Math Review Notes—Real Analysis

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1 Real Analysis

These are my notes from Math 4650: Analysis I at Cal State LA as well as Prof. Steven Heilman's notes from Math 541A at USC.

1.1 Midterm 1

1.1.1 Homework 1

Definition 1.1. Let $S \subseteq \mathbb{R}$. We say that S is **bounded from above** if $\exists\, b \in \mathbb{R}$ where

$$s \leq b \;\forall\; s \in S$$

If this is the case, we call b an **upper bound** of S.

If $b \leq c$ for all upper bounds c of S, we call b the **supremum** of S: $b = \sup(S)$.

Definition 1.2. We say that S is **bounded from below** if $\exists\, a \in \mathbb{R}$ where

$$s \geq a \;\forall\; s \in S$$

If this is the case, we call a a **lower bound** of S.

If $a \geq d$ for all lower bounds d of S, we call a the **infimum** of S: $a = \inf(S)$.

Proposition 1. Useful Sup/Inf Fact: Let $S \in \mathbb{R}$, $S \neq \emptyset$.

(1) Suppose S is bounded from above by an element b. Then $b = \sup(S) \iff \forall\, \epsilon > 0 \;\exists\, x \in S$ with

$$b - \epsilon < x \leq b$$

(2) Suppose S is bounded from below by an element a. Then $a = \inf(S) \iff \forall\, \epsilon > 0 \;\exists\, x \in S$ with

$$a \leq x < a + \epsilon$$

Completeness Axiom: Let S be a nonempty subset of $\mathbb{R}$. If S is bounded from above, then $\sup(S)$ exists. If S is bounded from below, then $\inf(S)$ exists.

Facts about absolute value:

- **Proposition 2.** $|x - y| < \epsilon \iff y - \epsilon < x < y + \epsilon$.

  Proof. In notes 08/23. □

- **Proposition 3.** $|ab| = |a||b|$.

Proof.

$$|ab| = \begin{cases} ab & ab \geq 0 \\ -ab & ab < 0 \end{cases} = \begin{cases} ab & a \geq 0, b \geq 0 \\ -ab & a \geq 0, b < 0 \\ -ab & a < 0, b \geq 0 \\ ab & a < 0, b < 0 \end{cases} = \begin{cases} ab & a \geq 0, b \geq 0 \\ a(-b) & a \geq 0, b < 0 \\ (-a)b & a < 0, b \geq 0 \\ (-a)(-b) & a < 0, b < 0 \end{cases}$$

$$= \begin{cases} |a||b| & a \geq 0, b \geq 0 \\ |a||b| & a \geq 0, b < 0 \\ |a||b| & a < 0, b \geq 0 \\ |a||b| & a < 0, b < 0 \end{cases} \implies |ab| = |a||b|$$

□

- **Proposition 4.** Let $\epsilon > 0$. Then $|a| < \epsilon \iff -\epsilon < a < \epsilon$.

  Proof. Follows from Proposition 2 if $x = a$, $y = 0$. □

- **Proposition 5.** $-|a| \leq a \leq |a|$

  Proof. Follows from Proposition 2 if $x = a$, $y = 0$, $\epsilon = |a|$. □

- **Theorem 6. Triangle Inequality:** $|a + b| \leq |a| + |b|$.

  Proof. In notes 08/23. □

  Corollary 6.1. Triangle Inequality: $|a - b| \leq |a| + |b|$.

  Proof. Follows from Theorem 6, let $b = -b$.

  □

  Remark. See also Theorem **??**.

- **Proposition 7.** $|\ |a| - |b|\ | \leq |a - b|$.

  Proof. By Proposition 2, $|\ |a| - |b|\ | \leq |a - b|$ if and only if

$$|b| - |a - b| \leq |a| \leq |b| + |a - b| \tag{1}$$

The left half of (1) is true by the Triangle Inequality (Theorem 6):

$$|b| = |a - (a - b)| \leq |a| + |a - b| \iff |b| \leq |a| + |a - b| \iff |b| - |a - b| \leq |a|$$

The right half of (1) is also true by the Triangle Inequality (Theorem 6):

$$|a| = |b + a - b| \leq |b| + |a - b|$$

Therefore

$$|\ |a| - |b|\ | \leq |a - b|.$$

□

Proof. (Alternative proof.) Note that by the Triangle Inequality (Theorem 6),

$$|a| = |a - b + b| \leq |a - b| + |b| \implies |a| - |b| \leq |a - b|$$

Also,

$$|b| = |b - a + a| \leq |b - a| + |a| \implies -|b - a| \leq |a| - |b| \implies -|a - b| \leq |a| - |b|$$

where the last step follows from Proposition 9. Therefore

$$-|a - b| \leq |a| - |b| \leq |a - b|$$

and by Proposition 2,

$$|\ |a| - |b|\ | \leq |a - b|.$$

□

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Proposition 8. If $a < x < b$ and $a < y < b$ then $|x - y| < b - a$.

Proof.

$$y > a \implies -y < -a \implies b - y < b - a$$

$$b > y \implies b - y = |b - y| \implies \boxed{|b - y| < b - a}$$

By the Triangle Inequality (Theorem 6),

$$|x - y| = |x - b + b - y| \leq |x - b| + |b - y|$$

Since $b < x$, $|x - b| > 0$. Therefore $\boxed{|x - y| < |b - y|}$.

$$\implies |x - y| < |b - y| < b - a$$

$$\implies |x - y| < b - a$$

□

Proof. (Alternative proof.) Break into two cases.

- **Case 1:** $x \geq y$. Then $|x-y| = x-y$. We know $a < x < b \implies 0 < x-a < b-a$.

$$a < y \implies -a > -y \implies x-a > x-y \implies x-y < x-a < b-a$$

$$\implies \boxed{|x-y| < b-a}$$

- **Case 2:** $x < y$. Then $|x-y| = y-x$. We know $a < y < b \implies 0 < y-a < b-a$.

$$a < x \implies -a > -x \implies y-a > y-x \implies y-x < y-a < b-a$$

$$\implies \boxed{|x-y| < b-a}$$

□

- **Proposition 9.** $|a-b| = |b-a|$

  Proof. $|a-b| = |(-1)(b-a)| = |-1||b-a| = |b-a|$, where the second-to-last step follows from Proposition 2.

  □

1.1.2 Homework 2

Definition 1.3. A sequence (a_n) of real numbers is said to **converge** to a **limit** $L \in \mathbb{R}$ if $\forall\ \epsilon > 0\ \exists\ N > 0$ where

$$n \geq N \implies |a_n - L| < \epsilon$$

We say that (a_n) **diverges** if it does not converge.

Definition 1.4. A sequence (a_n) of real numbers is **bounded** if $\exists\ M > 0$ where $\forall\ n \in \mathbb{N}$

$$|a_n| \leq M.$$

Theorem 10. If (a_n) converges then (a_n) is bounded.

Definition 1.5. Let (a_n) be a sequence of real numbers. We say that (a_n) is a **Cauchy sequence** if $\forall\ \epsilon > 0\ \exists\ N$ where

$$n, m \geq N \implies |a_n - a_m| < \epsilon$$

Theorem 11. (a_n) is Cauchy if and only if (a_n) converges.

Corollary 11.1. If (a_n) is Cauchy then (a_n) is bounded.

Theorem 12. Suppose that $\{a_n\}$ is a Cauchy sequence. Then $\{a_n\}$ is bounded.

Proof. Let $\epsilon = 1$. Since (a_n) is Cauchy, $\exists\ N > 0 \mid n, m \geq N \implies$

$$|a_n - a_m| < 1$$

So, $n \geq N \implies$

$$|a_n - a_N| < 1 \iff a_N - 1 < a_n < a_N + 1 \implies |a_n| < |a_N + 1| \leq |a_N| + 1$$

Let $M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a_N| + 1\}$. Then $|a_n| \leq M\ \forall\ n \geq 1$. Therefore (a_n) is bounded.

□

Theorem 13. (Squeeze theorem.) Suppose that $\{a_n\}, \{b_n\}$, and $\{c_n\}$ are sequences of real numbers such that $a_n \leq b_n \leq c_n$ for all n. If both $\{a_n\}$ and $\{c_n\}$ converge to L, then $\{b_n\}$ converges to L.

Proof. Let $\epsilon > 0$. (a_n) converges to $L \implies$

$$\forall\ \epsilon > 0\ \exists\ N_A \mid n \geq N_A \implies |a_n - L| < \epsilon$$

(c_n) converges to $L \implies$

$$\forall\ \epsilon > 0\ \exists\ N_C \mid n \geq N_C \implies |c_n - L| < \epsilon$$

Let $N = \max\{N_A, N_C\}$. Then by one of our absolute values rules, $n \geq N \implies$

$$|a_n - L| < \epsilon \iff L - \epsilon < a_n < L + \epsilon$$

$$|c_n - L| < \epsilon \iff L - \epsilon < c_n < L + \epsilon$$

Therefore since $a_n \leq b_n \leq c_n$,

$$L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon \implies L - \epsilon < b_n < L + \epsilon \iff |b_n - L| < \epsilon$$

Therefore (b_n) converges to L.

□

Theorem 14. Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers such that $a_n \leq b_n$ for all n. If $\{a_n\}$ and $\{b_n\}$ converge to A and B respectively, then $A \leq B$.

Proof. Suppose $A > B$. Then let $\epsilon = \frac{A-B}{4} > 0$. (a_n) converges to $A \implies$

$$\exists\ N_A \mid n \geq N_A \implies |a_n - A| < \epsilon \iff A - \epsilon < a_n < A + \epsilon$$

(b_n) converges to $B \implies$

$$\exists\ N_B \mid n \geq N_B \implies |b_n - B| < \epsilon \iff B - \epsilon < b_n < B + \epsilon$$

Then if $n > \max\{N_A, N_B\}$,

$$A - \epsilon < a_n < A + \epsilon \iff A - \frac{A-B}{4} < a_n < A + \frac{A-B}{4} \iff \frac{3A}{4} + \frac{B}{4} < a_n < \frac{5A}{4} - \frac{B}{4}$$

$$B - \epsilon < b_n < B + \epsilon \iff B - \frac{A-B}{4} < b_n < B + \frac{A-B}{4} \iff \frac{5B}{4} - \frac{A}{4} < b_n < \frac{3B}{4} + \frac{A}{4}$$

This implies

$$b_n < \frac{3B}{4} + \frac{A}{4} = \frac{B}{4} + \frac{A}{4} + \frac{2B}{4} < \frac{B}{4} + \frac{A}{4} + \frac{2A}{4} = \frac{3A}{4} + \frac{B}{4} < a_n$$

Contradiction, since it is given that $a_n \leq b_n\ \forall\ n$. Therefore $A \leq B$. $\square$

1.2 Midterm 2

1.2.1 Homework 3

Definition 1.6. (Limits of functions at infinity.) Let f be a real-valued function defined on some set D where D contains an interval of the form (a, ∞). Let $L \in \mathbb{R}$. We say

$$\lim_{x \to \infty} f(x) = L$$

if $\forall\ \epsilon > 0\ \exists\ N \in \mathbb{R}$ where

$$x \geq N \implies |f(x) - L| < \epsilon.$$

Definition 1.7. Let $D \subseteq \mathbb{R}$. Let $a \in \mathbb{R}$. We say that a is a **limit point** (or "cluster point," or "accumulation point") of D if $\forall\ \delta > 0\ \exists\ x \in D$ where

$$x \neq a \text{ and } |x - a| < \delta$$

(Note that a may or may not be contained in D.)

Definition 1.8. (Limit of a function at a.): Let $D \subseteq \mathbb{R}$ and $f : d \to \mathbb{R}$. Let a be a limit point of D. Let $x \in D$. We say that f has a *limit as x tends to a* if $\exists\ L \in \mathbb{R}$ where $\forall\ \epsilon > 0\ \exists\ \delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$

and we write

$$\lim_{x \to a} f(x) = L$$

Proposition 15. (Properties of Limits.) Let $D \in \mathbb{R}$ and let a be a limit point of D. Suppose $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$. Let $\alpha \in \mathbb{R}$.

(1) If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ then

(a)
$$\lim_{x \to a} \alpha = \alpha$$

(b)
$$\lim_{x \to a} [f(x) + g(x)] = L + M$$

(c)
$$\lim_{x \to a} [f(x) - g(x)] = L - M$$

(d)
$$\lim_{x \to a} [f(x) \cdot g(x)] = L \cdot M$$

(e)
$$\lim_{x \to a} [\alpha \cdot f(x)] = \alpha \cdot L$$

(2) If $h : D \to \mathbb{R}$ and $h(x) \neq 0 \ \forall \ x \in D$ and $\lim_{x\to a} h(x) = H \neq 0$, then

$$\lim_{x\to a} \frac{1}{h(x)} = \frac{1}{H}$$

Note that properties (2) and (1)(d) combined imply

$$\lim_{x\to a} \frac{f(x)}{h(x)} = \frac{L}{H}$$

1.2.2 Homework 4

Definition 1.9. (Continuity.) Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ and $a \in D$. Then f is **continuous** at a if $\lim_{x\to a} f(x)$ exists and

$$\lim_{x\to a} f(x) = f(a)$$

Remark. if f is continuous at a, then we can say $\forall \ \epsilon > 0 \ \exists \ \delta > 0$ such that

$$|x - a| < \delta \implies |f(x) - L| < \epsilon$$

that is, we don't need to say $0 < |x - a| < \delta$.

Definition 1.10. If $B \subseteq D$, then f is **continuous on B** if f is continuous at every $b \in B$.

Theorem 16. (Intermediate Value Theorem.) Let f be continuous on $[a, b]$ and suppose $f(a) < f(b)$. $\forall \ d$ such that

$$f(a) < d < f(b)$$

$\exists \ c \in \mathbb{R}$ where

$$a < c < b, \ f(c) = d.$$

1.3 Final

1.3.1 Homework 5

Definition 1.11. Let $S \subseteq \mathbb{R}$. We say $x \in \mathbb{R}$ is an **interior point** of S if there exists an open interval (a, b) where

$$x \in (a, b) \text{ and } (a, b) \subseteq S.$$

Definition 1.12. (Open sets.) Let $S \subseteq \mathbb{R}$. We say S is **open** if every $x \in S$ is an interior point of S.

Definition 1.13. (Closed sets.) Let $S \subseteq \mathbb{R}$. We say S is **closed** if $\mathbb{R} \setminus S$ is open.

Theorem 17. A set is closed if and only if it contains all of its limit points.

(Facts about open and closed sets.) Suppose $a, b \in \mathbb{R}$. Then

- **Proposition 18.** (a, ∞) is open.

  Proof. Let $x \in (a, \infty)$. Since $x > a$, $\exists\ \epsilon > 0 \mid a + \epsilon = x$. Then $a = x - \epsilon < x - \frac{\epsilon}{2} < x < x + \frac{\epsilon}{2} < \infty$. Therefore $x \in (x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}) \subseteq (a, \infty)$, so (a, ∞) is open. □

- **Proposition 19.** $(-\infty, b)$ is open.

  Proof. Let $x \in (-\infty, b)$. Since $x < b$, $\exists\ \epsilon > 0 \mid b - \epsilon = x$. Then $-\infty < x - \frac{\epsilon}{2} < x < x + \frac{\epsilon}{2} < x + \epsilon = b$. Therefore $x \in (x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}) \subseteq (-\infty, b)$, so $(-\infty, b)$ is open. □

- **Proposition 20.** (a, b) is open.

  Proof. In class notes. □

- **Proposition 21.** If $a < b$, then $[a, b]$ is closed.

  Proof. Consider $\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$. By Proposition 19, $(-\infty, a)$ is open. By Proposition ra.hw5.5b, (b, ∞) is open. By Proposition 22, the union of two open sets is open. Therefore $\mathbb{R} \setminus [a, b]$ is open, so $[a, b]$ is closed. □

- **Proposition 22.** If A and B are open, then $A \cup B$ is open.

  Proof. Since A is open, $\forall\ x_A \in A\ \exists\ (a_A, b_A) \subseteq A \mid x_A \in (a_A, b_A)$. Since B is open, $\forall\ x_B \in B\ \exists\ (a_B, b_B) \subseteq B \mid x_B \in (a_B, b_B)$.

  Let $x \in A \cup B$. If $x \in A$, then per above $\exists\ (a_A, b_A) \subseteq A \subseteq A \cup B \mid x_A \in (a_A, b_A)$. If $x \in B$, then per above $\exists\ (a_B, b_B) \subseteq B \subseteq A \cup B \mid x_B \in (a_B, b_B)$. Therefore $A \cup B$ is open. □

- **Proposition 23.** If A and B are open, then $A \cap B$ is open.

  Proof. Since A is open, $\forall\ x_A \in A\ \exists\ (a_A, b_A) \subseteq A \mid x_A \in (a_A, b_A)$. Since B is open, $\forall\ x_B \in B\ \exists\ (a_B, b_B) \subseteq B \mid x_B \in (a_B, b_B)$.

  Let $x \in A \cap B$. Then $x \in A$ and $x \in B$, so $\exists\ (a_A, b_A) \subseteq A \mid x_A \in (a_A, b_A)$, and $\exists\ (a_B, b_B) \subseteq B \mid x_B \in (a_B, b_B)$. Let $a = \max\{a_A, a_B\}$, and $b = \min\{b_A, b_B\}$. Since $x > a$ and $x < b$, $x \in (a, b)$. Since $(a, b) \subseteq (a_A, b_A) \subseteq A$ and $(a, b) \subseteq (a_B, b_B) \subseteq B$, $(a, b) \subseteq A \cap B$. Therefore $A \cap B$ is open. □

- **Proposition 24.** If A and B are closed, then $A \cup B$ is closed.

  Proof. Since A is closed, $\mathbb{R} \setminus A$ is open. Since B is closed, $\mathbb{R} \setminus B$ is open. $\mathbb{R} \setminus (A \cup B) = (\mathbb{R} \setminus A) \cap (\mathbb{R} \setminus B)$. By Proposition 23 the intersection of two open sets is open. Therefore $\mathbb{R} \setminus (A \cup B)$ is open, so $A \cup B$ is closed.

  □

- **Proposition 25.** If A and B are closed, then $A \cap B$ is closed.

  Proof. Since A is closed, $\mathbb{R} \setminus A$ is open. Since B is closed, $\mathbb{R} \setminus B$ is open. $\mathbb{R} \setminus (A \cap B) = (\mathbb{R} \setminus A) \cup (\mathbb{R} \setminus B)$. By Proposition 22, the union of two open sets is open. Therefore $\mathbb{R} \setminus (A \cap B)$ is open, so $A \cap B$ is closed.

  □

- **Proposition 26.** $\mathbb{R}$ is open and closed.

  Proof. Let $\epsilon > 0$. Let $x \in \mathbb{R}$. Then $x - \epsilon, x + \epsilon \in \mathbb{R}$, and $x \in (x - \epsilon, x + \epsilon)$. Therefore $\mathbb{R}$ is open. $\mathbb{R}$ is closed because by Proposition 27, $\mathbb{R} \setminus \mathbb{R} = \emptyset$ is open. □

- **Proposition 27.** $\emptyset$ is open and closed.

  Proof. To show that a set S is open, we most show that $\forall\ x \in S\ \exists\ S' \subseteq S \mid x \in S'$ where S' is open. Since there are no $x \in \emptyset$, this condition is satisfied for $\emptyset$. $\emptyset$ is closed because per Proposition 26, $\mathbb{R} \setminus \emptyset = \mathbb{R}$ is open.

  □

Proposition 28. Let $x_1, x_2, \ldots, x_n$ be real numbers. Let S be the finite set $S = \{x_1, x_2, \ldots, x_n\}$. Then S is closed.

Proof. Consider $\mathbb{R} \setminus S = (-\infty, x_1) \cup (x_1, x_2) \cup \ldots \cup (x_{n-1}, x_n) \cup (x_n, \infty)$. $(-\infty, x_1)$ is open by Proposition 19. (x_n, ∞) is open by Proposition 18.

Consider (x_i, x_{i+1}) where $i \in \{1, 2, 3, \ldots, n-1\}$. Let $x \in (x_i, x_{i+1})$. Then since $x > x_i$ and $x < x_{i+1}$, $\exists\, \epsilon > 0 \mid x_i + \epsilon = x$ and $\exists\, \delta > 0 \mid x_{i+1} - \delta = x$. Then $x_i = x - \epsilon < x - \frac{\epsilon}{2} < x < x + \frac{\delta}{2} < x + \delta = x_{i+1}$. Therefore $x \in (x - \epsilon, x + \delta) \subseteq (x_i, x_{i+1})$, so (x_i, x_{i+1}) is open.

Finally, since $\mathbb{R} \setminus S$, by Proposition 18 (and induction) $\mathbb{R} \setminus S$ is open. Therefore S is closed. □

Proposition 29. Let $x_1, x_2, \ldots, x_n$ be real numbers. Let S be the finite set $S = \{x_1, x_2, \ldots, x_n\}$. Then S has no limit points.

Proof. Per Definition 1.7, we seek to show that (1) $\forall\, x_i \in S\ \exists\, \delta_i$ such that $\forall\, x_j \in D (x_j \neq x_i)$

$$|x_j - x_i| \geq \delta_i$$

and (2) $\forall\, x \in \mathbb{R} \setminus S\ \exists\, \delta_x$ such that $\forall\, x_i \in D$

$$|x_i - x| \geq \delta_x$$

(1) Let $x_i \in S$. Let $\delta_i = \frac{1}{2} \min\{|x_i - x_k| \mid x_k \neq x_i\}$. Then $\forall\, x_j \neq x_i \in S$,

$$|x_i - x_j| \geq |x_i - x_k| > \delta_i$$

(2) Let $x \in \mathbb{R} \setminus S$. Let $\delta_x = \frac{1}{2} \min\{|x - x_i| \mid x_i \in S\}$. Then

$$|x_i - x| \geq \min\{|x - x_i| \mid x_i \in S\} > \delta_x$$

□

Definition 1.14. Let $S \subseteq \mathbb{R}$. An **open cover** of S is a collection $X = \{\mathcal{O}_\alpha \mid \alpha \in I\}$ where each set $\mathcal{O}_\alpha$ is an open subset of $\mathbb{R}$ such that

$$S \subseteq \bigcup_{\alpha \in I} \mathcal{O}_\alpha$$

(Here I is some set that indexes the $\mathcal{O}_\alpha$).

Definition 1.15. If $X' \subseteq X$ such that

$$S \subseteq \bigcup_{\mathcal{O}_\alpha \in X'} \mathcal{O}_\alpha$$

then X' is called a **subcover** of S contained in X. In addition, if X' is finite then we call X' a **finite subcover** of S contained in X.

Definition 1.16. (Compactness.) Let $S \subseteq \mathbb{R}$. We say that S is **compact** if every open cover of S contains a finite subcover.

Definition 1.17. Let $S \subseteq \mathbb{R}$. We say that S is **bounded** if $\exists\, M > 0$ where $S \subseteq [-M, M]$.

Remark. S is bounded if and only if $|s| \leq M\ \forall\, s \in S$.

Theorem 30. (Heine-Borel Theorem.) Let $S \subseteq \mathbb{R}$. S is compact if and only if S is closed and bounded.

Proposition 31. Let $x_1, x_2, \ldots, x_n$ be real numbers. Let S be the finite set $S = \{x_1, x_2, \ldots, x_n\}$. Then S is compact.

Proof. Let $\{O_\alpha\}$ be an open cover of S. By definition of open cover, $\forall\ i\ \exists\ O_{\alpha_i}$ such that $x_i \in O_{\alpha_i}$. Thus, $\{O_{\alpha_1}, O_{\alpha_2}, \ldots, O_{\alpha_n}\}$ is a finite subcover of S. □

Proposition 32. Let A and B be compact subsets of $\mathbb{R}$. Then $A \cap B$ is compact.

Proof. Since $A \cap B \subseteq A$, $A \cap B \subseteq [-M_A, M_A]$. Therefore $A \cap B$ is bounded.

Since $A \cap B$ is closed and bounded, by the Heine-Borel Theorem (Theorem ra.heine-borel.thm), $A \cap B$ is compact.

□

Proposition 33. Let A and B be compact subsets of $\mathbb{R}$. Then $A \cup B$ is compact.

Proof. Let $M = \max\{M_A, M_B\}$. Note that $[-M_A, M_A] \subseteq [M, M]$ and $[-M_B, M_B] \subseteq [-M, M]$. This implies $A \subseteq [-M, M]$ and $B \subseteq [-M, M]$. Therefore $A \cup B \subseteq [-M, M]$.

Since $A \cup B$ is closed and bounded, by the Heine-Borel Theorem (Theorem ra.heine-borel.thm), $A \cup B$ is compact.

□

Theorem 34. Let $f : D \to \mathbb{R}$ be continuous on D. If $X \subseteq D$ and X is compact (closed and bounded), then

$$f(\bar{x}) = \{f(x) \mid x \in X\}$$

is compact (closed and bounded).

Corollary 34.1. Suppose $f : D \to \mathbb{R}$ where D is closed and bounded. Then there exists $a, b \in D$ where $f(a)$ is the min of f on D and $f(b)$ is the max of f on D.

1.3.2 Homework 6

Definition 1.18. (Uniform Continuity.) Let $D \subseteq \mathbb{R}$ and let $f : D \to \mathbb{R}$. We say that f is **uniformly continuous** on D if $\forall\ \epsilon > 0\ \exists\ \delta > 0$ where

$$x, y \in D \text{ and } 0 < |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Theorem 35. (Uniform continuity implies continuity.) Suppose $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$. If f is uniformly continuous on D, then f is continuous at every $a \in D$.

1.4 More Theorems

Theorem 36. Fubini's Theorem. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that $\int\int_{\mathbb{R}^2} |h(x,y)|dxdy < \infty$. Then

$$\int\int_{\mathbb{R}^2} h(x,y)dxdy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(x,y)dx \right) dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(x,y)dy \right) dx$$

1.5 Problems from Practice Math GRE Subject Tests

38. Let A and B be nonempty subsets of $\mathbb{R}$ and let $f : A \to B$ be a function. If $C \subseteq A$ and $D \subseteq B$, which of the following must be true?

(A) $C \subseteq f^{-1}\left(f(C)\right)$

(B) $D \subseteq f\left(f^{-1}(D)\right)$

(C) $f^{-1}\left(f(C)\right) \subseteq C$

Solution 38. (A) Neither of the equalities should hold – these are in fact nonsense statements, as one side lies in A and the other in B. To unravel the remaining two sets,

$$f^{-1}(f(C)) = \{x \in A : f(x) \in f(C)\}, \quad f(f^{-1}(D)) = f(\{y \in A : f(y) \in D\})$$

Clearly the second set must always be contained in D, but not the other way around. Similarly the first set certainly contains all $c \in C$ (as $f(c) \in f(C)$) but not the other way around.

47. The function $f : \mathbb{R} \to \mathbb{R}$ is defined as follows.

$$f(x) = \begin{cases} 3x^2 & \text{if } x \in \mathbb{Q} \\ -5x^2 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Which of the following is true?

(A) f is discontinuous at all $x \in \mathbb{R}$.

(B) f is continuous only at $x = 0$ and differentiable only at $x = 0$.

(C) f is continuous only at $x = 0$ and nondifferentiable at all $x \in \mathbb{R}$.

(D) f is continuous at all $x \in \mathbb{Q}$ and nondifferentiable at all $x \in \mathbb{R}$.

(E) f is continuous at all $x \notin \mathbb{Q}$ and nondifferentiable at all $x \in \mathbb{R}$.

Solution 47. (B) A classic kind of problem. We are clearly continuous and differentiable at 0. Anywhere else, near a rational number there is an irrational number and vice versa. Therefore there can be no continuity anywhere but at 0, and hence no differentiability either.

57. For each positive integer n, let x_n be a real number in the open interval $\left(0, \frac{1}{n}\right)$. Which of the following statements must be true?

I. $\lim_{n\to\infty} x_n = 0$

II. If f is a continuous real-valued function defined on $(0, 1)$, then $\{f(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence.

III. If g is a uniformly continuous real-valued function defined on $(0, 1)$, then $\lim_{n\to\infty} g(x_n)$ exists.

(A) I only (B) I and II only (C) I and III only (D) II and III only (E) I, II, and III

Solution 57. (C) I is true, since $\lim_{n\to\infty} x_n$ must be bounded between 0 and $\lim_{n\to\infty} 1/n = 0$. Unfortunately, x_n does not converge inside $(0, 1)$. There is no reason therefore that $f(x_n)$ should be a convergent sequence – suppose that $f(x) = 1/x$, so that $f(x_n)$ is certainly not Cauchy. However, if g is uniformly continuous, then g extends to a continuous function on $[0, 1]$. Now x_n is a convergent sequence, so $\lim_{n\to\infty} g(x_n) = g(\lim_{n\to\infty} x_n) = g(0)$ exists.

60. A real-valued function f defined on $\mathbb{R}$ has the following property.

For every positive number ϵ, there exists a positive number δ such that
$|f(x) - f(1)| \geq \epsilon$ whenever $|x - 1| \geq \delta$.

This property is equivalent to which of the following statements about f?

(A) f is continuous at $x = 1$.

(B) f is discontinuous at $x = 1$.

(C) f is unbounded.

(D) $\lim_{|x|\to\infty} |f(x)| = \infty$

(E) $\int_0^{\infty} |f(x)|\,dx = \infty$

Solution 60. (D) While it looks like this is the opposite of continuity, that should read 'there exists $\varepsilon > 0$'. What the statement says is that we not only get arbitrarily far away from $f(1)$, but we must for all x sufficiently far away from 1. So as $|x|$ gets very large, so does $|f(x)|$.

63. For any nonempty sets A and B of real numbers, let $A \cdot B$ be the set defined by

$$A \cdot B = \{xy : x \in A \text{ and } y \in B\}.$$

If A and B are nonempty bounded sets of real numbers and if $\sup(A) > \sup(B)$, then $\sup(A \cdot B) =$

(A) $\sup(A)\sup(B)$

(B) $\sup(A)\inf(B)$

(C) $\max\{\sup(A)\sup(B),\ \inf(A)\inf(B)\}$

(D) $\max\{\sup(A)\sup(B),\ \sup(A)\inf(B)\}$

(E) $\max\{\sup(A)\sup(B),\ \inf(A)\sup(B),\ \inf(A)\inf(B)\}$

Solution 63. (E) The supremum is either going to be the product of the two largest positive numbers in A and B or the product of the two smallest negative numbers in A and B. That means we should look for $\sup \cdot \sup$ or $\inf \cdot \inf$. However, it might be the case that B contains only negative numbers and A contains only positive numbers. Then the largest value in $A \cdot B$ will be attained by the smallest positive element of A and the largest negative element of B, giving us our third option: $\inf A \cdot \sup B$.