Math 541A - Mathematical Statistics: Homework 7

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Exercise 1.

Exercise 2.

Exercise 3. First,

$$\mathbb{E}(\overline{X}^2) = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^n X_i\right)^2 = \frac{1}{n^2}\mathbb{E}\left(\sum_{i=1}^n X_i^2 + 2\sum_{1\le i<j\le n} X_iX_j\right) = \frac{1}{n^2}\left[\sum_{i=1}^n \mathbb{E}\left(X_i^2\right) + 2\binom{n}{2}\mu^2\right]$$

$$= \frac{1}{n^2}\sum_{i=1}^n \left(\mathrm{Var}(X_i) + \mathbb{E}(X_i)^2\right) + \frac{2}{n^2}\frac{n(n-1)}{2}\mu^2 = \frac{1}{n^2}\cdot n\left(\sigma^2+\mu^2\right) + \frac{n-1}{n}\mu^2$$

$$= \mu^2 + \frac{\sigma^2}{n}.$$

Therefore since S^2 is unbiased for σ^2, $Y := \overline{X}^2 - n^{-1}S^2$ is unbiased for μ^2:

$$\mathbb{E}(Y) = \mathbb{E}\left(\overline{X}^2 - n^{-1}S^2\right) = \mu^2 + \frac{\sigma^2}{n} - \frac{\sigma^2}{n} = \mu^2.$$

Note that Y is a function of the complete sufficient statistic $Z := (\overline{X}, S^2)$; that is, $Y = h(Z)$ where $h\big((t_1,t_2)\big) = t_1^2 - n^{-1}t_2$. Therefore by Lehmann-Scheffe, $\mathbb{E}_\theta(Y \mid Z) = \mathbb{E}_\theta(h(Z) \mid Z) = h(Z) = Y$ is UMVU for μ^2.

Exercise 4.

Theorem 1 (**Rao-Blackwell; Theorem 6.4 in Math 541A notes**)**.** Let Z be a sufficient statistic for $\{f_\theta : \theta \in \Theta\}$ and let Y be an estimator for $g(\theta)$. Define $W := \mathbb{E}_\theta(Y \mid Z)$. Let $\theta \in \Theta$. Then

$$\mathrm{Var}_\theta(W) \le \mathrm{Var}_\theta(Y).$$

Further, let $r(\theta, y) < \infty$ and such that $\ell(\theta, y)$ is convex in y. Then

$$r(\theta, W) \le r(\theta, Y).$$

Proof (just of risk part). Note that since Z is sufficient, W does not depend on θ. By the Conditional Jensen's Inequality and using the convexity of $\ell(\theta, y)$ in y,

$$\ell(\theta, w) = \ell(\theta, \mathbb{E}_{\tilde{\theta}}(Y \mid Z)) \le \mathbb{E}_{\tilde{\theta}}[\ell(\theta, Y) \mid Z].$$

Take expectations of both sides to get

$$\mathbb{E}_{\tilde{\theta}}\ell(\theta, w) = r(\theta, W) \le \mathbb{E}_{\tilde{\theta}}\mathbb{E}_{\tilde{\theta}}[\ell(\theta, Y) \mid Z] = \mathbb{E}_{\tilde{\theta}}\ell(\theta, Y) = r(\theta, Y).$$

If $\ell(\theta, y)$ is strictly convex in y then this inequality is strict, unless Y is a function of Z. If Y is a function of Z, then $\mathbb{E}_\theta(Y \mid Z) = Y$, so $W = Y$.

$\square$

Proof (Variance part, then risk part as consequence). First we will show that $\mathrm{Var}_\theta(Y) \geq \mathrm{Var}_\theta(Y \mid Z)$, where Z is a sufficient statistic for $g(\theta)$. Then we will show that this implies that $r(\theta, Y \mid Z) \leq r(\theta, Y)$ when the loss function is mean squared error.

Let $W := \mathbb{E}(Y \mid Z)$. Using the given identities and the Law of Total Variance,

$$\mathrm{Var}_\theta(Y) = \mathbb{E}_\theta[\mathrm{Var}_\theta(Y|Z)] + \mathrm{Var}_\theta[\mathbb{E}_\theta(Y|Z)] = \mathbb{E}_\theta[\mathrm{Var}_\theta(Y|Z)] + \mathrm{Var}_\theta(W)$$

Note that $\mathrm{Var}_\theta(Y|Z)] \geq 0 \implies \mathbb{E}_\theta[\mathrm{Var}_\theta(Y|Z)] \geq 0$. Therefore we have

$$\mathrm{Var}_\theta(Y) \geq \mathrm{Var}_\theta(W) \tag{1}$$

as desired. Next, let $\mu = \mathbb{E}(Y)$. Then we have

$$\mathbb{E}(Y - g(\theta))^2 = \mathbb{E}(Y - \mu + \mu - g(\theta))^2 = \mathbb{E}\big[(Y-\mu)^2 + (\mu - g(\theta))^2 + 2(Y-\mu)(\mu - g(\theta))\big]$$

$$= \mathbb{E}\big[(Y-\mu)^2\big] + \mathbb{E}\big[(\mu - g(\theta))^2\big] + 2\mathbb{E}\big[(Y-\mu)(\mu - g(\theta))\big] = \mathrm{Var}(Y) + (\mu - g(\theta))^2$$

Since $\mathbb{E}(Y - g(\theta))^2 = r(\theta, Y)$, we have

$$\mathrm{Var}(Y) = r(\theta, Y) - (\mu - g(\theta))^2 \tag{2}$$

where $\mathbb{E}\big[(\mu - g(\theta))^2\big] = (\mu - g(\theta))^2$ because both quantities are constants. Similarly, since $\mathbb{E}(W) = \mathbb{E}[\mathbb{E}_\theta(Y \mid Z)] = \mathbb{E}(Y) = \mu$, we have

$$\mathbb{E}(W - g(\theta))^2 = \mathbb{E}(W - \mu + \mu - g(\theta))^2 = \mathbb{E}\big[(W-\mu)^2 + (\mu - g(\theta))^2 + 2(W-\mu)(\mu - g(\theta))\big]$$

$$= \mathbb{E}\big[(W-\mu)^2\big] + \mathbb{E}\big[(\mu - g(\theta))^2\big] + 2\mathbb{E}\big[(W-\mu)(\mu - g(\theta))\big] = \mathrm{Var}(W) + (\mu - g(\theta))^2$$

$$\iff \mathrm{Var}(W) = r(\theta, W) - (\mu - g(\theta))^2. \tag{3}$$

Finally, substituting (2) and (3) into (1) yields

$$r(\theta, Y) - (\mu - g(\theta))^2 \geq r(\theta, W) - (\mu - g(\theta))^2 \iff r(\theta, Y) \geq r(\theta, W).$$

□