Math Review Notes—Probability

Gregory Faletto

Contents

Last updated November 22, 2018

1 Probability

These are my notes from taking Math 505A at USC and the textbook *Probability and Random Processes* (Grimmet and Stirkazer) 3rd edition.

1.1 To Know for Math 505A Midterm 1 (Discrete Random Variables)

1.1.1 Definitions

Definition 1.1. The **probability mass function** of a discrete random variable X is the function $f : \mathbb{R} \to [0,1]$ given by $f(x) = \Pr(X = x)$.

Definition 1.2. The **(cumulative) distribution function** of a discrete random variable F is given by

$$F(x) = \sum_{i: x_i \leq x} f(x_i)$$

Definition 1.3. The **joint probability mass function** $f : \mathbb{R}^2 \to [0,1]$ of two discrete random variables X and Y is given by

$$f(x,y) = \Pr(X = x \cap Y = y)$$

Definition 1.4. The **joint distribution function** $F : \mathbb{R}^2 \to [0,1]$ is given by

$$F(x,y) = \Pr(X \leq x \cap Y \leq y)$$

Definition 1.5. If $\Pr(B) > 0$ then the **conditional probability** that A occurs given that B occurs is defined to be

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Definition 1.6. Two random variables X and Y are **independent** if and only if $\Pr(X \cap Y) = \Pr(X)\Pr(Y)$.

Theorem 1. (Law of total probability). If X is a random variable and Y is a discrete random variable taking on values $y_1, y_2, \ldots, y_n$, then $\Pr(X) = \sum_i \Pr(X \mid Y = y_i) \cdot \Pr(Y = y_i)$. (Can be used to prove independence.)

Definition 1.7. Two random variables X and Y are **uncorrelated** if $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.

Proposition 2. (a) Two random variables are uncorrelated if and only if their covariance $\operatorname{Cov}(X,Y) = \mathbb{E}\big[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))\big] = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$ equals 0.

(b) If X and Y are independent then they are uncorrelated.

Theorem 3. If X and Y are independent and $g, h : \mathbb{R} \to \mathbb{R}$, then $g(X)$ and $h(Y)$ are also independent.

1.1.2 Conditioning

Definition 1.8. The **conditional distribution function** of Y given $X = x$, written $F_{Y|X}(\cdot \mid x)$, is defined by

$$F_{Y|X}(y \mid x) = \Pr(Y \leq y \mid X = x)$$

Definition 1.9. The **conditional probability mass function** of Y given $X = x$, written $f_{Y|X}(\cdot \mid x)$, is defined by

$$f_{Y|X}(y \mid x) = \Pr(Y = y \mid X = x)$$

Proposition 4. Iterated expectations:

- $\mathbb{E}\big[\mathbb{E}(Y \mid X)\big] = \mathbb{E}(Y)$
- $\mathbb{E}\big[(X \mid Y) \mid Z\big] = \mathbb{E}(X \mid Y)$
- $\mathbb{E}(E(XY \mid Y)) = \mathbb{E}(Y\mathbb{E}(X \mid Y))$

Definition 1.10. Conditional Variance: $\mathrm{Var}(X \mid Y) = \mathbb{E}\big[(X - \mathbb{E}(X \mid Y))^2 \mid Y\big]$

1.1.3 Odds and Ends

Proposition 5. Inclusion-Exclusion Principle:

(a)
$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{1 \leq i_1 < \ldots < i_k \leq m} \Pr\left(A_{i1} \cap \ldots \cap A_{ik}\right) \right)$$

(b)
$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{1 \leq i_1 < \ldots < i_k \leq m} |A_{i1} \cap \ldots \cap A_{ik}| \right)$$

Theorem 6. Sums of random variables. If X and Y are independent then

$$\Pr(X + Y = z) = f_{X+Y}(z) = \sum_x f_X(x) f_Y(z - x) = \sum_y f_X(z - y) f_Y(y)$$

Proposition 7. Variance-Covariance Expansion. Let $X_1, \ldots, X_n$ be random variables. If $\mathbb{E}\left|X_k\right|^2 < \infty$, then

$$\mathrm{Var}(X_1 + \ldots + X_n) = \sum_k \mathrm{Var}(X_k) + \sum_{k \neq m} \sum_m \mathrm{Cov}(X_k, X_m)$$

1.1.4 Methods for Calculating Quantities

- Expectation
  - **Definition 1.11.** $\mathbb{E}(X) = \sum_x x \Pr(X = x)$
  - **Theorem 8.** (a) $\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$
    (b) If $X \geq 0$ then $\mathbb{E}(X) \geq 0$
  - **Theorem 9. Law of the Unconscious Statistician:** If X has mass function f, and $g : \mathbb{R} \to \mathbb{R}$, then
    $$\mathbb{E}(g(X)) = \sum_x g(x) f(x)$$
  - **Proposition 10.** Expectation is a linear operator: $\mathbb{E}\big(\sum_i X_i\big) = \sum_i \mathbb{E}(X_i)$
- Variance
  - **Definition 1.12.** $\mathrm{Var}(X) = \mathbb{E}(X - \mathbb{E}(X))^2$
  - **Proposition 11. (Useful reformulation:)** $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$
  - **Theorem 12. (Some useful results):**
    (a) $\mathrm{Var}(aX) = a^2 \mathrm{Var}(X)$
    (b) $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X, Y)$
    (c) $\mathrm{Var}(aX \pm bY) = a^2 \mathrm{Var}(X) + b^2 \mathrm{Var}(Y) \pm 2ab\mathrm{Cov}(X, Y)$
  - **Theorem 13. Law of Total variance:** $\mathrm{Var}(X) = \mathrm{Var}\big(\mathbb{E}(X \mid Y)\big) + \mathbb{E}\big(\mathrm{Var}(X \mid Y)\big)$
- Covariance
  - **Definition 1.13.** $\mathrm{Cov}(X, Y) = \mathbb{E}\big[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))\big]$
  - **Proposition 14. (Useful reformulation):** $\mathrm{Cov}(X) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$
  - **Definition 1.14.** Conditional covariance:
    $$\mathrm{Cov}(X, Y \mid Z) = \mathbb{E}(XY \mid Z) - \mathbb{E}(X \mid Z)\mathbb{E}(Y \mid Z) = \mathbb{E}\big[(X - \mathbb{E}(X \mid Z))(Y - \mathbb{E}(Y \mid Z)) \mid Z\big]$$
  - **Theorem 15. Law of Total Covariance:**
    $$\mathrm{Cov}(X, Y) = \mathbb{E}\big(\mathrm{Cov}(X, Y \mid Z)\big) + \mathrm{Cov}\big(\mathbb{E}(X \mid Z), \mathbb{E}(Y \mid Z)\big)$$

1.1.5 Discrete Random Variable Distributions

Binomial: Binomial(n, p) (sum of n Bernoulli random variables)

- Mass function: $\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$
- Distribution: $\Pr(X \leq k) = \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i}$
- Expectation: $\mathbb{E}(X) = np$
- Variance: $\mathrm{Var}(X) = np(1-p)$

Poisson: Poisson(λ): an approximation of the binomial distribution for n very large, p very small, $np \to \lambda \in (0, \infty)$.

- Mass function:
$$\Pr(X = k) = \frac{e^{-\lambda} \lambda^k}{k!}$$
- Distribution: $\Pr(X \leq k) = \sum_{i=0}^{k} \frac{e^{-\lambda} \lambda^i}{i!}$
- Expectation: $\mathbb{E}(X) = \lambda$ (derive from basic definitions)
- Variance: $\mathrm{Var}(X) = \lambda$

Geometric: $\mathrm{G}_1(p)$: the number of Bernoulli trials before the first success.

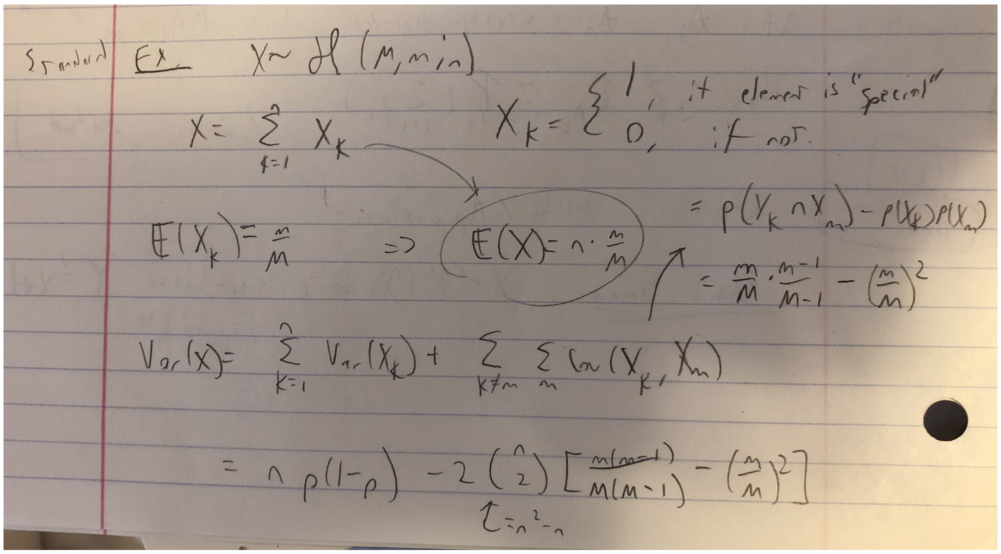
- Mass function: $\Pr(X = k) = p(1-p)^{k-1}$
- Distribution: $\Pr(X \leq k) = \sum_{i=1}^{k} p(1-p)^{k-1}$
- Expectation: $\mathbb{E}(X) = 1/p$
- Variance: $\mathrm{Var}(X) = (1-p)/p^2$

Negative binomial: NB(r, p): The number of Bernoulli trials required for r successes. (Can be derived as the sum of r identically distributed geometric random variables.)

- Mass function: $\Pr(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$
- Distribution: $\Pr(X \leq k) = \sum_{i=r}^{k} \binom{i-1}{r-1} p^r (1-p)^{i-r}$
- Expectation: $\mathbb{E}(X) =$
- Variance: $\mathrm{Var}(X) =$

Hypergeometric: Hypergeometric(N, M, K): When drawing a sample of size K from a group of N items, M of which are special, the number of special items retrieved.

- Mass function:
$$\Pr(X = k) = \frac{\binom{M}{k}\binom{N-M}{K-k}}{\binom{N}{K}}$$
- Distribution:
$$\Pr(X \le k) = \sum_{i=0}^{k} \frac{\binom{M}{i}\binom{N-M}{K-i}}{\binom{N}{K}}$$
- Expectation: $\mathbb{E}(X) =$ (find by indicator method)



- Variance: $\mathrm{Var}(X) =$ (find by indicator method)

1.1.6 Indicator Method

Proposition 16. If $\mathbf{1}_{A_k}$ is an indicator then

(a)
$$\mathrm{Cov}(\mathbf{1}_{A_k}, \mathbf{1}_{A_m}) = \mathbb{E}(\mathbf{1}_{A_k}\mathbf{1}_{A_m}) - \mathbb{E}(\mathbf{1}_{A_k})\mathbb{E}(\mathbf{1}_{A_m}) = \Pr(A_k \cap A_m) - \Pr(A_k)\Pr(A_m)$$

(b)
$$\mathrm{Var}(\mathbf{1}_{A_k}) = \mathbb{E}(\mathbf{1}_{A_k}^2) = \mathbb{E}(\mathbf{1}_{A_k})^2 = \Pr(A_k) - (\Pr(A_k))^2$$

Theorem 17. *X is independent of Y if and only if X is independent of $\mathbf{1}_A$, $A \in Y$.*

Example problems: 505A Homework 3 problem 9(a)

Worked examples in p. 56 - 59 of Grimmett and Stirzaker 3rd edition.

1.1.7 Linear transformations of random variables

1.1.8 Poisson Paradigm (Poisson approximation for indicator method)

Theorem 18. (Theorem 4.12.9, p. 129 of Grimmett and Stirzaker.) *Let A_i be an event. If $X = \sum_{i=1}^{m} \mathbf{1}_{A_i}$ where $\mathbf{1}_{A_i}$ is an indicator variable for A_i, and the A_i are only weakly dependent on each other, then*

$$\text{As } m \to \infty, \quad X \sim \text{Poisson}(\mathbb{E}(X))$$

More specifically, let B_i be n independent Bernoulli random variables with probabilities p_i. If $Y = \sum_{i=1}^n B_i$ then

$$\text{As } n \to \infty, \quad Y \sim \text{Poisson}\left(\mathbb{E}\left(\sum_i B_i\right)\right) = \text{Poisson}\left(\sum_i \mathbb{E}B_i\right) = \text{Poisson}\left(\sum_i p_i\right)$$

1.1.9 Asymptotic Distributions

Proposition 19.

$$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

Theorem 20. Stirling's Formula:

$$n! \approx n^n e^{-n}\sqrt{2\pi n}$$

1.2 Worked problems

1.2.1 Example Problems That Will Likely Appear on Midterm

Fall 2011 Problem 1 (same as HW1 problem 5; similar to HW3 problem 2(5); likely to be question 1 on the midterm.) True or false: if A and B are events such that $0 < \Pr(A) < 1$ and $\Pr(B \mid A) = \Pr(B \mid A^c)$, then A and B are independent.

Solution. A and B are independent if and only if

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$$

We know that

$$\Pr(B) = \Pr(B|A) \cdot \Pr(A) + \Pr(B|A^c) \cdot \Pr(A^c)$$

$$= \Pr(B|A) \cdot \Pr(A) + \Pr(B|A) \cdot (1 - \Pr(A)) = \Pr(B|A) \cdot \Pr(A) + \Pr(B|A) - \Pr(B|A) \cdot \Pr(A)$$

$$= \Pr(B|A)$$

Also, we know that since $\Pr(A) \neq 0$,

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)}$$

Per above $\Pr(B|A) = \Pr(B)$, so we have

$$\Pr(B) = \frac{\Pr(A \cap B)}{\Pr(A)}$$

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$$

which is what we were trying to prove. So the answer is $\boxed{\text{true.}}$

Similar problem: HW3 Problem 2(5). Verify: $\mathbb{E}(X \mid Y) = \mathbb{E}(X)$ if X and Y are independent.

Solution. X and Y are independent if and only if

$$\Pr(X \cap Y) = \Pr(X) \cdot \Pr(Y) \iff \Pr(X = x \cap Y = y) = \Pr(X = x) \Pr(Y = y)$$

$$\iff \Pr(X = x \mid Y = y) \cdot \Pr(Y = y) = \Pr(X = x) \Pr(Y = y) \iff \Pr(X = x \mid Y = y) = \Pr(X = x)$$

$$\implies E(X \mid Y) = \sum_x x \cdot \Pr(X = x \mid Y = y) = \sum_x x \cdot \Pr(X = x) = \mathbb{E}(X)$$

Fall 2014 Problem 1 (likely to be question 2 on the midterm). Let A and B be two events with $0 < \Pr(A) < 1,\ 0 < \Pr(B) < 1$. Define the random variables $\xi = \xi(\omega)$ and $\eta = \eta(\omega)$ by

$$\xi(\omega) = \begin{cases} 5 & \text{if } \omega \in A \\ -7 & \text{if } \omega \notin A \end{cases}, \quad \eta(\omega) = \begin{cases} 2 & \text{if } \omega \in B \\ 3 & \text{if } \omega \notin B \end{cases}$$

True or false: the events A and B are independent if and only if the random variables ξ and η are uncorrelated?

Solution. ($\implies$) Suppose A and B are independent. Then ξ and η are uncorrelated if and only if $\mathbb{E}(\xi\eta) = \mathbb{E}(\xi)\mathbb{E}(\eta)$. We can write $\xi = 5 \cdot \mathbf{1}_A - 7 \cdot \mathbf{1}_{A^c}$ and $\eta = 2 \cdot \mathbf{1}_B + 3 \cdot \mathbf{1}_{B^c}$. So we have

$$\xi\eta = (5 \cdot \mathbf{1}_A - 7 \cdot \mathbf{1}_{A^c})(2 \cdot \mathbf{1}_B + 3 \cdot \mathbf{1}_{B^c}) = 10 \cdot \mathbf{1}_{A \cap B} + 15 \cdot \mathbf{1}_{A \cap B^c} - 14 \cdot \mathbf{1}_{A^c \cap B} - 21 \cdot \mathbf{1}_{A^c \cap B^c}$$

$$\implies \mathbb{E}(\xi\eta) = 10 \Pr(A \cap B) + 15 \Pr(A \cap B^c) - 14 \Pr(A^c \cap B) - 21 \Pr(A^c \cap B^c)$$

Then

$$\mathbb{E}(\xi)\mathbb{E}(\eta) = (5 \Pr(A) - 7 \Pr(A^c))(2 \Pr(B) + 3 \Pr(B^c))$$

$$= 10\Pr(A \cap B) + 15\Pr(A \cap B^c) - 14\Pr(A^c \cap B) - 21\Pr(A^c \cap B^c) = \mathbb{E}(\xi\eta)$$

where the second-to-last step follows from the independence of A and B. Therefore η and ξ are uncorrelated.

($\Longleftarrow$) Now suppose η and ξ are uncorrelated. Then ξ and η are independent if and only if $\Pr(\xi \cap \eta) = \Pr(\xi)\Pr(\eta)$. Define

$$\alpha(\omega) = \xi(\omega) + 7 = \begin{cases} 12 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}, \qquad \beta(\omega) = \eta(\omega) - 3 = \begin{cases} -1 & \text{if } \omega \in B \\ 0 & \text{if } \omega \notin B \end{cases}$$

Then we have

$$(\alpha\beta)(\omega) = \begin{cases} -12 & \text{if } \omega \in A \cap B \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\mathbb{E}(\xi\eta) = \mathbb{E}\big[(\alpha - 7)(\beta + 3)\big] = \mathbb{E}(\alpha\beta) + 3\mathbb{E}(\alpha) - 7\mathbb{E}(\beta) - 21$$

$$\mathbb{E}(\xi)\mathbb{E}(\eta) = (\mathbb{E}(\alpha) - 7)(\mathbb{E}(\beta) + 3) = \mathbb{E}(\alpha)\mathbb{E}(\beta) - 7\mathbb{E}(\beta) + 3\mathbb{E}(\alpha) - 21$$

Since by assumption $\mathbb{E}(\xi\eta) = \mathbb{E}(\xi)\mathbb{E}(\eta)$, this yields $\mathbb{E}(\alpha\beta) = \mathbb{E}(\alpha)\mathbb{E}(\beta)$. But

$$\mathbb{E}(\alpha\beta) = -12\Pr(A \cap B), \qquad \mathbb{E}(\alpha)\mathbb{E}(\beta) = 12\Pr(A)(-1)\Pr(B) = -12\Pr(A)\Pr(B)$$

Therefore $\Pr(\xi \cap \eta) = \Pr(\xi)\Pr(\eta)$ and ξ and η are independent.

HW1 Problem 8. Two people, A and B, are involved in a duel. The rules are simple: shoot at each other once; if at least one is hit, the duel is over, if both miss, repeat (go to the next round), and so on. Denote by p_A and p_B the probabilities that A hits B and B hits A with one shot, and assume that that hitting/missing is independent from round to round. Compute the probabilities of the following events: (a) the duel ends and A is not hit; (b) the duel ends and both are hit; (c) the duel ends after round number n; (d) the duel ends after round number n GIVEN that A is not hit; (e) the duel ends after n rounds GIVEN that both are hit; (f) the duel goes on forever.

Solution.

(a) Let A_k denote the event that the duel is ended by A shooting B in the kth round (with neither person being shot in the first $k-1$ rounds). Note that $\{A_k | k = 1, 2, \ldots\}$ are all mutually exclusive. Therefore the probability of the duel ending without A being hit is $\sum_{k=1}^{\infty} A_k$. Because the probabilities in each round are constant and independent,

$$A_k = (1-p_A)^{k-1}p_A(1-p_B)^k$$

So the probability that the duel ends and A is not hit is

$$\sum_{k=1}^{\infty} A_k = \sum_{k=1}^{\infty}(1-p_A)^{k-1}p_A(1-p_B)^k = p_A(1-p_B)\sum_{k=1}^{\infty}(1-p_A)^{k-1}(1-p_B)^{k-1}$$

This is an infinite geometric series. Since the ratio $(1-p_A)(1-p_B)$ has absolute value less than 1, the sum can be calculated.

$$\sum_{k=1}^{\infty} A_k = p_A(1-p_B)\cdot\frac{1}{1-(1-p_A)(1-p_B)} = \frac{p_A(1-p_B)}{p_A+p_B-p_Ap_B} = \boxed{\frac{p_A(1-p_B)}{p_A(1-p_B)+p_B}}$$

(b) Similar to part (a). Let C_k denote the event that the duel is ended with both players being shot in the kth round (with neither person being shot in the first $k-1$ rounds). Again, $\{C_k | k = 1, 2, \ldots\}$ are all mutually exclusive, so the probability of the duel ending in these circumstances is $\sum_{k=1}^{\infty} C_k$. We have

$$C_k = (1-p_A)^{k-1}p_A(1-p_B)^{k-1}p_B$$

$$\sum_{k=1}^{\infty} C_k = \sum_{k=1}^{\infty}(1-p_A)^{k-1}p_A(1-p_B)^{k-1}p_B = p_Ap_B\sum_{k=1}^{\infty}(1-p_A)^{k-1}(1-p_B)^{k-1}$$

$$= p_Ap_B\cdot\frac{1}{1-(1-p_A)(1-p_B)} = \boxed{\frac{p_Ap_B}{p_A+p_B-p_Ap_B}}$$

Note that this value is less than the answer from part (a) if $p_B < \frac{1}{2}$ and greater if $p_B > \frac{1}{2}$

(c) Let B_k denote the event that the duel is ended by B shooting A in the kth round (with neither person being shot in the first $k-1$ rounds), with

$$B_k = (1-p_A)^k p_B(1-p_B)^{k-1}$$

Let A_k and C_k be defined as above. Note that $\{A_k | k = 1, 2, \ldots\}, \{B_k | k = 1, 2, \ldots\}, \{C_k | k = 1, 2, \ldots\}$ are all mutually exclusive, and that the event that the duel ends in round n is $\{A_n \cup B_n \cup C_n\}$. So the probability of the duel ending in round n is

$$\Pr(A_n \cup B_n \cup C_n) = \Pr(A_n) + \Pr(B_n) + \Pr(C_n)$$

$$= (1-p_A)^{n-1}p_A(1-p_B)^n + (1-p_A)^n p_B(1-p_B)^{n-1} + (1-p_A)^{n-1}p_A(1-p_B)^{n-1}p_B$$

$$= (1-p_A)^{n-1}(1-p_B)^{n-1}\left[p_A(1-p_B) + (1-p_A)p_B + p_Ap_B\right]$$

$$= \boxed{(1-p_A)^{n-1}(1-p_B)^{n-1}(p_A+p_B-p_Ap_B)}$$

(d) Let A_k, B_k, C_k be defined as above. The event that the duel ends at round n without A being hit is given by $\{A_n\}$.

$$\Pr(A_n) = \boxed{(1-p_A)^{n-1}p_A(1-p_B)^n}$$

(e) Let A_k, B_k, C_k be defined as above. The event that the duel ends at round n with both players being hit is given by $\{C_n\}$.

$$\Pr(C_n) = \boxed{(1-p_A)^{n-1}p_A(1-p_B)^{n-1}p_B}$$

(f) Let A_k, B_k, C_k be defined as above. The probability that the duel never ends is equal to 1 - the probability that the duel ends at some point, which is $\{A_k | k = 1,2,\ldots\} \cup \{B_k | k = 1,2,\ldots\} \cup \{C_k | k = 1,2,\ldots\}$. Since all of these events are mutually exclusive, we have

$$1 - \Pr(\{A_k|k=1,2,\ldots\} \cup \{B_k|k=1,2,\ldots\} \cup \{C_k|k=1,2,\ldots\}) = 1 - \sum_{k=1}^{\infty}\big(A_k + B_k + C_k\big)$$

$$= 1 - \sum_{k=1}^{\infty}\big((1-p_A)^{k-1}p_A(1-p_B)^k + (1-p_A)^k p_B(1-p_B)^{k-1} + (1-p_A)^{k-1}p_A(1-p_B)^{k-1}p_B\big)$$

$$= 1 - \big[p_A(1-p_B) + (1-p_A)p_B + p_Ap_B\big]\sum_{k=1}^{\infty}(1-p_A)^{k-1}(1-p_B)^{k-1}$$

$$= 1 - \big[p_A 1 - p_Ap_B) + p_B - p_A)p_B + p_Ap_B\big] \cdot \frac{1}{1-(1-p_A)(1-p_B)}$$

$$= 1 - \frac{p_A - p_Ap_B + p_B - p_Ap_B + p_Ap_B}{p_A + p_B - p_Ap_B} = 1 - \frac{p_A + p_B - p_Ap_B}{p_A + p_B - p_Ap_B} = \boxed{0}$$

Similar: HW3 Problem 2 (parts 1 - 4). Verify:

(1) $\mathbb{E}(\mathbb{E}(X \mid Y)) = \mathbb{E}(X)$

(2) $\mathbb{E}(g(Y)X \mid Y) = g(Y)\mathbb{E}(X \mid Y)$

(3) $\mathrm{Cov}(\mathbb{E}(X \mid Y), Y) = \mathrm{Cov}(X, Y)$

(4) Y and $X - \mathbb{E}(X \mid Y)$ are uncorrelated.

Solution.

(1)

$$\mathbb{E}\big(\mathbb{E}(X \mid Y)\big) = \sum_y \mathbb{E}(X \mid Y)\Pr(Y=y) = \sum_y \left[\sum_x x \cdot \Pr(X = x \mid Y = y)\Pr(Y=y)\right]$$

$$= \sum_y \left[\sum_x x \cdot \Pr(X = x \cap Y = y)\right] = \sum_y \left[\sum_x x \cdot \Pr(Y = y \mid X = x) \cdot \Pr(X = x)\right]$$

$$= \sum_x \left[x \cdot \Pr(X = x) \cdot \sum_y \big(\Pr(Y = y \mid X = x) \big) \right] = \sum_x \left[x \cdot \Pr(X = x) \cdot 1 \right]$$

$$= \mathbb{E}(X)$$

(2) 2

(3)

$$\mathrm{Cov}(\mathbb{E}(X \mid Y), Y) = \mathbb{E}\Big(\Big[\mathbb{E}(X \mid Y) - \mathbb{E}\big(\mathbb{E}(X \mid Y)\big) \Big] \Big[Y - \mathbb{E}(Y) \Big] \Big)$$

$$= \mathbb{E}\Big(\Big[\mathbb{E}(X \mid Y) - \mathbb{E}(X) \Big] \Big[Y - \mathbb{E}(Y) \Big] \Big) = \mathbb{E}\Big(\mathbb{E}(X \mid Y)Y - \mathbb{E}(X)Y - \mathbb{E}(X \mid Y)\mathbb{E}(Y) + \mathbb{E}(X)\mathbb{E}(Y) \Big)$$

$$= \mathbb{E}\big(\mathbb{E}(X \mid Y)Y\big) - \mathbb{E}(X)\mathbb{E}(Y) - \mathbb{E}(Y)\mathbb{E}\big(\mathbb{E}(X \mid Y)\big) + \mathbb{E}(X)\mathbb{E}(Y) = \mathbb{E}\big(\mathbb{E}(X \mid Y)Y\big) - \mathbb{E}(Y)\mathbb{E}(X)$$

$$= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = \mathrm{Cov}(X, Y)$$

(4) Y and $X - \mathbb{E}(X \mid Y)$ are uncorrelated if and only if $\mathrm{Cov}\big(Y, X - \mathbb{E}(X \mid Y)\big) = 0 \iff \mathbb{E}(Y \cdot \big[X - \mathbb{E}(X \mid Y)\big]) - \mathbb{E}(Y)\mathbb{E}(X - \mathbb{E}(X \mid Y)) = 0$.

$$\mathbb{E}(Y \cdot \big[X - \mathbb{E}(X \mid Y)\big]) - \mathbb{E}(Y)\mathbb{E}(X - \mathbb{E}(X \mid Y)) = \mathbb{E}(YX - Y\mathbb{E}(X \mid Y)) - \mathbb{E}(Y)\mathbb{E}(X) + \mathbb{E}(Y)\mathbb{E}\big(\mathbb{E}(X \mid Y)\big)\big)$$

$$= \mathbb{E}(YX) - \mathbb{E}\big(Y\mathbb{E}(X \mid Y)\big) - \mathbb{E}(Y)\mathbb{E}(X) + \mathbb{E}(Y)\mathbb{E}(X) = \mathbb{E}(YX) - \mathbb{E}(YX) = 0$$

Remaining problems are likely to be indicator method.

1.2.2 Problems we did in class that professor mentioned

Matching: n objects belong in n places. If placed randomly, what is the probability of at least one match?

Let A_k = match for object k. Want: $P\left(\bigcup_{k=1}^{n} A\right)$

$$P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{(n-k)!}{n!}$$

# of groups of k objects that could be matched correctly

probability that k objects placed correctly (other $n-k$ placed randomly)

$$= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!} = 1 - \sum_{k=0}^{n} \frac{(-1)^k}{k!} \to 1 - \frac{1}{e}$$

Standard Example 2 Matches (n objects, n places)

X = # of matches

$X_k = 1$ if objects matched at correct location

$$X = \sum_{k=1}^{n} X_k$$

$P(X_k = 1) = \frac{1}{n}$ $\quad \mathbb{E}(X) = 1$

$P(X_k = 1, X_m = 1) = \frac{1}{n} \cdot \frac{1}{n-1}$

$$\mathrm{Var}(X) = \underbrace{n \cdot \frac{1}{n}\left(1 - \frac{1}{n}\right)}_{np(1-p)} + n(n-1)\left(\frac{1}{n} \cdot \frac{1}{n-1} - \frac{1}{n^2}\right)$$

Variance Problem 09/21 If $\mathbb{E}(X \mid Y) = Y, \mathbb{E}(Y \mid X) = X, \mathbb{E}(X^2) < \infty, \mathbb{E}(Y^2) < \infty$, show $\mathbb{E}(X - Y)^2 = 0$ (or equivalently, show $\Pr(X = Y) = 1$).

Solution.

$$\mathbb{E}(X - Y)^2 = \mathbb{E}(X^2 - 2XY + Y^2) = \mathbb{E}(X^2) - 2\mathbb{E}(XY) + \mathbb{E}(Y^2)$$

$$\mathbb{E}(XY) = \mathbb{E}(\mathbb{E}(XY \mid Y)) = \mathbb{E}(Y\mathbb{E}(X \mid Y)) = \mathbb{E}(Y \cdot Y) = \mathbb{E}(Y^2)$$

Also,

$$\mathbb{E}(XY) = \mathbb{E}((XY \mid X)) = \mathbb{E}(X\mathbb{E}(Y \mid X)) = \mathbb{E}(X \cdot X) = \mathbb{E}(X^2)$$

Therefore

$$\mathbb{E}(X - Y)^2 = 0$$

Spring 2018 Problem 2 (did not complete)

2. Consider positions 1 to n arranged in a circle, so that 2 comes after 1, 3 comes after 2, ..., n comes after $n-1$, and 1 comes after n. Similarly, take 1 to n as values, with cyclic order, and consider all $n!$ ways to assign values to positions, bijectively, with all $n!$ possibilities equally likely. For $i = 1$ to n, let X_i be the indicator that position i and the one following are filled in with two consecutive values in increasing order, and define

$$S_n = \sum_{i=1}^{n} X_i, \quad T_n = \sum_{i=1}^{n} iX_i$$

For example, with $n = 6$ and the the circular arrangement 314562, we get $X_3 = 1$ since 45 are consecutive in increasing order, and similarly $X_4 = X_6 = 1$, so that $S_6 = 3, T_6 = 13$.

a) Compute the mean and the variance of S_n.

b) Compute the mean and the variance of T_n.

Fall 2008 Problem 2 (HW1 Problem 10). Consider a lottery with n^2, tickets, of which only n tickets win prizes. Let p_n be the probability that, out of n randomly selected tickets, at least one wins a prize. Compute $\lim_{n\to\infty} p_n$.

Solution. There are $\binom{n^2}{n}$ possible sets of n tickets. The number of these sets that do not contain at least one winner (that is, they only contain members of the $n^2 - n$ losing tickets) is $\binom{n^2-n}{n}$. Therefore the probability of selecting a set of n tickets that contains at least one winner is

$$p_n = 1 - \binom{n^2 - n}{n} \Big/ \binom{n^2}{n} = 1 - \frac{(n^2-n)!}{n!(n^2-n-n)!} \Big/ \frac{(n^2)!}{(n^2-n)!n!} = 1 - \frac{(n^2-n)!}{n!(n^2-2n)!} \cdot \frac{(n^2-n)!n!}{(n^2)!}$$

$$= 1 - \frac{(n^2-n)!}{(n^2-2n)!} \cdot \frac{(n^2-n)!}{(n^2)!} = 1 - \prod_{i=0}^{n-1}(n^2-n-i) \Big/ \prod_{i=0}^{n-1}(n^2-i) = 1 - \prod_{i=0}^{n-1} \frac{n^2-n-i}{n^2-i}$$

$$= 1 - \prod_{i=0}^{n-1}\left(\frac{n^2-i}{n^2-i} - \frac{n}{n^2-i}\right) = 1 - \prod_{i=0}^{n-1}\left(1 - \frac{n}{n^2-i}\right)$$

Therefore

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty}\left[1-\prod_{i=0}^{n-1}\left(1-\frac{n}{n^2-i}\right)\right] = 1-\lim_{n\to\infty}\prod_{i=0}^{n}\left(1-\frac{n}{n^2-i}\right) = 1-\lim_{n\to\infty}\prod_{i=0}^{n}\left(1-\frac{n\cdot\frac{1}{n}}{\frac{n^2}{n}-\frac{i}{n}}\right)$$

$$= 1-\lim_{n\to\infty}\prod_{i=0}^{n}\left(1-\frac{1}{n-\frac{i}{n}}\right) = 1-\lim_{n\to\infty}\prod_{i=0}^{n}\left(1-\frac{1}{n}\right) = 1-\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \boxed{1-\exp(-1)}$$

1.2.3 Problems we did on homework

Fall 2017 Problem 3 (HW3 Problem 8—almost full solution)

Problem 8. Let $U_1, U_2, \ldots$ be iid random variables, uniformly distributed on $[0,1]$, and let N be a Poisson random variable with mean value equal to one. Assume that N is independent of $U_1, U_2, \ldots$ and define

$$Y = \begin{cases} 0, & \text{if } N = 0, \\ \max\limits_{1\le i\le N} U_i, & \text{if } N > 0. \end{cases}$$

Compute the expected value of Y.

Solution.

Since Y is a function of N, let $Y = y(N)$. By the Law of the Unconscious Statistician (Theorem 9),

$$\mathbb{E}(Y) = \mathbb{E}\big(\mathbb{E}(Y \mid N)\big) = \mathbb{E}\big(\mathbb{E}(\max_{1\le i\le N} U_i \mid N = n)\big)$$

Let $Z_n = \max_{1\le i\le n} U_i$. The cdf of Z_n can be calculated as follows:

$$\Pr(Z_n \le x) = \Pr(\max_{1\le i\le n} U_i \le x) = \Pr(U_1 \le x \cap U_2 \le x \cap \ldots \cap U_n \le x) = x^n$$

for $x \in [0,1]$. Therefore the pdf of Z_n is its derivative, nx^{n-1}. So we have

$$\mathbb{E}(\max_{1\le i\le N} U_i \mid N = n) = \mathbb{E}(Z_n) = \int_0^1 x n x^{n-1} dx = n\int_0^1 x^n dx = n\frac{x^{n+1}}{n+1}\bigg|_0^1 = \frac{n}{n+1}$$

Plugging this into the expression for $\mathbb{E}(Y)$ yields

$$\mathbb{E}(Y) = \mathbb{E}\left(\frac{N}{N+1}\right) = \sum_{n=1}^{\infty}\frac{n}{n+1}\Pr(N=n) = \sum_{n=1}^{\infty}\frac{n}{n+1}\frac{\exp(-1)1^n}{n!} = \boxed{\frac{1}{e}\sum_{n=1}^{\infty}\frac{n}{(n+1)!}}$$

Fall 2013 Problem 3/Spring 2011 Problem 2 (HW3 Problem 9; coupon collector problem)
Only parts I didn't do: Let D be the event that no box receives more than 1 ball. Fix $a \in (0,1)$. If both $n, d \to \infty$ together, what relation must they satisfy in order to have $\Pr(D) \to a$?

HW3 Problem 9. Consider n (different) balls placed at random in m boxes so that each of m^n configurations is equally likely.

(a) Compute the expected value and the variance of the number of empty boxes.

(b) Show that if $\lim_{m,n\to\infty} m\exp(-n/m) = \lambda \in (0,\infty)$, then, in the same limit, the number of empty boxes has Poisson distribution with parameter λ.

(c) For $k \geq 1$ such that $k+3 \leq m$, define the event A_k that the boxes $k, k+1, k+2, k+3$ are empty. Assuming that $m > 8$, compute $\Pr(A_1 \cup A_3 \cup A_5)$. How will the answer change if $m = 8$?

(d) Now imagine that the balls are dropped one-by-one (with each ball equally likely to go into any of the m boxes, independent of all other balls), and denote by N_m the minimal number of balls required to fill all the boxes. Compute $\mathbb{E}(N_m), \mathrm{Var}(N_m)$ and

$$\lim_{m\to\infty} \Pr\left(\frac{N_m - m\log m}{m} \leq x\right)$$

Solution.

(a) Let A_i be the event that the ith box is empty. Let $\mathbf{1}_{A_i}$ be the indicator for A_i. Then $X = \sum_{i=1}^m \mathbf{1}_{A_i}$.

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_{i=1}^m \mathbf{1}_{A_i}\right) = \sum_{i=1}^m \left(\mathbb{E}\mathbf{1}_{A_i}\right) = \sum_{i=1}^m \Pr(A_i) = \sum_{i=1}^m \left(\frac{m-1}{m}\right)^n = \boxed{\frac{(m-1)^n}{m^{n-1}}}$$

$$\mathrm{Var}(X) = \mathrm{Var}\left(\sum_{i=1}^m \mathbf{1}_{A_i}\right) = \sum_{i=1}^m \mathrm{Var}(\mathbf{1}_{A_i}) + 2\sum_{1\leq i<j\leq m} \mathrm{Cov}(\mathbf{1}_{A_i}, \mathbf{1}_{A_j})$$

$$\mathrm{Var}(\mathbf{1}_{A_i}, \mathbf{1}_{A_j}) = \mathbb{E}(\mathbf{1}_{A_i}\mathbf{1}_{A_i}) - \mathbb{E}(\mathbf{1}_{A_i})^2 = \Pr(A_i \cap A_i) - \Pr(A_i)^2 = \left(\frac{m-1}{m}\right)^n - \left(\frac{m-1}{m}\right)^{2n}$$

$$\mathrm{Cov}(\mathbf{1}_{A_i}, \mathbf{1}_{A_j}) = \mathbb{E}(\mathbf{1}_{A_i}\mathbf{1}_{A_j}) - \mathbb{E}(\mathbf{1}_{A_i})\mathbb{E}(\mathbf{1}_{A_j}) = \Pr(A_i \cap A_j) - \Pr(A_i)\Pr(A_j) = \left(\frac{m-2}{m}\right)^n - \left(\frac{m-1}{m}\right)^{2n}$$

$$\implies \mathrm{Var}(X) = m \cdot \left[\left(\frac{m-1}{m}\right)^n - \left(\frac{m-1}{m}\right)^{2n}\right] + \frac{m!}{(m-2)!}\left[\left(\frac{m-2}{m}\right)^n - \left(\frac{m-1}{m}\right)^{2n}\right]$$

$$= \frac{(m-1)^n}{m^{n-1}} - \frac{(m-1)^{2n}}{m^{2n-1}} + (m^2 - m)\left[\left(\frac{m-2}{m}\right)^n - \left(\frac{m-1}{m}\right)^{2n}\right]$$

$$\boxed{\mathrm{Var}(X) = \frac{(m-1)^n}{m^{n-1}} - \frac{(m-1)^{2n}}{m^{2n-1}} + (m-1)\left[\frac{(m-2)^n}{m^{n-1}} - \frac{(m-1)^{2n}}{m^{2n-1}}\right]}$$

(b) Note that

$$X = \sum_{i=1}^m \mathbf{1}_{A_i}$$

and that the A_i are only weakly dependent on each other, especially as m and n increase. Therefore as $m, n \to \infty$, the Poisson paradigm suggests $X \sim \text{Poisson}(\mathbb{E}(X))$. We have

$$\mathbb{E}(X) = \frac{(m-1)^n}{m^{n-1}}$$

so

$$\lim_{n,m\to\infty} \mathbb{E}(X) = \lim_{n,m\to\infty} m \cdot \left(\frac{m-1}{m}\right)^n = \lim_{n,m\to\infty} m \cdot \left(1 - \frac{1}{m}\right)^n = \lim_{n,m\to\infty} m \cdot \left[\left(1 - \frac{1}{m}\right)^m\right]^{n/m}$$

$$\approx \lim_{n,m\to\infty} m \cdot \left[e^{-1}\right]^{n/m} = \lim_{n,m\to\infty} me^{-n/m}$$

Using

$$\lim_{m,n\to\infty} m \exp(-n/m) = \lambda \in (0, \infty)$$

we have $\boxed{X \sim \text{Poisson}(\lambda) \text{ as } m, n \to \infty}$.

(c)

$$\Pr(A_1 \cup A_3 \cup A_5) = \Pr(A_1) + \Pr(A_3) + \Pr(A_5) - \Pr(A_1 \cap A_3) - \Pr(A_1 \cap A_5) - \Pr(A_3 \cap A_5) + \Pr(A_1 \cap A_3 \cap A_5)$$

We have

$$\Pr(A_1) = \Pr(A_3) = \Pr(A_5) = \left(\frac{m-4}{m}\right)^n$$

$$\Pr(A_1 \cap A_3) = \Pr(A_3 \cap A_5) = \left(\frac{m-6}{m}\right)^n$$

$$\Pr(A_1 \cap A_5) = \Pr(A_1 \cap A_3 \cap A_5) = \left(\frac{m-8}{m}\right)^n$$

Therefore

$$\Pr(A_1 \cup A_3 \cup A_5) = 3\left(\frac{m-4}{m}\right)^n - 2\left(\frac{m-6}{m}\right)^n = \boxed{\frac{3(m-4)^n - 2(m-6)^n}{m^n}}$$

(d) N_m is the minimal number of balls required to fill all the boxes. Let T_i be the number of balls that have to be dropped to fill the ith box after $i-1$ boxes have been filled. The probability of filling a new box after $i-1$ boxes have been filled is $\frac{m-(i-1)}{m}$. Therefore T_i has a geometric distribution with $E(T_i) = \frac{m}{m-(i-1)}$. Since $N_m = \sum_{i=1}^m T_i$, we have

$$\mathbb{E}(N_m) = \mathbb{E}\left(\sum_{i=1}^m T_i\right) = \sum_{i=1}^m \mathbb{E}(T_i) = \sum_{i=1}^m \frac{m}{m-(i-1)} = \boxed{m \sum_{i=1}^m \frac{1}{i}}$$

Because the T_i are independent, we have

$$\text{Var}(N_m) = \text{Var}\bigg(\sum_{i=1}^m T_i\bigg) = \sum_{i=1}^m \text{Var}(T_i) = \sum_{i=1}^m \left(1 - \frac{m-(i-1)}{m}\right) \Big/ \left(\frac{m-(i-1)}{m}\right)^2$$

$$= \sum_{i=1}^m \frac{i-1}{m} \cdot \left(\frac{m}{m-(i-1)}\right)^2 = \boxed{m \sum_{i=1}^m \frac{i-1}{[m-(i-1)]^2}}$$

Finally, to find

$$\lim_{m \to \infty} \Pr\left(\frac{N_m - m \log m}{m} \leq x\right)$$

begin by noting that we can also express N_m as

$$\Pr(N_m \leq k) = \Pr\big(X_{m,k} = 0\big)$$

where $X_{m,k}$ is defined as X is in part (b) with k being the number of balls that have been dropped so far, $k \in \mathbb{N} \geq m$. (For $k < m$, $\Pr(N_m \leq k) = 0$.)

Again, let $A_{i,k}$ be the event that the ith box is empty after dropping k balls. Then because $X_{m,k} = \sum_{i=1}^m \mathbf{1}_{A_{i,k}}$ and the $A_{i,k}$ are only weakly dependent on each other (especially as m becomes large), the Poisson paradigm again suggests that as $m \to \infty$, $X_{m,k} \sim \text{Poisson}(\lambda_k)$ where $\lambda_k = \mathbb{E}(X_{m,k})$ is defined as above. Therefore we have

$$\lim_{m \to \infty} \Pr\left(\frac{N_m - m \log m}{m} \leq x\right) = \lim_{m \to \infty} \Pr\big(N_m \leq xm + m \log m\big) = \lim_{m \to \infty} \Pr\big(X_{m, xm + m \log m}$$

$$= 0\big) \approx \frac{\exp(-\lambda_{xm + m \log m}) \cdot \lambda_{xm + m \log m}^0}{0!} = \exp\big(-\lambda_{xm + m \log m}\big)$$

And we have

$$\lambda_{xm + m \log m} = \lim_{m \to \infty} m \exp\left(-\frac{xm + m \log m}{m}\right) = \lim_{m \to \infty} m \exp\big(-x - \log m\big) = \lim_{m \to \infty} m/m \exp(-x)$$

$$= \exp(-x)$$

which yields

$$\boxed{\lim_{m \to \infty} \Pr\left(\frac{N_m - m \log m}{m} \leq x\right) = \exp(\exp(-x))}$$

Fall 2012 Problem 1 (HW2 Problem 10/HW 1 Problem 9) Only part I didn't do: Find the mean and variance of $S_n = X_1 + \ldots + X_n$, the total number of white balls added to the urn up to time n.

HW1 Problem 9. An urn contains b black and w white balls. At each step, a ball is removed from the urn at random and then put back together with one more ball of the same color. Compute the probability p_n to get a black ball on step $n, n \geq 1$.

Solution. Step 1:

$$p_1 = \frac{b}{b+w}$$

Step 2: We need to separately consider the cases where a black ball was selected on step 1 (with probability p_1) or a white ball (with probability $1 - p_1$).

$$p_2 = p_1 \cdot \frac{b+1}{b+w+1} + (1-p_1) \cdot \frac{b}{b+w+1} = p_1\left(\frac{b+1}{b+w+1} - \frac{b}{b+w+1}\right) + \frac{b}{b+w+1}$$

$$= p_1\left(\frac{1}{b+w+1} + \frac{1}{p_1}\frac{b}{b+w+1}\right) = p_1\left(\frac{1}{b+w+1} + \frac{b+w}{b}\frac{b}{b+w+1}\right)$$

$$= p_1\left(\frac{b+w+1}{b+w+1}\right) = p_1$$

$$\implies p_2 = p_1 = \frac{b}{b+w}$$

Step 3: Regardless of the previous steps, there are now $b + w + 2$ balls in the urn. Since we know that $p_1 = p_2$, the probability that we have selected k black balls so far (and thus, the probability that there are currently $b + k$ black balls in the urn) is given by

$$\Pr(k \text{ balls chosen in first 2 rounds}) = \binom{2}{k} p_1^k (1-p_1)^{2-k} = \binom{2}{k}\left(\frac{b}{b+w}\right)^k\left(\frac{w}{b+w}\right)^{2-k}$$

$$= \binom{2}{k}\frac{b^k w^{2-k}}{(b+w)^2}$$

for $k \in \{0, 1, 2\}$. Given that we have selected k black balls so far, the probability of selecting a black ball this time is $\frac{b+k}{b+w+2}$. Therefore the probability of selecting a black ball this round is

$$p_3 = \sum_{k=0}^{2}\binom{2}{k}\frac{b^k w^{2-k}}{(b+w)^2}\frac{b+k}{b+w+2} = \frac{1}{(b+w+2)(b+w)^2}\sum_{k=0}^{2}\binom{2}{k}(b+k)b^k w^{2-k}$$

$$= \frac{1}{(b+w+2)(b+w)^2}\left(\binom{2}{0}bw^2 + \binom{2}{1}(b+1)bw + \binom{2}{2}(b+2)b^2\right)$$

$$= \frac{bw^2 + 2(b+1)bw + (b+2)b^2}{(b+w+2)(b+w)^2} = \frac{b}{b+w}\left(\frac{w^2 + 2bw + 2w + b^2 + 2b}{b^2 + bw + 2b + wb + w^2 + 2w}\right)$$

$$= \frac{b}{b+w}\left(\frac{w^2 + 2bw + 2w + b^2 + 2b}{b^2 + 2bw + 2b + w^2 + 2w}\right) = \frac{b}{b+w} = p_1$$

There seems to be a clear pattern here. Let's find the general formula by induction.

Step $n+1$**:** Assume that the probability of choosing a black ball on steps $1, 2, \ldots, n$ was $\frac{b}{b+w}$ each time.

(a bunch of boring stuff, then it worked.)

HW2 Problem 10. Random variables $(X_1, \ldots, X_n)$ are called *exchangeable* if $\Pr(X_1 = x_1, \ldots, X_n = x_n) = \Pr(X_{\tau(1)} = x_1, \ldots, X_{\tau(n)} = x_n)$ for all real numbers $x_1, \ldots, x_n$ and every permutation τ of the set $\{1, \ldots, n\}$. In the setting of Problem 9 from Homework 1, let $X_k = 1$ if a white ball is drawn on step k, and $X_k = 0$ otherwise. Show that the random variables $X_1, \ldots, X_n$ are exchangeable for every $n \geq 2$.

Solution. For $n = 2$: There are two cases which we must show are equal to show exchangeability:

$$\Pr(X_1 = 0, X_2 = 1) = \Pr(X_1 = 1, X_2 = 0)$$

First,

$$\Pr(X_1 = 0, X_2 = 1) = \Pr(\text{black first}) \Pr(\text{white second} \mid \text{black first}) = \left(\frac{b}{b+w}\right)\left(\frac{w}{b+w+1}\right)$$

$$\left(\frac{w}{b+w}\right)\left(\frac{b}{b+w+1}\right) = \Pr(X_1 = 1, X_2 = 0)$$

which proves exchangeability for $n = 2$. In the general case, we seek to show that $X_1, \ldots, X_n$ are exchangeable. That is, in all $n+1$ unordered sets $\mathbb{X}_k = \{x_{1k}, x_{2k}, \ldots, x_{nk} \mid x_{ik} \in \{0, 1\}, \sum_i x_{ik} = k\}$, in all $\binom{n}{k}$ permutations of $\mathbb{X}_k$,

$$\Pr(\mathbb{X}_{kj} = \Pr(\mathbb{X}_{kj'}$$

where j and j' denote different permutations of $\mathbb{X}_k$. That is,

$$\Pr(X_1 = x_{1k}, X_2 = x_{2k}, \ldots, X_n = x_{nk}) = \Pr(X_{j_1} = x_{1k}, X_{j_2} = x_{2k}, \ldots, X_{j_n} = x_{nk})$$

where $j_1, j_2, \ldots, j_n$ index the permuted variables. Consider $\mathbb{X}_{kj^*}$ where all k white balls are chosen first and all $n-k$ black balls are chosen last. We have

$$\Pr(\mathbb{X}_{kj^*}) = \prod_{i=1}^{k}\left(\frac{w+i-1}{b+w+i-1}\right) \cdot \prod_{i=k+1}^{n}\left(\frac{b+i-k-1}{b+w+i-1}\right)$$

$$= \prod_{i=1}^{n}\left(\frac{1}{b+w+i-1}\right) \cdot \left[\prod_{i=1}^{k}(w+i-1) \prod_{i=k+1}^{n}(b+i-k-1)\right] = \prod_{i=1}^{n}\left(\frac{1}{b+w+i-1}\right) \cdot \left[\prod_{i=1}^{k}(w+i-1) \prod_{i'=1}^{n-k}(b+i'-1)\right]$$

It is easy to see that the leftmost product will always equal the product of the denominators, regardless of the permutation, since one ball is added to the urn after every draw. Similarly, regardless of permutation, the numerator of the probability of drawing the ith white ball will always equal $w + i - 1$, the number of white balls already in the urn. Likewise, the numerator of the probability of drawing the i'th black ball is always $b + i' - 1$. Because multiplication is commutative, all permutations of these numbers will have equal products. Therefore $\Pr(\mathbb{X}_{kj^*}) = \Pr(\mathbb{X}_{kj}$ for all k. That is,

$$\Pr(X_1 = x_1, \ldots, X_n = x_n) = \Pr(X_{\tau(1)} = x_1, \ldots, X_{\tau(n)} = x_n)$$

for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$, all $n \in \mathbb{Z}$ such that $n \geq 2$, all permutations τ.

1.3 To Know for Math 505A Midterm 2

1.3.1 Definitions

Definition 1.15. A random variable X is **continuous** if its distribution function $F(x) = \Pr(X \leq x)$ can be written as

$$F(x) = \int_{-\infty}^{x} f(u)du$$

for some integrable $f : \mathbb{R} \to [0, \infty)$.

Definition 1.16. The function f is called the **(probability) density function** of the continuous random variable X.

Proposition 21. If X has pdf $f_X(x)$, then for $\mu \in \mathbb{R}$, $\sigma > 0$,

$$h(x) = \frac{1}{\sigma} f_X\left(\frac{x - \mu}{\sigma}\right)$$

is a pdf. In this setting μ is sometimes called a "location parameter" and σ is called a "scale parameter."

Definition 1.17. The **joint distribution function** of X and Y is the function $F : \mathbb{R}^2 \to [0, 1]$ given by

$$F(x, y) = \Pr(X \leq x \cap Y \leq y)$$

Definition 1.18. The random variables X and Y are **jointly continuous** with **joint (probability) density function** $f : \mathbb{R}^2 \to [0, \infty)$ if

$$F(x, y) = \int_{v=-\infty}^{y} \int_{u=-\infty}^{x} f(u, v)dudv \text{ for each } x, y \in \mathbb{R}$$

Definition 1.19. Two continuous random variables are **independent** if and only if $\{X \leq x\}$ and $\{Y \leq y\}$ are independent events for all $x, y \in \mathbb{R}$.

Ways to show independence:

- Use Definition 1.19: show that $\Pr(X \leq x \cap Y \leq y) = \Pr(X \leq x) \Pr(Y \leq y)$ for all $x, y \in \mathbb{R}$.
-

  Theorem 22. The random variables X and Y are independent if and only if $F(x, y) = F_X(x)F_Y(x)$ for all $x, y \in \mathbb{R}$.
-

  Proposition 23. For continuous random variables, the previous condition is equivalent to requiring $f(x, y) = f_X(x)f_Y(y)$.
-

  Theorem 24. If two variables are bivariate normal, they are independent if and only if their covariance

  $$\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x, y)dxdy$$

  is equal to 0.
- Characteristic functions:

  Theorem 25. X and Y are independent if and only if $\phi_{X,Y}(s, t) = \phi_X(s)\phi_Y(t)$.

Theorem 26. (Theorem 4.2.3, Grimmett and Stirzaker.) Let X and Y be random variables, and let $g, h : \mathbb{R} \to \mathbb{R}$. If X and Y are independent, then so are $g(X)$ and $h(Y)$.

1.3.2 Probability-Generating Functions

Definition 1.20.

$$G_X(s) = \mathbb{E}(s^X)$$

Theorem 27. Some useful properties:

(a) $\mathbb{E}(X) = G'_X(1)$, $\mathbb{E}[X(X-1)\cdots(X-k+1)] = G^{(k)}(1)$

(b) If X and Y are independent then $G_{X+Y}(s) = G_X(s)G_Y(s)$.

1.3.3 Moment-Generating Functions

Definition 1.21.

$$M_X(t) = \mathbb{E}(e^{tX})$$

Theorem 28. Some useful properties:

(a) $\mathbb{E}(X) = M'_X(0)$, $\mathbb{E}(X^k) = M^{(k)}(0)$

(b) If X and Y are independent then $M_{X+Y}(t) = M_X(t)M_Y(t)$.

1.3.4 Characteristic Functions

Definition 1.22.

$$\phi_X(t) = \mathbb{E}(e^{itX})$$

Proposition 29. Necessary and sufficient conditions for a function to be a characteristic function:

(a) $\phi_X(0) = 1$

(b) $|\phi(t)| \leq 1 \ \forall \ t$

(c) ϕ is uniformly continuous on $\mathbb{R}$

(d) ϕ is positive semidefinite; that is,

$$\sum_{i,j} \phi(t_j - t_k) z_j \overline{z}_k \geq 0 \text{ for all real } t_1, t_2, \ldots, t_n \text{ and complex } z_1, z_2, \ldots, z_n$$

Or, equivalently, or every set of real numbers $t_1, t_2, \ldots, t_n$, the matrix $\phi(t_i - t_j), i, j \in \{1, 2, \ldots, n\}$ is Hermitian and nonnegative definite.

Remark. Relationship between characteristic functions and probability and moment generating functions:

$$\phi_X(t) = M_X(it) = G_X(e^{it})$$

Theorem 30. Some useful properties:

(a) $X \perp\!\!\!\perp Y \implies \phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$

(b) $Y = aX + b \implies \phi_Y(t) = e^{itb}\phi_X(at)$

(c) $\phi_X^{(k)}(0) = i^k \mathbb{E}(X^k)$

(d) $\phi_{X,Y}(s,t) = \mathbb{E}(e^{isX}e^{itY})$

(e) $X \perp\!\!\!\perp Y \iff \phi_{X,Y}(s,t) = \phi_X(s)\phi_Y(t)$

Theorem 31. Other facts from notes on course website

(a) If $\phi(t)$ is even, $\phi(0) = 1$, ϕ is convex for $t > 0$, and $\lim_{t\to\infty} \phi(t) = 0$, then ϕ is a characteristic function of an absolutely continuous random variable.

(b) If ϕ is a characteristic function and $\phi(t) = 1 + o(t^2), t \to 0$, then $\phi(t) = 1$ for all t. The random variable with such a characteristic function must have zero mean and zero variance. In particular, if $r > 2$, then $\exp(-|t|^r)$ is not a characteristic function.

(c) If $\phi(t) = e^{p(t)}$ is a characteristic function and $p = p(t)$ is a polynomial, then the degree of p is at most 2. For example, $e^{t^2 - t^4}$ is not a characteristic function.

(d) If ξ is absolutely continuous, then $\lim_{|t|\to\infty} |\phi_\xi(t)| = 0$ (Riemann-Lebesgue).

(e) If $\int_{-\infty}^{\infty} |\phi_\xi(t)|\, dt < \infty$, then ξ is absolutely continuous with pdf

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-itx)\phi(t)dt$$

1.3.5 Continuous Random Variable Distributions

Uniform: $\mathrm{U}(a, b)$

- Probability density function:

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

- Cumulative distribution function:

$$F(x) = \Pr(X \leq x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a < x \leq b \\ 1 & x > b \end{cases}$$

- Probability-generating function:
- Moment-generating function:
- Characteristic function:
- Expectation: $\mathbb{E}(X) = (b-a)/2$
- Variance: $\mathrm{Var}(X) = (b-a)^2/12$

Normal: $\mathcal{N}(\mu, \sigma^2)$

- Probability density function:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(- \frac{(x-\mu)^2}{2\sigma^2}\right)$$

- Cumulative distribution function: $F(x) = \Pr(X \leq x) =$
- Probability-generating function:
- Moment-generating function:
- Characteristic function: $\phi(t) = \exp(i\mu t - (1/2)\sigma^2 t^2)$. Standard normal: $\phi(t) = \exp((-1/2)t^2)$.
- Expectation: $\mathbb{E}(X) = \mu$
- Variance: $\mathrm{Var}(X) = \sigma^2$

Gamma: $\Gamma(\alpha, \beta)$ (**note: this parameterization is a little unusual; more commonly β is expressed as the reciprocal of how it appears here.**)

- Probability density function:

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} = \frac{1}{\Gamma(\alpha, \beta)} x^{\alpha-1} e^{-x/\beta}$$

- Cumulative distribution function: $F(x) = \Pr(X \leq x) =$
- Probability-generating function:
- Moment-generating function:
- Characteristic function:
- Expectation: $\mathbb{E}(X) = \alpha\beta$
- Variance: $\text{Var}(X) = \alpha\beta^2$

χ^2_n: special case of a gamma distribution: $\Gamma(n/2, 2)$. Also the sum of n independent standard normally distributed variables.

- Probability density function:

$$f(x) = \frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} e^{-x/2} = \frac{1}{\Gamma(n/2, 2)} x^{n/2-1} e^{-x/2}$$

- Cumulative distribution function: $F(x) = \Pr(X \leq x) =$
- Probability-generating function:
- Moment-generating function:
- Characteristic function:
- Expectation: $\mathbb{E}(X) = n/2 \cdot 2 = n$
- Variance: $\text{Var}(X) = n/2 \cdot 2^2 = 2n$

Exponential: (special case of a gamma distribution: $\Gamma(1, \beta)$. Also a special case of a Weibull distribution with $\beta = 1$.)

- Probability density function: $f(x) = \frac{1}{\beta} \exp(-x/\beta) = \lambda e^{-\lambda x}$
- Cumulative distribution function: $F(x) = \Pr(X \leq x) = 1 - e^{-\lambda x}$
- Probability-generating function:
- Moment-generating function:
- Characteristic function:
- Expectation: $\mathbb{E}(X) = \beta = \lambda^{-1}$
- Variance: $\text{Var}(X) = \beta^2 = \lambda^{-2}$

Cauchy:

- Probability density function:

$$f(x) = \frac{1}{\pi(1 + x^2)} \text{ (standard Cauchy) }, f(x) = \frac{1}{\pi\sigma(1 + (x - \mu)^2/\sigma^2)} \text{ (general)}$$

- Cumulative distribution function: $F(x) = \Pr(X \leq x) =$
- Probability-generating function:
- Moment-generating function:
- Characteristic function:
- Expectation: does not exist
- Variance: does not exist (Cauchy distribution has no moments.)

Beta: Recall:

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$$

$$\implies \frac{B(\alpha + 1, \beta)}{B(\alpha, \beta)} = \frac{\Gamma(\alpha + 1)\Gamma(\beta)}{\Gamma(\alpha + 1 + \beta)} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} = \frac{\alpha}{\alpha + \beta}$$

- Probability density function: $f(x) =$
- Cumulative distribution function: $F(x) = \Pr(X \leq x) =$
- Probability-generating function:
- Moment-generating function:
- Characteristic function:
- Expectation: $\mathbb{E}(X) =$
- Variance: $\mathrm{Var}(X) =$

t_n:

- Probability density function:

$$f(x) = \frac{\Gamma((n+1)/2)}{\sqrt{n\pi} \cdot \Gamma(n/2)} \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2}$$

- Cumulative distribution function: $F(x) = \Pr(X \leq x) =$
- Probability-generating function:
- Moment-generating function:
- Characteristic function:
- Expectation: $\mathbb{E}(X) = 0$
- Variance: $\mathrm{Var}(X) = n/(n-2)$

Weibull:

- Probability density function: $f(x) = \alpha\beta x^{\beta-1}\exp(-\alpha x^{\beta})$
- Cumulative distribution function: $F(x) = \Pr(X \leq x) = 1 - \exp(-\alpha x^{\beta})$
- Probability-generating function:
- Moment-generating function:
- Characteristic function:
- Expectation: $\mathbb{E}(X) =$
- Variance: $\mathrm{Var}(X) =$

1.3.6 Multivariate Gaussian (Normal) Distributions

Definition 1.23. From http://pluto.huji.ac.il/~pchiga/teaching/MathStat/SIAnotes2013.pdf (definition 2b6): A random vector $X = (X_1, X_2)$ is Gaussian with mean $\mu = (\mu_1, \mu_2)$ and the covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

if it has a joint pdf of the form

$$f_X(x) = \frac{1}{2\pi\sigma_2\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2}\frac{1}{1-\rho^2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2)} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_1} + \frac{(x_2-\mu_2)^2}{\sigma_2^2)} \right)\right]$$

for $x \in \mathbb{R}^2$.

Proposition 32. From http://pluto.huji.ac.il/~pchiga/teaching/MathStat/SIAnotes2013.pdf (Proposition 3c1): Let X be a Gaussian random variable in $\mathbb{R}^2$ as in Definition 2b6. Then $f_{X_1|X_2}(x_1; x_2)$ is Gaussian with the (conditional) mean

$$\mathbb{E}(X_1 \mid X_2 = x_2) = \mu_1 + \frac{\rho\sigma_1}{\sigma_2}(x_2 - \mu_2)$$

and the (conditional) variance

$$\mathrm{Var}(X_1 \mid X_2 = x_2) = \sigma_1^2(1-\rho^2)$$

Recall Theorem 24: if two variables are bivariate normal, they are independent if and only if their covariance

$$\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy$$

equals 0.

Theorem 33. For a bivariate normal distribution

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \right)$$

the conditional distribution of X_1 given X_2 is

$$X_1 \mid X_2 = x_2 \sim \mathcal{N}\left(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(x_2 - \mu_2), (1-\rho^2)\sigma_1^2 \right)$$

Remark. Note that this matches the OLS coefficients in the univariate case. In other words, the univariate OLS formula can be derived using only this fact.

1.4 Worked problems

1.4.1 Example Problems That Will Likely Appear on Midterm

(1) **Question:** Fix $p \in (0,1)$ and consider independent Poisson random variables $X_k, k \geq 1$ with

$$\mathbb{E}X_k = \frac{p^k}{k}$$

Verify that the sum $\sum_{k=1}^{\infty} kX_k$ converges with probability one and determine the distribution of the random variable $Y = \sum_{k=1}^{\infty} kX_k$.

Solution. Melike's solution (use for midterm): We have $\mathbb{E}[kX_k] = p^k$ and $\sum_{k=1}^{\infty} p^k = p/(1-p) < \infty$, and $\text{Var}(kX_k) = kp^k$ and

$$\sum_{k=1}^{\infty} kp^k = p\sum_{k=1}^{\infty} kp^{k-1} = p\frac{\mathrm{d}}{\mathrm{d}p}\sum_{k=1}^{\infty} p^k = p\frac{\mathrm{d}}{\mathrm{d}p}\frac{p}{1-p} = p \cdot \frac{(1-p) - p(-1)}{(1-p)^2} = \frac{p}{(1-p)^2} < \infty$$

Since the sequence $\{Y_k|\}_{k\geq 1}$ is independent, by Kolmogorov's Two Series Theorem (Theorem **??**: "Let $X_1, X_2, \ldots$ be independent random variables with $\mathbb{E}(X_n) = \mu_n$ and $\text{Var}(X_n) = \sigma_n^2$ such that $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$. Then $\sum_{n=1}^{\infty} X_n$ converges in $\mathbb{R}$ almost surely."), we conclude that $\sum_{k=1}^{\infty} kX_k$ converges almost surely.

To find the distribution of Y, let X be a Poisson random variable and consider its probability generating function:

$$G_X(s) = \mathbb{E}(s^X) = \sum_{k=0}^{\infty} s^k e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda s)^k}{k!} = e^{-\lambda}e^{\lambda s} = e^{\lambda(s-1)}$$

So $\mathbb{E}(s^{X_k}) = \exp\left(\frac{p^k}{k}(s-1)\right)$ and $\mathbb{E}(s^{kX_k}) = \mathbb{E}[(s^k)^{X_k}] = \exp\left(\frac{p^k}{k}(s^k-1)\right)$. Then define $Y_n = \sum_{k=1}^{n} kX_k$ and consider

$$G_{Y_n}(s) = \mathbb{E}(s^{Y_n}) = \mathbb{E}\left(\prod_{k=1}^{n} s^{kX_k}\right) = \prod_{k=1}^{n}\mathbb{E}\left(s^{kX_k}\right) = \prod_{k=1}^{n}\exp\left(\frac{p^k}{k}(s^k-1)\right) = \exp\left(\sum_{k=1}^{n}\frac{p^k}{k}(s^k-1)\right)$$

$$= \exp\left(\sum_{k=1}^{n} \frac{(ps)^k}{k} - \sum_{k=1}^{n} \frac{p^k}{k}\right)$$

Now, by taking limits as $n \to \infty$ (since we are allowed to take limit inside of expectation here), we get

$$G_Y(s) = \mathbb{E}(s^Y) = \exp\left(\sum_{k=1}^{\infty} \frac{(ps)^k}{k} - \sum_{k=1}^{\infty} \frac{p^k}{k}\right) = \exp\left(\int \sum_{k=1}^{\infty} (ps)^{k-1} dp - \int \sum_{k=1}^{\infty} p^{k-1} dp\right)$$

$$= \exp\left(\int \frac{1}{1-ps} dp - \int \frac{1}{1-p} dp\right) = \exp(-\log(1-ps) + \log(1-p)), \quad -1 \leq ps < 1 \text{ and } -1 \leq p < 1$$

$$= \frac{1-p}{1-ps}, \quad -1 \leq ps < 1$$

Since we know $\Pr(X = k) = \frac{G_X^{(k)}(0)}{k!}$, we have

$$G_Y(s) = \frac{1-p}{1-sp},\ G'(s) = \frac{p(1-p)}{(1-sp)^2},\ G''(s) = \frac{2p^2(1-p)}{(1-sp)^3},\ G^{(3)}(s) = \frac{3 \cdot 2p^3(1-p)}{(1-sp)^3}, \ldots$$

$$G^{(k)}(s) = \frac{k! p^k (1-p)}{(1-sp)^k} \text{ for } k = 0, 1, 2, \ldots$$

So we have

$$\Pr(Y = k) = (1-p)p^k, \quad k = 0, 1, 2, \ldots$$

which means $Y \sim G_1(p) - 1$.

(2) **Note: we worked through an example problem like this on Friday. Should probably fix solution, and use geometric**

Question:

Let X, Y, Z be independent uniform on $(0, 1)$. Compute the cdfs of XY, X/Y, and XY/Z.

Solution (may not be the way Lototsky suggested, consider revising).

Using the information from part (a), and the fact that $f_X(x) = 1$ (for $x \in [0, 1]$) and likewise for $f_Y(y)$:

- XY:

$$F_{XY}(z) = \int_0^{\infty} f_X(x) \int_{-\infty}^{z/x} f_Y(y) dy dx - \int_{-\infty}^{0} f_X(x) \int_{\infty}^{z/x} f_Y(y) dy dx$$

$$= \int_0^1 \left[(z/x)\mathbf{1}_{\{0 < z/x \leq 1\}} + \mathbf{1}_{\{z/x > 1\}}\right] dx = \int_0^1 \left[(z/x)\mathbf{1}_{\{z \leq x\}} + \mathbf{1}_{\{z > x\}}\right] dx = \int_0^z dx + \int_z^1 (z/x) dx$$

$$= z + z\log(x)\Big|_z^1 = z + z\log(1) - z\log(z) = z(1 - \log(z))$$

$$\implies \boxed{F_{XY}(z) = \begin{cases} 0 & z \leq 0 \\ z(1 - \log(z)) & 0 < z \leq 1 \\ 1 & z > 1 \end{cases}}$$

- X/Y:

$$F_{X/Y}(z) = \int_0^\infty f_Y(y) \int_{-\infty}^{zy} f_X(x) dx dy - \int_{-\infty}^0 f_Y(y) \int_\infty^{zy} f_X(x) dx dy$$

$$= \int_0^1 \left[zy\mathbf{1}_{\{0<zy\le 1\}} + \mathbf{1}_{\{zy>1\}}\right] dy = \int_0^1 \left[zy\mathbf{1}_{\{y>0 \cap y\le 1/z\}} + \mathbf{1}_{\{y>1/z\}}\right] dy = \int_0^{1/z} zy \cdot dy + \int_{1/z}^1 dy$$

$$= \frac{zy^2}{2}\bigg|_0^{1/z} + (1 - 1/z) = \frac{z}{2z^2} + 1 - \frac{2}{2z} = 1 - \frac{1}{2z}$$

$$\implies F_{XY}(z) = \begin{cases} 0 & z \le 0 \\ 1 - \frac{1}{2z} & 0 < 1/z \le 1 \end{cases}$$

$$= \boxed{\begin{cases} 0 & z \le 0 \\ z/2 & 0 < z \le 1 \\ 1 - \frac{1}{2z} & z > 1 \end{cases}}$$

- XY/Z: Consider this the cdf of the quotient of $W = XY$ and Z.

$$F_U(u) = \int_0^\infty f_Z(z) \int_{-\infty}^{uz} f_W(w) dw dz - \int_{-\infty}^0 f_Z(z) \int_\infty^{uz} f_W(w) dw dz$$

$$= \int_0^1 \int_0^{uz} -\log(w)\mathbf{1}_{\{0<uz\le 1\}} dw dz = \int_0^1 -\left[w\log(w) - w\right]_0^{uz} \mathbf{1}_{\{0<z\le 1/u\}} dz$$

$$= \int_0^{1/u} uz\left[1 - \log(uz)\right] dz = \frac{u}{4} z^2\big(3 - 2\log(uz)\big)\bigg|_0^{1/u} = \frac{u}{4u^2}\big(3 - 2\log(1)\big) - 0 = \frac{3}{4u}$$

$$\implies \boxed{F_{XY/Z}(u) = \begin{cases} 0 & u \le 0 \\ \frac{3}{4u} & 0 < u \le 3/4 \\ 1 & u > 3/4 \end{cases}}$$

(3) **Note: we worked through an example problem like this on Friday.**

Question from Friday: Let X, Y be distributed exponentially with mean 1. What is the probability distribution of $X/(X+Y)$?

Solution. Find the cdf:

$$\Pr\left(\frac{X}{X+Y} < t\right) = \Pr(X < tX + tY) = \Pr\left(Y > \frac{X(1-t)}{t}\right)$$

To find $\Pr(Y > aX)$, graph the line aX