).



Voice style transfer [demo] (van den Oord et al, 2017).



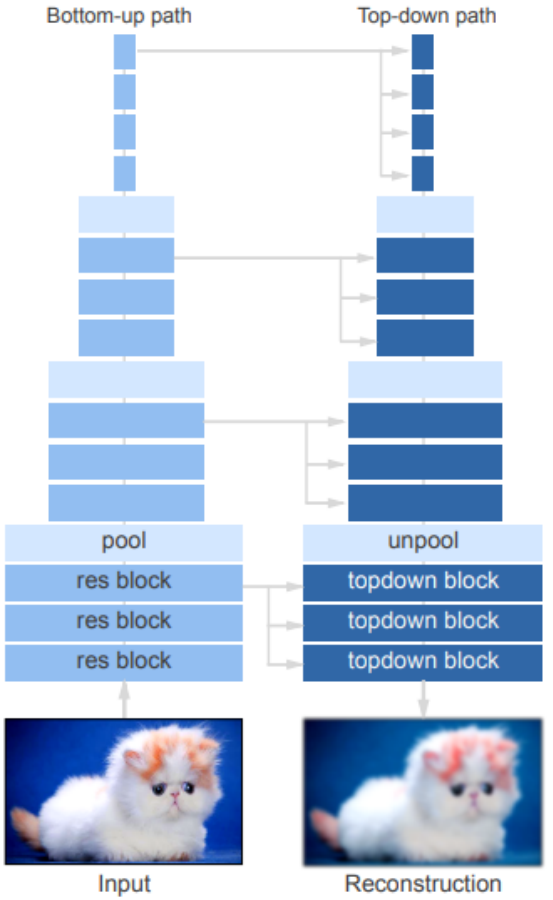
Design of new molecules with desired chemical properties (Gomez-Bombarelli et al, 2016).

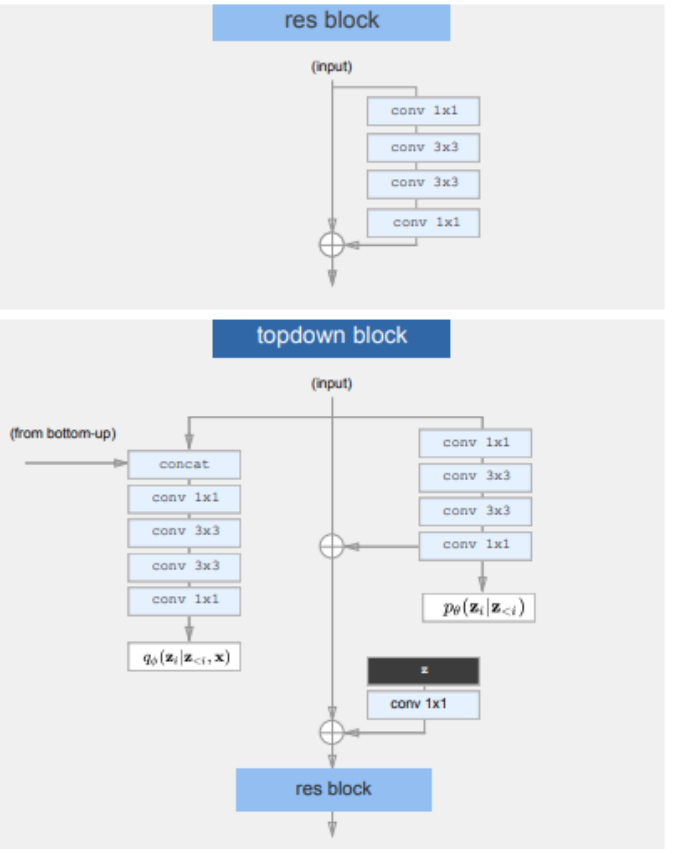


Bridging the **simulation-to-reality** gap (Inoue et al, 2017).

Hierarchical VAEs

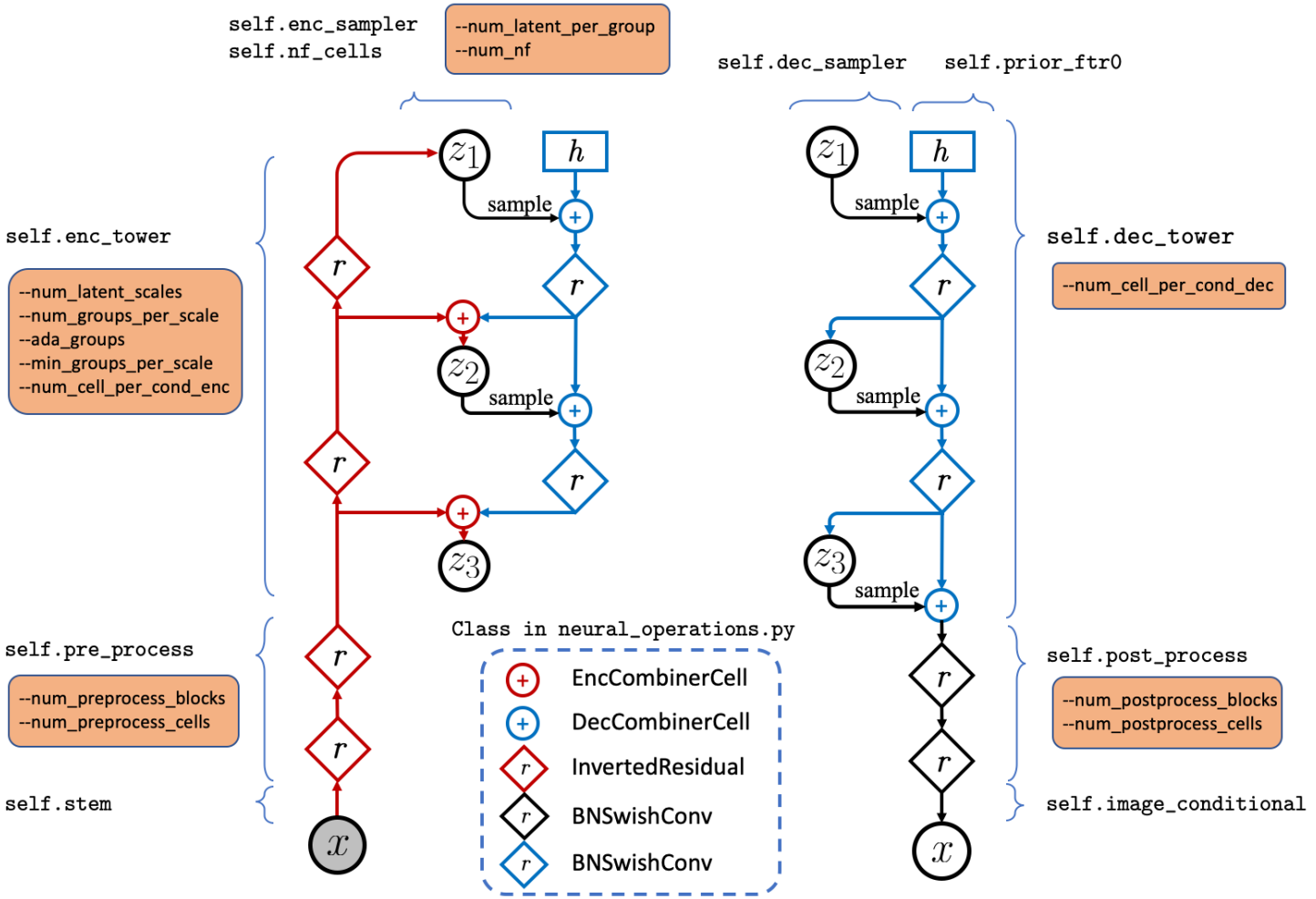
See side notes.





VDVAE: Very Deep VAEs (Child, 2020-2021).

VDVAE samples (Child, 2020-2021).



NVAE: A Deep Hierarchical Variational Autoencoder (Vahdat and Kautz, 2020).

NVAE samples (Vahdat and Kautz, 2020).

NVAE: Random walks in latent space. (Vahdat and Kautz, 2020)

The end.