Math Review Notes—Asymptotics and Convergence

Gregory Faletto

Contents

Last updated February 18, 2019

1 Asymptotics and Convergence

These notes are based on my notes from chapter 8 of *Time Series and Panel Data Econometrics* (1st edition) by M. Hashem Pesaran and coursework for Economics 613: Economic and Financial Time Series I at USC, as well as Math 505A and Math 541A at USC and chapter 7 from *Probability and Random Processes* (Grimmett and Stirzaker) 3rd edition.

1.1 Preliminaries (5.9 and 7.1, Grimmett and Stirzaker)

Definition 1.1. Definition 7.1.4, Grimmett and Stirzaker. If for all $x \in [0,1]$ the sequence $\{f_n(x)\}$ of real numbers satisfies $f_n(x) \to f(x)$ as $n \to \infty$ then we say $f_n \to f$ **pointwise.**

Remark. In practice pointwise convergence is often not useful for functions because a sequence of functions may be continuous while its limit is not. For instance, consider $\{f_n : f_n = x^n \ \forall x \in [0,1]\}$. Then f_n is continuous for all n but

$$\lim_{n\to\infty} f_n = \begin{cases} 0 & x \leq 1 \\ 1 & x = 1 \end{cases}$$

Instead, the following definition is often more useful.

Definition 1.2. (from class notes.) We say that f_n **uniformly converges to** f **on** $[a,b]$ if for every $\epsilon > 0$ there exists N such that for every $n > N$,

$$\forall x \in [a,b] \ |f_n(x) - f(x)| < \epsilon$$

Definition 1.3. (Definition 7.1.5, Grimmett and Stirzaker.) Let V be a collection of functions mapping $[0,1]$ into $\mathbb{R}$ and assume V is endowed with a function $\|\cdot\| : V \to \mathbb{R}$ satisfying

(a) $\|f\| \geq 0$ for all $f \in V$

(b) $\|f\| = 0$ if and only if f is the zero function (or equivalent to it)

(c) $\|af\| = |a| \cdot \|f\|$ for all $a \in \mathbb{R}$, $f \in V$

(d) $\|f + g\| \leq \|f\| + \|g\|$ (Triangle Inequality)

The function $\|\cdot\|$ is called a **norm**. If $\{f_n\}$ is a sequence of members of V then we say that $f_n \to f$ **with respect to the norm** $\|\cdot\|$ if $\|f_n - f\| \to 0$ as $n \to \infty$.

Definition 1.4. (Definition 7.16, Grimmett and Stirzaker.) Let $\epsilon > 0$ be prescribed, and define the distance between two functions $g, h : [0,1] \to \mathbb{R}$ by

$$d_\epsilon(g,h) = \int_E dx$$

where $E = \{u \in [0,1] : |g(u) - h(u)| > \epsilon\}$. We say that $f_n \to f$ **in measure** if

$$d_\epsilon(f_n, f) \to 0 \text{ as } n \to \infty \text{ for all } \epsilon > 0$$

Theorem 1. Inversion Theorem (Theorem 5.9.2, Grimmett and Stirzaker). Let X have distribution function F and characteristic function ϕ. Define $\overline{F} : \mathbb{R} \to [0, 1]$ by

$$\overline{F}(x) = \frac{1}{2}\big[F(x) + \lim_{y \to x^-} F(y)\big]$$

Then

$$\overline{F}(b) - \overline{F}(a) = \lim_{N \to \infty} \int_{-N}^{N} \frac{\exp(-iat) - \exp(-ibt)}{2\pi i t} \cdot \phi(t) dt$$

Proof. See Kingman and Taylor (1966). □

Corollary 1.1. Corollary 5.9.3. Random variables X and Y have the same characteristic function if and only if they have the same distribution function.

Proof. **Available in Grimmett and Stirzaker section 5.9, pp. 189 - 190.** □

Definition 1.5. (Definition 5.9.4, Grimmett and Stirzaker.) We say that the sequence $F_1, F_2, \ldots$ of distribution functions **converges** to the distribution function F (written $F_n \to F$) if $F(x) = \lim_{n\to\infty} F_n(x)$ at each point x where F is continuous.

Theorem 2. Continuity theorem (Thereom 5.9.5; in notes from Friday 10/26, Lecture 28). Supose that $F_1, F_2, \ldots$ is a sequence of distribution functions with corresponding characteristic functions $\phi_1, \phi_2, \ldots$.

(a) If $F_n(x) \to F(x)$ for some distribution function F with characteristic function ϕ (at x where F is continuous), then $\phi_n(t) \to \phi(t)$ for all t.

(b) Conversely, if $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ exists and $\phi(t)$ is continuous at $t = 0$, then ϕ is the characteristic function of some distribution function F, and $F_n \to F$.

Proof. See Kingman and Taylor (1966). □

1.2 Inequalities (8.6 of Pesaran)

Inequalities

- Probabilities
  -
    Lemma 3. Markov's Inequality (Grimmett and Stirzaker p. 311, 319)): Let $X : \Omega \to [-\infty, \infty]$ be a random variable. Then for all $a > 0$,

    $$\Pr(|X| \geq a) \leq \frac{\mathbb{E}(|X|)}{a}$$

Proof. Note $t \cdot \mathbf{1}_{\{|X| \geq t\}} \leq |X|$, where $\mathbf{1}$ is the indicator function. Dividing both sides by t and taking expectations, we have

$$\mathbb{E}(\mathbf{1}_{\{|X|\geq t\}}) \leq \frac{\mathbb{E}|X|}{t} \iff \Pr(|X|) \geq t) \leq \frac{\mathbb{E}|X|}{t}, \qquad \forall\, t > 0.$$

$\square$

Corollary 3.1. If n is a positive integer, then

$$\Pr(|X| \geq t) \leq \frac{\mathbb{E}(|X|^n)}{t^n} \quad \forall t > 0$$

Proof. By Markov's Inequality (Theorem 3),

$$\Pr(|X| \geq t) = \Pr(|X|^n \geq t^n) \leq \frac{\mathbb{E}(|X|^n)}{t^n}$$

$\square$

–

Theorem 4. Chebyshev's Inequality: (probability p. 319) Let $X : \Omega \to [-\infty, \infty]$ be an (integrable) random variable with $\mathbb{E}(X^2) < \infty$.Then for any real number $k > 0$

$$\Pr\left(|X - \mathbb{E}(X)| \geq k\sqrt{\text{Var}(X)}\right) \leq \frac{1}{k^2}$$

This can also be written as

$$\Pr(|X - \mathbb{E}(X)| \geq k) \leq \frac{\text{Var}(X)}{k^2}$$

(Can be used to demonstrate consistency of estimators: if we can show that as $T \to \infty$ $\text{Var}(X) = \sigma^2 \to 0$, then this implies $\Pr\left(|X - \mu| \geq k\sigma\right) \to 0$ as $T \to \infty$, showing consistency.)

–

Theorem 5. Chernoff For $x \geq 0$, $a > 0$, $\forall\ t > 0$,

$$\Pr(X \geq a) = \Pr(e^{tX} \geq e^{ta}) \leq \frac{\mathbb{E}(e^{tX})}{e^{ta}}$$

- **Moments**

–

Theorem 6. Cauchy-Schwarz (and **Bunyakovsky).** If X and Y are random variables with finite variance then

$$\mathbb{E}(XY)^2 \leq \mathbb{E}(X^2)\mathbb{E}(Y^2)$$

Note that this can is a corollary of Theorem 11 with $p = q = 2$. We can also prove this theorem on its own in a different one. We first prove a useful result.

Lemma 7. If $\text{Var}(X) = 0$ then X is almost surely constant; that is, $\Pr(X = a) = 1$ for some $a \in \mathbb{R}$.

Proof. Note that because $\text{Var}(X) = 0 < \infty$, we know that $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$ exist. We have

$$\text{Var}(X) = \mathbb{E}\big[(X - \mathbb{E}(X))^2\big] = 0$$

Let $Y = (X - \mathbb{E}(X))^2$. Note that $Y = (X - \mathbb{E}(X))^2 \geq 0$ and that $\mathbb{E}(Y) = \text{Var}(X) = 0$. Therefore $\Pr(Y = 0) = 1$, so $\Pr(Y \neq 0) = 0$. To see why, in the case that X is discrete,

$$\mathbb{E}(Y) = \sum_{k=0}^{\infty} k \cdot \Pr(Y = k) = \text{Var}(X) = 0$$

which is true if and only if $\Pr(Y = k) = 0$ for all $k > 0$. Since we already showed that $Pr(Y < 0) = 0$, it follows that $\Pr(Y = 0) = 1$. In the continuous case,

$$\mathbb{E}(Y) = \int_0^{\infty} y \cdot f_Y(y) dy = \text{Var}(X) = 0$$

which implies that $f_Y(x) = 0$ for all $x > 0$. Again, since $\Pr(Y < 0) = 0$, we have $\Pr(Y \neq 0) = 0$. But $Y = 0 \iff X = \mathbb{E}(X)$ so we have $\Pr(X = \mathbb{E}(X)) = 1$. □

Remark. Note that Lemma 7 along with Proposition **??** imply that X has variance 0 if and only if it is (almost surely) constant.

We are now ready to prove the Cauchy-Schwarz Inequality.

Proof. if $\mathbb{E}(X^2) = 0$ or $\mathbb{E}(Y^2) = 0$, the Cauchy-Schwarz Inequality follows immediately. To see why, suppose without loss of generality that $\mathbb{E}(X^2) = 0$. Then the right side is 0. Also, $0 \leq \text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = -\mathbb{E}(X)^2$. Since $\mathbb{E}(X)^2 \geq 0$, we must have $E(X)^2 = 0$ and therefore $\text{Var}(X) = 0$. Therefore by Lemma 7, X is almost surely constant, which means that $\text{Cov}(X, Y) = 0$.

In the case that $\mathbb{E}(X^2) > 0$ and $\mathbb{E}(Y^2) > 0$, for $a, b \in \mathbb{R}$, let $Z = aX - bY$. Then

$$0 \leq \mathbb{E}(Z^2) = a^2\mathbb{E}(X^2) - 2ab\mathbb{E}(XY) + b^2\mathbb{E}(Y^2) \tag{1}$$

The right side of (1) is quadratic in a. Because it is greater than or equal to zero, it has at most one real root, which means its discriminant must be non-positive. That is, if $b \neq 0$,

$$(-2b\mathbb{E}(XY))^2 - 4b^2\mathbb{E}(X^2)\mathbb{E}(Y^2) \leq 0 \iff \mathbb{E}(XY)^2 - \mathbb{E}(X^2)\mathbb{E}(Y^2) \leq 0$$

which yields the result. Note that equality holds if and only if $\Pr(aX = bY) = 1$ because the discriminant is zero if and only if the quadratic has a real root, which occurs if and only if

$$\mathbb{E}\big[(aX - bY)^2\big] = 0$$

which is true if and only if $\Pr(aX = bY) = 1$ by by Lemma 7 and Proposition **??**. □

- **Krylov**
-

Definition 1.6. Let $\phi : \mathbb{R} \to \mathbb{R}$. We say that ϕ is **convex** if for any $x, y \in \mathbb{R}$ and for any $t \in [0, 1]$, we have

$$\phi(tx + (1 - t)y) \leq t\phi(x) + (1 - t)\phi(y)$$

Theorem 8 (Jensen's Inequality, from Math 541A. Also Grimmett and Stirzaker p.181, 349). Let $X : \Omega \to [-\infty, \infty]$ be a random variable. Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex. If $\mathbb{E}|X| < \infty$ and $\mathbb{E}|\phi(X)| < \infty$, then

$$\phi(\mathbb{E}X) \leq \mathbb{E}\phi(X).$$

(See also Theorem **??**.)

For the definition of convexity, see Definition **??**. To prove this result, we will first prove some other useful results.

Lemma 9 (Result from Math 541A Homework 2)**.** The slope of a convex function is nondecreasing. More formally, let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function. For any $x \in \mathbb{R}$, let

$$M_R := \{\frac{\phi(c) - \phi(x)}{c - x} : c > x\}, \quad M_L := \{\frac{\phi(x) - \phi(b)}{x - b} : b > x\}$$

be the slopes of the secant lines through ϕ using points to the right and left of x, respectively. Then for any $m \in M_R$, $p \in M_L$ we have $m \geq p$. (See also Lemma **??**.)

Proof. Fix $x \in \mathbb{R}$. Let $m \in M_R, p \in M_L$. By definition, there exist $b < x < c$ such that

$$m = \frac{\phi(c) - \phi(x)}{c - x}, \quad p = \frac{\phi(x) - \phi(b)}{x - b}.$$

Let $t \in (0, 1)$ such that

$$tb + (1 - t)c = x. \tag{2}$$

Then we have

$$m \geq p \iff \frac{\phi(c) - \phi(x)}{c - x} \geq \frac{\phi(x) - \phi(b)}{x - b} \iff (x - b)(\phi(c) - \phi(x)) \geq (c - x)(\phi(x) - \phi(b))$$

$$\iff (x - b)\phi(c) + b\phi(x) \geq c\phi(x) - (c - x)\phi(b)$$

From (2), we have $x - b = tb + (1 - t)c - b = (t - 1)b + (1 - t)c = (1 - t)(c - b)$ and $t(b - c) = x - c \iff t(c - b) = c - x$. Therefore

$$(x - b)\phi(c) + b\phi(x) \geq c\phi(x) - (c - x)\phi(b) \iff (1 - t)(c - b)\phi(c) + b\phi(x) \geq c\phi(x) - t(c - b)\phi(b)$$

$$\iff (1 - t)(c - b)\phi(c) \geq (c - b)\phi(x) - t(c - b)\phi(b) \iff (1 - t)\phi(c) + t\phi(b) \geq \phi(x)$$

But $t\phi(b) + (1 - t)\phi(c) \geq \phi(x)$ since ϕ is convex. Therefore $m \geq p$.

□

Theorem 10 (Result from 541A Homework 2). Let $\phi : \mathbb{R} \to \mathbb{R}$. Then ϕ is convex if and only if: for any $y \in \mathbb{R}$, there exists a constant a and there exists a function $L : \mathbb{R} \to \mathbb{R}$ defined by $L(x) = a(x - y) + \phi(y)$, $x \in \mathbb{R}$, such that $L(y) = \phi(y)$ and such that $L(x) \leq \phi(x)$ for all $x \in \mathbb{R}$. (In the case that ϕ is differentiable, the latter condition says that ϕ lies above all of its tangent lines. See also Theorem **??**.)

Proof. $\Longrightarrow$: As in Lemma 9, let

$$M_R := \{\frac{\phi(c)-\phi(y)}{c-y} : c > y\}, \quad M_L := \{\frac{\phi(y)-\phi(b)}{y-b} : b > y\}$$

be the slopes of the secant lines through ϕ using points to the right and left of y, respectively. Then by Lemma 9, for any $m \in M_R$, $p \in M_L$ we have $m \geq p$, so we can choose some $a_0 \in \mathbb{R}$ such that $p \leq a_0 \leq m$ for all $p \in M_L, m \in M_R$. Then let $L : \mathbb{R} \to \mathbb{R}$ be defined by $L(x) = a_0(x-y) + \phi(y)$, $x \in \mathbb{R}$. Note that $L(y) = \phi(y)$.

We argue that $L(x) \leq \phi(x)$ for all $x \in \mathbb{R}$ by contradiction. Suppose there is some $z \in \mathbb{R}$ with $L(z) > \phi(z)$. Note that $z \neq y$ because we have already shown that $L(y) = \phi(y)$. Then we have

$$L(z) > \phi(z) \iff a_0(z-y) + \phi(y) > \phi(z) \tag{3}$$

If $z > y$, then we can solve (3) for a_0 as follows:

$$a_0 > \frac{\phi(z)-\phi(y)}{z-y}$$

But $z \in M_R$, so we have $\frac{\phi(z)-\phi(y)}{z-y} > a_0$. Contradiction. If $z < y$, we solve (3) for a_0 as follows:

$$a_0 < \frac{\phi(z)-\phi(y)}{z-y} = \frac{\phi(y)-\phi(z)}{y-z}$$

But $z \in M_L$, so we have $\frac{\phi(y)-\phi(z)}{y-z} < a_0$. Contradiction. Therefore for every $z \in \mathbb{R}$ we have $L(z) \leq \phi(z)$ as desired.

$\Longleftarrow$: Now suppose that for any $y \in \mathbb{R}$ there exists a constant a and a function $L : \mathbb{R} \to \mathbb{R}$ defined by $L(x) = a(x-y) + \phi(y)$, $x \in \mathbb{R}$, such that $L(y) = \phi(y)$ and such that $L(x) \leq \phi(x)$ for all $x \in \mathbb{R}$.

Fix $b, c \in \mathbb{R}$ and let $t \in (0,1)$. Set $y := tb + (1-t)c$. Then by assumption we can write

$$a(b-y) + \phi(y) \leq \phi(b), \quad a(c-y) + \phi(y) \leq \phi(c)$$

Multiply by $t > 0$ and $(1-t) > 0$ respectively to yield

$$ta(b-y) + t\phi(y) \leq t\phi(b), \quad (1-t)a(c-y) + (1-t)\phi(y) \leq (1-t)\phi(c) \tag{4}$$

Note that

$$ta(b-y) + (1-t)a(c-y) = a(tb - ty + (c - y - ct + yt)) = a(tb + c - y - ct)$$

$$= a(tb + c(1-t) - tb - (1-t)c) = 0$$

So adding the inequalities in (4) yields

$$t\phi(y) + (1-t)\phi(y) \leq t\phi(b) + (1-t)\phi(c) \iff \phi(y) \leq t\phi(b) + (1-t)\phi(c)$$

$$\iff \phi(tb + (1-t)c) \leq t\phi(b) + (1-t)\phi(c).$$

$\square$

We are now ready to prove Jensen's Inequality.

Proof. Note that from Theorem 10, for any $y \in \mathbb{R}$ there exists a constant a and a function L such that

$$a(x-y)+\phi(y) \leq \phi(x) \quad \forall x \in \mathbb{R}$$

Letting $y = \mathbb{E}(X)$ we have

$$a(X-\mathbb{E}X)+\phi(\mathbb{E}X) \leq \phi(X)$$

Since expectations preserve inequalities,

$$\mathbb{E}[a(X-\mathbb{E}X)+\phi(\mathbb{E}X)] \leq \mathbb{E}\phi(X)$$

But

$$\mathbb{E}[a(X-\mathbb{E}X)+\phi(\mathbb{E}X)] = a(\mathbb{E}X-\mathbb{E}X)+\mathbb{E}(\phi(\mathbb{E}X)) = \phi(\mathbb{E}X)$$

which yields

$$\phi(\mathbb{E}X) \leq \mathbb{E}\phi(X).$$

□

Corollary 10.1 (**Jensen's Inequality: EE 588 Formulation**). f is convex if and only if

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}$$

for all $a, b \in \mathbf{dom}(f)$. (See also Corollary 10.1.)

Proof. Follows from Theorem 8 if X is a discrete random variable that equals a or b each with probability $1/2$ and $\phi(X) = f(X)$. Note that $\phi(X)$ is convex.

□

Corollary 10.2. Triangle Inequality. Let $X : \Omega \to [-\infty, \infty]$ be a random variable. If $\mathbb{E}|X| < \infty$, then $|\mathbb{E}(X)| \leq \mathbb{E}(|X|)$.

Proof. Note that $u(x) = |x|$ is convex by the definition of convexity: for any $x, y \in \mathbb{R}$ and for any $t \in (0,1)$, we have

$$u(tx+(1-t)y) = |tx+(1-t)y| \leq \ldots = t|x|+(1-t)|y| = tu(x)+(1-t)u(y).$$

Then the result follows immediately from Jensen's Inequality using $u(X) = X^2$:

$$|\mathbb{E}X| \leq \mathbb{E}|X|$$

□

Corollary 10.3. If $\mathbb{E}(X^2) < \infty$, then $\mathbb{E}(|X|) < \infty$, so $\mathbb{E}(X) \in \mathbb{R}$.

Proof. Follows immediately from Jensen's Inequality using $u(X) = X^2$. □

Corollary 10.4. Let $n \in \mathbb{N}$. If $\mathbb{E}(|X|^n) < \infty$, then $\mathbb{E}(|X|^{n-1}) < \infty$.

–

Theorem 11. Hölder (Grimmett and Stirzaker p. p. 143, 319) Generalization of Cauchy-Schwarz. For $p, q > 1$ satisfying $1/p + 1/q = 1$ we have

$$\mathbb{E}(|XY|) \leq (\mathbb{E}(|X^p|))^{1/p}(\mathbb{E}(|X^q|))^{1/q}$$

Proof. Assume without loss of generality that $\|X\|_p = \|Y\|_q = 1$. Also, the case $p = 1, q = \infty$ follows from the triangle inequality, so we assume $1 < p < \infty$. From concavity of the log function, we have

$$\log\left((x^p)^{1/p}(y^q)^{1/q}\right) = (1/p)\log\left(x^p\right) + (1/q)\log\left(y^q\right)$$

$$\leq \log\left(\frac{1}{p}x^p + \frac{1}{q}y^q\right)$$

$$\implies (x^p)^{1/p}(y^q)^{1/q} \leq \frac{1}{p}x^p + \frac{1}{q}y^q$$

Fixing an $\omega \in \Omega$, we have

$$|X(\omega)Y(\omega)| = (|X(\omega)|^p)^{1/p}(|Y(\omega)|^q)^{1/q} \leq \frac{1}{p}|X(\omega)|^p + \frac{1}{q}|Y(\omega)|^q$$

Integrating we have...

□

–

Theorem 12. Minkowski (Grimmett and Stirzaker p. p. 143) For $p \geq 1$,

$$[\mathbb{E}(|X+Y|^p)]^{1/p} \leq (\mathbb{E}|X^p|)^{1/p} + (\mathbb{E}|Y^p|)^{1/p}$$

– Useful for showing lower order moments are finite (e.g. finite variance implies finite mean).

Lemma 13. Lyapunov's Inequality (Grimmett and Stirzaker p. 143). For $0 < r \leq s < \infty$,

$$\mathbb{E}(|X|^r)^{1/r} \leq \mathbb{E}(|X|^s)^{1/s}$$

–

Theorem 14. Triangle Inequality: Let $X, Y : \Omega \to \mathbb{R}$ be random variables. Let $1 \leq p \leq \infty$. Then

$$\|X + Y\|_p \leq \|X\|_p + \|Y\|_p, 1 \leq p \leq \infty$$

Proof. The case $p = \infty$ follows from the scalar triangle inequality, so assume $1 \leq p < \infty$. By scaling, we may assume $\|X\|_p = 1 - t, \|Y\|_p = t$, for some $t \in (0, 1)$ (zeroes and infinities being trivial). Define $V := X/(1-t), W := Y/t$. Then by convexity of $x \to |x|^p$ on $\mathbb{R}$,

$$|(1-t)V(\omega) + t(W(\omega)|^p \leq (1-t)|V(\omega)|^p + t|W(\omega)|^p$$

Take expectation of both sides:

$$\mathbb{E}|X + Y|^p \leq (a-t)^{1-p}\mathbb{E}(|X|^p) + t^{1-p}\mathbb{E}(|Y|^p)$$

Since $\|X\|_p = t, \|Y\|_p = 1-t$, we have that the right side is $1 - t + t = 1$. (Note: $\|Y\|_p = t, \mathbb{E}|Y|^p = t^p, \|X\|_p = 1-t$ Therefore

$$(\mathbb{E}|X+Y|^p)^{1/p} = \|X+Y\|_p \leq 1$$

□

Remark. See also Theorem **??** and Corollary **??**.

Monotone convergence theorem.

Dominated Convergence Theorem (Theorem **??**).

1.3 Modes of Convergence (7.2 of Grimmett and Stirzaker, 8.2 and 8.4 of Pesaran)

Let $\{X_n\} = \{X_1, X_2, \ldots\}$ and X be random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 1.7. Convergence in probability. $\{X_n\}$ is said to **converge in probability** to X if

- Grimmett and Strizaker definition:

$$\lim_{n\to\infty} \Pr(|X_n - X| > \epsilon) = 0, \text{ for every } \epsilon > 0$$

- Pesaran definition:

$$\lim_{n\to\infty} \Pr(|X_n - X| < \epsilon) = 1, \text{ for every } \epsilon > 0$$

- More formal (from Math 541A):

$$\forall \epsilon > 0, \lim_{n\to\infty} \Pr(\{\omega \in \Omega : |X_n(\omega) - X(\omega)| > \epsilon) = 0$$

Remark. This mode of convergence is also often denoted by $X_n \xrightarrow{p} X$ and when X is a fixed constant it is referred to as the **probability limit of** X_n, written as $Plim(X_n) = x$, as $n \to \infty$.

The above concept is readily extended to multivariate cases where $\{\boldsymbol{X}_n, n = 1, 2, \ldots\}$ denote m-dimensional vectors of random variables. Then the condition is

$$\lim_{n\to\infty} \Pr(\|\boldsymbol{X}_n - \boldsymbol{X}\| < \epsilon) = 1, \text{ for every } \epsilon > 0$$

where $\|\cdot\|$ denotes an appropriate norm (say ℓ_2). Convergence in probability is often referred to as "weak convergence" (in contrast to convergence with probability 1, below).

Definition 1.8. Convergence with probability 1 or almost surely. The sequence of random variables $\{X_n\}$ is said to **converge with probability 1** (or **almost surely**) to X if

- (505A class notes definition)

$$\Pr\left(\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}\right) = 1$$

(Note: pointwise convergence can hardly ever be shown here and is not useful.)

- Grimmett and Stirzaker textbook definition:

$$\Pr\big(\{\omega \in \Omega : X_n(\omega) \to X(\omega) \text{ as } n \to \infty\}\big) = 1$$

- Pesaran textbook definition:

$$\Pr\left(\lim_{n\to\infty} X_n = X\right) = 1$$

Remark. This is often written as $X_n \xrightarrow{w.p.1} X$ or $X_n \xrightarrow{a.s.} X$. An equivalent condition for convergence with probability 1 is given by

$$\lim_{n\to\infty} \Pr(|X_m - X| < \epsilon, \text{ for all } m \geq= n) = 1, \text{ for every } \epsilon > 0$$

which shows that convergence in probability is a special case of convergence with probability 1 (obtained by setting $m = n$). Convergence with probability 1 is stronger than convergence in probability and is often referred to as "strong convergence."

Definition 1.9. Convergence in r-th mean or **convergence in ℓ_p.** $X_n \to X$ **in rth mean** (or **in ℓ_p**) where $r \geq 1$ (or $0 < p \leq \infty$) if $\mathbb{E}|X_n^r| < \infty$ for all n and

$$\lim_{n\to\infty} \mathbb{E}(|X_n - X|^r) = 0$$

or if $\|X\|_p < \infty$ and

$$\lim_{n\to\infty} \|X_n - X\|_p = 0$$

Remark. Recall that $\|X\|_p := (\mathbb{E}(X)^p)^{1/p}$ if $0 < p < \infty$ and $\|X\|_\infty := \inf\{c > 0 : \Pr(|X| leqc) = 1\}$. Note that if $p < 1$, $\|\cdot\|_p$ is no longer a norm because it does not satisfy the Triangle Inequality (Corollary 10.2 and Theorem 14), but this property still holds. Convergence in rth mean is often written $X_n \xrightarrow{r} X$.

Definition 1.10. Convergence in Distribution. Let $X_1, X_2, \ldots$ have distribution functions $F_1(\cdot), F_2(\cdot), \ldots$ respectively. Then X_n is said to **converge in distribution to** X if

$$\lim_{n\to\infty} \Pr(X_n \leq u) = \Pr(X \leq u)$$

for all u at which $F_X(x) = \Pr(X \leq x)$ is continuous. This can also be written

$$\lim_{n\to\infty} F_n(u) = F(u)$$

for all u at which F is continuous.

Remark. Convergence in distribution is usually denoted by $X_n \xrightarrow{d} X$, $X_n \xrightarrow{L} X$, or $F_n \implies F$. By the Continuity Theorem (Theorem 2, section 1.1), this is equivalent to

$$\lim_{n\to\infty} \phi_{X_n}(t) = \phi_X(t), \quad t \in \mathbb{R}.$$

Note that the random variables are allowed to have different domains.

Definition 1.11. (Convergence in distribution for vector-valued random variables.) We say that random variables $Y^{(1)}, \ldots, : \Omega \to \mathbb{R}^d$ **converge in distribution** to $Y : \Omega \to \mathbb{R}^d$ if for all $v \in \mathbb{R}^d$, $\langle v, Y^{(1)} \rangle, \langle v, Y^{(2)} \rangle, \ldots$ converges in distribution to $\langle v, Y \rangle$.

Theorem 15. (Theorem 7.2.3, Grimmett and Stirzaker.) The following implications hold:

- $(X_n \xrightarrow{a.s.} X) \implies (X_n \xrightarrow{p} X)$
- $(X_n \xrightarrow{r} X) \implies (X_n \xrightarrow{p} X)$ for any $r \geq 1$
- $(X_n \xrightarrow{p} X) \implies (X_n \xrightarrow{d} X)$

Also, if $r > s \geq 1$, then $(X_n \xrightarrow{r} X) \implies (X_n \xrightarrow{s} X)$. No other implications hold in general.

Theorem 16. Some exceptions (Theorem 7.2.4).

- If $X_n \xrightarrow{d} c$ where c is constant, then $X_n \xrightarrow{p} c$.
- If $X_n \xrightarrow{p} X$ and $\Pr(|X_n| \leq k) = 0$ for all n and some k, then $X_n \xrightarrow{r} X$ for all $r \geq 1$.
- If $P_n(\epsilon) = \Pr(|X_n - X| > \epsilon)$ satisfies $\sum_n P_n(\epsilon) < \infty$ for all $\epsilon > 0$, then $X_n \xrightarrow{a.s.} X$.

Proof. (Part (c).) Let $A_n(\epsilon) = \{|X_n - X| > \epsilon\}$ (so that $P_n(\epsilon) = \Pr[A_n(\epsilon))]$, and let $B_m(\epsilon) = \bigcup_{n \geq m} A_n(\epsilon)$. Then

$$\Pr(B_m(\epsilon)) \leq \sum_{n=m}^{\infty} \Pr(A_n(\epsilon))$$

so $\lim_{m\to\infty} \Pr(B_m(\epsilon)) = 0$ whenever $\sum_n \Pr(A_n(\epsilon)) < \infty$. See also Lemma 18 part (b). $\square$

1.4 More on convergence (7.2 of Grimmet and Stirzaker)

Other theorems to include: Fatou's Lemma, Fubini's Theorem, Kolmogorov's Maximal Inequality, Kolmogorov Three-Series Test, Lindeberg Feller Central Limit Theorem, **this and more at beginning of Mike's 505A qual solutions.**

Definition 1.12. Cauchy Convergence. We say that the sequence $\{X_n : n \geq 1\}$ of random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is **almost surely Cauchy convergent** if

$$\Pr\left(\{\omega \in \Omega : X_m(\omega) - X_n(\omega) \to 0 \text{ as } m, n \to \infty\}\right) = 1$$

That is, the set of points ω of the sample space for which the real sequence $\{X_n(\omega) : n \geq 1\}$ is Cauchy convergent is an event having probability 1.

Lemma 17. (Lemma 7.2.6 from Grimmett and Stirzaker)

(a) If $r > s \geq 1$ and $X_n \xrightarrow{r} X$, then $X_n \xrightarrow{s} X$.

(b) If $X_n \xrightarrow{1} X$ then $X_n \xrightarrow{p} X$.

The converse assertions fail in general.

Proof. (a) Using Lyapunov's Inequality (Lemma 13), if $r > s \geq 1$

$$\left[\mathbb{E}(|X_n - X|^s)\right]^{1/s} \leq \left[\mathbb{E}(|X_n - X|^r)\right]^{1/r}$$

Therefore if $X_n \xrightarrow{r} X$ (meaning $\lim_{n\to\infty} \mathbb{E}(|X_n - X|^r) = 0$), (then $\lim_{n\to\infty} \mathbb{E}(|X_n - X|^s) = 0$, so $X_n \xrightarrow{s} X$. We show the converse fails by counterexample:

$$X_n = \begin{cases} n & \text{with probability } n^{(-1/2)(r+s)} \\ 0 & \text{with probability } 1 - n^{(-1/2)(r+s)} \end{cases}$$

Then $\mathbb{E}|X_n^s| = n^{(1/2)(s-r)} \to 0$ and $\mathbb{E}|X_n^r| = n^{(1/2)(r-s)} \to \infty$.

(b) By Markov's Inequality (Lemma 3),

$$\Pr(|X_n - X| > \epsilon) \leq \frac{\mathbb{E}|X_n - X|}{\epsilon} \quad \text{for all } \epsilon > 0$$

Therefore if $X_n \xrightarrow{1} X$; that is, $\lim_{n\to\infty} \mathbb{E}(|X_n - X|) = 0$, then $\lim_{n\to\infty} \Pr(|X_n - X| > \epsilon) = 0$ for every $\epsilon > 0$, so $X_n \xrightarrow{p} X$.

To see the converse fails, define an independent sequence $\{X_n\}$ by

$$X_n = \begin{cases} n^3 & \text{with probability } n^{-2} \\ 0 & \text{with probability } 1 - n^{-2} \end{cases}$$

Then $\Pr(|X| > \epsilon) = n^{-2}$ for all large n, and so $X_n \xrightarrow{p} 0$. However, $\mathbb{E}|X_n| = n \to \infty$.

□

Lemma 18. (Lemma 7.2.10, Grimmett and Stirzaker.) Let $A_n(\epsilon) = \{|X_n - X| > \epsilon\}$ and $B_m(\epsilon) = \cup_{n\geq m} A_n(\epsilon)$. Then:

(a) $X_n \xrightarrow{a.s.} X$ if and only if $\Pr(B_m(\epsilon)) \to 0$ as $m \to \infty$ for all $\epsilon > 0$.

(b) $X_n \xrightarrow{a.s.} X$ if $\sum_n \Pr(A_n(\epsilon)) < \infty$ for all $\epsilon > 0$.

(c) If $X_n \xrightarrow{a.s.} X$ then $X_n \xrightarrow{p} X$, but the converse fails in general.

Proof. (a)

(b) As for Theorem 16 part (c).

(c) To see the converse fails, define an independent sequence $\{X_n\}$ by

$$X_n = \begin{cases} 1 & \text{with probability } n^{-1} \\ 0 & \text{with probability } 1 - n^{-1} \end{cases}$$

Clearly $X_n \xrightarrow{p} 0$. However, if $0 < \epsilon < 1$,

$$\Pr(B_m(\epsilon)) = 1 - \lim_{r\to\infty} \Pr(X_n = 0 \text{ for all } n \text{ such that } m \leq n \leq r) \text{ (by Lemma 1.3.5)}$$

$$= 1 - \left(1 - \frac{1}{m}\right)\left(1 - \frac{1}{m+1}\right)\cdots \text{ (by independence)}$$

$$= 1 - \lim_{M\to\infty}\left(\frac{m-1}{m}\cdot\frac{m}{m+1}\cdot\frac{m+1}{m+2}\cdots\frac{M}{M+1}\right)$$

$$= 1 - \lim_{M\to\infty}\frac{m-1}{M+1} = 1$$

and so $\{X_n\}$ does not converge almost surely.

□

Lemma 19. (Lemma 7.2.12, Grimmett and Stirzaker.) There exist sequences which

(a) converge almost surely but not in mean,

(b) converge in mean but not almost surely.

Proof. (a) As for Lemma 17 part (b).

□

Theorem 20. (Theorem 7.2.13, Grimmett and Stirzaker.) If $X_n \xrightarrow{p} X$, there exists a non-random increasing sequence of integers $n_1, n_2, \ldots$ such that $X_{n_i} \xrightarrow{a.s.} X$ as $i \to \infty$.

Theorem 21. Skorokhod's representation theorem (Theorem 7.2.14, Grimmett and Stirzaker). If $\{X_n\}$ and X with distribution functions $\{F_n\}$ and F are such that $X_n \xrightarrow{d} X$ (or equivalently, $F_n \to F$) as $n \to \infty$, then there exists a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and random variables $\{Y_n\}$ and Y mapping Ω' into $\mathbb{R}$ such that

(a) $\{Y_n\}$ and Y have distribution functions $\{F_n\}$ and F

(b) $Y_n \xrightarrow{a.s.} Y$ as $n \to \infty$

Therefore, although X_n may fail to converge to X in any mode other than in distribution, there exists a sequence $\{Y_n\}$ such that Y_n is distributed identically to X_n for every n, which converges almost surely to a copy of X.

Theorem 22. (Theorem 7.2.19, Grimmett and Stirzaker; same as Portmanteau Theorem?) The following three statements are equivalent:

(a) $X_n \xrightarrow{d} X$

(b) $\mathbb{E}[g(X_n)] \to \mathbb{E}[g(X)]$ for all bounded continuous functions g.

(c) $\mathbb{E}[g(X_n)] \to \mathbb{E}[g(X)]$ for all functions g of the form $g(x) = f(x)\mathbf{1}_{[a,b]}(x)$ where f is continuous on $[a,b]$ and a and b are points of continuity of the distribution function of the random variable X.

Theorem 23. (Grimmett and Stirzaker Theorem 7.3.9.)

(a) If $X_n \xrightarrow{a.s.} X$ and $Y_n \xrightarrow{a.s.} Y$ then $X_n + Y_n \xrightarrow{a.s.} X + Y$.

(b) If $X_n \xrightarrow{r} X$ and $Y_n \xrightarrow{r} Y$ then $X_n + Y_n \xrightarrow{r} X + Y$.

(c) If $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$ then $X_n + Y_n \xrightarrow{p} X + Y$.

(d) It is not in general true that $X_n + Y_n \xrightarrow{d} X + Y$ whenever $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$.

Theorem 24. Borel-Cantelli lemmas (Grimmett and Stirzaker Theorem 7.3.10.) Let $\{A_n\}$ be an infinite sequence of events from some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $A = \bigcap_n \bigcup_{m=n}^\infty A_m \ = \limsup_{n\to\infty} A_n = \{A_n \text{ i.o.}\}$ be the event that infinitely many of the A_n occur. Then:

(a) $\Pr(A) = 0$ if $\sum_n \Pr(A_n) < \infty$

(b) $\Pr(A) = 1$ if $\sum_n \Pr(A_n) = \infty$ and $A_1, A_2, \ldots$ are independent events.

Proof. (a) We have that $A \subseteq \bigcup_{m=n}^\infty A_n$ for all n, so

$$\Pr(A) \leq \sum_{m=n}^{\infty} \Pr(A_m) \to 0 \text{ as } n \to \infty$$

whenever $\sum_n \Pr(A_n) < \infty$.

(b) One can confirm that

$$A^c = \bigcup_n \bigcap_{m=n}^{\infty} A_m^c$$

But

$$\Pr\left(\bigcap_{m=n}^{\infty} A_m^c\right) = \lim_{r\to\infty} \Pr\left(\bigcap_{m=n}^{r} A_m^c\right) = \prod_{m=n}^{\infty} [1 - \Pr(A_m)] \text{ (by independence) } \leq \prod_{m=n}^{\infty} \exp(-\Pr(A_m))$$

$$= \exp\left(-\sum_{m=n}^{\infty} \Pr(A_m)\right) = 0$$

whenever $\sum_n \Pr(A_n) = \infty$, where the fourth step follows since $1 - x \leq e^{-x}$ if $x \geq 0$. Thus

$$\Pr(A^c) = \lim_{n\to\infty} \Pr\left(\bigcap_{m=n}^{\infty} A_m^c\right) = 0$$

so $\Pr(A) = 1$.

$\square$

Theorem 25. Kolmogorov's Two-Series Theorem. Let $X_1, X_2, \ldots$ be independent random variables with $\mathbb{E}(X_n) = \mu_n$ and $\text{Var}(X_n) = \sigma_n^2$ such that $\sum_{n=1}^\infty \mu_n < \infty$ and $\sum_{n=1}^\infty \sigma_n^2 < \infty$. Then $\sum_{n=1}^\infty X_n$ converges in $\mathbb{R}$ almost surely.

Proof. Available on wikipedia, https://en.wikipedia.org/wiki/Kolmogorov%27s_two-series_theorem. □

1.4.1 Slutsky's Convergence Theorems (8.4.1 of Pesaran, 7.3 of Grimmett and Stirzaker)

Theorem 26. Theorem 6 of Pesaran, Section 8.4.1, p. 173. Let $\{x_t, y_t\}, t = 1, 2, \ldots$ be a sequence of pairs of random variables with $y_t \xrightarrow{d} y$ and $|y_t - x_t| \xrightarrow{p} 0$. Then $x_t \xrightarrow{d} y$.

Theorem 27. Theorem 7 in Pesaran, on p.318 (section 7.3) of Grimmett and Stirzaker. (Section 8.4.1, p. 174) If $x_t \xrightarrow{d} x$ and $y_t \xrightarrow{p} c$ where c is a finite constant, then

(i) $x_t + y_t \xrightarrow{d} x + c$

(ii) $y_t x_t \xrightarrow{d} cx$

(iii) $x_t / y_t \xrightarrow{d} x/c$, if $c \neq 0$.

Theorem 28. on p.318 (section 7.3) of Grimmett and Stirzaker. Suppose that $X_n \xrightarrow{d} 0$ and $Y_n \xrightarrow{p} Y$, and let $g : \mathbb{R}^2 \to \mathbb{R}$ be such that $g(x, y)$ is a continuous function of y for all x, and $g(x, y)$ is continuous at $x = 0$ for all y. Then $g(X_n, Y_n) \xrightarrow{p} g(0, Y)$.

Theorem 29. Continuous Mapping Theorem (Theorem 9 of Pesaran, Section 8.4.1, p. 176: convergence properties of transformed sequences.) Suppose $\{\boldsymbol{x}_t\}$, $\{\boldsymbol{y}_t\}$, $\boldsymbol{x}$, and $\boldsymbol{y}$ are $m \times 1$ vectors of random variables on a probability space, and let $\boldsymbol{g}(\cdot)$ be a continuous vector-valued function. (Alternatively, suppose g has the set of discontinuity points D_g such that $\Pr(X \in D_g) = 0$.) Then

(i) $\boldsymbol{x}_t \xrightarrow{a.s.} x \implies \boldsymbol{g}(\boldsymbol{x}_t) \xrightarrow{a.s.} \boldsymbol{g}(\boldsymbol{x})$

(ii) $\boldsymbol{x}_t \xrightarrow{p} x \implies \boldsymbol{g}(\boldsymbol{x}_t) \xrightarrow{p} \boldsymbol{g}(\boldsymbol{x})$

(iii) $\boldsymbol{x}_t \xrightarrow{d} x \implies \boldsymbol{g}(\boldsymbol{x}_t) \xrightarrow{d} \boldsymbol{g}(\boldsymbol{x})$

(iv) $\boldsymbol{x}_t - \boldsymbol{y}_t \xrightarrow{p} \boldsymbol{0}$ and $\boldsymbol{y}_t \xrightarrow{d} \boldsymbol{y} \implies \boldsymbol{g}(\boldsymbol{x}_t) - \boldsymbol{g}(\boldsymbol{y}_t) \xrightarrow{d} \boldsymbol{0}(\boldsymbol{x})$

Proof. See Serfling (1980) or Rao (1973). □

See also:

Theorem 30. (Theorem 7.2.18, Grimmett and Stirzaker.) If $X_n \xrightarrow{d} X$ and $g : \mathbb{R} \to \mathbb{R}$ is continuous, then $g(X_n) \xrightarrow{d} g(X)$.

1.5 Stochastic orders $\mathcal{O}_p(\cdot)$ and $o_p(\cdot)$ (Pesaran 8.5)

Definition 1.13 (**Pesaran 8.5 Definition 6.**)**.** Let $\{a_t\}$ be a sequence of positive numbers and $\{x_t\}$ be a sequence of random variables. Then

(i) $x_t = \mathcal{O}_p(a_t)$, or x_t/a_t is bounded in probability, if for every $\epsilon > 0$ there exist real numbers M_ϵ and N_ϵ such that

$$\Pr\left(\frac{|x_t|}{a_t} > M_\epsilon\right) < \epsilon, \quad \text{for } t > N_\epsilon$$

(ii) $x_t = o_p(a_t)$ if

$$\frac{\boldsymbol{x}_t}{a_t} \xrightarrow{p} 0$$

Definition 1.14 (Ross ISE 620 Definition)**.** We say that $f(x)$ is $o(h)$ if $\lim_{h\to 0} f(h)/h = 0$.

1.6 Laws of Large Numbers and Central Limit Theorems (Pesaran 8.6; Grimmett and Stirzaker 7.4, 7.5)

Theorem 31. Weak Law of Large Numbers (Khinchine) (Pesaran 8.6 Theorem 10, Grimmett and Stirzaker Theorem 7.4.7, 541A notes Theorem 2.10). Suppose that $\{X_k\}$ is a sequence of (i) independent (ii) identically distributed random variables with (iii) constant means, i.e., $\mathbb{E}(X_k) = \mu < \infty$. Then

$$\overline{X}_k = \frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{p} \mu$$

Theorem 32. Weak Law of Large Numbers (Chebyshev) (Pesaran Section 8.6, p. 178, Theorem 11.) Let $\{X_k\}$ be a sequence of random variables. If (i) $\mathbb{E}(X_k) = \mu_k$, (ii) $\text{Var}(X_k) = \sigma_k^2$, and (iii) $\text{Cov}(X_k, X_j) = 0,\ k \neq j$, and (iv)

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \sigma_k^2 < \infty$$

then we have $\overline{X}_n - \overline{\mu}_n \xrightarrow{p} 0$, where $\overline{\mu}_n = n^{-1}\sum_{k=1}^{n} \mu_k$.

Theorem 33. Strong Law of Large Numbers (Grimmett and Stirzakker Theorem 7.4.3). Let $\{X_k\}$ be a sequence of (i) independent (ii) identically distributed random variables with (iii) $\mathbb{E}(X_k) = \mu$ and (iv) $\mathbb{E}(X_k^2) < \infty$. Then

$$\frac{1}{n}\sum_{k=1}^{n} X_k \to \mu \text{ almost surely and in mean square.}$$

Theorem 34. Strong Law of Large Numbers (Grimmett and Stirzakker Theorem 7.5.1, 541A notes Theorem 2.11). Let $\{X_k\}$ be a sequence of (i) independent (ii) identically distributed random variables. Then if and only if (iii) $\mathbb{E}|X_k| < \infty$,

$$\frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{a.s.} \mu$$

Theorem 35. Strong Law of Large Numbers 1 (Kolmogorov) (Pesaran 8.8 Theorem 12). Let $\{X_k\}$ be a sequence of (i) independent random variables with (ii) $\mathbb{E}(X_k) = \mu_k < \infty$ and (ii) $\text{Var}(X_k) = \sigma_k^2$ such that (iii)

$$\sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^2} < \infty$$

Then $\overline{X}_n - \overline{\mu}_n \xrightarrow{wp1} 0$. If the independence assumption (i) is replaced by a lack of correlation (i.e. $\text{Cov}(X_k, X_j) = 0, k \neq j$), the convergence of $\overline{X}_n - \overline{\mu}_n$ with probability one requires the stronger condition

$$\sum_{k=1}^{\infty} \frac{\sigma_k^2 (\log k)^2}{k^2} < \infty$$

Theorem 36. Strong Law of Large Numbers 2 (Pesaran 8.8 Theorem 13) Suppose that $X_1, X_2, \ldots$ are (i) independent random variables, and that (ii) $\mathbb{E}(X_k) = 0$, (iii) $\mathbb{E}(X_k^4) \leq M \;\forall\; k$ where M is an arbitrary positive constant. Then

$$\overline{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{a.s.} 0$$

Theorem 37. Central Limit Theorem (Grimmett and Stirzaker theorem 5.10.4.) Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables with finite mean μ and finite non-zero variance σ^2, and let $S_n = \sum_{i=1}^n X_i$. Then

$$\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \xrightarrow{d} \mathcal{N}(0, 1)$$

Theorem 38. (Berry-Esseen Central Limit Theorem.) There exists $c > 0$ such that the following holds. Let $X_1, X_2, \ldots$ be i.i.d. real-valued random variables with mean zero, variance 1, and $\mathbb{E}(|X_1|)^3 < \infty$. Let Z be a standard Gaussian random variable. Then for any $n \geq 1$,

$$\sup_{t \in \mathbb{R}} \left| \Pr\left(\frac{X_1 + \ldots + X_n}{\sqrt{n}} \leq t \right) - \Pr(Z \leq t) \right| \leq c \cdot \frac{\mathbb{E}(|X_1|^3)}{\sqrt{n}}$$

Remark. You can look up what the c is; Heilman doesn't think it's any bigger than around 10.

Theorem 39. (Central Limit Theorem in $\mathbb{R}^d$, Heilman notes Theorem 2.33.) Let $X^{(1)}, X^{(2)}, \ldots$ be a sequence of independent identically distributed $\mathbb{R}^d$-valued random variables. (Notation: we write $X^{(1)} = (X_1^{(1)}, \ldots, X_d^{(1)})$.) Assume $\mathbb{E}(X^{(n)} = \boldsymbol{\mu}$ for all $n \geq 1$ and for any $1 \leq i < j \leq d$, all of the covariances

$$a_{ij} = \mathbb{E}\big[(X_i^{(1)} - \mathbb{E}(X_i^{(1)})(X_j^{(1)} - \mathbb{E}(X_j^{(1)})\big]$$

are finite. Let $\boldsymbol{S}_n = \sum_{i=1}^n X^{(i)}$. Then as $n \to \infty$,

$$\frac{\boldsymbol{S}_n - n\boldsymbol{\mu}}{\sqrt{n\sigma^2}} \xrightarrow{d} \mathcal{N}(\boldsymbol{\mu}, [a_{ij}])$$

Theorem 40. (Grimmett and Stirzaker theorem 5.10.5.) Let $X_1, X_2, \ldots$ be independent random variables satisfying $\mathbb{E}(X_j) = 0$, $\text{Var}(X_j) = \sigma_j^2$, $\mathbb{E}|X_j^3| < \infty$ such that

$$\lim_{n\to\infty} \frac{1}{\sigma(n)^3} \sum_{j=1}^n \mathbb{E}|X_j^3| = 0$$

where $\sigma(n)^2 = \text{Var}\big(\sum_{j=1}^n X_j\big) = \sum_{j=1}^n \sigma_j^2$. Then

$$\frac{1}{\sigma(n)} \sum_{j=1}^n X_j \xrightarrow{d} \mathcal{N}(0,1)$$

Proof. See Loeve (1977, p. 287) and Grimmett and Stirzaker Problem 5.12.40. □

Lemma 41. Lindeberg's Condition: Let $\{X_k\}$ be a sequence of independent (not necessarily identically distributed) random variables with expectations μ_k and finite variances σ_k^2. Let $s_n^2 = \sum_{k=1}^n \sigma_k^2$. If such a sequence of independent random variables X_k satisfies the condition

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^n \mathbb{E}\big[(X_k - \mu_k)^2) \cdot \mathbf{1}_{\{|X_k - \mu_k| > \epsilon s_n\}}\big] = 0$$

for all $\epsilon > 0$ then the central limit theorem holds; that is, the random variables

$$Z_n = \frac{1}{s_n} \sum_{k=1}^n (X_k - \mu_k)$$

converge in distrbution to $\mathcal{N}(0,1)$ as $n \to \infty$.

1.7 The case of dependent and heterogeneously distributed observations (Pesaran 8.8)

Theorem 42. Central limit theorem for martingale difference sequences (Pesaran 8.8 Theorem 28). Let $\{x_t\}$ be a martingale difference sequence with respect to the information set Ω_t. Let $\overline{\sigma}_T^2 = \text{Var}(\sqrt{T}\overline{x}_T) = T^{-1}\sum_{t=1}^T \sigma_t^2$. If $\mathbb{E}(|x_t|^r) < K < \infty$ for any $r > 2$ and for all t, and

$$\frac{1}{T}\sum_{t=1}^T x_t^2 - \overline{\sigma}_T^2 \xrightarrow{p} 0$$

then $\sqrt{T}\overline{x}_T/\overline{\sigma}_T \xrightarrow{d} \mathcal{N}(0,1)$.

1.8 Worked Examples from Math 505A Midterm 2

(1) (a) **Fall 2010 Problem 1.** Let X_k, $k \geq 1$, be i.i.d. random variables with mean 1 and variance 1. Show that the limit

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n X_k}{\sum_{k=1}^n X_k^2}$$

exists in an appropriate sense, and identify the limit.

(b) **Not included on midterm or final.** Let $(X_j)_{j\geq 1}$ be i.i.d. uniform on $(-1,1)$. Let

$$Y_n = \frac{\sum_{j=1}^n X_j}{\sum_{j=1}^n X_j^2 + \sum_{j=1}^n X_j^3}$$

Prove that $\lim_{n\to\infty} \sqrt{n} Y_n$ exists in an appropriate sense, and identify the limit.

Solution.

(a)

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n X_k}{\sum_{k=1}^n X_k^2} = \lim_{n\to\infty} \frac{n^{-1}\sum_{k=1}^n X_k}{n^{-1}\sum_{k=1}^n X_k^2}$$

Since $X_1, X_2, \ldots$ are i.i.d., $E(X_1^2) = \text{Var}(X_1) + (\mathbb{E}(X_1))^2 = 2 < \infty$, we have

$$n^{-1}\sum_{k=1}^n X_k \xrightarrow{a.s.} \mathbb{E}(X_1) = 1 \text{ as } n \to \infty$$

by Theorem 33 (Strong Law of Large Numbers). Also, $X_1^2, X_2^2, \ldots$ are clearly identically distributed, and are independent by Theorem 4.2.3 ("If X and Y are independent, then so are $g(X)$ and $g(Y)$."). It is clear also that $\mathbb{E}(|X_1^2|) = \mathbb{E}(X_1^2) = \text{Var}(X_1) + \mathbb{E}(X_1)^2 = 1 + 1 = 2 < \infty$. Therefore by Theorem 34 (Strong Law of Large Numbers),

$$n^{-1}\sum_{k=1}^n X_k^2 \xrightarrow{a.s.} \mathbb{E}(X_1^2) = 2 \text{ as } n \to \infty$$

(From here I had two different ways of finishing the problem.)

- Because we have almost sure convergence in the numerator and denominator, by the Continuous Mapping Theorem (Theorem 29),

$$\lim_{n\to\infty} \frac{n^{-1}\sum_{k=1}^n X_k}{n^{-1}\sum_{k=1}^n X_k^2} = \frac{\lim_{n\to\infty} n^{-1}\sum_{k=1}^n X_k}{\lim_{n\to\infty} n^{-1}\sum_{k=1}^n X_k^2} \xrightarrow{a.s.} \boxed{\frac{1}{2}}$$

- Then, using one of Slutsky's convergence theorems (Theorem 27: "If $x_t \xrightarrow{d} x$ and $y_t \xrightarrow{p} c$ where c is a finite constant, then $x_t/y_t \xrightarrow{d} x/c$, if $c \neq 0$."), we have

$$\frac{n^{-1}\sum_{k=1}^n X_k}{n^{-1}\sum_{k=1}^n X_k^2} \xrightarrow{d} \frac{\mathbb{E}(X_1)}{\mathbb{E}(X_1^2)} = \frac{\mathbb{E}(X_1)}{\text{Var}(X_1) + \mathbb{E}(X_1)^2} = \frac{1}{1+1} = \frac{1}{2}$$

  But then, by Theorem 16 (Theorem 7.2.4(a) in Grimmett and Stirzaker: "If $X_n \xrightarrow{d} c$ where c is constant, then $X_n \xrightarrow{p} c$."), we have $\frac{n^{-1}\sum_{k=1}^n X_k}{n^{-1}\sum_{k=1}^n X_k^2} \xrightarrow{p} 1/2$.

(b) **(Not included on midterm or final.)**

$$Y_n = \frac{\sum_{j=1}^n X_j}{\sum_{j=1}^n X_j^2 + \sum_{j=1}^n X_j^3} = \frac{n^{-1}\sum_{j=1}^n X_j}{n^{-1}\sum_{j=1}^n X_j^2 + n^{-1}\sum_{j=1}^n X_j^3}$$

Note that $\mathbb{E}(X_1) = 0, \mathbb{E}(X_1^2) = \text{Var}(X_1) + \mathbb{E}(X_1)^2 = (1 - -1)^2/12 + 0^2 = 1/3, \mathbb{E}(X_1^3) = (1/2)\int_{-1}^1 x^3 dx = 0$. (We derived the formulae for the first three moments of a uniform distribution on Homework 4 problem 2(2).)

$$\implies \sqrt{n}Y_n = \frac{\sqrt{1/3}\big(\sum_{j=1}^n X_j - n\mathbb{E}(X_1)\big)/\sqrt{n \cdot 1/3}}{n^{-1}\sum_{j=1}^n X_j^2 + n^{-1}\sum_{j=1}^n X_j^3}$$

By the Central Limit Theorem (Theorem 37),

$$\frac{\sum_{j=1}^n X_j - n\mathbb{E}(X_1)}{\sqrt{n \cdot 1/3}} \xrightarrow{d} \mathcal{N}(0,1)$$

By the Law of Large Numbers (Theorem 34), since $\mathbb{E}(|X_1^2|) = \mathbb{E}(X_1^2) = 1/3 < \infty$,

$$\frac{1}{n}\sum_{j=1}^n X_j^2 \xrightarrow{a.s.} \mathbb{E}(X_1^2) = 1/3$$

By the Law of Large Numbers (Theorem 34), since $\mathbb{E}(|X_1^3|) = (1/2)\int_{-1}^1 |x^3| dx = \int_0^1 x^3 dx = 1/4 < \infty$,

$$\frac{1}{n}\sum_{j=1}^n X_j^3 \xrightarrow{a.s.} \mathbb{E}(X_1^3) = 0$$

In the denominator, since we have almost sure convergence, the regular rules of calculus/real analysis apply. That is, using the above results,

$$n^{-1}\sum_{j=1}^n X_j^2 + n^{-1}\sum_{j=1}^n X_j^3 \xrightarrow{a.s.} 1/3$$

Therefore

$$\sqrt{n}Y_n = \frac{\sqrt{1/3}\big(\sum_{j=1}^n X_j - n\mathbb{E}(X_1)\big)/\sqrt{n \cdot 1/3}}{n^{-1}\sum_{j=1}^n X_j^2 + n^{-1}\sum_{j=1}^n X_j^3} \xrightarrow{d} \frac{\sqrt{1/3}}{1/3}\mathcal{N}(0,1) = \boxed{\mathcal{N}(0,3)}$$

(2) **Fall 2010 Problem 2.** Fix $p \in (0,1)$ and consider independent Poisson random variables $X_k, k \geq 1$ with

$$\mathbb{E}X_k = \frac{p^k}{k}$$

Verify that the sum $\sum_{k=1}^\infty kX_k$ converges with probability one and determine the distribution of the random variable $Y = \sum_{k=1}^\infty kX_k$.

Solution. Melike's solution (use for midterm): We have $\mathbb{E}[kX_k] = p^k$ and $\sum_{k=1}^\infty p^k = p/(1-p) < \infty$, and $\text{Var}(kX_k) = kp^k$ and

$$\sum_{k=1}^\infty kp^k = p\sum_{k=1}^\infty kp^{k-1} = p\frac{\mathrm{d}}{\mathrm{d}p}\sum_{k=1}^\infty p^k = p\frac{\mathrm{d}}{\mathrm{d}p}\frac{p}{1-p} = p \cdot \frac{(1-p) - p(-1)}{(1-p)^2} = \frac{p}{(1-p)^2} < \infty$$

Since the sequence $\{Y_k|\}_{k\geq 1}$ is independent, by Kolmogorov's Two Series Theorem (Theorem 25: "Let $X_1, X_2, \ldots$ be independent random variables with $\mathbb{E}(X_n) = \mu_n$ and $\text{Var}(X_n) = \sigma_n^2$ such that $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$. Then $\sum_{n=1}^{\infty} X_n$ converges in $\mathbb{R}$ almost surely."), we conclude that $\sum_{k=1}^{\infty} kX_k$ converges almost surely.

To find the distribution of Y, let X be a Poisson random variable and consider its probability generating function:

$$G_X(s) = \mathbb{E}(s^X) = \sum_{k=0}^{\infty} s^k e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda s)^k}{k!} = e^{-\lambda} e^{\lambda s} = e^{\lambda(s-1)}$$

So $\mathbb{E}(s^{X_k}) = \exp\left(\frac{p^k}{k}(s-1)\right)$ and $\mathbb{E}(s^{kX_k}) = \mathbb{E}[(s^k)^{X_k}] = \exp\left(\frac{p^k}{k}(s^k-1)\right)$. Then define $Y_n = \sum_{k=1}^{n} kX_k$ and consider

$$G_{Y_n}(s) = \mathbb{E}(s^{Y_n}) = \mathbb{E}\left(\prod_{k=1}^{n} s^{kX_k}\right) = \prod_{k=1}^{n} \mathbb{E}\left(s^{kX_k}\right) = \prod_{k=1}^{n} \exp\left(\frac{p^k}{k}(s^k-1)\right) = \exp\left(\sum_{k=1}^{n} \frac{p^k}{k}(s^k-1)\right)$$

$$= \exp\left(\sum_{k=1}^{n} \frac{(ps)^k}{k} - \sum_{k=1}^{n} \frac{p^k}{k}\right)$$

Now, by taking limits as $n \to \infty$ (since we are allowed to take limit inside of expectation here), we get

$$G_Y(s) = \mathbb{E}(s^Y) = \exp\left(\sum_{k=1}^{\infty} \frac{(ps)^k}{k} - \sum_{k=1}^{\infty} \frac{p^k}{k}\right) = \exp\left(\int \sum_{k=1}^{\infty} (ps)^{k-1} dp - \int \sum_{k=1}^{\infty} p^{k-1} dp\right)$$

$$= \exp\left(\int \frac{1}{1-ps} dp - \int \frac{1}{1-p} dp\right) = \exp(-\log(1-ps) + \log(1-p)), \quad -1 \leq ps < 1 \text{ and } -1 \leq p < 1$$

$$= \frac{1-p}{1-ps}, \quad -1 \leq ps < 1$$

Since we know $\Pr(X = k) = \frac{G_X^{(k)}(0)}{k!}$, we have

$$G_Y(s) = \frac{1-p}{1-sp}, \; G'(s) = \frac{p(1-p)}{(1-sp)^2}, \; G''(s) = \frac{2p^2(1-p)}{(1-sp)^3}, \; G^{(3)}(s) = \frac{3 \cdot 2p^3(1-p)}{(1-sp)^3}, \ldots$$

$$G^{(k)}(s) = \frac{k! p^k (1-p)}{(1-sp)^k} \text{ for } k = 0, 1, 2, \ldots$$

So we have

$$\Pr(Y = k) = (1-p)p^k, \quad k = 0, 1, 2, \ldots$$

$$= \Pr(G_1(1-p) = k+1) = \Pr(G_1(1-p) - 1 = k)$$

which means $Y \sim G_1(1-p) - 1$.

(3) **Spring 2017 Problem 3.**

(a) Consider the sequence $\{X_k, k \geq 1\}$ of random variables such that X_1 is uniform on $(0, 1)$ and, given X_k, the distribution of X_{k+1} is uniform on $(0, CX_k)$, where $\sqrt{3} < C < 2$.

(i) For $n \geq 1$, compute the conditional expectation $\mathbb{E}(X_{n+1}^r \mid X_n)$.

(ii) For $n \geq 1$, compute $\mathbb{E}(X_n^r)$.

(iii) Show that $\lim_{x\to\infty} X_n = 0$ in ℓ_1 and with probability one, but not in ℓ_2.

(iv) Investigate the same questions for all other values of $C > 0$.

(b) Let $a > 0$, let $X_n, n \geq 1$ be i.i.d. random variables that are uniform on $(0, a)$, and let $Y_n = \prod_{k=1}^n X_k$. Determine, with a proof, all values of a for which $\lim_{n\to\infty} Y_n = 0$ with probability one.

Solution.

(a) (i) We have that $X_{n+1} \mid X_n \sim U(0, CX_n)$. Therefore

$$\mathbb{E}(X_{n+1}^r \mid X_n) = \frac{1}{CX_n} \int_0^{CX_n} x^r dx = \frac{1}{CX_n} \cdot \left. \frac{x^{r+1}}{r+1} \right|_0^{CX_n} = \frac{C^r X_n^r}{r+1}$$

$$\implies \mathbb{E}(X_{n+1}^r) = \mathbb{E}[\mathbb{E}(X_{n+1}^r \mid X_n)] = \frac{C^r}{r+1} \cdot \mathbb{E}(X_n^r)$$

$$\implies \boxed{\mathbb{E}(X_{n+1}^r \mid X_n) = \frac{C^r}{r+1} X_n^r}$$

(ii) Note that $E(X_1^r) = \int_0^1 x^r dr = 1/(r+1)$. Therefore

$$\mathbb{E}(X_{n+1}^r) = \frac{C^r}{r+1} \cdot \mathbb{E}(X_n^r) = \left(\frac{C^r}{r+1} \right)^n \cdot \mathbb{E}(X_1^r) = \boxed{\left(\frac{C^r}{r+1} \right)^n \cdot \frac{1}{r+1}}$$

(iii) We would like to show that $X_n \xrightarrow{w.p.1} 0$ and that $X_n \xrightarrow{1} 0$, but that the same result does not follow for the ℓ_2 norm.

- **Convergence with probability one:** We seek to show that $\Pr\big(\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = 0\}\big) = 1$. By Markov's Inequality (Lemma 3), we have

$$\Pr(|X_n| \geq a) \leq \frac{\mathbb{E}(X_n)}{a} \quad \forall\, a > 0$$

$$\iff \Pr(|X_n| \geq a) \leq \left(\frac{C^1}{1+1} \right)^{n-1} \cdot \frac{1}{1+1} \cdot \frac{1}{a} = \left(\frac{C}{2} \right)^{n-1} \cdot \frac{1}{2a} \quad \forall\, a > 0$$

Since $\sqrt{3} < C < 2$, $\sqrt{3}/2 < C/2 < 1$. Since $X_n \in [0, CX_{n-1}]$, $X_n \geq 0$, so $|X_n| = X_n$. Therefore we have

$$\Pr(\lim_{n\to\infty} |X_n| \geq a) = \Pr(\lim_{n\to\infty} X_n \geq a) \leq \lim_{n\to\infty} \left(\frac{C}{2} \right)^{n-1} \cdot \frac{1}{2a} = 0 \quad \forall\, a > 0$$

Since $|X_n| \geq 0$, this implies that $\Pr(\lim_{n\to\infty} X_n = 0) = \Pr\big(\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = 0\}\big) = 1$, so by the Borel-Cantelli Lemma (Theorem 24), X_n converges to 0 with probability 1.

- **Convergence in ℓ_1 norm:** We seek to show that $\lim_{n\to\infty} \mathbb{E}(|X_n|) = 0$. Since $X_n \in [0, CX_{n-1}]$, $X_n \geq 0$, so $|X_n| = X_n$. Therefore

$$\lim_{n\to\infty} \mathbb{E}(|X_n|) = \lim_{n\to\infty} \mathbb{E}(X_n) = \lim_{n\to\infty} \left(\frac{C}{2}\right)^{n-1} \cdot \frac{1}{2}$$

Since $\sqrt{3} < C < 2$, $\sqrt{3}/2 < C/2 < 1$, so $C/2 < 1$. Therefore we have

$$\lim_{n\to\infty} \mathbb{E}(|X_n|) = \lim_{n\to\infty} \left(\frac{C}{2}\right)^{n-1} \cdot \frac{1}{2} = 0$$

so X_n converges to 0 in 1st mean.

- **Convergence in ℓ_2 norm:** We seek to show that $\lim_{n\to\infty} \mathbb{E}(|X_n|^2) \neq 0$. We have

$$\lim_{n\to\infty} \mathbb{E}(|X_n|^2) = \lim_{n\to\infty} \mathbb{E}(X_n^2) = \lim_{n\to\infty} \left(\frac{C^2}{3}\right)^{n-1} \cdot \frac{1}{3}$$

Since $\sqrt{3} < C < 2$, $3/3 < C^2/3 < 4/3$, so $C^2/3 > 1$. Therefore we have

$$\lim_{n\to\infty} \mathbb{E}(|X_n|^2) = \lim_{n\to\infty} \left(\frac{C^2}{3}\right)^{n-1} \cdot \frac{1}{3} = \infty \neq 0$$

so X_n does not converge to 0 in 2nd mean.

(iv) From the above, it is clear that for convergence with probability one or in 1st mean we require $0 < C/2 < 1$ and for convergence in second mean we require $0 < C^2/3 < 1$. For $0 < C < \sqrt{3}$, we see that X_n would converge to zero in 2nd mean since this would imply that $0 < C^2/3 < 1$. It would also still converge to 0 in 1st mean (and with probability 1) since we would have $(0 < C/2 < \sqrt{3}/2 < 1$.
For $C = \sqrt{3}$, X_n would still converge to 0 with probability one and in 1st mean for the same reasons. However, it would not converge in 2nd mean because we would have

$$\lim_{n\to\infty} \mathbb{E}(|X_n|^2) = \lim_{n\to\infty} \left(\frac{\sqrt{3}^2}{3}\right)^{n-1} \cdot \frac{1}{3} = \frac{1}{3} \neq 0$$

For $C \geq 2$, it would diverge in all three cases, since in this case $C/2 \geq 2/2 = 1$ and $C^2/3 \geq 4/3 > 1$.

(b) **Probably won't be on midterm.** Note that

$$\lim_{n\to\infty} Y_n = \lim_{n\to\infty} \prod_{k=1}^{n} X_k = 0 \iff \log(Y_n) = \log\left(\prod_{k=1}^{n} X_k\right) = \sum_{k=1}^{n} \log(X_k) \to -\infty$$

Note that

$$\mathbb{E}[\log(Y_n)] = \mathbb{E}\left(\sum_{k=1}^{n} \log(X_k)\right) = \sum_{k=1}^{n} \mathbb{E}[\log(X_k)] = \sum_{k=1}^{n} \mathbb{E}[\log(X_1)] = \sum_{k=1}^{n} \int_0^a (\log(x)/a)dx$$
$$= \sum_{k=1}^{n} \frac{1}{a}\left[x \log x - x\right]_0^a = \sum_{k=1}^{n} \frac{a \log a - a}{a} = \sum_{k=1}^{n} (\log(a) - 1) = n(\log(a) - 1)$$

As $n \to \infty$ we have

$$\mathbb{E}[\log(Y_n)] = \begin{cases} -\infty & a < e \\ 0 & a = e \\ \infty & a > e \end{cases}$$

Since $\mathbb{E}[\log(Y_n)] \to \infty$ for $a < e$, we have $\lim_{n\to\infty} Y_n = 0$ for $a < 3$. Therefore

$$\boxed{\lim_{n\to\infty} Y_n = \lim_{n\to\infty} \prod_{k=1}^{n} X_k = 0 \iff a < e.}$$