Math 425B Midterm 1 Cheat Sheet

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1 Function Spaces: Uniform Convergence and C^0 (Chapter 4 of Pugh [2015], Section 4.1)

Definition 1.1 (**Pointwise convergence; from Section 4.1 of Pugh [2015]**)**.** Let $\mathcal{X}$ be a set, let $(\mathcal{Y}, d)$ be a metric space. Let $f_n : \mathcal{X} \to \mathcal{Y}$ be functions for $n \geq 1$. Let $f : \mathcal{X} \to \mathcal{Y}$ be another function. We say f_n **converges to** f **pointwise** if for all $x \in \mathcal{X}$, the sequence $\{f_n(x)\}_{n=1}^{\infty}$ converges to $f(x)$ as a sequence of points in $\mathcal{Y}$. I.e., if for all $\epsilon > 0$ and for all $x \in \mathcal{X}$, there exists N (maybe depending on ϵ and x) such that for $n \geq N$, we have $d(f_n(x), f(x)) < \epsilon$.

Definition 1.2 (**Uniform convergence; from Section 4.1 of Pugh [2015]**)**.** Let $\mathcal{X}$ be a set, let $(\mathcal{Y}, d)$ be a metric space. Let $f_n : \mathcal{X} \to \mathcal{Y}$ be functions for $n \geq 1$. Let $f : \mathcal{X} \to \mathcal{Y}$ be another function. We say f_n **converges uniformly to** f if for all $\epsilon > 0$, there exists N (maybe depending on ϵ bit not x) such that for $n \geq N$ and for all $x \in \mathcal{X}$, we have $d(f_n(x), f(x)) < \epsilon$.

Remark. Note that uniform convergence implies pointwise convergence. This fact implies that uniform limits are unique, since limits in metric space $(\mathcal{Y}, d)$ are unique, so the pointwise limits of sequences of functions are unique.

Example 1.1 (**Example from 4.1 of pointwise convergence but not uniform convergence.**)**.**

Proposition 1. In Example **??**, f_n does not converge uniformly to f.

Proof. Suppose f_n converges uniformly to 0 (the pointwise limit). Take $\epsilon = 1/2$. Can choose $N \geq 1$ such that $|x^n - 0| < 1/2$ for all $n \geq N$, and for all $x \in (0, 1)$. Let $x = (3/4)^{1/N}$. (Nth roots exist by IVT.) Then $|x^N - 0| = 3/4$ which is not less than $1/2$.

□

Theorem 2 (**Cauchy 1821; Theorem 4.1 from Pugh [2015]**)**.** Let (X, d) and (Y, d') be metric spaces. Let $f_n : X \to Y$ be continuous at $x_0 \in X$ for all $n \geq 1$. Suppose f_n converges uniformly to f for some $f : X \to Y$. Then f is continuous at x_0. (This implies that if all f_n are continuous everywhere and f_n converges uniformly to f then f is continuous everywhere.)

What about if you only have pointwise convergence? Then this theorem won't work. Consider the saw-tooth wave example: the limit is not continuous. There exist easier counterexamples as well.

Example 1.2 (From Section 4.1, p. 212 of Pugh [2015])**.** Let $X = [0, 1]$, $Y = \mathbb{R}$, $f_n(x) = x^n$. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0, & 0 \leq x < 1, \\ 1, & x = 1 \end{cases}$$

As before, f_n converges to f pointwise but not uniformly. f_n is continuous on $[0, 1]$ for all n but f is not. See Figure **??**.

Exercise 1. Let X be a set, (Y, d) be a metric space. Let $x_0 \in X$ be fixed. Let $f : X \to Y$ be a function. Then the following are equivalent:

1. There exists $m \in \mathbb{R}$ such that $d(f(x_0), f(x)) \leq m$ for all $x \in X$.

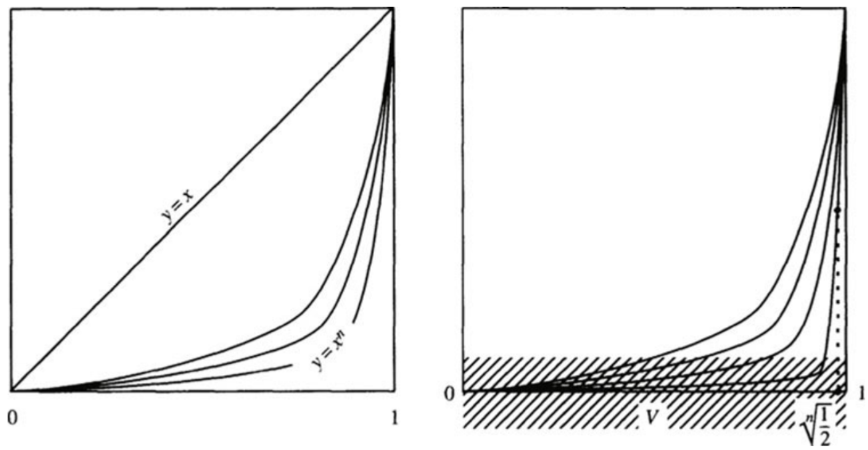

Figure 1: Figure 88 from Section 4.1, p. 213 of Pugh [2015]

2. There exists $x_1 \in X$ and $m \in \mathbb{R}$ such that $d(f(x_1), f(x_2)) \leq m$ for all $x_2 \in X$.

3. There exists $m \in \mathbb{R}$ such that $d(f(x_1), f(x_2)) \leq m$ for all $x_1, x_2 \in X$.

Definition 1.3. If $f : X \to (Y, d)$ satisfies the statements in Exercise **??**, then f is called **bounded.**

Definition 1.4. Let X be a set and (Y, d) be a metric space. Define $C_b(X, Y)$ as the set of bounded functions from X to Y.

Definition 1.5. For $f, g \in C_b(X, Y)$, let $d_\infty(f, g) := \sup_{x \in X} d(f(x), g(x))$.

Theorem 3 (Theorem 4.2 in Pugh [2015]). X set, (Y, d) metric space. If $f_n \in C_b(X, Y)$ for $n \geq 1$ and $f \in C_b(X, Y)$, then f_n converges uniformly to f if and only if $f_n \xrightarrow{d_\infty} f$).

Proposition 4 (Uniform convergence preserves boundedness)**.** If $f_n : X \to Y$ is bounded for all n and f_n converges uniformly to f, then f is bounded.

Corollary 4.1 (similar to Theorem 4.2 in Pugh [2015]). If $f_n \in C_b(X, Y)$ for $n \geq 1$, then (f_n) converges uniformly if and only if (f_n) converges in $C_b(X, Y)$. (functional analysis perspective of uniform convergence.)

Definition 1.6. X set, (Y, d) metric space. A sequence of functions $f_n : X \to Y$ is **uniformly Cauchy** if for every $\epsilon > 0$ there exists $N \geq 1$ such that for $n, m \geq N$ we have $d(f_n(x), f_m(x)) < \epsilon$ for all $x \in X$.

Theorem 5. X set, (Y, d) metric space. If (Y, d) is complete, then any uniformly Cauchy sequence of functions $f_n : X \to Y$ is uniformly convergent.

Definition 1.7. Assume both (X, d) and (Y, d') are metric spaces. $(C^0(X, Y) \subset C_b(X, Y)$ is the set of bounded continuous functions $X \to Y$. We can restrict d_∞ to $C_0(X, Y)$ and get a metric subspace.

Corollary 5.1 (Corollary to Theorem ??). $C^0(X, Y)$ is a closed subset of $C_b(X, Y)$.

References

C. Pugh. *Real Mathematical Analysis*. Undergraduate Texts in Mathematics. Springer International Publishing, 2015. ISBN 9783319177717. URL https://books.google.com/books?id=2NVJCgAAQBAJ.