Math Review Notes—Miscellaneous

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Last updated March 20, 2019

1 Miscellaneous

1.1 Set Theory

Proposition 1 (Math 541A Homework Problem). Let A, B, Ω be sets. Let $u : \Omega \to A$ and let $t : \Omega \to B$. Assume that, for every $x, y \in \Omega$, if $u(x) = u(y)$, then $t(x) = t(y)$. Show that there exists a function $s : A \to B$ such that

$$t = s(u).$$

Proof. Let $B' \subseteq B$ be the image of Ω under t (so that t is surjective onto B'). Fix $x \in \Omega$ and let $y \in \Omega$ range over $\{\Omega \setminus x\}$. Let $Y \subseteq \Omega$ be the set of values such that $t(y) = t(x)$ for all $y \in Y$. Then let s map $u(y) \in A$ to $t(y) = t(x) \in B$ for every $y \in Y$.

(Note that if x is not the only value in Ω that t maps to $t(x)$, then Y contains elements other than x; otherwise, $Y = \{x\}$. In either case this mapping is fine. Note that $u(y_1)$ does not necessarily equal $u(y_2)$ for every $y_1 \neq y_2 \in Y$, but again this does not pose any difficulties for the mapping.)

Since this argument is true for every $x \in \Omega$, we can argue by contradiction that s is surjective onto B'. Let $A' \subseteq A$ be the image of Ω under u (so that u is surjective onto A'). Suppose there is some $b \in B'$ such that there is no unused $a \in A'$ to correspond to it. That is, there are some $y, z \in \Omega$ such that $t(z) \neq t(y)$ but $u(z) = u(y)$. In that case the mapping s would map $u(z)$ and $u(y)$ both to the same value in B', so one of the values $t(z)$ or $t(y)$ would necessarily be missed. But by assumption there is no $z \in \Omega$ such that $t(z) \neq t(y)$ but $u(z) = u(y)$. (Note the contrapositive of the assumption: "for every $x, y \in \Omega$, if $t(x) \neq t(y)$, then $u(x) \neq u(y)$.")

□

Remark. By showing this mapping exists, we have shown that the cardinality of B' is less than or equal to the cardinality of A'.

1.2 Other

6. Which of the following circles has the greatest number of points of intersection with the parabola $x^2 = y + 4$?

(A) $x^2 + y^2 = 1$

(B) $x^2 + y^2 = 2$

(C) $x^2 + y^2 = 9$

(D) $x^2 + y^2 = 16$

(E) $x^2 + y^2 = 25$

Solution 6. (C) We can try to do this algebraically, but non-algebraically is simpler. Graphing $y = x^2 - 4$ shows that the graph crosses the x-axis at ± 2. Therefore a circle of radius 1 or $\sqrt{2}$ will not intersect the parabola at all. A circle of radius 3 will intersect four times – twice above and twice below the x-axis. A circle of radius 4 will only intersect at one point below the x-axis (and twice above), and a circle of radius 5 will only intersect at the two points above.

19. If z is a complex variable and $\overline{z}$ denotes the complex conjugate of z, what is $\lim\limits_{z \to 0} \frac{(\overline{z})^2}{z^2}$?

(A) 0 (B) 1 (C) i (D) ∞ (E) The limit does not exist.

Solution 19. (E) Let us represent $z = a + bi$. Then our limit becomes

$$\lim_{(a,b)\to 0} \frac{(a-bi)^2}{(a+bi)^2} = \lim_{(a,b)\to 0} \frac{a^2 - b^2 - 2abi}{a^2 - b^2 + 2abi}.$$

If we let $a = 0$ (for instance), it is easy to see that the limit is equal to 1. However, if we let $a = b$, then our limit becomes

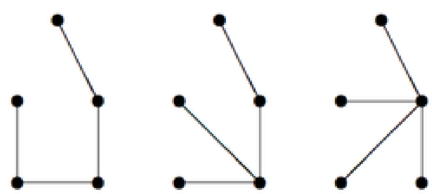
$$\lim_{a\to 0} \frac{-2a^2 i}{2a^2 i} = -1.$$

Therefore the limit does not exist.

29. A tree is a connected graph with no cycles. How many nonisomorphic trees with 5 vertices exist?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution 29. (C) It's probably easiest to draw this out for yourself. The maximum degree of any vertex is 2, 3, or 4. If there is a vertex of degree 4, then our tree looks like a star. If the maximum degree of any vertex is 2, then we have a straight line. In the middle case, we obtain a 3-pointed star to which we attach one more vertex – the choice of branch yields isomorphic graphs. See Figure 1.



38. The maximum number of acute angles in a convex 10-gon in the Euclidean plane is

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution 38. (C) The total angle measure of a 10-gon is $180 \cdot 8 = 1440°$. If the polygon is to be convex, all angles must be less than $180°$. If we have 5 acute angles, then the remaining 5 angles would have to make up for $> 1440 - 5 \cdot 90 = 990$ degrees. This is impossible to do and remain convex. If we have 4 acute angles, the remaining 6 angles need to make up for $> 1440 - 4 \cdot 90 = 1080$ degrees. This is our edge case, so the answer must be 3 acute angles.

45. How many positive numbers x satisfy the equation $\cos(97x) = x$?

(A) 1 (B) 15 (C) 31 (D) 49 (E) 96

Solution 45. (C) Certainly our solutions are concentrated in $[0, 1]$. We know that every $2\pi/97$ units in x, we get another period of $\cos(97x)$, and each period must meet $y = x$ twice. Therefore there are

$$\frac{1}{2\pi/97} = \frac{97}{2\pi} \approx \frac{97}{6.3} \approx 15$$

periods in $[0, 1]$ and about 30 meetings. There's only one answer in that range, so we'll stick with it.