Isabelle vs. Agda

The formalization of the terms and reduction rules of the λ-Y calculus presented here is a locally nameless presentation due to Aydemir et al. (2008). The basic definitions of λ-terms and β-reduction were borrowed from an implementation of the λ-calculus with the associated Church Rosser proof in Agda, by Mu (2011).

The proofs of confluence/Church Rosser were formalized using the paper by R. Pollack (1995), which describes a coarser proof of Church Rosser than the one formalized by Mu (2011). This proof uses the notion of a maximal parallel reduction, introduced by Takahashi (1995) to simplify the inductive proof of confluence.

Programs as proofs

One of the most obvious differences between Agda and Isabelle is the treatment of functions and proofs in both languages. Whilst in Isabelle, there is always a clear syntactic distinction between programs and proofs, Agda's richer dependent-type system allows constructing proofs as programs. This distinction is especially apparent in inductive proofs, which have a completely distinct syntax in Isabelle. As proofs are not objects which can be directly manipulated in Isabelle, to modify the proof goal, user commands such as `apply rule` or `by auto` are used:

```
lemma subst_fresh: "x ∉ FV t ⇒ t[x ::= u] = t"
apply (induct t)
by auto
```

In the proof above, the command `apply (induct t)` takes a proof object with the goal `x ∉ FV t ⇒ t[x ::= u] = t`, and applies the induction principle for `t`, generating 5 new proof obligations:

```
1. ⋀xa. x ∉ FV (FVar xa) ⇒ FVar xa [x ::= u] = FVar xa
2. ⋀xa. x ∉ FV (BVar xa) ⇒ BVar xa [x ::= u] = BVar xa
3. ⋀t1 t2.
     (x ∉ FV t1 ⇒ t1 [x ::= u] = t1) ⇒
     (x ∉ FV t2 ⇒ t2 [x ::= u] = t2) ⇒
     x ∉ FV (App t1 t2) ⇒ App t1 t2 [x ::= u] = App t1 t2
4. ⋀t. (x ∉ FV t ⇒ t [x ::= u] = t) ⇒ x ∉ FV (Lam t) ⇒
     Lam t [x ::= u] = Lam t
5. ⋀xa. x ∉ FV (Y xa) ⇒ Y xa [x ::= u] = Y xa
```

These are then discharged by the call to `auto`.

In comparison, the Agda proof exposes the proof objects to the user directly. Instead of using commands modifying the proof state, one begins with a definition of the lemma in the following shape:

```
subst-fresh : ∀ x t u -> (x∉FVt : x ∉ (FV t)) -> (t [ x ::= u ]) ≡ t
subst-fresh x t u x∉FVt = ?
```

The `?` acts as a 'hole' which the user needs to fill in to construct the proof. Using the emacs/atom agda-mode, once can apply a case split o `t`, corresponding to the `apply (induct t)` call in Isabelle:

```
subst-fresh : ∀ x t u -> (x∉FVt : x ∉ (FV t)) -> (t [ x ::= u ]) ≡ t
subst-fresh x (bv i) u x∉FVt = {!   0!}
subst-fresh x (fv x₁) u x∉FVt = {!   1!}
subst-fresh x (lam t) u x∉FVt = {!   2!}
subst-fresh x (app t t₁) u x∉FVt = {!   3!}
subst-fresh x (Y t₁) u x∉FVt = {!   4!}
```

When the above definition is compiled, Agda generates 5 goals needed to 'fill' each hole:

```
?0  :  (bv i [ x ::= u ]) ≡ bv i
?1  :  (fv x₁ [ x ::= u ]) ≡ fv x₁
```

```
?2  :  (lam t [ x ::= u ]) ≡ lam t
?3  :  (app t t₁ [ x ::= u ]) ≡ app t t₁
?4  :  (Y t₁ [ x ::= u ]) ≡ Y t₁
```

As one can see, there is a clear correspondence between the 5 generated goals in Isabelle and the cases of the Agda proof above.

Due to this correspondence, reasoning in both systems is largely similar, whereas in Isabelle, one modifies the proof indirectly by issuing commands to modify proof goals, in Agda, one generates proofs directly by writing a program-as-proof, which satisfies the type constraints given in the definition.

Automation

As seen previously, Isabelle includes several automatic provers of varying complexity, including `simp`, `auto`, `blast`, `metis` and others. These are tactics/programs which automatically apply rewrite-rules until the goal is discharged. If the tactic fails to discharge a goal within a set number of steps, it stops and lets the user direct the proof. The use of tactics in Isabelle is common to discharge trivial proof steps, which usually follow form simple rewriting of definitions or case analysis of certain variables.
For example, the proof goal

```
⋀xa. x ∉ FV (FVar xa) ⟹ FVar xa [x ::= u] = FVar xa
```

will be discharged by first unfolding the definition of substitution for `FVar`, where

```
(FVar xa)[x ::= u] = (if xa = x then u else FVar xa)
```

and then deriving `x ≠ xa` from the assumption `x ∉ FV (FVar xa)`. Doing these steps explicitly, quickly becomes cumbersome, as one needs to constantly look up the names of lemmas for trivial rewrite rules like `(?c ∈ {}) = False`.

Unlike Isabelle, Agda does not include nearly as much automation. The only proof search tool included with Agda is Agsy, which is similar, albeit weaker than the `simp` tactic. It may therefore seem that Agda should will be much more cumbersome to reason in than Isabelle. This, however, turned out not to be the case, due to Agda's type system and the powerful pattern matching available.... ...expand

Pattern matching

Automatic inference vs.

```
show ?case unfolding 1
using 1(2) apply (cases rule:pbeta.cases)
apply simp
```

References

Aydemir, Brian, Arthur Charguéraud, Benjamin C. Pierce, Randy Pollack, and Stephanie Weirich. 2008. “Engineering Formal Metatheory.” In *Proceedings of the 35th Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, 3–15. POPL ’08. New York, NY, USA: ACM. doi:10.1145/1328438.1328443.

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Pollack, Robert. 1995. “Polishing up the Tait-Martin-Löf Proof of the Church-Rosser Theorem.”

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