Math Review Notes

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1 Introduction

These are notes I've collected on various math topics. I originally created this document to prepare for the GRE Math Subject test. Since then I've expanded it as I've reviewed concepts from past classes and reinforced concepts from new classes. This document is very much a work in progress, with many typos, omissions to be filled in, and probably errors. Nonetheless, I share this document in case it's useful to anyone else as a reference.

I use many sources throughout this document, which I either cite at the beginning of the section (for sources I use broadly) or as I use them (for sources I use for one or two isolated results).

2 Linear Algebra

These are my notes from taking EE 588 at USC, Math 541A at USC, and various other sources which I mostly cite within the text.

2.1 Properties of Projection Matrices

i. Formula:

$$P = A(A^T A)^{-1} A^T$$

(Note that if A is an invertible (square) matrix, then $P = A(A^T A)^{-1} A^T = AA^{-1}(A^T)^{-1} A^T = I$.)

The projection matrix projects any vector b into the column space of A. In other words, $p = Pb$ is the component of b in the column space, and the error $e = b - Pb$ is the component in the orthogonal complement. ($I - P$ is also a projection matrix. It projects b onto the orthogonal complement, and the projection is $b - Pb = e$).

(Note that if A is an invertible (square) matrix, then its column space is all of $\mathbb{R}^n$, so b is already in the column space of A.)

ii. The projection matrix is **idempotent**: it equals its square–$P^2 = P$.

iii. The projection matrix is **symmetric**: it equals its transpose–$P^T = P$.

iv. Conversely, **any symmetric idempotent matrix represents a projection**. P is unique for a given subspace.

v. If A is an $m \times n$ matrix with rank n, then $\text{rank}(P) = n$. The eigenvalues of P consist of n ones and $m - n$ zeroes. P always contains n independent eigenvectors and is thus diagonalizable.

Suppose A is a square nonsingular matrix and λ is an eigenvalue of A. Then λ^{-1} is an eigenvalue of the matrix A^{-1}.

The trace of an idempotent matrix with rank r is r.

2.2 Eigenvalues, Eigenvectors, Diagonalization, Symmetric Matrices

Notes on Diagonalization

Suppose the $n \times n$ matrix A has n linearly independent eigenvectors. If these eigenvectors are the columns of a matrix S, then $S^{-1}AS$ is a diagonal matrix Λ. The eigenvalues of A are on the diagonal of Λ:

$$S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

We call S the **eigenvector matrix** and Λ the **eigenvalue matrix**.

1. If the matrix A has no repeated eigenvalues, then its n eigenvectors are automatically independent. Therefore **any matrix with n distinct eigenvalues can be diagonalized**.

2. **The diagonalizing matrix S is not unique**. An eigenvector x can be multiplied by a constant and remains an eigenvector. We can multiply the columns of S by any nonzero constants and produce a new diagonalizing S. Repeated eigenvalues leave even more freedom in S (columns with identical eigenvalues can be interchanged).

   (Note that for the trivial example $A = I$, any invertible S will do. $S^{-1}IS$ is always diagonal, and Λ is just I. **All vectors are eigenvectors of the identity.**)

3. **Other matrices S will not produce a diagonal** Λ. Since $\Lambda = S^{-1}AS$, S must satisfy $S\Lambda = AS$. Suppose the first column of S is y. Then the first column of $S\Lambda$ is $\lambda_1 y$. If this is to agree with the first column of AS, which by matrix multiplication is Ay, then y must be an eigenvector: $Ay = \lambda_1 y$.

   (Note that the *order* of the eigenvectors in S and the eigenvalues in Λ must match.)

4. Not all matrices posses n linearly independent eigenvectors, so **not all matrices are diagonalizable**.

   Diagonalizability of A depends on having enough (n) independent eigenvectors. Invertibility of A depends on having nonzero eigenvalues.

   There is no connection between diagonalizability (n independent eigenvectors) and invertibility (no zero eigenvalues). The only indication given by the eigenvalues is that diagonalization can fail only if there are repeated eigenvalues. (But even then, it does not always fail–e.g. I.)

   The test is to check, for an eigenvalue that is repeated p times, whether there are p independent eigenvectors– in other words, whether $A - \lambda$ has rank $n - p$.

5. **Projection matrices always contain n independent eigenvectors and thus are always diagonalizable**.

Eigenvalues of Symmetric Matrices: If A is symmetric, then it has the following properties:

1. A has exactly n (not necessarily distinct) eigenvalues

2. There exists a set of n eigenvectors, one for each eigenvalue, that are mutually orthogonal (even if the eigenvalues are not distinct).

Eigenvalues of the Inverse of a Matrix: Suppose A is a square nonsingular matrix and λ is an eigenvalue of A. Then λ^{-1} is an eigenvalue of the matrix A^{-1}. Proof: Note that since A is nonsingular, A^{-1} exists and λ is nonnegative for all eigenvalues of A. Let λ be an eigenvalue of A and let $x \neq 0$ be an eigenvector of A for λ. Suppose A is n by n. Then we have

$$A^{-1}x = A^{-1}\lambda^{-1}\lambda x = \lambda^{-1}A^{-1}\lambda x = \lambda^{-1}A^{-1}Ax = \lambda^{-1}x$$

The inverse of a symmetric matrix is symmetric. Proof: Let A be a symmetric matrix.

$$I = I'$$

$$AA^{-1} = (AA^{-1})'$$

$$A^{-1}A = (A^{-1})'A'$$

$$A^{-1}AA^{-1} = (A^{-1})'AA^{-1}$$

$$A^{-1} = (A^{-1})'$$

2.3 Positive Definite Matrices

For any real invertible matrix A, the product $A'A$ is a positive definite matrix. (Proof: Let z be a non-zero vector. We want $z'A'Az > 0 \forall z$. Note that $z'A'Az = (Az)'(Az)$. Because A is invertible and $z \neq 0$, $Az \neq 0$, so $(Az)'(Az) > 0$.)

Let $A \in \mathbb{R}^{m \times n}$ with $m \geq n$ and let $\text{rank}(A) = n$ (that is, A has full column rank). Then $A'A$ is a positive definite matrix. (Proof: Let z be a non-zero vector. We want $z'A'Az > 0 \forall z$. Note that $z'A'Az = (Az)'(Az)$. Because A has full column rank (and n linearly independent columns) and $z \neq 0$, $Az \neq 0$, so $(Az)'(Az) > 0$.)

Every positive definite matrix is invertible and its inverse is also positive definite.

2.4 Matrix Decompositions

Schur complement, Schur decomposition: For information, see Section 12.2.

QR decompositon

Orthogonal Decomposition

Spectral Decomposition (eigenvalue decomposition

Generalized eigenvalue decomposition

Singular value decomposition and Pseudo-inverse

Jordan decomposition

Cholesky decomposition

2.5 Inverting Matrices

Theorem 2.1 (**Woodbury Matrix Identity** (or **Sherman-Morrison-Woodbury formula**))**.** For $A \in \mathbb{R}^{n \times n}$, $U \in \mathbb{R}^{n \times k}$, $C \in \mathbb{R}^{k \times k}$, and $V \in \mathbb{R}^{v \times n}$,

$$(A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}.$$

Theorem 2.2 (**Binomial Inverse Thereom**)**.**

2.6 Other

Frobenius norm

From appendix of Time Series:

Quadratic forms

Special matrices

Difference Equations

2.7 Practice Problems

[**The Power Method**] This exercise gives an algorithm for finding the eigenvectors and eigenvalues of a symmetric matrix. In modern statistics, this is often a useful thing to do. The Power Method described below is not the best algorithm for this task, but it is perhaps the easiest to describe and analyze.

Let A be an $n \times n$ real symmetric matrix. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the (unknown) eigenvalues of A, and let $v_1, \ldots, v_n \in \mathbb{R}^n$ be the corresponding (unknown) eigenvectors of A such that $|v_i| = 1$ and such that $Av_i = \lambda_i v_i$ for all $1 \leq i \leq n$.

Given A, our first goal is to find v_1 and λ_1. For simplicity, assume that $1/2 < \lambda_1 < 1$, and $0 \leq \lambda_n \leq \cdots \leq \lambda_2 < 1/4$. Suppose we have found a vector $v \in \mathbb{R}^n$ such that $|v| = 1$ and $|\langle v, v_1 \rangle| > 1/n$. Let k be a positive integer. Show that

$$A^k v$$

approximates v_1 well as k becomes large. More specifically, show that for all $k \geq 1$,

$$|A^k v - \langle v, v_1 \rangle \lambda_1^k v_1|^2 \leq \frac{n-1}{16^k}.$$

(Hint: use the spectral theorem for symmetric matrices.)

Solution. Since the eigenvectors for A are orthogonal, they form a basis for $\mathbb{R}^n$, so for any $v \in \mathbb{R}^n$ we have $v = \sum_{i=1}^n c_i v_i$ for some $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$. It also follows then that $\langle v, v_1 \rangle = \langle \sum_{i=1}^n c_i v_i, v_1 \rangle = c_1 v_1' v_1 = c_1$. And finally, since $\|v\| = 1$ and $\|v_i\| = 1$ for all i, clearly we have $-1 \leq c_i \leq 1$. Using these facts, we have

$$\|A^k v - \langle v, v_1 \rangle \lambda_1^k v_1\|^2 = \|\sum_{i=1}^n \lambda_i^k c_i v_i - \langle v, v_1 \rangle \lambda_1^k v_1\|^2 = \|\sum_{i=1}^n \lambda_i^k c_i v_i - \lambda_1^k c_1 v_1\|^2 = \|\sum_{i=2}^n \lambda_i^k c_i v_i\|^2$$

$$= \sum_{i=2}^n \lambda_i^{2k} c_i^2 v_i' v_i = \sum_{i=2}^n \lambda_i^{2k} c_i^2$$

Since by assumption $0 \leq \lambda_n \leq \ldots \leq \lambda_2 \leq 1/4$, $\lambda_i^{2k} \leq 1/16^k$ for all i, so we have

$$\|A^k v - \langle v, v_1 \rangle \lambda_1^k v_1\|^2 \leq \frac{1}{16^k} \sum_{i=2}^n c_i^2$$

Since $-1 \leq c_i \leq 1 \implies 0 \leq c_i^2 \leq 1$, we have $\sum_{i=2}^n c_i^2 \leq n-1$, so this can be written as

$$\boxed{\|A^k v - \langle v, v_1\rangle \lambda_1^k v_1\|^2 \leq \frac{n-1}{16^k}}$$

Remark. Since $|\langle v, v_1\rangle|\lambda_1^k > 2^{-k}/n$, this inequality implies that $A^k v$ is approximately an eigenvector of A with eigenvalue λ_1. That is, by the triangle inequality,

$$|A(A^k v) - \lambda_1(A^k v)| \leq |A^{k+1}v - \langle v, v_1\rangle \lambda_1^{k+1} v_1| + \lambda_1 |\langle v, v_1\rangle \lambda_1^k v_1 - A^k v| \leq 2\frac{\sqrt{n-1}}{4^k}.$$

Moreover, by the reverse triangle inequality,

$$|A^k v| = |A^k v - \langle v, v_1\rangle \lambda_1^k v_1 + \langle v, v_1\rangle \lambda_1^k v_1| \geq \frac{1}{n}2^{-k} - \frac{\sqrt{n-1}}{4^k}.$$

If we take k to be large (say $k > 10 \log n$), and if we define $z : equals A^k v$, then z is approximately an eigenvector of A, that is

$$|A\frac{A^k v||A^k v| - \lambda_1 \frac{A^k v}{|A^k v|}|}{\leq} 4n^{3/2}2^{-k} \leq 4n^{-4}.$$

And to approximately find the first eigenvalue λ_1, we simply compute

$$\frac{z^T A z}{z^T z}.$$

That is, we have approximately found the first eigenvector and eigenvalue of A.

To find the second eigenvector and eigenvalue, we can repeat the above procedure, where we start by choosing v such that $\langle v, v_1\rangle = 0$, $|v| = 1$ and $|\langle v, v_2\rangle| > 1/(10\sqrt{n})$. To find the third eigenvector and eigenvalue, we can repeat the above procedure, where we start by choosing v such that $\langle v, v_1\rangle = \langle v, v_2\rangle = 0$, $|v| = 1$ and $|\langle v, v_3\rangle| > 1/(10\sqrt{n})$. And so on.

Google's PageRank algorithm uses the power method to rank websites very rapidly. In particular, they let n be the number of websites on the internet (so that n is roughly 10^9). They then define an $n \times n$ matrix C where $C_{ij} = 1$ if there is a hyperlink between websites i and j, and $C_{ij} = 0$ otherwise. Then, they let B be an $n \times n$ matrix such that B_{ij} is 1 divided by the number of 1's in the i^{th} row of C, if $C_{ij} = 1$, and $B_{ij} = 0$ otherwise. Finally, they define

$$A = (.85)B + (.15)D/n$$

where D is an $n \times n$ matrix all of whose entries are 1.

The power method finds the eigenvector v_1 of A, and the size of the i^{th} entry of v_1 is proportional to the "rank" of website i.

12. Let A be a 2×2 matrix for which there is a constant k such that the sum of the entries in each row and each column is k. Which of the following must be an eigenvector of A ?

    I. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

    II. $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

    III. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

    (A) I only (B) II only (C) III only (D) I and II only (E) I, II, and III

Solution 12. (C) This condition makes the matrix of the form

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

There is no reason that $a = 0$ or $b = 0$, so there is no reason $(1,0)$ or $(0,1)$ should be eigenvectors. But it is easy to verify that $(1,1)$ must be.

24. Consider the system of linear equations

$$\begin{aligned} w + 3x + \ 2y + \ 2z &= 0 \\ w + 4x + \ \ y \qquad\ \ &= 0 \\ 3w + 5x + 10y + 14z &= 0 \\ 2w + 5x + \ 5y + \ 6z &= 0 \end{aligned}$$

with solutions of the form (w, x, y, z), where w, x, y, and z are real. Which of the following statements is FALSE?

(A) The system is consistent.

(B) The system has infinitely many solutions.

(C) The sum of any two solutions is a solution.

(D) $(-5, 1, 1, 0)$ is a solution.

(E) Every solution is a scalar multiple of $(-5, 1, 1, 0)$.

Solution 24. (E) Looking at our answers, we can verify directly that $(-5,1,1,0)$ is a solution. Any multiple of $(-5,1,1,0)$ is also a solution, which shows that (A), (B), (C), and (D) are all true – leaving only (E). Another solution, for example, is $(0,2,-8,5)$

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

34. Which of the following statements about the real matrix shown above is FALSE?

(A) A is invertible.

(B) If $\mathbf{x} \in \mathbb{R}^5$ and $A\mathbf{x} = \mathbf{x}$, then $\mathbf{x} = \mathbf{0}$.

(C) The last row of A^2 is $(0 \quad 0 \quad 0 \quad 0 \quad 25)$.

(D) A can be transformed into the 5×5 identity matrix by a sequence of elementary row operations.

(E) $\det(A) = 120$

Solution 34. (B) An upper triangular matrix is easily verified to be invertible so long as its diagonal entries are all nonzero. Specifically, $\det A$ is still the product of its diagonal entries, so (E) and (D) and (A) are all true. (C) can easily be verified to be true by computing that the bottom-right corner is 25 (the product of upper triangular matrices still being upper triangular). This leaves (B). (B) can be checked directly to be false: if we let $x = (1, 0, 0, 0, 0)$, then $Ax = x$.

37. Let V be a finite-dimensional real vector space and let P be a linear transformation of V such that $P^2 = P$. Which of the following must be true?

I. P is invertible.

II. P is diagonalizable.

III. P is either the identity transformation or the zero transformation.

(A) None (B) I only (C) II only (D) III only (E) II and III

Solution 37. (C) $P^2 = P$ means that P is projection onto some subspace. There is no reason to believe that this should be invertible, but it should definitely be diagonalisable (with eigenbasis some basis of that subspace). III also need not be true if the subspace is anything proper or nontrivial.

50. Let A be a real 2×2 matrix. Which of the following statements must be true?

I. All of the entries of A^2 are nonnegative.

II. The determinant of A^2 is nonnegative.

III. If A has two distinct eigenvalues, then A^2 has two distinct eigenvalues.

(A) I only (B) II only (C) III only (D) II and III only (E) I, II, and III

Solution 50. (B) There is no reason that all the entires of A^2 need to be nonnegative. Its determinant must be nonnegative though: $\det(A^2) = (\det A)^2$. For III, suppose A is the diagonal matrix with entires $\pm\lambda$. Then those are its eigenvalues, and they are distinct so long as $\lambda \neq 0$. But A^2 has only one eigenvalue: λ^2.

51. Which of the following is an <u>orthonormal</u> basis for the column space of the real matrix $\begin{pmatrix} 1 & -1 & 2 & -3 \\ -1 & 1 & -3 & 2 \\ 2 & -2 & 5 & -5 \end{pmatrix}$?

(A) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

(B) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

(C) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix} \right\}$

(D) $\left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} \right\}$

(E) $\left\{ \begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right\}$

Solution 51. (E) The basis (C) is not orthogonal and (D) is not normal, so we can rule those out. We can throw out the first column, since it is the negation of the second. A little bit of math shows that the remaining 3×3 matrix has determinant 0, so the rank of our column space is 2. That leaves only (A) and (E), but (A) cannot be correct. Our column space contains vectors that have nonzero third entry, so cannot lie in the span of that basis.

3 Calculus

These notes include some screenshots from Wikipedia as well as from *Calculus* by Gilbert Strang, available at https://ocw.mit.edu/ans7870/resources/Strang/Edited/Calculus/Calculus.pdf. I also used parts from some other resources which I mention when they arise.

3.1 Differentiation and common derivatives and integrals to know

Theorem 3.1 (**Clairaut's Theorem**)**.** Let $z = f(x, y)$ be a two variable real-valued function that is defined on a disk $\mathcal{D}$ that contains the point (a, b). Then if $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$ are continuous on $\mathcal{D}$, $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x}, \ x > 0$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x}, \ x \neq 0$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left(\log_a(x)\right) = \frac{1}{x \ln a}, \ x > 0$$

$$\int \tan u \, du = \ln|\sec u| + c$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + c$$

$$\int \frac{1}{a^2+u^2} du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + c$$

$$\int \frac{1}{\sqrt{a^2-u^2}} du = \sin^{-1}\left(\frac{u}{a}\right) + c$$

$$\int \ln u \, du = u \ln(u) - u + c$$

$$\int \sinh x \, dx = \cosh x + C \qquad \int \cosh x \, dx = \sinh x + C$$

3.2 Matrix Differentiation

Recommended resource: "Matrix Differentiation (and some other stuff)" by Randal J. Barnes (Department of Civil Engineering, University of Minnesota). Available for download at https://atmos.washington.edu/~dennis/MatrixCalculus.pdf.

More information not contained in that pdf (from the appendix of *Convex Optimization* by Stephen Boyd and Lieven Vandenberghe, available for free download at https://web.stanford.edu/~boyd/cvxbook/):

Chain rule. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in \textbf{int dom} f$ and $g : \mathbb{R}^m \to \mathbb{R}^p$ is differentiable at $f(x) \in \textbf{int dom } g$. Define the composition $h : \mathbb{R}^n \to \mathbb{R}^p$ by $h(z) = g(f(x))$. Then

$$Dh(x) = Dg(f(x))Df(x)$$

In particular, if $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$,

$$\nabla h(x) = g'(f(x))\nabla f(x)$$

Example with an affine function. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, $A \in \mathbb{R}^{n\times p}$, and $b \in \mathbb{R}^n$. Define $g : \mathbb{R}^p \to \mathbb{R}^m$ as $g(x) = f(Ax + b)$ with **dom** $g = \{x \mid Ax + b \in$ **dom** $f\}$. Then

$$\nabla g(x) = A^T \nabla f(Ax + b)$$

Example 2. Consider $f : \mathbb{R}^n \to \mathbb{R}$ where

$$f(x) = \log \sum_{i=1}^{m} \exp(a_i^T x + b_i) =$$

where $a_1, \ldots, a_m \in \mathbb{R}^n$ and $b_1, \ldots, b_m \in \mathbb{R}$. Note that $f(\cdot)$ can be expressed as a composition of $Ax + b$ (where $A \in \mathbb{R}^{m\times n}$ has rows $a_1^T, \ldots, a_m^T$) and the function $g : \mathbb{R}^m \to \mathbb{R}$ given by $g(y) = \log(\sum_{i=1}^m \exp(y_i)$. We have

$$\nabla g(y) = \left[\sum_{i=1}^{m} e^{y_i} \right]^{-1} \big(\exp(y_1) \ldots \exp(y_m) \big)^T$$

so applying the chain rule yields

$$\nabla f(x) = \left[\sum_{i=1}^{m} \exp(a_i^T x + b_i) \right]^{-1} A^T z$$

where $z_i = \exp(a_i^T x + b_i),\ i = 1, \ldots, m$.

Hessians. The Hessian matrix of $f : \mathbb{R}^n \to \mathbb{R}$ is denoted by $\nabla^2 f(x)$ and is given by

$$\nabla^2 f(x)_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j},\ i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n$$

The quadratic function

$$f(x_0) + \nabla f(x_0)^T (x - x_0) + \frac{1}{2}(x - x_0)^T \nabla^2 f(x_0)(x - x_0)$$

is called the **second-order approximation of** f **near** x_0.

Chain rule for second derivative. A chain rule for the second derivative is difficult in general. Here are some special cases.

Composition with scalar function. Suppose $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, and $h(x) = g(f(x))$. We have

$$\nabla^2 h(x) = g'(f(x))\nabla^2 f(x) + g''(f(x))\nabla f(x)\nabla f(x)^T$$

Composition with affine function. Suppose $f : \mathbb{R} \to \mathbb{R}$, $a \in \mathbb{R}^m$, $b \in \mathbb{R}$. Define $g : \mathbb{R}^m \to \mathbb{R}$ by $g(x) = f(a^T x + b)$. Then

$$\nabla^2 g(x) = a^T \nabla^2 f(a^T x + b)a$$

More generally, suppose $f : \mathbb{R}^n \to \mathbb{R}$, $A \in \mathbb{R}^{n\times m}$, and $b \in \mathbb{R}^n$. Define $g : \mathbb{R}^m \to \mathbb{R}$ by $g(x) = f(Ax + b)$. Then

$$\nabla^2 g(x) = A^T \nabla^2 f(Ax + b)A$$

3.3 Some theorems in higher dimensions

Proposition 3.2 (**Change of Variables**)**.** If U is a "nice" subset of $\mathbb{R}^2$ and ϕ is an injective differentiable function on U, then

$$\int_{\phi(U)} f(u, v)dudv = \int_U f(\phi(x, y))|J\phi(x, y)|dxdy$$

where $J\phi(x, y)$ is the Jacobian of ϕ at (x, y).

Taylor's Theorem (first order). (borrowed from https://www.rose-hulman.edu/~bryan/lottamath/mtaylor.pdf) Consider a function $f : \mathbb{R}^n \to \mathbb{R}$. Let $a \in \mathbb{R}^n$ be a fixed point. Then Taylor's Theorem states:

If $f(x)$ is differentiable on an open ball B around a and $x \in B$ then

$$f(x) = f(a) + \nabla f(b)^T (x - a)$$

for some b on the line segment joining a and x.

This can also be expressed as follows. *Let $x, y \in \mathbb{R}^n$. If $f(x)$ is continuously differentiable, then*

$$f(y) = f(x) + \nabla f(tx + (1 - t)y)^T (y - x)$$

for some $t \in [0, 1]$.

Proof. Consider $g(z) = f(zy + (1 - z)x)$. If f is differentiable then so is g. Then by the Mean Value Theorem, for some $t \in (0, 1)$ we have $g(1) - g(0) = g'(t)$. By the chain rule,

$$g'(t) = \nabla f(x + t(y - x))^T (y - x)$$

Using $g(1) = f(y)$ and $g(0) = f(x)$, we have

$$\iff \nabla f(tx + (1 - t)y)^T (y - x) = g(1) - g(0) = f(y) - f(x)$$

$\square$

Taylor's Theorem (second order). Consider a function $f : \mathbb{R}^n \to \mathbb{R}$. Let $a \in \mathbb{R}^n$ be a fixed point. Then Taylor's Theorem states:

If $f(x)$ is twice differentiable on an open ball B around a and $x \in B$ then

$$f(x) = f(a) + (x - a)^T \nabla f(a) + \frac{1}{2}(x - a)^T \nabla^2 f(b)(x - a)$$

for some b on the line segment joining a and x.

This can also be expressed as follows. *Let $x, y \in \mathbb{R}^n$. If $f(x)$ is twice continuously differentiable, then*

$$f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(ty + (1 - t)x)(y - x)$$

for some $t \in [0, 1]$.

Proof. Consider $g(z) = f(zy + (1 - z)x)$. If f is differentiable then so is g. Then by the second order case of Taylor's Theorem in one dimension, for some $t \in (0, 1)$ we have $g(1) = g(0) + g'(0) + (1/2)g''(t)$. By the chain rule,

$$g''(t) = \frac{\partial}{\partial t} \nabla f(x + t(y - x))^T (y - x) = (y - x)^T \nabla^2 f(x + t(y - x))^T (y - x)$$

Using this result along with $g(1) = f(y)$, $g(0) = f(x)$, and $g'(0) = \nabla f(x)^T (y - x)$, we have

$$f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(x + t(y - x))^T (y - x)$$

$\square$

3.4 Optimizing functions of several variables

Functions of two variables [edit]

Suppose that $f(x, y)$ is a differentiable real function of two variables whose second partial derivatives exist. The Hessian matrix H of f is the 2×2 matrix of partial derivatives of f:

$$H(x,y) = \begin{pmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{pmatrix}.$$

Define $D(x, y)$ to be the determinant

$$D(x,y) = \det(H(x,y)) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2 ,$$

of H. Finally, suppose that (a, b) is a critical point of f (that is, $f_x(a, b) = f_y(a, b) = 0$). Then the second partial derivative test asserts the following:[1]

1. If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$ then (a,b) is a local minimum of f.
2. If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$ then (a,b) is a local maximum of f.
3. If $D(a,b) < 0$ then (a,b) is a saddle point of f.
4. If $D(a,b) = 0$ then the second derivative test is inconclusive, and the point (a, b) could be any of a minimum, maximum or saddle point.

Functions of many variables [edit]

For a function f of two or more variables, there is a generalization of the rule above. In this context, instead of examining the determinant of the Hessian matrix, one must look at the eigenvalues of the Hessian matrix at the critical point. The following test can be applied at any critical point $(a, b, ...)$ for which the Hessian matrix is invertible:

1. If the Hessian is positive definite (equivalently, has all eigenvalues positive) at $(a, b, ...)$, then f attains a local minimum at $(a, b, ...)$.
2. If the Hessian is negative definite (equivalently, has all eigenvalues negative) at $(a, b, ...)$, then f attains a local maximum at $(a, b, ...)$.
3. If the Hessian has both positive and negative eigenvalues then $(a, b, ...)$ is a saddle point for f (and in fact this is true even if $(a, b, ...)$ is degenerate).

3.5 Lagrange Multipliers

: to flesh out! http://tutorial.math.lamar.edu/Classes/CalcIII/LagrangeMultipliers.aspx

3.6 Line Integrals

(p. 555 of Strang book)

Suppose a force in two-dimensional space is given by $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j}$. Then the work done by this force on a particle moving along a curve C is given by

$$W = \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{R} = \int_C M\mathrm{d}x + N\mathrm{d}y$$

Along a curve in three-dimensional space the work done by a three-dimensional force $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j} + P\boldsymbol{j}$ is given by

$$W = \int_C \boldsymbol{F} \cdot \boldsymbol{T}\mathrm{d}s = \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{R} = \int_C M\mathrm{d}x + N\mathrm{d}y + P\mathrm{d}z$$

where the tangent vector $\boldsymbol{T}$ is given by

$$\boldsymbol{T} = \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}s}$$

Green's Theorem: Suppose the region R is bounded by the simple closed piecewise smooth curve C. Then an integral over R equals a line integral around C:

$$\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{R} = \oint_C M\mathrm{d}x + N\mathrm{d}y = \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathrm{d}x \ \mathrm{d}y$$

Line integrals chapter! http://tutorial.math.lamar.edu/Classes/CalcIII/LineIntegralsIntro.aspx

Surface inegrals chapter! http://tutorial.math.lamar.edu/Classes/CalcIII/SurfaceIntegralsIntro.aspx

3.7 Miscellaneous

13A The tangent plane at (x_0, y_0, z_0) has the same slopes as the surface $z = f(x, y)$. The equation of the tangent plane (a linear equation) is

$$z - z_0 = \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0). \qquad (1)$$

The normal vector $\mathbf{N}$ to that plane has components $(\partial f/\partial x)_0$, $(\partial f/\partial y)_0$, -1.

13B The tangent plane to the surface $F(x, y, z) = c$ has the linear equation

$$\left(\frac{\partial F}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial F}{\partial y}\right)_0 (y - y_0) + \left(\frac{\partial F}{\partial z}\right)_0 (z - z_0) = 0. \qquad (7)$$

The normal vector is $\mathbf{N} = \left(\frac{\partial F}{\partial x}\right)_0 \mathbf{i} + \left(\frac{\partial F}{\partial y}\right)_0 \mathbf{j} + \left(\frac{\partial F}{\partial z}\right)_0 \mathbf{k}$.

$$dz = (\partial z/\partial x)_0 dx + (\partial z/\partial y)_0 dy \quad \text{or} \quad df = f_x dx + f_y dy. \qquad (10)$$

This is the ***total differential***. All letters dz and df and dw can be used, but ∂z and ∂f are not used. Differentials suggest small movements in x and y; then dz is the resulting movement in z. On the tangent plane, equation (10) holds exactly.

The **directional derivative**, denoted $D_v f(x,y)$, is a derivative of a multivariable function in the direction of a vector $\boldsymbol{v}$. It is the scalar projection of the gradient onto $\boldsymbol{v}$.

$$D_v f(x,y) = \text{comp}_v \nabla f(x,y) = \frac{\nabla f(x,y) \cdot \boldsymbol{v}}{|\boldsymbol{v}|}$$

3.8 Practice Problems

13F The directional derivative is $D_{\mathbf{u}} f = (\text{grad } f) \cdot \mathbf{u}$. The level direction is perpendicular to grad f, since $D_{\mathbf{u}} f = 0$. ***The slope $D_{\mathbf{u}} f$ is largest when u is parallel to*** grad f. That maximum slope is the length $|\text{grad } f| = \sqrt{f_x^2 + f_y^2}$:

$$\textit{for} \quad \mathbf{u} = \frac{\text{grad } f}{|\text{grad } f|} \quad \textit{the slope is} \quad (\text{grad } f) \cdot \mathbf{u} = \frac{|\text{grad } f|^2}{|\text{grad } f|} = |\text{grad } f|.$$

$$\int_C g(x,y)\, ds = \text{limit of} \sum_{i=1}^{N} g(x_i, y_i) \Delta s_i \quad \text{as} \quad (\Delta s)_{\max} \to 0.$$

The differential ds becomes $(ds/dt)dt$. Everything changes over to t:

$$\int g(x,y) ds = \int_{t=a}^{t=b} g(x(t), y(t)) \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt.$$

19. Let f and g be twice-differentiable real-valued functions defined on $\mathbb{R}$. If $f'(x) > g'(x)$ for all $x > 0$, which of the following inequalities must be true for all $x > 0$?

(A) $f(x) > g(x)$

(B) $f''(x) > g''(x)$

(C) $f(x) - f(0) > g(x) - g(0)$

(D) $f'(x) - f'(0) > g'(x) - g'(0)$

(E) $f''(x) - f''(0) > g''(x) - g''(0)$

Solution 19. (C) There is no reason that $f(x) > g(x)$, or that $f''(x) > g''(x)$. But we do know that

$$\int_0^x f'(t)\, dt > \int_0^x g'(t)\, dt \implies f(x) - f(0) > g(x) - g(0).$$

This is precisely an answer.

22. What is the volume of the solid in xyz-space bounded by the surfaces $y = x^2$, $y = 2 - x^2$, $z = 0$, and $z = y + 3$?

(A) $\frac{8}{3}$ (B) $\frac{16}{3}$ (C) $\frac{32}{3}$ (D) $\frac{104}{105}$ (E) $\frac{208}{105}$

Solution 22. (C) It looks like our x-coordinates are running over $[-1, 1]$, with y depending on x and z depending on y. To find the volume of the solid, we just need to integrate the constant function 1. We must therefore compute

$$\begin{aligned}
\int_{-1}^{1}\int_{x^2}^{2-x^2}\int_{0}^{y+3} 1\,dz\,dy\,dx &= \int_{-1}^{1}\int_{x^2}^{2-x^2} y+3\,dy\,dx \\
&= \int_{-1}^{1} \left((2-x^2)^2/2 + 3(2-x^2)\right) - \left((x^2)^2/2 + 3(x^2)\right)\,dx \\
&= \int_{-1}^{1} 8 - 8x^2\,dx \\
&= 8x - 8x^3/3\Big|_{-1}^{1} = (8 - 8/3) - (-8 + 8/3) = 32/3.
\end{aligned}$$

24. Let h be the function defined by $h(x) = \int_0^{x^2} e^{x+t}\,dt$ for all real numbers x. Then $h'(1) =$

(A) $e - 1$ (B) e^2 (C) $e^2 - e$ (D) $2e^2$ (E) $3e^2 - e$

Solution 24. (E) We can actually just integrate this, and not worry about differentiation under the integral.

$$\int_0^{x^2} e^{x+t}\,dt = e^x \int_0^{x^2} e^t\,dt = e^x(e^{x^2} - 1) = e^{x^2+x} - e^x.$$

Then deriving that,

$$h'(x) = (2x+1)e^{x^2+x} - e^x,$$

whence our result follows immediately.

26. Let $f(x, y) = x^2 - 2xy + y^3$ for all real x and y. Which of the following is true?

(A) f has all of its relative extrema on the line $x = y$.

(B) f has all of its relative extrema on the parabola $x = y^2$.

(C) f has a relative minimum at $(0, 0)$.

(D) f has an absolute minimum at $\left(\frac{2}{3}, \frac{2}{3}\right)$.

(E) f has an absolute minimum at $(1, 1)$.

Solution 26. (A) We are concerned about its extrema, we should find some partial derivatives.

$$f_x = 2x - 2y, \quad f_y = -2x + 3y^2.$$

We would like to know when they are both zero. The first equation gives us $x = y$ and the second gives us $2x = 3y^2$, so that

$$2y = 3y^2 \implies (3y-2)y = 0 \implies y = 0, 2/3.$$

Therefore our solutions are $(0,0)$ and $(2/3, 2/3)$. Indeed, our relative extrema are all on the line $x = y$. To do some more checking (which you should not do on the actual test),

$$f_{xx} = 2, \quad f_{yy} = 6y, \quad f_{xy} = f_{yx} = -2.$$

Then the determinant of the Hessian is $12y - 4$. This shows that $(0,0)$ is a saddle point. There is no reason that $(2/3, 2/3)$ is an absolute minimum without further verification, and $(1,1)$ needn't be an extreme point.

27. Consider the two planes $x + 3y - 2z = 7$ and $2x + y - 3z = 0$ in $\mathbb{R}^3$. Which of the following sets is the intersection of these planes?

(A) $\varnothing$

(B) $\{(0, 3, 1)\}$

(C) $\{(x, y, z) : x = t,\ y = 3t,\ z = 7 - 2t,\ t \in \mathbb{R}\}$

(D) $\{(x, y, z) : x = 7t,\ y = 3 + t,\ z = 1 + 5t,\ t \in \mathbb{R}\}$

(E) $\{(x, y, z) : x - 2y - z = -7\}$

Solution 27. (D) First, we know that the intersection of two planes in $\mathbb{R}^3$ should be either a plane or a line. In our case, the two planes are definitely not the same, so we will obtain a line. The slope of the line can be found by taking the cross product of the normal vectors of the two planes in question.

$$(1,3,-2) \times (2,1,-3) = \det \begin{bmatrix} i & j & k \\ 1 & 3 & -2 \\ 2 & 1 & -3 \end{bmatrix} = (-7,-1,-5).$$

The only solution corresponding to this slope is (D), as the coefficients of t in (x,y,z) are $(7,1,5)$.

32. $\dfrac{d}{dx}\displaystyle\int_{x^3}^{x^4} e^{t^2}\,dt =$

(A) $e^{x^6}\left(e^{x^8 - x^6} - 1\right)$ (B) $4x^3 e^{x^8}$ (C) $\dfrac{1}{\sqrt{1 - e^{x^2}}}$ (D) $\dfrac{e^{x^2}}{x^2} - 1$ (E) $x^2 e^{x^6}\left(4xe^{x^8 - x^6} - 3\right)$

Solution 32. (E) We can sort this out in two steps and apply the fundamental theorem to each.

$$\frac{d}{dx}\left(\int_{x^3}^{0} e^{t^2}\,dx + \int_0^{x^4} e^{t^2}\,dx\right)$$

For the first,

$$\frac{d}{dx}\int_{x^3}^{0} e^{t^2}\,dx = -\frac{d}{dx}\int_0^{x^3} e^{t^2}\,dx = -3x^2e^{x^6}$$

For the second,

$$\frac{d}{dx}\int_0^{x^4} e^{t^2}\,dx = 4x^3e^{x^8}.$$

All told, our integral is $x^2e^{x^6}(4xe^{x^8-x^6}-3)$.

41. Let ℓ be the line that is the intersection of the planes $x+y+z=3$ and $x-y+z=5$ in $\mathbb{R}^3$. An equation of the plane that contains $(0,0,0)$ and is perpendicular to ℓ is

(A) $x-z=0$

(B) $x+y+z=0$

(C) $x-y-z=0$

(D) $x+z=0$

(E) $x+y-z=0$

Solution 41. (A) The first plane is determined by the normal vector $(1,1,1)$, and the second determined by $(1,-1,1)$. Therefore the slope of ℓ is determined by a vector perpendicular to those, i.e. the cross product.

$$(1,1,1)\times(1,-1,1) = \det\begin{bmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -1 & 1\end{bmatrix} = (2,0,-2).$$

41. Let C be the circle $x^2+y^2=1$ oriented counterclockwise in the xy-plane. What is the value of the line integral $\oint_C (2x-y)\,dx+(x+3y)\,dy$?

(A) 0 (B) 1 (C) $\frac{\pi}{2}$ (D) π (E) 2π

Solution 41. (E) This is a classic Green's theorem problem.

$$\oint_{\partial D} L\,dx + M\,dy = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right)dx\,dy.$$

In our case,

$$\oint_C (2x-y)\,dx + (x+3y)\,dy = \iint_D (1+1)\,dx\,dy = 2A,$$

where A is the area of the unit circle, i.e. π.

So that is the slope of ℓ. We need this to be the normal vector for the plane in question, so it seems that $(1, 0, -1)$ is our best bet (out of the given options).

$$y' + xy = x$$
$$y(0) = -1$$

44. If y is a real-valued function defined on the real line and satisfying the initial value problem above, then $\lim_{x \to -\infty} y(x) =$

(A) 0 (B) 1 (C) -1 (D) ∞ (E) $-\infty$

Solution 44. (B) Putting it in simpler terms,

$$\frac{dy}{dx} + xy = x \implies \frac{dy}{dx} = x(1-y) \implies \frac{dy}{1-y} = x\,dx.$$

Integrating both sides, we obtain

$$-\log(1-y) = x^2/2 + C' \implies 1 - y = Ce^{-x^2/2} \implies y = 1 - Ce^{-x^2/2}.$$

Solving the initial value problem gives $C = 2$. Furthermore, as $x \to -\infty$, the second term above vanishes so we get 1 in the limit.

48. Let g be the function defined by $g(x, y, z) = 3x^2y + z$ for all real x, y, and z. Which of the following is the best approximation of the directional derivative of g at the point $(0, 0, \pi)$ in the direction of the vector $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$? (Note: $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the standard basis vectors in $\mathbb{R}^3$.)

(A) 0.2 (B) 0.8 (C) 1.4 (D) 2.0 (E) 2.6

Solution 48. (B) It would be good to recall the formula for the directional derivative. We take the gradient of the function then take its scalar product with the normalised vector in the direction we want. To begin,

$$\nabla g = (6xy, 3x^2, 1).$$

At the point $(0, 0, \pi)$, we have $\nabla g = (0, 0, 1)$. That works out pretty well for us. The normalised version of the vector $(1, 2, 3)$ is $(1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14})$. Dotting this with $(0, 0, 1)$ gives $3/\sqrt{14}$, and since $\sqrt{14} = 3.5$ or so our answer should be closer to 0.8 than 0.2.

48. Consider the theorem: If f and f' are both strictly increasing real-valued functions on the interval $(0, \infty)$, then $\lim_{x \to \infty} f(x) = \infty$. The following argument is suggested as a proof of this theorem.

(1) By the Mean Value Theorem, there is a c_1 in the interval $(1, 2)$ such that

$$f'(c_1) = \frac{f(2) - f(1)}{2 - 1} = f(2) - f(1) > 0.$$

(2) For each $x > 2$, there is a c_x in $(2, x)$ such that $\frac{f(x) - f(2)}{x - 2} = f'(c_x)$.

(3) For each $x > 2$, $\frac{f(x) - f(2)}{x - 2} = f'(c_x) > f'(c_1)$ since f' is strictly increasing.

(4) For each $x > 2$, $f(x) > f(2) + (x - 2)f'(c_1)$.

(5) $\lim_{x \to \infty} f(x) = \infty$

Which of the following statements is true?

(A) The argument is valid.

(B) The argument is not valid since the hypotheses of the Mean Value Theorem are not satisfied in (1) and (2).

(C) The argument is not valid since (3) is not valid.

(D) The argument is not valid since (4) cannot be deduced from the previous steps.

(E) The argument is not valid since (4) does not imply (5).

Solution 48. (A) The only issue here seems to be that (4) implies that $f(x)$ gets very large so long as $f'(c_1)$ is positive. But we know that it is, since f is a strictly increasing function. Therefore everything is satisfactory.

4 Differential Equations

61. A tank initially contains a salt solution of 3 grams of salt dissolved in 100 liters of water. A salt solution containing 0.02 grams of salt per liter of water is sprayed into the tank at a rate of 4 liters per minute. The sprayed solution is continually mixed with the salt solution in the tank, and the mixture flows out of the tank at a rate of 4 liters per minute. If the mixing is instantaneous, how many grams of salt are in the tank after 100 minutes have elapsed?

(A) 2 (B) $2 - e^{-2}$ (C) $2 + e^{-2}$ (D) $2 - e^{-4}$ (E) $2 + e^{-4}$

Solution 61. (E) We can set this up as a differential equation. Let s denote the amount of salt in the tank, and let t denote time. We have the initial condition of $s(0) = 3$. $s'(t)$ depends on two factors: the salt flowing in and the salt flowing out. The salt flows in constantly at a rate of 0.08 grams per minute, and the salt flows out at a rate of $4 \cdot (s/100) = s/25$ grams per minute. Therefore

$$s'(t) = \frac{ds}{dt} = 0.08 - s(t)/25 \implies \frac{ds}{dt} = 0.04(2 - s) \implies \frac{ds}{2 - s} = 0.04\, dt.$$

Doing the usual calculus,

$$-\log(2 - s) = 0.04t + C' \implies 2 - s = Ce^{-0.04t} \implies s(t) = 2 - Ce^{-0.04t}.$$

The initial condition tells us that $C = -1$, so $s(t) = 2 + e^{-0.04t}$. Plugging in $t = 100$ gives our answer.

5 Real Analysis

These are my notes from Math 4650: Analysis I at Cal State LA as well as Prof. Steven Heilman's notes from Math 541A at USC.

5.1 Midterm 1

5.1.1 Homework 1

Definition 5.1. Let $S \subseteq \mathbb{R}$. We say that S is **bounded from above** if $\exists\ b \in \mathbb{R}$ where

$$s \leq b \ \forall \ s \in S$$

If this is the case, we call b an **upper bound** of S.

If $b \leq c$ for all upper bounds c of S, we call b the **supremum** of S: $b = \sup(S)$.

Definition 5.2. We say that S is **bounded from below** if $\exists\ a \in \mathbb{R}$ where

$$s \geq a \ \forall \ s \in S$$

If this is the case, we call a a **lower bound** of S.

If $a \geq d$ for all lower bounds d of S, we call a the **infimum** of S: $a = \inf(S)$.

Proposition 5.1. Useful Sup/Inf Fact: Let $S \in \mathbb{R}$, $S \neq \emptyset$.

(1) Suppose S is bounded from above by an element b. Then $b = \sup(S) \iff \forall\ \epsilon > 0\ \exists\ x \in S$ with

$$b - \epsilon < x \leq b$$

(2) Suppose S is bounded from below by an element a. Then $a = \inf(S) \iff \forall\ \epsilon > 0\ \exists\ x \in S$ with

$$a \leq x < a + \epsilon$$

Completeness Axiom: Let S be a nonempty subset of $\mathbb{R}$. If S is bounded from above, then $\sup(S)$ exists. If S is bounded from below, then $\inf(S)$ exists.

Facts about absolute value:

- **Proposition 5.2.** $|x - y| < \epsilon \iff y - \epsilon < x < y + \epsilon$.

  Proof. In notes 08/23. ☐

- **Proposition 5.3.** $|ab| = |a||b|$.

Proof.

$$|ab| = \begin{cases} ab & ab \geq 0 \\ -ab & ab < 0 \end{cases} = \begin{cases} ab & a \geq 0, b \geq 0 \\ -ab & a \geq 0, b < 0 \\ -ab & a < 0, b \geq 0 \\ ab & a < 0, b < 0 \end{cases} = \begin{cases} ab & a \geq 0, b \geq 0 \\ a(-b) & a \geq 0, b < 0 \\ (-a)b & a < 0, b \geq 0 \\ (-a)(-b) & a < 0, b < 0 \end{cases}$$

$$= \begin{cases} |a||b| & a \geq 0, b \geq 0 \\ |a||b| & a \geq 0, b < 0 \\ |a||b| & a < 0, b \geq 0 \\ |a||b| & a < 0, b < 0 \end{cases} \implies |ab| = |a||b|$$

□

- **Proposition 5.4.** Let $\epsilon > 0$. Then $|a| < \epsilon \iff -\epsilon < a < \epsilon$.

  Proof. Follows from Proposition 5.2 if $x = a$, $y = 0$. □

- **Proposition 5.5.** $-|a| \leq a \leq |a|$

  Proof. Follows from Proposition 5.2 if $x = a$, $y = 0$, $\epsilon = |a|$. □

- **Theorem 5.6. Triangle Inequality:** $|a + b| \leq |a| + |b|$.

  Proof. In notes 08/23. □

  Corollary 5.6.1. Triangle Inequality: $|a - b| \leq |a| + |b|$.

  Proof. Follows from Theorem 5.6, let $b = -b$.

  □

  Remark. See also Theorem 8.13.

- **Proposition 5.7.** $|\ |a| - |b|\ | \leq |a - b|$.

  Proof. By Proposition 5.2, $|\ |a| - |b|\ | \leq |a - b|$ if and only if

$$|b| - |a - b| \leq |a| \leq |b| + |a - b| \tag{5.1}$$

The left half of (5.1) is true by the Triangle Inequality (Theorem 5.6):

$$|b| = |a - (a - b)| \leq |a| + |a - b| \iff |b| \leq |a| + |a - b| \iff |b| - |a - b| \leq |a|$$

The right half of (5.1) is also true by the Triangle Inequality (Theorem 5.6):

$$|a| = |b + a - b| \leq |b| + |a - b|$$

Therefore

$$|\ |a| - |b|\ | \leq |a - b|.$$

□

Proof. (Alternative proof.) Note that by the Triangle Inequality (Theorem 5.6),

$$|a| = |a - b + b| \leq |a - b| + |b| \implies |a| - |b| \leq |a - b|$$

Also,

$$|b| = |b - a + a| \leq |b - a| + |a| \implies -|b - a| \leq |a| - |b| \implies -|a - b| \leq |a| - |b|$$

where the last step follows from Proposition 5.9. Therefore

$$-|a - b| \leq |a| - |b| \leq |a - b|$$

and by Proposition 5.2,

$$|\ |a| - |b|\ | \leq |a - b|.$$

□

- **Proposition 5.8.** If $a < x < b$ and $a < y < b$ then $|x - y| < b - a$.

Proof.

$$y > a \implies -y < -a \implies b - y < b - a$$

$$b > y \implies b - y = |b - y| \implies \boxed{|b - y| < b - a}$$

By the Triangle Inequality (Theorem 5.6),

$$|x - y| = |x - b + b - y| \leq |x - b| + |b - y|$$

Since $b < x$, $|x - b| > 0$. Therefore $\boxed{|x - y| < |b - y|}$.

$$\implies |x - y| < |b - y| < b - a$$

$$\implies |x - y| < b - a$$

□

Proof. (Alternative proof.) Break into two cases.

– **Case 1:** $x \geq y$. Then $|x-y| = x-y$. We know $a < x < b \implies 0 < x-a < b-a$.

$$a < y \implies -a > -y \implies x-a > x-y \implies x-y < x-a < b-a$$

$$\implies \boxed{|x-y| < b-a}$$

– **Case 2:** $x < y$. Then $|x-y| = y-x$. We know $a < y < b \implies 0 < y-a < b-a$.

$$a < x \implies -a > -x \implies y-a > y-x \implies y-x < y-a < b-a$$

$$\implies \boxed{|x-y| < b-a}$$

□

•

Proposition 5.9. $|a-b| = |b-a|$

Proof. $|a-b| = |(-1)(b-a)| = |-1||b-a| = |b-a|$, where the second-to-last step follows from Proposition 5.2.

□

5.1.2 Homework 2

Definition 5.3. A sequence (a_n) of real numbers is said to **converge** to a **limit** $L \in \mathbb{R}$ if $\forall\ \epsilon > 0\ \exists\ N > 0$ where

$$n \geq N \implies |a_n - L| < \epsilon$$

We say that (a_n) **diverges** if it does not converge.

Definition 5.4. A sequence (a_n) of real numbers is **bounded** if $\exists\ M > 0$ where $\forall\ n \in \mathbb{N}$

$$|a_n| \leq M.$$

Theorem 5.10. If (a_n) converges then (a_n) is bounded.

Definition 5.5. Let (a_n) be a sequence of real numbers. We say that (a_n) is a **Cauchy sequence** if $\forall\ \epsilon > 0\ \exists\ N$ where

$$n, m \geq N \implies |a_n - a_m| < \epsilon$$

Theorem 5.11. (a_n) is Cauchy if and only if (a_n) converges.

Corollary 5.11.1. If (a_n) is Cauchy then (a_n) is bounded.

Proof. Let $\epsilon = 1$. Since (a_n) is Cauchy, $\exists\ N > 0 \mid n, m \geq N \implies$

$$|a_n - a_m| < 1$$

So, $n \geq N \implies$

$$|a_n - a_N| < 1 \iff a_N - 1 < a_n < a_N + 1 \implies |a_n| < |a_N + 1| \leq |a_N| + 1$$

Let $M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a_N| + 1\}$. Then $|a_n| \leq M\ \forall\ n \geq 1$. Therefore (a_n) is bounded.

□

Theorem 5.12. (Squeeze theorem.) Suppose that $\{a_n\}, \{b_n\}$, and $\{c_n\}$ are sequences of real numbers such that $a_n \leq b_n \leq c_n$ for all n. If both $\{a_n\}$ and $\{c_n\}$ converge to L, then $\{b_n\}$ converges to L.

Proof. Let $\epsilon > 0$. (a_n) converges to $L \implies$

$$\forall\ \epsilon > 0\ \exists\ N_A \mid n \geq N_A \implies |a_n - L| < \epsilon$$

(c_n) converges to $L \implies$

$$\forall\ \epsilon > 0\ \exists\ N_C \mid n \geq N_C \implies |c_n - L| < \epsilon$$

Let $N = \max\{N_A, N_C\}$. Then by one of our absolute values rules, $n \geq N \implies$

$$|a_n - L| < \epsilon \iff L - \epsilon < a_n < L + \epsilon$$

$$|c_n - L| < \epsilon \iff L - \epsilon < c_n < L + \epsilon$$

Therefore since $a_n \leq b_n \leq c_n$,

$$L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon \implies L - \epsilon < b_n < L + \epsilon \iff |b_n - L| < \epsilon$$

Therefore (b_n) converges to L.

□

Theorem 5.13. Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers such that $a_n \leq b_n$ for all n. If $\{a_n\}$ and $\{b_n\}$ converge to A and B respectively, then $A \leq B$.

Proof. Suppose $A > B$. Then let $\epsilon = \frac{A-B}{4} > 0$. (a_n) converges to $A \implies$

$$\exists\ N_A \mid n \geq N_A \implies |a_n - A| < \epsilon \iff A - \epsilon < a_n < A + \epsilon$$

(b_n) converges to $B \implies$

$$\exists\ N_B \mid n \geq N_B \implies |b_n - B| < \epsilon \iff B - \epsilon < b_n < B + \epsilon$$

Then if $n > \max\{N_A, N_B\}$,

$$A - \epsilon < a_n < A + \epsilon \iff A - \frac{A-B}{4} < a_n < A + \frac{A-B}{4} \iff \frac{3A}{4} + \frac{B}{4} < a_n < \frac{5A}{4} - \frac{B}{4}$$

$$B - \epsilon < b_n < B + \epsilon \iff B - \frac{A-B}{4} < b_n < B + \frac{A-B}{4} \iff \frac{5B}{4} - \frac{A}{4} < b_n < \frac{3B}{4} + \frac{A}{4}$$

This implies

$$b_n < \frac{3B}{4} + \frac{A}{4} = \frac{B}{4} + \frac{A}{4} + \frac{2B}{4} < \frac{B}{4} + \frac{A}{4} + \frac{2A}{4} = \frac{3A}{4} + \frac{B}{4} < a_n$$

Contradiction, since it is given that $a_n \leq b_n\ \forall\ n$. Therefore $A \leq B$. $\square$

5.2 Midterm 2

5.2.1 Homework 3

Definition 5.6. (Limits of functions at infinity.) Let f be a real-valued function defined on some set D where D contains an interval of the form (a, ∞). Let $L \in \mathbb{R}$. We say

$$\lim_{x \to \infty} f(x) = L$$

if $\forall\ \epsilon > 0\ \exists\ N \in \mathbb{R}$ where

$$x \geq N \implies |f(x) - L| < \epsilon.$$

Definition 5.7. Let $D \subseteq \mathbb{R}$. Let $a \in \mathbb{R}$. We say that a is a **limit point** (or "cluster point," or "accumulation point") of D if $\forall\ \delta > 0\ \exists\ x \in D$ where

$$x \neq a \text{ and } |x - a| < \delta$$

(Note that a may or may not be contained in D.)

Definition 5.8. (Limit of a function at a.): Let $D \subseteq \mathbb{R}$ and $f : d \to \mathbb{R}$. Let a be a limit point of D. Let $x \in D$. We say that f has a *limit as x tends to a* if $\exists\ L \in \mathbb{R}$ where $\forall\ \epsilon > 0\ \exists\ \delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$

and we write

$$\lim_{x \to a} f(x) = L$$

Proposition 5.14. (Properties of Limits.) Let $D \in \mathbb{R}$ and let a be a limit point of D. Suppose $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$. Let $\alpha \in \mathbb{R}$.

(1) If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ then

(a)
$$\lim_{x \to a} \alpha = \alpha$$

(b)
$$\lim_{x \to a} [f(x) + g(x)] = L + M$$

(c)
$$\lim_{x \to a} [f(x) - g(x)] = L - M$$

(d)
$$\lim_{x \to a} [f(x) \cdot g(x)] = L \cdot M$$

(e)
$$\lim_{x \to a} [\alpha \cdot f(x)] = \alpha \cdot L$$

(2) If $h : D \to \mathbb{R}$ and $h(x) \neq 0 \ \forall \ x \in D$ and $\lim_{x\to a} h(x) = H \neq 0$, then

$$\lim_{x\to a} \frac{1}{h(x)} = \frac{1}{H}$$

Note that properties (2) and (1)(d) combined imply

$$\lim_{x\to a} \frac{f(x)}{h(x)} = \frac{L}{H}$$

5.2.2 Homework 4

Definition 5.9. (Continuity.) Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ and $a \in D$. Then f is **continuous** at a if $\lim_{x\to a} f(x)$ exists and

$$\lim_{x\to a} f(x) = f(a)$$

Remark. if f is continuous at a, then we can say $\forall \ \epsilon > 0 \ \exists \ \delta > 0$ such that

$$|x - a| < \delta \implies |f(x) - L| < \epsilon$$

that is, we don't need to say $0 < |x - a| < \delta$.

Definition 5.10. If $B \subseteq D$, then f is **continuous on B** if f is continuous at every $b \in B$.

Theorem 5.15. (Intermediate Value Theorem.) Let f be continuous on $[a, b]$ and suppose $f(a) < f(b)$. $\forall \ d$ such that

$$f(a) < d < f(b)$$

$\exists \ c \in \mathbb{R}$ where

$$a < c < b, \ f(c) = d.$$

5.3 Final

5.3.1 Homework 5

Definition 5.11. Let $S \subseteq \mathbb{R}$. We say $x \in \mathbb{R}$ is an **interior point** of S if there exists an open interval (a, b) where

$$x \in (a, b) \text{ and } (a, b) \subseteq S.$$

Definition 5.12. (Open sets.) Let $S \subseteq \mathbb{R}$. We say S is **open** if every $x \in S$ is an interior point of S.

Definition 5.13. (Closed sets.) Let $S \subseteq \mathbb{R}$. We say S is **closed** if $\mathbb{R} \setminus S$ is open.

Theorem 5.16. A set is closed if and only if it contains all of its limit points.

(Facts about open and closed sets.) Suppose $a, b \in \mathbb{R}$. Then

- **Proposition 5.17.** (a, ∞) is open.

  Proof. Let $x \in (a, \infty)$. Since $x > a$, $\exists\ \epsilon > 0 \mid a + \epsilon = x$. Then $a = x - \epsilon < x - \frac{\epsilon}{2} < x < x + \frac{\epsilon}{2} < \infty$. Therefore $x \in (x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}) \subseteq (a, \infty)$, so (a, ∞) is open. □

- **Proposition 5.18.** $(-\infty, b)$ is open.

  Proof. Let $x \in (-\infty, b)$. Since $x < b$, $\exists\ \epsilon > 0 \mid b - \epsilon = x$. Then $-\infty < x - \frac{\epsilon}{2} < x < x + \frac{\epsilon}{2} < x + \epsilon = b$. Therefore $x \in (x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}) \subseteq (-\infty, b)$, so $(-\infty, b)$ is open. □

- **Proposition 5.19.** (a, b) is open.

  Proof. In class notes. □

- **Proposition 5.20.** If $a < b$, then $[a, b]$ is closed.

  Proof. Consider $\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$. By Proposition 5.18, $(-\infty, a)$ is open. By Proposition ra.hw5.5b, (b, ∞) is open. By Proposition 5.21, the union of two open sets is open. Therefore $\mathbb{R} \setminus [a, b]$ is open, so $[a, b]$ is closed. □

- **Proposition 5.21.** If A and B are open, then $A \cup B$ is open.

  Proof. Since A is open, $\forall\ x_A \in A\ \exists\ (a_A, b_A) \subseteq A \mid x_A \in (a_A, b_A)$. Since B is open, $\forall\ x_B \in B\ \exists\ (a_B, b_B) \subseteq B \mid x_B \in (a_B, b_B)$.

  Let $x \in A \cup B$. If $x \in A$, then per above $\exists\ (a_A, b_A) \subseteq A \subseteq A \cup B \mid x_A \in (a_A, b_A)$. If $x \in B$, then per above $\exists\ (a_B, b_B) \subseteq B \subseteq A \cup B \mid x_B \in (a_B, b_B)$. Therefore $A \cup B$ is open. □

- **Proposition 5.22.** If A and B are open, then $A \cap B$ is open.

  Proof. Since A is open, $\forall\ x_A \in A\ \exists\ (a_A, b_A) \subseteq A \mid x_A \in (a_A, b_A)$. Since B is open, $\forall\ x_B \in B\ \exists\ (a_B, b_B) \subseteq B \mid x_B \in (a_B, b_B)$.

  Let $x \in A \cap B$. Then $x \in A$ and $x \in B$, so $\exists\ (a_A, b_A) \subseteq A \mid x_A \in (a_A, b_A)$, and $\exists\ (a_B, b_B) \subseteq B \mid x_B \in (a_B, b_B)$. Let $a = \max\{a_A, a_B\}$, and $b = \min\{b_A, b_B\}$. Since $x > a$ and $x < b$, $x \in (a, b)$. Since $(a, b) \subseteq (a_A, b_A) \subseteq A$ and $(a, b) \subseteq (a_B, b_B) \subseteq B$, $(a, b) \subseteq A \cap B$. Therefore $A \cap B$ is open. □

- **Proposition 5.23.** If A and B are closed, then $A \cup B$ is closed.

  Proof. Since A is closed, $\mathbb{R} \setminus A$ is open. Since B is closed, $\mathbb{R} \setminus B$ is open. $\mathbb{R} \setminus (A \cup B) = (\mathbb{R} \setminus A) \cap (\mathbb{R} \setminus B)$. By Proposition 5.22 the intersection of two open sets is open. Therefore $\mathbb{R} \setminus (A \cup B)$ is open, so $A \cup B$ is closed.

  □

- **Proposition 5.24.** If A and B are closed, then $A \cap B$ is closed.

  Proof. Since A is closed, $\mathbb{R} \setminus A$ is open. Since B is closed, $\mathbb{R} \setminus B$ is open. $\mathbb{R} \setminus (A \cap B) = (\mathbb{R} \setminus A) \cup (\mathbb{R} \setminus B)$. By Proposition 5.21, the union of two open sets is open. Therefore $\mathbb{R} \setminus (A \cap B)$ is open, so $A \cap B$ is closed.

  □

- **Proposition 5.25.** $\mathbb{R}$ is open and closed.

  Proof. Let $\epsilon > 0$. Let $x \in \mathbb{R}$. Then $x - \epsilon, x + \epsilon \in \mathbb{R}$, and $x \in (x - \epsilon, x + \epsilon)$. Therefore $\mathbb{R}$ is open. $\mathbb{R}$ is closed because by Proposition 5.26, $\mathbb{R} \setminus \mathbb{R} = \emptyset$ is open. □

- **Proposition 5.26.** $\emptyset$ is open and closed.

  Proof. To show that a set S is open, we most show that $\forall\ x \in S\ \exists\ S' \subseteq S \mid x \in S'$ where S' is open. Since there are no $x \in \emptyset$, this condition is satisfied for $\emptyset$. $\emptyset$ is closed because per Proposition 5.25, $\mathbb{R} \setminus \emptyset = \mathbb{R}$ is open.

  □

Proposition 5.27. Let $x_1, x_2, \ldots, x_n$ be real numbers. Let S be the finite set $S = \{x_1, x_2, \ldots, x_n\}$. Then S is closed.

Proof. Consider $\mathbb{R} \setminus S = (-\infty, x_1) \cup (x_1, x_2) \cup \ldots \cup (x_{n-1}, x_n) \cup (x_n, \infty)$. $(-\infty, x_1)$ is open by Proposition 5.18. (x_n, ∞) is open by Proposition 5.17.

Consider (x_i, x_{i+1}) where $i \in \{1, 2, 3, \ldots, n-1\}$. Let $x \in (x_i, x_{i+1})$. Then since $x > x_i$ and $x < x_{i+1}$, $\exists\, \epsilon > 0 \mid x_i + \epsilon = x$ and $\exists\, \delta > 0 \mid x_{i+1} - \delta = x$. Then $x_i = x - \epsilon < x - \frac{\epsilon}{2} < x < x + \frac{\delta}{2} < x + \delta = x_{i+1}$. Therefore $x \in (x - \epsilon, x + \delta) \subseteq (x_i, x_{i+1})$, so (x_i, x_{i+1}) is open.

Finally, since $\mathbb{R} \setminus S$, by Proposition 5.17 (and induction) $\mathbb{R} \setminus S$ is open. Therefore S is closed. □

Proposition 5.28. Let $x_1, x_2, \ldots, x_n$ be real numbers. Let S be the finite set $S = \{x_1, x_2, \ldots, x_n\}$. Then S has no limit points.

Proof. Per Definition 5.7, we seek to show that (1) $\forall\, x_i \in S\ \exists\, \delta_i$ such that $\forall\, x_j \in D (x_j \neq x_i)$

$$|x_j - x_i| \geq \delta_i$$

and (2) $\forall\, x \in \mathbb{R} \setminus S\ \exists\, \delta_x$ such that $\forall\, x_i \in D$

$$|x_i - x| \geq \delta_x$$

(1) Let $x_i \in S$. Let $\delta_i = \frac{1}{2} \min\{|x_i - x_k| \mid x_k \neq x_i\}$. Then $\forall\, x_j \neq x_i \in S$,

$$|x_i - x_j| \geq |x_i - x_k| > \delta_i$$

(2) Let $x \in \mathbb{R} \setminus S$. Let $\delta_x = \frac{1}{2} \min\{|x - x_i| \mid x_i \in S\}$. Then

$$|x_i - x| \geq \min\{|x - x_i| \mid x_i \in S\} > \delta_x$$

□

Definition 5.14. Let $S \subseteq \mathbb{R}$. An **open cover** of S is a collection $X = \{\mathcal{O}_\alpha \mid \alpha \in I\}$ where each set $\mathcal{O}_\alpha$ is an open subset of $\mathbb{R}$ such that

$$S \subseteq \bigcup_{\alpha \in I} \mathcal{O}_\alpha$$

(Here I is some set that indexes the $\mathcal{O}_\alpha$).

Definition 5.15. If $X' \subseteq X$ such that

$$S \subseteq \bigcup_{\mathcal{O}_\alpha \in X'} \mathcal{O}_\alpha$$

then X' is called a **subcover** of S contained in X. In addition, if X' is finite then we call X' a **finite subcover** of S contained in X.

Definition 5.16. (Compactness.) Let $S \subseteq \mathbb{R}$. We say that S is **compact** if every open cover of S contains a finite subcover.

Definition 5.17. Let $S \subseteq \mathbb{R}$. We say that S is **bounded** if $\exists\, M > 0$ where $S \subseteq [-M, M]$.

Remark. S is bounded if and only if $|s| \leq M\ \forall\, s \in S$.

Theorem 5.29. (Heine-Borel Theorem.) Let $S \subseteq \mathbb{R}$. S is compact if and only if S is closed and bounded.

Proposition 5.30. Let $x_1, x_2, \ldots, x_n$ be real numbers. Let S be the finite set $S = \{x_1, x_2, \ldots, x_n\}$. Then S is compact.

Proof. Let $\{O_\alpha\}$ be an open cover of S. By definition of open cover, $\forall\ i\ \exists\ O_{\alpha_i}$ such that $x_i \in O_{\alpha_i}$. Thus, $\{O_{\alpha_1}, O_{\alpha_2}, \ldots, O_{\alpha_n}\}$ is a finite subcover of S. □

Proposition 5.31. Let A and B be compact subsets of $\mathbb{R}$. Then $A \cap B$ is compact.

Proof. Since $A \cap B \subseteq A$, $A \cap B \subseteq [-M_A, M_A]$. Therefore $A \cap B$ is bounded.

Since $A \cap B$ is closed and bounded, by the Heine-Borel Theorem (Theorem ra.heine-borel.thm), $A \cap B$ is compact.

□

Proposition 5.32. Let A and B be compact subsets of $\mathbb{R}$. Then $A \cup B$ is compact.

Proof. Let $M = \max\{M_A, M_B\}$. Note that $[-M_A, M_A] \subseteq [M, M]$ and $[-M_B, M_B] \subseteq [-M, M]$. This implies $A \subseteq [-M, M]$ and $B \subseteq [-M, M]$. Therefore $A \cup B \subseteq [-M, M]$.

Since $A \cup B$ is closed and bounded, by the Heine-Borel Theorem (Theorem ra.heine-borel.thm), $A \cup B$ is compact.

□

Theorem 5.33. Let $f : D \to \mathbb{R}$ be continuous on D. If $X \subseteq D$ and X is compact (closed and bounded), then

$$f(\bar{x}) = \{f(x) \mid x \in X\}$$

is compact (closed and bounded).

Corollary 5.33.1. Suppose $f : D \to \mathbb{R}$ where D is closed and bounded. Then there exists $a, b \in D$ where $f(a)$ is the min of f on D and $f(b)$ is the max of f on D.

5.3.2 Homework 6

Definition 5.18. (Uniform Continuity.) Let $D \subseteq \mathbb{R}$ and let $f : D \to \mathbb{R}$. We say that f is **uniformly continuous** on D if $\forall\ \epsilon > 0\ \exists\ \delta > 0$ where

$$x, y \in D \text{ and } 0 < |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Theorem 5.34. (Uniform continuity implies continuity.) Suppose $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$. If f is uniformly continuous on D, then f is continuous at every $a \in D$.

5.4 More Theorems

Theorem 5.35. Fubini's Theorem. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that $\int\int_{\mathbb{R}^2} |h(x,y)|dxdy < \infty$. Then

$$\int\int_{\mathbb{R}^2} h(x,y)dxdy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(x,y)dx \right) dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(x,y)dy \right) dx$$

5.5 Problems from Practice Math GRE Subject Tests

38. Let A and B be nonempty subsets of $\mathbb{R}$ and let $f : A \to B$ be a function. If $C \subseteq A$ and $D \subseteq B$, which of the following must be true?

(A) $C \subseteq f^{-1}\left(f(C)\right)$

(B) $D \subseteq f\left(f^{-1}(D)\right)$

(C) $f^{-1}\left(f(C)\right) \subseteq C$

Solution 38. (A) Neither of the equalities should hold – these are in fact nonsense statements, as one side lies in A and the other in B. To unravel the remaining two sets,

$$f^{-1}(f(C)) = \{x \in A : f(x) \in f(C)\}, \quad f(f^{-1}(D)) = f(\{y \in A : f(y) \in D\})$$

Clearly the second set must always be contained in D, but not the other way around. Similarly the first set certainly contains all $c \in C$ (as $f(c) \in f(C)$) but not the other way around.

47. The function $f : \mathbb{R} \to \mathbb{R}$ is defined as follows.

$$f(x) = \begin{cases} 3x^2 & \text{if } x \in \mathbb{Q} \\ -5x^2 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Which of the following is true?

(A) f is discontinuous at all $x \in \mathbb{R}$.

(B) f is continuous only at $x = 0$ and differentiable only at $x = 0$.

(C) f is continuous only at $x = 0$ and nondifferentiable at all $x \in \mathbb{R}$.

(D) f is continuous at all $x \in \mathbb{Q}$ and nondifferentiable at all $x \in \mathbb{R}$.

(E) f is continuous at all $x \notin \mathbb{Q}$ and nondifferentiable at all $x \in \mathbb{R}$.

Solution 47. (B) A classic kind of problem. We are clearly continuous and differentiable at 0. Anywhere else, near a rational number there is an irrational number and vice versa. Therefore there can be no continuity anywhere but at 0, and hence no differentiability either.

57. For each positive integer n, let x_n be a real number in the open interval $\left(0, \frac{1}{n}\right)$. Which of the following statements must be true?

I. $\lim_{n\to\infty} x_n = 0$

II. If f is a continuous real-valued function defined on $(0, 1)$, then $\{f(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence.

III. If g is a uniformly continuous real-valued function defined on $(0, 1)$, then $\lim_{n\to\infty} g(x_n)$ exists.

(A) I only (B) I and II only (C) I and III only (D) II and III only (E) I, II, and III

Solution 57. (C) I is true, since $\lim_{n\to\infty} x_n$ must be bounded between 0 and $\lim_{n\to\infty} 1/n = 0$. Unfortunately, x_n does not converge inside $(0, 1)$. There is no reason therefore that $f(x_n)$ should be a convergent sequence – suppose that $f(x) = 1/x$, so that $f(x_n)$ is certainly not Cauchy. However, if g is uniformly continuous, then g extends to a continuous function on $[0, 1]$. Now x_n is a convergent sequence, so $\lim_{n\to\infty} g(x_n) = g(\lim_{n\to\infty} x_n) = g(0)$ exists.

60. A real-valued function f defined on $\mathbb{R}$ has the following property.

For every positive number ϵ, there exists a positive number δ such that
$$|f(x) - f(1)| \geq \epsilon \text{ whenever } |x - 1| \geq \delta.$$

This property is equivalent to which of the following statements about f?

(A) f is continuous at $x = 1$.

(B) f is discontinuous at $x = 1$.

(C) f is unbounded.

(D) $\lim_{|x|\to\infty} |f(x)| = \infty$

(E) $\int_0^\infty |f(x)|\,dx = \infty$

Solution 60. (D) While it looks like this is the opposite of continuity, that should read 'there exists $\varepsilon > 0$'. What the statement says is that we not only get arbitrarily far away from $f(1)$, but we must for all x sufficiently far away from 1. So as $|x|$ gets very large, so does $|f(x)|$.

63. For any nonempty sets A and B of real numbers, let $A \cdot B$ be the set defined by

$$A \cdot B = \{xy : x \in A \text{ and } y \in B\}.$$

If A and B are nonempty bounded sets of real numbers and if $\sup(A) > \sup(B)$, then $\sup(A \cdot B) =$

(A) $\sup(A)\sup(B)$

(B) $\sup(A)\inf(B)$

(C) $\max\{\sup(A)\sup(B),\ \inf(A)\inf(B)\}$

(D) $\max\{\sup(A)\sup(B),\ \sup(A)\inf(B)\}$

(E) $\max\{\sup(A)\sup(B),\ \inf(A)\sup(B),\ \inf(A)\inf(B)\}$

Solution 63. (E) The supremum is either going to be the product of the two largest positive numbers in A and B or the product of the two smallest negative numbers in A and B. That means we should look for $\sup \cdot \sup$ or $\inf \cdot \inf$. However, it might be the case that B contains only negative numbers and A contains only positive numbers. Then the largest value in $A \cdot B$ will be attained by the smallest positive element of A and the largest negative element of B, giving us our third option: $\inf A \cdot \sup B$.

6 Probability

These are my notes from taking Math 505A at USC taught by Sergey Lototsky, Math 541A at USC taught by Steven Heilman, ISE 620 at USC taught by Sheldon Ross (as well as the corresponding textbooks *Introduction to Probability Models* [Ross, 2014] and *Stochastic Processes* [Ross, 2008] by Sheldon Ross) and the textbook *Probability and Random Processes* (Grimmett and Stirzaker) 3rd edition [Grimmett and Stirzaker, 2001], Statistics 100B at UCLA taught by Nicolas Christou, as well as a few other sources I cite within the text.

6.1 To Know for Math 505A Midterm 1 (Discrete Random Variables)

6.1.1 Definitions

Definition 6.1. The **probability mass function** of a discrete random variable X is the function $f : \mathbb{R} \to [0,1]$ given by $f(x) = \Pr(X = x)$.

Definition 6.2. The **(cumulative) distribution function** of a discrete random variable F is given by

$$F(x) = \sum_{i: x_i \leq x} f(x_i)$$

Definition 6.3. The **joint probability mass function** $f : \mathbb{R}^2 \to [0,1]$ of two discrete random variables X and Y is given by

$$f(x,y) = \Pr(X = x \cap Y = y)$$

Definition 6.4. The **joint distribution function** $F : \mathbb{R}^2 \to [0,1]$ is given by

$$F(x,y) = \Pr(X \leq x \cap Y \leq y)$$

Definition 6.5. If $\Pr(B) > 0$ then the **conditional probability** that A occurs given that B occurs is defined to be

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Definition 6.6. (Independent sets.) Let $A_1, A_2, \ldots$ be subsets of a sample space Ω, and let $\mathbb{P}$ be a probability law on Ω. We say that $A_1, A_2, \ldots$ are **independent** if for any finite subset S of $\{1, 2, \ldots\}$, we have

$$\mathbb{P}\left(\bigcap_{i \in S} A_i \right) = \prod_{i \in S} \mathbb{P}(A_i)$$

Definition 6.7. (Notation.) Let $X : \Omega \to \mathbb{R}$ be a random variable. Let $B \subseteq \mathbb{R}$. We define $\{X \in B\} := \{\omega \in \Omega : X(\omega) \in B\}$.

Definition 6.8. (Independence of random variables.) Random variables $X_1, X_2, \ldots$ are **independent** if for every $B_1, B_2, \ldots \subseteq \mathbb{R}$, the events $\{X_1 \in B_1\}, \{X_2 \in B_2\}, \ldots$ are independent; that is,

$$\mathbb{P}\left(\bigcap_{i=1}^{n}\{X_i \in B_i\}\right) = \prod_{i=1}^{n} \mathbb{P}(\{X_i \in B_i\})$$

Remark. A more informal definition is as follows: Two random variables X and Y are **independent** if and only if $\Pr(X \cap Y) = \Pr(X)\Pr(Y)$.

Theorem 6.1. (Law of total probability). If X is a random variable and Y is a discrete random variable taking on values $y_1, y_2, \ldots, y_n$, then $\Pr(X) = \sum_i \Pr(X \mid Y = y_i) \cdot \Pr(Y = y_i)$. (Can be used to prove independence.)

Lemma 6.2. Let $B_1, A_1, A_2, \ldots, A_m$ be subsets of a sample space Ω and let $\mathbb{P}$ be a probability law on Ω. Let $X_1, X_2, \ldots, X_m$, and Z be random variables. If Z is independent of X_i for all $i \in 1, \ldots, m$, then

$$\mathbb{P}\left(\bigcap_{i=1}^{m}\{X_i \in A_i\} \cap Z \in B_1\right) = \mathbb{P}\left(\bigcap_{i=1}^{m}\{X_i \in A_i\}\right)\mathbb{P}(Z \in B_1).$$

That is, $\bigcap_{i=1}^{m} X_i$ is independent of Z.

Proof. Note that by the definition of conditional probability,

$$\mathbb{P}\left(\bigcap_{i=1}^{m} X_i \in A_i \mid Z \in B_1\right) = \frac{\mathbb{P}(\bigcap_{i=1}^{m}\{X_i \in A_i\} \cap Z \in B_1)}{\mathbb{P}(Z \in B_1)}$$

$$= \frac{\prod_{j=1}^{m-1}\left[\mathbb{P}(X_j \in A_j \mid \bigcap_{k=j+1}^{m}\{X_k \in A_k\} \cap Z \in B_1)\right] \cdot \mathbb{P}(X_m \in A_m \cap Z \in B_1)}{\mathbb{P}(Z \in B_1)}$$

$$= \frac{\prod_{j=1}^{m-1}\left[\mathbb{P}(X_j \in A_j \mid \bigcap_{k=j+1}^{m}\{X_k \in A_k\}, Z \in B_1)\right] \cdot \mathbb{P}(X_m \in A_m \mid Z \in B_1)\mathbb{P}(Z \in B_1)}{\mathbb{P}(Z \in B_1)}$$

Canceling and using the independence of X_i and Z for all i, we have

$$= \prod_{j=1}^{m-1}\left[\mathbb{P}(X_j \in A_j \mid \bigcap_{k=j+1}^{m}\{X_k \in A_k\}\right] \cdot \mathbb{P}(X_m \in A_m) = \mathbb{P}\left(\bigcap_{i=1}^{m}\{X_i \in A_i\}\right)$$

Then the result follows since

$$\mathbb{P}\left(\bigcap_{i=1}^{m}\{X_i \in A_i\} \cap Z \in B_1\right) = \mathbb{P}\left(\bigcap_{i=1}^{m} X_i \in A_i \mid Z \in B_1\right)\mathbb{P}(Z \in B_1) = \mathbb{P}\left(\bigcap_{i=1}^{m}\{X_i \in A_i\}\right)\mathbb{P}(Z \in B_1).$$

□

Definition 6.9. Two random variables X and Y are **uncorrelated** if $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.

Proposition 6.3. (a) Two random variables are uncorrelated if and only if their covariance $\text{Cov}(X,Y) = \mathbb{E}\big[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))\big] = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$ equals 0.

(b) If X and Y are independent then they are uncorrelated.

Theorem 6.4. If X and Y are independent and $g, h : \mathbb{R} \to \mathbb{R}$, then $g(X)$ and $h(Y)$ are also independent.

Definition 6.10. We say that a nonnegative random variable X is **lattice** with period d if

$$\sum_{n=0}^{\infty} \Pr(X = nd) = 1$$

and d is the largest number that satisfies this equation.

Remark. Most common discrete random variables are lattice, but not all discrete random variables are. For example, the discrete random variable X such that

$$\Pr(X = 1/i) = p_i > 0, \qquad i \geq 1$$

such that $\sum_{i=1}^{\infty} p_i = 1$ is not lattice. Another example is

$$X = \begin{cases} 1 & \text{with probability } 1/2 \\ \sqrt{2} & \text{with probability } 1/2 \end{cases}$$

because there is no d such that there are integers m and n so that

$$1 = md$$

$$\sqrt{2} = nd.$$

(If there were, then we would have

$$\frac{n}{m} = \frac{nd}{md} = \sqrt{2}$$

but there are no integers n and m such that this is true, since $\sqrt{2}$ is irrational.)

6.1.2 Conditioning

Definition 6.11. The **conditional distribution function** of Y given $X = x$, written $F_{Y|X}(\cdot \mid x)$, is defined by

$$F_{Y|X}(y \mid x) = \Pr(Y \leq y \mid X = x)$$

Definition 6.12. The **conditional probability mass function** of Y given $X = x$, written $f_{Y|X}(\cdot \mid x)$, is defined by

$$f_{Y|X}(y \mid x) = \Pr(Y = y \mid X = x)$$

Theorem 6.5. Iterated expectations:

(i) $\mathbb{E}\big[\mathbb{E}(X \mid Y)\big] = \mathbb{E}(X)$ **(Law of Total Expectation)**

(ii) $\mathbb{E}\big[(X \mid Y) \mid Z\big] = \mathbb{E}(X \mid Y)$

(iii) $\mathbb{E}(E(XY \mid Y)) = \mathbb{E}(Y\mathbb{E}(X \mid Y))$

Proof. (i) Discrete case:

$$\mathbb{E}\big[\mathbb{E}(X \mid Y)\big] = \sum_y \mathbb{E}(X \mid Y = y)\Pr(Y = y) = \sum_y \sum_x x \Pr(X = x \mid Y = y)\Pr(Y = y)$$

$$= \sum_y \sum_x x \Pr(X = x \cap Y = y) = \sum_x x \sum_y \Pr(X = x \cap Y = y) = \sum_x x \Pr(X = x) = \mathbb{E}(X)$$

Continuous case:

$$\mathbb{E}\big[\mathbb{E}(X \mid Y)\big] = \int_{-\infty}^{\infty} \mathbb{E}(X \mid Y = y) f_Y(y) dy = \text{ (by definition 1.75) } \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x f_{X|Y}(x \mid y) dx \right) f_Y(y) dy$$

$$\text{(by Fubini's Theorem, Theorem 5.35) } \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x f_{X|Y}(x \mid y) f_Y(y) dy \right) dx = \int_{-\infty}^{\infty} x f_X(x) dx = \mathbb{E}(X)$$

□

Definition 6.13. Conditional Variance: $\mathrm{Var}(X \mid Y) = \mathbb{E}\big[(X - \mathbb{E}(X \mid Y))^2 \mid Y\big]$

Theorem 6.6 (**Conditional Expectation as a Random Variable**). (i) Let X, Y be random variables such that (X, Y) is uniformly distributed on the triangle $\{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x + y \leq 1\}$. Then

$$\mathbb{E}(X|Y) = \frac{1}{2}(1 - Y).$$

(ii) Total Expectation Theorem:

$$\mathbb{E}(\mathbb{E}(X|Y)) = \mathbb{E}(X).$$

- If X is a random variable, and if $f(t) := \mathbb{E}(X - t)^2$, $t \in \mathbb{R}$, then the function $f : \mathbb{R} \to \mathbb{R}$ is uniquely minimized when $t = \mathbb{E}X$. A similar minimizing property holds for conditional expectation. Let $h : \mathbb{R} \to \mathbb{R}$. Then the quantity $\mathbb{E}(X - h(Y))^2$ is minimized among all functions $h : \mathbb{R} \to \mathbb{R}$ when $h(Y) = \mathbb{E}(X|Y)$.

(iii)

$$\mathbb{E}(Xh(Y)|Y) = h(Y)\mathbb{E}(X|Y).$$

$$\mathbb{E}([\mathbb{E}(X|h(Y)]\,|Y) = \mathbb{E}(X|h(Y)).$$

(iv)

$$\mathbb{E}(X|X) = X.$$

$$\mathbb{E}(X + Y|Z) = \mathbb{E}(X|Z) + \mathbb{E}(Y|Z).$$

(v) If Z is independent of X and Y, then

$$\mathbb{E}(X|Y,Z) = \mathbb{E}(X|Y).$$

(Here $\mathbb{E}(X|Y,Z)$ is notation for $\mathbb{E}(X|(Y,Z))$ where (Y,Z) is interpreted as a random vector, so that X is conditioned on the random vector (Y,Z).)

Proof. (i) Note that since $0 \leq x$ and $x + y \leq 1$, conditional on $y = y \geq 0$ x is uniformly distributed on $[0, 1-y]$. That is,

$$\Pr(X \leq x \mid Y = y) = \begin{cases} 0 & x < 0 \\ x/(1-y) & 0 \leq x < 1-y \\ 1 & x \geq 1-y \end{cases}$$

Since the expected value of a random variable uniformly distributed on $[a,b]$ is $(a+b)/2$, it follows that $\mathbb{E}(X \mid Y = y) = (0 + 1 - y)/2 = (1/2)(1-y)$. Therefore $\boxed{\mathbb{E}(X \mid Y) = \frac{1}{2}(1-Y)}$.

(ii) • Discrete case:

$$\mathbb{E}\big[\mathbb{E}(X \mid Y)\big] = \sum_y \mathbb{E}(X \mid Y = y)\Pr(Y = y) = \sum_y \sum_x x \Pr(X = x \mid Y = y)\Pr(Y = y)$$

$$= \sum_y \sum_x x \Pr(X = x \cap Y = y) = \sum_x x \sum_y \Pr(X = x \cap Y = y) = \sum_x x \Pr(X = x) = \mathbb{E}(X)$$

• Continuous case:

$$\mathbb{E}\big[\mathbb{E}(X \mid Y)\big] = \int_{-\infty}^{\infty} \mathbb{E}(X \mid Y = y) f_Y(y) dy = \text{ (by definition 1.75) } \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x f_{X|Y}(x \mid y) dx \right) f_Y(y) dy$$

$$\text{(by Fubini's Theorem) } \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x f_{X|Y}(x \mid y) f_Y(y) dy \right) dx = \int_{-\infty}^{\infty} x f_X(x) dx = \mathbb{E}(X)$$

• Next we will show that the quantity $\mathbb{E}(X - h(Y))^2$ is minimized among all functions $h : \mathbb{R} \to \mathbb{R}$ when $h(Y) = \mathbb{E}(X \mid Y)$. We seek

$$\arg \min_{\{h:\mathbb{R}\to\mathbb{R}\}} \mathbb{E}(X - h(Y))^2 = \arg \min_{\{h:\mathbb{R}\to\mathbb{R}\}} \big[\mathbb{E}(X^2) - 2\mathbb{E}[h(Y)]\mathbb{E}(X) + \mathbb{E}[h(Y)^2]\big].$$

This expression is quadratic in $\mathbb{E}[h(Y)]$. Differentiating with respect to $\mathbb{E}[h(Y)]$ and setting equal to 0, we have

$$2\mathbb{E}[h(Y)] - 2\mathbb{E}(X) = 0 \iff \mathbb{E}[h(Y)] = \mathbb{E}(X) \implies \boxed{\arg \min_{\{h:\mathbb{R}\to\mathbb{R}\}} \mathbb{E}(X - h(Y))^2 = \mathbb{E}(X \mid Y)}$$

(iii) • Discrete case:

$$\mathbb{E}(Xh(Y)|Y) = \sum_{x\in\mathbb{R}} x \cdot h(Y) \cdot \Pr(X = x \mid Y) = h(Y) \sum_{x\in\mathbb{R}} x \cdot \Pr(X = x \mid Y) = h(Y)\mathbb{E}(X \mid Y).$$

Continuous case:

$$\mathbb{E}(Xh(Y)|Y) = \int_{x\in\mathbb{R}} x \cdot h(Y) \cdot f_{X|Y}(x) = h(Y) \int_{x\in\mathbb{R}} x \cdot f_{X|Y}(x) = h(Y)\mathbb{E}(X \mid Y).$$

- Discrete case: Note that

$$\mathbb{E}[X \mid h(Y)] = \sum_{x\in\mathbb{R}} x \Pr(X = x \mid h(Y)) = \sum_{x\in\mathbb{R}} x\mathbb{E}[\mathbf{1}_{\{X=x\}} \mid h(Y)]$$

where $\mathbf{1}_{\{X=x\}}$ is an indicator variable for X taking on the value x. Note that $\mathbb{E}[\mathbf{1}_{\{X=x\}} \mid h(Y)]$ is a function of Y (and a random variable). Then we have

$$\mathbb{E}([\mathbb{E}(X \mid h(Y)] \mid Y) = \mathbb{E}\left(\sum_{x\in\mathbb{R}} x\mathbb{E}[\mathbf{1}_{\{X=x\}} \mid h(Y)] \mid Y\right) = \sum_{x\in\mathbb{R}} \mathbb{E}[x\mathbb{E}[\mathbf{1}_{\{X=x\}} \mid h(Y)] \mid Y]$$

$$= \text{(by the previous result)} \sum_{x\in\mathbb{R}} \mathbb{E}[\mathbf{1}_{\{X=x\}} \mid h(Y)]\mathbb{E}[x \mid Y] = \sum_{x\in\mathbb{R}} \Pr(X = x \mid h(Y)) \cdot x = \mathbb{E}(X|h(Y)).$$

Continuous case: Note that

$$\mathbb{E}[X \mid h(Y)] = \int_{x\in\mathbb{R}} x f_{X|h(Y)}(x)dx.$$

Note that for a fixed x, $f_{X|h(Y)}(x)$ is a function of Y (and a random variable). Then we have

$$\mathbb{E}([\mathbb{E}(X \mid h(Y)] \mid Y) = \mathbb{E}\left(\int_{x\in\mathbb{R}} x f_{X|h(Y)}(x)dx \mid Y\right) = \int_{x\in\mathbb{R}} \mathbb{E}[x \cdot f_{X|h(Y)}(x) \mid Y]dx$$

$$= \text{(by the previous result)} \int_{x\in\mathbb{R}} f_{X|h(Y)}(x)\mathbb{E}[x \mid Y]dx = \int_{x\in\mathbb{R}} f_{X|h(Y)}(x) \cdot x = \mathbb{E}(X|h(Y)).$$

(iv)
- Discrete case:

$$\mathbb{E}(X|X) = \sum_{x\in\mathbb{R}} x \Pr(X = x \mid X)$$

Note that

$$\Pr(X = x \mid X) = \begin{cases} 1 & x = X \\ 0 & \text{otherwise} \end{cases}$$

so we have

$$\mathbb{E}(X|X) = \sum_{x\in\mathbb{R}} x \Pr(X = x \mid X) = \ldots + 0 + 0 + X + 0 + 0 + \ldots = X.$$

Continuous case:

$$\mathbb{E}(X|X) = \int_{x\in\mathbb{R}} x \cdot dF_X$$

Note that

$$\Pr(X \leq x \mid X) = F_{X|X}(x) = \begin{cases} 0 & x < X \\ 1 & x \geq X \end{cases}$$

so we have

$$\mathbb{E}(X|X) = \int_{x \in \mathbb{R}} x \cdot dF_X = X.$$

- Discrete case:

$$\mathbb{E}(X + Y \mid Z = z) = \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{Z}} (x + y) \Pr(X = x, Y = y \mid Z)$$

$$= \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{Z}} x \Pr(X = x, Y = y \mid Z) + \sum_{x \ \in \mathbb{Z}} \sum_{y \in \mathbb{Z}} y \Pr(X = x, Y = y \mid Z)$$

$$= \sum_{x \in \mathbb{R}} x \Pr(X = x \mid Z) + \sum_{y \in \mathbb{Z}} y \Pr(Y = y \mid Z) = \mathbb{E}(X|Z) + \mathbb{E}(Y|Z).$$

Continuous case:

$$\mathbb{E}(X + Y \mid Z = z) = \int_{x \in \mathbb{R}} \int_{y \in \mathbb{R}} (x + y) \Pr(X = x, Y = y \mid Z) dy dx$$

$$= \int_{x \in \mathbb{R}} \int_{y \in \mathbb{R}} x \Pr(X = x, Y = y \mid Z) dy dx + \int_{x \in \mathbb{R}} \int_{y \in \mathbb{R}} y \Pr(X = x, Y = y \mid Z) dy dx$$

$$= \int_{x \in \mathbb{R}} x \Pr(X = x \mid Z) dx + \int_{y \in \mathbb{R}} y \Pr(Y = y \mid Z) dy = \mathbb{E}(X|Z) + \mathbb{E}(Y|Z).$$

(v) Note by the definition of conditional probability that

$$\mathbb{P}(X = x \mid Y = y \cap Z = z) = \frac{\mathbb{P}(X = x \cap \{Y = y \cap Z = z\})}{\mathbb{P}(Y = y \cap Z = z)}$$

Using the independence of X and Y from Z and Lemma 6.2, we can express this as

$$= \frac{\mathbb{P}(X = x \cap Y = y)\mathbb{P}(Z = z)}{\mathbb{P}(Y = y)\mathbb{P}(Z = z)} = \frac{\mathbb{P}(X = x \cap Y = y)}{\mathbb{P}(Y = y)} = \mathbb{P}(X = x \mid Y = y)$$

by the definition of conditional probability. So $\mathbb{P}(X = x \mid Y = y, Z) = \mathbb{P}(X = x, Y = y)$. Therefore we have (in the discrete case)

$$\mathbb{E}(X \mid Y = y, Z) = \sum_{x \in \mathbb{R}} x\mathbb{P}(X = x \mid Y = y, Z) = \sum_{x \in \mathbb{R}} x\mathbb{P}(X = x, Y = y) = \mathbb{E}(X \mid Y = y) = g(y)$$

which implies that $\mathbb{E}(X \mid Y, Z) = g(Y) = \mathbb{E}(X \mid Y)$. In the continuous case, note that

$$F_{X|Y,Z}(x) = \mathbb{P}(X \leq x \mid Y, Z) = \mathbb{P}(X \leq x \mid Y) = F_{X|Y}(x)$$

by Lemma 6.2. Therefore we have

$$\mathbb{E}(X \mid Y, Z) = \int_{x \in \mathbb{R}} x dF_{X \mid Y, Z}(x) = \int_{x \in \mathbb{R}} x dF_{X \mid Y}(x) = \mathbb{E}(X \mid Y).$$

□

Corollary 6.6.1.

$\mathbb{E}(h(Y) \mid Y) = h(Y)$.

Proof. Use the first result in part (iii) of Theorem 6.6 with $X = 1$ and note that $\mathbb{E}(1 \mid Y) = 1$.

□

Corollary 6.6.2.

$\mathbb{E}[\mathbb{E}(X \mid Y)) \mid Y] = \mathbb{E}(X \mid Y)$.

Proof. Use the second result in part (iii) of Theorem 6.6 with $h(Y) = Y$.

□

Lemma 6.7. If $X \geq Z$ then $\mathbb{E}(X|Y) \geq \mathbb{E}(Z|Y)$.

Proof. Suppose that X and Z are nonnegative. Note that

$$\mathbb{E}(X \mid Y) = \int_0^\infty \Pr(X > t \mid Y) dt \geq \int_0^\infty \Pr(Z > t \mid Y) dt = \mathbb{E}(Z \mid Y)$$

where the third step follows since $X \geq Z$. If X and Z are not nonnegative, then

$$\begin{aligned} \mathbb{E}(X \mid Y) &= \mathbb{E}(\max\{X, 0\} \mid Y) - \mathbb{E}(\max\{-X, 0\} \mid Y) \\ &= \int_0^\infty \Pr(\max\{X, 0\} > t \mid Y) dt - \int_0^\infty \Pr(\max\{-X, 0\} > t \mid Y) dt \\ &\geq \int_0^\infty \Pr(\max\{Z, 0\} > t \mid Y) dt - \int_0^\infty \Pr(\max\{-Z, 0\} > t \mid Y) dt = \mathbb{E}(Z \mid Y) \end{aligned}$$

where the inequality follows since $X \geq Z$.

□

6.1.3 Convolution

Theorem 6.8. Sums of random variables. If X and Y are independent then

$$\Pr(X + Y = z) = f_{X+Y}(z) = \sum_x f_X(x) f_Y(z - x) = \sum_y f_X(z - y) f_Y(y)$$

Remark. **Convolution on the integers.** Let X, Y be independent integer-valued random variables. Let $t \in \mathbb{Z}$.

$$\Pr(X+Y=t) = \sum_{j,k \in \mathbb{Z}: j+k=t} \Pr(X=j, Y=k) = \sum_{j \in F} \Pr(X=j, Y=t-j) = \sum_{j \in \mathbb{Z}} \Pr(X=j)\Pr(Y=t-j)$$

$$= \sum_{j \in \mathbb{Z}} p_X(j) p_Y(t-j)$$

Definition 6.14. Let $g, h : \mathbb{Z} \to \mathbb{R}$ be functions. The **convolution** of g and h, denoted by $g * h$, is a function $g * h : \mathbb{Z} \to \mathbb{R}$ defined by

$$(g * h)(t) = \sum_{j \in \mathbb{Z}} g(j) h(t-j) \quad \forall t \in \mathbb{Z}$$

Definition 6.15. **(Convolution on the real line.)** Let $g, h : \mathbb{R} \to \mathbb{R}$ b functions. The **convolution** of g and h, denoted by $g * h$, is a function $g * h : \mathbb{R} \to \mathbb{R}$ defined by

$$(g * h)(t) = \int_{-\infty}^{\infty} g(x) h(t-x) dx, \quad \forall t \in \mathbb{R}$$

Proposition 6.9. Let X, Y be two continuous independent random variables such that $\Pr(X+Y \leq t)$ is differentiable with respect to $t \in \mathbb{R}$. Then

$$f_{X+Y}(t) = (f_X * f_Y)(t), \quad \forall t \in \mathbb{R}$$

Proof.

$$\Pr(X+Y \leq t) = \int_{\{(x,y) \in \mathbb{R}^2 : x+y \leq t\}} f_{X,Y}(x,y) dx dy = \int_{x=-\infty}^{x=\infty} \int_{y=-\infty}^{y=t-x} f_X(x) f_Y(y) dy dx$$

Then, since $\Pr(X+Y \leq t)$ is differentiable with respect to t, we have by the Fundamental Theorem of Calculus

$$f_{X+Y}(t) = \frac{\mathrm{d}}{\mathrm{d}t} \Pr(X+Y \leq t) = \int_{x=-\infty}^{x=\infty} f_X(x) \frac{\mathrm{d}}{\mathrm{d}t} \int_{y=-\infty}^{y=t-x} f_Y(y) dy dx = \int_{x=-\infty}^{x=\infty} f_X(x) f_Y(t-x) dx$$

□

Example 6.1. Let X, Y be independent standard Gaussian random variables. Then by Proposition 6.9, $X+Y$ has density

$$f_{X+Y}(t) = \int_{\infty}^{\infty} f_X(x) f_Y(t-x) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-(t-x)^2/2} dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-x^2+tx-t^2/2} dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(x-t/2)^2 - t^2/4 - t^2/2} dx$$

$$= e^{-t^2/4}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(x-t/2)^2}dx = \frac{1}{2\pi}e^{-t^2/4}\int_{-\infty}^{\infty} e^{-x^2}dx$$

Let $x = y/\sqrt{2}, dx = dy/\sqrt{2}$.

$$= \frac{1}{2\pi}e^{-t^2/4}\int_{-\infty}^{\infty} e^{-y^2/2}\cdot\frac{1}{\sqrt{2}}dy = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2}}e^{-t^2/4}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2/2}dy$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2}}e^{-t^2/4}\frac{1}{\sqrt{2\pi}}$$

which is the density for a Gaussian random variable distributed as $\mathcal{N}(0,1)$.

Definition 6.16. (Convoluted cdfs; Ross's in-class definition from ISE 620.) Suppose we have random variables X and Y with cdfs F_X and F_Y and pdfs f_X and f_Y. Then

$$(F_Y * F_X)(t) = \Pr(X+Y \leq t) = \int_{-\infty}^{\infty} \Pr(X+Y\leq t \mid X = x)f_X(x)dx$$

$$= \int_{-\infty}^{\infty} F_Y(t-x)f_X(x)dx$$

6.1.4 Compound Random Variables

Definition 6.17. Let $\{X_i\}$ be i.i.d. random variables. Let N be a random variable taking on positive integer values. Let

$$S = \sum_{i=1}^{N} X_i$$

Then S is a **compound random variable**.

Proposition 6.10. (Wald's Equation.) Let $\{X_i\}$ be i.i.d. random variables with mean $\mathbb{E}(X)$. Let N be a random variable taking on positive integer values, and let $S = \sum_{i=1}^{N} X_i$. Then $\mathbb{E}(S) = \mathbb{E}(N)\mathbb{E}(X)$.

Proof.

$$\mathbb{E}(S \mid N) = \mathbb{E}\left(\sum_{i=1}^{N} X_i\right) = \sum_{i=1}^{N}\mathbb{E}(X_i) = N\mathbb{E}(X)$$

$$\mathbb{E}(S) = \mathbb{E}(\mathbb{E}[S \mid N]) = \mathbb{E}(N\mathbb{E}(X)) = \mathbb{E}(N)\mathbb{E}(X)$$

□

Proposition 6.11. Let $\{X_i\}$ be i.i.d. random variables with mean $\mathbb{E}(X)$. Let N be a random variable taking on positive integer values, and let $S = \sum_{i=1}^{N} X_i$. Then $\text{Cov}(N, S) = \mathbb{E}(X)\text{Var}(N)$.

Proof. Let $\mathbb{E}(X_i) = \mathbb{E}(X)$ for all i. We will use the result from Wald's Equation (Proposition 6.10: $E(S) = \mathbb{E}(X)\mathbb{E}(N)$). We have

$$\text{Cov}(N, S) = \mathbb{E}(NS) - \mathbb{E}(N)\mathbb{E}(S) = \mathbb{E}[\mathbb{E}(NS \mid N)] - \mathbb{E}(N)\mathbb{E}(N)\mathbb{E}(X) = \mathbb{E}[N\mathbb{E}(S \mid N)] - \mathbb{E}(N)^2\mathbb{E}(X) \quad (6.1)$$

Note that

$$\mathbb{E}(S \mid N) = \mathbb{E}\left(\sum_{i=1}^{N} X_i\right) = \sum_{i=1}^{N} \mathbb{E}(X_i) = N\mathbb{E}(X) \implies \mathbb{E}[N\mathbb{E}(S \mid N)] = \mathbb{E}(N^2\mathbb{E}(X)) = \mathbb{E}(X)\mathbb{E}(N^2)$$

Plugging this into (6.1) yields

$$\text{Cov}(N, S) = \mathbb{E}(X)\mathbb{E}(N^2) - \mathbb{E}(N)^2\mathbb{E}(X) = \mathbb{E}(X)\left[\mathbb{E}(N^2) - \mathbb{E}(N)^2\right] = \mathbb{E}(X)\text{Var}(N)$$

□

Definition 6.18 (Compound Poisson random variable). Let N be a Poisson random variable. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables that are also independent of N. Then

$$S := \sum_{i=1}^{N} X_i$$

is called a **compound Poisson random variable.**

Proposition 6.12 (Variance of a compound Poisson random variable, from Ross *Introduction to Probability Models*). For a compound Poisson random variable $S = \sum_{i=1}^{N} X_i$ having $\mathbb{E}(N) = \lambda, \mathbb{E}(X_i) = \mu, \text{Var}(X_i) = \sigma^2$,

$$\text{Var}(S) = \lambda\sigma^2 + \lambda\mu^2 = \lambda\mathbb{E}(X^2).$$

6.1.5 Odds and Ends

Proposition 6.13. Inclusion-Exclusion Principle:

(a)

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n}(-1)^{k+1}\left(\sum_{1 \le i_1 < \ldots < i_k \le m} \Pr\left(A_{i1} \cap \ldots \cap A_{ik}\right)\right)$$

(b)

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n}(-1)^{k+1}\left(\sum_{1 \le i_1 < \ldots < i_k \le m} |A_{i1} \cap \ldots \cap A_{ik}|\right)$$

To prove this, we will first prove the Multi-Binomial Theorem.

Lemma 6.14. (Multi-Binomial Theorem)

$$\prod_{i=1}^{d}(x_i+y_i)^{n_i} = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\cdots\sum_{k_d=0}^{n_d}\binom{n_1}{k_1}x_1^{k_1}y_1^{n_1-k_1}\binom{n_2}{k_2}x_2^{k_2}y_2^{n_2-k_2}\cdots\binom{n_d}{k_d}x_d^{k_d}y_d^{n_d-k_d}$$

Proof. □

We are now ready to prove the Inclusion-Exclusion Principle.

Proof. (Proof of (a).) Begin by noting that

$$\mathbf{1}_{\{\cup_{i=1}^{n}A_i\}} = 1-\prod_{i=1}^{n}(1-\mathbf{1}_{\{A_i\}}) \tag{6.2}$$

because the expression on the right will equal 1 if at least one term in the product equals 0 (that is, if $\mathbf{1}_{\{A_i\}} = 1$ for some $i \in 1,\ldots,n$) and will equal 0 if every term in the product equals 1 (if $\mathbf{1}_{\{A_i\}} = 0$ for every $i \in 1,\ldots,n$), which is exactly what we want. Expanding the right side of (6.2) using the Multi-Binomial Theorem (Lemma 6.14), we have

$$= 1-\prod_{i=1}^{n}(1-\mathbf{1}_{\{A_i\}}) = 1-(1-\mathbf{1}_{\{A_1\}})(1-\mathbf{1}_{\{A_2\}})\cdots(1-\mathbf{1}_{\{A_n\}})$$

$$= 1-\left[1+\sum_{k=1}^{n}(-1)^k\left(\sum_{1\le i_1<\ldots<i_k\le k}\mathbf{1}_{\{A_{i_1}\}}\cdots\mathbf{1}_{\{A_{i_k}\}}\right)\right] = -1\cdot\sum_{k=1}^{n}(-1)^k\left(\sum_{1\le i_1<\ldots<i_k\le k}\mathbf{1}_{\{A_{i_1}\cap\ldots\cap A_{i_k}\}}\right)$$

$$= \sum_{k=1}^{n}(-1)^{k+1}\left(\sum_{1\le i_1<\ldots<i_k\le k}\mathbf{1}_{\{A_{i_1}\cap\ldots\cap A_{i_k}\}}\right)$$

$$\implies \mathbf{1}_{\{\cup_{i=1}^{n}A_i\}} = \sum_{k=1}^{n}(-1)^{k+1}\left(\sum_{1\le i_1<\ldots<i_k\le k}\mathbf{1}_{\{A_{i_1}\cap\ldots\cap A_{i_k}\}}\right) \tag{6.3}$$

Taking expectations of both sides of (6.3) yields

$$\Pr(\cup_{i=1}^{n}A_i) = \mathbb{E}\left[\sum_{k=1}^{n}(-1)^{k+1}\left(\sum_{1\le i_1<\ldots<i_k\le k}\mathbf{1}_{\{A_{i_1}\cap\ldots\cap A_{i_k}\}}\right)\right]$$

$$= \sum_{k=1}^{n}(-1)^{k+1}\left(\sum_{1\le i_1<\ldots<i_k\le k}\mathbb{E}[\mathbf{1}_{\{A_{i_1}\cap\ldots\cap A_{i_k}\}}]\right)$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{1 \le i_1 < \ldots < i_k \le k} \Pr\left(A_{i_1} \cap \ldots \cap A_{i_k}\right) \right)$$

□

Proposition 6.15. (Proposition 1.6.1 in Sheldon Ross *A First Course in Probability.*) There are $\binom{n-1}{r-1}$ distinct positive integer-valued vectors $(x_1, x_2, \ldots, x_r), x_i > 0 \; \forall \; i$ satisfying the equation $x_1 + x_2 + \ldots + x_r = n$.

Proof. (Not rigorous, but a justification.) Imagine we have n indistinguishable objects to allocate to r people. We lay out the n objects and take $r-1$ sticks to place in the $n-1$ spaces between them. The first person gets all the objects to the left of the leftmost stick, the second person gets the objects between the leftmost and second leftmost stick, and so on, until the last person gets all the objects to the right of the rightmost stick. The constraint that x_i be positive is equivalent to saying that each person must receive at least one object. Therefore we must place each stick in a different place. There are $\binom{n-1}{r-1}$ ways to do this.

□

Proposition 6.16. (Proposition 1.6.2 in Sheldon Ross *A First Course in Probability.*) There are $\binom{n+r-1}{r-1}$ distinct nonnegative integer-valued vectors $(x_1, x_2, \ldots, x_r), x_i \ge 0 \; \forall \; i$ satisfying the equation $x_1 + x_2 + \ldots + x_r = n$.

Proof. We would like to solve the problem

$$x_1 + x_2 + \ldots + x_r = n, x_i \ge 0 \; \forall i$$

Note that we can transform this problem in the following way:

$$x_1 + 1 + x_2 + 1 + \ldots + x_r + 1 = n + 1 \cdot r, x_i + 1 \ge 1 \; \forall i$$

Letting $y_i = x_i + 1$, we have the equivalent system

$$y_1 + y_2 + \ldots + y_r = n + r, y_i \ge 1 \; \forall i$$

Since $y_i \ge 1 \iff y_i > 0$, by Proposition 6.15, the number of distinct solutions to this equation is $\binom{n+r-1}{r-1}$.

□

Proposition 6.17. More generally, if we desire solutions to $x_1 + x_2 + \ldots + x_r = n$ such that $x_i \ge k \in \mathbb{N}$, then $\binom{n+r\cdot(1-k)-1}{r-1} = \binom{n+r-1-rk}{r-1}$ solutions are possible.

Proof. We can construct a similar argument to that used in the proof of Proposition 6.16 by adding $r \cdot (1-k)$ to each side of $x_1 + x_2 + \ldots + x_r = n$:

$$x_1 + 1 - k + x_2 + 1 - k + \ldots + x_r + 1 - k = n + r \cdot (1-k), \;\; x_i \ge k \; \forall i$$

Then substitute $y_i = x_i + 1 - k$ to yield

$$y_1 + y_2 + \ldots + y_r = n + r \cdot (1 - k),\ y_i + k - 1 \geq k\ \forall i$$

then apply Proposition 6.15 noting that $y_i + k - 1 \geq k \iff y_i \geq 1 \iff x_i \geq k$ to yield the result. ☐

Proposition 6.18. Even more generally, suppose we desire solutions to $x_1 + x_2 + \ldots + x_r = n$ such that $x_1 \geq k_1 \in \mathbb{N}, x_2 \geq k_2 \in \mathbb{N}, \ldots, x_r \geq k_r \in \mathbb{N}$. Then

$$\binom{n + \sum_{i=1}^r (1 - k_i) - 1}{r - 1} = \binom{n + r - 1 - \sum_{i=1}^r k_i}{r - 1}$$

solutions are possible.

Proof. Very similar to the proof of Proposition 6.17. Add $\sum_{i=1}^r (1-k_i)$ to each side of $x_1+x_2+\ldots+x_r = n$:

$$x_1 + 1 - k_1 + x_2 + 1 - k_2 + \ldots + x_r + 1 - k_r = n + \sum_{i=1}^r (1 - k_i),\ x_i \geq k_i\ \forall i$$

Then substitute $y_i = x_i + 1 - k_i$ to yield

$$y_1 + y_2 + \ldots + y_r = n + \sum_{i=1}^r (1 - k_i),\ y_i + k_i - 1 \geq k_i\ \forall i$$

Finally, apply Proposition 6.15 noting that $y_i + k_i - 1 \geq k_i \iff y_i \geq 1 \iff x_i \geq k_i$ to yield the result. ☐

Proposition 6.19. Suppose we desire solutions to $x_1 + x_2 + \ldots + x_r = n$ such that $\tau \leq r$ of the $\{x_i\}$ exceed some threshold k. For example, if we have $x_1 \geq k, x_2 \geq k, \ldots, x_\tau \geq k$, with $x_{\tau+1}, \ldots, x_r$ taking on arbitrary values, then the condition is satisfied. (The particular x_i that exceed k does not matter, as long as τ of them exceed k). Then

$$\binom{r}{\tau}\binom{n + r - 1 - k\tau}{r - 1}$$

solutions are possible.

Proof. By Proposition 6.18, the number of ways this condition can be met for a particular set of τ variables x_i is $\binom{n+r-1-k\tau}{r-1}$. Since there are $\binom{r}{\tau}$ ways to choose which τ variables will exceed k, the result follows. ☐

6.1.6 Methods for Calculating Quantities

- Expectation
  -

Definition 6.19. (Math 541A definition 1.37.) Let Ω be a sample space, let $\mathbb{P}$ be a probability law on Ω. Let X be a random variable on Ω. Assume that X takes on nonnegative values; that is, $X : \Omega \to [0, \infty)$. We define the **expected value** of X by

$$\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X > t)dt$$

In analytic notation, $\mathbb{E}(X) = \int_\Omega X(\omega)d\mathbb{P}(\omega)$. More generally, if $g : [0 \to \infty) \to [0 \to \infty)$ is a differentiable function such that g' is continuous and $g(0) = 0$, we define

$$\mathbb{E}(g(X)) = \int_0^\infty g'(t)\mathbb{P}(X > t)dt$$

For a general random variable X, if $\mathbb{E}(\max\{X, 0\} < \infty)$ and if $\mathbb{E}(\max\{-X, 0\}) < \infty$, we then define $\mathbb{E}(X) = \mathbb{E}(\max\{X, 0\}) - \mathbb{E}(\max\{-X, 0\})$. Otherwise, we say that $\mathbb{E}(X)$ is undefined.

Definition 6.20. In the above, taking $g(t) = t^n$ for any positive integer n, for any $t \geq 0$, we have

$$\mathbb{E}(X^n) = \int_0^\infty nt^{n-1}\mathbb{P}(X > t)dt$$

Remark. If we assume that the expected value and the integral on $\mathbb{R}$ can be commuted, then the following derivation of the formula for $\mathbb{E}(g(X))$ can be given. From the Fundamental Theorem of Calculus, we have

$$g(X) = \int_0^X g'(t)dt = \int_0^\infty g'(t)\mathbf{1}_{\{X>t\}}dt$$

where $\mathbf{1}_{\{X>t\}}$ is an indicator variable. Therefore

$$\mathbb{E}(g(X)) = \int_0^\infty g'(t)\mathbf{1}_{\{X>t\}}dt = \int_0^\infty g'(t)\mathbb{E}(\mathbf{1}_{\{X>t\}}) = \int_0^\infty g'(t)\mathbb{P}(X > t)dt.$$

–

Definition 6.21. (Discrete random variables.) $\mathbb{E}(X) = \sum_x x \Pr(X = x)$

–

Theorem 6.20. (a) $\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$
(b) If $X \geq 0$ then $\mathbb{E}(X) \geq 0$

–

Theorem 6.21. Expectation of sums is sum of expectations if sum is finite or if sum is infinite and all variables are positive, but not necessarily otherwise.

Example 6.2 (Example 4.18 in *Introduction to Probability Models*). Consider a sequence of discrete random variables $X_1, X_2, \ldots$ such that $X_i = 1$ with probability $1/2$ and $X_i = -1$ with probability $1/2$. Let N be a stopping time:

$$N = \min\{n : X_1, + \ldots + X_n = 1\} \implies 1 = X_1 + \ldots + X_N.$$

Let $I\{i \leq N\}$ be an indicator variable for $i \leq N$. So

$$1 = \sum_{i=1}^N X_i = \sum_{i=1}^\infty X_i I\{i \leq N\}.$$

Taking expectations we have

$$1 = \mathbb{E} \sum_{i=1}^{\infty} X_i I\{i \leq N\}.$$

Note that $N \geq i$ if and only if you have not yet stopped after the first $i-1$ games, so it depends on the results of the previous games but not on X_i. So $I\{i \leq N\}$ and X_i are independent. Therefore

$$\mathbb{E}[X_i I\{i \leq N\} = \mathbb{E}(X_i)\mathbb{E}(I\{i \leq N\}) = 0$$

since $\mathbb{E}(X_i) = 0$. So

$$\sum_{i=1}^{\infty} \mathbb{E}[X_i I\{i \leq N\}] = \sum_{i=1}^{\infty} 0 = 0$$

which means that

$$\mathbb{E} \sum_{i=1}^{\infty} X_i I\{i \leq N\} \neq \sum_{i=1}^{\infty} \mathbb{E}[X_i I\{i \leq N\}].$$

See also Example 7.37 in the Stochastic Processes notes.

–

Theorem 6.22. Law of the Unconscious Statistician: If X has mass function f, and $g : \mathbb{R} \to \mathbb{R}$, then

$$\mathbb{E}(g(X)) = \sum_x g(x) f(x)$$

–

Theorem 6.23. Expectation is a linear operator: $\mathbb{E}\big(\sum_i X_i\big) = \sum_i \mathbb{E}(X_i)$

–

Theorem 6.24. (Layer cake formulation.) If N is a discrete random variable taking on non-negative values, then $\mathbb{E}(N) = \sum_{i=1}^{\infty} \Pr(N \geq i)$.

Proof. Let $\mathbf{1}_{\{i \leq N\}}$ be an indicator variable. Then

$$N = \sum_{i=1}^{\infty} \mathbf{1}_{\{i \leq N\}}.$$

Take expectations of both sides to yield the result.

□

Remark. Also note that we can derive this result from Definition 6.19:

$$\mathbb{E}(X) = \int_0^{\infty} \Pr(X > t) dt = \sum_{k=1}^{\infty} \int_{k-1}^{k} \Pr(X > t) dt = \sum_{k=1}^{\infty} \Pr(X \geq k) = \sum_{k=0}^{\infty} \Pr(X > k).$$

Using Fubini's Theorem (Theorem 5.35) to rearrange the sum, we can arrive at

$$\mathbb{E}(X) = \sum_{k=0}^{\infty} \Pr(X > k) = \sum_{k=0}^{\infty} \sum_{j=k+1}^{\infty} \Pr(X = j) = \sum_{0 \leq k < j < infty} \Pr(X = j)$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{j} \mathbb{P}(X = j) = \sum_{j=0}^{\infty} j \Pr(X = j)$$

which is the usual definition for a discrete random variable.

– For the conditional expectation of one Gaussian random variable on another when the covariance or correlation between them is known, see Proposition 6.69. For the conditional expectation of a set of Gaussian random variables on another set when the covariance matrix is known, see Proposition 6.70.

- Variance

  –

  Definition 6.22. $\text{Var}(X) = \mathbb{E}(X - \mathbb{E}(X))^2$

  –

  Proposition 6.25. (Useful reformulation:) $\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$

  –

  Theorem 6.26. (Some useful results):

  (a) $\text{Var}(aX) = a^2 \text{Var}(X)$
  (b) $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$
  (c) $\text{Var}(aX \pm bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y) \pm 2ab\text{Cov}(X, Y)$
  (d) **Variance-Covariance Expansion.** $\text{Var}\left(\sum_{i=1}^n X_i \right) = \sum_{i=1}^n \text{Var}(X_i) + 2 \sum_{1 \le i < j \le n} \text{Cov}(X_i, X_j)$

  –

  Definition 6.23. Conditional variance:

  $$\text{Var}(X \mid Y) = \mathbb{E}\big[(X - \mathbb{E}(X \mid Y))^2 \mid Y\big]$$

  –

  Theorem 6.27. Law of Total Variance: $\text{Var}(X) = \text{Var}\big(\mathbb{E}(X \mid Y)\big) + \mathbb{E}\big(\text{Var}(X \mid Y)\big)$

  Proof.

  $$\text{Var}\big(\mathbb{E}(X \mid Y)\big) = \mathbb{E}[(\mathbb{E}(X \mid Y))^2] - [\mathbb{E}(\mathbb{E}(X \mid Y))]^2 = \mathbb{E}[(\mathbb{E}(X \mid Y))^2] - \mathbb{E}(X)^2 \tag{6.4}$$

  $$\text{Var}(X \mid Y) = \mathbb{E}(X^2 \mid Y) - (\mathbb{E}(X \mid Y))^2 \implies \mathbb{E}[\text{Var}(X \mid Y)] = \mathbb{E}(X^2) - \mathbb{E}[(\mathbb{E}(X \mid Y))^2] \tag{6.5}$$

  Adding together (6.4) and (6.5) yields

  $$\text{Var}\big(\mathbb{E}(X \mid Y)\big) + \mathbb{E}\big(\text{Var}(X \mid Y)\big) = \mathbb{E}[(\mathbb{E}(X \mid Y))^2] - \mathbb{E}(X)^2 + \mathbb{E}(X^2) - \mathbb{E}[(\mathbb{E}(X \mid Y))^2]$$

  $$= \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \text{Var}(X)$$

  □

  Corollary 6.27.1. (Rao-Blackwell Theorem.) $\text{Var}(X) \ge \text{Var}(\mathbb{E}(X \mid Y))$

Proof. Follows immediately from Theorem 6.27 by noting that since the variance is nonnegative, $\mathbb{E}\big(\text{Var}(X \mid Y)\big) \geq 0$.

$\square$

–

Proposition 6.28. If $c \in \mathbb{R}$, then $\text{Var}(c) = 0$.

– For the conditional variance of one Gaussian random variable on another when the covariance or correlation between them is known, see Proposition 6.69. For the conditional variance of a set of Gaussian random variables on another set when the covariance matrix is known, see Proposition 6.70.

- Covariance

–

Definition 6.24. $\text{Cov}(X, Y) = \mathbb{E}\big[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))\big]$

–

Proposition 6.29. (Useful reformulation): $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$

–

Theorem 6.30. (Some useful results):

(a) $\text{Cov}(aX, BY) = ab\text{Cov}(X, Y)$

(b) $\text{Cov}(X, X) = \text{Var}(X)$

(c) $\text{Cov}(aX + bY) = ac\text{Var}(X) + bd\text{Var}(Y) + (ad + bc)\text{Cov}(X, Y)$

(d) $\text{Cov}(X_1 + X_2, Y) = \text{Cov}(X_1, Y) + \text{Cov}(X_2, Y)$

(e) $\text{Cov}\left(\sum_{i=1}^n X_i, Y\right) = \sum_{i=1}^n \text{Cov}(X_i, Y)$

(f) $\text{Cov}\left(\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j\right) = \sum_{i=1}^n \sum_{j=1}^m \text{Cov}(X_i, Y_j)$

–

Definition 6.25. Conditional covariance:

$$\text{Cov}(X, Y \mid Z) = \mathbb{E}(XY \mid Z) - \mathbb{E}(X \mid Z)\mathbb{E}(Y \mid Z) = \mathbb{E}\big[(X - \mathbb{E}(X \mid Z))(Y - \mathbb{E}(Y \mid Z)) \mid Z\big]$$

–

Theorem 6.31. Law of Total Covariance:

$$\text{Cov}(X, Y) = \mathbb{E}\big(\text{Cov}(X, Y \mid Z)\big) + \text{Cov}\big(\mathbb{E}(X \mid Z), \mathbb{E}(Y \mid Z)\big)$$

6.1.7 Discrete Random Variable Distributions

Binomial: Binomial(n, p) (sum of n Bernoulli random variables)

- Mass function: $\Pr(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$
- Distribution: $\Pr(X \leq k) = \sum_{i=0}^k \binom{n}{i} p^i (1 - p)^{n-i}$

- Expectation: $\mathbb{E}(X) = np$
- Variance: $\text{Var}(X) = np(1-p)$

Multinomial:

Definition 6.26 (Multinomial($n, p_1, p_2, \ldots, p_{r-1}$) **distribution).** Suppose that n independent trials, each of which results in either outcome $1, 2, \ldots, r$ with respective probabilities $p_1, p_2, \ldots, p_r$ (with $\sum_i p_i = 1$), are performed. Let N_i denote the number of trials resulting in outcome i. Then the joint distribution of $N_1, \ldots, N_r$ is called the **multinomial distribution.**

- Mass function:

$$\Pr(\boldsymbol{N} = \boldsymbol{\mathcal{N}}) = \binom{n}{\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_r} \prod_{i=1}^{r} p_i^{\mathcal{N}_i} = \frac{n!}{\mathcal{N}_1! \mathcal{N}_2! \ldots \mathcal{N}_r!} \prod_{i=1}^{r} p_i^{\mathcal{N}_i}$$

Proof. For $r = 2$, we have the binomial distribution: $\Pr(N_1 = x_1 = \binom{n}{x_1} p_1^{x_1}(1-p_1)^{n-x_1}$; $\Pr(N_2 = x_2 = \binom{n}{x_2} p_2^{x_2}(1-p_2)^{n-x_2}$. But in the general case, it is still true that N_i is a binomial random variable for all i. So in general we have $\Pr(N_i = x_i) = \binom{n}{x_i} p_i^{x_i}(1-p_i)^{n_i - x_i}, \quad i\ in\{1, 2, \ldots, r\}$.

We would like the distribution of the random vector $\boldsymbol{N} = (N_1, \ldots, N_r)$. First consider the case where $n = 1$; that is, the Multinouli distribution. Note that in this case, we have $\Pr(\boldsymbol{N} = \boldsymbol{\mathcal{N}}) = \prod_{i=1}^{r} p_i^{\mathcal{N}_i}$, where $\mathcal{N}_i$ are the entries of the observed vector $\boldsymbol{\mathcal{N}}$ ($\mathcal{N}_i \in \{0, 1\}$.

Now consider an arbitrary $n \in \mathbb{N}$. Then $\boldsymbol{N}$ is the sum of n Multinouli random variables. Just as before, the probability of a particular $\boldsymbol{\mathcal{N}}$ obtained in a particular ordering is $\Pr(\boldsymbol{N} = \boldsymbol{\mathcal{N}}) = \prod_{i=1}^{r} p_i^{\mathcal{N}_i}$. However, we must also consider the number of possible orderings in which these successes could have occurred. This number of orderings is exactly equal to the multinomial coefficient,

$$\binom{n}{\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_r} = \frac{n!}{\mathcal{N}_1! \mathcal{N}_2! \ldots \mathcal{N}_r!}$$

Therefore the joint probability mass function for $\boldsymbol{N}$ is

$$\Pr(\boldsymbol{N} = \boldsymbol{\mathcal{N}}) = \binom{n}{\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_r} \prod_{i=1}^{r} p_i^{\mathcal{N}_i} = \frac{n!}{\mathcal{N}_1! \mathcal{N}_2! \ldots \mathcal{N}_r!} \prod_{i=1}^{r} p_i^{\mathcal{N}_i}$$

$\square$

- Distribution: $\Pr(X \leq k) =$
- Expectation: $\mathbb{E}(X) =$
- Variance: $\text{Var}(X) =$

Proposition 6.32. $\text{Cov}(N_i, N_j) = -np_i p_j$.

Proof. For a Multinoulli random variable, we have

$$\text{Cov}(N_i, N_j) = \mathbb{E}(N_i N_j) - \mathbb{E}(N_i)\mathbb{E}(N_j) = 0 - p_i p_j = \ldots = -p_i p_j$$

(where $\mathbb{E}(N_i)\mathbb{E}(N_j) = 0$ because at least one of them must equal 0). Since a multinomial random variable is the sum of n independent Multinoulli random variables, in the general case we have

$$\text{Cov}(N_i, N_j) = -np_i p_j$$

□

Poisson: Poisson(λ): an approximation of the binomial distribution for n very large, p very small, $np \to \lambda \in (0, \infty)$.

- Mass function:
$$\Pr(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$
- Distribution: $\Pr(X \leq k) = \sum_{i=0}^{k} \frac{e^{-\lambda}\lambda^i}{i!}$
- Expectation: $\mathbb{E}(X) = \lambda$ (derive from basic definitions)
- Variance: $\text{Var}(X) = \lambda$
- Moment-generating function: $M_X(t) = e^{\lambda(e^t - 1)}$

Proposition 6.33. Let $X \sim \text{Binomial}(n, p)$. Then

$$\lim_{n\to\infty,\ p\to 0,\ np\to\lambda} X \sim \text{Poisson}(np).$$

Proof.
$$\lim_{n\to\infty,\ p\to 0,\ np\to\lambda} \binom{n}{k} p^k (1-p)^{n-k} = \lim_{n\to\infty,\ p\to 0,\ np\to\lambda} \frac{n!}{(n-k)!k!} p^k (1-p)^{n-k}$$

$$= \frac{1}{k!} \cdot \lim_{n\to\infty,\ p\to 0,\ np\to\lambda} (1-p)^{n-k} p^k \prod_{i=0}^{k-1} (n-i) = \frac{1}{k!} \cdot \lim_{n\to\infty,\ p\to 0,\ np\to\lambda} \left(1 - \frac{np}{n}\right)^{n-k} p^k \prod_{i=0}^{k-1} (n-i)$$

Using $\lim_{n\to\infty}(1 - \lambda/n)^n = \exp(\lambda)$, and letting $\lambda = np$, we have

$$= \frac{\exp(-np)(np)^k}{k!} = \boxed{\frac{\exp(-\lambda)\lambda^k}{k!}} \sim \text{Poisson}(np)$$

□

Proposition 6.34. Let $X \sim \text{Poisson}(\lambda)$. If λ is sufficiently large (say $\lambda > 20$, then we can use the approximation

$$X \sim \mathcal{N}(\lambda, \lambda)$$

Proof. (Informal justification.) By Proposition 6.33, a Poisson distribution can be thought of as a close approximation of a binomial distribution. Since a binomial distribution can be approximated by a normal distribution for n large and np not too large, the same is true of a Poisson distribution. ☐

Geometric: $\mathrm{G}_1(p)$: the number of Bernoulli trials before the first success.

- Mass function: $\Pr(X = k) = p(1-p)^{k-1}$
- Distribution: $\Pr(X \leq k) = \sum_{i=1}^{k} p(1-p)^{k-1}$
- Expectation: $\mathbb{E}(X) = 1/p$
- Variance: $\mathrm{Var}(X) = (1-p)/p^2$

Negative binomial: $\mathrm{NB}(r, p)$: The number of Bernoulli trials required for r successes. (Can be derived as the sum of r identically distributed geometric random variables.)

- Mass function: $\Pr(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$
- Distribution: $\Pr(X \leq k) = \sum_{i=r}^{k} \binom{i-1}{r-1} p^r (1-p)^{i-r}$
- Expectation: $\mathbb{E}(X) = \frac{r}{p}$
- Variance: $\mathrm{Var}(X) = \frac{r(1-p)}{p^2}$
- Moment-generating function:

$$M_X(t) = \frac{(pe^t)^r}{[1-(1-p)e^t]^r}$$

Hypergeometric: Hypergeometric(N, M, K): When drawing a sample of size K from a group of N items, M of which are special, X is the number of special items retrieved.

- Mass function:
$$\Pr(X = k) = \frac{\binom{M}{k}\binom{N-M}{K-k}}{\binom{N}{K}}$$
- Distribution:
$$\Pr(X \leq k) = \sum_{i=0}^{k} \frac{\binom{M}{i}\binom{N-M}{K-i}}{\binom{N}{K}}$$
- Expectation: $\mathbb{E}(X) = \frac{MK}{N}$

Proof. Let Y_j be an indicator variable for special item j being selected. Note that $X = \sum_{j=1}^M Y_j$ and that

$$\mathbb{E}(Y_j) = \frac{\binom{1}{1}\binom{N-1}{K-1}}{\binom{N}{K}} = \frac{1 \cdot \frac{(N-1)!}{(N-K)!(K-1)!}}{\frac{N!}{(N-K)!K!}} = \frac{K}{N},$$

so we have

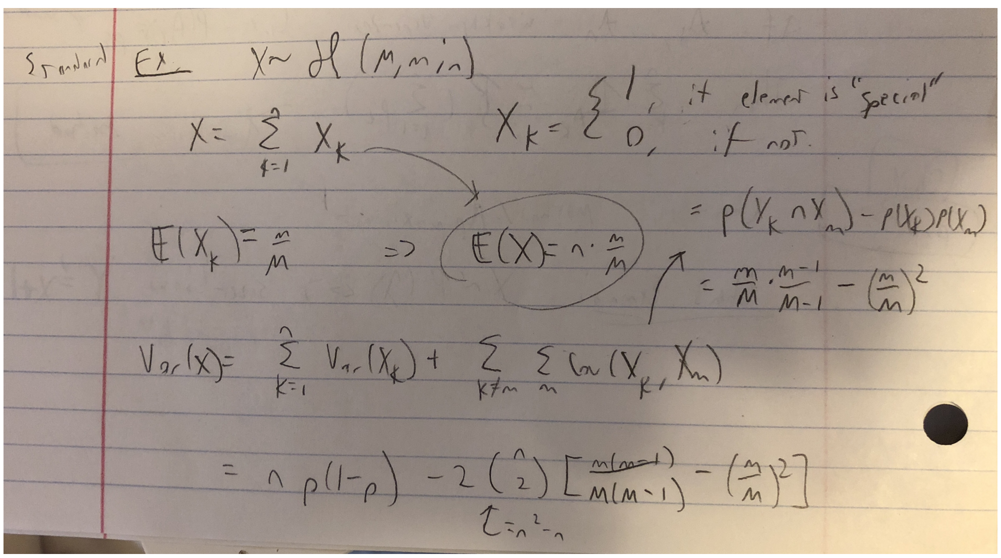
$$\mathbb{E}(X) = \mathbb{E}\bigg(\sum_{j=1}^M Y_j\bigg) = \sum_{j=1}^M \mathbb{E}(Y_j) = \frac{MK}{N}.$$

Alternative proof:

Let Z_i be an indicator variable for the ith selected item to be special. Then $X = \sum_{i=1}^K Z_i$ and $\mathbb{E}(Z_i) = \frac{M}{N}$, so we have

$$\mathbb{E}(X) = \mathbb{E}\bigg(\sum_{i=1}^K Z_i\bigg) = \sum_{i=1}^K \mathbb{E}(Z_i) = \frac{MK}{N}.$$

Variance: $\text{Var}(X) =$ (find by indicator method. Proof from notes, I think this may have errors:)



□

Proposition 6.35. Let $X \sim \text{Hypergeometric}(N, M, K)$. Then

$$\lim_{M,N\to\infty, M/N\to p} \Pr(X = k) \sim \text{Binomial}(K, p)$$

Proof.

$$\lim_{M,N\to\infty, M/N\to p} p_k(M, N, K) = \lim_{M,N\to\infty, M/N\to p} \frac{\binom{M}{k}\binom{N-M}{K-k}}{\binom{N}{K}}$$

$$= \lim_{M,N\to\infty, M/N\to p} \frac{M!(N-M)!/[k!(M-k)!(K-k)!(N-M-K+k)!]}{N!/[K!(N-K)!]}$$

$$= \lim_{M,N\to\infty, M/N\to p} \frac{K!}{(K-k)!k!} \cdot \frac{M!(N-M)!(N-K)!}{N!(M-k)!(N-M-K+k)!}$$

$$= \lim_{M,N\to\infty, M/N\to p} \binom{K}{k} \cdot \frac{M!/(M-k)!}{N!/(N-k)!} \cdot \frac{(N-M)!(N-K)!}{(N-k)!(N-M-(K-k))!}$$

$$= \binom{K}{k} \lim_{M,N\to\infty, M/N\to p} \frac{M!/(M-k)!}{N!/(N-k)!} \cdot \frac{(N-M)!/(N-M-(K-k))!}{(N-K+(K-k))!/(N-K)!}$$

$$= \binom{K}{k} \lim_{M,N\to\infty, M/N\to p} \prod_{i=0}^{k-1} \frac{M-i}{N-i} \cdot \prod_{j=0}^{K-k-1} \frac{N-M-j}{N-K+1+j}$$

$$= \binom{K}{k} \left(\frac{M}{N}\right)^k \left(\frac{N-M}{N}\right)^{K-k}$$

$$= \binom{K}{k} \left(\frac{M}{N}\right)^k \left(1-\frac{M}{N}\right)^{K-k} = \binom{K}{k} p^k (1-p)^{K-k}$$

□

6.1.8 Indicator Method

Proposition 6.36. If $\mathbf{1}_{A_k}$ is an indicator then

(a)
$$\operatorname{Cov}(\mathbf{1}_{A_k}, \mathbf{1}_{A_m}) = \mathbb{E}(\mathbf{1}_{A_k}\mathbf{1}_{A_m}) - \mathbb{E}(\mathbf{1}_{A_k})\mathbb{E}(\mathbf{1}_{A_m}) = \Pr(A_k \cap A_m) - \Pr(A_k)\Pr(A_m)$$

(b)
$$\operatorname{Var}(\mathbf{1}_{A_k}) = \mathbb{E}(\mathbf{1}_{A_k}^2) = \mathbb{E}(\mathbf{1}_{A_k})^2 = \Pr(A_k) - (\Pr(A_k))^2$$

Theorem 6.37. *X is independent of Y if and only if X is independent of $\mathbf{1}_A$, $A \in Y$.*

Example problems: 505A Homework 3 problem 9(a)

Worked examples in p. 56 - 59 of Grimmett and Stirzaker 3rd edition.

6.1.9 Linear transformations of random variables

6.1.10 Poisson Paradigm (Poisson approximation for indicator method)

Theorem 6.38. (Theorem 4.12.9, p. 129 of Grimmett and Stirzaker.) *Let A_i be an event. If $X = \sum_{i=1}^m \mathbf{1}_{A_i}$ where $\mathbf{1}_{A_i}$ is an indicator variable for A_i, and the A_i are only weakly dependent on each other, then*

$$\text{As } m \to \infty, \quad X \sim \text{Poisson}(\mathbb{E}(X))$$

More specifically, let B_i be n independent Bernoulli random variables with probabilities p_i. If $Y = \sum_{i=1}^n B_i$ then

$$\text{As } n \to \infty, \quad Y \sim \text{Poisson}\left(\mathbb{E}\left(\sum_i B_i\right)\right) = \text{Poisson}\left(\sum_i \mathbb{E}B_i\right) = \text{Poisson}\left(\sum_i p_i\right)$$

Proof. Full proof available in Grimmett and Stirzaker, section 4.12, page 129. A justification of the first claim is as follows: if the A_i are independent and $\Pr(A_i) = p \; \forall i$, then $X \sim \text{Binomial}(m, p)$. Then by Proposition 6.33, the result follows. It turns out that this result holds up if the probabilities are not necessarily identical (but all small) and the variables are not necessarily independent (but only weakly dependent). □

Solution. Alternative Solution to Exercise 1 (Matching Problem):

Let X be the number of matches. Let $P_n = \Pr(X = 0)$ given that there are n people. Let Y be an indicator variable for the first person who receives their sandwich receiving the correct one. Note that

$$P_n = \Pr(X = 0) = \Pr(X = 0 \mid Y = 1)(1/n) + \Pr(X = 0 \mid Y = 0)(n-1)/n.$$

Note that if the first person didn't match ($Y = 0$), we have $n-2$ people with their hats left, but one of the remaining $n-1$ people can't get their sandwich because it was taken by the first person. Therefore

$$\Pr(X = 0 \mid Y = 0) = \Pr(\{\text{the extra person selects the first person's hat, and the rest of the people pick the wrong hat}\})$$

$$+ \Pr(\{\text{the extra person does not select the first person's hat, and the rest of the people don't have any matches}\})$$

$$= \frac{1}{n-1} \cdot P_{n-2} + P_{n-1}$$

This yields

$$P_n = \frac{1}{n} P_{n-2} + \frac{n-1}{n} P_{n-1} \iff P_n - P_{n-1} = -\frac{1}{n}(P_{n-1} - P_{n-2})$$

which is a recursive formula. Now we seek to find a closed form solution. We have

$$P_3 - P_2 = -\frac{1}{3}(P_2 - P_1) = -\frac{1}{3!} \iff P_3 = P_2 - \frac{1}{3!} = \frac{1}{2!} - \frac{1}{3!}$$

$$P_4 - P_3 = -\frac{1}{4}(P_3 - P_2) = \frac{1}{4!} \iff P_4 = P_3 + \frac{1}{4!} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}$$

$$\vdots$$

$$P_n = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + \frac{(-1)^n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \to e^{-1} \text{ as } n \to \infty$$

so we see that for large n we approximately have $P_n \sim \text{Poisson}(1)$.

Now we consider $\Pr(X = k)$. Consider a set of k people who all have matches and no one else matches. The probability that everyone in a set of k people match is $(n-k)!/n!$. The probability that none of the other $n-k$ people match is P_{n-k} per above. Since there are $\binom{n}{k}$ ways to chose groups of k people, we have

$$\Pr(X = k) = \binom{n}{k} \frac{(n-k)!}{n!} \cdot P_{n-k} = \frac{P_{n-k}}{k!} \xrightarrow{n \to \infty} \frac{e^{-1} 1^k}{k!}$$

so again we see that for large n we approximately have $P_n \sim \text{Poisson}(1)$.

6.1.11 Asymptotic Distributions

Proposition 6.39.

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

Theorem 6.40. Stirling's Formula:

$$n! \sim n^n e^{-n} \sqrt{2\pi n}$$

That is,

$$\lim_{n \to \infty} \frac{n^n e^{-n} \sqrt{2\pi n}}{n!} = 1.$$

6.2 Worked problems

6.2.1 Example Problems That Will Likely Appear on Midterm (and Final)

(1) (a) **Fall 2011 Problem 1 (same as HW1 problem 5; similar to HW3 problem 2(5).)** Let A and B be events such that $0 < \Pr(A) < 1$. Show that if $\Pr(B \mid A) = \Pr(B \mid A^c)$, then A and B are independent.

(b) Let X and Y be two discrete random variables, each taking only two possible values. Show that if $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$ then X and Y are independent.

Solution.

(a) A and B are independent if and only if

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$$

We know that

$$\Pr(B) = \Pr(B|A) \cdot \Pr(A) + \Pr(B|A^c) \cdot \Pr(A^c)$$

$$= \Pr(B|A) \cdot \Pr(A) + \Pr(B|A) \cdot (1 - \Pr(A)) = \Pr(B|A) \cdot \Pr(A) + \Pr(B|A) - \Pr(B|A) \cdot \Pr(A)$$

$$= \Pr(B|A)$$

Also, we know that since $\Pr(A) \neq 0$,

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)}$$

Per above $\Pr(B|A) = \Pr(B)$, so we have

$$\Pr(B) = \frac{\Pr(A \cap B)}{\Pr(A)}$$

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$$

which is what we were trying to prove. So the answer is $\boxed{\text{true.}}$

(b) Without loss of generality, let X and Y have mass functions

$$X = \begin{cases} x_1 & \text{with probability } \Pr(A) \\ x_2 & \text{with probability } \Pr(A^c) \end{cases}$$

$$Y = \begin{cases} y_1 & \text{with probability } \Pr(B) \\ y_2 & \text{with probability } \Pr(B^c) \end{cases}$$

Then $X \perp\!\!\!\perp Y \iff \Pr(A \cap B) = \Pr(A)\Pr(B)$. Let $\alpha = X - x_2$, $\beta = Y - y_2$; that is,

$$\alpha = \begin{cases} x_1 - x_2 & \text{with probability } \Pr(A) \\ 0 & \text{with probability } \Pr(A^c) \end{cases}$$

$$\beta = \begin{cases} y_1 - y_2 & \text{with probability } \Pr(B) \\ 0 & \text{with probability } \Pr(B^c) \end{cases}$$

Then we have

- $\mathbb{E}(\alpha) = (x_1 - x_2)\Pr(A)$
- $\mathbb{E}(\beta) = (y_1 - y_2)\Pr(B)$
- $\mathbb{E}(\alpha\beta) = (x_1 - x_2)(y_1 - y_2)\Pr(A \cap B)$

which we can use to obtain

$$\mathbb{E}(XY) = \mathbb{E}\big[(\alpha + x_2)(\beta + y_2)\big] = \mathbb{E}(\alpha\beta) + y_2\mathbb{E}(\alpha) + x_2\mathbb{E}(\beta) + x_2 y_2$$

$$= (x_1 - x_2)(y_1 - y_2)\Pr(A \cap B) + y_2(x_1 - x_2)\Pr(A) + x_2(y_1 - y_2)\Pr(B) + x_2 y_2 \tag{6.6}$$

$$\mathbb{E}(X)\mathbb{E}(Y) = \mathbb{E}(\alpha + x_2)\mathbb{E}(\beta + y_2) = \big[x_2 + (x_1 - x_2)\Pr(A)\big]\big[y_2 + (y_1 - y_2)\Pr(B)\big]$$

$$= x_2y_2 + x_2(y_1 - y_2)\Pr(B) + y_2(x_1 - x_2)\Pr(A) + (x_1 - x_2)(y_1 - y_2)\Pr(A)\Pr(B) \tag{6.7}$$

Using $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$, we set (6.6) and (6.7) equal to each other. Canceling terms appearing in both yields

$$(x_1 - x_2)(y_1 - y_2)\Pr(A \cap B) = (x_1 - x_2)(y_1 - y_2)\Pr(A)\Pr(B) \iff \Pr(A \cap B) = \Pr(A)\Pr(B)$$

which proves the independence of X and Y if $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.

(2) **Fall 2013 Qual Problem 1.** Consider a sequence of independent tosses of a pair of fair dice. Compute the probability that the sum 4 will occur before the sum 5.

Solution. Let Y_k be the outcome of the kth toss. Let X_{k1} be the number of the first die on the kth toss and X_{k2} be the outcome of the second die. Note that

$$\Pr(Y_k = 4) = \Pr(X_{k1} = 1 \cap X_{k2} = 3) + \Pr(X_{k1} = 2 \cap X_{k2} = 2) + \Pr(X_{k1} = 3 \cap X_{k2} = 1) = 3 \cdot \frac{1}{36} = \frac{1}{12}$$

$$\Pr(Y_k = 5) = \Pr(X_{k1} = 1 \cap X_{k2} = 4) + \Pr(X_{k1} = 2 \cap X_{k2} = 3) + \Pr(X_{k1} = 3 \cap X_{k2} = 2)$$

$$+ \Pr(X_{k1} = 4 \cap X_{k2} = 1) = 4 \cdot \frac{1}{36} = \frac{1}{9}$$

Let A_k be the event that $Y_k = 4$ and $Y_j \neq 4$ or 5, $j = 1, \ldots, k-1$. Note that all A_k are mutually exclusive and

$$\Pr(A_k) = \frac{1}{12} \cdot \left(1 - \frac{3+4}{36}\right)^{k-1} = \frac{1}{12} \cdot \left(\frac{29}{36}\right)^{k-1}.$$

Then

$$\Pr(\{\text{roll a 4 before a 5}\}) = \Pr\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \Pr(A_k) = \frac{1}{12}\sum_{k=1}^{\infty}\left(\frac{29}{36}\right)^{k-1} = \frac{1/12}{1 - 29/36}$$

$$= \frac{3}{36-29} = \boxed{\frac{3}{7}}$$

(3) **Spring 2013 Problem 1, in notes from 09/21.** Let X and Y be random variables such that $\mathbb{E}(X \mid Y) = Y, \mathbb{E}(Y \mid X) = X, \mathbb{E}(X^2) < \infty, \mathbb{E}(Y^2) < \infty$. Show that $\mathbb{E}(X - Y)^2 = 0$ (or equivalently, show $\Pr(X = Y) = 1$).

Solution.

$$\mathbb{E}(X - Y)^2 = \mathbb{E}(X^2 - 2XY + Y^2) = \mathbb{E}(X^2) - 2\mathbb{E}(XY) + \mathbb{E}(Y^2)$$

$$\mathbb{E}(XY) = \mathbb{E}(\mathbb{E}(XY \mid Y)) = \mathbb{E}(Y\mathbb{E}(X \mid Y)) = \mathbb{E}(Y \cdot Y) = \mathbb{E}(Y^2)$$

Also,

$$\mathbb{E}(XY) = \mathbb{E}(\mathbb{E}(XY \mid X)) = \mathbb{E}(X\mathbb{E}(Y \mid X)) = \mathbb{E}(X \cdot X) = \mathbb{E}(X^2)$$

Therefore

$$\mathbb{E}(X - Y)^2 = 0$$

(4) **Fall 2016 Problem 4.** Consider a group of $n \geq 4$ people, among whom are Alice, Bob, Charles, and Diana, standing in a row. Assume that all possible orderings of the n people are equally likely.

(a) Compute the probability that Charles stands somewhere between Alice and Bob. (Note: this does not mean that the three are necessarily adjacent; there can be other people between Alice and Bob.)

(b) Compute the probability that Diana stands somewhere between Alice and Bob given that Charles stands somewhere between Alice and Bob.

(c) Let X be the number of people who stand between Alice and Bob. Compute the expected value and the variance of X. (Note: Alice and Bob themselves are not counted in this number.)

Solution.

(a) For this part, n does not make a difference; all we need to know is the ordering of A, B, and C. This is because conditional on a specific ordering of A, B, and C, all arrangements of everyone else are equally likely; and conversely, the a particular ordering of A, B, and C is independent of which three particular slots are made available to them. Examining the permutations of A, B, and C, two of them have C in the middle, so the answer is $2/6 = \boxed{1/3}$.

(b) Similarly, the answer is independent of n, so we work with $n = 4$. All possible orderings with Charles between Alice and Bob are as follows:

$$ACDB,\ ADCB,\ BCDA,\ BDCA,\ ACBD,\ BCAD,\ DACB,\ DBCA$$

The first four of these have Diana between Alice and Bob, so the answer is $4/8 = \boxed{1/2}$.

(c) Let I_k be an indicator variable for the event that person k is between A and B. By the result from part (a), $\mathbb{E}(I_k) = 1/3$. Then we have

$$\text{Var}(I_k) = \mathbb{E}(I_k^2) - \mathbb{E}(I_k)^2 = 1^2 \cdot \Pr(I_k = 1) - \frac{1}{9} = \frac{1}{3} - \frac{1}{9} = \frac{2}{9}$$

Noting that the four arrangements above with Charles and Diana in between Alice and Bob are the only ones where this will be the case of the $4! = 24$ possible orderings, we have

$$\mathbb{E}(I_k I_j) = \frac{4}{24} = \frac{1}{6}$$

so

$$\text{Cov}(I_j, I_k) = \mathbb{E}(I_k I_j) - \mathbb{E}(I_i)\mathbb{E}(I_j) = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$$

Therefore

$$\mathbb{E}(X) = \mathbb{E}(\sum_{i=1}^{n-2} I_k) = \sum_{i=1}^{n-2} \frac{1}{3} = \frac{n-2}{3}$$

$$\mathrm{Var}(X) = \mathrm{Var}\left(\sum_{i=1}^{n-2} I_k\right) = \sum_{i=1}^{n-2} \mathrm{Var}(I_k) + 2 \sum_{1 \le j < k \le n-2} \mathrm{Cov}(I_j, I_k) = \frac{2(n-2)}{9} + \left[(n-2)^2 - (n-2)\right] \cdot \frac{1}{18}$$

$$= \frac{4(n-2)}{18} + \frac{(n-2)(n-3)}{18} = \boxed{\frac{(n-2)(n+1)}{18}}$$

(5) **Spring 2015 Problem 2.** A deck of 52 cards is shuffled thoroughly. Someone goes through all 52 cards, scoring 1 point each time 2 cards of the same value are consecutive (that is, when two consecutive cards have the same rank but different suits). For example, the sequence 9H, 8H, 7D, 6C, 7S, 7H, 7C scores 2 points, one for the 7H following the 7S, one for the 7C following the 7H. Let X be the total score.

(a) Compute $\mathbb{E}(X)$.

(b) Compute $\Pr(X = 39)$. (Note that there are 13 different ranks and you cannot score more than 3 per rank.)

(c) In the line below, circle the number that you think is the closest to the value $\Pr(X = 0)$ and briefly explain your choice:

$$\frac{1}{1000}, \ \frac{1}{500}, \ \frac{1}{100}, \ \frac{1}{50}, \ \frac{1}{20}, \ \frac{1}{10}, \ \frac{1}{5}, \ \frac{1}{2}$$

Solution.

(a) Start by assuming the permutation is cyclic (that is, after the last card you go back to the beginning). Let Y be the number of matches in this situation. Let A_i be the event that the ith card is followed by a match. Then $\Pr(A_i) = 3/51 = 1/17$, so

$$Y = \sum_{k=1}^{52} \mathbf{1}_{\{A_i\}} \implies \mathbb{E}(Y) = \sum_{i=1}^{52} \mathbb{E}(\mathbf{1}_{\{A_i\}}) = \sum_{i=1}^{52} \Pr(A_i) = 52 \cdot 1/17 = 52/17$$

Note that $\mathbb{E}(Y) = \mathbb{E}(X) + \mathbb{E}(\mathbf{1}_{\{A_i\}})$ because if the permutation is cyclic then you have one extra opportunity to mach at the end.

$$\implies \mathbb{E}(X) = \frac{52}{17} - \frac{1}{17} = \frac{51}{17} = \boxed{3}$$

(b) $\Pr(X = 39) = \Pr(\{\text{all possible matches occur}\})$, so in this event all cards of the same rank are clustered together. There are 13 clusters of 4 cards, so there are 13! ways to order the clusters and 4! ways to order the cards within each cluster. Therefore

$$\boxed{\Pr(X = 39) = \frac{13!(4!)^{13}}{52!}}$$

(c) Because A_i are only weakly dependent and $\Pr(A_i)$ is small for all A_i, we can use the Poisson approximation (see Section 6.1.10); that is, $X \sim \mathrm{Poisson}(\mathbb{E}(X)) = \mathrm{Poisson}(3)$. Therefore

$$\Pr(X = 0) \approx \frac{e^{-3} \cdot 3^0}{0!} = \frac{1}{e^3} \approx \frac{1}{2.8^3} \approx \boxed{\frac{1}{20}}$$

6.2.2 Problems we did in class that professor mentioned

(Fall 2014 Problem 1) (Variation of Midterm problem 1 above) Let A and B be two events with $0 < \Pr(A) < 1,\ 0 < \Pr(B) < 1$. Define the random variables $\xi = \xi(\omega)$ and $\eta = \eta(\omega)$ by

$$\xi(\omega) = \begin{cases} 5 & \text{if } \omega \in A \\ -7 & \text{if } \omega \notin A \end{cases}, \qquad \eta(\omega) = \begin{cases} 2 & \text{if } \omega \in B \\ 3 & \text{if } \omega \notin B \end{cases}$$

True or false: the events A and B are independent if and only if the random variables ξ and η are uncorrelated?

Solution. ($\implies$) Suppose A and B are independent. Then ξ and η are uncorrelated if and only if $\mathbb{E}(\xi\eta) = \mathbb{E}(\xi)\mathbb{E}(\eta)$. We can write $\xi = 5 \cdot \mathbf{1}_A - 7 \cdot \mathbf{1}_{A^c}$ and $\eta = 2 \cdot \mathbf{1}_B + 3 \cdot \mathbf{1}_{B^c}$. So we have

$$\xi\eta = (5 \cdot \mathbf{1}_A - 7 \cdot \mathbf{1}_{A^c})(2 \cdot \mathbf{1}_B + 3 \cdot \mathbf{1}_{B^c}) = 10 \cdot \mathbf{1}_{A \cap B} + 15 \cdot \mathbf{1}_{A \cap B^c} - 14 \cdot \mathbf{1}_{A^c \cap B} - 21 \cdot \mathbf{1}_{A^c \cap B^c}$$

$$\implies \mathbb{E}(\xi\eta) = 10 \Pr(A \cap B) + 15 \Pr(A \cap B^c) - 14 \Pr(A^c \cap B) - 21 \Pr(A^c \cap B^c)$$

Then

$$\mathbb{E}(\xi)\mathbb{E}(\eta) = (5 \Pr(A) - 7 \Pr(A^c))(2 \Pr(B) + 3 \Pr(B^c))$$

$$= 10 \Pr(A \cap B) + 15 \Pr(A \cap B^c) - 14 \Pr(A^c \cap B) - 21 \Pr(A^c \cap B^c) = \mathbb{E}(\xi\eta)$$

where the second-to-last step follows from the independence of A and B. Therefore η and ξ are uncorrelated.

($\impliedby$) Now suppose η and ξ are uncorrelated. Then ξ and η are independent if and only if $\Pr(\xi \cap \eta) = \Pr(\xi)\Pr(\eta)$. Define

$$\alpha(\omega) = \xi(\omega) + 7 = \begin{cases} 12 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}, \qquad \beta(\omega) = \eta(\omega) - 3 = \begin{cases} -1 & \text{if } \omega \in B \\ 0 & \text{if } \omega \notin B \end{cases}$$

Then we have

$$(\alpha\beta)(\omega) = \begin{cases} -12 & \text{if } \omega \in A \cap B \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\mathbb{E}(\xi\eta) = \mathbb{E}\big[(\alpha - 7)(\beta + 3)\big] = \mathbb{E}(\alpha\beta) + 3\mathbb{E}(\alpha) - 7\mathbb{E}(\beta) - 21$$

$$\mathbb{E}(\xi)\mathbb{E}(\eta) = (\mathbb{E}(\alpha) - 7)(\mathbb{E}(\beta) + 3) = \mathbb{E}(\alpha)\mathbb{E}(\beta) - 7\mathbb{E}(\beta) + 3\mathbb{E}(\alpha) - 21$$

Since by assumption $\mathbb{E}(\xi\eta) = \mathbb{E}(\xi)\mathbb{E}(\eta)$, this yields $\mathbb{E}(\alpha\beta) = \mathbb{E}(\alpha)\mathbb{E}(\beta)$. But

$$\mathbb{E}(\alpha\beta) = -12\Pr(A \cap B), \qquad \mathbb{E}(\alpha)\mathbb{E}(\beta) = 12\Pr(A)(-1)\Pr(B) = -12\Pr(A)\Pr(B)$$

Therefore $\Pr(\xi \cap \eta) = \Pr(\xi)\Pr(\eta)$ and ξ and η are independent.

Exercise 1. Example (Letter/envelope matching problem; sometimes referred to as Montmort's matching problem). An assistant brings n sandwiches for n employees at a company. Each employee ordered a unique sandwich, but unfortunately the assistant forgot to ask that the sandwiches be labeled, so they are all indistinguishable, wrapped in the same paper. The assistant plans to distribute one sandwich to each employee and hope for the best. Let X be the number of sandwiches that are delivered to the correct person.

(a) What is the probability of at least one match; that is, $\Pr(X \geq 1)$?

(b) What is the probability of r correct matches?

(c) What $\mathbb{E}(X)$?

(d) What is $\text{Var}(X)$?

Solution.

(a) Let A_k be an indicator variable for the event that sandwich k is matched to the correct employee. Then

$$\Pr(X \geq 1) = \Pr\left(\bigcup_{k=1}^{n} A_k\right)$$

Consider that if there are k correct matches, there are $\binom{n}{k}$ sets of k sandwiches that could be correctly distributed. Also, the probability of a particular set of k sandwiches being correctly distributed is $(n-k)!/n!$. So we have

$$\Pr(X = k) = \binom{n}{k}\frac{(n-k)!}{n!}$$

Therefore by the Inclusion-Exclusion Principle (Proposition 6.13),

$$\Pr\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n}(-1)^{k-1}\Pr(X = k) = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{(n-k)!}{n!} = \sum_{k=1}^{n}(-1)^{k-1}\frac{n!}{(n-k)!k!}\frac{(n-k)!}{n!}$$

$$= \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!} = \frac{(-1)^0}{0!} - \sum_{k=0}^{n}\frac{(-1)^k}{k!} = \boxed{1 - \sum_{k=0}^{n}\frac{(-1)^k}{k!}}$$

As $n \to \infty$, we have

$$1 - \sum_{k=0}^{n} \frac{(-1)^k}{k!} \to 1 - e^{-1} = \boxed{1 - \frac{1}{e}}$$

(b) Clearly there is only one way to match all the sandwiches correctly, so $\Pr(X = r \mid r = n) = 1/n!$. Also, note that it is impossible to match all but one sandwich, so $\Pr(X = r \mid r = n - 1) = 0$. Only the cases for $r \leq n - 2$ are nontrivial. Using a similar argument as part (a), we see that for any set of m sandwiches, the probability that at least one was correctly distributed is

$$\Pr\left(\bigcup_{k=1}^{m} A_k\right) = \sum_{k=1}^{m} (-1)^{k-1} \frac{(n-k)!}{n!}$$

and that the probability that *any* set of m sandwiches contained at least one correct match is

$$\sum_{k=1}^{m} (-1)^{k-1} \binom{n}{k} \frac{(n-k)!}{n!} = \sum_{k=1}^{m} (-1)^{k-1} \frac{n!}{(n-k)!k!} \frac{(n-k)!}{n!}$$

$$= \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k!} = 1 + \sum_{k=2}^{m} \frac{(-1)^{k-1}}{k!} = 1 - \sum_{k=2}^{m} \frac{(-1)^k}{k!}$$

So for $m \geq 2$, the probability of *no* correct matches is $\sum_{k=2}^{m} \frac{(-1)^k}{k!}$ if $n \geq 2$, and of course 0 if $n = 1$. Therefore the probability of r matches is the probability of any one set of r sandwiches all matching and none of the remaining $n - r$ sandwiches matching times the number of sets of r sandwiches; that is,

$$\Pr(X = r \mid r \leq n - 2) = \binom{n}{r} \cdot \frac{(n-r)!}{n!} \cdot \left(\sum_{k=2}^{n-r} \frac{(-1)^k}{k!}\right) = \frac{r!}{(n-r)!r!} \cdot \frac{(n-r)!}{n!} \sum_{k=2}^{n-r} \frac{(-1)^k}{k!}$$

$$= \frac{1}{r!} \sum_{k=2}^{n-r} \frac{(-1)^k}{k!}$$

Therefore we have

$$\boxed{\Pr(X = r) = \begin{cases} \frac{1}{r!} \sum_{k=2}^{n-r} \frac{(-1)^k}{k!} & r \leq n - 2 \\ 0 & r = n - 1 \\ \frac{1}{r!} & r = n \end{cases}}$$

(c)

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mathbb{E}(A_k) = \sum_{k=1}^{n} \Pr(A_k = 1) = n \cdot \frac{1}{n} = \boxed{1}$$

(d)

$$\operatorname{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$\mathbb{E}(X^2) = \mathbb{E}\left(\sum_{k=1}^{n} A_k\right)^2 = \mathbb{E}\left(\sum_{k=1}^{n} A_k^2 + 2 \sum_{1 \leq i < j \leq n} A_i A_j\right) = \sum_{k=1}^{n} \mathbb{E}(A_k^2) + 2 \sum_{1 \leq i < j \leq n} \mathbb{E}(A_i A_j)$$

Because

$$\mathbb{E}(A_k^2) = 1^2 \cdot \Pr(A_k = 1) = \frac{1}{n}$$

$$\mathbb{E}(A_i A_j) = \Pr(A_j = 1 \cap A_j = 1) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)} = \frac{1}{2} \cdot \binom{n}{2}^{-1}$$

we have

$$\mathbb{E}(X^2) = \sum_{k=1}^{n} \frac{1}{n} + 2 \sum_{1 \leq i < j \leq n} \frac{1}{n(n-1)} = 1 + 2 \cdot \binom{n}{2} \cdot \frac{1}{2} \cdot \binom{n}{2}^{-1} = 2$$

$$\implies \mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = 2 - 1 = \boxed{1}$$

HW3 Problem 2(5). Verify: $\mathbb{E}(X \mid Y) = \mathbb{E}(X)$ if X and Y are independent.

Solution. X and Y are independent if and only if

$$\Pr(X \cap Y) = \Pr(X) \cdot \Pr(Y) \iff \Pr(X = x \cap Y = y) = \Pr(X = x) \Pr(Y = y)$$

$$\iff \Pr(X = x \mid Y = y) \cdot \Pr(Y = y) = \Pr(X = x) \Pr(Y = y) \iff \Pr(X = x \mid Y = y) = \Pr(X = x)$$

$$\implies E(X \mid Y) = \sum_x x \cdot \Pr(X = x \mid Y = y) = \sum_x x \cdot \Pr(X = x) = \mathbb{E}(X)$$

HW3 Problem 2 (parts 1 - 4). Verify:

(1) $\mathbb{E}(\mathbb{E}(X \mid Y)) = \mathbb{E}(X)$

(2) $\mathbb{E}(g(Y)X \mid Y) = g(Y)\mathbb{E}(X \mid Y)$

(3) $\mathrm{Cov}(\mathbb{E}(X \mid Y), Y) = \mathrm{Cov}(X, Y)$

(4) Y and $X - \mathbb{E}(X \mid Y)$ are uncorrelated.

Solution.

(1)
$$\mathbb{E}\big(\mathbb{E}(X \mid Y)\big) = \sum_y \mathbb{E}(X \mid Y) \Pr(Y = y) = \sum_y \left[\sum_x x \cdot \Pr(X = x \mid Y = y) \Pr(Y = y)\right]$$

$$= \sum_y \left[\sum_x x \cdot \Pr(X = x \cap Y = y)\right] = \sum_y \left[\sum_x x \cdot \Pr(Y = y \mid X = x) \cdot \Pr(X = x)\right]$$

$$= \sum_x \left[x \cdot \Pr(X = x) \cdot \sum_y \big(\Pr(Y = y \mid X = x) \big) \right] = \sum_x \left[x \cdot \Pr(X = x) \cdot 1 \right]$$

$$= \mathbb{E}(X)$$

(2) 2

(3)

$$\mathrm{Cov}(\mathbb{E}(X \mid Y), Y) = \mathbb{E}\bigg(\Big[\mathbb{E}(X \mid Y) - \mathbb{E}\big(\mathbb{E}(X \mid Y)\big) \Big] \Big[Y - \mathbb{E}(Y) \Big] \bigg)$$

$$= \mathbb{E}\bigg(\Big[\mathbb{E}(X \mid Y) - \mathbb{E}(X) \Big] \Big[Y - \mathbb{E}(Y) \Big] \bigg) = \mathbb{E}\Big(\mathbb{E}(X \mid Y)Y - \mathbb{E}(X)Y - \mathbb{E}(X \mid Y)\mathbb{E}(Y) + \mathbb{E}(X)\mathbb{E}(Y) \Big)$$

$$= \mathbb{E}\big(\mathbb{E}(X \mid Y)Y\big) - \mathbb{E}(X)\mathbb{E}(Y) - \mathbb{E}(Y)\mathbb{E}\big(\mathbb{E}(X \mid Y)\big) + \mathbb{E}(X)\mathbb{E}(Y) = \mathbb{E}\big(\mathbb{E}(X \mid Y)Y\big) - \mathbb{E}(Y)\mathbb{E}(X)$$

$$= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = \mathrm{Cov}(X, Y)$$

(4) Y and $X - \mathbb{E}(X \mid Y)$ are uncorrelated if and only if $\mathrm{Cov}\big(Y, X - \mathbb{E}(X \mid Y)\big) = 0 \iff \mathbb{E}(Y \cdot \big[X - \mathbb{E}(X \mid Y)\big]) - \mathbb{E}(Y)\mathbb{E}(X - \mathbb{E}(X \mid Y)) = 0$.

$$\mathbb{E}(Y \cdot \big[X - \mathbb{E}(X \mid Y)\big]) - \mathbb{E}(Y)\mathbb{E}(X - \mathbb{E}(X \mid Y)) = \mathbb{E}(YX - Y\mathbb{E}(X \mid Y)) - \mathbb{E}(Y)\mathbb{E}(X) + \mathbb{E}(Y)\mathbb{E}\big(\mathbb{E}(X \mid Y)\big))$$

$$= \mathbb{E}(YX) - \mathbb{E}\big(Y\mathbb{E}(X \mid Y)\big) - \mathbb{E}(Y)\mathbb{E}(X) + \mathbb{E}(Y)\mathbb{E}(X) = \mathbb{E}(YX) - \mathbb{E}(YX) = 0$$

Spring 2018 Problem 2 (did not complete)

2. Consider positions 1 to n arranged in a circle, so that 2 comes after 1, 3 comes after 2, ..., n comes after $n-1$, and 1 comes after n. Similarly, take 1 to n as values, with cyclic order, and consider all $n!$ ways to assign values to positions, bijectively, with all $n!$ possibilities equally likely. For $i = 1$ to n, let X_i be the indicator that position i and the one following are filled in with two consecutive values in increasing order, and define

$$S_n = \sum_{i=1}^n X_i, \quad T_n = \sum_{i=1}^n iX_i$$

For example, with $n = 6$ and the the circular arrangement 314562, we get $X_3 = 1$ since 45 are consecutive in increasing order, and similarly $X_4 = X_6 = 1$, so that $S_6 = 3, T_6 = 13$.

a) Compute the mean and the variance of S_n.
b) Compute the mean and the variance of T_n.

Fall 2008 Problem 2 (HW1 Problem 10). Consider a lottery with n^2, tickets, of which only n tickets win prizes. Let p_n be the probability that, out of n randomly selected tickets, at least one wins a prize. Compute $\lim_{n\to\infty} p_n$.

Solution. There are $\binom{n^2}{n}$ possible sets of n tickets. The number of these sets that do not contain at least one winner (that is, they only contain members of the $n^2 - n$ losing tickets) is $\binom{n^2-n}{n}$. Therefore the probability of selecting a set of n tickets that contains at least one winner is

$$p_n = 1 - \binom{n^2-n}{n} \Big/ \binom{n^2}{n} = 1 - \frac{(n^2-n)!}{n!(n^2-n-n)!} \Big/ \frac{(n^2)!}{(n^2-n)!n!} = 1 - \frac{(n^2-n)!}{n!(n^2-2n)!} \cdot \frac{(n^2-n)!n!}{(n^2)!}$$

$$= 1 - \frac{(n^2-n)!}{(n^2-2n)!} \cdot \frac{(n^2-n)!}{(n^2)!} = 1 - \prod_{i=0}^{n-1} (n^2-n-i) \Big/ \prod_{i=0}^{n-1} (n^2-i) = 1 - \prod_{i=0}^{n-1} \frac{n^2-n-i}{n^2-i}$$

$$= 1 - \prod_{i=0}^{n-1} \left(\frac{n^2-i}{n^2-i} - \frac{n}{n^2-i} \right) = 1 - \prod_{i=0}^{n-1} \left(1 - \frac{n}{n^2-i} \right)$$

Therefore

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} \left[1 - \prod_{i=0}^{n-1} \left(1 - \frac{n}{n^2-i} \right) \right] = 1 - \lim_{n\to\infty} \prod_{i=0}^{n} \left(1 - \frac{n}{n^2-i} \right) = 1 - \lim_{n\to\infty} \prod_{i=0}^{n} \left(1 - \frac{n \cdot \frac{1}{n}}{\frac{n^2}{n} - \frac{i}{n}} \right)$$

$$= 1 - \lim_{n\to\infty} \prod_{i=0}^{n} \left(1 - \frac{1}{n - \frac{i}{n}} \right) = 1 - \lim_{n\to\infty} \prod_{i=0}^{n} \left(1 - \frac{1}{n} \right) = 1 - \lim_{n\to\infty} \left(1 - \frac{1}{n} \right)^n = \boxed{1 - \exp(-1)}$$

6.2.3 Problems we did on homework

Fall 2017 Problem 2 (Homework 3 Problem 6). An urn contains $2n$ balls, coming in pairs: two balls are labeled "1", two balls are labeled "2", ..., two balls are labeled "n". A sample of size n is taken without replacement. Denote by N the number of pairs in the sample. Compute the expected value and the variance of N. **You do not need to simplify the expression for the variance.**

Solution. Let X_k be an indicator variable for both balls labeled k being in the sample. Note that

$$\mathbb{E}(X_k) = \Pr(X_k = 1) = \frac{\binom{2n-2}{n-2}}{\binom{2n}{n}} = \frac{(2n-2)!}{(n-2)!n!} \Big/ \frac{(2n)!}{n!n!} = \frac{(2n-2)!n!}{(2n)!(n-2)!} = \frac{n(n-1)}{2n(2n-1)} = \frac{n-1}{2(2n-1)}$$

Now since $N = \sum_{k=1}^{n} X_k$, we have

$$\mathbb{E}(N) = \mathbb{E}\left(\sum_{k=1}^{n} X_k \right) = \sum_{k=1}^{n} \mathbb{E}(X_k) = \boxed{\frac{n(n-1)}{2(2n-1)}}$$

To obtain the variance, note that

$$\mathbb{E}(N^2) = \mathbb{E}\bigg(\sum_{k=1}^n X_k\bigg)^2 = \mathbb{E}\bigg(\sum_{k=1}^n X_k^2 + 2\sum_{1\leq i<j\leq n} X_i X_j\bigg) = \sum_{k=1}^n \mathbb{E}(X_k^2) + 2\sum_{1\leq i<j\leq n} \mathbb{E}(X_i X_j)$$

Because

$$\mathbb{E}(X_k^2) = 1^2 \cdot \Pr(X_k = 1) = \mathbb{E}(X_k) = \frac{n-1}{2(2n-1)}$$

$$\mathbb{E}(X_i X_j) = \Pr(X_i = 1 \cap X_j = 1) = \frac{\binom{2n-4}{n-4}}{\binom{2n}{n}} = \frac{(2n-4)!}{(n-4)!n!}\Bigg/\frac{(2n)!}{n!n!} = \frac{(2n-4)!n!}{(2n)!(n-4)!}$$

$$= \frac{n(n-1)(n-2)(n-3)}{2n(2n-1)(2n-2)(2n-3)} = \frac{(n-1)(n-2)(n-3)}{2(2n-1)(2n-2)(2n-3)}$$

we have

$$\mathbb{E}(N^2) = \sum_{k=1}^n \frac{n-1}{2(2n-1)} + 2\sum_{1\leq i<j\leq n} \frac{(n-1)(n-2)(n-3)}{2(2n-1)(2n-2)(2n-3)} = \frac{n(n-1)}{2(2n-1)} + 2\binom{n}{2}\frac{(2n-4)!n!}{(2n)!(n-4)!}$$

$$= \frac{n(n-1)}{2(2n-1)} + \frac{n!}{(n-2)!}\cdot\frac{(2n-4)!n!}{(2n)!(n-4)!} = \frac{n(n-1)}{2(2n-1)} + n(n-1)\cdot\frac{(n-1)(n-2)(n-3)}{2(2n-1)(2n-2)(2n-3)}$$

$$= \frac{n(n-1)}{2(2n-1)} + \frac{n(n-1)^2(n-2)(n-3)}{2(2n-1)(2n-2)(2n-3)}$$

$$\implies \operatorname{Var}(N) = \mathbb{E}(N^2) - \mathbb{E}(N)^2 = \boxed{\frac{n(n-1)}{2(2n-1)} + \frac{n(n-1)^2(n-2)(n-3)}{2(2n-1)(2n-2)(2n-3)} - \frac{n^2(n-1)^2}{4(2n-1)^2}}$$

Fall 2017 Problem 3 (HW3 Problem 8—almost full solution)

Let $U_1, U_2, \ldots$ be iid random variables, uniformly distributed on $[0,1]$, and let N be a Poisson random variable with mean value equal to 1. Assume that N is independent of $U_1, U_2, \ldots$ and define

$$Y = \begin{cases} 0 & \text{if } N = 0 \\ \max_{1\leq i\leq N} U_i & \text{if } N > 0 \end{cases}$$

Compute the expected value of Y.

Solution. Since Y is a function of N, let $Y = y(N)$. By the Law of the Unconscious Statistician,

$$\mathbb{E}(Y) = \mathbb{E}\big(\mathbb{E}(Y \mid N)\big) = \mathbb{E}\big(\mathbb{E}(\max_{1\leq i\leq N} U_i \mid N = n)\big)$$

Let $Z_n = \max_{1 \leq i \leq n} U_i$. The cdf of Z_n can be calculated as follows:

$$\Pr(Z_n \leq x) = \Pr(\max_{1 \leq i \leq n} U_i \leq x) = \Pr(U_1 \leq x \cap U_2 \leq x \cap \ldots \cap U_n \leq x) = x^n$$

for $x \in [0, 1]$. Therefore the pdf of Z_n is its derivative, nx^{n-1}. So we have

$$\mathbb{E}(\max_{1 \leq i \leq N} U_i \mid N = n) = \mathbb{E}(Z_n) = \int_0^1 x n x^{n-1} dx = n \int_0^1 x^n dx = n \frac{x^{n+1}}{n+1}\bigg|_0^1 = \frac{n}{n+1}$$

Plugging this into the expression for $\mathbb{E}(Y)$ yields

$$\mathbb{E}(Y) = \mathbb{E}\big(\mathbb{E}(Y \mid N)\big) = \sum_{n=0}^{\infty} \frac{n}{n+1} \Pr(N = n) = \sum_{n=1}^{\infty} \frac{n}{n+1} \frac{\exp(-1)1^n}{n!}$$

$$= \frac{1}{e} \sum_{n=1}^{\infty} \frac{n+1-1}{(n+1)!} = \frac{1}{e}\left(\sum_{n=1}^{\infty} \frac{n+1}{(n+1)!} - \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \right) = \frac{1}{e}\left(\sum_{n=1}^{\infty} \frac{1}{n!} - \sum_{m=2}^{\infty} \frac{1}{m!} \right)$$

$$= \frac{1}{e}\big[e - 1 - (e - 1 - 1)\big] = \boxed{\frac{1}{e}}$$

Fall 2013 Problem 3/Spring 2011 Problem 2 (HW3 Problem 9; coupon collector problem) Only parts I didn't do: Let D be the event that no box receives more than 1 ball. Fix $a \in (0, 1)$. If both $n, d \to \infty$ together, what relation must they satisfy in order to have $\Pr(D) \to a$?

HW3 Problem 9. Consider n (different) balls placed at random in m boxes so that each of m^n configurations is equally likely.

(a) Compute the expected value and the variance of the number of empty boxes.

(b) Show that if $\lim_{m,n\to\infty} m \exp(-n/m) = \lambda \in (0, \infty)$, then, in the same limit, the number of empty boxes has Poisson distribution with parameter λ.

(c) For $k \geq 1$ such that $k + 3 \leq m$, define the event A_k that the boxes $k, k + 1, k + 2, k + 3$ are empty. Assuming that $m > 8$, compute $\Pr(A_1 \cup A_3 \cup A_5)$. How will the answer change if $m = 8$?

(d) Now imagine that the balls are dropped one-by-one (with each ball equally likely to go into any of the m boxes, independent of all other balls), and denote by N_m the minimal number of balls required to fill all the boxes. Compute $\mathbb{E}(N_m), \mathrm{Var}(N_m)$ and

$$\lim_{m\to\infty} \Pr\left(\frac{N_m - m \log m}{m} \leq x \right)$$

(e) Suppose we instead place an unlimited number of balls into the m boxes until we have k consecutive balls land in the same box (it doesn't matter which box). What is the expected number of balls we will drop until this happens?

Solution.

(a) Let A_i be the event that the ith box is empty. Let $\mathbf{1}_{A_i}$ be the indicator for A_i. Then $X = \sum_{i=1}^m \mathbf{1}_{A_i}$.

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_{i=1}^m \mathbf{1}_{A_i} \right) = \sum_{i=1}^m \left(\mathbb{E}\mathbf{1}_{A_i} \right) = \sum_{i=1}^m \Pr(A_i) = \sum_{i=1}^m \left(\frac{m-1}{m} \right)^n = \boxed{\frac{(m-1)^n}{m^{n-1}}}$$

$$\operatorname{Var}(X) = \operatorname{Var}\left(\sum_{i=1}^m \mathbf{1}_{A_i} \right) = \sum_{i=1}^m \operatorname{Var}(\mathbf{1}_{A_i}) + 2 \sum_{1 \le i < j \le m} \operatorname{Cov}(\mathbf{1}_{A_i}, \mathbf{1}_{A_j})$$

$$\operatorname{Var}(\mathbf{1}_{A_i}, \mathbf{1}_{A_j}) = \mathbb{E}(\mathbf{1}_{A_i}\mathbf{1}_{A_i}) - \mathbb{E}(\mathbf{1}_{A_i})^2 = \Pr(A_i \cap A_i) - \Pr(A_i)^2 = \left(\frac{m-1}{m} \right)^n - \left(\frac{m-1}{m} \right)^{2n}$$

$$\operatorname{Cov}(\mathbf{1}_{A_i}, \mathbf{1}_{A_j}) = \mathbb{E}(\mathbf{1}_{A_i}\mathbf{1}_{A_j}) - \mathbb{E}(\mathbf{1}_{A_i})\mathbb{E}(\mathbf{1}_{A_j}) = \Pr(A_i \cap A_j) - \Pr(A_i)\Pr(A_j) = \left(\frac{m-2}{m} \right)^n - \left(\frac{m-1}{m} \right)^{2n}$$

$$\implies \operatorname{Var}(X) = m \cdot \left[\left(\frac{m-1}{m} \right)^n - \left(\frac{m-1}{m} \right)^{2n} \right] + \frac{m!}{(m-2)!} \left[\left(\frac{m-2}{m} \right)^n - \left(\frac{m-1}{m} \right)^{2n} \right]$$

$$= \frac{(m-1)^n}{m^{n-1}} - \frac{(m-1)^{2n}}{m^{2n-1}} + (m^2 - m) \left[\left(\frac{m-2}{m} \right)^n - \left(\frac{m-1}{m} \right)^{2n} \right]$$

$$\boxed{\operatorname{Var}(X) = \frac{(m-1)^n}{m^{n-1}} - \frac{(m-1)^{2n}}{m^{2n-1}} + (m-1) \left[\frac{(m-2)^n}{m^{n-1}} - \frac{(m-1)^{2n}}{m^{2n-1}} \right]}$$

(b) Note that

$$X = \sum_{i=1}^m \mathbf{1}_{A_i}$$

and that the A_i are only weakly dependent on each other, especially as m and n increase. Therefore as $m, n \to \infty$, the Poisson paradigm (see Section 6.1.10) suggests $X \sim \operatorname{Poisson}(\mathbb{E}(X))$. We have

$$\mathbb{E}(X) = \frac{(m-1)^n}{m^{n-1}}$$

so

$$\lim_{n,m \to \infty} \mathbb{E}(X) = \lim_{n,m \to \infty} m \cdot \left(\frac{m-1}{m} \right)^n = \lim_{n,m \to \infty} m \cdot \left(1 - \frac{1}{m} \right)^n = \lim_{n,m \to \infty} m \cdot \left[\left(1 - \frac{1}{m} \right)^m \right]^{n/m}$$

$$\approx \lim_{n,m \to \infty} m \cdot \left[e^{-1} \right]^{n/m} = \lim_{n,m \to \infty} m e^{-n/m}$$

Using

$$\lim_{m,n \to \infty} m \exp(-n/m) = \lambda \in (0, \infty)$$

we have $\boxed{X \sim \operatorname{Poisson}(\lambda) \text{ as } m, n \to \infty}$.

(c)

$$\Pr(A_1 \cup A_3 \cup A_5) = \Pr(A_1) + \Pr(A_3) + \Pr(A_5) - \Pr(A_1 \cap A_3) - \Pr(A_1 \cap A_5) - \Pr(A_3 \cap A_5) + \Pr(A_1 \cap A_3 \cap A_5)$$

We have

$$\Pr(A_1) = \Pr(A_3) = \Pr(A_5) = \left(\frac{m-4}{m}\right)^n$$

$$\Pr(A_1 \cap A_3) = \Pr(A_3 \cap A_5) = \left(\frac{m-6}{m}\right)^n$$

$$\Pr(A_1 \cap A_5) = \Pr(A_1 \cap A_3 \cap A_5) = \left(\frac{m-8}{m}\right)^n$$

Therefore

$$\Pr(A_1 \cup A_3 \cup A_5) = 3\left(\frac{m-4}{m}\right)^n - 2\left(\frac{m-6}{m}\right)^n = \boxed{\frac{3(m-4)^n - 2(m-6)^n}{m^n}}$$

(d) N_m is the minimal number of balls required to fill all the boxes. Let T_i be the number of balls that have to be dropped to fill the ith box after $i-1$ boxes have been filled. The probability of filling a new box after $i-1$ boxes have been filled is $\frac{m-(i-1)}{m}$. (Note that T_1 should be identically 1 regardless of $m \geq 1$; this checks out using this expression.) Therefore T_i has a geometric distribution with $E(T_i) = \frac{m}{m-(i-1)}$. Since $N_m = \sum_{i=1}^m T_i$, we have

$$\mathbb{E}(N_m) = \mathbb{E}\left(\sum_{i=1}^m T_i\right) = \sum_{i=1}^m \mathbb{E}(T_i) = \sum_{i=1}^m \frac{m}{m-(i-1)} = \boxed{m\sum_{i=1}^m \frac{1}{i}}$$

Because the T_i are independent, we have

$$\operatorname{Var}(N_m) = \operatorname{Var}\left(\sum_{i=1}^m T_i\right) = \sum_{i=1}^m \operatorname{Var}(T_i) = \sum_{i=1}^m \left(1 - \frac{m-(i-1)}{m}\right) \Big/ \left(\frac{m-(i-1)}{m}\right)^2$$

$$= \sum_{i=1}^m \frac{i-1}{m} \cdot \left(\frac{m}{m-(i-1)}\right)^2 = \boxed{m\sum_{i=1}^m \frac{i-1}{[m-(i-1)]^2}}$$

Finally, to find

$$\lim_{m\to\infty} \Pr\left(\frac{N_m - m\log m}{m} \leq x\right)$$

begin by noting that we can also express N_m as

$$\Pr(N_m \leq k) = \Pr\left(X_{m,k} = 0\right)$$

where $X_{m,k}$ is defined as X is in part (b) with k being the number of balls that have been dropped so far, $k \in \mathbb{N} \geq m$. (For $k < m$, $\Pr(N_m \leq k) = 0$.)

Again, let $A_{i,k}$ be the event that the ith box is empty after dropping k balls. Then because $X_{m,k} = \sum_{i=1}^m \mathbf{1}_{A_{i,k}}$ and the $A_{i,k}$ are only weakly dependent on each other (especially as m becomes large), the Poisson paradigm (see Section 6.1.10) again suggests that as $m \to \infty$, $X_{m,k} \sim \text{Poisson}(\lambda_k)$ where $\lambda_k = \mathbb{E}(X_{m,k})$ is defined as above. Therefore we have

$$\lim_{m\to\infty} \Pr\left(\frac{N_m - m\log m}{m} \leq x\right) = \lim_{m\to\infty} \Pr\big(N_m \leq xm + m\log m\big) = \lim_{m\to\infty} \Pr\big(X_{m,xm+m\log m}$$

$$= 0) \approx \frac{\exp(-\lambda_{xm+m\log m}) \cdot \lambda_{xm+m\log m}^0}{0!} = \exp\big(-\lambda_{xm+m\log m}\big)$$

And we have

$$\lambda_{xm+m\log m} = \lim_{m\to\infty} m\exp\left(-\frac{xm+m\log m}{m}\right) = \lim_{m\to\infty} m\exp\big(-x-\log m\big) = \lim_{m\to\infty} m/m\exp(-x)$$

$$= \exp(-x)$$

which yields

$$\boxed{\lim_{m\to\infty} \Pr\left(\frac{N_m - m\log m}{m} \leq x\right) = \exp(\exp(-x))}$$

(e) Let $N = N_k$ be the number of balls that are dropped until k consecutive balls land in the same box, and likewise for N_{k-1}. Suppose we have already observed $k-1$ consecutive outcomes (of any kind) in N_{k-1} trials. Then we finish on the next term (by having another consecutive outcome) with probability $1/m$. Otherwise we have a different outcome and then repeat the same process again. So we have

$$\mathbb{E}(N_k \mid N_{k-1}) = N_{k-1} + 1 \cdot \frac{1}{m} + \mathbb{E}(N_k) \cdot \left(1 - \frac{1}{m}\right)$$

Therefore

$$\mathbb{E}(N) = \mathbb{E}(N_k) = \mathbb{E}\big[\mathbb{E}(N_k \mid N_{k-1})\big] = \mathbb{E}(N_{k-1}) + \frac{1}{m} + \left(1 - \frac{1}{m}\right)\mathbb{E}(N_k)$$

$$\iff \frac{1}{m}\mathbb{E}(N_k) = \mathbb{E}(N_{k-1}) + \frac{1}{m} \iff \mathbb{E}(N_k) = m\mathbb{E}(N_{k-1}) + 1$$

We have a recursive formula. Note that $\mathbb{E}(N_1) = 1$ because the number of trials until there is 1 consecutive outcome of any kind is simply 1. We can then calculate as follows:

$$\mathbb{E}(N_2) = m\mathbb{E}(N_{2-1}) + 1 = m + 1$$

$$\mathbb{E}(N_3) = m\mathbb{E}(N_{3-1}) + 1 = m(m+1) + 1 = 1 + m + m^2$$

$$\mathbb{E}(N_4) = m\mathbb{E}(N_{4-1}) + 1 = m(1 + m + m^2) + 1 = 1 + m + m^2 + m^3$$

$$\vdots$$

$$\mathbb{E}(N_k) = \sum_{i=0}^{k-1} m^i = \frac{1 \cdot (1 - m^k)}{1 - m} = \boxed{\frac{m^k - 1}{m - 1}}$$

Fall 2012 Problem 1 (HW2 Problem 10/HW 1 Problem 9) Only part I didn't do: Find the mean and variance of $S_n = X_1 + \ldots + X_n$, the total number of white balls added to the urn up to time n.

HW1 Problem 9. An urn contains b black and w white balls. At each step, a ball is removed from the urn at random and then put back together with one more ball of the same color. Compute the probability p_n to get a black ball on step $n, n \geq 1$.

Solution. Step 1:

$$p_1 = \frac{b}{b + w}$$

Step 2: We need to separately consider the cases where a black ball was selected on step 1 (with probability p_1) or a white ball (with probability $1 - p_1$).

$$p_2 = p_1 \cdot \frac{b+1}{b+w+1} + (1 - p_1) \cdot \frac{b}{b+w+1} = p_1 \left(\frac{b+1}{b+w+1} - \frac{b}{b+w+1} \right) + \frac{b}{b+w+1}$$

$$= p_1 \left(\frac{1}{b+w+1} + \frac{1}{p_1} \frac{b}{b+w+1} \right) = p_1 \left(\frac{1}{b+w+1} + \frac{b+w}{b} \frac{b}{b+w+1} \right)$$

$$= p_1 \left(\frac{b+w+1}{b+w+1} \right) = p_1$$

$$\implies p_2 = p_1 = \frac{b}{b+w}$$

Step 3: Regardless of the previous steps, there are now $b + w + 2$ balls in the urn. Since we know that $p_1 = p_2$, the probability that we have selected k black balls so far (and thus, the probability that there are currently $b + k$ black balls in the urn) is given by

$$\Pr(k \text{ balls chosen in first 2 rounds}) = \binom{2}{k} p_1^k (1 - p_1)^{2-k} = \binom{2}{k} \left(\frac{b}{b+w} \right)^k \left(\frac{w}{b+w} \right)^{2-k}$$

$$= \binom{2}{k} \frac{b^k w^{2-k}}{(b+w)^2}$$

for $k \in \{0, 1, 2\}$. Given that we have selected k black balls so far, the probability of selecting a black ball this time is $\frac{b+k}{b+w+2}$. Therefore the probability of selecting a black ball this round is

$$p_3 = \sum_{k=0}^{2} \binom{2}{k} \frac{b^k w^{2-k}}{(b+w)^2} \frac{b+k}{b+w+2} = \frac{1}{(b+w+2)(b+w)^2} \sum_{k=0}^{2} \binom{2}{k} (b+k) b^k w^{2-k}$$

$$= \frac{1}{(b+w+2)(b+w)^2} \left(\binom{2}{0} bw^2 + \binom{2}{1} (b+1)bw + \binom{2}{2} (b+2)b^2 \right)$$

$$= \frac{bw^2 + 2(b+1)bw + (b+2)b^2}{(b+w+2)(b+w)^2} = \frac{b}{b+w} \left(\frac{w^2 + 2bw + 2w + b^2 + 2b}{b^2 + bw + 2b + wb + w^2 + 2w} \right)$$

$$= \frac{b}{b+w} \left(\frac{w^2 + 2bw + 2w + b^2 + 2b}{b^2 + 2bw + 2b + w^2 + 2w} \right) = \frac{b}{b+w} = p_1$$

There seems to be a clear pattern here. Let's find the general formula by induction.

Step $n+1$: Assume that the probability of choosing a black ball on steps $1, 2, \ldots, n$ was $\frac{b}{b+w}$ each time.

(a bunch of boring stuff, then it worked.)

HW2 Problem 10. Random variables $(X_1, \ldots, X_n)$ are called *exchangeable* if $\Pr(X_1 = x_1, \ldots, X_n = x_n) = \Pr(X_{\tau(1)} = x_1, \ldots, X_{\tau(n)} = x_n)$ for all real numbers $x_1, \ldots, x_n$ and every permutation τ of the set $\{1, \ldots, n\}$. In the setting of Problem 9 from Homework 1, let $X_k = 1$ if a white ball is drawn on step k, and $X_k = 0$ otherwise. Show that the random variables $X_1, \ldots, X_n$ are exchangeable for every $n \geq 2$.

Solution. For $n = 2$: There are two cases which we must show are equal to show exchangeability:

$$\Pr(X_1 = 0, X_2 = 1) = \Pr(X_1 = 1, X_2 = 0)$$

First,

$$\Pr(X_1 = 0, X_2 = 1) = \Pr(\text{black first}) \Pr(\text{white second} \mid \text{black first}) = \left(\frac{b}{b+w} \right) \left(\frac{w}{b+w+1} \right)$$

$$\left(\frac{w}{b+w} \right) \left(\frac{b}{b+w+1} \right) = \Pr(X_1 = 1, X_2 = 0)$$

which proves exchangeability for $n = 2$. In the general case, we seek to show that $X_1, \ldots, X_n$ are exchangeable. That is, in all $n+1$ unordered sets $\mathbb{X}_k = \{x_{1k}, x_{2k}, \ldots, x_{nk} \mid x_{ik} \in \{0, 1\}, \sum_i x_{ik} = k\}$, in all $\binom{n}{k}$ permutations of $\mathbb{X}_k$,

$$\Pr(\mathbb{X}_{kj} = \Pr(\mathbb{X}_{kj'}$$

where j and j' denote different permutations of $\mathbb{X}_k$. That is,

$$\Pr(X_1 = x_{1k}, X_2 = x_{2k}, \dots, X_n = x_{nk}) = \Pr(X_{j_1} = x_{1k}, X_{j_2} = x_{2k}, \dots, X_{j_n} = x_{nk})$$

where $j_1, j_2, \dots, j_n$ index the permuted variables. Consider $\mathbb{X}_{kj^*}$ where all k white balls are chosen first and all $n-k$ black balls are chosen last. We have

$$\Pr(\mathbb{X}_{kj^*}) = \prod_{i=1}^{k} \left(\frac{w+i-1}{b+w+i-1} \right) \cdot \prod_{i=k+1}^{n} \left(\frac{b+i-k-1}{b+w+i-1} \right)$$

$$= \prod_{i=1}^{n} \left(\frac{1}{b+w+i-1} \right) \cdot \left[\prod_{i=1}^{k} (w+i-1) \prod_{i=k+1}^{n} (b+i-k-1) \right] = \prod_{i=1}^{n} \left(\frac{1}{b+w+i-1} \right) \cdot \left[\prod_{i=1}^{k} (w+i-1) \prod_{i'=1}^{n-k} (b+i'-1) \right]$$

It is easy to see that the leftmost product will always equal the product of the denominators, regardless of the permutation, since one ball is added to the urn after every draw. Similarly, regardless of permutation, the numerator of the probability of drawing the ith white ball will always equal $w+i-1$, the number of white balls already in the urn. Likewise, the numerator of the probability of drawing the i'th black ball is always $b+i'-1$. Because multiplication is commutative, all permutations of these numbers will have equal products. Therefore $\Pr(\mathbb{X}_{kj^*}) = \Pr(\mathbb{X}_{kj}$ for all k. That is,

$$\Pr(X_1 = x_1, \dots, X_n = x_n) = \Pr(X_{\tau(1)} = x_1, \dots, X_{\tau(n)} = x_n)$$

for all $(x_1, \dots, x_n) \in \mathbb{R}^n$, all $n \in \mathbb{Z}$ such that $n \geq 2$, all permutations τ.

Homework 2 Problem 2. Consider the function

$$f(x) = \begin{cases} C(2x - x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Could f be a distribution function? If so, determine C.

(b) Could f be a probability density function? If so, determine C.

Solution.

(a) If f is a distribution function, $\lim_{x \to -\infty} f(x) = 0$, $\lim_{x \to \infty} f(x) = 1$, and $f'(x) \geq 0 \ \forall \ x \in \mathbb{R}$. f clearly does not meet the second or third conditions and is therefore $\boxed{\text{not a distribution function.}}$

(b) If f is a density function then $\int_{-\infty}^{\infty} f(x)dx = 1$ and $f(x) \geq 0 \ \forall x \in \mathbb{R}$.

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{2} C(2x - x^2)dx = C\left[x^2 - \frac{x^3}{3}\right]_0^2 = C\left(4 - \frac{8}{3} - 0\right) = C \cdot \frac{4}{3}$$

$$= 1 \iff C = \frac{3}{4}$$

Next we check that f is always nonnegative. It equals zero except on $(0, 2)$.

$$\frac{3}{4}(2x - x^2) \geq 0 \iff x(2-x) \geq 0 \iff x \in (0,2)$$

Therefore f is nonnegative $\forall x \in \mathbb{R}$, so $\boxed{f \text{ is a probability density function if } C = \frac{3}{4}.}$

HW1 Problem 8. Two people, A and B, are involved in a duel. The rules are simple: shoot at each other once; if at least one is hit, the duel is over, if both miss, repeat (go to the next round), and so on. Denote by p_A and p_B the probabilities that A hits B and B hits A with one shot, and assume that that hitting/missing is independent from round to round. Compute the probabilities of the following events: (a) the duel ends and A is not hit; (b) the duel ends and both are hit; (c) the duel ends after round number n; (d) the duel ends after round number n GIVEN that A is not hit; (e) the duel ends after n rounds GIVEN that both are hit; (f) the duel goes on forever.

Solution.

(a) Let A_k denote the event that the duel is ended by A shooting B in the kth round (with neither person being shot in the first $k-1$ rounds). Note that $\{A_k | k = 1, 2, \ldots\}$ are all mutually exclusive. Therefore the probability of the duel ending without A being hit is $\sum_{k=1}^{\infty} A_k$. Because the probabilities in each round are constant and independent,

$$A_k = (1-p_A)^{k-1} p_A (1-p_B)^k$$

So the probability that the duel ends and A is not hit is

$$\sum_{k=1}^{\infty} A_k = \sum_{k=1}^{\infty} (1-p_A)^{k-1} p_A (1-p_B)^k = p_A(1-p_B) \sum_{k=1}^{\infty} (1-p_A)^{k-1} (1-p_B)^{k-1}$$

This is an infinite geometric series. Since the ratio $(1-p_A)(1-p_B)$ has absolute value less than 1, the sum can be calculated.

$$\sum_{k=1}^{\infty} A_k = p_A(1-p_B) \cdot \frac{1}{1-(1-p_A)(1-p_B)} = \frac{p_A(1-p_B)}{p_A + p_B - p_A p_B} = \boxed{\frac{p_A(1-p_B)}{p_A(1-p_B) + p_B}}$$

(b) Similar to part (a). Let C_k denote the event that the duel is ended with both players being shot in the kth round (with neither person being shot in the first $k-1$ rounds). Again, $\{C_k | k = 1, 2, \ldots\}$ are all mutually exclusive, so the probability of the duel ending in these circumstances is $\sum_{k=1}^{\infty} C_k$. We have

$$C_k = (1-p_A)^{k-1} p_A (1-p_B)^{k-1} p_B$$

$$\sum_{k=1}^{\infty} C_k = \sum_{k=1}^{\infty} (1-p_A)^{k-1} p_A (1-p_B)^{k-1} p_B = p_A p_B \sum_{k=1}^{\infty} (1-p_A)^{k-1} (1-p_B)^{k-1}$$

$$= p_A p_B \cdot \frac{1}{1-(1-p_A)(1-p_B)} = \boxed{\frac{p_A p_B}{p_A + p_B - p_A p_B}}$$

Note that this value is less than the answer from part (a) if $p_B < \frac{1}{2}$ and greater if $p_B > \frac{1}{2}$

(c) Let B_k denote the event that the duel is ended by B shooting A in the kth round (with neither person being shot in the first $k-1$ rounds), with

$$B_k = (1-p_A)^k p_B (1-p_B)^{k-1}$$

Let A_k and C_k be defined as above. Note that $\{A_k | k = 1, 2, \ldots\}, \{B_k | k = 1, 2, \ldots\}, \{C_k | k = 1, 2, \ldots\}$ are all mutually exclusive, and that the event that the duel ends in round n is $\{A_n \cup B_n \cup C_n\}$. So the probability of the duel ending in round n is

$$\Pr(A_n \cup B_n \cup C_n) = \Pr(A_n) + \Pr(B_n) + \Pr(C_n)$$

$$= (1-p_A)^{n-1} p_A (1-p_B)^n + (1-p_A)^n p_B (1-p_B)^{n-1} + (1-p_A)^{n-1} p_A (1-p_B)^{n-1} p_B$$

$$= (1-p_A)^{n-1}(1-p_B)^{n-1}\big[p_A(1-p_B) + (1-p_A)p_B + p_A p_B\big]$$

$$= \boxed{(1-p_A)^{n-1}(1-p_B)^{n-1}(p_A + p_B - p_A p_B)}$$

(d) Let A_k, B_k, C_k be defined as above. The event that the duel ends at round n without A being hit is given by $\{A_n\}$.

$$\Pr(A_n) = \boxed{(1-p_A)^{n-1} p_A (1-p_B)^n}$$

(e) Let A_k, B_k, C_k be defined as above. The event that the duel ends at round n with both players being hit is given by $\{C_n\}$.

$$\Pr(C_n) = \boxed{(1-p_A)^{n-1} p_A (1-p_B)^{n-1} p_B}$$

(f) Let A_k, B_k, C_k be defined as above. The probability that the duel never ends is equal to 1 - the probability that the duel ends at some point, which is $\{A_k | k = 1, 2, \ldots\} \cup \{B_k | k = 1, 2, \ldots\} \cup \{C_k | k = 1, 2, \ldots\}$. Since all of these events are mutually exclusive, we have

$$1 - \Pr(\{A_k | k = 1, 2, \ldots\} \cup \{B_k | k = 1, 2, \ldots\} \cup \{C_k | k = 1, 2, \ldots\}) = 1 - \sum_{k=1}^{\infty} \big(A_k + B_k + C_k\big)$$

$$= 1 - \sum_{k=1}^{\infty} \big((1-p_A)^{k-1} p_A (1-p_B)^k + (1-p_A)^k p_B (1-p_B)^{k-1} + (1-p_A)^{k-1} p_A (1-p_B)^{k-1} p_B\big)$$

$$= 1 - \big[p_A(1-p_B) + (1-p_A)p_B + p_A p_B\big] \sum_{k=1}^{\infty} (1-p_A)^{k-1}(1-p_B)^{k-1}$$

$$= 1 - \big[p_A 1 - p_A p_B) + p_B - p_A)p_B + p_A p_B\big] \cdot \frac{1}{1-(1-p_A)(1-p_B)}$$

$$= 1 - \frac{p_A - p_A p_B + p_B - p_A p_B + p_A p_B}{p_A + p_B - p_A p_B} = 1 - \frac{p_A + p_B - p_A p_B}{p_A + p_B - p_A p_B} = \boxed{0}$$

Homework 1 Problem 1.

(I) Seven different gifts are distributed among 10 children. How many different outcomes are possible if every child can receive (a) at most one gift, (b) at most two gifts, (c) any number of gifts?

(II) Answer the same questions if the gifts are identical (but the children are still different).

Solution.

(I) (a) $\binom{10}{7}7! = \boxed{604,800}$

(b) Clearly all outcomes that satisfy part (I)(a) also satisfy these conditions, so we start with $\binom{10}{7}7! = 604,800$ possible outcomes. In addition, the following outcomes are possible:

(i) **A set of 6 children receive gifts; one child receives two gifts.** There are $\binom{10}{6}$ ways to pick a group of 6 children to receive the gifts. Next, there are $\binom{6}{1} = 6$ ways to choose which child receives two gifts. Finally, there are $7!/2!$ unique ways to distribute the gifts among the children once a particular partition is chosen (since order matters for all of the gifts except for the two that are received by the same child).

(ii) **A set of 5 children receive gifts; two children receive two gifts.** There are $\binom{10}{5}$ ways to pick a group of 5 children to receive the gifts. Next, there are $\binom{5}{2}$ ways to choose which of these children receive one gift and which receive two. Finally, there are $7!/(2!2!)$ unique ways to distribute the gifts among the children once a particular partition is chosen (since order matters for all of the gifts except for the two batches of two gifts that are received by the same child).
(Note that without the restriction that a child can receive at most two gifts, another possibility is that 1 child could receive 3 gifts, but that wouldn't work in this case.)

(iii) **A set of 4 children receive gifts; three children each receive two gifts.** There are $\binom{10}{4}$ ways to pick a group of 4 children to receive the gifts. Next, there are $\binom{4}{3} = 4$ ways to choose which of these children receive one gift and which receive two. Finally, there are $7!/(2!2!2!)$ unique ways to distribute the gifts among the children once a particular partition is chosen (since order matters for all of the gifts except for the three batches of two gifts that are received by the same child).
(Again, there are other possibilities for 4 children to receive 7 gifts, but none that satisfy the condition that no child receives more than 2 gifts.)

Clearly each of these outcomes are mutually exclusive. Therefore the answer is

$$\binom{10}{7}7! + \binom{10}{6} \cdot \binom{6}{1} \cdot \frac{7!}{2!} + \binom{10}{5} \cdot \binom{5}{2} \cdot \frac{7!}{2!2!} + \binom{10}{4} \cdot \binom{4}{3} \cdot \frac{7!}{2!2!2!}$$

$$= 7! \cdot \left(\frac{10!}{3!} + \frac{10!}{6!4!} \cdot 6 \cdot \frac{1}{2} + \frac{10!}{5!5!} \cdot \frac{5!}{3!2!} \cdot \frac{1}{4} + \frac{10!}{4!6!} \cdot \frac{4!}{3!} \cdot \frac{1}{8} \right)$$

$$= 7!10! \cdot \left(\frac{1}{3!} + \frac{1}{6!4!} \cdot \frac{6}{2} + \frac{1}{5!} \cdot \frac{1}{3!2!} \cdot \frac{1}{4} + \frac{1}{6!3! \cdot 8} \right)$$

$$= \boxed{7,484,400}$$

(c) $10^7 = \boxed{10,000,000}$

(II) (a) $\binom{10}{7} = \boxed{120}$

(b) Clearly all outcomes that satisfy part (I)(a) also satisfy these conditions, so we start with $\binom{10}{7} =$ 120 possible outcomes. In addition, the following outcomes are possible:

(i) A set of 6 children receive gifts; one child receives two gifts (6 distinct ways this could happen for each set of 6 children).

(ii) A set of 5 children receive gifts; two children receive two gifts ($\binom{5}{2}$ distinct ways this could happen for each set of 5 children).

(iii) A set of 4 children receive gifts; three children each receive two gifts (4 distinct ways this could happen for each set of 4 children).

Clearly each of these outcomes are mutually exclusive. Therefore the answer is

$$\binom{10}{7} + \binom{10}{6} \cdot \binom{6}{7-6} + \binom{10}{5} \cdot \binom{5}{7-5} + \binom{10}{4} \cdot \binom{4}{7-4} = \boxed{4,740}$$

(c) By Proposition 6.16, the number of nonnegative integer-valued vectors $(x_1, x_2, \ldots, x_r)$ satisfying the equation

$$x_1 + x_2 + \ldots + x_r = n$$

is equal to $\binom{n+r-1}{r-1} = \binom{7+10-1}{10-1} = \boxed{11,440}$.

Homework 1 Problem 2.

(I) 20 different gifts are distributed among seven children. How many different outcomes are possible if every child can receive (a) at least one gift, (b) at least two gifts, (c) any number of gifts?

(II) Answer the same questions if the gifts are identical (but the children are still different).

(III) Now try to generalize problems (1) and (2).

Solution.

(I) (a) There are 7^{20} possible allocations of gifts if we have no restrictions. If one child doesn't get a gift, there are $\binom{7}{1}$ ways to choose which child that is and 6^{20} subsequent allocations of gifts. Likewise, there are $\binom{7}{2} \cdot (7-2)^{20}$ ways to allocate the gifts if two children don't receive gifts, $\binom{7}{3} \cdot (7-3)^{20}$ ways if three children don't receive gifts, $\binom{7}{4} \cdot (7-4)^{20}$ ways if four children don't receive gifts, $\binom{7}{5} \cdot (7-5)^{20}$ ways if five children don't receive gifts, and $\binom{7}{6} \cdot (7-6)^{20}$ ways if six children don't receive gifts.

Let A_i denote the number of allocations in which i children do not receive gifts. In order to make the calculation, we must use the Inclusion-Exclusion principle (Proposition 6.13) (because, for example, some of the allocations in which three children don't receive gifts include allocations where four or more children don't receive gifts, and we don't want to double-count). Therefore the number of ways that at least one child can not receive a gift (i.e. the complement of every child receiving at least one gift) is

$$\left| \bigcup_{i=1}^{6} A_i \right| = \sum_{i=1}^{6} |A_i| - \sum_{1 \le i < j \le 6} |A_i \cap A_j| + \sum_{1 \le i < j < k \le 6} |A_i \cap A_j \cap A_k| - \ldots$$

$$+(-1)^{6-1}\left|A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_6\right|$$

Fortunately, these allocations are nested in the sense that all the allocations where e.g. 5 children do not receive gifts are a subset of all the allocations where 4 children do not receive gifts; that is

$$A_6 \subset A_5 \subset A_4 \subset A_3 \subset A_2 \subset A_1$$

which implies e.g.

$$A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_6 = A_6,$$

$$\sum_{1 \le i < j \le 6} |A_i \cap A_j| = 5|A_6| + 4|A_5| + 3|A_4| + 2|A_3| + |A_2|$$

So we have

$$\left|\bigcup_{i=1}^{6} A_i\right| = |A_6| + |A_5| + |A_4| + |A_3| + |A_2| + |A_1| - (5|A_6| + 4|A_5| + 3|A_4| + 2|A_3| + |A_2|)$$

$$+(4|A_6| + 3|A_5| + 2|A_4| + |A_3|) - (3|A_6| + 2|A_5| + |A_4|) + \ldots - |A_6|$$

$$= |A_1| - |A_2| + |A_3| - |A_4| + |A_5| - |A_6|$$

$$= \binom{7}{1} \cdot 6^{20} - \binom{7}{2} \cdot (7-2)^{20} + \binom{7}{3} \cdot (7-3)^{20} - \binom{7}{4} \cdot (7-4)^{20}$$

$$+\binom{7}{5} \cdot (7-5)^{20} - \binom{7}{6} \cdot (7-6)^{20}$$

The final answer is

$$7^{20} - \left|\bigcup_{i=1}^{6} A_i\right| = 7^{20} - \binom{7}{1} \cdot 6^{20} + \binom{7}{2} \cdot (7-2)^{20} - \binom{7}{3} \cdot (7-3)^{20} + \binom{7}{4} \cdot (7-4)^{20}$$

$$-\binom{7}{5} \cdot (7-5)^{20} + \binom{7}{6} \cdot (7-6)^{20} \approx \boxed{5.616 \cdot 10^{16}}$$

(b) Similar to above, but more complicated. The complement of every child receiving at least two gifts is that at least one child doesn't receive a gift (same as above) or at least one child only receives one gift. So we start from the baseline answer above, and subtract out all the possible allocations in which at least one child receives one gift.

If one child only receives one gift (and the rest receive more than one), there are $\binom{7}{1}$ ways to choose which child that is, $\binom{20}{1}$ ways to choose which gift that child receives, and 6^{20-1} allocations of the remaining gifts. If two children receive only one gift, there are $\binom{7}{2}$ ways to choose which children those are, $\binom{20}{2} \cdot 2!$ ways to choose which gifts those children get and distribute them among those children, and $(7-2)^{20-2}$ ways to allocate the remaining gifts. Likewise, if three children receive only one gift there are $\binom{7}{3}\binom{20}{3} \cdot 3! \cdot (7-3)^{20-3}$ ways to allocate the gifts,

$\binom{7}{4}\binom{20}{4} \cdot 4! \cdot (7-4)^{20-4}$ ways if four children receive only one gift, $\binom{7}{5}\binom{20}{5} \cdot 5! \cdot (7-5)^{20-5}$ ways if five children receive only one gift, and $\binom{7}{6}\binom{20}{6} \cdot 6!(7-6)^{20-6}$ ways if six children don't receive gifts.

Let B_j be the event that j children receive only one gift. Note that $B_1 \cap A_i$ is nonempty $\forall i < 7-1$, $B_2 \cap A_i$ is nonempty $\forall i < 7-2$, and in general, $B_j \cap A_i$ is nonempty $\forall i < 7-j, j \in \{1, 2, \ldots, 6\}$. Applying the Inclusion-Exclusion Principle (Proposition 6.13) in a similar way as in part (I)(a), the answer is

$$\boxed{7^{20} - \left|\bigcup_{i=1}^{6} A_i\right| - \left|\bigcup_{j=1}^{6} B_j\right| + \sum_{i \in \{1,\ldots,6\}, j \in \{1,\ldots,6\}} \left|A_i \bigcap B_j\right|}$$

Per part (I)(a), the first two terms approximately equal $5.616 \cdot 10^16$. Clearly

$$\bigcup_{i \in \{1,\ldots,6\}, j \in \{1,\ldots,6\}} \left(A_i \bigcap B_j\right) \subset \bigcup_{j=1}^{6} B_j$$

which implies

$$-\left|\bigcup_{j=1}^{6} B_j\right| + \left|A_i \bigcap_{i \in \{1,\ldots,6\}, j \in \{1,\ldots,6\}} B_j\right| < 0$$

so the answer to this part will be less than $5.616 \cdot 10^16$, which makes sense.

Calculating $\left|\bigcup_{j=1}^{6} B_j\right|$ is not too difficult using Inclusion-Exclusion:

$$\left|\bigcup_{j=1}^{6} B_j\right| = \sum_{j=1}^{6} |B_j| - \sum_{1 \le j < k \le 6} |B_j \cap B_k| + \sum_{1 \le j < k < \ell \le 6} |B_j \cap B_k \cap B_\ell| - \ldots$$

$$+(-1)^{6-1} |B_1 \cap B_2 \cap B_3 \cap \cdots \cap B_6|$$

where since

$$B_6 \subset B_5 \subset B_4 \subset B_3 \subset B_2 \subset B_1$$

which implies e.g.

$$B_1 \cap B_2 \cap B_3 \cap \cdots \cap B_6 = B_6,$$

$$\sum_{1 \le j < k \le 6} |B_j \cap B_j| = 5|B_6| + 4|B_5| + 3|B_4| + 2|B_3| + |B_2|$$

we have

$$\left|\bigcup_{j=1}^{6} B_j\right| = |B_6| + |B_5| + |B_4| + |B_3| + |B_2| + |B_1| - (5|B_6| + 4|B_5| + 3|B_4| + 2|B_3| + |B_2|)$$

$$+(4|B_6| + 3|B_5| + 2|B_4| + |B_3|) - (3|B_6| + 2|B_5| + |B_4|) + \ldots - |B_6|$$

$$= |B_1| - |B_2| + |B_3| - |B_4| + |B_5| - |B_6|$$

$$= \binom{7}{1}\binom{20}{1}\cdot(7-1)^{20-1} - \binom{7}{2}\binom{20}{2}\cdot 2!\cdot(7-2)^{20-2} + \binom{7}{3}\binom{20}{3}\cdot 3!\cdot(7-3)^{20-3} - \binom{7}{4}\binom{20}{4}\cdot 4!\cdot(7-4)^{20-4}$$

$$+\binom{7}{5}\binom{20}{5}\cdot 5!\cdot(7-5)^{20-5} - \binom{7}{6}\binom{20}{6}\cdot 6!(7-6)^{20-6}$$

$$\boxed{\approx 5.846 \cdot 10^{16}}$$

However, calculating

$$\sum_{i\in\{1,\ldots,6\}, j\in\{1,\ldots,6\}} \left|A_i \bigcap B_j\right|$$

is very difficult because, for example, $B_2 \cap A_3$ is nonempty but $B_2 \not\subset A_3$ and $A_3 \not\subset B_2$.

(c) $\boxed{7^{20} \approx 7.979 \cdot 10^{16}}$

(II) (a) By Proposition 6.15, there are $\binom{19}{6} = \boxed{27,132}$ ways to do this.

(b) Similar to Problem 1 part (II)(c), if the vector $(x_1, x_2, \ldots, x_7)$ represents the number of gifts given to each child, we would like a solution such that

$$x_1 + x_2 + \ldots + x_7 = 20, x_i \geq 2\ \forall i$$

By Proposition 6.17, the number of possible allocations under these conditions, is $\binom{20+7\cdot(1-2)-1}{7-1} = \binom{12}{6} = \boxed{924}$.

(c) By Proposition 6.16, the number of nonnegative integer-valued vectors $(x_1, x_2, \ldots, x_r)$ satisfying the equation

$$x_1 + x_2 + \ldots + x_r = n$$

is equal to $\binom{n+r-1}{r-1}$. In distributing 20 identical gifts to 7 different children, we can imagine the vector $(x_1, x_2, \ldots, x_{10})$ represents the number of gifts given to each child (where x_i is a nonnegative integer for all i). So we have $n = 20$ and $r = 7$. Therefore the number of possible allocations is

$$\binom{20+7-1}{7-1} = \boxed{165,765,600}$$

(III) Generalization of 1(I): If there are g distinguishable gifts and $c \geq g$ children, the number of distinct allocations if each child can receive

(a) at most one gift is $\binom{c}{g}g!$.

(b) at most two gifts is

$$\sum_{i=c-g+1}^{g} \binom{c}{i} \cdot \binom{i}{g-i} \cdot \frac{g!}{(2!)^{g-i}}$$

(c) any number of gifts is c^g.

Generalization of 1(II): If there are g identical gifts and $c \geq g$ children, the number of distinct allocations if each child can receive

(a) at most one gift is $\binom{c}{g}$.

(b) at most two gifts is

$$\sum_{i=c-g+1}^{g} \binom{c}{i} \cdot \binom{i}{g-i}$$

(c) any number of gifts is $\binom{g+c-1}{c-1}$

Generalization of 2(I): If there are g distinguishable gifts and $c \leq g$ children, the number of distinct allocations if each child must receive

(a) at least one gift is

$$c^g - \sum_{i=1}^{c-1} (-1)^{i+1} \binom{c}{i} \cdot (c-i)^g$$

(b) at least two gifts is

$$c^g - \sum_{i=1}^{c-1} (-1)^{i+1} \binom{c}{i} \cdot (c-i)^g - \sum_{i=1}^{c-1} (-1)^{i+1} \binom{c}{i} \binom{g}{i} \cdot i! \cdot (c-i)^{g-i}$$

(c) any number of gifts is c^g

Generalization of 2(II): If there are g identical gifts and $c \leq g$ children, the number of distinct allocations if each child must receive

(a) at least one gift is

$$\binom{g-1}{c-1}$$

(b) at least two gifts is

$$\binom{g-c-1}{c-1}$$

(c) any number of gifts is

$$\binom{g+c-1}{c-1}$$

Homework 1 Problem 4. You have \$20K to invest, and have a choice of stocks, bonds, mutual funds, or a CD. Investments must be made in multiples of \$1K, and there are minimal amounts to be invested: \$2K in stocks, \$2K in bonds, \$3K in mutual funds, and \$4K in the CD. Count the number of choices in each situation: (a) You want to invest in all four, (b) you want to invest in at least three out of four.

Solution.

(a) If the vector $(x_S, x_B, x_{MF}, x_{CD})$ represents the amount of money (in thousands of dollars) invested in each instrument, we would like a solution such that

$$x_S + x_B + x_{MF} + x_{CD} = 20$$

where

$$x_S \geq 2, x_B \geq 2, x_{MF} \geq 3, x_{CD} \geq 4$$

In a way similar to the proof for Proposition 6.17, note that we can transform this problem in the following way:

$$x_S - 1 + x_B - 1 + x_{MF} - 2 + x_{CD} - 3 = 20 - (1 + 1 + 2 + 3)$$

where

$$x_S - 1 \geq 1, x_B - 1 \geq 1, x_{MF} - 2 \geq 1, x_{CD} - 3 \geq 1$$

Letting $y_S = x_S - 1, y_B = x_B - 1, y_{MF} = x_{MF} - 2, y_{CD} = x_{CD} - 3$, we have the equivalent system

$$y_S + y_B + y_{MF} + y_{CD} = 13, y \geq 1 \ \forall y$$

By Proposition 6.15, the number of distinct solutions to this equation, and therefore the number of possible allocations under these conditions, is $\binom{13-1}{4-1} = \boxed{220}$.

(b) Enumerate the $\binom{4}{3} = 4$ possibilities.

(i) **Invest in stocks, bonds, and mutual funds.**

$$x_S + x_B + x_{MF} = 202$$

where

$$x_S \geq 2, x_B \geq 2, x_{MF} \geq 3$$

Note that we can transform this problem in the following way:

$$x_S - 1 + x_B - 1 + x_{MF} - 2 = 20 - (1 + 1 + 2)$$

where

$$x_S - 1 \geq 1, x_B - 1 \geq 1, x_{MF} - 2 \geq 1$$

Letting $y_S = x_S - 1, y_B = x_B - 1, y_{MF} = x_{MF} - 2$, we have the equivalent system

$$y_S + y_B + y_{MF} = 16, y \geq 1 \ \forall y$$

Therefore the number of possible allocations under these conditions is $\binom{16-1}{3-1} = \boxed{105}$.

(ii) **Invest in stocks, bonds, and CDs.**

$$x_S + x_B + x_{CD} = 20$$

where

$$x_S \geq 2, x_B \geq 2, x_{CD} \geq 4$$

Note that we can transform this problem in the following way:

$$x_S - 1 + x_B - 1 + x_{CD} - 3 = 20 - (1 + 1 + 3)$$

where

$$x_S - 1 \geq 1, x_B - 1 \geq 1, x_{CD} - 3 \geq 1$$

Letting $y_S = x_S - 1, y_B = x_B - 1, y_{CD} = x_{CD} - 3$, we have the equivalent system

$$y_S + y_B + y_{CD} = 15, y \geq 1 \ \forall y$$

Therefore the number of possible allocations under these conditions is $\binom{15-1}{3-1} = \boxed{91}$.

(iii) **Invest in stocks, mutual funds, and CDs.**

$$x_S + x_{MF} + x_{CD} = 2$$

where

$$x_S \geq 2, x_{MF} \geq 3, x_{CD} \geq 4$$

Note that we can transform this problem in the following way:

$$x_S - 1 + x_{MF} - 2 + x_{CD} - 3 = 20 - (1 + 2 + 3)$$

where

$$x_S - 1 \geq 1, x_{MF} - 2 \geq 1, x_{CD} - 3 \geq 1$$

Letting $y_S = x_S - 1, y_{MF} = x_{MF} - 2, y_{CD} = x_{CD} - 3$, we have the equivalent system

$$y_S + y_{MF} + y_{CD} = 14, y \geq 1 \ \forall y$$

Therefore the number of possible allocations under these conditions is $\binom{14-1}{3-1} = \boxed{78}$.

(iv) **Invest in bonds, mutual funds, and CDs.**

$$x_B + x_{MF} + x_{CD} = 2$$

where

$$x_B \geq 2, x_{MF} \geq 3, x_{CD} \geq 4$$

Note that we can transform this problem in the following way:

$$x_B - 1 + x_{MF} - 2 + x_{CD} - 3 = 20 - (1 + 2 + 3)$$

where

$$x_B - 1 \geq 1, x_{MF} - 2 \geq 1, x_{CD} - 3 \geq 1$$

Letting $y_B = x_B - 1, y_{MF} = x_{MF} - 2, y_{CD} = x_{CD} - 3$, we have the equivalent system

$$y_B + y_{MF} + y_{CD} = 14, y \geq 1 \ \forall y$$

therefore the number of possible allocations under these conditions is $\binom{14-1}{3-1} = \boxed{78}$.

(v) **Invest in all four:** per part 4(a), there are $\boxed{220}$ ways to do this.

Note that all of these possibilities are mutually exclusive. Therefore the total number is

$$\binom{16-1}{3-1} + \binom{15-1}{3-1} + \binom{14-1}{3-1} + \binom{14-1}{3-1} + \binom{13-1}{4-1} = 105 + 91 + 78 + 78 + 220 = \boxed{572}$$

6.2.4 DSO Statistics Group Screening Exam Problems

Exercise 2 (2017 DSO Statistics Group In-Class Screening Exam, Question 1). Let $X_1, X_2, \ldots, X_k$ be independent standard normal random variables and $\gamma_1(t), \ldots, \gamma_k(t)$ infinitely differentiable functions of a real variable defined on a closed, bounded interval, such that $\sum_{i=1}^k \gamma_i^2(t) = 1$ for all t. Let $Z(t) = \sum_{i=1}^k \gamma_i(t) X_i$. Let $\dot{Z}(t), \ddot{Z}(t)$, etc. denote first, second, etc. derivatives of $Z(t)$ with respect to t.

(a) Show that $\operatorname{Cov}(Z(t), \dot{Z}(t)) = 0$.

(b) Evaluate $\mathbb{E}(Z(t) \mid \ddot{Z}(t))$ in terms of $\ddot{Z}(t)$ and expressions of the form

$$\sum_{i=1}^k (\gamma_i(t))^a (\delta^m \gamma_i(t)/\delta t^m)^b,$$

for some a, b, m values.

Solution.

(a)

$$\dot{Z}(t) = \frac{\partial}{\partial t} \sum_{i=1}^k \gamma_i(t) X_i = \sum_{i=1}^k \dot{\gamma}_i(t) X_i$$

$$\implies \mathbb{E}\left(\dot{Z}(t)\right) = \sum_{i=1}^k \mathbb{E}\left(\dot{\gamma}_i(t) X_i\right) = 0$$

$$\implies \operatorname{Cov}(Z(t), \dot{Z}(t)) = \mathbb{E}\big[(Z(t) - \mathbb{E}[Z(t)])(\dot{Z}(t) - \mathbb{E}[\dot{Z}(t)])\big] = \mathbb{E}\big[Z(t)\dot{Z}(t)\big]$$

$$= \mathbb{E}\left[\left(\sum_{i=1}^k \gamma_i(t) X_i\right)\left(\sum_{i=1}^k \dot{\gamma}_i(t) X_i\right)\right] = \sum_{i=1}^k \mathbb{E}\left(\gamma_i(t)\dot{\gamma}_i(t) X_i^2\right) + 0 = \sum_{i=1}^k \gamma_i(t)\dot{\gamma}_i(t)$$

since $\mathbb{E}(X_i^2) = 1$. But

$$\sum_{i=1}^k \gamma_i \dot{\gamma}_i(t) = 0 \tag{6.8}$$

because

$$\sum_{i=1}^k \gamma_i^2(t) = 1 \iff \frac{\partial}{\partial t}\left(\sum_{i=1}^k \gamma_i^2(t)\right) = 0 \iff 2\sum_{i=1}^k \gamma_i(t)\dot{\gamma}_i(t) = 0,$$

so the conclusion follows.

(b)

$$Z(t) = \sum_{i=1}^k \gamma_i(t) X_i \implies Z(t) \sim \mathcal{N}\left(0, \sum_{i=1}^k \gamma_i^2(t)\right) = \mathcal{N}(0, 1)$$

$$\ddot{Z}(t) = \sum_{i=1}^k \ddot{\gamma}_i(t) X_i \implies \ddot{Z}(t) \sim \mathcal{N}\left(0, \sum_{i=1}^k \ddot{\gamma}_i^2(t)\right)$$

Also, we have

$$\operatorname{Cov}(Z(t), \ddot{Z}(t)) = \mathbb{E}\big[(Z(t) - \mathbb{E}[Z(t)])(\ddot{Z}(t) - \mathbb{E}[\ddot{Z}(t)])\big] = \mathbb{E}\big[Z(t)\ddot{Z}(t)\big]$$

$$= \mathbb{E}\left[\left(\sum_{i=1}^k \gamma_i(t) X_i\right)\left(\sum_{i=1}^k \ddot{\gamma}_i(t) X_i\right)\right] = \sum_{i=1}^k \mathbb{E}\left(\gamma_i(t)\ddot{\gamma}_i(t) X_i^2\right) + 0 = \sum_{i=1}^k \gamma_i(t)\ddot{\gamma}_i(t) = -\sum_{i=1}^k \dot{\gamma}_i^2(t)$$

because differentiating (6.8) yields

$$\frac{\partial}{\partial t}\sum_{i=1}^k \gamma_i \dot{\gamma}_i(t) = 0 \iff \sum_{i=1}^k \left(\dot{\gamma}_i^2(t) + \gamma_i(t)\ddot{\gamma}_i(t)\right) = 0 \iff \sum_{i=1}^k \gamma_i(t)\ddot{\gamma}_i(t) = -\sum_{i=1}^k \dot{\gamma}_i^2(t).$$

Since both distributions are Gaussian, we have

$$\mathbb{E}(Z(t) \mid \ddot{Z}(t)) = \mathbb{E}[Z(t)] + \frac{\operatorname{Cov}[Z(t), \ddot{Z}(t)]}{\operatorname{Var}[\ddot{Z}(t)]}(\ddot{Z}(t) - \mathbb{E}[\ddot{Z}(t)]) = \left[\sum_{i=1}^k \ddot{\gamma}_i^2(t)\right]^{-1} \cdot \left[-\sum_{i=1}^k \dot{\gamma}_i^2(t)\right] \ddot{Z}(t).$$

Exercise 3 (2018 DSO Statistics Group In-Class Screening Exam, Question 1). (a) For each $x > 0$, let $M(x)$ be a real-valued random variable and set $M(0) = 0$. Assume that the random function $M(x)$ is monotone non-decreasing on $[0, \infty)$. Define $T(y) := \inf\{x \geq 0 : M(x) \geq y\}$. Suppose that $e^{-1}T(y)$ converges in distribution to an Exponential(λ) random variable when $y \to \infty$.

(i) Find non-random $a(x)$ and $b(x) > 0$ such that $(M(x) - a(x))/b(x)$ converges in distribution for $x \to \infty$ to a non-degenerate random variable.

(ii) Provide the distribution function of the limit random variable. What is the name of this distribution?

(b) Let $X \sim \text{Bin}(n, p)$. Find $\mathbb{E}(1/(i + X))$, for $i = 1, 2$. Hint: Recall that $\int x^\alpha dx = x^{a+1}/(a+1) + C$ for $a \neq -1$.

Solution.

(a) (i) We have

$$T(y) = \inf\{x \geq 0 : M(x) \geq y\}; \qquad \lim_{y\to\infty} \Pr\left(e^{-y}T(y) \leq a\right) = 1 - e^{-\lambda a} \qquad \forall a \geq 0.$$

Note that

$$\lim_{y\to\infty} \Pr\left(e^{-y}T(y) \leq a\right) = \lim_{y\to\infty} \Pr(T(y) \leq ae^y)$$

Because $T(y) = \inf\{x \geq 0 : M(x) \geq y\}$ and $M(\cdot)$ is montonically increasing, we have the inequality $M(z) \geq y$ for all $z \geq x$. Therefore $T(y) \leq ae^y \iff M(ae^y) \geq y$. Let $z = ae^y \implies y = \log(z/a)$; then we have

$$\lim_{y\to\infty} \Pr(T(y) \le ae^y) = \lim_{y\to\infty} \Pr(M(ae^y) \ge y) = \lim_{z\to\infty} \Pr(M(z) \ge \log(z) - \log(a))$$

Let $b = -\log a$ to get

$$= \lim_{z\to\infty} \Pr(M(z) - \log(z) \ge b) = 1 - e^{-\lambda a} \implies \lim_{z\to\infty} \Pr(M(z) - \log(z) \ge b) = 1 - e^{-\lambda e^{-b}}$$

$$\iff \lim_{z\to\infty} \Pr(M(z) - \log(z) \le b) = e^{-\lambda e^{-b}}$$

(ii) The distribution function is $F(x) = e^{-\lambda e^{-x}}$, a Gumbel distribution with parameter λ. This is a proper cdf because $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$.

(b) Using hint:

$$\int_0^1 t^x \, dt = \left[\frac{t^{x+1}}{x+1}\right]_0^1 = \frac{1}{x+1}$$

$$\iff \mathbb{E}\left[\int_0^1 t^X \, dt\right] = \mathbb{E}\left(\frac{1}{X+1}\right) \iff \int_0^1 \mathbb{E}(t^X) \, dt = \mathbb{E}\left(\frac{1}{X+1}\right)$$

$$\iff \int_0^1 \sum_{i=0}^n t^i \Pr(X = i) \, dt = \mathbb{E}\left(\frac{1}{X+1}\right) \iff \int_0^1 \sum_{i=0}^n t^i \binom{n}{i} p^i (1-p)^{n-i} \, dt = \mathbb{E}\left(\frac{1}{X+1}\right)$$

$$\iff (1-p)^n \int_0^1 \sum_{i=0}^n \binom{n}{i} \left(\frac{tp}{1-p}\right)^i \, dt = \mathbb{E}\left(\frac{1}{X+1}\right)$$

$$\iff (1-p)^n \int_0^1 \left(1 + \frac{tp}{1-p}\right)^n \, dt = \mathbb{E}\left(\frac{1}{X+1}\right)$$

$$\iff \int_0^1 (1-p+tp)^n \, dt = \mathbb{E}\left(\frac{1}{X+1}\right)$$

Let $u = 1 - p + tp \implies du = p \, dt$. Then we can write

$$\frac{1}{p}\int_{1-p}^1 u^n \, du = \mathbb{E}\left(\frac{1}{X+1}\right) \iff \frac{1}{p}\left[\frac{u^{n+1}}{n+1}\right]_{1-p}^1 = \mathbb{E}\left(\frac{1}{X+1}\right) \iff \mathbb{E}\left(\frac{1}{X+1}\right) = \frac{1}{p}\left[\frac{1-(1-p)^{n+1}}{n+1}\right]$$

Now we consider $\mathbb{E}\left[\frac{1}{X+2}\right]$.

$$\int_0^1 t^{x+1} \, dt = \left[\frac{t^{x+2}}{x+2}\right]_0^1 = \frac{1}{x+2}$$

$$\iff \mathbb{E}\left[\int_0^1 t^{X+1} \, dt\right] = \mathbb{E}\left(\frac{1}{X+2}\right) \iff \int_0^1 \mathbb{E}(t^{X+1}) \, dt = \mathbb{E}\left(\frac{1}{X+2}\right)$$

$$\iff \int_0^1 \sum_{i=0}^n t^{i+1} \Pr(X = i) \, dt = \mathbb{E}\left(\frac{1}{X+2}\right) \iff \int_0^1 \sum_{i=0}^n t^{i+1} \binom{n}{i} p^i (1-p)^{n-i} \, dt = \mathbb{E}\left(\frac{1}{X+2}\right)$$

$$\iff (1-p)^n \int_0^1 t \sum_{i=0}^{n} \binom{n}{i} \left(\frac{tp}{1-p} \right)^i \, dt = \mathbb{E}\left(\frac{1}{X+2} \right)$$

$$\iff (1-p)^n \int_0^1 t \left(1 + \frac{tp}{1-p} \right)^n \, dt = \mathbb{E}\left(\frac{1}{X+2} \right)$$

$$\iff \int_0^1 t\,(1-p+tp)^n \, dt = \mathbb{E}\left(\frac{1}{X+2} \right)$$

Let $u = 1-p+tp \implies du = p\,dt$ and $t = (u+p-1)/p$. Then we can write

$$\frac{1}{p^2}\int_{1-p}^1 u^n(u+p-1)\,du = \mathbb{E}\left(\frac{1}{X+2} \right) \iff \frac{1}{p^2}\int_{1-p}^1 [u^{n+1} + (p-1)u^n]\,du = \mathbb{E}\left(\frac{1}{X+2} \right)$$

$$\iff \frac{1}{p^2}\left[\frac{u^{n+2}}{n+2} + (p-1)\frac{u^{n+1}}{n+1} \right]_{1-p}^1 = \mathbb{E}\left(\frac{1}{X+2} \right)$$

$$\iff \mathbb{E}\left(\frac{1}{X+2} \right) = \frac{1}{p^2}\left[\frac{1-(1-p)^{n+2}}{n+2} - (1-p)\frac{1-(1-p)^{n+1}}{n+1} \right]$$

6.3 To Know for Math 505A Midterm 2

6.3.1 Definitions

Definition 6.27. A random variable X is **continuous** if its distribution function $F(x) = \Pr(X \le x)$ can be written as

$$F(x) = \int_{-\infty}^{x} f(u)du$$

for some integrable $f : \mathbb{R} \to [0, \infty)$.

Definition 6.28. The function f is called the **(probability) density function** of the continuous random variable X.

Proposition 6.41. If X has pdf $f_X(x)$, then for $\mu \in \mathbb{R}$, $\sigma > 0$,

$$h(x) = \frac{1}{\sigma} f_X\left(\frac{x-\mu}{\sigma} \right)$$

is a pdf. In this setting μ is sometimes called a "location parameter" and σ is called a "scale parameter."

Definition 6.29. The **joint distribution function** of X and Y is the function $F : \mathbb{R}^2 \to [0, 1]$ given by

$$F(x, y) = \Pr(X \le x \cap Y \le y)$$

Definition 6.30. The random variables X and Y are **jointly continuous** with **joint (probability) density function** $f : \mathbb{R}^2 \to [0, \infty)$ if

$$F(x, y) = \int_{v=-\infty}^{y} \int_{u=-\infty}^{x} f(u, v) du dv \text{ for each } x, y \in \mathbb{R}$$

Definition 6.31. Two continuous random variables are **independent** if and only if $\{X \leq x\}$ and $\{Y \leq y\}$ are independent events for all $x, y \in \mathbb{R}$.

Ways to show independence:

- Use Definition 6.31: show that $\Pr(X \leq x \cap Y \leq y) = \Pr(X \leq x) \Pr(Y \leq y)$ for all $x, y \in \mathbb{R}$.
- **Theorem 6.42.** The random variables X and Y are independent if and only if $F(x, y) = F_X(x) F_Y(x)$ for all $x, y \in \mathbb{R}$.
- **Proposition 6.43.** For continuous random variables, the previous condition is equivalent to requiring $f(x, y) = f_X(x) f_Y(y)$.
- **Theorem 6.44.** If two variables are bivariate normal, they are independent if and only if their covariance

  $$\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x, y) dx dy$$

  is equal to 0.

  Theorem 6.45. (Change of variables.) Let X_1, Y_1 be random variables with joint PDF f_{X_1,Y_1}. Let X_2, Y_2 be random variables with joint PDF f_{X_2,Y_2}. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ and let $S : \mathbb{R}^2 \to \mathbb{R}^2$ so that $ST(x, y) = (x, y)$ and $TS(x, y) = (x, y)$ for every $(x, y) \in \mathbb{R}^2$. Let $J(x, y)$ denote the determinant of the Jacobian of S at (x, y). Then

  $$f_{X_2,Y_2}(x, y) = f_{X_1,Y_1}(S(x, y)) \left| J(x, y) \right|.$$

  Proof. If the transformation from X_1, Y_1 to X_2, Y_2 is given by $S(X_1, Y_1) = (X_2, Y_2)$, then the change of variables formula from calculus is as follows:

  $$\int \int_A f_{X_1,Y_1}(x, y) dx dy = \int \int_B f_{X_1,Y_1}\big(S(x, y)\big) \left| J(x, y) \right| dx dy$$

  where $A \subseteq \text{domain}(f_{X_1,Y_1}(\cdot))$, B is the transformation of the region A under S, and $|J(x, y)|$ is the Jacobian of S at (x, y). It follows from the definition of joint pdfs that the integrand on the right is the joint pdf of (X_2, Y_2); that is,

  $$f_{X_2,Y_2}(x, y) = f_{X_1,Y_1}\big(S(x, y)\big) \left| J(x, y) \right|.$$

  $\square$

- Characteristic functions:

  Theorem 6.46. X and Y are independent if and only if $\phi_{X,Y}(s,t) = \phi_X(s)\phi_Y(t)$.

Theorem 6.47. (Theorem 4.2.3, Grimmett and Stirzaker.) Let X and Y be random variables, and let $g, h : \mathbb{R} \to \mathbb{R}$. If X and Y are independent, then so are $g(X)$ and $h(Y)$.

Definition 6.32. The **correlation coefficient** between random variables X and Y is given by

$$\rho = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

Theorem 6.48. The correlation coefficient satisfies $|\rho| \leq 1$.

Proof. Apply the Cauchy-Schwarz Inequality (Theorem 8.6) to $X - \mathbb{E}(X)$ and $Y - \mathbb{E}(Y)$:

$$\text{Cov}(X,Y)^2 = \left(\mathbb{E}\left[\left(X - \mathbb{E}(X)\right)\left(Y - \mathbb{E}(Y)\right)\right]^2\right)^2 \leq \mathbb{E}\left[\left(X - \mathbb{E}(X)\right)^2\right]\mathbb{E}\left[\left(Y - \mathbb{E}(Y)\right)^2\right] = \text{Var}(X)\text{Var}(Y)$$

$$\iff \frac{\text{Cov}(X,Y)^2}{\text{Var}(X)\text{Var}(Y)} \leq 1 \iff \rho^2 \leq 1 \iff |\rho| \leq 1$$

□

Theorem 6.49. (Stein identity.) Let X be a standard Gaussian random variable, so that X has density $x \to e^{-x^2/2}/\sqrt{2\pi}$, $\forall\, x \in \mathbb{R}$. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that g and g' have polynomial volume growth. That is, $\exists\, a, b > 0$ such that $|g(x)|, |g'(x)| \leq a(1 + |x|)^b$, $\forall\, x \in \mathbb{R}$. Then

$$\mathbb{E}Xg(X) = \mathbb{E}g'(X).$$

Proof. Examining the left side, we have

$$\mathbb{E}(Xg(X)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xg(x)e^{-x^2/2}dx$$

We will use integration by parts with $u = g(x) \implies du = g'(x)dx,\quad dv = xe^{-x^2/2}dx \implies v = -e^{-x^2/2}$ to yield the result:

$$\mathbb{E}(Xg(X)) = -g(x)e^{-x^2/2} + \int_{-\infty}^{\infty} g'(x)e^{-x^2/2}dx = \mathbb{E}(g'(X)).$$

□

Example. Using Theorem 6.49, recursively compute $\mathbb{E}X^k$ for any positive integer k. Alternatively, for any $t > 0$, show that $\mathbb{E}e^{tX} = e^{t^2/2}$, i.e. compute the **moment generating function** of X. Then, using $\frac{d^k}{dt^k}|_{t=0}\mathbb{E}e^{tX} = \mathbb{E}X^k$ and using the power series expansion of the exponential, compute $\mathbb{E}X^k$ directly from the identity $\mathbb{E}e^{tX} = e^{t^2/2}$.

Solution. We can use this result to recursively calculate $\mathbb{E}(X^k)$ for any positive integer k. Suppose we have $\mathbb{E}(X^{k-1})$. Letting $g(X) = X^{k-1} \implies g'(X) = (k-1)X^{k-2}$, we have

$$\mathbb{E}(X^k) = \mathbb{E}(X \cdot X^{k-1}) = \mathbb{E}(Xg(X)) = \mathbb{E}(g'(X)) \iff \boxed{\mathbb{E}(X^k) = (k-1)\mathbb{E}(X^{k-2})}.$$

Since $X \sim \mathcal{N}(0,1)$, we have $\mathbb{E}(X) = 0, \quad \mathbb{E}(X^2) = \text{Var}(X) + \mathbb{E}(X)^2 = 1 + 0 = 1.$ Therefore we have

$$\mathbb{E}(X^k) = \begin{cases} \prod_{i=1}^{(k-1)/2}(k-(2i-1))\mathbb{E}(X) & k \text{ is odd} \\ \prod_{i=1}^{k/2}(k-(2i-1))\mathbb{E}(X^2) & k \text{ is even} \end{cases}$$

$$\boxed{\mathbb{E}(X^k) = \begin{cases} 0 & k \text{ is odd} \\ \prod_{i=1}^{k/2}(k-(2i-1)) & k \text{ is even} \end{cases}}$$

6.3.2 Probability-Generating Functions

Definition 6.33.

$$G_X(s) = \mathbb{E}(s^X)$$

Theorem 6.50. Some useful properties:

(a) $\mathbb{E}(X) = G_X'(1)$, $\mathbb{E}[X(X-1)\cdots(X-k+1)] = G^{(k)}(1)$

(b) If X and Y are independent then $G_{X+Y}(s) = G_X(s)G_Y(s)$.

6.3.3 Moment-Generating Functions

Definition 6.34.

$$M_X(t) = \mathbb{E}(e^{tX})$$

Theorem 6.51 (Some useful properties). (a) $\mathbb{E}(X) = M_X'(0)$, $\mathbb{E}(X^k) = M^{(k)}(0)$

(b) If $X_1, X_2, \ldots, X_n$ are independent then $M_{X_1+ldots+X_n}(t) = \prod_{i=1}^n M_{X_i}(t)$.

Proof. (a)

(b)

$$M_{X_1+X_2+\ldots+X_n}(t) = \mathbb{E}\exp\left(t\sum_{i=1}^n X_i\right) = \mathbb{E}\prod_{i=1}^n e^{tX_i} = \prod_{i=1}^n \mathbb{E}e^{tX_i} = \prod_{i=1}^n M_{X_i}(t).$$

□

6.3.4 Characteristic Functions

Definition 6.35.

$$\phi_X(t) = \mathbb{E}(e^{itX})$$

Proposition 6.52. Necessary and sufficient conditions for a function to be a characteristic function:

(a) $\phi_X(0) = 1$

(b) $|\phi(t)| \leq 1 \; \forall \; t$

(c) ϕ is uniformly continuous on $\mathbb{R}$

(d) ϕ is positive semidefinite; that is,

$$\sum_{i,j} \phi(t_j - t_k) z_j \overline{z}_k \geq 0 \text{ for all real } t_1, t_2, \ldots, t_n \text{ and complex } z_1, z_2, \ldots, z_n$$

Or, equivalently, or every set of real numbers $t_1, t_2, \ldots, t_n$, the matrix $\phi(t_i - t_j), i, j \in \{1, 2, \ldots, n\}$ is Hermitian and nonnegative definite.

Remark. Relationship between characteristic functions and probability and moment-generating functions:

$$\phi_X(t) = M_X(it) = G_X(e^{it})$$

Theorem 6.53. Some useful properties:

(a) $X \perp\!\!\!\perp Y \implies \phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$

(b) $Y = aX + b \implies \phi_Y(t) = e^{itb}\phi_X(at)$

(c) $\phi_X^{(k)}(0) = i^k \mathbb{E}(X^k)$

(d) $\phi_{X,Y}(s,t) = \mathbb{E}(e^{isX}e^{itY})$

(e) $X \perp\!\!\!\perp Y \iff \phi_{X,Y}(s,t) = \phi_X(s)\phi_Y(t)$

Theorem 6.54. Other facts from notes on course website

(a) If $\phi(t)$ is even, $\phi(0) = 1$, ϕ is convex for $t > 0$, and $\lim_{t \to \infty} \phi(t) = 0$, then ϕ is a characteristic function of an absolutely continuous random variable.

(b) If ϕ is a characteristic function and $\phi(t) = 1 + o(t^2), t \to 0$, then $\phi(t) = 1$ for all t. The random variable with such a characteristic function must have zero mean and zero variance. In particular, if $r > 2$, then $\exp(-|t|^r)$ is not a characteristic function.

(c) If $\phi(t) = e^{p(t)}$ is a characteristic function and $p = p(t)$ is a polynomial, then the degree of p is at most 2. For example, $e^{t^2 - t^4}$ is not a characteristic function.

(d) If ξ is absolutely continuous, then $\lim_{|t| \to \infty} |\phi_\xi(t)| = 0$ (Riemann-Lebesgue).

(e) If $\int_{-\infty}^{\infty} |\phi_\xi(t)| \, dt < \infty$, then ξ is absolutely continuous with pdf

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-itx)\phi(t)dt$$

6.3.5 Continuous Random Variable Distributions

Uniform: $\mathrm{U}(a,b)$

- Probability density function:

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

- Cumulative distribution function:

$$F(x) = \Pr(X \leq x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a < x \leq b \\ 1 & x > b \end{cases}$$

- Probability-generating function:
- Moment-generating function:

$$M_X(t) = \frac{1}{(b-a)t}\big[\exp(bt) - \exp(at)\big]$$

Proof.

$$M_X(t) = \mathbb{E}(\exp(tX)) = \int_a^b \frac{1}{b-a} \cdot \exp(tx)dx = \frac{1}{b-a}\left[\frac{1}{t}\exp(tx)\right]_a^b = \frac{1}{(b-a)t}\big[\exp(bt) - \exp(at)\big]$$

□

- Characteristic function:

$$\frac{2}{(b-a)t}\sin\left(\frac{1}{2}(b-a)t\right)\exp\left(i(a+b)\frac{t}{2}\right)$$

- Expectation: $\mathbb{E}(X) = (b-a)/2$
- Variance: $\mathrm{Var}(X) = (b-a)^2/12$

Proposition 6.55. If $X \sim U(0,1)$, then $Y = -\log(X) \sim$ Exponential(1).

Proof.

$$\Pr(Y \leq y) = \Pr(-\log(X) \leq y) = \Pr(\log(X) \geq -y) = \Pr(X \geq e^{-y}) = \int_{\exp(-y)}^{\infty} f_X(t)dt = \int_{\exp(-y)}^{1} dt$$

$$= 1 - e^{-y}$$

which is the cdf for an exponential distribution with mean 1. □

Normal (or **Gaussian**): $\mathcal{N}(\mu, \sigma^2)$

- Probability density function:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(- \frac{(x-\mu)^2}{2\sigma^2} \right)$$

- Cumulative distribution function: $F(x) = \Pr(X \leq x) =$
- Probability-generating function:
- Moment-generating function: $M_X(t) = \exp(\mu t + \sigma^2 t^2/2)$
- Characteristic function: $\phi(t) = \exp(i\mu t - (1/2)\sigma^2 t^2)$. Standard normal: $\phi(t) = \exp((-1/2)t^2)$.
- Expectation: $\mathbb{E}(X) = \mu$
- Variance: $\mathrm{Var}(X) = \sigma^2$

Theorem 6.56. Let $X := \Omega \to \mathbb{R}^n$ be a random Gaussian variable with the **standard Gaussian distribution**:

$$\Pr(X \in A) := \int_A e^{-(x_1^2 + \cdots + x_n^2)/2} dx (2\pi)^{-n/2}, \qquad \forall\, A \subset \mathbb{R}^n \text{ measurable.}$$

Let $v_1, \ldots, v_m$ be vectors in $\mathbb{R}^n$. Let $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the standard inner product on $\mathbb{R}^n$, so that $\langle x, y \rangle := \sum_{i=1}^n x_i y_i$ for any $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

Let $v \in \mathbb{R}^n$. Then $\langle X, v \rangle$ is a mean zero Gaussian with variance $\langle v, v \rangle$.

Remark. Similar to Exercise 6.1. You can then use an induction argument to prove the case for the sum of arbitrarily many Gaussian random variables.

Proof. Let $X = (X_1, X_2, \ldots, X_n)$ and $v = (v_1, v_2, \ldots, v_n)$. Then $\langle X, v \rangle = \sum_{i=1}^n X_i v_i$. Let $Y_i = v_i X_i$, so $\langle X, v \rangle = \sum_{i=1}^n Y_i$. Note that $Y_i \sim \mathcal{N}(0, v_i^2)$. Recall that the moment-generating function of a Gaussian random variable is $\mathbb{E}e^{tX} = e^{\mu t + \sigma^2 t^2/2}$, so we have $M_{Y_i}(t) = \exp(v_i^2 t^2/2)$. Next we seek the distribution of $\sum_{i=1}^n v_i X_i = \sum_{i=1}^n Y_i$. From Proposition 1.67, we have

$$M_{Y_1 + Y_2 + \ldots + Y_n}(t) = \mathbb{E} \exp\left(t \sum_{i=1}^n Y_i \right) = \mathbb{E} \prod_{i=1}^n e^{tY_i} = \prod_{i=1}^n \mathbb{E} e^{t y_i} = \prod_{i=1}^n M_{Y_i}(t).$$

Then

$$\prod_{i=1}^n M_{Y_i}(t) = \prod_{i=1}^n \exp(v_i^2 t^2/2) = \exp\left(\frac{t^2}{2} \sum_{i=1}^n v_i^2 \right) = \exp\left(\frac{t^2}{2} \cdot \langle v, v \rangle \right)$$

which is the same as the moment-generating function for a mean zero Gaussian random variable with variance $\langle v, v \rangle$. By the provided uniqueness result from Problem 6 ("If Y and Z are two random variables whose MGFs coincide in a neighborhood of 0 ($\exists \delta > 0$ for which $M_Y(u) = M_Z(u) < \infty$ for all $u \in [-\delta, \delta]$), then Y and Z have the same distribution."), the result follows.

$\square$

Proposition 6.57. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random sample from $\mathcal{N}(\mu, \sigma)$. Then the sum of these n observations $T = \sum_{i=1}^n X_i$ also follows the normal distribution.

Proof.

$$T = \sum_{i=1}^{n} X_i = X_1 + X_2 + \ldots + X_n$$

Since X_i is $\mathcal{N}(\mu, \sigma)$, we have

$$M_{X_i}(t) = \exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$$

Since all observations are independent, we have

$$M_T(t) = \prod_{i=1}^{n} M_{X_i}(t) = \left(\exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)\right)^n = \boxed{\exp\left(n\mu t + \frac{t^2 n\sigma^2}{2}\right)}$$

which is the moment generating function for a normal distribution with mean $n\mu$ and standard deviation $\sigma\sqrt{n}$.

□

Lognormal: If the random variable X follows the normal distribution with $\mu = 0$, $\sigma^2 = 1$ and $Y = e^X$, then Y has a lognormal distribution.

- Probability density function:

$$f_X(x) = \frac{1}{x\sqrt{2\pi}} \exp\left(\frac{-1}{2}(\log(x))^2\right)$$

Proof.

$$F_Y(y) = \Pr(Y \le y) = \Pr(e^X \le y) = \Pr(X \le \log(y)) \implies F_Y(y) = F_X(\log(y))$$

$$\implies f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_X(\log(y)) = f_X(\log(y))\frac{1}{y}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}x^2} \implies f_Y(y) = \frac{1}{y\sqrt{2\pi}} \exp\left(\frac{-1}{2}(\log(y))^2\right)$$

□

- Cumulative distribution function: $F(x) = \Pr(X \le x) =$
- Probability-generating function:
- Moment-generating function: $M_X(t) =$
- Characteristic function: $\phi(t) =$.

- Expectation: $\mathbb{E}(X) =$
- Variance: $\text{Var}(X) =$

Gamma: $\Gamma(\alpha, \beta)$

- Probability density function:

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} = \frac{1}{\Gamma(\alpha, \beta)} x^{\alpha-1} e^{-x/\beta}$$

- Cumulative distribution function:

$$F(x) = \Pr(X \leq x) =$$

- Probability-generating function:
- Moment-generating function:

$$\left(\frac{1/\beta}{1/\beta - t} \right)^\alpha = \left(\frac{1}{1 - \beta t} \right)^\alpha$$

Proof.

$$\mathbb{E}(e^{tX}) = \int_{\mathbb{R}} e^{tx} f(x) dx = \int_0^\infty e^{tx} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} dx = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-x(1/\beta - t)} dx$$

Using integration by parts, one can find the identity

$$\int_0^\infty x^a e^{-bx} dx = \frac{\Gamma(a+1)}{b^{a+1}}.$$

Using this yields

$$\mathbb{E}(e^{tX}) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \frac{\Gamma(\alpha)}{(1/\beta - t)^\alpha} \boxed{= \left(\frac{1/\beta}{1/\beta - t} \right)^\alpha = \left(\frac{1}{1 - \beta t} \right)^\alpha}$$

$\square$

- Characteristic function:
- Expectation: $\mathbb{E}(X) = \alpha\beta$
- Variance: $\text{Var}(X) = \alpha\beta^2$

Proposition 6.58. Let $X_i \sim \text{Gamma}(\alpha_i, \beta)$ for $i = 1, 2, \ldots, n$. Then

$$\sum_{i=1}^n X_i \sim \text{Gamma}\left(\sum_{i=1}^n \alpha_i, \beta \right).$$

Proof. Using the moment-generating function for a Gamma distribution as well as Theorem 6.51(b), we have

$$M_{X_1+\ldots+X_n}(t) = \prod_{i=1}^n M_{X_i}(t) = \prod_{i=1}^n \left(\frac{1}{1-\beta t}\right)^{\alpha_i} = \left(\frac{1}{1-\beta t}\right)^{\sum_{i=1}^n \alpha_i}$$

which is the same as the moment-generating function for a $\Gamma(\sum_{i=1}^n \alpha_i, \beta)$ distribution. By the provided uniqueness result ("If Y and Z are two random variables whose MGFs coincide in a neighborhood of 0 ($\exists \delta > 0$ for which $M_Y(u) = M_Z(u) < \infty$ for all $u \in [-\delta, \delta]$), then Y and Z have the same distribution."), the result follows. $\square$

Proposition 6.59. Let $X \sim \text{Gamma}(\alpha, \beta)$. Then as $\beta \to \infty$, $X \xrightarrow{d} \mathcal{N}(\alpha\beta, \alpha\beta^2)$.

Proof. See http://www.math.wm.edu/~leemis/chart/UDR/PDFs/GammaNormal1.pdf. $\square$

Proposition 6.60 (Stats 100B homework problem). The method of moments estimators for α and β are

$$\hat{\alpha} = \frac{n\bar{x}^2}{\sum_{i=1}^n (X_i^2 - \bar{x}^2)}$$

$$\hat{\beta} = \frac{1}{n\bar{x}} \sum_{i=1}^n (X_i^2 - \bar{x}^2)$$

Proof.

$$\mathbb{E}(X) = \alpha\beta \implies \hat{\alpha}\hat{\beta} = \bar{x}$$

$$\hat{\alpha} = \frac{\bar{x}}{\hat{\beta}}$$

$$\text{Var}(X) = \alpha\beta^2 = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

$$\implies \hat{\alpha}\hat{\beta}^2 = \frac{1}{n}\sum_{i=1}^n X_i^2 - \bar{x}^2$$

$$\frac{\bar{x}}{\hat{\beta}}\hat{\beta}^2 = \frac{1}{n}\sum_{i=1}^n X_i^2 - \bar{x}^2$$

$$\bar{x}\hat{\beta} = \frac{1}{n}\sum_{i=1}^n X_i^2 - \bar{x}^2$$

$$\hat{\beta} = \frac{1}{n\bar{x}}\sum_{i=1}^n X_i^2 - \bar{x} = \frac{1}{n\bar{x}}\sum_{i=1}^n X_i^2 - \frac{n\bar{x}^2}{n\bar{x}}$$

$$\hat{\beta} = \frac{1}{n\bar{x}} \sum_{i=1}^{n} (X_i^2 - \bar{x}^2)$$

$$\implies \hat{\alpha} = \frac{\bar{x}}{\hat{\beta}} = \frac{n\bar{x}^2}{\sum_{i=1}^{n} (X_i^2 - \bar{x}^2)}$$

□

Proposition 6.61 (**Some useful formulae and integrals related to the gamma function**).

- Definition:

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \, dx$$

- Beta function (see also the information on the Beta distribution):

$$B(z_1, z_2) := \int_0^1 t^{z_1 - 1} (1-t)^{z_2 - 1} \, dt = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1 + z_2)}$$

- Generalization of the factorial function:

$$\Gamma(z+1) = z\Gamma(z)$$

  (in particular, for any integer $n \geq 1$, $\Gamma(n) = (n-1)!$.)

- $\Gamma(1/2) = \sqrt{\pi}$.
- Using integration by parts, one can find the identity

$$\int_0^\infty x^a e^{-bx} dx = \frac{\Gamma(a+1)}{b^{a+1}}.$$

- **Euler's Reflection Formula**: For $z \notin \mathbb{Z}$,

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$$

- **Legendre Duplication Formula:**

$$\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$$

- For all $z \in \mathbb{C}$,

$$\lim_{n \to \infty} \frac{\Gamma(n+z)}{\Gamma(n)n^z} = 1.$$

- From 2018 DSO Statistics Screening Exam:

$$\lim_{x \to 0} x\Gamma(x) = 1$$

- See Theorem 6.40 (Stirling's Formula)

$$\Gamma(n+1) \sim n^n e^{-n}\sqrt{2\pi n}$$

That is,

$$\lim_{n\to\infty} \frac{n^n e^{-n}\sqrt{2\pi n}}{\Gamma(n+1)} = 1.$$

χ^2_n: special case of a gamma distribution: $\Gamma(n/2, 2)$. Also the sum of n independent standard normally distributed variables.

- Probability density function:

$$f(x) = \frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} e^{-x/2} = \frac{1}{\Gamma(n/2,2)} x^{n/2-1} e^{-x/2}$$

For χ^2_1: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x/2} x^{-1/2}$. For χ^2_2: $f(x) = \frac{1}{2} e^{-x^2}, x > 0$.

Proof. See notes from Math 541A. □

- Cumulative distribution function: $F(x) = \Pr(X \leq x) =$
- Probability-generating function:
- Moment-generating function:
- Characteristic function:
- Expectation: $\mathbb{E}(X) = n/2 \cdot 2 = n$
- Variance: $\text{Var}(X) = n/2 \cdot 2^2 = 2n$

Exponential: (special case of a gamma distribution: $\Gamma(1, \beta)$. Also a special case of a Weibull distribution with $\beta = 1$.)

- Probability density function: $f(x) = \frac{1}{\beta} \exp(-x/\beta) = \lambda e^{-\lambda x}$
- Cumulative distribution function: $F(x) = \Pr(X \leq x) = 1 - e^{-\lambda x}$
- Probability-generating function:
- Moment-generating function: $\frac{\lambda}{\lambda - t}$
- Characteristic function:
- Expectation: $\mathbb{E}(X) = \beta = \lambda^{-1}$

Proof.

$$\mathbb{E}(X) = \int_0^\infty \overline{F}(t)dt = \int_0^\infty e^{-\lambda t} dt = \frac{1}{\lambda}$$

□

- Variance: $\text{Var}(X) = \beta^2 = \lambda^{-2}$

  Proof. See Definition 6.20 above for the definition of $E(X^n)$. Then using that:

$$\mathbb{E}(X^2) = 2\int_0^\infty te^{-\lambda t}dt = \frac{2}{\lambda}\int_0^\infty \lambda te^{-\lambda t}dt = \frac{2}{\lambda}\mathbb{E}(X) = \frac{2}{\lambda^2}$$

  Then use $\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$ to yield the result. □

  Remark. Note also the general case:

$$\mathbb{E}(X^n) = n\int_0^\infty t^{n-1}\overline{F}(t)dt = n\int_0^\infty t^{n-1}e^{-\lambda t}dt = \frac{n}{\lambda}\mathbb{E}(X^{n-1})$$

  For much more on exponential random variables, see Section 6.3.6.

Cauchy:

- Probability density function:

$$f(x) = \frac{1}{\pi(1+x^2)} \text{ (standard Cauchy) }, f(x) = \frac{1}{\pi\sigma(1+(x-\mu)^2/\sigma^2)} \text{ (general)}$$

- Cumulative distribution function: $F(x) = \Pr(X \leq x) =$
- Probability-generating function:
- Moment-generating function:
- Characteristic function:
- Expectation: does not exist
- Variance: does not exist (Cauchy distribution has no moments.)

Beta: Recall:

$$B(\alpha, \beta) := \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$\implies \frac{B(\alpha+1, \beta)}{B(\alpha, \beta)} = \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+1+\beta)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} = \frac{\alpha}{\alpha+\beta}$$

- Probability density function:

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}(x)^{\alpha-1}(1-x))^{\beta-1}$$

- Cumulative distribution function: $F(x) = \Pr(X \leq x) =$
- Probability-generating function:
- Moment-generating function:

$$1 + \sum_{k=1}^\infty \left(\prod_{r=0}^{k-1} \frac{\alpha+r}{\alpha+\beta+r} \right) \frac{t^k}{k!}$$

- Characteristic function:
- Expectation: $\mathbb{E}(X) = \frac{\alpha}{\alpha+\beta}$

– Variance:

$$\mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

t_n:

– Probability density function:

$$f(x) = \frac{\Gamma((n+1)/2)}{\sqrt{n\pi}\cdot\Gamma(n/2)}\left(1+\frac{x^2}{n}\right)^{-(n+1)/2}$$

– Cumulative distribution function: $F(x) = \Pr(X \le x) =$

– Probability-generating function:

– Moment-generating function:

– Characteristic function:

– Expectation: $\mathbb{E}(X) = 0$

– Variance: $\mathrm{Var}(X) = n/(n-2)$

Snedecor's F-distribution:

– Probability density function:

$$f(x) = \frac{t^{(p/2)-1}(p/q)^{p/2}\Gamma((p+q)/2)}{\Gamma(p/2)\Gamma(q/2)}\Big(1+t(p/q)\Big)^{-(p+q)/2}, \qquad \forall\, t>0$$

$$= p^{p/2}q^{q/2}\cdot\frac{\Gamma([p+q]/2)}{\Gamma(p/2)\Gamma(q/2)}\cdot\frac{t^{p/2-1}}{(pt+q)^{(p+q)/2}}$$

– Cumulative distribution function: $F(x) =$

– Probability-generating function:

– Moment-generating function:

– Characteristic function:

– Expectation: $\mathbb{E}(X) =$

– Variance: $\mathrm{Var}(X) =$

Proposition 6.62. Let X be a chi squared random variables with p degrees of freedom. Let Y be a chi squared random variable with q degrees of freedom, with X and Y independent. Then $(X/p)/(Y/q)$ is an F-distributed random variable with p and q degrees of freedom.

Proof. Let $c = p/2$, $d = q/2$. Note that

$$f_X(x) = \frac{1}{\Gamma(c)2^c}x^{c-1}e^{-x/2}, \quad x>0$$

$$f_Y(y) = \frac{1}{\Gamma(d)2^d}y^{d-1}e^{-y/2}, \quad x>0$$

We first seek $F_{X/Y}(t)$. Note that $F_{X/Y}(t) = \Pr(X/Y \le t) = \Pr(X \le tY)$. Thinking about this graphically, we can calculate this as

$$\Pr(X \leq tY) = \int_0^\infty \int_0^{ty} f_{X,Y}(x,y) dx dy$$

By assumption, X and Y are independent, so

$$f_{X,Y}(x,y) = \frac{1}{\Gamma(c)\Gamma(d)2^{c+d}} x^{c-1} y^{d-1} e^{-x/2} e^{-y/2}$$

Plugging this in we have

$$\Pr(X \leq tY) = \frac{1}{\Gamma(c)\Gamma(d)2^{c+d}} \int_0^\infty \int_0^{ty} x^{c-1} y^{d-1} e^{-x/2} e^{-y/2} dx dy$$

$$= \frac{1}{\Gamma(c)\Gamma(d)2^{c+d}} \int_0^\infty \int_0^{ty} x^{c-1} e^{-x/2} dx \; y^{d-1} e^{-y/2} dy$$

Rather than solving this integral, we next differentiate with respect to t:

$$f_{X/Y}(t) = \frac{\mathrm{d}}{\mathrm{dt}} F_{X/Y}(t) = \frac{1}{\Gamma(c)\Gamma(d)2^{c+d}} \int_0^\infty y(ty)^{c-1} e^{-ty/2} \; y^{d-1} e^{-y/2} dy$$

$$= \frac{t^{c-1}}{\Gamma(c)\Gamma(d)2^{c+d}} \int_0^\infty y^{c+d-1} e^{-y(t+1)/2} dy \tag{6.9}$$

Compare the integrand of (6.9) to the pdf of a Gamma distributed random variable

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$$

to see that with some manipulations we can express the integrand of (6.9) as the pdf of a Gamma distributed random variable with parameters $\alpha = c + d, \beta = 2/(t + 1)$:

$$f_{X/Y}(t) = \frac{t^{c-1}\Gamma(c+d)}{\Gamma(c)\Gamma(d)(t+1)^{c+d}} \int_0^\infty \frac{1}{[2/(t+1)]^{c+d}\Gamma(c+d)} y^{c+d-1} e^{-y(t+1)/2} dy$$

Then the integral becomes 1, so we have

$$f_{X/Y}(t) = \frac{t^{c-1}\Gamma(c+d)}{\Gamma(c)\Gamma(d)(t+1)^{c+d}}$$

Substitute back in $c = p/2$, $d = q/2$:

$$f_{X/Y}(t) = \frac{\Gamma([p+q]/2)}{\Gamma(p/2)\Gamma(q/2)} \cdot \frac{t^{p/2-1}}{(t+1)^{(p+q)/2}}$$

Now take into account the constants:

$$f_{(X/p)/(Y/q)}(t) = \frac{p}{q} f_{X/Y}\left(\frac{p}{q}t\right) = \frac{p}{q} \cdot \frac{\Gamma([p+q]/2)}{\Gamma(p/2)\Gamma(q/2)} \cdot \frac{(pt/q)^{p/2-1}}{(pt/q+1)^{(p+q)/2}}$$

$$= \frac{t^{(p/2)-1}(p/q)^{p/2}\Gamma((p+q)/2)}{\Gamma(p/2)\Gamma(q/2)} \Big(1 + t(p/q)\Big)^{-(p+q)/2}$$

(or this can be expressed as)

$$= \left(\frac{p}{q}\right)^{p/2} \cdot \frac{\Gamma([p+q]/2)}{\Gamma(p/2)\Gamma(q/2)} \cdot \frac{t^{p/2-1}q^{(p+q)/2}}{(pt+q)^{(p+q)/2}} = p^{p/2}q^{q/2} \cdot \frac{\Gamma([p+q]/2)}{\Gamma(p/2)\Gamma(q/2)} \cdot \frac{t^{p/2-1}}{(pt+q)^{(p+q)/2}}$$

□

Weibull:

- Probability density function: $f(x) = \alpha\beta x^{\beta-1}\exp(-\alpha x^{\beta})$
- Cumulative distribution function: $F(x) = \Pr(X \leq x) = 1 - \exp(-\alpha x^{\beta})$
- Probability-generating function:
- Moment-generating function:
- Characteristic function:
- Expectation: $\mathbb{E}(X) =$
- Variance: $\mathrm{Var}(X) =$

Pareto: (parameters α, x_m)

- Probability density function: $f(x) = \alpha x_m^{\alpha}/x^{\alpha+1}$ for $x \geq 1, 0$ otherwise.
- Cumulative distribution function: $F(x) = 1 - (x_m/x)^{\alpha}$ for $x \geq 1, 0$ otherwise.
- Probability-generating function:
- Moment-generating function:
- Characteristic function:
- Expectation: $\mathbb{E}(X) = \alpha x_m/(\alpha - 1)$ for $\alpha > 1, \infty$ otherwise.
- Variance: $\mathrm{Var}(X) = \frac{x_m^2\alpha}{(\alpha-1)^2(\alpha-2)}$ for $\alpha > 2$, ∞ otherwise.

6.3.6 More on Exponential Random Variables

Remark. Recall Proposition 6.55: If $X \sim U(0,1)$, then $Y = -\log(X) \sim \text{Exponential}(1)$.

Proposition 6.63. Let X be a random variable. Then X is exponentially distributed if and only if X has the **memoryless** property; that is,

$$\Pr(X > s + t \mid X > t) = \Pr(X > s) \quad \forall\ s, t \geq 0$$

or, equivalently,

$$\Pr(X > s + t) = \Pr(X > s)\Pr(X > t) \quad \forall\ s, t \geq 0$$

Proof. See Ross *Introduction to Probability Models* section 5.2.2.

□

Remark. Exponential distributions can be derived as follows: Let X be a nonnegative random variable with cdf F and pdf f. Define the **failure** or **hazard rate**

$$\lambda(t) = \frac{f(t)}{1 - F(t)}$$

The intuition for the hazard rate is as follows: think of X as the lifetime and consider the probability that a unit with age t fails within some timespan $(t, t+h)$ with h small; that is, $\Pr(t < X < t+h \mid X > t)$. Letting $1 - F(t) = \overline{F}(t)$, we have

$$\Pr(t < X < t + h \mid X > t) = \frac{\Pr(t < X < t + h)}{\Pr(X > t)} = \frac{\int_t^{t+h} f(s)ds}{\overline{F}(t)} \approx \frac{f(t)}{\overline{F}(t)} \text{ for small h}$$

If this quantity is constant at λ (i.e. the process is "memoryless," see Proposition 6.63), we have an exponential distribution:

$$\lambda = \frac{f(t)}{\overline{F}(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}}$$

Indeed, a process is memoryless if and only if it is exponential. To see this, note that

$$\int_0^t \lambda(s)ds = \int_0^t \frac{f(s)}{1 - F(s)}ds$$

Letting $u = 1 - F(s) \implies du = -f(s)ds$, we have

$$= \int_{u=1}^{u=\overline{F}(t)} -\frac{du}{u}ds = \int_{\overline{F}(t)}^{1} \frac{du}{u}ds = \log(1) - \log(\overline{F}(t)) = -\log \overline{F}(t)$$

$$\implies \int_0^t \lambda(s)ds = -\log \overline{F}(t) \implies \boxed{\overline{F}(t) = \exp\left(- \int_0^t \lambda(s)ds \right)}$$

If the process is memoryless, we must have $\lambda(s) = \lambda$ (constant). Then $\overline{F}(t) = \exp(-\lambda t)$ and we have the exponential distribution. See the Weibull distribution below for a more general case.

Proposition 6.64. Let X be an exponential random variable. Then

(a) $\mathbb{E}(X - s \mid X > s) = \mathbb{E}(X)$

(b) $\mathbb{E}(X - s) = \Pr(X > s)\mathbb{E}(X)$ and

Proof. (a) By the memoryless property (Proposition 6.63), $\Pr(X > t + s \mid X > s) = \Pr(X - s > t \mid X > s) = \Pr(X > t)$. Then we have

$$\mathbb{E}(X - s \mid X > s) = \int_0^\infty \Pr(X - s > t \mid X > s)dt = \int_0^\infty \Pr(X > t)dt = \mathbb{E}(X)$$

(b) By the memoryless property (Proposition 6.63), $\Pr(X > t + s) = \Pr(X - s > t) = \Pr(X > t)\Pr(X > s)$. Then we have

$$\mathbb{E}(X - s) = \int_0^\infty \Pr(X > t)\Pr(X > s)dt = \Pr(X > s) \int_0^\infty \Pr(X > t)dt = \Pr(X > s)\mathbb{E}(X)$$

□

Proposition 6.65. Let X be an exponential random variable. Then $X - t \mid X > t$ is identically distributed as X.

Proof. Because X is memoryless, we know by Proposition 6.63 that $\Pr(X > s + t \mid X > t) = \Pr(X > s)$, which is to say $\Pr(X \leq s + t \mid X > t) = \Pr(X \leq s) \iff \Pr(X - t \leq s \mid X > t) = \Pr(X \leq s)$; that is, $X - t \mid X > t$ and X have identical distributions.

□

Proposition 6.66. Let X be an exponential random variable. Then $\mathbb{E}[X^2 \mid X > t] = \mathbb{E}[(X + t)^2]$.

Proof. By Proposition 6.65, X and $X - t \mid X > t$ have identical distributions. That means they have identical variances, so

$$\operatorname{Var}(X - t \mid X > t) = \operatorname{Var}(X) \iff \mathbb{E}[(X - t)^2 \mid X > t] - [\mathbb{E}(X - t \mid X > t)]^2 = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2.$$

Using $\mathbb{E}(X - t \mid X > t) = \mathbb{E}(X)$ we have

$$\mathbb{E}[(X - t)^2 \mid X > t] - [\mathbb{E}(X)]^2 = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 \iff \mathbb{E}[(X - t)^2 \mid X > t] = \mathbb{E}(X^2)$$

$$\mathbb{E}[(X - t)^2 \mid X > t] - [\mathbb{E}(X)]^2 = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 \iff \mathbb{E}[X^2 - 2tX + t^2 \mid X > t] = \mathbb{E}(X^2)$$

$$\iff \mathbb{E}[X^2 \mid X > t] - 2t\mathbb{E}(X \mid X > t) + t^2 = \mathbb{E}(X^2) \iff \mathbb{E}[X^2 \mid X > t] = \mathbb{E}(X^2) + 2t\mathbb{E}(X \mid X > t) - t^2$$

Using $\mathbb{E}(X \mid X > t) = t + \mathbb{E}(X)$, we have

$$\mathbb{E}[X^2 \mid X > t] = \mathbb{E}(X^2) + 2t(t + \mathbb{E}(X)) - t^2 = \mathbb{E}(X^2) + 2t\mathbb{E}(X) + t^2 = \mathbb{E}[(X + t)^2]$$

□

Example 6.3. (ISE 620): Enter a bank with one teller, 5 present upon your arrival, all service times are exponential with parameter 1. T is time you spend in system.

$$\mathbb{E}(T) = R + \sum_{i=1}^{t} S_i \implies \mathbb{E}(T) = \mathbb{E}(R) + \sum_{i=1}^{5} \mathbb{E}(S_i)$$

$$\mathbb{E}(T) = \mathbb{E}(R) + \frac{5}{\lambda} = \frac{6}{\lambda}$$

Example 6.4. (ISE 620): Suppose $X \sim \text{Exponential}(\lambda)$ and Y Exponential(μ). Then

$$\Pr(X < Y) = \int_0^\infty \Pr(Y > X \mid X = x) \cdot \lambda e^{-\lambda x} dx = \int_0^\infty e^{-\mu x} \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{-(\lambda + \mu)x} dx = \boxed{\frac{\lambda}{\lambda + \mu}}$$

Remark. (ISE 620). The following is an intuitive explanation for why $\min\{X, Y\}$ is distributed exponentially if X and Y are exponential: Suppose we require two machines with failure times X and Y distributed exponentially, and we need both to work. If at time t they are both still working, the remaining expected time is the same as initially because of the memoryless property (Proposition 6.63). Thus $\min\{X, Y\}$ is also memoryless, and exponential. Do the math:

$$\Pr(\min\{X, Y\} > t) = \Pr(X > t.Y > t) = e^{-\lambda t}e^{-\mu t} = e^{-(\lambda+\mu)t}$$

The same is not true for maximum:

$$\Pr(\max\{X, Y\} > t) = (1 - e^{-\lambda t})(1 - e^{-\mu t})$$

which is not exponential. Intuitively, if we only need one machine to be working, just because we are working at time t does not mean we are in the same position as we were initially; one machine could have failed. Now the general case: $X_1, \ldots, X_n \sim \text{Exponential}(\lambda_i)$.

$$\Pr(\min\{X_i\} > t) = \Pr(X_1 > t \cap \ldots \cap X_n > t) = \prod_{i=1}^{n} \Pr(X_i > t) = \exp\left(t \sum_{i=1}^{n} \lambda_i\right)$$

so the minimum of an arbitrary number of exponential random variables is distributed exponentially. Now

$$\Pr(X_i < \min_{j \neq i}\{X_j\}) = \Pr(X_i < Z)$$

where $Z \sim \text{Exponential}(\sum_{j\neq i} \lambda_j$. So by the other example, this probability is $\lambda_i / \sum_{j=1}^{n} \lambda_j$.

Proposition 6.67. (ISE 620). If X and Y are independent exponential random variables, $\min\{X, Y\}$ is independent of whether X or Y is smaller. That is

$$\Pr(\min\{X, Y\} > t \mid X < Y) = \Pr(\min\{X, Y\} > t).$$

Proof.

$$\Pr(\min\{X, Y\} > t \mid X < Y) = \frac{\Pr(Y > X > t)}{\Pr X < Y)} = \frac{1}{\lambda/(\lambda + \mu)} \cdot \int_0^\infty \Pr(Y > X > t \mid X = s)\lambda e^{-\lambda s} ds$$

$$= \frac{\lambda + \mu}{\lambda} \int_t^\infty e^{-\mu s} \lambda e^{-\lambda s} ds = \int_t^\infty (\lambda + \mu)e^{-(\lambda+\mu)s} ds = e^{-(\lambda+\mu)t} = \Pr(\min\{X, Y\} > t)$$

□

Example 6.5. (ISE 620). You have two servers with exponential service times with parameters μ_1 and μ_2. You wait until the first person is done serving a customer, then you are served by that server. T is the time you spend in the system. What is $E(T)$?

$$\mathbb{E}(T) = \mathbb{E}(T \mid \mu_1)\Pr(\mu_1) + \mathbb{E}(T \mid \mu_2)\Pr(\mu_2) = \mathbb{E}(T \mid \mu_1)\frac{\mu_1}{\mu_1+\mu_2} + \mathbb{E}(T \mid \mu_2)\frac{\mu_2}{\mu_1+\mu_2}$$

$$= \left(\frac{1}{\mu_1} + \mathbb{E}(X_1 \mid X_1 < X_2)\right)\frac{\mu_1}{\mu_1+\mu_2} + \left(\frac{1}{\mu_2} + \mathbb{E}(X_2 \mid X_2 < X_1)\right)\frac{\mu_2}{\mu_1+\mu_2}$$

$$= \left(\frac{1}{\mu_1} + \mathbb{E}(\min\{X_1, X_2\})\right)\frac{\mu_1}{\mu_1+\mu_2} + \left(\frac{1}{\mu_2} + \mathbb{E}(\min\{X_1, X_2\})\right)\frac{\mu_2}{\mu_1+\mu_2}$$

$$= \left(\frac{1}{\mu_1} + \frac{1}{\mu_1+\mu_2}\right)\frac{\mu_1}{\mu_1+\mu_2} + \left(\frac{1}{\mu_2} + \frac{1}{\mu_1+\mu_2}\right)\frac{\mu_2}{\mu_1+\mu_2}$$

Proposition 6.68. (Equation (5.5) in Sheldon Ross *Introduction to Probability Models*.) For independent exponential variables T_1 and T_2 with means $1/\lambda_1$ and $1/\lambda_2$, $\Pr(T_2 < T_1) = \frac{\lambda_2}{\lambda_1+\lambda_2}$.

Proof.

$$\Pr(T_2 < T_1) = \int_0^\infty \Pr(T_2 < T_1 \mid T_2 = t)\lambda_2 e^{-\lambda_2 t} dt = \int_0^\infty \Pr(t < T_1)\lambda_2 e^{-\lambda_2 t} dt = \int_0^\infty e^{-\lambda_1 t}\lambda_2 e^{-\lambda_2 t} dt$$

$$= \int_0^\infty \lambda_2 e^{-(\lambda_1+\lambda_2)t} dt = \frac{\lambda_2}{\lambda_1+\lambda_2}$$

□

6.3.7 Multivariate Gaussian (Normal) Distributions

Definition 6.36. From http://pluto.huji.ac.il/~pchiga/teaching/MathStat/SIAnotes2013.pdf (definition 2b6): A random vector $X = (X_1, X_2)$ is Gaussian with mean $\mu = (\mu_1, \mu_2)$ and the covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

if it has a joint pdf of the form

$$f_X(x) = \frac{1}{2\pi\sigma_2\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2}\frac{1}{1-\rho^2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2)} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_1} + \frac{(x_2-\mu_2)^2}{\sigma_2^2)}\right)\right]$$

for $x \in \mathbb{R}^2$.

Definition 6.37. (From Heilman Math 541A; Vector-valued Gaussian random variables.) $Z := (Z_1, \ldots, Z_d) \in \mathbb{R}^d$ is a **Gaussian random vector** if for all $v \in \mathbb{R}^d$, $\langle v, Z\rangle$ is a Gaussian random variable (in the usual sense). Equivalently, any linear combination of $Z_1, \ldots, Z_d$ is a Gaussian random variable. Also, tZ has covariance matrix $a_{ij} = \mathbb{E}\big[(Z_i - \mathbb{E}(Z_i))(Z_j - \mathbb{E}(Z_j)\big], 1 \le i < j \le d$.

Definition 6.38. A random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_p)$ is Gaussian with mean $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_p)$ and covariance matrix $\boldsymbol{\Sigma} = \left[\boldsymbol{\sigma}_{ij}\right]$ if it has a joint pdf of the form

$$f_{\boldsymbol{\mu},\boldsymbol{\Sigma}}(\boldsymbol{x}) = (2\pi)^{-p/2} |\boldsymbol{\Sigma}|^{-1/2} \cdot \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right).$$

Proposition 6.69. [**Conditional distribution of one Gaussian random variable on another. From** http://pluto.huji.ac.il/~pchiga/teaching/MathStat/SIAnotes2013.pdf **(Proposition 3c1)**]

Let X be a Gaussian random variable in $\mathbb{R}^2$ such that

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}\right) = \mathcal{N}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \sigma_{12}^2 \\ \sigma_{12}^2 & \sigma_2^2 \end{bmatrix}\right)$$

Then $f_{X_1|X_2}(x_1; x_2)$ is Gaussian with the (conditional) mean

$$\mathbb{E}(X_1 \mid X_2 = x_2) = \mu_1 + \frac{\rho\sigma_1}{\sigma_2}(x_2 - \mu_2) = \mu_1 + \frac{\sigma_{12}^2}{\sigma_2^2}(x_2 - \mu_2)$$

and the (conditional) variance

$$\mathrm{Var}(X_1 \mid X_2 = x_2) = \sigma_1^2(1-\rho^2) = \sigma_1^2 - \frac{\sigma_{12}^4}{\sigma_2^2}$$

That is, the conditional distribution of X_1 given $X_2 = x_2$ is

$$X_1 \mid X_2 = x_2 \sim \mathcal{N}\left(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(x_2 - \mu_2), (1-\rho^2)\sigma_1^2\right) = \mathcal{N}\left(\mu_1 + \frac{\sigma_{12}^2}{\sigma_2^2}(x_2 - \mu_2), \sigma_1^2 - \frac{\sigma_{12}^4}{\sigma_2^2}\right)$$

Proposition 6.70 (**Generalization of Proposition 6.69; from** https://en.wikipedia.org/wiki/Multivariate_norm
Suppose

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{12}^T & \boldsymbol{\Sigma}_{22} \end{bmatrix}\right)$$

where $\boldsymbol{\mu}_1 \in \mathbb{R}^q$, $\boldsymbol{\mu}_2 \in \mathbb{R}^{p-q}$, $\boldsymbol{\Sigma}_{11} \in \mathbb{R}^{q \times q}$, $\boldsymbol{\Sigma}_{12} \in \mathbb{R}^{(p-q)\times q}$, and $\boldsymbol{\Sigma}_{22} \in \mathbb{R}^{(p-q)\times(p-q)}$. Then

$$\{\boldsymbol{X}_1 \mid \boldsymbol{X}_2\} \sim \mathcal{N}\left(\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\left(\boldsymbol{X}_2 - \boldsymbol{\mu}_2\right), \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{12}^T\right) \tag{6.10}$$

Remark. Note that the conditional covariance matrix in (6.10) is the Schur complement (see Sections 12.2 and 2.4) of $\boldsymbol{\Sigma}_{22}$ in $\boldsymbol{\Sigma}$.

Remark. A similar Berry-Esseen theorem (Theorem 8.38) exists for multivariate Gaussian distributions.

Remark. Note that this matches the OLS coefficients in the univariate case. In other words, the univariate OLS formula can be derived using only this fact.

Recall Theorem 6.44: if two variables are bivariate normal, they are independent if and only if their covariance

$$\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy$$

equals 0.

Proposition 6.71 (Stats 100B homework problem). Let $(X_i, Y_i), i = 1, 2, \ldots, n$, be a random sample from a bivariate normal distribution, where $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ are independent.. Then the joint moment generating function of $(\bar{X}, \bar{Y})$ is

$$\exp\left(\boldsymbol{t}'\boldsymbol{\mu} + \frac{1}{2}\boldsymbol{t}'\frac{1}{n}\boldsymbol{\Sigma}\boldsymbol{t}\right) \sim \boxed{\mathcal{N}\left(\boldsymbol{\mu}, \frac{1}{n}\boldsymbol{\Sigma}\right)}$$

and $(\bar{X}, \bar{Y})$ is bivariate normal with mean $\boldsymbol{\mu}$ and variance-covariance matrix $\frac{1}{n}\boldsymbol{\Sigma}$.

Proof. Let $\boldsymbol{W_i} = (X_i, Y_i)$. Then the moment-generating function of $\boldsymbol{W_i} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is given by

$$M_{\boldsymbol{W_i}}(\boldsymbol{t}) = \exp(t'\boldsymbol{\mu} + \frac{1}{2}\boldsymbol{t}'\boldsymbol{\Sigma}\boldsymbol{t})$$

where $\boldsymbol{\mu}$ is a two-dimensional column vector and $\boldsymbol{\Sigma}$ is a two-by-two matrix. Then

$$(\bar{X}, \bar{Y}) = \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{W_i}(\boldsymbol{t}) = \sum_{i=1}^{n}\frac{1}{n}\boldsymbol{W_i}(\boldsymbol{t})$$

Since $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ are independent,

$$M_{(\bar{X},\bar{Y})}(\boldsymbol{t}) = \prod_{i=1}^{n} M_{\boldsymbol{W_i}}\left(\frac{1}{n}\boldsymbol{t}\right)$$

$$= \left[\exp\left(\frac{1}{n}\boldsymbol{t}'\boldsymbol{\mu} + \frac{1}{2}\frac{1}{n}\boldsymbol{t}'\boldsymbol{\Sigma}\frac{1}{n}\boldsymbol{t}\right)\right]^n = \exp\left(n\frac{1}{n}\boldsymbol{t}'\boldsymbol{\mu} + \frac{1}{2}n\frac{1}{n^2}\boldsymbol{t}'\boldsymbol{\Sigma}\boldsymbol{t}\right)$$

$$= \exp\left(\boldsymbol{t}'\boldsymbol{\mu} + \frac{1}{2}\boldsymbol{t}'\frac{1}{n}\boldsymbol{\Sigma}\boldsymbol{t}\right) \sim \boxed{\mathcal{N}\left(\boldsymbol{\mu}, \frac{1}{n}\boldsymbol{\Sigma}\right)}$$

Thus $(\bar{X}, \bar{Y})$ is bivariate normal with mean $\boldsymbol{\mu}$ and variance-covariance matrix $\frac{1}{n}\boldsymbol{\Sigma}$.

□

6.4 Exponential Families

Definition 6.39. Informally, an **exponential family** is a family of probability distributions that depends on a parameter $w \in \mathbb{R}^k$.

Formally, let n, k be positive integers and let μ be a *measure* on $\mathbb{R}^n$ (that is, a probability law that does not necessarily sum to 1). Let $t_1, \ldots, t_k : \mathbb{R}^n \to \mathbb{R}$. Let $h : \mathbb{R}^n \to [0, \infty]$, and assume h is not identically zero. For any $w = (w_1, \ldots, w_k) \in \mathbb{R}^k$, define

$$a(w) := \log \left[\int_{\mathbb{R}^n} h(x) \exp \left(\sum_{i=1}^k w_i t_i(x) \right) d\mu(x) \right], \quad \forall x \in \mathbb{R}^n$$

The set $\{w \in \mathbb{R}^k\}$ is called the **natural parameter space**. On this set, the function

$$f_w(x) := h(x) \exp \left(\sum_{i=1}^k w_i t_i(x) - a(w) \right), \quad \forall x \in \mathbb{R}^n$$

satisfies $\int_{\mathbb{R}^n} f_w(x) d\mu(s) = 1$ (by the definition of $a(w)$). (Why?

$$\int_{\mathbb{R}^n} f_w(x) d\mu(s) = \frac{\int_{\mathbb{R}^n} h(x) \exp \left(\int_{i=1}^k w_i t_i(x) \right) d\mu(x)}{\int_{\mathbb{R}^n} h(x) \exp \left(\int_{i=1}^k w_i t_i(x) \right) d\mu(x)} = 1.)$$

So, the set of functions (which can be interpreted as probability density functions, or as probability mass functions according to μ) $\{f_w : \theta \in \Theta : a(w(\theta)) < \infty\}$ is called a **k-parameter exponential family in canonical form.**

More generally, let $\Theta \in \mathbb{R}^k$ be any set an let $w : \Theta \to \mathbb{R}^k$. We define a **$k$-parameter exponential family** to be a set of functions $\{f_\theta : \theta \in \Theta\}$, where

$$f_\theta(x) := h(x) \exp \left(\sum_{i=1}^k w_i(\theta) t_i(x) - a(w(\theta)) \right), \quad \forall x \in \mathbb{R}^n$$

satisfies

Also, an exponential family is called **curved** if the dimension of Θ is less than k.

Example 6.6. A Gaussian random variable has mean μ and standard deviation σ, and is in an exponential family:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad \mu \in \mathbb{R}, \sigma > 0$$

In this case, $k = 2$.

$$\vdots$$

(Continuation of example 3.2.) If we instead write this in canonical form, we get

$$a(w) = \frac{\mu^2}{2\sigma^2} + \log(\sigma) = \left(\frac{\mu}{\sigma^2}\right)^2 \left[(-2)\frac{(-1)}{\sigma^2}\right]^{-1} - \frac{1}{2}\log\left((-2)\frac{(-1)}{2\sigma^2}\right) = \frac{w_1^2}{4w_2} - \frac{1}{2}\log(-2w_2)$$

That is, in this case you can get rid of the thetas and write this in canonical form. Define

$$f_w(x) = h(x) \exp\left(\sum_{i=1}^{2} w_i t_i(x) - a(w)\right), \quad \forall x \in \mathbb{R}$$

where w ranges in $\{(w_1, w_2) \in \mathbb{R}^2 : w_2 > 0\}$.

Remark. (notes remark 3.4) **(location family.)** Let X be a random variable with density $f : \mathbb{R} \to \mathbb{R}$. Let $\mu \in \mathbb{R}$. Then the densites $\{f(x+\mu\}_{\mu\in\mathbb{R}}$ are called a **location family** of X. This family may or may not be an exponential family.

Remark. (notes remark 3.6) **(Scale family.)** Let X be a random variable with density $f : \mathbb{R} \to \mathbb{R}$. Let $\sigma > 0$. The family of densities $\{\sigma^{-1} f(x/\sigma)\}_{\sigma>0}$ is called a **scale family.** Note

$$\int_{\mathbb{R}} \sigma^{-1} f(x/\sigma) dx = \int_{\mathbb{R}} f(y) dy = 1$$

(subsituting $y = x/\sigma, dy = dx/\sigma$).

Remark. (notes remark 3.7) **(Location and scale family.)** The family of densities $\{\sigma^{-1} f((x+\mu)/\sigma)\}_{\sigma>0,\mu\in\mathbb{R}}$ is called a **location and scale family.** This family may or may not be an exponential family (although Gaussian random variables are one example where it is.)

Exercise 4. Try to write a binomial random variable with parameters n and p as a two-parameter exponential family. (Note: it's impossible to do, but instructive to try.)

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} = \frac{n!}{(n-x)!x!} \exp\big(x \log(p)\big)(1-p)^{n-x} = \dots$$

It turns out you can do it with one parameter with p, but not with two parameters with n.

Example 6.7 (**Example 3.15 in 541A notes**)**.** Write a binomial random variable with parameters n and p as an exponential family (when n is fixed), then take derivatives in p.

Recall

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad 0 \text{ for any other } x.$$

Keep n fixed and look at p.

$$\binom{n}{x} p^x (1-p)^{n-x} = \binom{n}{x} \exp\big(x \log p + (n-x)\log(1-p)\big) = \binom{n}{x} \exp\big[x \log(p/(1-p)) - (-1)n \log(1-p)\big]$$

Define $h : \mathbb{R} \to \mathbb{R}$ so that

$$h(x) = \begin{cases} \binom{n}{x} & 0 \le x \le n \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}$$

Let $\theta := p, \Theta := (0,1)$, $t(x) := x$, $w(\theta) := \log(\theta/(1-\theta)), a(w(\theta)) = -n\log(1-\theta)$.

So, $f_\theta(x) := h(x) \exp\big[w(\theta)t(x) - a(w(\theta))\big], \forall x \in \mathbb{R}$.

Now we will take derivatives. As in previous example (last class),

$$e^{-a(w(\theta))} \frac{\partial}{\partial \theta_1} e^{a(w(\theta))} = \mathbb{E}_\theta \left(\sum_{i=1}^{k} \frac{\partial w_i}{\partial \theta_1} t_i \right)$$

So in this case

$$e^{-a(w(\theta))} \frac{\partial}{\partial \theta_1} e^{a(w(\theta))} = \mathbb{E}_\theta \left(\frac{\mathrm{d}}{\mathrm{d}\theta} w(\theta) t \right).$$

Plugging in yields

$$(1-\theta)^n \frac{\mathrm{d}}{\mathrm{d}\theta} (1-\theta)^{-n} = \frac{n}{1-\theta}$$

on the left side and

$$= \left(\frac{1}{\theta} + \frac{1}{1-\theta} \right) \mathbb{E}_\theta(X) = \mathbb{E}_\theta(X) \left(\frac{1}{\theta(1-\theta)} \right)$$

on the right side which yields

$$\mathbb{E}_\theta(X) = \frac{n}{1-\theta}(\theta(1-\theta)) = n\theta \implies \mathbb{E}(X) = pn$$

Remark. This is a probability mass function since it is defined only on integers.

6.4.1 Differential identities (Generalization of Moment-Generating Functions)

Recall that the moment-generating function for a Gaussian random variable is

$$\mathbb{E}(e^{tX}) = e^{t^2/2} \quad \forall t \in \mathbb{R}$$

Consequently,

$$\left.\frac{\mathrm{d}^m}{\mathrm{d}t^m}\right|_{t=0} \mathbb{E}(e^{tX}) = \mathbb{E}(X^m)$$

for any integer $m > 0$. We can do a similar thing for an exponential family—we can differentiate the parameters of exponential families and find out information about the exponential family. As in Definition 6.39, let

$$a(w) := \log \int_{\mathbb{R}^n} h(x) \exp\left(\sum_{i=1}^k w_i t_i(x)\right) d\mu(x)$$

Define

$$W := \{w \in \mathbb{R}^k : a(w) < \infty\}.$$

Question: Is $a(w)$ differentiable?

Lemma 6.72 (Lemma 3.8 in 541A notes)**.** The function $a(w)$ is continuous and has continuous partial derivatives of all orders on the interior of W. Moreover, we can compute these derivatives by differentiating under the integral sign.

Proof. We prove only the case of a first order partial derivative. Consider the case of the partial derivative with respect to w_1 at w in the interior of W. Let $e_1 = (1, 0 \ldots, 0) \in \mathbb{R}^k$. Since the exponential function is analytic, it suffices to show that the partial derivative of $e^{a(w)}$ exists in the direction e_1. We form the difference quotient for $e^{a(w)}$ as follows:

$$\frac{\exp[a(w + \epsilon e_1)] - \exp[a(w)]}{\epsilon} = \frac{1}{\epsilon} \int_{\mathbb{R}^n} h(x) \left[\exp\left(\epsilon t_1(x) + \sum_{i=1}^k w_i t_i(x)\right) - \exp\left(\sum_{i=1}^k w_i t_i(x)\right)\right] d\mu(x)$$

$$= \int_{\mathbb{R}^n} h(x) \frac{\exp[\epsilon t_1(x)] - 1}{\epsilon} \exp\left(\sum_{i=1}^k w_i t_i(x)\right) d\mu(x). \tag{6.11}$$

By the Mean Value Theorem,

$$\frac{|e^a - 1|}{a} = \frac{|e^a - e^0|}{a - 0} \leq e^a \text{ if } a > 0, \text{ or } 1 \text{ if } a < 0$$

so we have

$$|e^a - 1| \leq |a| \max\{1, e^a\} \leq |a|e^{|a|} \leq e^{2|a|} \leq e^{2a} + e^{-2a} \quad \forall a \in \mathbb{R}.$$

In particular, if $a = \epsilon t_1(x)$, we have

$$|e^{\epsilon t_1(x)} - 1| \leq e^{2\epsilon t_1(x)} + e^{-2\epsilon t_1(x)} \tag{6.12}$$

Therefore, examining the integrand of (6.11), we have

$$\left| h(x) \frac{\exp[\epsilon t_1(x)] - 1}{\epsilon} \exp\left(\sum_{i=1}^{k} w_i t_i(x) \right) \right| \leq h(x) \left| \frac{\exp[\epsilon t_1(x)] - 1}{\epsilon} \right| \exp\left(\sum_{i=1}^{k} w_i t_i(x) \right)$$

$$\leq \text{ (by (6.12)) } h(x)\big(e^{2\epsilon t_1(x)} + e^{-2\epsilon t_1(x)}\big) \exp\left(\sum_{i=1}^{k} w_i t_i(x) \right)$$

$$\vdots$$

$$= h(x)|t_1(x)| \exp\left(\epsilon t_1(x) + \sum_{i=1}^{k} w_i t_i(x) \right) + h(x)|t_1(x)| \exp\left(- \epsilon t_1(x) + \sum_{i=1}^{k} w_i t_i(x) \right) \tag{6.13}$$

If both of the expressions in (6.13) always have finite expected value uniformly for all $\epsilon > 0$, we will be done by the Dominated Convergence Theorem (Theorem 6.73). (Assuming those things are bounded uniformly in expectation then the limit of the integrals is the integral of the limits.)

□

Remark. Notes from proof of Lemma 3.8.

$\left|\frac{e^a - 1}{a}\right| \leq e^a + e^{-a}$. Then

$$\left| h(x) \exp\left(\epsilon t_1(x) + \sum_{i=1}^{k} w_i t_i(x) \right) - \exp\left(\sum_{i=1}^{k} w_i t_i(x) \right) \right| \leq h(x) \exp\left(\sum_{i=1}^{k} w_i t_i(x) \right) |t_1(x)| \Big(\exp(\epsilon t_1(x) + \exp(-\epsilon t_1(x) \Big)$$

Theorem 6.73 (Dominated convergence theorem (Math 541A))**.** Let $X_1, X_2, \ldots : \Omega \to \mathbb{R}$ and $Y : \Omega \to [0, \infty)$ such that $|X_i| \leq Y$ for all $i \geq 1$ and $\mathbb{E}(Y) < \infty$. Assume $X_1, X_2, \ldots$ converges almost surely to $X : \Omega \to \mathbb{R}$. Then

$$\lim_{i \to \infty} \mathbb{E}(x_i) = \mathbb{E}\big(\lim_{i \to \infty} X_i \big) = \mathbb{E}(X)$$

Corollary 6.73.1 (Corollary 3.11 in 541A notes.)**.** Let $\epsilon > 0$. Let $X : \Omega \to \mathbb{R}$ be a random variable such that $\mathbb{E}(e^{wX}) < \infty$ for all $w \in (-\epsilon, \epsilon)$. Then for any integer $n \geq 1$, $\mathbb{E}(X^n)$ exists and

$$\left. \frac{\mathrm{d}^n}{\mathrm{d}w^n} \right|_{w=0} e^{wX} = \mathbb{E}(X^n).$$

Proof. Apply Lemma 6.72 when $\mu = \mathbb{P}, h = 1, k = 1, t(x) = x$: we see that

$$a(w) = \log \int_{\mathbb{R}^n} e^{wx} d\mathbb{P}(x)$$

$\square$

Remark. Notes from example 3.13.

$$e^{-a(w)} \frac{\partial}{\partial w_2} e^{a(w)} = \int_{\mathbb{R}} t_1(x) h(x) \exp\left(\sum_{i=1}^{k} w_i t_i(x) - a(w) \right) dx$$

$$= \int_{\mathbb{R}} t_1(x) f_x(x) dx \tag{6.14}$$

So by chain rule

$$e^{-a(w(\theta))} \frac{\partial}{\partial \theta_1} e^{a(w(\theta))} = e^{-a(w(\theta))} \sum_i \frac{\partial e^{a(w)}}{\partial w_i} \frac{\partial w_i}{\partial \theta_1} \tag{6.15}$$

$$= (\text{by } (6.14)) \sum_{i=1}^{k} \frac{\partial w_i}{\partial \theta_1} \mathbb{E}_\theta t_i = \mathbb{E}_\theta \left(\sum_{i=1}^{k} t_i \frac{\partial w_i}{\partial \theta_1} \right)$$

Example 6.8. Recall Example 3.3. Gaussian, mean μ, variance σ^2. $\theta = (\theta_1, \theta_2) = (\mu, \sigma^2)$, $t_1(x) = x, t_2(x) = x^2$k $w_1(\theta) = \theta_1/\theta_2 = \mu/\sigma^2$, $w_2(\theta) = -1/(2\sigma^2) = -1/(2\theta_2)$, $a(w(\theta)) = \theta_1^2/(2\theta_2) + (1/2)\log(\theta_2) = \mu^2/(2\sigma^2) + \log(\sigma^2)$.

Complete both sides of (6.15).

$$e^{-a(w(\theta))} \frac{\partial}{\partial \theta_1} e^{a(w(\theta))} = \frac{\theta_1}{\theta_2} = \frac{\mu}{\sigma^2}$$

$$\frac{\partial}{\partial \theta_1} e^{a(w(\theta))} = \frac{\mathrm{d}}{\mathrm{d}\theta_1} \sqrt{\theta_2} \exp\left(\frac{\theta_1^2}{2\theta_2} \right) = \frac{\theta_1}{\theta_2} \exp(a(w(\theta))$$

Right side:

$$\frac{\mathrm{d}w_1}{\mathrm{d}\theta_2} = \frac{1}{\theta_2}, \quad \frac{\mathrm{d}w_2}{\mathrm{d}\theta_1} = 0$$

$$\implies \mathbb{E}_\theta \left(\sum_{i=1}^{k} t_i \frac{\mathrm{d}w_i}{\mathrm{d}\theta_1} \right) = \mathbb{E}_\theta (t_1 \frac{\mathrm{d}w_1}{\mathrm{d}\theta_1} + t_2 \frac{\mathrm{d}w_2}{\mathrm{d}\theta_1})$$

$$= \mathbb{E}_\theta(x/\theta_2) \implies \mathbb{E}_\theta(x) = \theta_1 = \mu$$

Theorem 6.74 (Theorem 3.4.2 from Casella and Berger). If X is a random variable in an exponential family, then

$$\mathbb{E}\left(\sum_{i=1}^{k}\frac{\partial w_i(\theta)}{\partial \theta_j}t_i(X)\right) = \frac{\partial}{\partial \theta_j}a(w(\theta)). \tag{6.16}$$

Example 6.9. Recall Example 3.3 again: Gaussian, mean μ, variance σ^2. $\theta = (\theta_1, \theta_2) = (\mu, \sigma^2)$, $t_1(x) = x, t_2(x) = x^2$k $w_1(\theta) = \theta_1/\theta_2 = \mu/\sigma^2$, $w_2(\theta) = -1/(2\sigma^2) = -1/(2\theta_2)$, $a(w(\theta)) = \theta_1^2/(2\theta_2) + (1/2)\log(\theta_2) = \mu^2/(2\sigma^2) + \log(\sigma^2)$.

Complete both sides of (6.16).

$$\frac{\partial}{\partial \theta_1}a(w(\theta)) = \frac{\partial}{\partial \theta_1}\big(\theta_1^2/(2\theta_2) + (1/2)\log(\theta_2)\big) = \frac{\theta_1}{\theta_2}.$$

Left side:

$$\mathbb{E}\left(\sum_{i=1}^{k}\frac{\partial w_i(\theta)}{\partial \theta_1}t_i(X)\right) = \mathbb{E}\left(\frac{\partial}{\partial \theta_1}w_1(\theta)t_1(X) + \frac{\partial}{\partial \theta_1}w_2(\theta)t_2(X)\right) = \mathbb{E}\left(\frac{\partial}{\partial \theta_1}\frac{\theta_1}{\theta_2}\cdot x - \frac{1}{2}\frac{\partial}{\partial \theta_1}\frac{1}{\theta_2}x^2\right)$$

$$= \frac{1}{\theta_2}\mathbb{E}(x)$$

$$\implies \frac{1}{\theta_2}\mathbb{E}(x) = \frac{\theta_1}{\theta_2} \implies \mathbb{E}(X) = \theta_1 = \mu.$$

Repeating by taking the partial derivatives with respect to θ_2 instead:

$$\frac{\partial}{\partial \theta_2}a(w(\theta)) = \frac{\partial}{\partial \theta_2}\big(\theta_1^2/(2\theta_2) + (1/2)\log(\theta_2)\big) = \frac{\theta_1^2}{2}\cdot\frac{-1}{\theta_2^2} + \frac{1}{2\theta_2} = \frac{\theta_2 - \theta_1^2}{2\theta_2^2}$$

Left side:

$$\mathbb{E}\left(\sum_{i=1}^{k}\frac{\partial w_i(\theta)}{\partial \theta_2}t_i(X)\right) = \mathbb{E}\left(\frac{\partial}{\partial \theta_2}w_1(\theta)t_1(X) + \frac{\partial}{\partial \theta_2}w_2(\theta)t_2(X)\right) = \mathbb{E}\left(\frac{\partial}{\partial \theta_2}\frac{\theta_1}{\theta_2}\cdot x - \frac{1}{2}\frac{\partial}{\partial \theta_2}\frac{1}{\theta_2}x^2\right)$$

$$= \mathbb{E}\left(-\frac{\theta_1}{\theta_2^2}\cdot x + \frac{1}{2\theta_2^2}x^2\right) = -\frac{\theta_1}{\theta_2^2}\mathbb{E}(X) + \frac{1}{2\theta_2^2}\mathbb{E}(X^2) = -\frac{\theta_1^2}{\theta_2^2} + \frac{1}{2\theta_2^2}\mathbb{E}(X^2)$$

$$\implies -\frac{\theta_1^2}{\theta_2^2} + \frac{1}{2\theta_2^2}\mathbb{E}(X^2) = \frac{\theta_2 - \theta_1^2}{2\theta_2^2} \iff -2\theta_1^2 + \mathbb{E}(X^2) = \theta_2 - \theta_1^2 \iff \mathbb{E}(X)^2 = \theta_2 + \theta_1^2 = \sigma^2 + \mu^2.$$

6.5 KL Divergence (DSO 607)

Let $f(\cdot \mid \theta) : \mathbb{R}^k \to \mathbb{R}^n$ be a family of parameterized probability densities with $\theta \in \Theta$. Suppose the true model is parameterized by $\theta_0 \in \mathbb{R}^s$, and we are interested in comparing a different model parameterized by $\theta \in \mathbb{R}^k$ to this model. The likelihood ratio $\tau : \mathbb{R}^n \to \mathbb{R}$ defined by $\tau(x; \theta, \theta_0) := f(x \mid \theta)/f(x \mid \theta_0)$ is a good tool for this comparison. Define the **discrimination** between θ and θ_0 at x to be $\Phi(\tau(x; \theta, \theta_0))$ for some function $\Phi : \mathbb{R} \to \mathbb{R}$, and define the **mean discrimination** $D(\theta; \theta_0) : \mathbb{R}^k \to \mathbb{R}$ between θ and θ_0 as

$$D(\theta; \theta_0) := \int_{\mathbb{R}^n} \Phi(\tau(x; \theta, \theta_0)) \cdot f(x \mid \theta_0) \; dx = \mathbb{E}_{\theta_0} \left[\Phi(\tau(X; \theta, \theta_0)) \right].$$

To choose Φ so that $D(\theta; \theta_0)$ behaves like a distance, we would like $\Phi(1) = 0$ since this covers the case where $\theta = \theta_0$, because under the assumption that $f(x \mid \theta) = f(x \mid \theta_0) \; \forall x \in \mathbb{R}^n \iff \theta = \theta_0$, we have $\tau(x; \theta, \theta_0) = 1 \; \forall x \in \mathbb{R}^n \iff \theta = \theta_0$. In particular, we would like this to be the minimum of the function; that is, $\tau''(1; \theta, \theta_0) > 0$. Lastly, we would also like $\tau'(x; \theta, \theta_0) > 0 \; \forall x \in \mathbb{R}^n$. One such choice is $\Phi(t) := -2\log(t)$. This yields the **Kullback-Leibler (KL) divergence** $I(\theta; \theta_0) : \mathbb{R}^k \to \mathbb{R}$

$$I(\theta; \theta_0) := -2 \int_{\mathbb{R}^n} \log \left(\frac{f(x \mid \theta)}{f(x \mid \theta_0)} \right) \cdot f(x \mid \theta_0) \; dx = 2 \int_{\mathbb{R}^n} \left[\log\left(f(x \mid \theta_0)\right) - \log\left(f(x \mid \theta)\right) \right] \cdot f(x \mid \theta_0) \; dx$$

$$= 2\mathbb{E}_{\theta_0} \left[\log\left(f(X \mid \theta_0)\right) \right] - 2\mathbb{E}_{\theta_0} \left[\log\left(f(X \mid \theta)\right) \right].$$

Of course, in practice we will estimate θ_0 as best as we can, by maximizing the **probabilistic negentropy**

$$\mathbb{E}_Z I(\theta; \hat{\theta}_0(Z)) := 2\mathbb{E}_{\theta_0} \left[\log\left(f(X \mid \theta_0)\right] - 2\mathbb{E}_{\theta_0, Z} \left[\log \left(f \left(X \mid \hat{\theta}_0(Z) \right) \right) \right] \right.$$

where Z describes the probability distribution of the sample data and $\hat{\theta}_0(Z)$ is our estimate of θ_0 from the data.

Way we wrote this in DSO 607: KL Divergence of density f (estimated) from density g (true model/distribution):

$$I(g; f) = \int [\log g(x)] g(z) dz - \int [\log f(x)] g(z) dz = \mathbb{E}_g \log(g(z)) - \mathbb{E}_g (\log(f(z))$$

One application of KL Divergence is AIC for model selection; see Section **??**.

6.6 Worked problems

6.6.1 Example Problems That Will Likely Appear on Midterm (and Final)

(1) Let X be uniform on $[1, 5]$, let Y be uniform on $[0, 1]$, and assume that X and Y are independent.

(a) Compute the probability density function of the product XY.

(b) **Only part included on midterm.** Compute the cumulative distribution function of the ratio X/Y.

(c) Compute the characteristic function of the sum $X+Y$.

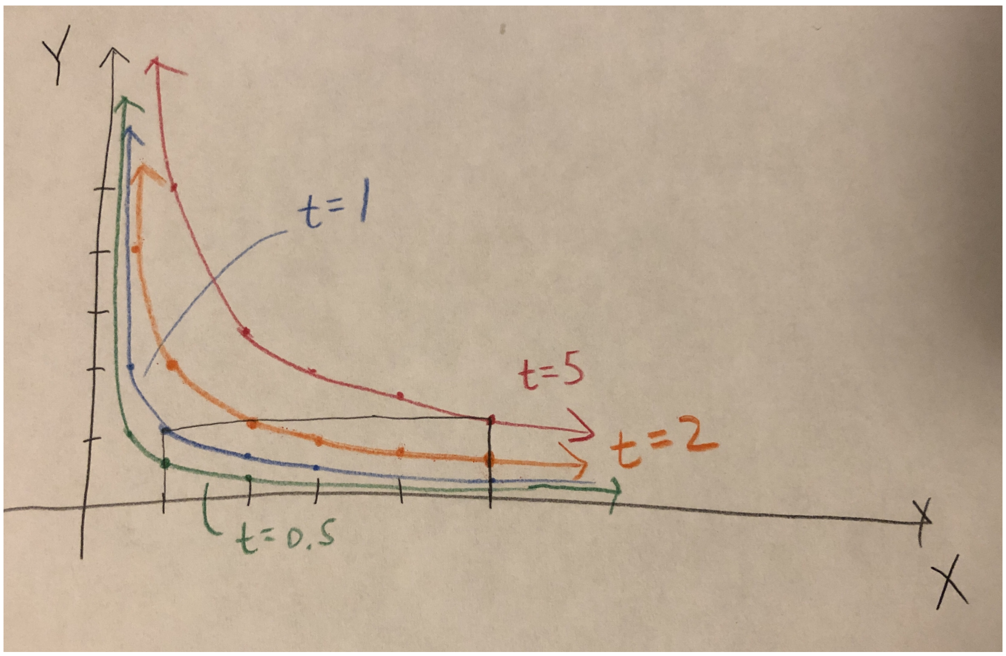
(d) Compute the moment-generating function of the random variable $X - \ln(Y)$.

Solution.

(a) We will find the cdf and then differentiate to yield the pdf. Observe that

$$F_{XY}(t) = \Pr(XY \le t) = \Pr(Y \le tX^{-1})$$

Plotting the density function of XY along with plots of $F_{XY}(t)$ as a function of X for various values of t, we have the following:



Now since both X and Y are distributed uniformly, for a given t, $\Pr(Y \le tX^{-1})$ is the area under the curve and in the rectangle, weighted by $1/4$ since the rectangle has total area 4 but total probability 1. It is clear that the four regimes we need to consider are (1) $t < 0$, (2) $0 \le t < 1$, (3) $1 \le t < 5$, and (4) $t \ge 5$.

(1) $t < 0$: The curve lies below the rectangle, so there is no area below the curve and in the rectangle. Therefore $\boxed{\Pr(Y \le tX^{-1} \mid t < 0) = 0}$. (This is also clear since tX^{-1} would be a negative number and Y is nonnegative.)

(2) $0 \le t < 1$: Integrating the relevant area, we have

$$\Pr(Y \le tX^{-1} \mid 0 \le t < 1) = \frac{1}{4}\int_1^5 \frac{t}{x}dx = \frac{t}{4}\big[\log(x)\big]_1^5 = \boxed{\frac{t}{4}\log(5)}$$

(3) $1 \le t < 5$: In this case, the area is a rectangle of height 1 and width $t-1$ plus the area under the curve from t to 5.

$$\Pr(Y \le tX^{-1} \mid 1 \le t < 5) = \frac{1}{4}\left(1\cdot(t-1)+\int_t^5 \frac{t}{x}dx\right) = \frac{1}{4}\left(t-1+t\big[\log(x)\big]_t^5\right) = \boxed{\frac{1}{4}\big[t\big(1+\log(5/t)\big)-1\big]}$$

(4) $t \geq 5$: In this case, the entire rectangle lies below the curve. Therefore $\boxed{\Pr(Y \le tX^{-1} \mid t \geq 5) = 1}$. So we have

$$F_{XY}(t) = \begin{cases} 0 & t < 0 \\ \frac{t}{4}\log(5) & 0 \le t < 1 \\ \frac{1}{4}\big[t\big(1+\log(5/t)\big)-1\big] & 1 \le t < 5 \\ 1 & t \geq 5 \end{cases}$$

Finally, differentiating yields

$$\boxed{f_{XY}(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{4}\log(5) & 0 \le t < 1 \\ \frac{1}{4}\log\left(\frac{5}{t}\right) & 1 \le t < 5 \\ 0 & t \geq 5 \end{cases}}$$

since

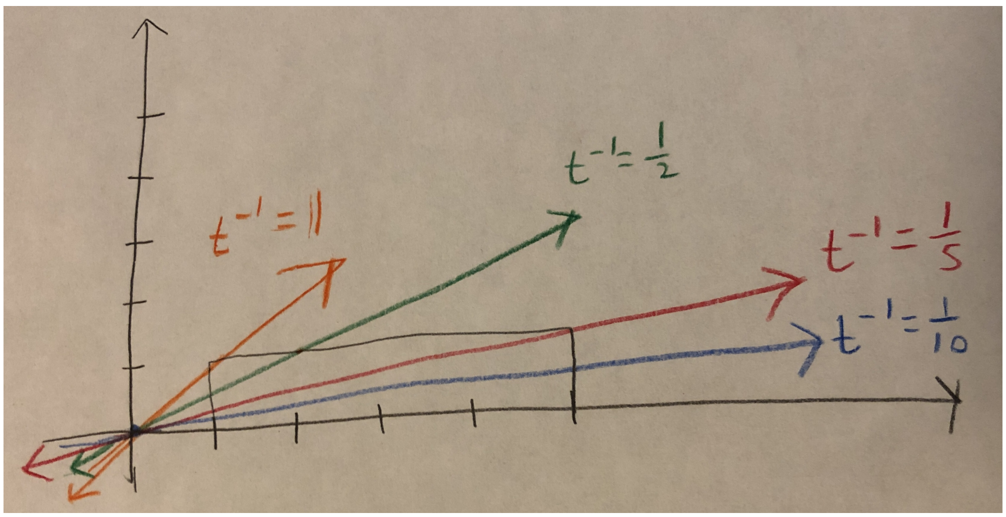
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{4}\big[t\big(1+\log(5/t)\big)-1\big]\right) = \frac{1}{4}\left[1+\log(5/t)+t\left(\frac{1}{5/t}\cdot -5\cdot t^{-2}\right)\right] = \frac{1}{4}\left[1+\log(5/t)+t\left(t\cdot -1\cdot t^{-2}\right)\right]$$

$$= \frac{1}{4}\log\left(\frac{5}{t}\right)$$

(b) **Only part included on midterm.** We will proceed in a similar way as part (a). Observe that

$$F_{X/Y}(t) = \Pr\left(\frac{X}{Y} \le t\right) = \Pr(Y \geq X/t)$$

Plotting the density function of X/Y along with plots of $F_{X/Y}(t)$ as a function of X for various values of t, we have the following:



Now since both X and Y are distributed uniformly, for a given t, $\Pr(Y \geq X/t)$ is the area above the curve and in the rectangle, weighted by $1/4$ since the rectangle has total area 4 but total probability 1. It is clear that the three regimes we need to consider are (1) $t^{-1} \geq 1 \iff t \leq 1$, (2) $1/5 \leq t^{-1} < 1 \iff 1 < t \leq 5$, and (3) $0 < t^{-1} < 1/5 \iff t > 5$.

(1) $t \leq 1$: The curve lies above the rectangle, so there is no area above the curve and in the rectangle. Therefore $\Pr(Y \geq X/t \mid t \leq 1) = 0$. (This is also clear since X/t would have to be greater than 1 and Y is less than or equal to 1.)

(2) $1 < t \leq 5$: The relevant area is the triangle above the green line in the rectangle. Note that it intersects the vertical line at $Y = 1/t$ and the horizontal line at $X = t$.

$$\Pr(Y \geq X/t \mid 1 < t \leq 5) = \frac{1}{4} \cdot \frac{1}{2}\left(1 - \frac{1}{t}\right)(t-1) = \frac{1}{8}\left(t - 1 - 1 + \frac{1}{t}\right) = \frac{1}{8}\left(t - 2 + \frac{1}{t}\right)$$

(3) $t > 5$: In this case, the area is a trapezoid above the blue line and in the rectangle. Note that the blue line intersects the left vertical line at $Y = 1/t$ and the right vertical line at $Y = 5/t$.

$$\Pr(Y \geq X/t \mid t > 5) = \frac{1}{4} \cdot \frac{1}{2} \cdot \left(1 - \frac{1}{t} + 1 - \frac{5}{t}\right) \cdot 4 = \frac{1}{2} \cdot \left(2 - \frac{6}{t}\right) = 1 - \frac{3}{t}$$

So we have

$$\boxed{F_{XY}(t) = \begin{cases} 0 & t \leq 1 \\ \frac{1}{8}\left(t - 2 + \frac{1}{t}\right) & 1 < t \leq 5 \\ 1 - \frac{3}{t} & t > 5 \end{cases}}$$

(c) The characteristic function for a uniform distribution on $[a, b]$ is

$$\frac{2}{(b-a)t} \sin\left(\frac{1}{2}(b-a)t\right) \exp\left(i(a+b)\frac{t}{2}\right).$$

Using the fact that $X \perp\!\!\!\perp Y \implies \phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$, we have

$$\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t) = \frac{2}{(5-1)t} \sin\left(\frac{1}{2}(5-1)t\right) \exp\left(i(5+1)\frac{t}{2}\right) \cdot \frac{2}{t} \sin\left(\frac{1}{2}t\right) \exp\left(i\frac{t}{2}\right)$$

$$= \frac{1}{2t} \sin(2t) \exp(3it) \cdot \frac{2}{t} \sin\left(\frac{1}{2}t\right) \exp\left(i\frac{t}{2}\right) = \boxed{\frac{1}{t^2} \exp\left(\frac{7}{2}it\right) \cdot \sin(2t) \sin\left(\frac{1}{2}t\right)}$$

(d) The moment-generating function for a uniform distribution on $[a, b]$ is

$$M_X(t) = \mathbb{E}(\exp(tX)) = \int_a^b \frac{1}{b-a} \cdot \exp(tx)dx = \frac{1}{b-a}\left[\frac{1}{t}\exp(tx)\right]_a^b = \frac{1}{(b-a)t}\big[\exp(bt) - \exp(at)\big]$$

Therefore the moment-generating function for X is $t^{-1}\big[\exp(t) - 1\big]$. Note that

$$\Pr(Y \leq y) = \Pr(-\log(X) \leq y) = \Pr(\log(X) \geq -y) = \Pr(X \geq e^{-y}) = \int_{\exp(-y)}^{\infty} dt = \int_{\exp(-y)}^{1} dt$$

Substituting $t = e^{-u}$ (so that we have $u = -\log(t)$, $dt = -e^{-u}du$, we have

$$\Pr(Y \leq y) = -\int_y^0 e^{-u} du = \left[e^{-u}\right]_y^0 = 1 - e^{-y}$$

which is the cdf for an exponential distribution with mean 1. Therefore $Y = -\log(X) \sim \text{Exponential}(1)$, so

$$M_Y(t) = \frac{1}{1-t}$$

Using the fact that if X and Y are independent then $M_{X+Y}(t) = M_X(t)M_Y(t)$, we have

$$M_{X+Y}(t) = M_X(t)M_Y(t) = \frac{\exp(t) - 1}{t} \cdot \frac{1}{1-t} = \boxed{\frac{\exp(t)-1}{t - t^2}}$$

(2) **Fall 2016 Problem 2.** Let X and Y be i.i.d. exponential with mean 1. Show that for every $t > 0$ the events $\{\omega : \min\{X, Y\} > t\}$ and $\{\omega : X < Y\}$ are independent.

Solution. Note that $\min\{X, Y\} > t \iff X > t \cap Y > t$.

- $\Pr(\min\{X, Y\} > t) = \Pr(X > t \cap Y > t) = \Pr(X > t)\Pr(Y > t)$

$$= \int_t^\infty e^{-x} dx \int_t^\infty e^{-y} dy = -e^{-x}\big|_t^\infty - e^{-y}\big|_t^\infty = \boxed{e^{-2t}}$$

- $\Pr(X < Y)$: Note that in Figure 2, the region that satisfies this condition is region G_1 plus G_2 (note that X and Y are nonnegative). Therefore we can find this probability by integrating the joint pdf over that region.

$$\Pr(X < Y) = \iint_{\{G_1 + G_2\}} f_{X,Y}(x, y) dx dy$$

Note that the joint pdf is the probability of the marginal pdfs since X and Y are independent.

$$= \int_0^\infty \int_x^\infty e^{-x-y} dy dx = \int_0^\infty e^{-x} \int_x^\infty e^{-y} dy dx$$

$$\int_x^\infty e^{-y} dy = -e^{-y}\big|_x^\infty = e^{-x}$$

$$\implies \Pr(X < Y) = \int_0^\infty e^{-2x} dx = -\frac{1}{2}e^{-2x}\big|_0^\infty = \boxed{\frac{1}{2}}$$

- $\Pr(X < Y \cap \min\{X, Y\} > t)$: Note that in Figure 2, the region that satisfies this condition is region G_1. Therefore we can find this probability by integrating the joint pdf over that region.

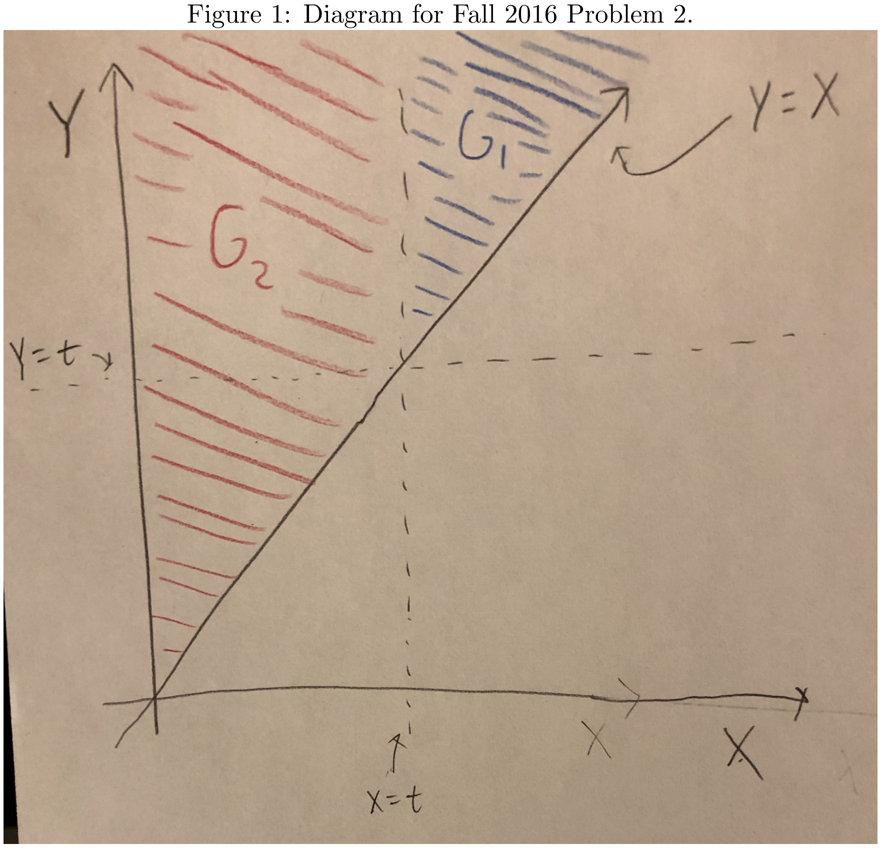
$$\Pr(X < Y \cap \min\{X, Y\} > t) = \iint_{G_1} f_{X,Y}(x, y) dx dy = \int_t^\infty \int_t^y e^{-x-y} dx dy = \int_t^\infty e^{-y} \int_t^y e^{-x} dx dy$$

$$\int_t^y e^{-x} dx = -e^{-x}\big|_t^y = -e^{-y} + e^{-t}$$

$$\implies \Pr(X < Y \cap \min\{X, Y\} > t) = \int_t^\infty e^{-y}\big(-e^{-y} + e^{-t}\big) dy = \int_t^\infty \big(e^{-t-y} - e^{-2y}\big) dy$$

$$= \frac{1}{2}e^{-2y} - e^{-t}e^{-y}\big|_t^\infty = -\frac{1}{2}e^{-2t} - e^{-2t} = \boxed{\frac{1}{2}e^{-2t}}$$

Figure 1: Diagram for Fall 2016 Problem 2.



Note that

$$\Pr(X < Y \cap \min\{X, Y\} > t) = \frac{1}{2} \cdot e^{-2t} = \Pr(X < Y)\Pr(\min\{X, Y\} > t)$$

Therefore the events $\{\omega : \min\{X, Y\} > t\}$ and $\{\omega : X < Y\}$ are independent for every $t > 0$.

(3) In a certain area, earthquakes happen at a frequency of one every four days. What is the probability that more than 100 earthquakes will occur in this area in one year (365 days)?

Solution. We can think of this as a Poisson process (see section 7.2) with $\lambda = 1/4$. Then there are two ways to obtain the answer: we can either examine the number of earthquakes in a 365 day period $N(365)$ and find the probability that $N(365) > 100$, or we can examine the number of days until the 101st earthquake T_{101} and find the probability that $T_{101} < 365$.

(i) **Number of earthquakes in 365 days:** Let $N(t)$ be the number of earthquakes that occur in t days after the start of this process. By Theorem 7.3, $N(t) \sim \text{Poisson}(t \cdot 1/4)$. Then

$$\Pr(N(t) > 100) = \sum_{j=101}^{\infty} \frac{(365 \cdot 1/4)^j \exp(-365 \cdot 1/4)}{j!}$$

To obtain an answer for this, we can use the normal approximation to a Poisson distribution (Proposition 6.34):

$$N(t) \sim \mathcal{N}(t/4, t/4) \implies \Pr(N(365) > 100) \approx \Pr\left(\mathcal{N}(0,1) > \frac{100.5 - 365/4}{\sqrt{365/4}}\right)$$

$$= \Pr\left(\mathcal{N}(0,1) > \frac{100.5 - 91.25}{\sqrt{91.25}}\right) \approx \Pr\left(\mathcal{N}(0,1) > \frac{9.25}{9.1}\right) \approx \boxed{0.1664}$$

(ii) **Number of days before 100th earthquake:** Let T_n be the number of days until the nth earthquake happens. By Corollary 7.4.1, $T_n \sim \text{Gamma}(n, 4)$. Then

$$\Pr(T_{101} < 365) = \int_0^{365} \frac{1}{\Gamma(101, 4)} x^{101-1} e^{-x/4} dx$$

To obtain an answer for this, we can use the normal approximation to a Gamma distribution (Proposition 6.59):

$$T_n \sim \mathcal{N}(404, 1616) \implies \Pr(T_{101} < 365) \approx \Pr\left(\mathcal{N}(0,1) < \frac{365 - 404}{\sqrt{1616}}\right) \approx \Pr\left(\mathcal{N}(0,1) < \frac{-40}{40}\right)$$

$$= \Pr\left(\mathcal{N}(0,1) < -1\right) \approx \boxed{0.1660}$$

6.6.2 More Problems From Homework

Homework 5 Problem 4.

Let $X_1, X_2, \ldots$ be i.i.d. having moment-generating functions $M_X = M_X(t), t \in (-\infty, \infty)$. Let N be an integer-valued random variable with moment-generating function $M_N = M_N(t), t \in (-\infty, \infty)$. Assume that N is independent of all X_k and define $S = \sum_{k=1}^{N} X_k$. Confirm that the random variable S has the moment-generating function $M_S = M_S(t)$ defined for all $t \in (-\infty, \infty)$ and

$$M_S(t) = M_N\big(M_X(t)\big)$$

Then use the result to derive the formulae

$$\mathbb{E}(S) = \mu_N \mu_X, \mathrm{Var}(S) = (\sigma_N^2 - \mu_N)\mu_X^2 + \mu_N \sigma_X^2$$

where $\mu_N = \mathbb{E}(N)$, $\mu_X = \mathbb{E}(X_1)$, $\sigma_N^2 = \mathrm{Var}(N)$, and $\sigma_X^2 = \mathrm{Var}(X_1)$. How will the above computations change if we use the characteristic function ϕ_X instead of the moment-generating function M_X?

Solution.

$$M_S(t) = \mathbb{E}(e^{tS}) = \mathbb{E}[\mathbb{E}(e^{tS} \mid N)] = \sum_{n=0}^{\infty} \mathbb{E}(e^{tS} \mid N = n) \Pr(N = n) = \sum_{n=0}^{\infty} \mathbb{E}(e^{t(X_1 + X_2 + \ldots + X_n)} \mid N = n) \Pr(N = n)$$

$$= \sum_{n=0}^{\infty} \mathbb{E}(e^{tX_1} e^{tX_2} \cdots e^{tX_n}) \Pr(N = n)$$

By independence of the X_i we have

$$= \sum_{n=0}^{\infty} \mathbb{E}(e^{tX_1}) \mathbb{E}(e^{tX_2}) \cdots \mathbb{E}(e^{tX_n}) \Pr(N = n)$$

which, since the X_i are i.i.d., can be written as

$$= \sum_{n=0}^{\infty} \mathbb{E}(e^{tX_1})^n \Pr(N = n) = \sum_{n=0}^{\infty} (M_X(t))^n \Pr(N = n)$$

But since $G_N(s) = \mathbb{E}(s^N) = \sum_{n=0}^{\infty} s^n \Pr(N = n)$, this can be written as

$$M_S(t) = G_N(M_X(t))$$

as desired. Note that

$$M_S'(t) = G_N'\big(M_X(t)\big) M_X'(t)$$

$$M_S''(t) = G_N''\big(M_X(t)\big)(M_X'(t))^2 + G_N'\big(M_X(t)\big) M_X''(t)$$

So we have

- $\mathbb{E}(S) = M_S'(0) = G_N'\big(M_X(0)\big) M_X'(0) = G_N'(1)\mathbb{E}(X_1) = \mathbb{E}(N)\mathbb{E}(X_1) = \mu_N \mu_X$

- $\text{Var}(S) = \mathbb{E}(S^2) - \mathbb{E}(S)^2 = M_S''(0) - (M_S'(0))^2$

$$= G_N''\big(M_X(0)\big)(M_X'(0))^2 + G_N'\big(M_X(0)\big)M_X''(0) - \mu_N^2\mu_X^2 = G_N''\big(1\big)\mathbb{E}(X_1)^2 + G_N'\big(1\big)\text{Var}(X_1) - \mu_N^2\mu_X^2$$

$$= \mathbb{E}[N(N-1)]\mathbb{E}(X_1)^2 + \mathbb{E}(N)\text{Var}(X_1) - \mu_N^2\mu_X^2 = \mathbb{E}[N^2 - N]\mathbb{E}(X_1)^2 + \mathbb{E}(N)\text{Var}(X_1) - \mu_N^2\mu_X^2$$

$$= [\mathbb{E}(N^2) - \mathbb{E}(N)^2 + \mathbb{E}(N)^2 - \mathbb{E}(N)]\mathbb{E}(X_1)^2 + \mathbb{E}(N)\text{Var}(X_1) - \mu_N^2\mu_X^2 =$$

$$= [\text{Var}(N) + \mathbb{E}(N)^2 - \mathbb{E}(N)]\mathbb{E}(X_1)^2 + \mathbb{E}(N)\text{Var}(X_1) - \mu_N^2\mu_X^2 = (\sigma_N^2 + \mu_N^2 - \mu_N)\mu_X^2 + \mu_N\sigma_X^2 - \mu_N^2\mu_X^2$$

$$= \boxed{(\sigma_N^2 - \mu_N)\mu_X^2 + \mu_N\sigma_X^2}$$

To use the characteristic function ϕ_X instead of the moment generating function M_X, we would do the following:

$$\phi_S(t) = \mathbb{E}(e^{itS}) = \mathbb{E}[\mathbb{E}(e^{itS} \mid N)] = \sum_{n=0}^{\infty} \mathbb{E}(e^{itS} \mid N = n)\Pr(N = n) = \sum_{n=0}^{\infty} \mathbb{E}(e^{it(X_1 + X_2 + \ldots + X_n)} \mid N = n)\Pr(N = n)$$

$$= \sum_{n=0}^{\infty} \mathbb{E}(e^{itX_1}e^{tX_2}\cdots e^{itX_n})\Pr(N = n)$$

By independence of the X_i we have

$$= \sum_{n=0}^{\infty} \mathbb{E}(e^{itX_1})\mathbb{E}(e^{itX_2})\cdots\mathbb{E}(e^{itX_n})\Pr(N = n)$$

which, since the X_i are i.i.d., can be written as

$$= \sum_{n=0}^{\infty} \mathbb{E}(e^{itX_1})^n \Pr(N = n) = \sum_{n=0}^{\infty} (\phi_X(t))^n \Pr(N = n)$$

But since $G_N(s) = \mathbb{E}(s^N) = \sum_{n=0}^{\infty} s^n \Pr(N = n)$, this can be written as

$$\phi_S(t) = G_N(\phi_X(t))$$

Homework 5 Problem 7.

(a) Let $X_1, X_2, \ldots, X_n$ be independent with mean zero and finite third moment. Prove that

$$\mathbb{E}(X_1 + \ldots + X_n)^3 = \mathbb{E}X_1^3 + \ldots + \mathbb{E}X_n^3$$

Solution.

(a) Let $\mathbb{E}(\exp(it_iX_i) = \phi_{X_i}(t_i)$. Let $S_n = \sum_{i=1}^n X_i$. Then by independence the characteristic function for S_n is

$$\mathbb{E}(\exp(itS_n)) = \phi_{S_n}(t) = \prod_{i=1}^n \phi_{X_i}(t)$$

Then

$$\mathbb{E}(X_1 + X_2 + \ldots + X_n)^3 = \mathbb{E}(S_n^3) = \phi_{S_n}^{(3)}(0)$$

$$= \sum_{i=1}^n \phi_{X_i}^{(3)}(0) \cdot \left(\prod_{j \in \{1,\ldots,n\}, j \neq i} \phi_{X_j}(0) \right) + C\left[\sum_{i=1}^n \cdot \left(\sum_{j \in \{1,\ldots,n\}, j \neq i} \phi_{X_i}^{(2)}(0)\phi_{X_j}^{(1)}(0) \right) \cdot \left(\prod_{k \in \{1,\ldots,n\}, k \neq i,j} \phi_{X_k}(0) \right) \right]$$

where C is some coefficient resulting from the multinomial expansion of S_n after repeated differentiation product rules. But because $\mathbb{E}(X_i) = 0$, $\phi_{X_i}^{(1)}(0) = 0 \ \forall i$, so the second term goes to 0. Therefore we have

$$\mathbb{E}(X_1 + X_2 + \ldots + X_n)^3 = \sum_{i=1}^n \phi_{X_i}^{(3)}(0) \cdot \left(\prod_{j \in \{1,\ldots,n\}, j \neq i} \phi_{X_j}(0) \right) = \sum_{i=1}^n \mathbb{E}(X_i^3) \cdot 1^{n-1} = \sum_{i=1}^n \mathbb{E}(X_i^3)$$

as desired.

Homework 6 Problem 10.

(a) For $p \in (0, 1)$, let $x(p)$ be the smallest number of people so that there is a better than $100 \cdot p\%$ chance to have at least two born on the same day. Find an approximate expression for $x(p)$, and sketch the graph of the function $x = x(p)$.

(b) Repeat part (a) when you want at least three people to share a birthday.

Solution.

(a) Let $f(x)$ be the probability of no matches in birthdays in a group of x people; that is,

$$f(x) = \frac{365 \cdot 364 \cdot 363 \cdots (365 - x + 1)}{365^x} = \frac{1}{365^x} \cdot \frac{365!}{(365 - x)!} = \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right) \cdots \left(1 - \frac{x-1}{365}\right)$$

Using the first order Taylor approximation $\exp(-k/x) \approx 1 - k/x$, we have

$$f(x) = \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right)\cdots\left(1 - \frac{x-1}{365}\right) \approx \exp(-1/365)\exp(-2/365)\cdots\exp(-(x-1)/365)$$

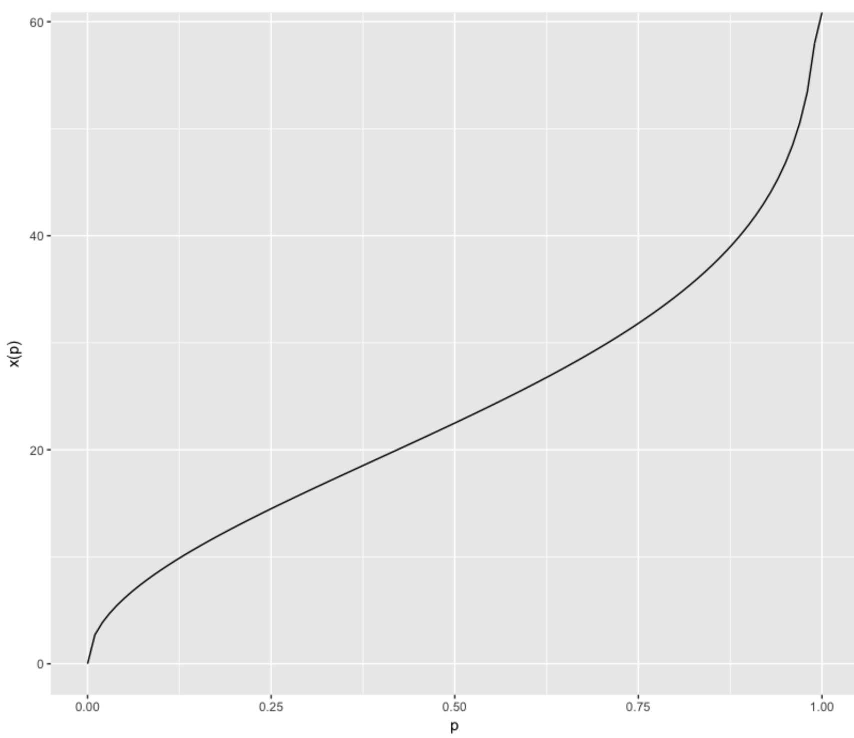
$$= e^{-(x^2-x)/(2\cdot 365)}$$

We want the probability of a match to be at least p; that is, $f(x) \leq 1 - p$. Setting this equal to $q = 1 - p$, we have

$$e^{-(x^2-x)/(2\cdot 365)} = q \iff -\frac{x^2 - x}{730} = \log(q) \iff x^2 - x + 730\log(q) = 0$$

$$\implies x = 0.5 + \sqrt{1/4 + 730\log(1/q)} \approx \boxed{\sqrt{2 \cdot 365 \log(1/q)}}$$

where we discard the negative root because we have to have a nonnegative number of people, and we don't worry about the decimals since this is an approximation and we have to round up to the nearest whole person anyway.



(b) For a group of three people, the Poisson approximation (see Section 6.1.10) is more convenient. The number of groups of 3 people in a room of x people is $\binom{x}{3}$. For a group of three people, the probability that all three have the same birthday is $1 \cdot 1/365 \cdot 1/365 = 365^{-2}$. Therefore we can think of the number of matches of three people as distributed Poisson with expectation $\binom{x}{3} \cdot 365^{-2}$. Then we have the probability of at least one "success" (triplet with three matched birthdays) is

$$1 - \frac{\exp(-\lambda)\lambda^0}{0!} = 1 - \exp\left(-\binom{x}{3} \cdot 365^{-2}\right)$$

We set this equal to p and solve:

$$p = 1 - \exp\left(- \binom{x}{3} \cdot 365^{-2} \right) \iff -\binom{x}{3} \cdot 365^{-2} = \log(1-p) \iff \frac{x!}{(x-3)!3!} = 365^2 \cdot \log\left(\frac{1}{1-p}\right)$$

$$\iff x(x-1)(x-2) = 6 \cdot 365^2 \cdot \log\left(\frac{1}{1-p}\right) \iff (x^2-x)(x-2) = x^3 - 3x^2 + 2x = 6 \cdot 365^2 \cdot \log\left(\frac{1}{1-p}\right)$$

This has a unique real solution, but it is hard to find.

Exercise 5. Let X, Y, Z be independent uniform on (0, 1). Compute the cdfs of XY, X/Y, and XY/Z.

Solution.

Using the information from part (a), and the fact that $f_X(x) = 1$ (for $x \in [0,1]$) and likewise for $f_Y(y)$:

- XY:

$$F_{XY}(z) = \int_0^\infty f_X(x) \int_{-\infty}^{z/x} f_Y(y) dy dx - \int_{-\infty}^0 f_X(x) \int_\infty^{z/x} f_Y(y) dy dx$$

$$= \int_0^1 \left[(z/x)\mathbf{1}_{\{0 < z/x \leq 1\}} + \mathbf{1}_{\{z/x > 1\}}\right] dx = \int_0^1 \left[(z/x)\mathbf{1}_{\{z \leq x\}} + \mathbf{1}_{\{z > x\}}\right] dx = \int_0^z dx + \int_z^1 (z/x) dx$$

$$= z + z\log(x)\Big|_z^1 = z + z\log(1) - z\log(z) = z(1 - \log(z))$$

$$\implies \boxed{F_{XY}(z) = \begin{cases} 0 & z \leq 0 \\ z(1-\log(z)) & 0 < z \leq 1 \\ 1 & z > 1 \end{cases}}$$

- X/Y:

$$F_{X/Y}(z) = \int_0^\infty f_Y(y) \int_{-\infty}^{zy} f_X(x) dx dy - \int_{-\infty}^0 f_Y(y) \int_\infty^{zy} f_X(x) dx dy$$

$$= \int_0^1 \left[zy\mathbf{1}_{\{0 < zy \leq 1\}} + \mathbf{1}_{\{zy > 1\}}\right] dy = \int_0^1 \left[zy\mathbf{1}_{\{y > 0 \cap y \leq 1/z\}} + \mathbf{1}_{\{y > 1/z\}}\right] dy = \int_0^{1/z} zy \cdot dy + \int_{1/z}^1 dy$$

$$= \frac{zy^2}{2}\Big|_0^{1/z} + (1 - 1/z) = \frac{z}{2z^2} + 1 - \frac{2}{2z} = 1 - \frac{1}{2z}$$

$$\implies F_{XY}(z) = \begin{cases} 0 & z \leq 0 \\ 1 - \frac{1}{2z} & 0 < 1/z \leq 1 \end{cases}$$

$$= \boxed{\begin{cases} 0 & z \leq 0 \\ z/2 & 0 < z \leq 1 \\ 1 - \frac{1}{2z} & z > 1 \end{cases}}$$

- XY/Z: Consider this the cdf of the quotient of $W = XY$ and Z.

$$F_U(u) = \int_0^\infty f_Z(z) \int_{-\infty}^{uz} f_W(w) dw dz - \int_{-\infty}^{0} f_Z(z) \int_{\infty}^{uz} f_W(w) dw dz$$

$$= \int_0^1 \int_0^{uz} -\log(w) \mathbf{1}_{\{0<uz\leq 1\}} dw dz = \int_0^1 -\big[w\log(w) - w\big]_0^{uz} \mathbf{1}_{\{0<z\leq 1/u\}} dz$$

$$= \int_0^{1/u} uz\big[1 - \log(uz)\big] dz = \frac{u}{4} z^2 \big(3 - 2\log(uz)\big)\Big|_0^{1/u} = \frac{u}{4u^2}\big(3 - 2\log(1)\big) - 0 = \frac{3}{4u}$$

$$\implies \boxed{F_{XY/Z}(u) = \begin{cases} 0 & u \leq 0 \\ \frac{3}{4u} & 0 < u \leq 3/4 \\ 1 & u > 3/4 \end{cases}}$$

7 Stochastic Processes

These notes are based on my notes from ISE 620 at USC taught by Sheldon Ross (along with the textbooks *Stochastic Processes* [Ross, 2008] and *Introduction to Probability Models* [Ross, 2014] by Sheldon Ross) *Time Series and Panel Data Econometrics* (1st edition) by M. Hashem Pesaran [Pesaran, 2015] and coursework for Economics 613: Economic and Financial Time Series I at USC, as well as notes from *Probability and Random Processes* by Grimmett and Stirzaker [Grimmett and Stirzaker, 2001].

7.1 Preliminaries

Definition 7.1. A **stochastic process** is a collection of random variables $X(t), t \geq 0$ (in the continuous case) or $X_1, X_2, \ldots$ in the discrete case such that ...

Definition 7.2. A stochastic process $\{N(t), t \geq 0\}$ is said to be a **counting process** if $N(t)$ represents the total number of events that have occurred up to time t. Hence, a counting process $N(t)$ must satisfy

(i) $N(t) \geq 0$

(ii) $N(t)$ is integer-valued.

(iii) If $s < t$ then $N(s) \leq N(t)$.

(iv) For $s < t$, $N(s) - N(t)$ equals the number of events that have occurred in the interval $(s, t]$.

Definition 7.3. We say a counting process $\{N(t), t \geq 0\}$ has **independent increments** if the numbers of events that occur in disjoint time intervals are independent; that is, for all $t_0 < t_1 < \ldots < t_n$, $N(t_1) - N(t_0), \ldots, N(t_n) - N(t_{n-1})$ are independent.

Definition 7.4. A counting process $\{N(t), t \geq 0\}$ has **stationary** increments if the distribution of the number of events that occur in any interval of time depends only on the length of the time interval. That is, if $N(t+s) - N(s)$ has a distribution that does not depend on s (is the same for all s; only depends on t).

7.2 Poisson Processes

Definition 7.5 (**Poisson Process, Grimmet and Stirzaker definition**). A **Poisson process with intensity** λ is a process $N = \{N(t) : t \geq 0\}$ taking values in $S = \{0, 1, 2, \ldots\}$ such that

(a) $N(0) = 0$; if $s < t$ then $N(s) \leq N(t)$.

(b) $\Pr(N(t+h) = n + m \mid N(t) = n) = \begin{cases} \lambda h + o(h) & \text{if } m = 1, \\ o(h) & \text{if } m > 1; \\ 1 - \lambda h + o(h) & \text{if } m = 0 \end{cases}$

(c) If $s < t$, the number $N(t) - N(s)$ of emissions in the interval $(s, t]$ is independent of the times of emissions during $[0, s]$.

Remark. λ can be interpreted as the average or long-run frequency of the Poisson process.

Definition 7.6 (Poisson Process, Ross definition, 2.1.2 in Stochastic Processes). The counting process $\{N(t) : t \geq 0\}$ is said to be a **Poisson process with rate** λ if

(a) $N(0) = 0$

(b) $\{N(t) : t \geq 0\}$ has independent increments

(c) $\Pr(N(t+h) - N(t) = 1) = \lambda h + o(h)$

(d) $\Pr(N(t+h) - N(t) \geq 2) = o(h)$

Remark. Note that a Poisson process has stationary increments.

Lemma 7.1 (ISE 620 Ross Lemma 1). Let $N(t)$ be a Poisson process. For a fixed time s, let $N_s(t) = N(s+t) - N(s)$ be the difference between two Poisson processes. Then $\{N_s(t), t \geq 0\}$ is a Poisson process.

Proof. We verify the conditions in Definition 7.6.

(a) $N_s(0) = N(s) - N(s) = 0$, QED.

(b) $\{N_s(t) : t \geq 0\}$ has independent increments, yes.

(c) $\Pr(N_s(t+h) - N_s(t) = 1) = \Pr(N(s+t+h) - N_s(s+t) = 1) = \lambda h + o(h)$, QED.

(d) $\Pr(N_s(t+h) - N_s(t) \geq 2) = (\Pr(N(s+t+h) - N_s(s+t) \geq 2) = o(h)$, QED.

□

Lemma 7.2 (ISE 620 Ross Lemma 2). Let $N(t)$ be a Poisson process. Then $P_0(t) = \Pr(N(t) = 0) = e^{-\lambda t}$.

Proof.

$$P_0(t+h) = \Pr(N(t+h) = 0) = \Pr(N(t+h) = 0, N(t) = 0)$$

$$= \Pr(N(t) = 0) \Pr(N(t+h) - N(t) = 0) \text{ (by independent increments)}$$

Using conditions (iii) and (iv) of Definition 7.6,

$$P_0(t+h) = P_0(t)(1 - \lambda h - o(h)) = P_0(t)(1 - \lambda h) + o(h)$$

$$\iff \frac{P_0(t+h) - P_0(t)}{h} = \frac{-\lambda h P_0(t)}{h} + \frac{o(h)}{h}$$

Taking the limit as $h \to 0$, we have

$$P_0'(t) = -\lambda P_0(t) \iff \frac{P_0'(t)}{P_0(t)} = -\lambda \iff \log(P_0(t)) = -\lambda t + C$$

$$\iff P_0(t) = \lambda e^{-\lambda t} \iff P_0(0) = 1$$

□

Theorem 7.3. [**Grimmett and Stirzaker theorem 6.8.2**] Let $N(t)$ be a Poisson process with intensity λ. Then $N(t)$ has the Poisson distribution with parameter λt; that is,

$$\Pr(N(t) = j) = \frac{(\lambda t)^j \exp(-\lambda t)}{j!}, \quad j = 0, 1, 2, \dots$$

Proof. See Grimmett and Stirzaker section 6.8.2, page 247. Ross proof:

$$\Pr(N(t) = n) = \frac{1}{\Pr(X_{n+1} = t - s \mid X_n = s)} \int_0^t \Pr(N(t) = n \mid S_n = s)\lambda e^{-\lambda s} \frac{(\lambda s0^{n-1}}{(n-1)!} ds = 1$$

$$= \Pr(X_{n+1} > t - s) = e^{-\lambda(t-s)}$$

$$\Pr(N(t) = n) = \int_0^t e^{-\lambda(t-s)} \lambda e^{-\lambda s} \frac{(\lambda s)^{n-1}}{(n-1)!} ds$$

$$= \frac{e^{-\lambda t}\lambda^n}{(n-1)!} \int_0^t s^{n-1} ds = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

□

Corollary 7.3.1. $N(t + s) - N(s) \sim \text{Poisson}(\lambda t)$

Proof. By Lemma 7.1, ?? is a Poisson process. . . .

□

Definition 7.7 (**Poisson Process, Ross definition, 2.1.1 in Stochastic Processes**)**.** The counting process $\{N(t) : t \geq 0\}$ is said to be a **Poisson process with rate** λ if

(a) $N(0) = 0$

(b) $\{N(t) : t \geq 0\}$ has independent increments

(c) The number of events in any interval of length t is Poisson distributed with mean λt. That is, for all $s, t \geq 0$,

$$\Pr(N(t + s) - N(s) = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \quad n = 0, 1, \dots$$

Remark. Note that it follows from condition (iii) in Definition 7.7 that a Poisson process has stationary increments and also that $\mathbb{E}(N(t)) = \lambda t$.

Definition 7.8 (**Ross Stochastic Processes definition, Section 2.2**). Let X_n denote the time between the $n-1$st and nth event in a Poisson process. The sequence $\{X_n, n \geq 1\}$ is called the **sequence of interarrival times**.

Definition 7.9 (**Grimmett and Stirzaker definition**). Let $N(t)$ be a Poisson process with intensity λ. Let $T_0, T_1, \ldots$ be given by

$$T_0 = 0, T_n = \inf\{t : N(t) = n\} \tag{7.1}$$

so that T_n is the time of the nth arrival. The **interarrival times** are the random variables $X_1, X_2, \ldots$ given by

$$X_n = T_n - T_{n-1}. \tag{7.2}$$

Remark. From knowledge of N, we can find the values of $X_1, X_2, \ldots$ by (7.1) and (7.2). Conversely, we can construct N from knowledge of the X_i by

$$T_n = \sum_{i=1}^{n} X_i, \quad N(t) = \max\{n : T_n \leq t\} \tag{7.3}$$

Theorem 7.4. (Grimmett and Stirzaker theorem 6.8.10.) Let $N(t)$ be a Poisson process with intensity λ. Let $T_0, T_1, \ldots$ be given by (7.1) and let X_n be given by (7.2). Then then random variables $\{X_n\}$ are independent, each having an exponential distribution with parameter λ.

Proof. See Grimmett and Stirzaker section 6.8.2, page 249. □

Corollary 7.4.1. Let $N(t)$ be a Poisson process with intensity λ. Let $T_0, T_1, \ldots$ be given by (7.1). Then $T_n \sim \text{Gamma}(n, \lambda^{-1})$.

Proof. By (7.3), $T_n = \sum_{i=1}^n X_i$. $X_i \sim \text{Exponential}(\lambda)$ by Theorem 7.4, which means $X_i \sim \text{Gamma}(1, \lambda^{-1})$. Then by Proposition 6.58, $T_n \sim \text{Gamma}(n, \lambda^{-1})$. □

Lemma 7.5 (**ISE 620 Ross in-class Lemma 3, Proposition 2.2.1 in Stochastic Processes**). Let $N(t)$ be a Poisson process. Let $X_1, X_2, \ldots$ be the interarrival times. Then $X_1, X_2, \ldots$ are independently and identically distributed as Exponential(λ). Since the sum of exponential distributions is Gamma, we have $S_n = \sum_{i=1}^n X_i \sim \text{Gamma}(n, 1/\lambda)$.

Proof. See proof on page 64 (section 2.2) of *Stochastic Processes*. Follows from Lemmas 7.1 and 7.2. □

Remark. If the arrival times are IID exponential, then the process is Poisson. (Will come later in notes.)

Remark.

$$X_1 < t \iff N(t) = t$$

$$\Pr(X_2 > t \mid X_a = s) = \Pr(N(S+t) - N(s) = 0 \mid X_1 = s) = \Pr(N(s+t) - N(s) = 0) = \Pr(N_s(t) = 0) = e^{-\lambda t}$$

Theorem 7.6. Suppose events occur in a Poisson process with parameter lambda, PP(λ). Let $N(t)$ be the number of events that occur by time t. Suppose each event is independent of all that has previously occurred; that is, each event is type $i = 1, \ldots, r$ with probability p_i, and $\sum_{i=1}^{r} p_i = 1$. Let $N_i(t)$ be the number of type i events that occur by time t. $\{N_i(t)\} \sim$ PP(λp_i). Moreover, these processes are independent for different i.

Proof. Verify axioms of Poisson process:

1. $N_i(0) = 0$
2. independent increments? Yes, the number of type i events in each interval is totally independent of what happened in previous intervals.
3. $\Pr(N_i(t+h) - N_i(t) = 1) = \Pr(N(t+h) - N(t) = 1, \text{ event is type } i + + \ldots$

   probability of two or more events (and then one of them is type i) is $o(h)$.

$$= \lambda h p_i + o(h)$$

4. $\Pr(N_i(t+h) - N_i(t) \geq 2) \leq \Pr(N(t_h) - N(t) \geq 2) = o(h)$

Therefore $N_i(t)$ is a Poisson process with rate λp_i. We could have also proven this by looking at the inter-arrival times and showing that they are i.i.d. exponential.

□

Proof: Alternative, stronger statement.

$$\Pr(N_1(t) = n_1, N_2(t) = n_2, \ldots, N_r(t) = n_r)$$

we want to show that these random variables are independent. Let $n = \sum_{i=1}^{r} n_i$. Then

$$= \Pr(N_1(t) = n_1, \ldots, N_r(t) = n_r \mid N(t) = n) \Pr(N(t) = n)$$

But this is a Multinomial distribution. So we have

$$= \frac{n!}{n_1! \cdots n_r!} \prod_{i=1}^{r} p_i^{n_i} \cdot e^{-\lambda t} \frac{(\lambda t)^{\sum_i n_i}}{n!}$$

$$\prod_{i=1}^{r} e^{-\lambda t p_i} \frac{(\lambda t p_i)^{n_i}}{n_i!}$$

And these are just Poisson probabilities.

Then using Proposition 7.7, we have independence (???) □

Proposition 7.7 (offhand claim Ross made)**.** If $\Pr(X = n, Y = m) = g(n)h(m)$ (that is, if probability can be broken into products of functions) then variables are independent.

Proof.

$$\Pr(X = n) = \sum_m g(n)h(m) = g(n)C_h$$

$$\Pr(Y = m) = h(m)\sum_n g(n) = h(m)C_g$$

$$1 = \sum_n \sum_m g(n)h(m) = C_g C_h$$

So this is true as long as $C_g C_h = 1$ which it does.

$\square$

Example 7.1. Suppose in particular $\lambda = 10$ so we expect 10 people to arrive every hour, either men or women. What is the expected number of women to arrive given that 7 men arrived? (5—women and men's arrivals are independent.)

Example 7.2 (**Coupon collecting problem, see also probability notes.**)**.** r types of coupons collected with probability $p_1, \ldots, p_r$. Let N be the number of coupons you collect until you have a complete set. What is $\mathbb{E}(N)$?

Solution. Let $m(\mathcal{S})$ be the mean number of draws required to obtain at least one coupon of type i for each $i \in \mathcal{S}$. Note that $m(\emptyset) = 0$, and

$$m(\mathcal{S}) = 1 + \sum_{i \in \mathcal{S}} p_i m(\mathcal{S} \setminus i) + \sum_{i \notin \mathcal{S}} p_i m(S)$$

$$\implies m(\mathcal{S}) = \frac{1 + \sum_{i \in \mathcal{S}} p_i m(\mathcal{S} \setminus i)}{\sum_{i \notin \mathcal{S}} p_i}$$

But unless r is small this is going to be hard to compute, so this doesn't really work.

Solution. New idea: Let N_i be the number of draws required to obtain types $1, \ldots, i$. We want N_r. Note that $\mathbb{E}(N_1) = 1/p_1$. We have $N_{i+1} = N_i + A$ where A is the additional time required.

$$\mathbb{E}(N_{i+1}) = \mathbb{E}(N_i) + \mathbb{E}(A)$$

if there is a type $i + 1$ in the original group, then $\mathbb{E}(A) = 0$. If not, note that

$$\mathbb{E}(A) = \begin{cases} 1/(p_i + 1) & \Pr(\{\text{a type } i + 1 \text{ coupon has not already been collected}\}) \\ 0 & Pr(\{\text{a type } i + 1 \text{ coupon has already been collected}\}) \end{cases}$$

$$\implies \mathbb{E}(A) = \frac{1}{p_i + 1} \cdot \Pr(\{\text{a type } i+1 \text{ coupon has not already been collected}\})$$

$$\implies \mathbb{E}(A) = \frac{1}{p_i + 1} \cdot \Pr(\{\text{type } i+1 \text{ is the last of types } 1, \ldots, i+1 \text{ to be collected}\})$$

$$= \frac{1}{p_i + 1} \cdot \frac{\sum_{j=1}^{i+1} p_j}{p_{i+1}}$$

this didn't work either.

Solution.

Let N_i be the number to get type i. So $N_i \sim \text{Geometric}(p_i)$. Then $N = \max\{N_i\}$. So

$$\Pr(N \leq k) = \Pr(N_1 \leq k, \ldots, N_r \leq k)$$

but you can't multiply out probabilities because N_i is not independent.

7.2.1 Poissonization Trick

Suppose we collect coupons at times distributed according to a Poisson process with rate $\lambda = 1$. Let event type i be the event of collecting a type i coupon. Let T_i be the time until collecting a type i coupon. Note that by Theorem 7.6 and Lemma 7.5, $T_i \sim \text{Exponential}(\lambda \cdot p_i) = \text{Exponential}(p_i)$. Then the time to collect at least one of every type is $T = \max_i\{T_i\}$. And by Theorem 7.6, the T_i are all independent. Return to the coupon collecting problem:

Solution. Recall the layer cake formulation for expected value:

$$\mathbb{E}(T) = \int_0^\infty \Pr(T > t) dt$$

We have (using independence of the T_i)

$$\Pr(T > t) = 1 - \Pr(T \leq t) = 1 - \Pr(T_i \leq t, \ldots, T_r \leq t) = 1 - \prod_{i=1}^{r} \Pr(T_i \leq t) = 1 - \prod_{i=1}^{r} (1 - e^{-p_i t})$$

$$\implies \boxed{\mathbb{E}(T) = \int_0^\infty \left(1 - \prod_{i=1}^{r} (1 - e^{-p_i t})\right) dt}$$

$$= \sum_i \frac{1}{p_i} - \sum_{i<j} \frac{1}{p_i + p_j} + \sum_{i,j,k} \frac{1}{[} p_i + p_j + p_k] - \cdots + \frac{(-1)^{r+1}}{p_1 + \ldots + p_r}$$

But we want $\mathbb{E}(N)$. Recall that T is the time of the Nth event. So

$$T = \sum_{i=1}^{N} X_i$$

where $X_1, X_2, \ldots$ are the interarrival times of a Poisson process with rate $\lambda = 1$. But N is independent of $X_1, X_2, \ldots$ (the type of coupons you get has nothing to do with the time between coupons). So T is a compound random variable. Therefore

$$\mathbb{E}(T) = \mathbb{E}(N)\mathbb{E}(X) = \mathbb{E}(N) \cdot \frac{1}{\lambda} = \mathbb{E}(N)$$

Therefore we have $\boxed{\mathbb{E}(N) = \int_0^\infty \left(1 - \prod_{i=1}^{r}(1 - e^{-p_i t})\right) dt}$.

Example 7.3. A type is a *singleton* if after you get a complete set, you have only one object of that type. Suppose we collect coupons until we have at least one of every type, and that all coupons are equally likely to be collected on each draw. What is the expected number of singletons when you stop?

Solution. Let X be the number of singletons when you stop. Let I_j be an indicator variable for type j being a singleton. Then

$$\mathbb{E}(X) = \mathbb{E}\left[\sum_{j=1}^{r} I_j\right] = \sum_{j=1}^{r} \mathbb{E}(I_j) = \sum_{j=1}^{r} \Pr(\{j \text{ is a singleton}\})$$

$$\vdots$$

$$= \frac{1}{r} \sum_{i=1}^{r} \frac{1}{r - i + 1}$$

$$\vdots$$

Let T_i be the time you get your first type i coupon. The probability we want is that at the time when you have at least one of every type except j, you have either 0 or 1 coupons of type j. That is, if $S_2^{(j)}$ is the time the second card of type j is selected, we must have $S_2^{(j)} > \max_{i \neq j}\{T_i\}$. So we seek $\Pr(S_2^{(j)} > \max_{i \neq j}\{T_i\})$. Note that $S_2^{(j)} \sim \text{Gamma}(2, p_j)$ by Corollary 7.4.1, so

$$f_{S_2^{(j)}}(s) = \frac{1}{p_j^{-2}} s^{2-1} e^{-p_j s} = p_j e^{-p_j s} p_j \cdot s.$$

So we have

$$\Pr(I_j = 1) = \Pr(S_2^{(j)} > \max_{i \neq j}\{T_i\}) = \int_0^\infty \Pr(\max_{i \neq j}\{T_i\} < S_2^{(j)} \mid S_2^{(j)} = s) \cdot f_{S_2^{(j)}}(s) ds$$

$$= \int_0^\infty \prod_{i \neq j} (1 - e^{-p_i s}) \cdot p_j e^{-p_j s} p_j \cdot s \cdot ds$$

which yields

$$\boxed{\mathbb{E}(X) = \sum_{j=1}^{r} p_j^2 \int_0^\infty e^{-p_j s} \cdot s \cdot \prod_{i \neq j} (1 - e^{-p_i s}) \cdot ds}$$

Remark. Per Example 7.4, suppose people arrive at a bus stop according to a Poisson process with rate λ. A bus arrives at (fixed) time T. Let W be the sum of the waiting times for everyone at the bus stop. Let S_j be the arrival time of the jth person. (Note that if X_i are the interarrival times then $S_j = \sum_{i=1}^{j} X_i$.) The number of people who get on the bus is $N(t)$, the Poisson counting process. So

$$W = \sum_{i=1}^{N(T)} (T - S_i) = N(T)T - \sum_{i=1}^{N(T)} S_i$$

Proposition 7.8. In a Poisson process, if we know that one event has occurred by time t, then the distribution of times of the event is uniform between 0 and t. That is,

$$S_1 < x \mid N(t) = 1 \sim \mathrm{U}(0, t)$$

Proof.

$$\Pr(S_1 < x \mid N(t) = 1) = \frac{\Pr(S_1 < x \cap N(t) = 1)}{\Pr(N(t) = 1)} = \frac{\Pr(N(x) = 1 \cap N(t) - N(x) = 0)}{\exp(-\lambda t)\lambda t}$$

$$= \frac{\Pr(N(x) = 1) \Pr(N(t) - N(x) 0}{\exp(-\lambda t)\lambda t} = \frac{\lambda x e^{-\lambda x} \cdot e^{-\lambda(t-x)}}{\exp(-\lambda t)\lambda t} = \frac{x}{t}$$

which is the cdf for a random variable distributed as $\mathrm{U}(0, t)$.

□

Remark. If $X_i \sim \mathrm{U}(0, t)$, then $f(x) = 1/t, \quad 0 < x < t$, so per Proposition 9.2,

$$f_{X_{(1)}, \ldots, X_{(n)}}(x_1, \ldots, x_n) = \frac{n!}{t^n}, \quad 0 < x < t$$

Theorem 7.9. In a Poisson process, if we know that n events have occurred by time t ($N(t) = n$), then the set of event times $\{S_1, \ldots, S_n\}$ are distributed as a set of n i.i.d. uniform random variables between 0 and t. That is, the unordered times are distributed uniformly on the interval.

More precisely: Given $N(t) = n$, $S_1, \ldots, S_n$ are distributed as order statistics of n i.i.d. $\mathrm{U}(0, t)$ random variables.

Proof. We will examine the density function

$$f_{S_1,\ldots,S_n,|N(t)=n}(t_1,\ldots,t_n), \quad 0 < t_1 < \ldots < t_n < t$$

Per the above remark, we want to show that this is $n!/t^n$. Note that by Bayes' Theorem

$$f_{S_1,\ldots,S_n,|N(t)=n}(t_1,\ldots,t_n) = \frac{f_{S_1,\ldots,S_n}(t_1,\ldots,t_n)\cdot \Pr(N(t)=n \mid S_1=t_1,\ldots,S_n=t_n)}{\Pr(N(t)=n)}. \tag{7.4}$$

Let $X_1, X_2, \ldots$ be the interarrival times. So the condition above is equivalent to $X_1 = t_1, X_2 = t_2 - t_1, \ldots, X_n = t_n - t_{n-1}$. Recall that all the $\{X_i\}$ are independent. So we have

$$f_{S_1,\ldots,S_n}(t_1,\ldots,t_n) = f_{X_1,\ldots,X_n}(t_1, t_2-t_1,\ldots,t_n-t_{n-1})$$

Also, we can interpret $\Pr(N(t) = n \mid S_1 = t_1,\ldots,S_n = t_n)$ as the probability that there are 0 arrivals between times t_n and t; that is,

$$\Pr(N(t)=n \mid S_1=t_1,\ldots,S_n=t_n) = e^{-\lambda(t-t_n)}$$

So we can write (7.4) as

$$f_{S_1,\ldots,S_n,|N(t)=n}(t_1,\ldots,t_n) = \frac{f_{X_1,\ldots,X_n}(t_2-t_1,\ldots,t_n-t_{n-1})\cdot e^{-\lambda(t-t_n)}}{e^{-\lambda t}(\lambda t)^n/n!}.$$

$$= \frac{\lambda e^{-\lambda t_1}\lambda e^{-\lambda(t_2-t_1)}\cdots\lambda e^{-\lambda(t_n-t_{n-1})}\cdot e^{-\lambda(t-t_n)}}{e^{-\lambda t}(\lambda t)^n/n!}.$$

$$= \frac{e^{-\lambda t}\lambda^n}{e^{-\lambda t}(\lambda t)^n/n!} = \frac{n!}{t^n}$$

□

Example 7.4. Suppose people arrive at a bus stop according to a Poisson process with rate λ. A bus arrives at (fixed) time T. What is the expected value of W, the sum of the waiting times for everyone at the bus stop?

Solution. Let S_j be the arrival time of the jth person. (Note that if X_i are the interarrival times then $S_j = \sum_{i=1}^j X_i$.) The number of people who get on the bus is $N(t)$, the Poisson counting process. So

$$W = \sum_{j=1}^{N(T)}(T - S_j) = N(T)T - \sum_{j=1}^{N(T)} S_j$$

$$\mathbb{E}(W) = \mathbb{E}(N(T)T) - \mathbb{E}\left(\sum_{j=1}^{N(T)} S_j\right) = \lambda T^2 - \mathbb{E}\left(\sum_{j=1}^{N(T)} S_j\right)$$

Note that by Theorem 7.9, if $\{U_i\}$ are i.i.d. uniform random variables on $[0, t]$,

$$\mathbb{E}\left(\sum_{j=1}^{N(T)} S_j \mid N(T) = n\right) = \mathbb{E}\left(\sum_{j=1}^{n} S_j \mid N(T) = n\right) = \mathbb{E}\left(\sum_{j=1}^{n} U_i\right);$$

that is, if we know that n arrivals have occurred by time T, then the (unordered) arrival times are distributed uniformly on $[0, T]$. But

$$\mathbb{E}\left(\sum_{j=1}^{n} U_i\right) = \sum_{j=1}^{n} \mathbb{E}U_i = \frac{nT}{2}$$

Also note that because the U_i is independent from $N(T)$ (the arrival time of the jth person has nothing to do with how many people arrive by time T), $\sum_{j=1}^{N(T)} S_j$ is a compound random variable, which means

$$\mathbb{E}\left(\sum_{j=1}^{N(T)} S_j\right) = \mathbb{E}\left[\mathbb{E}\left(\sum_{j=1}^{N(T)} S_j \mid N(T) = n\right)\right] = \mathbb{E}\left(\frac{N(T)T}{2}\right) = \boxed{\frac{\lambda T^2}{2}}$$

$\vdots$

Note that the sum of the ordered values is equal to the sum of the unordered values, so we have

$$\mathbb{E}\left(\sum_{i=1}^{n} X_{(i)}\right) = \mathbb{E}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathbb{E}(X_i) = n\mu$$

Example 7.5. Another thing: Let I_j be an indicator variable for the event $X_{(j)} < t$. We are interested in the distribution of $\sum_{j=1}^{n} I_j$.

Solution. Note that the I_j is not independent because if $I_1 = 0$ (so the smallest one is greater than t) then it must be the case that $I_2 = 0$ too. Also note that the sum of the ordered values less than t is the same as the sum of the unordered values less than t. So this is distributed as binomial with parameters n and $p = F(t)$.

Example 7.6. Suppose that shocks occur according to a Poisson process with rate λ. Let D_i be the damage caused by shock i, where $\{D_i\}$ are i.i.d. and independent of $\{N(t)\}$. The damages dissipate at an exponential rate α; that is, damage of value d has value de^{-as} after a time s. Damages are cumulative. What is the total damage by time t?

Solution. We have

$$D(t) = \sum_{i=1}^{N(t)} D_i e^{-a(t-S_i)}$$

so

$$\mathbb{E}(D(t) \mid N(t) = n) = \mathbb{E}\left(\sum_{i=1}^{N(t)} D_i e^{-a(t-S_i)} \mid N(t) = n \right) = \sum_{i=1}^{N(t)} \mathbb{E}\big(D_i e^{-a(t-S_i)} \mid N(t) = n\big)$$

$$= \sum_{i=1}^{N(t)} \mathbb{E}(D_i)\mathbb{E}\big(e^{-a(t-S_i)} \mid N(t) = n\big) = \mathbb{E}(D_i)e^{-at} \sum_{i=1}^{n} \mathbb{E}\big(e^{aS_i} \mid N(t) = n\big)$$

Let $\{U_i\}$ be i.i.d. random variables on $[0, t]$. Then by Theorem 7.9, we can write this as

$$= \mathbb{E}(D_i)e^{-at} \sum_{i=1}^{n} \mathbb{E}\big(e^{aU_i} \mid N(t) = n\big) = \mathbb{E}(D_i)e^{-at} \cdot n \int_0^t \frac{e^{ax}}{t} dx = \mathbb{E}(D_i)e^{-at} \cdot \frac{n}{at}(e^{at} - 1)$$

$$= \mathbb{E}(D_i) \cdot \frac{n}{at}(1 - e^{-at})$$

So finally we have

$$\mathbb{E}(D(t)) = \mathbb{E}[\mathbb{E}(D(t) \mid N(t) = n)] = \mathbb{E}\left[\mathbb{E}(D_i) \cdot \frac{N(t)}{at}(1 - e^{-at})\right] = \mathbb{E}(D_i) \cdot \frac{\mathbb{E}(N(t))}{at}(1 - e^{-at}) = \boxed{\mathbb{E}(D_i) \cdot \frac{\lambda}{a}(1 - e^{-at})}$$

7.2.2 Time Sampling Poisson Processes

Proposition 7.10 (**Time Sampling (Proposition 5.3 in Introduction to Probability Models)**). *Suppose we have events that happen as a Poisson process with rate λ. Each event is of type $1, \ldots, r$ independently of what has come before. An event at time s is type i with probability $P_i(s)$. Let $N_i(t)$ be the number of type i events by time t. Then $N_1(t), \ldots, N_r(t)$ are independent Poisson random variables with mean $\mathbb{E}[N_i(t)] = \lambda \int_0^t P_i(s)ds$.*

Proof. **I'm not sure this proof makes sense?** Note that

$$\Pr(N_1(t) = n_1, \ldots, N_r(t) = n_r) = \Pr(N_1(t) = n_1, \ldots, N_r(t) = n_r \mid N(t) = \sum_{i=1}^{r} n_i) \cdot \Pr\left(N(t) = \sum_{i=1}^{r} n_i\right).$$

Given $N(t) = n$, the n event times are i.i.d. $U(0, t)$ by Theorem 7.9. Let P_i be the probability that a particular event is of type i given a time t and the fact that $N(t) = n$. Then

$$P_i = \int_0^t P_i(s) \cdot f_U(s) ds = \int_0^t \frac{P_i(s)}{t} ds$$

Since $\Pr\left(N(t) = \sum_{i=1}^{r} n_i\right) = e^{-\lambda t}$, we have

$$\Pr(N_1(t) = n_1, \ldots, N_r(t) = n_r) = \frac{n!}{n_1! \cdots n_r!} P_1^{n_1} \cdot \ldots \cdot P_r^{n_r} e^{-\lambda t}$$

□

Remark. Notation for Queueing: in many cases we suppose times between successive arrivals are independent with a common distribution F. Then it is a renewal process. (If they are exponential, we get a Poisson process.) We have the following notation for a renewal process if k number of servers and G is the distribution of the service time:

$$F/G/k$$

If the distribution is exponential, we use M (for "memoryless" or "Markovian"). (E stands for Erlang, sum of i.i.d. exponentials).

Example 7.7. We have an $M/G/\infty$ queuing system. Arrivals are a Poisson process with rate λ. Fix t and let $X(t)$ be the number of people in the system at time t. Let $Y(t)$ be the number of people who have completed service at time t. Let the events be arrivals of customers, and call it a Type 1 event if the customer is still in the system at time t, and Type 2 if the customer completes service by time t (and Type 3 otherwise).

Suppose a customer arrives at time $s < t$. Then the customer is Type 1 if their service time exceeds $t - s$, which happens with probability $P_1(s) = \overline{G}(t - s)$. Similarly $P_2(s) = G(t - s)$. Let $N_1(t)$ be the number of Type I events that happen by time t and similarly for $N_2(t)$. Then by Proposition 7.10 we have

$$X(t) = N_1(t) \sim \text{Poisson}\left(\lambda \int_0^t \overline{G}(t - s) ds\right) = \text{Poisson}\left(\lambda \int_0^t \overline{G}(y) dy\right)$$

and

$$Y(t) = N_2(t) \sim \text{Poisson}\left(\lambda \int_0^t G(t - s) ds\right) = \text{Poisson}\left(\lambda \int_0^t G(y) dy\right)$$

which implies the independence of $X(t)$ and $Y(t)$.

7.2.3 Nonstationary Poisson Processes

Definition 7.10 (**Nonstationary Poisson process.**). The counting process $\{N(t), t \geq 0\}$ is said to be a **nonstationary Poisson process** (or **nonhomoegeneous Poisson process**) with intensity function $\lambda(t), t \geq 0$ if

(i) $N(0) = 0$.

(ii) $\{N(t)\}$ has independent increments.

(iii) $\Pr(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$.

(iv) $\Pr(N(t+h) - N(t) \geq 2) = o(h)$.

Remark. Note the similarities to Definition 7.6.

Lemma 7.11. Let $\{N(t), t \geq 0\}$ be a nonstationary Poisson process. Let $N_s(t) = N(s+t) - N(s)$. Then $\{N_s(t), t \geq 0\}$ is a nonstationary Poisson Process with intensity $\lambda_s(t) = \lambda(s+t)$.

Remark. Note the similarity to Lemma 7.1.

Proof. Note the parts of Definition 7.10 above:

(i) $N_s(0) = N(s) - N(s) = 0$.

(ii) $\{N_s(t)\} = \{N(s+t) - N(s)\}$ has independent increments since $N(t)$ has independent increments.

(iii) **Not sure about this part?** $\Pr(N_s(t+h) - N_s(t) = 1) = \Pr(N(s+t+h) - N(s) - [N(s+t) - N(s)] = 1) = \Pr(N(s+t+h) - N(s+t) = 1) = \lambda(t)h + o(h)$.

(iv) $\Pr(N_s(t+h) - N_s(t) \geq 2) = o(h)$ by similar argument to (iii).

$\square$

Definition 7.11. Let $m(t) = \int_0^t \lambda(s)ds$. Note that $m'(t) = \lambda(t)$. We call $m(t)$ the **mean value function.**

Remark. Note that the mean value function for $N_s(t)$ s

$$m_s(t) = \int_0^t \lambda_s(y)dy = \int_0^t \lambda(s+y)dy = \int_s^{s+t} \lambda(x)dx = m(t+s) - m(s).$$

Lemma 7.12. Let $N(t)$ be a nonstationary Poisson process. Then $\Pr(N(t) = 0) = e^{-m(t)}$ for any $t \geq 0$.

Remark. Note the similarity to Lemma 7.2.

Proof. Let $P(t) = \Pr(N(t) = 0)$. Then by independent increments

$$\Pr(N(t+h) = 0) = \Pr(N(t) = 0 \cap N(t+h) - N(t) = 0) = \Pr(N(t) = 0) \cdot \Pr(N(t+h) - N(t) = 0) \quad (7.5)$$

Let $P_0(t+h) = \Pr(N(t+h) = 0)$. Then using Definition 7.10 we have

$$\Pr(N(t+h) - N(t) = 0) = 1 - \Pr(N(t+h) - N(t) = 1) - \Pr(N(t+h) - N(t) \geq 2) = 1 - \lambda(t+h) + o(h)$$

so we can write (7.5) as

$$P_0(t+h) = P_0(t)[1 - \lambda(t+h) + o(h)]$$

$$\iff \frac{P_0(t+h) - P_0(t)}{h} = -\lambda(t+h)\frac{P_0(t)}{h} + \frac{o(h)}{h}$$

Taking the limit as $h \to 0$ yields

$$P_0'(t) = -\lambda(t)P_0(t) \iff \int_0^s \frac{P_0'(t)}{P_0(t)}ds = \int_0^s -\lambda(t)dt \iff \log P_0(t) = \int_0^s -\lambda(t)dt\Big]_0^s$$

$$\iff P_0(s) = e^{-m(s)}$$

□

Remark (Interarrival times). Let T_1 be the time of the first event. Note that $T_1 > t \iff N(t) = 0$. So

$$\overline{F}_{T_1}(t) = \Pr(T_1 > t) = \Pr(N(t) = 0) = e^{-m(t)}$$

which means

$$f_{T_1}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\big(-\overline{F}_{T_1}(t)\big) = m'(t)e^{-m(t)} = \lambda(t)e^{-m(t)}.$$

Note that because $\lambda(t)$ is not constant, the interarrival times are not i.i.d.

Theorem 7.13. Let $N(t)$ be the counting process for a nonstationary Poisson process. Then $N(t) \sim \text{Poisson}(m(t))$.

Proof. We must show that

$$\Pr(N(t) = n) = e^{-m(t)}\frac{(m(t))^n}{n!}, \quad n = 0, 1, \ldots$$

We have already shown that $\Pr(N(t) = 0) = e^{-m(t)}\frac{(m(t))^0}{0!} = e^{-m(t)}$ this is true when $n = 0$ in Lemma 7.12. We will show that this expression holds for $n = 1, 2, \ldots$ by induction. Assume for a fixed n

$$\Pr(N(t) = n) = e^{-m(t)}\frac{(m(t))^n}{n!}.$$

We seek

$$\Pr(N(t) = n+1) = \int_0^t \Pr(N(t) = n+1 \mid T_1 = s)f_{T_1}(s)ds \tag{7.6}$$

Note that (using the property of independent increments)

$$\Pr(N(t) = n+1 \mid T_1 = s) = \Pr(N(t) - N(s) = n \mid T_1 = s) = \Pr(N(t) - N(s) = n)$$

$$= \Pr(N_s(t-s) = n)$$

By Lemma 7.11 above, $N_s(\cdot)$ is a Poisson process. So using that and the induction hypothesis, we have

$$\Pr(N(t) = n+1 \mid T_1 = s) = e^{-m_s(t-s)} \frac{(m_s(t-s))^n}{n!} = e^{-[m(t)-m(s)]} \frac{[m(t)-m(s)]^n}{n!}.$$

Substituting this expression back in to (7.6) and using $f_{T_1}(t) = \lambda(t)e^{-m(t)}$, we have

$$\Pr(N(t) = n+1) = \int_0^t e^{-[m(t)-m(s)]} \frac{[m(t)-m(s)]^n}{n!} \cdot \lambda(s) e^{-m(s)} ds$$

$$= \frac{e^{-m(t)}}{n!} \int_0^t [m(t)-m(s)]^n \cdot \lambda(s) ds$$

Substituting $y = m(t) - m(s) \implies dy = -\lambda(s)ds$, we have

$$= \frac{e^{-m(t)}}{n!} \int_{m(t)}^0 -y^n dy = \frac{e^{-m(t)}}{n!} \int_0^{m(t)} y^n dy = e^{-m(t)} \frac{(m(t))^{n+1}}{(n+1)!}.$$

Therefore the result follows by induction. □

Corollary 7.13.1. $N(t+s) - N(s) \sim \text{Poisson}(m(t+s) - m(s)) = \text{Poisson}(\int_s^{s+t} \lambda(y)dy)$.

Proof. Follows almost immediately from Theorem 7.13, since if $N(t) \sim \text{Poisson}(m(t))$,

□

Proposition 7.14. Suppose events occur according to a Poisson process with rate λ. $\{N(t)\}$ is the counting process. An event at time s is type 1 with probability $p(s)$. Let $N_1(t)$ be the number of type 1 events by time t. Then $\{N_1(t)\}$ is a nonhomogeneous Poisson Process with intensity $\lambda(t) = \lambda p(t)$.

Remark. Note the similarity to Proposition 7.10.

Proof. Note the parts of Definition 7.10 above:

(i) $N_1(0) = 0$: yes.

(ii) $\{N_1(t)\}$ has independent increments: yes, follows pretty much immediately.

(iii) Since the probability of exactly one event in the interval that happens to be type 1 is $\lambda h P_1(t)$, we have $\Pr(N_1(t+h) - N_1(t) = 1) = \lambda h P_1(t) = \lambda(t)h + o(h)$.

(iv) $\Pr(N_1(t+h) - N_1(t) \geq 2) = o(h)$ by similar argument to (iii).

□

So now we have that $N_1(t) \sim \text{Poisson}(\int_0^t \lambda p_1(s) ds)$. So if every time an event happens it is of type i with a certain probability that varies over time, the counting process is for each is a nonstationary Poisson process, and all the processes are independent. (Another way to understand the time sampling result from before.)

What if the arrival process is nonstationary and the probability of a type i event is also time varying? $\lambda(t)$, $p(t)$. It's a nonstationary PP with intensity $\lambda(t)p(t)$. (Probability in an interval is $\lambda(t)hp(t) + o(h)$.

One more result about $M/G/\infty$ processes:

Theorem 7.15 (**Example 5.25 in *Introduction to Probability Models***)**.** The **departure process** from an $M/G/\infty$ queue is a nonstationary Poisson process with intensity function $\lambda(t) = \lambda G(t)$.

Proof. Let $D(t)$ be the number of departures by time t. Examine the axioms of Definition 7.10 above:

(i) $D(0) = 0$: yes.

(ii) $\{D(t)\}$ has independent increments: think of each arrival as an event. Types: type 1 if they depart in interval 1, 2 if they depart in 2, 3 if they depart in 3, 4 if they depart elsewhere, where 1, 2, and 3 are sequential nonoverlapping intervals. Note that for a given arrival time, the type of the event is dependent only on the service time. Since the arrival times are independent by assumption and the service times are also independent, the increments are independent.

(iii) $\Pr(D(t + h) - D(t) = 1)$: Call an event type 1 if they depart in the interval $(t, t + h)$. Then $P_1(s) = G(t + h + s) - G(t + s) = g(t - s) \cdot h + o(h)$. So

$$D(t+h)-D(t) \sim \text{Poisson}(\lambda h \int_0^t g(t-s)ds+o(h)) = \text{Poisson}(\lambda h \int_0^t g(y)dy+o(h)) \, \text{Poisson}(\lambda h G(t)+o(h))$$

which means $\Pr(D(t+h)-D(t) = 1) = \lambda G(t)he^{-\lambda G(t)h} =$ (by Taylor expansion of exp) $\lambda G(t)h+o(h)$ which is what we wanted to show.

(iv) $\Pr(D(t + h) - D(t) \geq 2) = 1 - (\Pr(D(t + h) - D(t) = 0) - (\Pr(D(t + h) - D(t) = 1) = 1 - e^{-\lambda G(t)h} - \lambda G(t)h + o(h) =$ (by Taylor expansion of exp) $1 + \lambda G(t)h - \lambda G(t)h + o(h) = o(h)$.

□

Remark. Notice in the limit as $t \to \infty$ it converges to a Poisson process with rate λ.

Example 7.8. Poisson process, event i is associated with reward V_i. $\{X(t), t \geq 0\}$ is amount of money you get at time t.

$$X(t) = \sum_{i=1}^{r} V_i N_i(t) \tag{7.7}$$

or

$$\mathbb{E}(X(t)) = \sum V_i \mathbb{E}(V_i(t)) = \sum V_i \lambda \alpha_i t = \lambda t \sum \alpha_i v_i = \lambda t \mathbb{E}(X)$$

nice thing about this representation is $V_i N_i(t)$ are independent, so variance is sum of variances. Using $\text{Var}(N_i(t) = \lambda\alpha_i t$,

$$\text{Var}(X(t)) = \sum_{i=1}^{r} V_i^2 \lambda\alpha_i t = \lambda t \sum V_i^2 \alpha_i = \lambda t \mathbb{E}(X^2)$$

so we verified the formulas we derived a different way.

Note that $N_i(t)$ is approximately Gaussian for large t (because Poisson goes to normal for large t). Therefore when t is large $X(t)$ is also approximately Gaussian.

7.2.4 Queueing Systems

Say we have a $M/G/1$ queuing process. That is, arrivals are $PP(\lambda)$, the service time has distribution G, and there is one server. Note that if no one is in line, the waiting time (length of an "idle period") until the next arrival is exponential (since it's a Poisson process). Consider the "busy periods," that is, the time from when a customer arrives until the time the next idle period starts.

Let B be the length of a busy period. Let S be the service time of the initial customer. Note that B is independent from what happened before. Let A be the additional time after the first customer is served until the next idle period so that $B = S + A$. Let $N(S)$ be the number of arrivals during S. Note that A depends on $N(S)$.

Suppose $N(S) = n$. If $n = 0$, then $A = 0$. If $n = 1, A$ has the same distribution a B_1. If $n = 2$, A has the same distribution as the sum of two Bs. And so on. This tells us we can say

$$B = S + \sum_{i=1}^{N(S)} B_i$$

If we condition on $S = s$,

$$\{B \mid S = s\} = s + \sum_{i=1}^{N(s)} B_i$$

Note that $\sum_{i=1}^{N(s)} B_i$ is a compound Poisson random variable. So

$$\mathbb{E}(B \mid S = s) = s + \mathbb{E}(N(s))\mathbb{E}(B_i) = s + \mathbb{E}(B)\lambda s$$

$$\text{Var}(B \mid S = s) = \lambda s \mathbb{E}(B^2)$$

or

$$\mathbb{E}(B \mid S) = S + \mathbb{E}(N(S))\mathbb{E}(B_i) = S + \mathbb{E}(B)\lambda S$$

$$\text{Var}(B \mid S) = \lambda S \mathbb{E}(B^2)$$

So

$$\mathbb{E}(B) = \mathbb{E}[\mathbb{E}(B \mid S)] = \mathbb{E}(S) + \lambda \mathbb{E}(S)\mathbb{E}(B) \implies \boxed{\mathbb{E}(B) = \frac{\mathbb{E}(S)}{1 - \lambda \mathbb{E}(S)}}$$

Note the similarity to the sum of an infinite geometric series. Similarly, if $\lambda \mathbb{E}(S) \geq 1 \iff \mathbb{E}(S) \geq 1/\lambda$ then the expected waiting time is infinite because the arrival times under the Poisson process ($1/\lambda$ in expectation) are faster in expectation than the service times. Also,

$$\text{Var}(B) = \lambda \mathbb{E}(S)\mathbb{E}(B^2) + (1 + \lambda \mathbb{E}(B))^2 \text{Var}(S)$$

Note that $\text{Var}(B) = \mathbb{E}(B)^2$, etc., then you work out the answer (**see textbook for complete answer.**)

End of Poisson processes

7.3 Renewal Processes

Definition 7.12 (**Stieltjes Integral**).

$$\int_a^b h(x)dF(x) = \lim_{n \to \infty} \sum_{i=1}^n h(x_i)(F(x_i) - F(x_{i-1}))$$

In particular, we often suppose that $F(x)$ is the distribution function for a random variable X. Then we have (if X is continuous)

$$\int_a^b h(x)dF(x) = \lim_{n \to \infty} \sum_{i=1}^n h(x_i)(\Pr(x_{i-1} < X \leq x_i) = \int_a^b h(x)f_X(x)dx = \mathbb{E}(h(x))$$

or if X is discrete

$$\int_a^b h(x)dF(x) = \lim_{n \to \infty} \sum_{i=1}^n h(x_i)(\Pr(x_{i-1} < X \leq x_i) = \mathbb{E}(h(x)).$$

Generalization of Poisson processes. A counting process where the interarrival times are i.i.d. Let $X_1, X_2, \ldots$ be i.i.d. non-negative random variables with distribution function F. We require $F(0) = \Pr(X \leq 0) = \Pr(X = 0) < 1$. Let

$$\mathbb{E}(X_1) = \mu = \int_0^\infty x dF(x) = \int_0^\infty \overline{F}(t) dt$$

where $0 < \mu \leq \infty$. Let $N(t)$ be the counting process (this is the largest value of n for which the nth event has occurred at time t, so $N(t) = \max\{n : S_n \leq t\}$). Let S_n be the arrival time for the nth event. Define $S_0 = 0, S_n = \sum_{i=1}^n X_n$.

Definition 7.13 (**Renewal process; Definition 3.1.1 in *Stochastic Processes***)**.** Let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of nonnegative independent random variables with a common distribution F. To avoid trivialities, suppose that $F(0) = \Pr(\{X_n = 0\}) < 1$. We shall interpret X_n as the time between the $(n-1)$st and nth event. Let

$$S_0 = 0, \qquad S_n = \sum_{i=1}^n X_i, n \geq 1$$

so that S_n is the time of the nth event. As the number of events by time t will equal the largest value of n for which the nth event occurs before or at time t, we have that $N(t)$, the number of events by time t, is given by

$$N(t) = \sup\{n : S_n \leq t\}.$$

The counting process $\{N(t), t \geq 0\}$ is called a **renewal process.** We can write this as $\{N(t), t \geq 0\}$ as a renewal process with intensity distribution F.

Remark. Let $\mu = \mathbb{E}(X_n) = \int_0^\infty x dF(x)$ denote the mean time between successive events and note that from the assumptions $X_n \geq 0$ and $F(0) < 1$ it follows that $0 < \mu \leq \infty$.

Every time an event occurs is a “renewal:’ from that time on, all the arrivals are i.i.d. (Between events it is unclear what is going on until the next event happens; depends on the nature of F. If we have a Poisson process, then it is memoryless, but otherwise it is not.)

Example 7.9 (**St. Petersburg Paradox**)**.** Idea: you play a game, you get the amount of money X. In order to play the game you have to pay, so what’s a fair amount to pay? Belief: $\mathbb{E}(X)$ (that’s a rational price).

Game: fair coin, flip until heads occurs. If it occurs on trial n, you win 2^n dollars. Note that

$$\mathbb{E}(X) = \sum_{i=1}^\infty 2^i \cdot \left(\frac{1}{2}\right)^i = \infty.$$

Proposition 7.16. With probability 1, $N(t) < \infty$ for all t.

Proof.

$$\frac{S_n}{n} = \frac{X_1 + \ldots + X_n}{n} \xrightarrow{a.s.} \mu > 0 \text{ as } n \to \infty$$

Because the denominator goes to ∞ and the ratio doesn't go to 0, we must have $S_n \xrightarrow{a.s.} \infty$. Therefore $\Pr(N(t) < \infty) = 1$ for all t.

□

Proposition 7.17. With probability 1, $N(\infty) = \lim_{t\to\infty} N(t) = \infty$. (There's never a last renewal—there will always be another.)

Proof. Recall **Boole's Inequality** (which can be proven in a similar manner to Theorem 7.59):

$$\Pr\left(\bigcup_{n=1}^{\infty}\{A_n\}\right) \leq \sum_{n=1}^{\infty}\Pr(A_n)$$

(with equality if the events are disjoint). Then

$$\Pr(N(\infty) < \infty) = \Pr(X_n = \infty) \text{ (for some } n) = \Pr\left(\bigcup_{n=1}^{\infty}\{X_n = \infty\}\right) \leq \sum_{n=1}^{\infty}\Pr(X_n = \infty) = 0$$

□

Definition 7.14 (Notation for n-fold self-convolution). Recall the notation for convolution: if $X \sim F \perp\!\!\!\perp Y \sim G$, we have $X + Y \sim F * G$. Then

$$S_n \sim F * F * \ldots * F(t).$$

Let

$$F_n(t) := F * F * \ldots * F(t)$$

the n-fold convolution of F with itself. (In general, this is very hard to get a closed-form notation for except in some simple cases. More of a theoretical construct than something practical to compute.)

Proposition 7.18. $\Pr(N(t) = n) = F_n(t) - F_{n+1}(t)$.

Proof.

$$N(t) \geq n \iff S_n \leq t.$$

$$\implies \Pr(N(t) \geq n) = \Pr(S_n \leq t)$$

Then we have

$$\Pr(N(t) = n) = \Pr(N(t) \geq n) - \Pr(N(t) \geq n + 1) = F_n(t) - F_{n+1}(t).$$

(Again, this is more of a theoretical construct than something practical to compute.)

□

Definition 7.15 (**Renewal function**)**.** Let $m(t) = \mathbb{E}(N(t))$. We call $m(t)$ the **renewal function.**

Remark.
$$m(t) = \sum_{n=1}^{\infty} \Pr(N(t) \geq n) = \sum_{n=1}^{\infty} F_n(t).$$

Lemma 7.19. $m(t) < \infty$ for all t.

Proof. Skipped in class, not very enlightening.

□

Proposition 7.20 (**Renewal Equation**)**.**
$$m(t) = F(t) + \int_0^t m(t-s)dF(s).$$

Proof.
$$m(t) = \mathbb{E}(N(t)) = \int_0^{\infty} \mathbb{E}(N(t) \mid X_1 = s)dF(s)$$

Note that

$$\mathbb{E}(N(t) \mid X_1 = s) = \begin{cases} 1 + m(t-s) & s \leq t \\ 0 & s > t \end{cases}$$

so we have

$$m(t) = \int_0^t (1 + m(t-s))dF(s) = F(t) + \int_0^t m(t-s)dF(s)$$

□

Remark. Remember we did a problem where $X_i \sim \text{Unif}(0,1)$ and asked how many you have to sum until the number of greater than 1. We ended up with $1/e$. This is the smallest value of n for which the event occured after time n (or something?)

$$N = \min\{X_1 + \ldots + X_n > 1\} = N(1) + 1$$

(the first event that occurred after time 1. We showed that this was equal to e basically by solving the renewal equation. Only feasible because of uniform distribution between 0 and 1.

Remark. Note that

$$S_{N(t)} := (\text{time of the most recent event before or at time } t),$$

$$S_{N(t+1)} := (\text{time of the first event after time } t).$$

At what rate do renewals occur?

Theorem 7.21 (Strong Law for Renewal Processes).

$$\frac{N(t)}{t} \xrightarrow{a.s.} \frac{1}{\mu}$$

(Intuitively: since X_i are the time between renewals, the average time between renewals is $\mathbb{E}(X_i) = \mu$. So the average rate of renewals is equal to one over the average time between events.)

Proof. Note that

$$S_{N(t)} \leq t < S_{N(t)+1}$$

$$\iff \frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}$$

As $t \to \infty$, the quantity on the left converges to the mean by the Strong Law of Large Numbers (see the proof of Proposition 7.16 above). For the quantity on the right, we have

$$\frac{S_{N(t)+1}}{N(t)} = \frac{S_{N(t)+1}}{N(t)+1} \cdot \frac{N(t)+1}{N(t)} \xrightarrow{a.s.} \mu \cdot 1$$

which means

$$\frac{t}{N(t)} \xrightarrow{a.s.} \mu \iff \frac{N(t)}{t} \xrightarrow{a.s.} \frac{1}{\mu}.$$

$\square$

Remark. In many applications: $X_1, X_2, \ldots$ are independent non-negative random variables. $X_1 \sim G$, $X_i \sim F, i \geq$. (That is, once the first event occurs we have a renewal process, but before then we don't.)

Definition 7.16 (Delayed renewal process). Define

$$N_d(t) := \max\{n : X_1 + \ldots + X_n \leq t\}.$$

We call $\{N_d(t), t \geq 0\}$ a **delayed renewal process.**

Remark. Almost all limiting results for renewal processes apply for delayed renewal processes as well. (In the long run, that first waiting period doesn't really make a difference.) For example, consider the Strong Law for Renewal Processes (Theorem 7.21) (quick note: let $N(s) = -1, s < 0.$):

$$N_d(t) = 1 + N(t - X_1) \iff \frac{N_d(t)}{t} = \frac{1}{t} + \frac{N(t-X_1)}{t-X_1}\frac{t-X_1}{t}$$

By the Strong Law for Renewal processes,

$$\frac{N(t-X_1)}{t-X_1} \xrightarrow{a.s.} \frac{1}{\mu_F}$$

where $\mu_F = \int_0^\infty \overline{F}(t)dt$. Since $\lim_{t\to\infty} \frac{t-X_1}{t} = 1$, we have the same result as for the Strong Law for Renewal Processes:

$$\frac{N_d(t)}{t} \xrightarrow{a.s.} \frac{1}{\mu_F}$$

Example 7.10. Arrivals to a single server queue according to a Poisson process with rate λ. However, they will only enter if the server is free when they arrive. Service time has distribution G. (This is often called a **loss model** or **Erlang loss model**, more generally with k servers.)

(a) At what rate to customers enter the system?

(b) What proportion of arrivals enter the system?

Solution.

(a) Note that the events of customers entering are a renewal process because everything probabilistically starts all over again when a customer arrived. Specifically it's a delayed renewal process because the time of the first arrival is exponential with rate λ, but the rest of them are more complicated because there must be a service and an arrival (the sum of two random variables). So the rate of arrivals is 1 over the expected time between events, or

$$\frac{1}{\mathbb{E}(S+I)}$$

where S is the service time and I is the interarrival time for the next customer. Let $\mathbb{E}(X) = \mu_G = \int_0^\infty \overline{G}(t)dt$. Then

$$\mathbb{E}(S+I) = \mu_G + \frac{1}{\lambda}$$

so the answer is

$$\frac{1}{\mu_G + \frac{1}{\lambda}} = \boxed{\frac{\lambda}{1+\lambda\mu_G}.}$$

(b) Rate of service arrivals divided by overall rates of arrivals:

$$\left.\frac{\lambda}{1+\lambda\mu_G}\right/\lambda = \boxed{\frac{1}{1+\lambda\mu_G}.}$$

Example 7.11. Suppose we have a bin with an infinite number of coins. Each bin has value p of landing on heads, and suppose the value of p for a randomly chosen coin is distributed as Uniform$(0,1)$. We draw coins and flip them. Every turn we can either draw a new coin or flip one of the coins we already have. If we want to maximize the proportion of coins that flip heads, what is the optimal strategy?

Solution. Consider a strategy of giving up on a coin if it comes up tails once. Every time you flip tails, the process renews. Let $N(n)$ be the number of tails in the first n flips. Then $N(n)$ is a renewal process. So we know that

$$\frac{N(n)}{n} \xrightarrow{a.s.} \frac{1}{\mathbb{E}(T)}$$

where T is the time between tails. Let P be a random variable for the probability of a coin flipping heads and note that

$$\mathbb{E}(T) = \int_0^1 \mathbb{E}(T \mid P = p)dp = \int_0^1 \frac{1}{1-p}dp = -\log(1-p)\Big|_0^1 = \infty$$

so the long-run proportion of coins that land heads is 1.

Proposition 7.22 (Homework problem; *Introduction to Probability Models* Ch. 7 Problem 30). For a renewal process, let $A(t) = t - S_{N(t)}$ be the age at time t. If $\mu < \infty$, then

$$\frac{A(t)}{t} \xrightarrow{w.p.1} 0.$$

Proof. We hope to show that

$$\frac{t - S_{N(t)}}{t} \xrightarrow{w.p.1} 0.$$

Note that

$$\frac{t - S_{N(t)}}{t} = \frac{t - S_{N(t)}}{N(t)} \cdot \frac{N(t)}{t} = \left(\frac{t}{N(t)} - \frac{S_{N(t)}}{N(t)}\right)\frac{N(t)}{t}$$

By the Strong Law for Renewal Processes (Theorem 7.21), $\frac{N(t)}{t} \xrightarrow{w.p.1} \mu^{-1}$, where $\mu = \mathbb{E}(X)$ is the expected interarrival time. Since the expected interarrival time is finite, $N(t) \to \infty$ as $t \to \infty$. Therefore

$$\frac{S_{N(t)}}{N(t)} = \frac{X_1 + \ldots + X_{N(t)}}{N(t)} \xrightarrow{w.p.1} \mu.$$

Lastly, since $N(t) > 0$ with probability 1 as $t \to \infty$ and $t > 0$, by the Continuous Mapping Theorem (Theorem asym.contmappthm),

$$\frac{t}{N(t)} = \left(\frac{N(t)}{t}\right)^{-1} \xrightarrow{w.p.1} (\mu^{-1})^{-1} = \mu.$$

Therefore by another application of the Continuous Mapping Theorem (Theorem asym.contmappthm),

$$\frac{t - S_{N(t)}}{t} = \left(\frac{t}{N(t)} - \frac{S_{N(t)}}{N(t)}\right)\frac{N(t)}{t} \xrightarrow{w.p.1} (\mu - \mu) \cdot \mu^{-1} = 0.$$

□

7.3.1 Stopping Times

Definition 7.17 (**Stopping times**). Let $X_1, X_2, \ldots$ be independent random variables. We say the non-negative integer valued random variable N is a **stopping time** for $X_1, X_2, \ldots$ if the event $\{N = n\}$ is independent of $X_{n+1}, X_{n+2}, \ldots$. (This definition is more general than saying it must depend on previous times because it could be random and not depend on anything (except the current one) or you could have a finite memory, etc.)

Theorem 7.23 (**Wald's Equation**). Suppose $X_1, X_2, \ldots$ are i.i.d. with $\mathbb{E}(X) < \infty$. Let N be a stopping time for $X_1, X_2, \ldots$ such that $\mathbb{E}(N) < \infty$. Then

$$\mathbb{E}\left(\sum_{i=1}^{N} X_i\right) = \mathbb{E}(N)\mathbb{E}(X).$$

Remark. Note the similarity to compound random variables. But the proof differs because N is not independent of the $\{X_i\}$.

Proof. Let I_i be an indicator variable for $I \leq N$. Then

$$\sum_{i=1}^{N} X_i = \sum_{i=1}^{\infty} X_i I_i \implies \mathbb{E}\left(\sum_{i=1}^{N} X_i\right) = \mathbb{E}\left(\sum_{i=1}^{\infty} X_i I_i\right) = \sum_{i=1}^{\infty} \mathbb{E}(X_i I_i)$$

Note that I_i depends on $X_1, \ldots, X_{i-1}$ but not on X_i (it only depends on whether you play the ith game, not the outcome of that game. You only decide whether to play the ith game based on the outcomes of the previous games.). Therefore $X_i \perp\!\!\!\perp I_i$. So we have

$$= \sum_{i=1}^{\infty} \mathbb{E}(X_i)\mathbb{E}(I_i) = \mathbb{E}(X)\sum_{i=1}^{\infty} \mathbb{E}(I_i) = \mathbb{E}(X)\sum_{i=1}^{\infty} \Pr(N \geq i) = \mathbb{E}(X)\mathbb{E}(N)$$

or we can write

$$\mathbb{E}(X)\sum_{i=1}^{\infty} \mathbb{E}(I_i) = \mathbb{E}(X)\mathbb{E}\left(\sum_{i=1}^{\infty} I_i\right) = \mathbb{E}(X)\mathbb{E}(N).$$

□

Remark. How do we justify

$$\mathbb{E}\left(\sum_{i=1}^{\infty} X_i I_i\right) = \sum_{i=1}^{\infty} \mathbb{E}(X_i I_i)?$$

Do the same thing again but replace all the X_i with $|X_i|$. Then the proof works, and Wald's Equation follows by Lebesgue's Dominated Convergence Theorem.

Example 7.12. Let

$$X_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{otherwise} \end{cases}$$

One stopping time could be $N_1 = \min\{n; X_1 + \ldots + X_n = k\}$. Then by Wald's Equation (Theorem 7.23),

$$\mathbb{E}\left(\sum_{i=1}^{N} X_i\right) = \mathbb{E}(N)\mathbb{E}(X) \iff k = \mathbb{E}(N)(p) \implies \mathbb{E}(N) = \frac{k}{p}.$$

Another could be $N_2 = \min\{n : X_{n-1} = X_n = 1\}$.

Example 7.13. Let

$$X_i = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p \end{cases}$$

with $p > 1/2$. Note that $\mathbb{E}(X_i) = 2p-1 > 0$. One stopping time could be $N = \min\{n : X_1+\ldots+X_n = 10\}$. (Note that by the Strong Law of Large Numbers, with probability 1 this will eventually happen.) Then by Wald's Equation (Theorem 7.23),

$$\mathbb{E}\left(\sum_{i=1}^{N} X_i\right) = \mathbb{E}(N)\mathbb{E}(X) \iff 10 = \mathbb{E}(N)(2p-1) \implies \mathbb{E}(N) = \frac{10}{2p-1}.$$

What's the mean amount of time until you're up by one dollar?

$$m = 1 + (1-p)\mathbb{E}(\text{ up } 2) = 1 + (1-p)2m$$

Possible stopping rule: stop when you're winning money.

$$1 = \sum_{i=1}^{N} X_i = \mathbb{E}(N)\mathbb{E}(X) = 0$$

but it turns out Wald's equation doesn't apply because $\mathbb{E}(N) = \infty$ (the mean number of plays to get ahead is infinite).

Example 7.14. Let

$$X_i = \begin{cases} 1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

with $p > 1/2$. Note that $\mathbb{E}(X_i) = 2p - 1 > 0$. One stopping time could be $N = \min\{n : X_1 + \ldots + X_n = 1\}$. It turns out from Markov chain theory that this will eventually happen with probability 1. We can't apply Wald's Equation (Theorem 7.23) because $\mathbb{E}(N)$ is not finite. Note that if we try, a contradiction results:

$$1 = \mathbb{E}(N) \cdot 0 = 0$$

Example 7.15. Let $U_i \sim \text{Uniform}(0, 1)$, U_i i.i.d. One stopping time is $N = \min\{n : U_n > 0.8\}$. Then by Wald's Equation (Theorem 7.23),

$$\mathbb{E}\left(\sum_{i=1}^{N} X_i \right) = \mathbb{E}(N)\mathbb{E}(X) = 10 \cdot 0 = 0$$

so the expected winnings when you stop is 0 (regardless of your stopping time, since it's a fair game).

Back to renewal theory: Let $X_1, X_2, \ldots$ be a sequence of random variables with interarrival times distributed as F. Note that $N(t) = 5 \iff X_1 + \ldots + X_5 \leq t, X_1 + \ldots + X_6 > t$ so this is not a stopping time. But we could stop at the first event that occurs after time t, so $N(t) + 1$ is a stopping time. Note that

$$N(t) + 1 = n \iff N(t) = n - 1, X_1 + \ldots + X_{n-1} \leq t, X_1 + \ldots + X_n > t.$$

(You can't say "I'll stop at the last event before t" because at the time of that event you wouldn't have been able to know it was the last event before t.)

Corollary 7.23.1 (Corollary to Wald's Equation).

$$\mathbb{E}\left(\sum_{i=1}^{N(t)+1} X_i \right) = \mu(m(t) + 1)$$

Theorem 7.24 (Elementary Renewal Theorem).

$$\frac{m(t)}{t} \xrightarrow{a.s.} \frac{1}{\mu}.$$

Proof. Note that

$$S_{N(t)+1} > t$$

We have

$$\mathbb{E}(S_{N(t)+1}) > t$$

Using Corollary 7.23.1,

$$\mu(m(t) + 1) > t \iff m(t) + 1 > \frac{t}{\mu} \iff \frac{m(t)}{t} > \frac{1}{\mu} - \frac{1}{t}$$

Then as $t \to \infty$

$$\liminf \frac{m(t)}{t} \geq \frac{1}{\mu}.$$

So if we can show $\liminf \frac{m(t)}{t} \leq \frac{1}{\mu}$, we're done. Assume $\Pr(X_i \leq M) = 1$ for some $M \in \mathbb{R}$. Then

$$S_{N(t)+1} < t + M \implies \mu(m(t)+1) < t + M \implies m(t) < \frac{t}{\mu} + \frac{M}{\mu} - 1$$

$$\implies \frac{m(t)}{t} < \frac{1}{\mu} + \frac{M}{t\mu} - \frac{1}{t}$$

$$\implies \limsup \frac{m(t)}{t} \leq \frac{1}{\mu}.$$

Now consider the general case. Consider $X_1, X_2, \ldots$. Let

$$X_i^* = \begin{cases} X_i & \text{if } X_i \leq M \\ M & \text{if } X_i > M \end{cases}$$

and let $N^*(t) = \max\{n : X_1^* + \ldots + X_n^* \leq t\}$. Then since $N^*(t)$ has smaller interarrival times than $N(t)$,

$$N(T) \leq N^*(t) \implies \frac{\mathbb{E}(N(t))}{t} \leq \frac{\mathbb{E}(N^*(t))}{t}$$

$$\implies \lim \frac{m(t)}{t} \leq \lim \frac{\mathbb{E}(N^*(t))}{t} = \frac{1}{\mathbb{E}(X_i^*)}$$

Note that (since $X_I^* = X_I$ if $X_i \leq M$)

$$\mathbb{E}(X_i^*) = \int_0^M x dF(x) + M\overline{F}(M) \to \mu + 0 = \mu \text{ as } M \to \infty$$

so

$$\lim \frac{m(t)}{t} \leq \frac{1}{\mathbb{E}(\min\{X, M\})}$$

which is true for every M. Letting $M \to \infty$, we have

$$\mathbb{E}(\min\{X, M\}) \to \mathbb{E}(X) \text{ as } M \to \infty$$

by Lebesgue's Monotone convergence theorem. ("if you have a sequence of variables $X_n \leq X_{n+1} \leq \ldots$ then $\mathbb{E}(\lim_{n\to\infty} X_n) = \lim_{n\to\infty} \mathbb{E}(X_n)$").

□

Remark. This technique is called "coupling:" relating something you don't know to something you do.

Remark. Why doesn't the Elementary Renewal Theorem follow from the Strong Law for Renewal Processes (Theorem 7.21)? Consider this counterexample: Let $U \sim \text{Uniform}(0, 1)$ and let X_n be a random variable such that

$$X_n = \begin{cases} n & U < 1/n \\ 0 & U > 1/n \end{cases}$$

Of course with probability 1, $U > 0$; that is, $U = \epsilon > 0$. Note that $X_n = 0$ for all n sufficiently large ($1/n < \epsilon$). So we see that $X_n \xrightarrow{a.s.} 0$. But for all n sufficiently large

$$\mathbb{E}(X_n) = n \cdot \frac{1}{n} = 1.$$

So we have

$$\lim_{n\to\infty} X_n = 0 = \mathbb{E}(\lim_{n\to\infty} X_n) \neq \lim_{n\to\infty} \mathbb{E}(X_n) = 1$$

In summary, just because $\frac{N(t)}{t} \xrightarrow{a.s.} \frac{1}{\mu}$ doesn't mean $\frac{m(t)}{t} \xrightarrow{a.s.} \frac{1}{\mu}$.

Proposition 7.25. Consider a delayed renewal process $N_d(t)$ and let $m_d(t) = \mathbb{E}(N_d(t))$. Recall that the first arrival has distribution G and the rest have distribution F. Then $m_d(t) \xrightarrow{a.s.} 1/\mu$.

Proof.

$$m_d(t) = \int_0^t \mathbb{E}(N_d(t) \mid X_1 = s) dG(s)$$

Note that

$$\mathbb{E}(N_d(t) \mid X_1 = s) = 1 + m(t - s)$$

so we have

$$m_d(t) = \int_0^t (1 + m(t - s)) dG(s) = G(t) + \int_0^t m(t - s) dG(s)$$

$$\iff \frac{m_d(t)}{t} = \frac{G(t)}{t} + \frac{1}{t} \int_0^t m(t - s) dG(s) \xrightarrow{a.s.} \frac{1}{\mu}$$

by using $m(y)/y \xrightarrow{a.s.} 1/\mu$. (Details can be fleshed out.)

□

Proposition 7.26. Suppose $X_1, X_2, \ldots$ are integer valued, and there is only one renewal at a time. Suppose $\Pr(X_i = 0) = 0$. Then

Proof. Note that

$$\frac{\mathbb{E}(N_d(t))}{n} \to \frac{1}{\mu}$$

by the Strong Law for Renewal Processes (Theorem 7.21). Note that

$$N_d(n) = \sum_{j=1}^{n} I_j$$

where I_j is an indicator variable for whether there is a renewal at time j. Then

$$\mathbb{E}(N_d(n)) = \sum_{j=1}^{n} \mathbb{E}(I_j) = \sum_{j=1}^{n} \Pr(I_j = 1)$$

Then by the Elementary Renewal Theorem (Theorem 7.24),

$$\frac{1}{n}\sum_{j=1}^{n} \Pr(I_j = 1) \xrightarrow{a.s.} \frac{1}{\mathbb{E}(X_i)}$$

□

Recall we say that $a_n \to a$ pointwise if $\lim_{n\to\infty} a_n = a$.

Definition 7.18 (Caesaro Convergence). We say a_n Caesaro converges to a if

$$\frac{a_1 + \ldots + a_n}{n} \to a \text{ as } n \to \infty.$$

Remark. Pointwise convergence implies Caesaro convergence, but not the other way around.

When does $\Pr(\{\text{renewal at } n\}) \to 1/\mu$? Let $\Pr(\{\text{renewal at } j\}) = P_j$ and suppose $P_n \to P^*$. This implies Caesaro convergence

$$\frac{P_1 + \ldots + P_n}{n} \to p^*$$

and $p^* = 1/\mu$ where μ is the expected time between renewals.

These examples are very confusing—look for explanations in textbook? Seems to be same as Section 7.9.2 in *Introduction to Probability Models.*

Example 7.16. Let $Y_1, Y_2, \ldots$ i.i.d. Suppose $p_j = \Pr(Y_i = j)$. We will keep doing draws until we observe the pattern 1, 2, 1, 3, 4, 1, 2, 1. Let N be the number of trials until this occurs. What is $\mathbb{E}(N)$?

Solution. Suppose we consider the process as continuing to happen forever even after a pattern appears, and treat a pattern arriving as an event. let $N(n)$ be the number of events by n. Is $\{N(n)\}$ a renewal process? That is, once an event occurs does everything start all over? Yes, so it's either an ordinary or delayed renewal process. Note that it's delayed because the last three digits of the pattern match the first three, so once a pattern has just happened, another one could happen in 5 draws. Note that

$$\Pr(\text{pattern appears at time } n) = P_1P_2\ldots = P_1^4P_2^2P_3P_4$$

so the expected time between renewals is $1/(P_1^4P_2^2P_3P_4)$, except that we need to take into account the fact that we already have the first three digits of the pattern if we just got one. Note that

$$\Pr(\text{pattern appears at time } n \mid \text{just had pattern}) = P_1^2P_2P_3P_4$$

We can look at this a different way. The time until we get a pattern T_p is the time until we get 1,2,1 T_{121} plus the time until we get the rest of the pattern T_{rest}.

$$T_p = T_{121} + T_{rest} \implies \mathbb{E}(T_p) = \mathbb{E}(T_{121}) + 1/(P_1^4P_2^2P_3P_4)$$

Then $\Pr(\text{renewal at } n) = P_1^2P_2$ so the expected time between renewals is $1/(P_1^2P_2)$. Then the time between renewals is the additional time between renewals given that you currently have 1, so

$$\mathbb{E}(\text{time between renewals}) = \mathbb{E}(\text{additional time when you have a 1}) = 1/(P_1^2P_2)$$

Do it again: time to get to 1,2,1 is time to get to 1 plus additional:

$$T_{121} = T_1 + T_{21} \implies \mathbb{E}(T_{121}) = \mathbb{E}(T_1) + /(P_1^2P_2) = 1/P_1 + /(P_1^2P_2)$$

putting this all together, we have

$$\mathbb{E}(T_p) = 1/P_1 + /(P_1^2P_2) + 1/(P_1^4P_2^2P_3P_4).$$

Example 7.17. Flips coins, get heads with probability P. Then T_k is the time to land k heads in a row. What is the expected time until k heads in a row?

Solution. Note that the probability of a renewal at n is p^k. So the expected time between renewals is equal to $1/p^k$ not taking into account that the next time we could use the heads we already have. Note that

$$T_k = T_{k-1} + T_rest$$

so

$$\mathbb{E}(T_k) = \mathbb{E}(T_{k=1} + 1/p^k = 1/p^k + \mathbb{E}(T_{k-2}) + 1/(p^{k-1}) + \dots$$

7.4 ISE 620 Midterm Solutions

Exercise 6.

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right)$$

Solution.

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \text{Var}\left(\sum_{i=1}^{n}(n+1-i)X_i\right) = \sum_{i=1}^{n} \frac{(n+1-i)^2}{\lambda^2} = \sum_{j=1}^{n} j^2/\lambda^2$$

or

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(S_i) + 2 \sum_{1 \leq i < j \leq n} \text{Cov}(S_i, S_j)$$

Note that

$$\text{Cov}(S_i, S_j) = \text{Cov}\left(S_i, S_i + \sum_{k=i+1}^{j} X_k\right) = \text{Var}(S_i) + 0 = i/\lambda^2$$

Exercise 7.

$$\text{Var}\left(\sum_{i=1}^{N(t)} S_i \mid N(t)\right)$$

Solution.

$$\text{Var}\left(\sum_{i=1}^{N(t)} S_i \mid N(t) = n\right) = \text{Var}\left(\sum_{i=1}^{n} U_i\right)$$

where $U_1, \dots, U_n \sim$ i.i.d. $U(0,t)$ and they are ordered. Since the sum of the ordered values is the same as the sum of the unordered values and they are i.i.d., we have

$$\text{Var}\left(\sum_{i=1}^{n} U_i\right) = \sum_{i=1}^{n} \text{Var}(U_i) = \frac{nt^2}{12}.$$

Exercise 8. A wins point with probability p_1 if A is server, A wins with probability p_2 if B serves. (a) serve next if you just won. (b) alternate serve.

Solution.

(a) A serving is a renewal. Suppose A just won a point. Let X be the number of games until A wins again. Frequency with which A wins converges to $1/\mathbb{E}(X)$ by the Strong Law for Renewal Processes (Theorem 7.21). Condition on who wins next game. Let Y be an indicator variable for A winning the next game. Then

$$\mathbb{E}(X) = \mathbb{E}(X \mid Y = 1)p_1 + \mathbb{E}(X \mid Y = 0)(1 - p_1)$$

Note that if $Y = 1$, $X = 1$. For $\mathbb{E}(X \mid Y = 0)$, B keeps serving until A wins, so the number of games until A wins is a geometric random variable with probability p_2, so the mean is $1/p_2$. So we have

$$\mathbb{E}(X) = 1 \cdot p_1 + (1 + 1/p_2)(1 - p_1) = 1 + \frac{1 - p_1}{p_2} = \frac{p_2 + 1 - p_1}{p_2}$$

and the rate is

$$\frac{p_2}{p_2 + 1 - p_1}$$

(b) Not a renewal—everything doesn't start all over again when A wins a point, because just because A just won doesn't tell us anything about who wins. A renewal would happen every time A wins on her serve. Let X be the number of games until the next renewal. Then the answer we seek is $1/\mathbb{E}(X)$. Let Y be an indicator variable for A winning on her next serve. Note that

$$\mathbb{E}(X) = \mathbb{E}(X \mid Y = 1)p_1 + \mathbb{E}(X \mid Y = 0)(1 - p_1)$$

Note that $\mathbb{E}(X \mid Y = 1) = 2$, $\mathbb{E}(X \mid Y = 0) = 2 + \mathbb{E}(X)$. So we have

$$\mathbb{E}(X) = 2p_1 + (2 + \mathbb{E}(X))(1 - p_1) \iff p_1\mathbb{E}(X) = 2p_1 + 2 - 2p_1 \iff \mathbb{E}(X) = 2/p_1$$

Do the same thing again for when A wins on B's serves. Then add them together.

A renewal would happen every time A wins on B's serve. Let Z be the number of games until the next renewal. Then the answer we seek is $1/\mathbb{E}(Z)$. Let Y be an indicator variable for A winning on B's next serve. Note that

$$\mathbb{E}(Z) = \mathbb{E}(Z \mid Y = 1)p_2 + \mathbb{E}(Z \mid Y = 0)(1 - p_2)$$

Note that $\mathbb{E}(Z \mid Y = 1) = 2$, $\mathbb{E}(Z \mid Y = 0) = 2 + \mathbb{E}(Z)$. So we have

$$\mathbb{E}(Z) = 2p_2 + (2 + \mathbb{E}(Z))(1 - p_2) \iff p_1\mathbb{E}(Z) = 2p_2 + 2 - 2p_2 \iff \mathbb{E}(Z) = 2/p_2$$

So the long-run proportion of games won by A is

$$\frac{1}{\mathbb{E}(X)} + \frac{1}{\mathbb{E}(Z)} = \frac{p_1 + p_2}{2}.$$

Exercise 9. $M/G/\infty$, $\quad \Pr(X(t) = j \mid N(t) = n)$.

Solution.

The arrival times are i.i.d. uniform on $(0, t)$. So each one has the same probability of being in the system at time t, and the number who are in the system is a binomial random variable.

$$\Pr(X(t) = j \mid N(t) = n) = \binom{n}{j} p^j (1-p)^{n-j}$$

where p is the probability of being in the system at time t:

$$p = \int_0^t \Pr(\text{in system} \mid \text{arrived at } s) \cdot (1/t) ds = \frac{1}{t} \int_0^t \overline{G}(t-s) ds.$$

Exercise 10. number of claims N poisson with mean lambda, claim amounts 'are uniformly distributed on $(0,1)$, we want $\mathbb{E}(\max\{X_1, \ldots, X_N\})$.

Solution. Note that

$$\Pr(\max\{X_1, \ldots, X_n\} \leq t) = t^n \implies \Pr(\max\{X_1, \ldots, X_n\} > t) = 1 - t^n$$

$$\implies \mathbb{E}(\max\{X_1, \ldots, X_n\}) = \int_0^1 (1 - t^n) dt = \frac{n}{n+1} = 1 - \frac{1}{n+1}$$

so this is the expected value if we condition on n. Therefore

$$\mathbb{E}(\max\{X_1, \ldots, X_N\}) = \mathbb{E}(\max\{X_1, \ldots, X_N\} \mid N = n) \Pr(N = n)$$

$$= \sum_{n=0}^{\infty} \left(1 - \frac{1}{n+1}\right) \cdot e^{-\lambda} \frac{\lambda^n}{n!} = \ldots$$

Exercise 11.

$$\mathbb{E}(X(t) \mid N(s) = n)$$

where

$$X(t) = \sum_{i=1}^{N(t)} X_i$$

Solution.

We have

$$X(t) \mid N(s) = n = \sum_{i=1}^{N(s)} X_i + \sum_{i=N(s)+1}^{N(t)} X_i = \sum_{i=1}^{n} X_i + \sum_{i=N(s)+1}^{N(t)} X_i$$

$$\implies \mathbb{E}(X(t) \mid N(s)) = \mathbb{E}\left(\sum_{i=1}^{n} X_i\right) + \mathbb{E}\left(\sum_{i=N(s)+1}^{N(t)} X_i\right) = n\mathbb{E}(X) + \mathbb{E}(X)\lambda(t-s)$$

where $\lambda(t-s)$ is the expected number of events between s and t.

7.5 Renewal Reward Processes (Section 4.7 in *Introduction to Probability Models*, 3.6 *Stochastic Processes*)

Definition 7.19 (Renewal reward process). Suppose we have a renewal process with interarrival times $X_1, X_2, \ldots$. Suppose we receive a reward R_i when renewal i occurs (after time X_i from the previous renewal). The reward is allowed to depend on what happens during the previous period X_i, but every time renewal happens, process starts again. So we assume the vectors $(X_i, R_i), i \geq 1$ are i.i.d. Let $R(t)$ be the total reward by time t; that is,

$$R(t) = \sum_{i=1}^{N(t)} R_i.$$

Then $\{R(t), t \geq 0\}$ is a **renewal reward process.**

Proposition 7.27 (Proposition 7.3 in *Introduction to Probability Models*; sometimes called Renewal Reward Theorem).

(a)

$$\frac{R(t)}{t} \xrightarrow{a.s.} \frac{\mathbb{E}(R)}{\mathbb{E}(X)}$$

(generalization of strong law)

(b)

$$\frac{\mathbb{E}(R(t))}{t} \xrightarrow{a.s.} \frac{\mathbb{E}(R)}{\mathbb{E}(X)}$$

Proof. (a)

$$\frac{R(t)}{t} = \frac{1}{t}\sum_{i=1}^{N(t)} R_i = \frac{1}{N(t)}\sum_{i=1}^{N(t)} R_i \cdot \frac{N(t)}{t}$$

Since $N(t) \to \infty$ as $t \to \infty$, by Strong Law of Large Numbers (Theorem 8.34)

$$\frac{1}{N(t)}\sum_{i=1}^{N(t)} R_i \xrightarrow{a.s.} \mathbb{E}(R)$$

By Strong Law for Renewal Processes (Theorem 7.21),

$$\frac{N(t)}{t} \xrightarrow{a.s.} 1/\mathbb{E}(X).$$

Then by the Continuous Mapping Theorem (Theorem 8.29), the result follows.

(b)

□

Example 7.18 (Same as exam question). A wins point with probability p_1 if A is server, A wins with probability p_2 if B serves. alternate serve.

Solution. Every time A serves is a renewal. 1 reward each time A wins a point. Expected earning per cycle is $p_1 + p_2$. Cycles are length 2. Works out to $(p_1 + p_2)/2$.

Proposition 7.28. We've assumed that the renewal is earned at the end of the cycle. But the result holds even if the reward is earned during the interval gradually, not all at once at the end of the renewal interval (assuming all rewards are nonnegative). So

$$\frac{R(t)}{t} \xrightarrow{a.s.} \frac{\mathbb{E}(R)}{\mathbb{E}(X)}.$$

Also,

$$\mathbb{E}\left[\frac{R(t)}{t}\right] \xrightarrow{a.s.} \frac{\mathbb{E}(R)}{\mathbb{E}(X)}.$$

Proof. $R(t)$ is the total reward you get by time t. So $\sum_{i=1}^{N(t)} R_i \leq R(t)$ (the inequality holds if you've earned a bit since the last renewal). We have

$$\sum_{i=1}^{N(t)} R_i \leq R(t) \leq \sum_{i=1}^{N(t)+1} R_i$$

Dividing through by t:

$$\frac{1}{t} \sum_{i=1}^{N(t)} R_i \leq \frac{R(t)}{t} \leq \frac{1}{N(t)+1} \sum_{i=1}^{N(t)+1} R_i \cdot \frac{N(t)+1}{t}$$

But by Proposition 7.27

$$\frac{1}{t} \sum_{i=1}^{N(t)} R_i \xrightarrow{a.s.} \mathbb{E}(R)/\mathbb{E}(X),$$

$$\frac{1}{N(t)+1} \sum_{i=1}^{N(t)+1} R_i \xrightarrow{a.s.} \mathbb{E}(R)$$

$$\frac{N(t)+1}{t} \xrightarrow{a.s.} 1/\mathbb{E}(X)$$

So by the Continuous Mapping Theorem,

$$\frac{R(t)}{t} \xrightarrow{a.s.} \frac{\mathbb{E}(R)}{\mathbb{E}(X)}.$$

□

Example 7.19 (**Similar to Example 7.15 in *Introduction to Probability Models***)**.** People arrive to train station in a Poisson process with rate λ. K is the cost to dispatch the train. Suppose we incur a cost c per unit of time waiting per customer. What dispatching policy minimizes the long-run average cost per unit time?

Solution.

The structure of the optimal policy is to wait until a certain number of people arrive and then dispatch a train (n policy). So we want to determine this number. (Another policy is a t policy: dispatch a train every t units of time. But it turns out the best n policy is better than the best t policy. We may be interested in the difference in minimal costs because t policies are easier to employ and more convenient for customers.)

(a) n**-policy:** Note that every time we dispatch a train, there is a renewal. At every renewal time there is a cost incurred. The long run average cost per unit time is the expected cost during a cycle divided by the expected time of a cycle. Note that the expected time between cycles is the expected amount of time for n people to arrive, which is simply n/λ. The cost of a cycle is

$$cX_2 + 2cX_3 + 3cX_4 + \ldots + (n-1)cX_n + K$$

so the expected cost is

$$\mathbb{E}(X)(c + 2c + 3c + \ldots + (n-1)c) + K = \frac{c}{\lambda}\frac{(n-1)n}{2} + K$$

Therefore taking the ratio, the average cost per cycle is

$$\left(\frac{1}{\lambda}\frac{c(n-1)n}{2} + K\right)\Big/\left(\frac{n}{\lambda}\right) = \frac{c(n-1)}{2} + \frac{\lambda K}{n}.$$

Now we want to pick n to minimize this quantity. Treat n as continuous and take the derivative.

$$\frac{c}{2} - \frac{\lambda K}{n^2} = 0 \iff n = \sqrt{\frac{2\lambda K}{c}}$$

which makes the minimum average cost

$$\frac{1}{2}\left(c\sqrt{\frac{2\lambda K}{c}} - c\right) + K\sqrt{\frac{c}{2\lambda K}} = \sqrt{2\lambda cK} - \frac{c}{2}.$$

(b) t**-policy:** Dispatch a train every t time units. Note that we have a renewal every time you dispatch a train. The average cost per unit time is the expected cost per cycle $\mathbb{E}(C)$ divided by t. Note that

$$C = c\sum_{i=1}^{N(t)}(t - S_i) = c\left(tN(t) - \sum_{i=1}^{N(t)} S_i\right) + K$$

$$\implies \mathbb{E}(C) = c\mathbb{E}\left(tN(t) - \sum_{i=1}^{N(t)} S_i\right) + K = c\left[\lambda t^2 - \mathbb{E}\left(\sum_{i=1}^{N(t)} S_i\right)\right] + K$$

Note that

$$\mathbb{E}\bigg(\sum_{i=1}^{N(t)} S_i \mid N(t)=n\bigg) = \mathbb{E}\bigg(\sum_{i=1}^{n} S_i\big) = \mathbb{E}\bigg(\sum_{i=1}^{n} U_i\big) = \frac{nt}{2} \implies \mathbb{E}\bigg(\sum_{i=1}^{N(t)} S_i \mid N(t)\bigg) = \frac{tN(t)}{2}$$

where $U_i \sim \mathrm{U}(0,t)$.

$$\implies \mathbb{E}(C) = c\lambda t^2 - c\mathbb{E}\bigg[\mathbb{E}\bigg(\sum_{i=1}^{N(t)} S_i \mid N(t)\bigg)\bigg] + K = c\lambda t^2 - c\mathbb{E}\bigg[\frac{tN(t)}{2}\bigg] + K = c\lambda t^2 - c\frac{\lambda t^2}{2} + K$$

$$\implies \mathbb{E}(C) = \frac{c\lambda t^2}{2} + K$$

Therefore the average cost per cycle is

$$\frac{1}{t}\cdot\bigg(\frac{c\lambda t^2}{2} + K\bigg) = \frac{\lambda ct}{2} + \frac{K}{t}$$

We seek the best t to minimize this cost. Take the derivative with respect to t.

$$\frac{\lambda c}{2} - \frac{K}{t^2} = 0 \implies t = \sqrt{\frac{2K}{\lambda c}}$$

So the minimum average cost is

$$\sqrt{\frac{2K}{\lambda c}}\frac{\lambda c}{2} + \frac{K}{t}\sqrt{\frac{\lambda c}{2K}} = K\sqrt{\frac{\lambda c}{2K}}.$$

Example 7.20. It costs C to buy a new car. The car lasts for a time L having distribution F before it fails. If the car fails, you incur cost K. Suppose you have a policy of buying a new car when the old car either fails or reaches a certain age t.

Solution. Note that a renewal occurs when you get a new car or when it fails. The cost per cycle is C if $L > t$, $C + K$ if $L \leq t$. The time for a cycle is $\min\{L, t\}$. Therefore the average cost per unit time is the expected cost of a cycle $C + K\Pr(L < t) = C + KF(t)$. The average lifetime is

$$\int_0^t xdF(x) + \int_t^\infty tdF(x) = \int_0^t xdF(x) + t\overline{F}(t).$$

Therefore the average cost per cycle is

$$\Big(c + KF(t)\Big)\Big/\bigg(\int_0^t xdF(x) + t\overline{F}(t)\bigg).$$

Definition 7.20 (**Age of a renewal process**)**.** We call $A(t) = t - S_{N(t)}$ the **age of the renewal process at time** t**.**

Definition 7.21 (**Excess (or residual) lifetime at time t**)**.** We call $Y(t) = S_{N(t)+1} - t$ be the **excess (or residual) lifetime of a renewal process at time** t**.**

Proposition 7.29. Let X be the interarrival time for a renewal reward process. Then

(a) with probability 1,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t A(s)ds = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)}.$$

(b) With probability 1,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t Y(s)ds = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)}.$$

(c) With probability 1,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{E}[A(s)]ds = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)}.$$

(d) With probability 1,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{E}[Y(s)]ds = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)}.$$

Proof. (a) Imagine we earn a reward at a rate equal to the age of the renewal process. Then the amount earned by time t is

$$\int_0^t A(s)ds.$$

When there is a new renewal, the process starts over again, so this is a renewal reward process. The reward earned during each cycle is

$$\int_0^X tdt = \frac{X^2}{2}.$$

Taking expectations and taking the ratio of these in the limit, we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t A(s)ds = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)}.$$

For more detail, see Example 7.18 in *Introduction to Probability Models*.

(b) Imagine we earn a reward at a rate equal to the residual lifetime of the renewal process. Then the amount earned by time t is

$$\int_0^t Y(s)ds.$$

When there is a new renewal, the process starts over again, so this is a renewal reward process. The reward earned during each cycle is

$$\int_0^X (X-t)dt = X^2 - \frac{X^2}{2} = \frac{X^2}{2}.$$

Taking expectations and taking the ratio of these in the limit, we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t A(s)ds = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)}.$$

For more detail, see Example 7.19 in *Introduction to Probability Models*.

(c) Similar to (a).

(d) Similar to (c).

□

Remark. This makes sense because if the process is infinitely long, if you look at it backwards in time the distribution of renewals is the same as if you look forward in time. But when you switch direction, the residual lifetimes and ages reverse meaning. Therefore since the distribution doesn't change depending on the direction you look at it from, the long-term expected residual lifetime and age ought to be equal.

Proposition 7.30 (Homework 7, *Introduction to Probability Models* Ch. 7 Problem 30).

$$\frac{A(t)}{t} \xrightarrow{a.s.} 0.$$

Proof. $A(t)$ is the time since the last renewal; that is, $A(t) = t - S_{N(t)}$. So we hope to show that with probability one,

$$\frac{t - S_{N(t)}}{t} \to 0 \text{ as } t \to \infty$$

(or, equivalently)

$$\frac{t - S_{N(t)}}{t} \xrightarrow{w.p,1} 0.$$

Note that

$$\frac{t - S_{N(t)}}{t} = \frac{t - S_{N(t)}}{N(t)} \cdot \frac{N(t)}{t} = \left(\frac{t}{N(t)} - \frac{S_{N(t)}}{N(t)}\right)\frac{N(t)}{t}$$

By the Strong Law for Renewal Processes (Theorem 7.21), $\frac{N(t)}{t} \xrightarrow{w.p.1} \mu^{-1}$, where $\mu = \mathbb{E}(X)$ is the expected interarrival time. Since the expected interarrival time is finite, $N(t) \to \infty$ as $t \to \infty$. Therefore

$$\frac{S_{N(t)}}{N(t)} = \frac{X_1 + \ldots + X_{N(t)}}{N(t)} \xrightarrow{w.p.1} \mu.$$

Lastly, since $N(t) > 0$ with probability 1 as $t \to \infty$ and $t > 0$, by the Continuous Mapping Theorem

$$\frac{t}{N(t)} = \left(\frac{N(t)}{t}\right)^{-1} \xrightarrow{w.p.1} (\mu^{-1})^{-1} = \mu.$$

Therefore by another application of the Continuous Mapping Theorem,

$$\frac{t - S_{N(t)}}{t} = \left(\frac{t}{N(t)} - \frac{S_{N(t)}}{N(t)}\right)\frac{N(t)}{t} \xrightarrow{w.p.1} (\mu - \mu) \cdot \mu^{-1} = 0.$$

□

Example 7.21 (Similar to Example 7.20 in *Introduction to Probability Models*). People arrive to a bus stop in a Poisson process with rate λ. Buses arrive in an independent renewal process after time T with distribution F where $\mathbb{E}(T) = \mu$.

(a) What is the long-run average number of people picked up by a bus when it arrives?

(b) Suppose busses arrive according to a Poisson process with rate λ; that is, $F(t) = 1 - e^{-\alpha t}$. What is the long-run average number of people waiting at any given time (averaged over all time)?

(c) What is the average amount of time a person waits for a bus (averaged over all people)? (Let W_i be the waiting time of person i. We seek $\lim_{n\to\infty} \sum_{i=1}^n W_i/n$.)

Solution.

(a) Every time a bus arrives, we have a renewal. The number of people picked up by a bus is $N(T)$. Note that $\{N(t)\} \sim PP(\lambda)$. Note that

$$\mathbb{E}(N(T) \mid T = t) = \mathbb{E}(N(t) \mid T = t) = \text{(by independence) } \mathbb{E}(N(t)) = \lambda t \implies \mathbb{E}(N(T) \mid T) = \lambda T$$

$$\implies \mathbb{E}(N(T)) = \mathbb{E}(\mathbb{E}(N(T) \mid T)) = \lambda \mathbb{E}(T) = \lambda\mu.$$

(b) Let $N_s(s)$ be the number of people waiting at time s. We seek

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t N_w(s)ds.$$

Imagine that at time s we earn at a rate $N_w(s)$. Then we want the average reward per unit time. Then this is a renewal reward process, where renewals happen when a bus arrive. The expected reward earned per cycle is

$$\frac{1}{\mathbb{E}(T)} \cdot \mathbb{E}\left(\int_0^T N(s)ds\right) \tag{7.8}$$

Note that

$$\mathbb{E}\left(\int_0^T N(s)ds \mid T = t\right) = \mathbb{E}\left(\int_0^t N(s)ds\right) = \text{(by Fubini's Theorem) } \int_0^t \mathbb{E}(N(s))ds = \int_0^t \lambda s ds = \frac{\lambda t^2}{2}$$

$$\implies \mathbb{E}\left(\int_0^T N(s)ds \mid T\right) = \frac{\lambda T^2}{2}$$

Therefore we have that the expected reward earned per cycle is (plugging into (7.8))

$$\frac{\lambda \mathbb{E}(T^2)}{2\mathbb{E}(T)}.$$

Because arrivals occur in a Poisson process ($F(t) = 1 - e^{-\alpha t}$), we have $\mathbb{E}(T) = \alpha^{-1}$, $\mathbb{E}(T^2) = 2\alpha^{-2}$. Therefore the expected reward earned per cycle is

$$\frac{\lambda 2\alpha^{-2}}{2\alpha^{-1}} = \frac{\lambda}{\alpha}.$$

(c) $\frac{\lambda}{\alpha}$.

Remark. Note that the average number of people picked up by a bus now matches the average number of people waiting at any given time. This is counterintuitive because you would think at the time the bus comes, the most people are there, so the average number of people picked up should be larger than the average number of people waiting in general.

This illustrates the **PASTA principle** and the **inspection paradox**. The paradox is resolved by the fact that the average in (b) is averaged over all time, while the average in (c) is averaged over all people.

Theorem 7.31. Queue, customers arrive in a renewal process with distribution F, so

$$\lambda^{-1} = \int_0^\infty \overline{F}(t)dt.$$

Each customer eventually leaves. Let L be the average number of customers in the system, averaged over all time. Let $X(s)$ be the number of customers in the system at time s. Note that

$$L = \lim_{t \to \infty} \frac{1}{t} \int_0^t X(s)ds.$$

Let W be the average time a customer spends in the system. Let W_i be the amount of time customer i spends in the system. Note that

$$W = \lim_{n \to \infty} \frac{W_1 + \ldots + W_n}{n}.$$

Then $L = \lambda W$.

Proof.

$$L = \lim_{t \to \infty} \frac{1}{t} \int_0^t X(s)ds$$

Suppose we are paid at any time at a rate equal to the number of customers in the system. Then we can interpret L as the average reward per unit time. We can consider a renewal as occurring upon the arrival of the first customer when the system was previously empty. Then we have a renewal reward process. Let T be the time between renewals. Assume the first arrival occurs at time $t = 0$ (so this is not a delayed renewal reward process). Note that

$$L = \frac{1}{\mathbb{E}(T)}\mathbb{E}\left(\int_0^T X(s)ds\right) \iff \mathbb{E}\left(\int_0^T X(s)ds\right) = L\mathbb{E}(T) \tag{7.9}$$

Note that

$$W = \lim_{n\to\infty} \frac{W_1 + \ldots + W_n}{n}$$

Let N be the number of customers served in a busy period. Then when customer $N+1$ arrives, the system is empty and there is a renewal, so T is the time of the $N+1$st arrival. So

$$T = \sum_{i=1}^{N} X_i$$

where X_i are the interarrival times (X_i is the time between the ith and $i+1$st arrival). Note that N depends on the interarrival times; if the interarrival times are small, N will be larger. So this is not a compound random variable, but we can use Wald's Equation (Theorem 7.23) if N is a stopping time for $X_1, \ldots, X_n$. Note that $N = 1$ if and only if X_1 is greater than the service time of the initial customer. And in general, $N = k$ depends only on the arrival times up to X_k. So N is a stopping time and we can use Wald's Equation. Therefore

$$\mathbb{E}(T) = \frac{\mathbb{E}(N)}{\lambda}. \tag{7.10}$$

Suppose each customer pays us 1 dollar per unit time. Then at time s we are earning at rate $X(s)$, so

$$\int_0^T X(s)ds$$

is the total amount you earn by time T. But we can also calculate this amount by adding up the amount of time each customer spends in the system:

$$\int_0^T X(s)ds = \sum_{i=1}^{N} W_i.$$

So we have (using (7.9) and (7.10))

$$\mathbb{E}\left(\int_0^T X(s)ds\right) = \mathbb{E}\sum_{i=1}^{N} W_i \iff L\mathbb{E}(T) = \mathbb{E}\sum_{i=1}^{N} W_i \iff L = \frac{\mathbb{E}(\sum_{i=1}^{N} W_i)}{\mathbb{E}(N)/\lambda}$$

$$= \text{(by Wald's Equation) } \frac{\lambda \mathbb{E}(N)\mathbb{E}(W_i)}{\mathbb{E}(N)} = \lambda W.$$

$\square$

Exercise 12. Queue, customers arrive in a renewal process with distribution F, so

$$\lambda^{-1} = \int_0^\infty \overline{F}(t)dt.$$

Each customer eventually leaves. Let L be the average number of customers in the system, averaged over all time. Let $X(s)$ be the number of customers in the system at time s. Note that

$$L = \lim_{t\to\infty} \frac{1}{t}\int_0^t X(s)ds.$$

Let W be the average time a customer spends in the system. Let W_i be the amount of time customer i spends in the system. Note that

$$W = \lim_{n\to\infty} \frac{W_1 + \ldots + W_n}{n}.$$

Solution.

7.5.1 Alternating Renewal Processes (Section 7.5.1 in *Introduction to Probability Models*

Definition 7.22 (Alternating renewal process). Suppose we have a process that is on for time Y_1, off for time Z_1, on for time Y_2, off for time Z_2, etc if (X_i, Y_i), $i \geq 1$ are i.i.d. Call $\{X(t), t \geq 0\}$ an **alternating renewal process**, where $X(t)$ is an indicator variable for the process being on at time t.

Proposition 7.32 (Proposition 7.4 in *Introduction to Probability Models*; similar to Theorem 7.33 (Theorem 3.4.4 in *Stochastic Processes*)).

(a) The long-run proportion of time that the system is on is

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t I\{X(s) = 1\}ds = \frac{\mathbb{E}(Y)}{\mathbb{E}(Y) + \mathbb{E}(Z)}.$$

(b)

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \Pr(X(s) = 1)ds = \frac{\mathbb{E}(Y)}{\mathbb{E}(Y) + \mathbb{E}(Z)}.$$

Proof. (a) Imagine we earn 1 dollar per unit time when the system is on and nothing when the system is off. Note that we have a renewal when the system turns back on after having been turned off. So this is a renewal reward process.

(b) Something about

$$\frac{R(t)}{t} \to \frac{\mathbb{E}(R)}{\mathbb{E}(T)}$$

and

$$\frac{\mathbb{E}R(t)}{t} \to \frac{\mathbb{E}(R)}{\mathbb{E}(T)}$$

□

Proposition 7.33 (Theorem 3.4.4 in *Stochastic Processes*; Similar to Proposition 7.32 (Proposition 7.4 in *Introduction to Probability Models*)). If the cycle distribution is not lattice (see Definition 6.10), then

$$\Pr(\text{on at } t) \to \frac{\mathbb{E}(Y)}{\mathbb{E}(Y) + \mathbb{E}(Z)} \text{ as } t \to \infty$$

where $\mathbb{E}(Y)$ is the expected length of time intervals when the system is on and $\mathbb{E}(Z)$ is the expected length of time intervals when the system is off.

Example 7.22. Suppose you have insurance and you pay at a rate r_1 until you have an accident. Then the rate is r_2 until s units pass without an accident, in which case you go back to paying rate r_1. Suppose accidents occur according to a Poisson process with rate λ. Then the probability you have to pay at rate r_1 is

$$\frac{\mathbb{E}(\text{on})}{\mathbb{E}(\text{on}) + \mathbb{E}(\text{off})}.$$

where we define the process to be "on" when your pay rate is r_1 and "off" when your pay rate is r_2. Note that

$$\mathbb{E}(\text{on}) = 1/\lambda$$

$$\mathbb{E}(\text{off}) = \int_0^\infty \mathbb{E}(\text{off} \mid T = x)\lambda e^{-\lambda x} dx$$

We have

$$\mathbb{E}(\text{off} \mid T = x) = \begin{cases} x + \mathbb{E}(\text{off}) & x < s \\ s & x > s \end{cases}$$

Substituting this in yields

$$\mathbb{E}(\text{off}) = \int_0^s (x + \mathbb{E}(\text{off}))\lambda e^{-\lambda x} dx + s\int_s^\infty \lambda e^{-\lambda x} dx = \int_0^s x\lambda e^{-\lambda x} dx + \mathbb{E}(\text{off})(1 - e^{-\lambda s}) + se^{-\lambda s}.$$

Recall that the age of the renewal process at time t is $A(t) - t - S_{N(t)}$ and the excess time at t is $Y(t) = S_{N(t)+1} - t$. We have already shown (in Proposition 7.29) that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t A(s)ds = \lim_{t\to\infty} \frac{1}{t}\int_0^t Y(s)ds = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)}.$$

Example 7.23 (Example 7.28 in *Introduction to Probability Models*). $M/G/\infty$ queue, (arrivals are a Poisson Process). Consider the system to be "on" if the system is empty and "off" when the system is not empty (busy period). Note that $\mathbb{E}(\text{on}) = \lambda^{-1}$. Let $\mathbb{E}(B)$ be the expected length of the "off" time (busy period). Find $\mathbb{E}(B)$.

Solution. A busy time starts when a first customer arrives. The long-run proportion of time the system is on is

$$\frac{\mathbb{E}(\text{on})}{\mathbb{E}(\text{on}) + \mathbb{E}(\text{off})}.$$

By Proposition 7.33, since this is nonlattice, this equals the limiting probability that the system is on at time t. That is,

$$\lim_{t\to\infty} \Pr(\text{on at } t) = \frac{\mathbb{E}(\text{on})}{\mathbb{E}(\text{on}) + \mathbb{E}(\text{off})}.$$

Let $X(t)$ be the number remaining in the system at time t. By Proposition 7.10,

$$X(T) \sim \text{Poisson}\left(\lambda \int_0^t \overline{G}(s)ds\right).$$

So

$$\lim_{t\to\infty} \Pr(\text{on at } t) = \Pr(X(t) = 0) = \exp\left(- \lambda \int_0^t \overline{G}(s)ds\right)$$

Plugging in we have **might have made a mistake here:** $S \sim G$, then $\lim_{t\to\infty} \Pr(\text{on at } t) = \mathbb{E}(S)$

$$\exp\left(- \lambda \int_0^t \overline{G}(s)ds\right) = \frac{\lambda^{-1}}{\lambda^{-1} + \mathbb{E}(\text{off})}.$$

think this bottom one below might be the right one

$$\exp\left(- \mathbb{E}(S)\right) = \frac{\lambda^{-1}}{\lambda^{-1} + \mathbb{E}(\text{off})}.$$

Definition 7.23 (Equilibrium distribution of a renewal process). We call

$$F_e(x) = \frac{1}{\mu}\int_0^x \overline{F}(t)\ dt$$

the **equilibrium distribution** of a renewal process with interarrival distribution F, where μ is the expected length of a cycle in the renewal process.

Proposition 7.34 (**Age of equilibrium distribution; restatement of Theorem 7.33 (Theorem 3.4.4 in *Stochastic Processes*); similar to Examples 7.26 and 7.27 in *Introduction to Probability Models*; Similar to Proposition 7.32 (Proposition 7.4 in *Introduction to Probability Models*)).** With probability 1, the long-run proportion of time that the age is less than x is

$$\lim_{s\to\infty} \frac{1}{s}\int_0^s I\{A(t) < x\}dt = F_e(x)$$

and the long-run proportion of the time that the excess is less than x

$$\lim_{s\to\infty} \frac{1}{s}\int_0^s I\{Y(t) < x\}dt = F_e(x)$$

both equal $F_e(x) = \frac{1}{\mu}\int_0^x \overline{F}(t)\ dt$. These are also equal to

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \Pr(A(s) < x)ds = F_e(x)$$

and

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \Pr(Y(s) < x)ds = F_e(x)$$

If F is not lattice, then

$$\lim \Pr(A(t) < a) = \lim \Pr(Y(t) < a) = F_e(t).$$

Proof. Say the system is “on” at time t if $A(t) < x$, “off” otherwise. A cycle is a renewal. So the system is “on” for the first x units of a renewal cycle, and then it is “off.” Once a renewal occurs, the age goes back to 0, so this is an alternating renewal process. The “on” time in a cycle is $\min\{x, X\}$ because it’s on for the first x units of time the item is in use of a cycle, unless the cycle is less than x, in which case it’s on for the whole length of the cycle. Therefore we have that the expected “on” time is

$$\frac{\mathbb{E}(\text{on time in cycle})}{\mu} = \frac{\mathbb{E}(\min\{x, X\})}{\mu} = F_e(x)$$

where μ is the average length of a cycle. Note that

$$\mathbb{E}(\min\{x, X\}) = \int_0^\infty \Pr(\min\{x, X\} > t)dt = \int_0^x \overline{F}(t)dt$$

so we have that the expected “on” time is

$$\frac{1}{\mu}\int_0^x \overline{F}(t)dt = F_e(x)$$

Now we will show that the expected excess is the same as the expected age. Say that the system is "on" at t if $Y(t) > x$ and "off" otherwise. A cycle happens every time a renewal occurs. So if a renewal cycle is longer than x, the system will be initially on and then become off for the last x time units of a renewal. So the off time is $\min\{x, X\}$. Again, by a similar argument as above we get the same result. (Intuitively this makes sense because the distribution of interarrival times is the same if you look backward in time as it is if you look forward in time.)

□

Recall Proposition 7.29: The average age and average excess both equal $\mathbb{E}(X^2)/(2\mathbb{E}(X))$. Note that

$$A(t) + Y(t) = S_{N(t)+1} - S_{N(t)} = X_{N(t)+1}.$$

Proposition 7.35 (**[Proposition 3.4.6 in *Stochastic Processes***). With probability 1, the average value of $X_{N(t)+1}$ is

$$\frac{\mathbb{E}(X^2)}{\mathbb{E}(x)} > \mathbb{E}(X)$$

Proof. Suppose $X_e \sim F_e$ where

$$F_e(x) = \frac{1}{\mu}\int_0^x \overline{F}(y)dy$$

Note that

$$\frac{\mathrm{d}F_e(x)}{\mathrm{d}x} = \frac{\overline{F}(x)}{\mu}$$

We have

$$\mathbb{E}(X_e) = \int_0^\infty x dF_e(x) = \int_0^\infty x\frac{\overline{F}(x)}{\mu}dx = \int_0^\infty \frac{x}{\mu}\int_x^\infty dF(y)dx = \text{(by Fubini's Theorem)} \int_0^\infty \int_0^y \frac{x}{\mu}dxdF(y)$$

$$= \int_0^\infty \frac{y^2}{2\mu}dF(y) = \frac{\mathbb{E}(X^2)}{2\mu}$$

□

Proposition 7.36 (Problem on homework 7; inspection paradox).

$$\Pr(X_{N(t)+1} > x) > \overline{F}(x).$$

7.5.2 Equilibrium renewal processes

Definition 7.24 (**Equilibrium renewal process**). Suppose we have a delayed renewal process where $X_1 \sim F_e$, $X_i \sim F \; \forall \; i > 1$. Then $\{N_e(t), t \geq 0\}$ is called an **equilibrium renewal process**. (the time until the first event has the equilibrium distribution.) We let $m_e(t) = \mathbb{E}(N_e(t))$.

Example 7.24. Suppose you arrive at some time t during a renewal process with finite expected renewal time μ. Then you will have to wait $Y(t)$ time until the next renewal. Let F_t be the distribution function of $Y(t)$. After that first arrival, all of the other arrival times will have distribution F. If the time t at which you start observing is very large, you observe an equilibrium renewal process (because $F_t \to F_e$ as $t \to \infty$).

Proposition 7.37 (**Theorem 3.5.2 from *Stochastic Processes***). Let $\{N_e(t), t \geq 0\}$ be an equilibrium renewal process. Let the excess at time t of $\{N_e(t)$ be $Y_e(t)$.

(1) $Y_e(t) \sim F_e$ for all t.

(2) $N_e(t+s) - N_e(t)$ has the same distribution for all t (this counting process has stationary increments—see Definition 7.4).

(3) $m_e(t) = t/\mu$.

Proof (More of an intuitive argument than a complete proof). (1) Consider an ordinary renewal process with interarrival times distributed as F. Suppose we start observing at a very large time t', so we are observing an equilibrium renewal process. That is, the time until the first event is distributed as F_e. We'd like to know the time until the next event after you've waited a time t. We have

$$Y_e(t) = Y(t + t') \sim F_e$$

since t' is very very large.

(2) Similar argument—after arriving after a very large amount of time, the distribution is in equilibrium, so it makes no difference when you start watching, just matters how large s is.

(3)

$$N_e(t+s) = N_e(t+s) - N_e(t) + N_e(t)$$

$$\iff \mathbb{E}(N_e(t+s)) = \mathbb{E}[N_e(t+s) - N_e(t)] + \mathbb{E}[N_e(t)] \iff m_e(t+s) = m_e(s) + m_e(t)$$

where $\mathbb{E}(N_e(t+s)) = m_e(t+s)$ by definition, $\mathbb{E}[N_e(t+s) - N_e(t)] = m_e(s)$ by the result from part (2), and $\mathbb{E}[N_e(t)] = m_e(t)$ by definition. It is the case that $f(x+y) = f(x) + f(y) \implies f(x) = cx$ for some constant c if f is measurable. Because of that, we have that

$$m_e(t) = ct$$

where c is some constant. By the Elementary Renewal Theorem (Theorem 7.24),

$$\frac{m_e(t)}{t} \to \frac{1}{\mu}$$

which implies that c is μ^{-1}.

$\square$

Proving that the previous statement is true for any measurable function requires measure theory, but it is easy to prove it is true if f is differentiable.

Lemma 7.38. If f is differentiable, $f(x+h) = f(x) + f(h)$.

Proof. Note that

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

Note that

$$\lim_{h \to 0} \frac{f(h)}{h} = f'(0)$$

Then $f'(x) = c$, $f(x) = cx + k$, so $f(x) = cx$.

□

Theorem 7.39 (Blackwell's Theorem (Theorem 3.4.1 in *Stochastic Processes*, p.110)). (i) If F is not lattice, then

$$m(t+a) - m(t) \to \frac{a}{\mu} \text{ as } t \to \infty$$

(ii) If F is lattice with period d, then

$$\mathbb{E}(\text{number of renewals at } nd) \to \frac{d}{\mu} \text{ as } n \to \infty.$$

Remark. If $\Pr(X_i = 0) = 0$ then then this means that $\Pr(\text{renewal at } nd) \to d/\mu$ as $n \to \infty$.

Proof (intuitive; rigorous proof is quite technical). (i) If you arrive at time t, then $\{N(s), s \geq 0\}$ is a delayed renewal process. That is, X_1 has the distribution of $Y(t) \sim F_t$, and the remaining $X_i \sim F$. So

$$m(t+a) - m(t) \ = \mathbb{E}[N_t(a)].$$

We already know that as $t \to \infty$, $F_t \to F_e$. So as $t \to \infty$, $m(t+a) - m(t) \ = \mathbb{E}[N_t(a)] \to \mathbb{E}[N_e(a)]$. But by Proposition 7.37, **the result follows (??).**

(ii) Suppose we have a renewal process where

$$\sum_{i=0}^{\infty} \Pr(X = i) = 1.$$

Then $m(n)$ is the expected number of renewals by time n. That is,

$$m(n) = \mathbb{E} \sum_{i=0}^{n} (\text{number of renewals at } i) = \sum_{i=0}^{n} \mathbb{E}(\text{number of renewals at } i)$$

Then by the Elementary Renewal Theorem (Theorem 7.24),

$$\frac{m(n)}{n} = \frac{1}{n}\sum_{i=0}^{n} \mathbb{E}(\text{number of renewals at } i) \to \frac{1}{\mu}$$

In general, this does not imply that

$$\mathbb{E}(\text{number of renewals at } nd) \to \frac{d}{\mu} \text{ as } n \to \infty.$$

For example, consider the discrete random variable

$$X_i = \begin{cases} 2 & \text{with probability } 1/2 \\ 4 & \text{with probability } 1/2 \end{cases}$$

Then

$$\mathbb{E}(\text{number of renewals at } n) = m(n) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(\text{number of renewals at } i) \to \frac{1}{\mu}$$

But of course all of these sums are 0 when i is odd. So we could also write this as

$$m(n) = \sum_{i=1}^{(} n/2)\frac{1}{n}\mathbb{E}(\text{number of renewals at } 2i) \to \frac{1}{\mu}$$

$$\iff m(n) = \frac{1}{2}\sum_{i=1}^{(} n/2)\frac{1}{n/2}\mathbb{E}(\text{number of renewals at } 2i) \to \frac{1}{\mu}$$

$$\iff m(n) = \sum_{i=1}^{(} n/2)\frac{1}{n/2}\mathbb{E}(\text{number of renewals at } 2i) \to \frac{2}{\mu}$$

$$\vdots$$

You get pointwise convergence in the lattice case with period d when d is the period.

□

Recall Proposition 7.29: The average age and average excess both equal $\mathbb{E}(X^2)/(2\mathbb{E}(X_e))$, where $X_e \sim F_e$.

Theorem 7.40 (Proposition 3.4.6 in *Stochastic Processes*). If F is not lattice,

$$\lim_{t\to\infty} \mathbb{E}(A(t)) = \lim_{t\to\infty} \mathbb{E}(Y(t)) = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)}.$$

Proof. Won't prove; requires Key Renewal Theorem, which is beyond scope of this course. Some notes: we know that $F_{Y(t)} \to F_e$, or $Y(t) \to Y(\infty)$ where $Y(\infty) \sim F_e$. Then

$$\lim \mathbb{E}(Y(t)) = (?\text{ can't justify now) } \mathbb{E}(\lim(Y(t))) = \mathbb{E}(X_\infty)$$

□

Corollary 7.40.1 (Corollary 3.4.7 in *Stochastic Processes*, p. 121). If $\mathbb{E}(X^2) < \infty$ and F is nonlattice, then

$$m(t) - \frac{t}{\mu} \to \frac{\mathbb{E}(X^2)}{2\mu^2} - 1 \text{ as } t \to \infty$$

Proof. Recall that by Wald's Equation (Theorem 7.23),

$$\mathbb{E}\left(\sum_{i=1}^{N(t)+1} X_i \right) = \mu(m(t) + 1)$$

So

$$\mu(m(t) + 1) = \mathbb{E}(t + Y(t)) = t + \mathbb{E}(Y(t)) \iff m(t) + 1 = \frac{t}{\mu} + \frac{\mathbb{E}(Y(t))}{\mu} \iff m(t) - \frac{t}{\mu} = \frac{\mathbb{E}(Y(t))}{\mu} - 1$$

$$\implies m(t) - \frac{t}{\mu} \to \frac{\mathbb{E}(X^2)}{2\mu^2} - 1 \text{ as } t \to \infty$$

$$\implies m(t) \approx \frac{t}{\mu} + \frac{\mathbb{E}(X^2)}{2\mu^2} - 1$$

□

Theorem 7.41 (Central Limit Theorem for Renewal Processes; Theorem 7.3 in *Introduction to Probability Models*). Let $N(t)$ be a renewal process. Then

$$N(t) \xrightarrow{d} \mathcal{N}\left(\frac{t}{\mu}, \frac{t\sigma^2}{\mu^3} \right)$$

where $\mu = \mathbb{E}(X_i)$ and $\sigma^2 = \text{Var}(X_i)$.

Remark. If the renewal process is a Poisson process with rate λ, then $F(x) = 1 - e^{-\lambda x}$. Then $\mu = \mathbb{E}(X) = \lambda^{-1}$, $\sigma^2 = \text{Var}(X) = \lambda^{-2}$, $t/\mu = \lambda t$, $t\sigma^2/\mu^2 = t\lambda^3/\lambda^2 = \lambda t$, which implies the limiting distribution is $\mathcal{N}(\lambda t, \lambda t)$, as expected.

Proof. Note that $N(t) < n$ if and only if the nth event occurred after t, or $X_1 + \ldots + X_n > t$. So

$$\Pr(N(t) < n) = \Pr(X_1 + \ldots + X_n > t)$$

Then by the Central Limit Theorem,

$$X_1 + \ldots + X_n \sim \mathcal{N}(n\mu, n\sigma^2) \text{ (approximately)}$$

$$\vdots$$

□

7.5.3 Regenerative Processes

Theorem 7.42 (Theorem 3.7.1 in *Stochastic Processes*). Let $\{X(t), t \geq 0\}$ be a stochastic process with state space $\{0, 1, 2, \ldots\}$ having the property that there exist time points at which the state restarts itself (with probability 1). Let $S_1, S_2, \ldots$ constitute the event times of a renewal process. We call $X(t)$ a **regenerative process**. So $\{S_1, S_2, \ldots\}$ constitute the event times of a renewal process. We say a cycle is completed every time a renewal occurs. Let $N(t) = \max\{n : S_n \leq t\}$ denote the number of cycles by time t.

If F, the distribution of a cycle, has a density over some interval, and if $\mathbb{E}(S_1) < \infty$, then

$$P_j = \lim_{t \to \infty} \Pr(X(t) = j) = \frac{\mathbb{E}(\text{amount of time in state } j \text{ in a cycle})}{\mathbb{E}(\text{time of a cycle})}.$$

7.5.4 Example of Renewal Reward Processes to Patterns

Example 7.25 (Seems to be similar to Section 7.9.2 in *Introduction to Probability Models*). Suppose we have i.i.d. data. Number i appears with probability p_i.We want to see the mean time until we see the pattern 1213121. An event is when the pattern appears. The next event is the next time the pattern appears, but we can't use data from the last event. Earn \$1 whenever the last 7 values are 1313121 (so you *are* allowed to use previous values for the reward process). That is, if the cycle is

$$\ldots 1213121 \mid 312\mathbf{1} \ldots 1213121 \mid$$

the events happen at the | marks, but in between there were two rewards (one at the second event, one at the bold 1). So this is a delayed renewal reward process $(X_i, R_i), i \geq 2$. Note that the limiting results will be the same for the delayed renewal process as for the normal one. So the expected average reward per unit time is the expected reward during the cycle divided by the expected time of a cycle;

$$\mathbb{E}(\text{average reward per unit time}) = \frac{\mathbb{E}(\text{reward during the cycle})}{\mathbb{E}(T)}$$

We want $\mathbb{E}(T)$, the expected length of a cycle. We know $\mathbb{E}$(average reward per unit time) is

$$\mathbb{E}(\text{average reward per unit time}) = p_1^4 p_2^2 p_3$$

Now we want $\mathbb{E}$(reward during the cycle). Suppose a cycle just ended (the pattern just appeared). Then we could earn a reward after 4 values (if the last 4 terms of the pattern appear again) after 6 values (if the last 6 terms of the pattern appear again). And we will earn a reward when the cycle ends (after the full pattern appears again). So if Y_i is the reward earned during after i new values,

$$\text{reward during the cycle} = Y_4 + Y_6 + 1 \implies \mathbb{E}(\text{reward during the cycle}) = \mathbb{E}(Y_4) + \mathbb{E}(Y_6) + 1 = p_1^2p_2p_3 + p_1^3p_2^2p_3 + 1$$

$$\implies \mathbb{E}(T) = \frac{p_1^2p_2p_3 + p_1^3p_2^2p_3 + 1}{p_1^4p_2^2p_3} = \frac{1}{p_1^2p_2} + \frac{1}{p_1} + \frac{1}{p_1^4p_2^2p_3}$$

Note that the mean time between renewals is $\frac{1}{p_1^4p_2^2p_3}$, the mean time to get 121 is $\frac{1}{p_1^2p_2}$, and the mean time to get 1 is $\frac{1}{p_1}$.

Example 7.26 (Similar to example 7.38 in *Introduction to Probability Models*). Suppose we are flipping coins, probability of flipping heads is p. The pattern is n heads in a row. Cycle is. a pattern without using data from last cycle. Get a reward of 1 each time the last n values were all heads. Then

$$p^n = \mathbb{E}(\text{average reward per unit time}) = \frac{\mathbb{E}(\text{reward during the cycle})}{\mathbb{E}(T)}$$

Note that

$$\text{reward during the cycle} = Y_1 + \ldots + Y_{n-1} + 1 \implies \mathbb{E}(\text{reward during the cycle}) = \sum_{i=1}^{n-1} \mathbb{E}(Y_i) + 1$$

$$= \sum_{i=1}^{n-1} p^i + 1$$

$$\implies p^n = \frac{1}{\mathbb{E}(T)} \left(\sum_{i=1}^{n-1} p^i + 1 \right) \implies \mathbb{E}(T) = p^{-n} \left(\sum_{i=1}^{n-1} p^i + 1 \right) = \left(\frac{1}{p}\right)^n + \left(\frac{1}{p}\right)^{n-1} + \ldots + \frac{1}{p}$$

7.6 Markov Chains (Chapter 4 of *Stochastic Processes*; Chapter 4 of *Introduction to Probability Models*)

Suppose we have $X_0, X_1, X_2, \ldots$, where X_n is the state of the system at time n. The set of possible values of states are the nonnegative integers (the number of possible states will be either finite or countable). In a Markov chain, the probability of reaching state j given that you are currently in state i is P_{ij}; that is,

$$\Pr(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = P_{ij}$$

That is, the next state depends only on the current state, not any of the previous states (all the information is contained in the current state). In other words, every starts over again when you reach state i; so there is a renewal every time you reach state i. We say a system has the **Markovian property** if given the present state X_n, the next state X_{n+1} is independent of the past $X_{n-1}, X_{n-2}, \ldots$.

Definition 7.25 (Markov chain). Let S be the set of possible values of a system. S must be either finite or countably infinite (in general we will take $S = \mathbb{N}$). Then $\{X_0, X_1, \ldots\}$ is said to be a **Markov chain** with transition probabilities $\Pr(X_{n+1} = j \mid X_n = i)P_{i,j}$, $i, j \in S$ if

$$\Pr(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = \Pr(X_{n+1} = j \mid X_n = i) = P_{ij}.$$

We call the matrix

$$P = \left[P_{ij}\right]$$

the **transition probability matrix.** Note that $\sum_j P_{ij} = 1$, so the the rows of the transition probability matrix all add to 1.

Remark. Given the initial state $\Pr(X_0 = i_0)$ and the probability transition matrix, all probabilities about $X_0, X_1, \ldots$ can (in theory) be determined. That is,

Note that

$$\Pr(X_n = i_n) = \sum_{i_0, i_1, \ldots, i_{n-1}} \Pr(X_0 = i_0) P_{i_0, i} P_{i_1, i_2} \ldots, P_{i_{n-1}, i_n}.$$

Example 7.27 (Examples 4.1 and 4.4 in *Introduction to Probability Models*). Suppose we have a state based on weather today and yesterday. Then probability of rain tomorrow is the following:

$$\begin{cases} dd \to r & \text{with probability } 0.3 \\ rd \to r & \text{with probability } 0.4 \\ dr \to r & \text{with probability } 0.6 \\ rr \to r & \text{with probability } 0.7 \end{cases}$$

The the transition matrix looks as follows:

$$P = \begin{array}{cc} & \begin{array}{cccc} dd & rd & dr & rr \end{array} \\ \begin{array}{c} dd \\ rd \\ dr \\ rr \end{array} & \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.6 & 0 & 0.4 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.3 & 0 & 0.7 \end{bmatrix} \end{array}$$

Example 7.28 (Random walk; examples 4.5 and 4.6 in *Introduction to Probability Models*).

$$P_{i,i+1} = p, P_{i,i-1} = 1 - p, \qquad S = \mathbb{Z}$$

One example is Gambler's Ruin: $X_0 = i$, stop when fortune is either 0 or N. (See Example 7.41 and several solutions in Grimmett and Stirzaker.)

Example 7.29 (Similar to example 4.10 in *Introduction to Probability Models*, similar to homework problem from Math 505A). Urn with 2 balls, red and blue. Each period, choose ball from urn at random. If red then replace with blue. If blue then replace with red with probability 0.7, with blue with probability 0.3. What is the long-run proportion of time the chosen ball is red?

Solution. Let X_n be an indicator variable for the nth ball chosen to be red. Note that if you just chose a red ball, it is not clear what the probability of choosing a red ball next is, because it depends on what color the other ball in the urn is. So consider the state of the Markov chain to be the number of red balls in the urn, and let X_n be the number of red balls in the urn before the nth withdrawal. Then we have the following transition matrix:

$$P = \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.5 & 0.5 \cdot 0.3 & 0.5 \cdot 0.7 \\ 0 & 1 & 0 \end{bmatrix}$$

Let π_j be the long-run proportion of the time the system is in state j. Note that every time we reach state j everything starts all over again, so that's a renewal (ordinary renewal process) if you are certain to eventually return to state j (we will assume this is true for now). We have

$$\pi_0 \cdot 0 + \pi_1 \cdot \frac{1}{2} + \pi_2 = \pi_2 + \frac{1}{2}\pi_1$$

is the long-run probability of choosing a red ball.

Example 7.30 (**Embedded Markov chain; example 4.1(a) in *Stochastic Processes***)**.** $M/G/1$ queueing process. The number of people currently in the system is not Markovian, but it would be if the service time were exponential (since then it wouldn't matter how long the customer had been in service). But the number of people in the system immediately after the nth service completion is Markovian. (Called an **embedded Markov chain** because it is a Markov chain if you only look at it at certain times.)

Note that

$$P_{0j} = a_j = \Pr(j \text{ arrivals during service time}) = \int_0^\infty \Pr(j \text{ arrivals during service time } t) dG(t)$$

$$= \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^j}{j!} dG(t)$$

More generally, note that for $i > 0, j \geq i - 1$

$$P_{ij} = a_{j-i+1} = \Pr(j{-}i{+}1 \text{ arrivals during service time}) = \int_0^\infty \Pr(j{-}i{+}1 \text{ arrivals during service time } t) dG(t)$$

$$= \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^(j-i+1)}{(j-i+1)!} dG(t).$$

Example 7.31 (**Example 4.1(b) in *Stochastic Processes***)**.** if X_n is the number of people in the system as seen by the nth arrival, then this is an (embedded) Markov chain.

$\vdots$

We want $P_{ij}, j > 0$. If n is the number of people served, then

$$i + 1 - n = j \iff n = i + 1 - j$$

Then

$$P_{ij} = \int \Pr(i + 1 - j \text{ services in time } T \mid T = s) dF(s) = \int e^{-\mu s} \frac{(\mu s)^{i+1-j}}{(1 + 1 - j)!} dF(s)$$

P_{i0} is different because if we end with 0 people, all of the previous people had to depart. But they could have departed at any time before the next person arrived. Imagine that when all the customers leave, the server keeps serving "imaginary" customers. Then P_{i0} is the probability of at least $i + 1$ customers (real or imaginary) being served.

Definition 7.26 (**n-step transition probabilities**).

$$P_{ij}^n = \Pr(X_n = j \mid X_0 = i)$$

7.6.1 Chapman-Kolmogorov Equations—section 4.2 of *Introduction to Probability Models*, section 4.2 of *Stochastic Processes* (p. 178 of pdf)

Proposition 7.43 (**Chapman-Kolmogorov Equations**).

Definition 7.27 (**Matrix of n-step transition probabilities**).

$$\boldsymbol{P}^{(n)} = \left[P_{ij}^n\right]$$

Example 7.32 (**Similar to example 4.10 in *Introduction to Probability Models***). Urn initially has 1 red ball 1 blue. At each stage choose a ball at random. If red, replace with blue. If blue, replace with blue with probability 1/3 or red with probability 2/3. What is the probability that the 5th ball selected is red?

Solution. Note that per the example from last time, knowing the ball you just drew is not sufficient to have a Markov chain (need to know color of other ball in urn to get probabilities of next state). So let X_n be the number of red balls in the urn after you draw the nth ball, and note that $X_0 = 1$. Note that there are three states. The probability transition matrix is

$$P = \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 \end{bmatrix}$$

Note that the probability the 5th ball selected is red is equal to

$$\sum_{i=1}^{2} \Pr(5th \text{ ball is red } \mid X_4 = i) P_{1i}^4 = 0 \cdot P_{10}^4 + \frac{1}{2} P_{11}^4 + 1 \cdot P_{12}^4$$

Now we calculate P^4.

$$P^2 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{9} & \frac{3}{9} & \frac{2}{9} \\ \frac{3}{12} & \frac{25}{36} & \frac{2}{36} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{bmatrix}$$

$$P^4 = P^2 P^2 = \begin{bmatrix} \frac{4}{9} & \frac{3}{9} & \frac{2}{9} \\ \frac{3}{12} & \frac{25}{36} & \frac{2}{36} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{4}{9} & \frac{3}{9} & \frac{2}{9} \\ \frac{3}{12} & \frac{25}{36} & \frac{2}{36} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{bmatrix} = \ldots$$

Example 7.33 (Example 4.12 from *Introduction to Probability Models*). Part (c) (not in book): N is the number of flips until either a run of 3 heads or 3 tails. $\Pr(N = 8)$?

Solution. Say start at state 0, state i if you are on a run of i heads for $i = 1, 2, 3$, and state i if you are on a state of $i - 3$ tails, $i = 4, 5, 6$. Then we have

$$P = \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} & P_{04} & P_{05} & P_{06} \\ P_{10} & P_{11} & P_{12} & P_{13} & P_{14} & P_{15} & P_{16} \\ P_{20} & P_{21} & P_{22} & P_{23} & P_{24} & P_{25} & P_{26} \\ P_{30} & P_{31} & P_{32} & P_{33} & P_{34} & P_{35} & P_{36} \\ P_{40} & P_{41} & P_{42} & P_{43} & P_{44} & P_{45} & P_{46} \\ P_{50} & P_{51} & P_{52} & P_{53} & P_{54} & P_{55} & P_{56} \\ P_{60} & P_{61} & P_{62} & P_{63} & P_{64} & P_{65} & P_{66} \end{bmatrix} = \begin{bmatrix} 0 & p & 0 & 0 & 1-p & 0 & 0 \\ 0 & 0 & p & 0 & 1-p & 0 & 0 \\ 0 & 0 & 0 & p & 1-p & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 1-p & 0 \\ 0 & p & 0 & 0 & 0 & 0 & 1-p \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

In general, if T_i is the number of transitions to enter state i even state 0 is 0, we can change state i into an absorbing state.

7.6.2 Classification of States (Section 4.3 of *Introduction to Probability Models*)

We say state j is *accessible* from state i if for some $n \geq 0$ $P_{ij}^n > 0$. Note that $P_{ij}^n = \Pr(X_n = j \mid X_0 = i)$. So j is accessible from i if and only if starting in i it is possible that the Markov chain is every in j. So if j is not accessible from i

$$\Pr(\text{ever in } j \mid X_0 = i) = \Pr\left(\bigcup_n \{X_n = j\} \mid X_0 = i \right) \leq \sum_n \Pr\big(\{X_n = j\} \mid X_0 = i\big) = 0$$

Two states i and j that are accessible to each other are said to *communicate*, and we write $i \leftrightarrow j$. Two states that communicate are said to be in the same *class*.

A Markov chain is **irreducible** if there is only one class.

Example 7.34.

$$P = \begin{bmatrix} 0.2 & 0.8 & 0 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 & 0 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0.1 & 0.9 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

Then the classes are $\{0,1\}$, $\{2\}$, and $\{3,4\}$. Note that for the classes $\{0,1\}$ and $\{3,4\}$ you will stay there forever—they are essentially Markov chains themselves. But if you start in state 2 you will eventually go to one of the other states and never go back. In contrast consider

$$P = \begin{bmatrix} 0.2 & 0.8 & 0 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0.9 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

where all three states have this property. We can merge classes $\{2\}$ and $\{3,4\}$ like this:

$$P = \begin{bmatrix} 0.2 & 0.8 & 0 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 & 0 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0.1 & 0.9 \\ 0 & 0 & 0.1 & 0.5 & 0.4 \end{bmatrix}$$

Let N_k be the number of transitions until you first enter state k. (discussion on p. 195 - 196 of *Introduction to Probability Models.*)

Example 7.35 (Example 4.12 in *Introduction to Probability Models*).

$$\Pr(N = 8) = P_{02}^7 p = P_{00}^5 p^3$$

Definition 7.28. For any states i and j in a Markov process, let f_{ij}^n be the probability that starting in i the first transition into j occurs at time n. Formally,

$$f_{ij}^0 = 0, \qquad f_{ij}^n = \Pr(X_n = j, X_k \neq j, k = 1, \ldots, n-1 \mid X_0 = i).$$

Let

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^n.$$

Then f_{ij} denotes the probability of ever making a transition into state j given that the process starts in i.

Proposition 7.44 (Homework problem; Problem 4.4 in *Stochastic Processes*).

$$P_{ij}^n = \sum_{k=0}^{n} f_{ij}^k P_{jj}^{n-k}.$$

Proof. Recall that P_{ij}^n is the probability that a process in state i will be in state j after n additional transitions. Also, f_{ij}^n is the probability that starting in I the first transition into j occurs at time n. Note that

$$P_{ij}^n = \sum_{k=0}^{n} \Pr(\text{first reach } j \text{ at time } k) \cdot \Pr(\text{end at } j \text{ starting from } j \text{ in } n-j \text{ steps}) = \sum_{k=0}^{n} f_{ij}^k P_{jj}^{n-k}.$$

□

Proposition 7.45 (Homework problem; Problem 4.5 in *Stochastic Processes*). Let

$$P_{ij/k} = \Pr(X_n = j, X_\ell \neq k, \ell = 1, \ldots, n-1 \mid X_0 = i).$$

(This is the probability that at time n we are in state j and we never reached state k between time 0 and n, given that we started in state i.) Then for $i \neq j$,

$$P_{ij}^n = \sum_{k=0}^{n} P_{ii}^k P_{ij/i}^{n-k}.$$

Proof. There are two classes of ways we can reach j starting from i: we either go back to i at some point and then end up at j, or we go straight to j without ever going back to i. If we go back to i for the last time at time $k \in \{1, \ldots, n-1\}$, we can end up at j at time n with probability $P_{ij/i}^{n-k}$. We end up back at state i for the last time at time k before time n (regardless of what happened before time k) with probability P_{ii}^k. Therefore,

$$\text{for } i \neq j, P_{ij}^n = \sum_{k=0}^{n} P_{ii}^k P_{ij/i}^{n-k}.$$

□

Definition 7.29 (Recurrent state). We call state i **recurrent** if starting in i the Markov Chain returns to i with probability 1.

Definition 7.30 (Transient state). We call state i **transient** if starting in i the probability the Markov Chain returns to i is less than 1.

Proposition 7.46 (Proposition 4.1 in *Introduction to Probability Models*, Proposition 4.2.3 in *Stochastic Processes*). In a Markov chain, state i is recurrent if

$$\sum_{n=1}^{\infty} P_{ii}^n = \infty$$

and transient if

$$\sum_{n=1}^{\infty} P_{ii}^n < \infty.$$

Proof. State i is recurrent if with probability 1 a process starting at i will eventually return to i. However, by the Markovian property it follows that this process will return again to i with probability 1, and so on. There is no last term, so with probability 1, the number of visits to i will be infinite.

On the other hand, suppose i is transient. Then each time the process returns to i there is a positive probability $1 - f_{ii}$ that it will never again return. That is,the probaiblity the total time in i is n periods given you start in i is

$$f_{ii}^{n_i - 1}(1 - f_{ii}).$$

Therefore the number of visits is geometric with finite mean $(1 - f_{ii})^{-1}$. Let I_n be an indicator variable for $X_n = i$. Then the total number of visits to state i is $\sum_{n=0}^{\infty} I_n$ and the expected number of visits if we start at i is

$$\mathbb{E} \sum_{n=0}^{\infty} \{I_n = 1 \mid X_0 = i\} = \sum_{n=0}^{\infty} \mathbb{E}\{I_n = 1 \mid X_0 = i\} = \sum_{n=0}^{\infty} \Pr\{X_n = i \mid X_0 = i\}$$

□

Corollary 7.46.1 (Corollary 4.2 in *Introduction to Probability Models*, Corollary 4.2.4 in *Stochastic Processes*). If state i is recurrent and state i communicates with state j, then state j is recurrent.

Proof. Let m and n be such that $P_{ij}^n < 0$, $P_{ji}^m > 0$ (these values exist since i and j communicate). Then for any $s \geq 0$,

$$P_{jj}^{m+n+s} \geq P_{ji}^m P_{ii}^s P_{ij}^n.$$

Therefore

$$\sum_{s=0}^{\infty} P_{jj}^{m+n+s} \geq P_{ji}^m P_{ij}^n \sum_{s=0}^{\infty} P_{ii}^s = \infty \implies \sum_{n=0}^{\infty} P_{jj}^n = \infty$$

where $\sum_{s=0}^{\infty} P_{ii}^s = \infty$ by definition of i being recurrent.

□

Corollary 7.46.2 (Corollary 4.2.5 in *Stochastic Processes*). If state i is recurrent and state i communicates with state j, then

$$f_{ij} = \Pr(\text{ever enter } j \mid X_0 = i) = 1$$

Proof. Suppose $X_0 = i$, and let n be such that $P_{ij}^n > 0$. Say that we miss opportunity 1 if $X_n \neq j$. If we miss opportunity 1, then let T_1 denote the next time we enter i. Note that T_1 is finite with probability 1 by Corollary 7.46.1. Say we miss opportunity 2 if $X_{T_1+n} \neq j$. If opportunity 2 is missed, let T_2 denote the next time we enter i and say we miss opportunity 3 if $X_{T_2+n} \neq j$, and so on. It is easy to see that the opportunity number of the first success is a geometric random variable with mean $1/P_{ij}^n$, and is thus finite with probability 1. The result follows since i being recurrent implies that the number of potential opportunities is infinite.

□

So we have the recurrence and transience are class properties (if one element in the class has them, they all do).

Example 7.36 (Simple random walk; example 4.18 in *Introduction to Probability Models*).

$$\frac{(2n)!}{n!n!} \frac{(2n)^{2n+1/2}e^{-2n}\sqrt{2\pi}}{n^{2n+1}e^{-2n}2\pi} = \frac{4^n}{\sqrt{n\pi}}$$

Homework problem: If R is a recurrent class, $i \in R$, $j \notin R$, then $P_{ij} = 0$. Why? Suppose $P_{ij} > 0$. But since i doesn't communicate with j, $P_{ij}^n = 0$ for all n. Contradiction.

7.6.3 Long-Run Proportions and Limiting Probabilities (Limit Theorems) (4.4 in *Introduction to Probability Models*, 4.3 in *Stochastic Processes*

Let π_j be the long-run proportion of the time you are in state j. That is,

$$\pi_j = \lim_{n\to\infty} \frac{N_n(j)}{n}$$

where $N_n(j)$ is the time in j during the first n periods. Then this equals $1/m_j$ where m_j is the expected time to re-enter j given that we started in j if state j is recurrent.

Bad approach to find:

$$m_j = \sum_k \mathbb{E}[\text{return to } j \mid X_1 = k]P_{jk} = P_{jj} + \sum_{k\neq j}(i + m_{kj})P_{jk} = 1 + \sum_{k\neq j} m_{kj}P_{jk}$$

$$m_{ij} = 1 + \sum_{k\neq j} P_{ik}m_{kj}$$

for $|S|$ states.

Proposition 7.47 (Proposition 4.4 in *Introduction to Probability Models*). If the Markov chain is irreducible and recurrent, then for any initial state,

$$\pi_j = \frac{1}{m_j}.$$

Proposition 7.48 (Proposition 4.5 in *Introduction to Probability Models*).

Theorem 7.49 (Theorem 4.1 in *Introduction to Probability Models*). In an irreducible Markov chain is positive recurrent, then the long-run proportions are the unique solution of the equations

$$\pi_j = \sum_i \pi_i P_{ij}, \qquad j \geq 1$$

$$\sum_j \pi_j = 1.$$

Moreover, if there is no solution of the preceding linear equations, then the Markov chain is either transient or null recurrent and all $pi_j = 0$.

Proof. π_i is the long-run proportion of time you are in state i; think of it as the long-run proportion of transitions that come from i. These transitions go to j with probability P_{ij}. So $\pi_i P_{ij}$ is the long-run proportion of transitions that go from i to j. If we sum over all i then we get the long-run proportion of transitions that go into j, which is the same as the proportion of time you're in j. So

$$\pi_j = \sum_i \pi_i P_{ij}$$

□

For an irreducible Markov chain, let $N_n(j)$ be the number of transitions into j by time n. Every time we reach state j we have a renewal. By the Strong Law for Renewal Processes (Theorem 7.21),

$$\frac{N_n(t)}{n} \xrightarrow{a.s.} \frac{1}{m_{jj}}$$

where m_{jj} is the expected number of transitions returning to j given that $X_0 = j$. By the Elementary Renewal Theorem (Theorem 7.24),

$$\lim_{s \to \infty} \frac{\mathbb{E}(N_n(s))}{n} = \frac{1}{m_{jj}}.$$

Note that

$$N_n(j) = \sum_{k=1}^{n} I_k$$

where I_k is an indicator variable for $X_k = j$.

$$\implies \frac{1}{n} \sum_{k=1}^{n} \Pr(X_k = j) \xrightarrow{a.s.} \frac{1}{m_{jj}}.$$

Let $\pi_j = \frac{1}{m_{jj}}$.

Definition 7.31. If state j is recurrent, call it **positive recurrent** if $m_{jj} < \infty$ and **null recurrent** if $m_{jj} = \infty$.

Remark. Being positive recurrent is equivalent to $\pi_j > 0$.

Proposition 7.50 (Proposition 4.3.2 in *Stochastic Processes* (p.185 of pdf)). Positive (null) recurrence is a class property.

Proof. Suppose state i is positive recurrent; that is $\pi_i > 0$. We will show that if it communicates with another state j, that state must be recurrent. Let n be such that $P_{ij}^n > 0$; such an n exists since i and j communicate.. Then $\pi_i P_{ij}^n$ is the long-run proportion of the time the chain is in state i and will be in state j n time periods later (or the proportion of time that the Markov chain is in state j and was in state i n time periods earlier). Since this is less than or equal to the proportion of time the Markov chain is in state j, we have

$$\pi_i P_{ij}^n \leq \pi_j$$

But since $\pi_i > 0$, we have

$$0 < \pi_i P_{ij}^n \leq \pi_j \implies \pi_j > 0.$$

□

Remark. This also shows that null recurrence is a class property (because if one element of the class is not positive recurrent, it must be that they are all not positive recurrent).

Note that if we have a Markov chain with $m < \infty$ states. we have to have

$$\sum_{j=1}^{m} N_n(0) = n \iff \frac{1}{m}\sum_{j=1}^{m} N_n(0) = 1$$

$$\implies 1 = \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{m} N_n(0) = \sum_{j=1}^{m} \lim_{n\to\infty} \frac{N_n(0)}{n} = \sum_{j=1}^{m} \pi_j$$

so all the π_j have to add up to 1.

Definition 7.32. The non-negative numbers $x_j, j \in S$ are said to be **stationary probabilities** of a Markov chain if

$$x_j = \sum_i x_i P_{ij}, \qquad i \in S$$

$$\sum_{j \in S} x_j = 1.$$

Definition 7.33 (p. 185 of *Stochastic Processes* pdf). A probability distribution $\{P_j, j \geq 0\}$ is said to be **stationary** for the Markov chain if

$$P_j = \sum_{i=0}^{\infty} P_i P_{ij}, \qquad j \geq 0.$$

Lemma 7.51. Suppose $\{x_i, i \in S\}$ is a stationary probability vector for a Markov chain. If $\Pr(X_0 = i) = x_i, i \in S$ $\Pr(X_n = j) = x_j, j \in S$ for all n.

Proof. We will prove this by induction on n for X_n. The case $n = 0$ is true by assumption. Assume $\Pr(X_n = i) = x_i, i \in S$. Then

$$\Pr(X_{n+1} = j) = \sum_{i \in S} \Pr(X_{n+1} = j \mid X_n = i) \Pr(X_n = i) = \sum_{i \in S} P_{ij} x_i = x_j$$

where the last step follows by Definition 7.32. $\square$

By the Elementary Renewal Theorem (Theorem 7.24),

$$\sum_{k=1}^{n} \frac{1}{n} \Pr(X_k = j) \xrightarrow{a.s.} \frac{1}{m_{jj}}.$$

Suppose the initial state is chosen randomly according to the probability vector.

Proposition 7.52 (Stationary probability vector is π_j). If x_j is a stationary probability vector then $x_j = \pi_j$, $j \in S$.

Proof. Let x_j be a stationary probability vector. Suppose $\Pr(X_0 = i) = x_i, i \in S$. Then $\Pr(X_k = j) = x_j$ for all k by Lemma 7.51. By the Elementary Renewal Theorem (Theorem 7.24),

$$\frac{1}{n} \Pr(X_k = j) \xrightarrow{a.s.} \pi_j.$$

But also

$$\lim_{n \to \infty} \frac{1}{n} \Pr(X_k = j) = x_j$$

so $\pi_j = x_j$.

$\square$

Definition 7.34 (Ergodic Markov chain state; p.174 of pdf of *Stochastic Processes*). A positive recurrent, aperiodic state is called **ergodic**.

Definition 7.35 (Ergodic Markov chain; p.188 of pdf of *Stochastic Processes*). We say a Markov chain is **ergodic** if all states are positive recurrent; that is,

$$\pi_j = \lim_{n \to \infty} P_{ij}^n > 0.$$

In this case, $\{\pi_j, j = 0, 1, 2, \ldots\}$ is a stationary distribution and there exists no other stationary distribution.

Definition 7.36. A Markov chain that can only return to a state in a multiple of $d > 1$ steps is said to be **periodic** and does not have limiting probabilities. That is, $P_{ij}^n = 0$ except when n is a multiple of d and d is the largest such integer.

An irreducible chain that is not periodic is said to be **aperiodic**.

Lemma 7.53. Periodicity is a class property. That is, if i communicates with j then i and j have the same period.

Proof. Requires non-trivial results from algebra to prove.

□

Also, $P_{jj}^{nd} \to d/m_{jj}$. For an aperiodic irreducible Marko chain, $P_{ij}^{n} \to 1/m_{jj} = \pi_j$. (Related to Blackwell's Theorem.)

Example 7.37 (Example 4.18 in *Introduction to Probability Models*). Stopping time:

$$N = \min\{n : X_1, + \ldots + X_n = 1\} \implies 1 = X_1 + \ldots + X_N$$

Apply Wald's Equation (Theorem 7.23):

$$1 = \mathbb{E}[X_1 + \ldots + X_N] = \mathbb{E}(X)\mathbb{E}(N) = 0$$

So Wald's equation leads to a contradiction, meaning its assumptions weren't satisfied, which must be because $\mathbb{E}(N) = \infty$. See also Example 6.2 in the Probability notes.

Example 7.38 (Example 4.24 in *Introduction to Probability Models*). Not an alternating renewal process because in general it depends on which "up" state you enter into. Typically use renewal reward processes to find average length of "up" and "down" periods.

Theorem 7.54 (Theorem 4.3.3 in *Stochastic Processes*, p. 186 of pdf). Suppose we have an irreducible aperiodic Markov chain. If it is either transient or null recurrent, then there are no solutions of the equations

$$x_j = \sum_{i \in S} x_i P_{ij}, \qquad j \in S,$$

$$\sum_j X_j = 1.$$

If it is positive recurrent, then $\{\pi_j, j \in S\}$ uniquely satisfy the preceding equations. (There is no irreducible finite state Markov chain.)

Proof. Suppose a Markov chain is positive recurrent. Fix a state 0 and consider the Markov chain to start a new cycle each time it transitions into state 0. Think of this as a renewal reward process: suppose you earn 1 each time a transition into state j occurs (renewal each time you reach state 0 again). Then the average reward per unit time is π_j. But this also equals

$$\frac{\mathbb{E}(N_j)}{\mathbb{E}(N)}$$

where N_j is the number of transitions into state j during a cycle and N is the number of transitions in a cycle, so $N = \sum_j N_j$. Let N_{ij} be the number of transitions from state i to j in a cycle, so $N_j = \sum_{i \in S} N_{ij}$. Then we have

$$\mathbb{E}(N_j) = \sum_{i \in S} \mathbb{E}(N_{ij})$$

Let I_k is an indicator variable for transitioning from i to j with $\mathbb{E}(I_k) = P_{ij}$. Then $N_{ij} = \sum_{k=1}^{N_i} I_k$. Note that N_i is a stopping time for $I_1, I_2, \ldots$ because it depends on what happened before but not on the future. Therefore by Wald's Equation (Theorem 7.23),

$$\mathbb{E}(N_{ij}) = \mathbb{E} \sum_{k=1}^{N_i} I_k = \mathbb{E}(I_k)\mathbb{E}(N_i) = \mathbb{E}(N_i)P_{ij}.$$

Using that we have

$$\mathbb{E}(N_j) = \sum_{i \in S} \mathbb{E}(N_{ij}) = \sum_{i \in S} \mathbb{E}(N_i)P_{ij}$$

$$\implies \frac{\mathbb{E}(N_j)}{\mathbb{E}(N)} = \sum_{i \in S} \frac{\mathbb{E}(N_i)}{\mathbb{E}(N)} P_{ij} \iff \pi_j = \sum_i \pi_i P_{ij}.$$

Also note that

$$\sum_j \pi_j = \sum_j \frac{\mathbb{E}(N_j)}{\mathbb{E}(N)} = \frac{\sum_j \mathbb{E}(N_j)}{\sum_j \mathbb{E}(N_j)} = 1.$$

□

Example 7.39 (Similar to section 4.1.1 in *Introduction to Probability Models*). Suppose we have a 2 state Markov chain for the weather. 0 means dry, 1 means rain. dry to dry with probability 0.7, rain to rain with probability 0.6. So

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

Then π_j are the unique solutions to

$$\pi_j = \sum_i \pi_i P_{ij}, \qquad \sum_j \pi_j = 1.$$

So we have

$$\pi_0 = \pi_0 P_{00} + \pi_1 P_{10}, \qquad \pi_1 = \pi_0 P_{01} + \pi_1 P_{11}, \qquad \pi_0 + \pi_1 = 1.$$

$$\implies \begin{bmatrix} (1-P_{00}) & -P_{10} \\ -P_{01} & (1-P_{11}) \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \pi_0 \\ \pi_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solving yields $\pi_0 = 4/7$, $\pi_1 = 3/7$.

Definition 7.37. A Markov chain is said to be **doubly stochastic** if the sums of the columns of the transition probability matrix also add to 1 (the rows must always add to 1). That is,

$$\sum_i P_{ij} = 1, \qquad \forall\, j \in S$$

Proposition 7.55. Suppose you have an irreducible Markov chain with $m < \infty$ states that is doubly stochastic. Then

$$\pi_j = \frac{1}{m}, \qquad j = 1, \ldots, m$$

That is, all the long-run proportions are the same.

Proof. Recall that π_j is the unique solution to $\pi_j = \sum_i \pi_i P_{ij}$ and that $\sum_j \pi_j = 1$. For $\pi_j = 1/m$, clearly the second condition is satisfied, so we need to verify the first state. That is we need to verify that

$$\frac{1}{m} = \frac{1}{m} \sum_i P_{ij} \iff \sum_i P_{ij} = 1$$

which is true if the Markov chain is doubly stochastic.

□

Remark. In practice if there is a Markov chain with more than three states and you are asked to find the long-run proportions, it is likely a doubly-stochastic Markov chain.

Example 7.40 (Example 4.25 in *Introduction to Probability Models*). Hotel, number of new guests every day is Poisson with mean λ. Each guest that stays the night will independently check out the next day with probability $p = 1 - \alpha$. (So the number of days each guest stays are independent geometric random variables.)

$$X_n = i \implies X_{n+1} = \text{Bin}(i, \alpha) * \text{Poisson}(\lambda)$$

where $*$ is being used for a sum to stress the independence of the random variables.

$$P_{ij} = \Pr(\text{Bin}(i, \alpha) + \text{Poisson}(\lambda) = j)$$

$$\vdots$$

Part (c): Suppose the number of people initially there X_0 is distributed Poisson(β). Then the number of that cohort who remains the next day is Poisson($\alpha\beta$). So $X_1 \sim$ Poisson($\alpha\beta + \lambda$). So to find the limiting distribution (where it's the same every day from now on), we need

$$\beta = \lambda + \alpha\beta \iff \beta = \frac{\lambda}{1-\alpha} \implies \pi_j = \exp\left(-\frac{\lambda}{1-\alpha}\right)\left(\frac{\lambda}{1-\alpha}\right)^j \Big/ j!$$

Proposition 7.56 (Proposition 4.4.2 in *Stochastic Processes*; relevant to Gambler's Ruin). If j is a recurrent state in a Markov chain, then the set of probabilities $\{f_{ij}, i \in T\}$ satisfies

$$f_{ij} = \sum_{k \in T} P_{ik} f_{kj} + \sum_{k \in R} P_{ik}, \qquad i \in T$$

where R denotes the set of states communicating with j.

Example 7.41 (Gambler's Ruin; Section 4.5.1 of *Introduction to Probability Models*, Example 4.4(A) in *Stochastic Processes* (p. 197 of pdf)). Also discussed drug testing example immediately afterward in section 4.5.1 of *Introduction to Probability Models*, and what happens as $N \to \infty$ (p. 199 of pdf of *Stochastic Processes*).

7.6.4 Branching Processes (Section 4.7 of *Introduction to Probability Models*, Section 4.5 of *Stochastic Processes*

Proposition 7.57 (Theorem 4.5.1 in *Stochastic Processes* (p. 203 of pdf)). (i) If $\mu \leq 1$, then $\pi_0 = 1$. (ii) If $\mu > 1$, $\pi_0 < 1$. (iii) π_0 is the smallest positive number satisfying

$$\pi_0 = \sum_{j=0}^{\infty} \pi_0^j P_j.$$

Proof. (i) Using

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i,$$

we have

$$\mathbb{E}(X_n \mid X_{n-1}) = X_{n-1} \cdot \sum_{j=0}^{\infty} j P_j = X_{n-1}\mu$$

$$\implies \mathbb{E}(X_n) = \mu \mathbb{E}(X_{n-1}) = \mu^2 \mathbb{E}(X_{n-2}) = \ldots = \mu^n \mathbb{E}(X_0) = \mu^n.$$

Suppose $\mu < 1$. By Markov's Inequality (Lemma 8.3),

$$\Pr(X_n \geq 1) \leq \mathbb{E}(X_n) = \mu^n \implies \lim_{n\to\infty} \Pr(X_n \geq 1) = 0.$$

When $\mu = 1$, can prove using a convexity argument and constraints $P_0 > 0, P_0 + P_1 < 1$ (see *Stochastic Processes*).

(ii) In *Stochastic Processes.*

(iii) Suppose $x > 0$ satisfies

$$x = \sum_{j=0}^{\infty} x^j P_j.$$

We will show by induction that $x \geq \Pr(X_n = 0 \mid X_0 = 1)$ for all n. To start we need to show that $x \geq \Pr(X_n = 0 \mid X_0 = 1)$. But

$$\Pr(X_01 = 0 \mid X_0 = 1) = P_0 \leq \sum_{j=0}^{\infty} x^j P_j = x$$

so that follows. Now we assume $x \geq \Pr(X_n = 0 \mid X_0 = 1)$. We would like to show that this implies $x \geq \Pr(X_{n+1} = 0 \mid X_0 = 1)$. Note that

$$\Pr(X_{n+1} = 0 \mid X_0 = 1) = \sum_{j=0}^{\infty} \Pr(X_{n+1} = 0 \mid X_1 = j, X_0 = 1) = \sum_{j=0}^{\infty} \Pr(X_{n+1} = 0 \mid X_1 = j) P_j$$

But $\Pr(X_{n+1} = 0 \mid X_1 = j) = (\Pr(X_{n+1} = 0 \mid X_1 = 1))^j$ for the following reason: if there are j people to begin with, imagine they each had their own independent branching process. The probability that you reach 0 eventually is equal to the joint probability that all j processes reach 0 eventually. So we have

$$\sum_{j=0}^{\infty} \Pr(X_{n+1} = 0 \mid X_1 = j) P_j = \sum_{j=0}^{\infty} P_j (\Pr(X_{n+1} = 0 \mid X_1 = 1))^j \leq \sum_{j=0}^{\infty} P_j x^j = x$$

where the second to last step follows by the inductive hypothesis.

□

7.6.5 A Markov Chain Model of Algorithmic Efficiency (Section 4.6.1 of *Stochastic Processes*)

Definition 7.38. A set of events $A_2, A_2, \ldots$ is said to be an **increasing sequence** if $A_n \subseteq A_{n+1}$ and a **decreasing sequence** if $A_n \supseteq A_{n+1}$. If $A_n, n \geq 1$ is increasing, define

$$\lim_{n \to \infty} A_n = \bigcup_{n=1}^{\infty} A_n.$$

If $A_n, n \geq 1$ is decreasing, define

$$\lim_{n \to \infty} A_n = \bigcap_{n=1}^{\infty} A_n.$$

The **limit infimum** is the set of all points contained in all but a finite number of $A_1, A_2, \ldots,$. The **limit supremum** is the set of all points contained in an infinite number of $A_1, A_2, \ldots$. Note that for an increasing sequence the limit infimum is a subset of the limit supremum.

Lemma 7.58. For any events $A_1, A_2, \ldots$ there are events $B_1, B_2, \ldots$ that are mutually exclusive ($B_i \cap B_j = \emptyset, i \neq j$) such that

$$B_i \cap B_j = \emptyset, \qquad \bigcup_{i=1}^{n} B_i = (\bigcup_{i=1}^{n} A_i \text{ for } n = 1, 2, \ldots, \qquad \bigcup_{i=1}^{\infty} B_i = (\bigcup_{i=1}^{\infty} A_i.$$

Proof. Let $B_1 = A_1$, $B_2 = A_2 \backslash A_1$, $B_3 = A_3 \backslash \{A_1 \cup A_2\}, \ldots, B_n = A_n \backslash \{A_1 \cup \ldots \cup A_n\} = A_n \backslash \left\{ \bigcup_{i=1}^{n-1} A_i \right\}$. □

Theorem 7.59. Probability is a continuous set function. That is, if A_n is an increasing or decreasing sequence then

$$\lim_{n \to \infty} \Pr(A_n) = \Pr\left(\bigcup_{n=1}^{\infty} A_n \right) = \Pr\left(\lim_{n \to \infty} A_n \right).$$

Proof. Let $A_1, A_2, \ldots$ be events. Per Lemma 7.58, define $B_1, B_2, \ldots$ such that $B_i \cap B_j = \emptyset$ and $\bigcup_{i=1}^{n} B_i = (\bigcup_{i=1}^{n} A_i$ for $n = 1, 2, \ldots$, $\bigcup_{i=1}^{\infty} B_i = (\bigcup_{i=1}^{\infty} A_i$. Suppose A_n is increasing, so that $A_n \subseteq A_{n+1}$. Then

$$\Pr\left(\lim_{n \to \infty} A_n \right) = \Pr\left(\bigcup_{i=1}^{\infty} A_i \right) = \Pr\left(\bigcup_{i=1}^{\infty} B_i \right) = \sum_{i=1}^{\infty} \Pr(B_i)$$

where the last equality follows since the B_i are disjoint. Continuing we have

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \Pr(B_i) = \lim_{n \to \infty} \Pr\left(\bigcup_{i=1}^{n} B_i \right) = \lim_{n \to \infty} \Pr\left(\bigcup_{i=1}^{n} A_i \right) = \lim_{n \to \infty} \Pr(A_n)$$

where the last step follows since A_n is an increasing sequence.

□

Remark. Can prove Boole's Inequality by a similar logic:

$$\Pr\left(\bigcup_{i=1}^{n} A_i \right) = \Pr\left(\bigcup_{i=1}^{n} B_i \right) = \sum_{i=1}^{n} \Pr(B_i) \leq \sum_{i=1}^{n} \Pr(A_i)$$

where the last step follows since $B_i \subseteq A_i$.

Remark. Consider the Gambler's Ruin problem (Example 7.41). Let A_N be the event of reaching 0 before N given $X_0 = i$. Note that $A_N \subseteq A_{N+1}$. So we have

$$\lim_{N \to \infty} \Pr(A_N) = \Pr\left(\lim_{N \to \infty} A_n \right) = \Pr\left(\bigcup_{N=1}^{\infty} A_n \right)$$

where the last quantity is clearly the probability that the gambler eventually goes to 0, matching our interpretation in Example 7.41.

Similarly, in the proof of Proposition 7.57 for branching processes, we claimed

$$\Pr(\text{population dies out}) = \lim_{n\to\infty} \Pr(X_n = 0).$$

Note that $\{X_n = 0\} \subset \{X_{n+1} = 0\}$ so

$$\lim_{n\to\infty} \{X_n = 0\} = \bigcup_{n=1}^{\infty} \{X_n = 0\}$$

again matching what we assumed.

7.6.6 Time Reversible Markov Chains (Section 4.8 of *Introduction to Probability Models*, Section 4.7 of *Stochastic Processes*

Proposition 7.60 (Unlabeled at beginning of each section). Given a stationary Markov chain $X_n, X_{n+1}, X_{n+2}, \ldots$, the reverse process $X_n, X_{n-1}.X_{n-2}, \ldots$ is a Markov chain with transition probabilities

$$Q_{ij} = P^*_{ij} = \frac{\pi_j P_{ji}}{\pi_i}.$$

Proof.

$$Q_{ij} = \Pr(X_m = j \mid X_{m+1} = i) = \frac{\Pr(X_m = j \cap X_{m+1} = i)}{\Pr(X_{m+1} = i)} = \frac{\Pr(X_m = j)\Pr(X_{m+1} = i \mid X_m = j)}{\Pr(X_{m+1} = i)}$$

$$= \frac{\pi_j P_{ji}}{\pi_i}. \tag{7.11}$$

□

Definition 7.39 (Time reversible Markov chain). Suppose for a Markov chain, the reverse transition probabilities Q_{ij} equal the corresponding forward transition probabilities P_{ij} for all i, j. Then the Markov chain is said to be **time reversible.** The condition $Q_{ij} = P_{ij}$ can also be written (per (7.11)) as

$$\pi_i P_{ij} = \pi_j P_{ji}, \qquad \forall\ i, j.$$

Theorem 7.61 (Theorem 4.7.2 in *Stochastic Processes* (p. 220 of pdf)). A stationary Markov chain is time reversible if and only if starting in state i, any path back to i has the same probability as the reverse path, for all i. That is, if

$$P_{i,i_1} P_{i_1,i_2} \cdots P_{i_k,i} = P_{i,i_k} P_{i_k,i_{k-1}} \cdots P_{i_1,i}$$

for all states $i, i_1, \ldots, i_k$.

Theorem 7.62 (Theorem 4.2 in *Introduction to Probability Models*). An ergodic (all states positive recurrent; see Definition 7.35) Markov chain for which $P_{ij} = 0$ whenever $P_{ji} = 0$ is time reversible if and only if starting in state i, any path back to i has the same probability as the reverse path, for all i. That is, if

$$P_{i,i_1} P_{i_1,i_2} \cdots P_{i_k,i} = P_{i,i_k} P_{i_k,i_{k-1}} \cdots P_{i_1,i}$$

for all states $i, i_1, \ldots, i_k$.

Proposition 7.63 (Proposition 4.9 in *Introduction to Probability Models*, Theorem 4.7.3 in *Stochastic Processes* (p. 222 of pdf)). Consider an irreducible Markov chain $\{X_n\}$ with transition probabilities P_{ij}. If $\{X_j, j \in S\}$ is stationary, and there exist positive numbers $x_i, i \geq 0$ and a transition probability matrix Q such that

$$x_i P_{ij} = x_j Q_{ji}, \qquad i \neq j$$

$$\sum x_i = 1$$

then the Markov chain is reversible, $\pi_j = x_j$ are the stationary probabilities for both the original and the reversed chain, and the Q_{ij} are the transition probabilities of the reversed chain.

Proof. Assume the above. we have

$$\sum_i x_i P_{ij} = \sum_i x_j P_{ji} = x_j$$

so these x_j satisfy the stationarity equations and the solution is unique, so $x_j = \pi_j$. □

Example 7.42 (Similar to Example 4.35 in *Introduction to Probability Models*, Gambler's Ruin-type problem).

Example 7.43 (Similar to Example 4.36 in *Introduction to Probability Models*, Proposition 4.7.1 in *Stochastic Processes* (p. 217 of pdf; weighted graph problem)).

Example 7.44 (Example 4.37 in *Introduction to Probability Models*, Example 4.7(b) in *Stochastic Processes* (p. 221 of pdf; list, items requested, then move up one position in the list; average position)). Note the similarity to *Introduction to Probability Models* problem 4.26 (from Homework 10).

7.6.7 Semi-Markov Processes (Section 4.8 of *Stochastic Processes* (p. 224 of pdf), Section 7.6 of *Introduction to Probability Models*

Definition 7.40 (Semi-Markov Process). Suppose the stochastic process $\{X(t), t \geq 0\}$ takes on values $\{0, 1, \ldots, M\}$. Then $\{X(t), t \geq 0\}$ is called a **semi-Markov process**.

Why semi-Markov? What happens next doesn't just depend on your current state, it also depends on how long you're been there. But it's semi-Markov because just after a transition you know everything.

7.7 ISE 620

Exercise 13. ("Best Prize" problem.) There are n prizes that are presented one at a time in a random order. Each prize has a defined value, and there is a defined ordering of the value of the prizes. Each time you see a prize, you only know the value of that prize relative to the prizes already seen—the value of later prizes remains unknown. Each time a prize is presented, we either accept it or reject it. You only get one prize, so once you accept it, you're done. Once you reject a prize, you can't get it back. Your goal is to maximize the probability of accepting the best prize. What do you do (what is the optimal policy)?

Solution. The only strategy that makes sense is to accept a prize if it is a *candidate*—that is, the best you've seen so far. You should never accept a prize that isn't a candidate unless you get to the last prize. Let P_n be the probability of getting the best prize. Then we expect P_n to approach 0 as n approaches infinity. Note that if it were good to accept the kth prize if it were a candidate, then it would definitely be better to accept the $k+1$th candidate if it were a candidate. So a good policy (k-policy) is to let k prizes go by, then accept the first candidate to come afterward.

Let X be the position of the best prize. We will condition on the best prize being in position i. Then

$$P_k(\text{best}) = \sum_{i=1}^{n} P_k(\text{best} \mid X = i) \Pr(X = i) = \frac{1}{n} \sum_{i=1}^{n} P_k(\text{best} \mid X = i)$$

Note that because in order for you to win, none of the prizes between position k and i can be candidates (or else you would accept them and not get the best prize). So the best of the first $i-1$ prizes must be among the first k prizes in order for you to get the best prize.

$$P_k(\text{best} \mid X = i) = \begin{cases} 0 & i \leq k \\ \Pr(\{\text{best of } 1, \ldots, i-1 \text{ is among the first } k\}) & k > k \end{cases}$$

and

$$\Pr(\{\text{best of } 1, \ldots, i-1 \text{ is among the first } k\}) = \frac{k}{i-1}$$

so

$$P_k(\text{best}) = \frac{1}{n} \sum_{i=k+1}^{n} \frac{k}{i-1} = \frac{k}{n} \sum_{j=k}^{n-1} \frac{1}{j} \approx \frac{k}{n} \int_{k}^{n-1} \frac{dx}{x} = \frac{k}{n} \log(x)\big|_k^{n-1} = \frac{k}{n} \log\left(\frac{n-1}{k}\right) \approx \boxed{\frac{k}{n} \log\left(\frac{n}{k}\right)}.$$

Now we want to choose k that maximizes this quantity. Generalize to letting x equal any real number in the expression

$$f(x) = \frac{x}{n} \log\left(\frac{n}{x}\right) \implies f'(x) = \frac{x}{n}\frac{x}{n}\left(\frac{-n}{x^2}\right) + \log\left(\frac{n}{x}\right)\frac{1}{n}$$

Setting equal to 0 we have

$$\frac{1}{n} = \frac{1}{n}\log\left(\frac{n}{x}\right) \implies \frac{n}{x} = e \implies \boxed{x = \frac{n}{e}}$$

so the optimal strategy is to let about $1/e$ of the prizes go by, then choose the first candidate to come by afterward.

Remark. The probability of getting the best prize is then

$$f\left(\frac{n}{e}\right) = \frac{1}{e}\log(e) = \frac{1}{e}$$

regardless of n!

Exercise 14. (Ballot problem.) Suppose we have candidates A and B with votes counted in random order. A has n votes, B has m with $n > m$. What is the probability that A is always ahead in the count at every stage of the vote?

Solution. Let $\Pr(\{A \text{ is always ahead}\}) = P(n,m) = P_{n,m}$. Then

$$P_{n,m} = \Pr(\{A \text{ is always ahead}\} \mid \{A \text{ receives the first vote}\}) \cdot \frac{n}{n+m}$$

$$+ \Pr(\{A \text{ is always ahead}\} \mid \{B \text{ receives the first vote}\}) \cdot \frac{m}{n+m}$$

$$= \Pr(\{A \text{ is always ahead}\} \mid \{A \text{ receives the first vote}\}) \cdot \frac{n}{n+m} + 0$$

$$= Q_{n-1,m} \cdot \frac{n}{n+m}$$

where Q represents the probability that A is never behind (since going forward ties would be ok because A starts out ahead). We have

$$Q_{nm} = \Pr(\{A \text{ is never behind}\}) = \frac{n}{n_m}\Pr(\{A \text{ is never behind}\} \mid \{A \text{ gets first vote}\})$$

Note that $\Pr(\{A \text{ is never behind}\} \mid \{A \text{ gets first vote}\}) = Q_{n-1,m-1}$. But now things are more complicated because A could afford to be behind by 1 going forward. So this strategy is not working. Try instead to condition on who gets the last vote.

$$P_{n,m} = \Pr(\{A \text{ is always ahead}\} \mid \{A \text{ receives the last vote}\}) \cdot \frac{n}{n+m}$$

$$+ \Pr(\{A \text{ is always ahead}\} \mid \{B \text{ receives the last vote}\}) \cdot \frac{m}{n+m}$$

$$= P_{n-1,m} \cdot \frac{n}{n+m} + P_{n,m-1} \cdot \frac{m}{n+m}$$

We can work this out recursively using the boundary conditions

$$P_{n,n} = 2, \quad P_{n,0} = 1, \ n > 0$$

We will try to guess the answer.

$$P_{2,1} = \Pr(\{A \text{ gets the first two votes}\}) = \frac{1}{3}$$

$$P_{n,1} = \Pr(\{\text{first two votes are for } A\}) = \frac{n}{n+1} \cdot \frac{n-1}{n} = \frac{n-1}{n+1}$$

$$P_{3,2} = \frac{3}{5} \cdot \frac{2}{4} \cdot \Pr(\{A \text{ is not the last of the remaining votes}\}) = \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} = \frac{1}{5}$$

$$P_{4,2} = \frac{4}{6} \cdot \frac{3}{5} \cdot \Pr(\{B \text{ does not get next two votes}\}) = \frac{4}{6} \cdot \frac{3}{5} \cdot \left(1 - \frac{2}{4} \cdot \frac{1}{3}\right) = \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{5}{6} = \frac{1}{3}$$

$$P_{4,3} = \frac{4}{7} \cdot \frac{3}{6} \cdot \Bigg(\Pr(\{\text{next is } A\}) \cdot \Pr(\{\text{not 3 } B \text{ votes in a row}\}) + \Pr(\{\text{next is } B\}) \cdot \Pr(\{$$

$$= \frac{4}{7} \cdot \frac{3}{6} \cdot \left(\frac{2}{5} \cdot \frac{3}{4} + \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} \right)$$

$$\vdots$$

It seems that

$$\boxed{P_{n,m}. = \frac{n-m}{n+m}}$$

We can argue this solution is unique or we can argue it is true by induction.

7.8 DSO Statistics Group Screening Exam Problems

Exercise 15 (2016 DSO Statistics Group In-Class Screening Exam, Question 1). Let $\{V_n\}$ be i.i.d. non-negative random variables. Fixing $r > 0$ and $q \in (0, 1]$, consider the sequence $W_0 = 1$ and $W_n = (qr + (1-q)V_n)W_{n-1}, n \geq 1$. A motivating example of W_n is recording the relative growth of a portfolio where a constant fraction q of one's wealth is re-invested each year in a risk-less asset that grows by r per year, with the remainder re-invested in a risky asset whose annual growth factors are random V_n.

(a) Show that $n^{-1} \log W_n \xrightarrow{a.s.} w(q)$, where $w(q) = \mathbb{E} \log(qr + (1-q)V_1)$.

(b) Show that $q \to w(q)$ is concave on $(0, 1]$.

(c) Using Jensen's Inequality, show that $w(q) \leq w(q)$ in case $\mathbb{E}V_1 \leq r$. Further, show that if $\mathbb{E}V_1^{-1} \leq r^{-1}$ then the almost sure convergence also applies for $q = 0$ and $w(q) \leq w(0)$.

Solution.

(a) Note that

$$W_1 = (qr + (1-q)V_1)W_0 = qr + (1-q)V_1$$

$$W_2 = (qr + (1-q)V_2)W_1 = (qr + (1-q)V_2)(qr + (1-q)V_1)$$

$$W_3 = (qr + (1-q)V_3)W_2 = (qr + (1-q)V_3)(qr + (1-q)V_2)(qr + (1-q)V_1) = \prod_{i=1}^{3}(qr + (1-q)V_i)$$

$$\vdots$$

$$W_n = \prod_{i=1}^{n}(qr + (1-q)V_i)$$

$$\implies n^{-1} \log(W_n) = \frac{1}{n}\sum_{i=1}^{n}(qr + (1-q)V_i)$$

$\{qr + (1-q)V_i\}$ is a sequence of i.i.d. nonnegative random variables, so by the Strong Law of Large Numbers, the conclusion follows if $\mathbb{E}[|qr + (1-q)V_1|] = \mathbb{E}[qr + (1-q)V_1] = qr + (1-q)\mathbb{E}(V_1) < \infty$.

(b) Since $w(q)$ is twice differentiable, a sufficient condition for concavity is $w''(q) \leq 0$ for all $q \in (0, 1]$.

$$w(q) = \mathbb{E} \log[qr + (1-q)V_1] = \mathbb{E} \log[q(r - V_1) + V_1]$$

$$w'(q) = \frac{\partial}{\partial q}\mathbb{E} \log[q(r - V_1) + V_1] = \mathbb{E}\left(\frac{\partial}{\partial q}\log[q(r - V_1) + V_1]\right) = \mathbb{E}\left(\frac{r - V_1}{q(r - V_1) + V_1}\right) \tag{7.12}$$

$$w''(q) = \mathbb{E}\left(\frac{\partial}{\partial q}\frac{r - V_1}{q(r - V_1) + V_1}\right) = \mathbb{E}\left((r - V_1) \cdot \frac{\partial}{\partial q}\left[q(r - V_1) + V_1\right]^{-1}\right)$$

$$= \mathbb{E}\left((r - V_1) \cdot (-1)\left[q(r - V_1) + V_1\right]^{-2} \cdot (r - V_1)\right) = -\mathbb{E}\left(\left[\frac{(r - V_1)}{q(r - V_1) + V_1}\right]^2\right) \tag{7.13}$$

This is -1 times the expectation of a nonnegative random variable, so by Markov's Inequality we have

$$\mathbb{E}\left(\left[\frac{(r - V_1)}{q(r - V_1) + V_1}\right]^2\right) \geq 0 \iff w''(q) \leq 0 \qquad \forall q \in (0, 1],$$

proving concavity.

(c) By Jensen's Inequality and the concavity of $q \to \log[qr + (1-q)V_1]$, we have

$$w(q) = \mathbb{E}\log[qr + (1-q)V_1] \le \log \mathbb{E}\left[qr + (1-q)V_1\right] = \log\left(qr + (1-q)\mathbb{E}[V_1]\right)$$

$$\le \log\left(qr + (1-q)r\right) = \log(r) = \mathbb{E}\log(r) = w(1).$$

where the third step used $\mathbb{E}(V_1) \le r$ and the second-to-last step used the fact that r is non-random. For $q = 0$, we have

$$n^{-1}\log(W_n) = \frac{1}{n}\sum_{i=1}^{n} V_i.$$

$\{V_i\}$ is a sequence of i.i.d. nonnegative random variables, so by the Strong Law of Large Numbers, almost sure convergence applies if $\mathbb{E}[|V_1|] = \mathbb{E}[V_1] < \infty$. **But this is the same condition as in the case $q \in (0, 1]$.**

To show $w(q) \le w(0)$, we will show that $w(q)$ is nonincreasing on $[0, 1]$. Recall from (7.12)

$$w'(q) = \mathbb{E}\left(\frac{r - V_1}{q(r - V_1) + V_1}\right)$$

We will show that (7.12) is upper-bounded by 0 on $[0, 1]$. Plugging in $q = 0$ yields

$$w'(0) = \mathbb{E}\left(\frac{r - V_1}{V_1}\right) = \mathbb{E}\left(rV_1^{-1} - 1\right) = r\mathbb{E}\left(V_1^{-1}\right) - 1 \le 1 - 1 = 0.$$

where we used the assumption $\mathbb{E}\left(V_1^{-1}\right) \le r$ on the second-to-last step. Recall that the second derivative (7.13) is continuous and nonpositive on $(0, 1]$. Therefore the first derivative (7.12) never exceeds $q(0) = 0$ on $[0, 1]$, so $w(q) \le w(0)$.

7.9 Simple Random Walk

Definition 7.41. Let $\{X_i\}$ be i.i.d. We have

$$X_i = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1 - p \end{cases}$$

Let $S_k = \sum_{i=1}^{k} X_i$. Then $\{S_k\}$ is a **simple random walk**. (It is simple because the outcomes are either 1 or -1.)

7.10 Martingales

Definition. Let $\{y_t\}_{t=0}^{\infty}$ be a sequence of random variables, and let Ω_t denote the information set available at date t, which at least contains $\{y_t, y_{t-1}, y_{t-2}, \ldots\}$. If $\mathbb{E}(y_t \mid \Omega_{t-1}) = y_{t-1}$ holds then $\{y_t\}$ is a martingale process with respect to Ω_t.

Definition. Let $\{y_t\}_{t=1}^{\infty}$ be a sequence of random variables, and let Ω_t denote the information set available at date t, which at least contains $\{y_t, y_{t-1}, y_{t-2}, \ldots\}$. If $\mathbb{E}(y_t \mid \Omega_{t-1}) = 0$, then $\{y_t\}$ is a martingale difference process with respect to Ω_t.

7.11 Brownian Motion

Appendix B.13, Brownian motion. A standard Brownian motion $b(\cdot)$ is a continuous-time stochastic process associating each date $a \in [0, 1]$ with the scalar $b(a)$ such that

(i) b(0) = 0

(ii) For any dates $0 \leq a_1 \leq a_2 \leq \ldots \leq a_k \leq 1$ the changes $[b(a_2) - b(a_1)]$, $[b(a_3) - b(a_2)], \ldots, [b(a_k) - b(a_k - 1)]$ are independent multivariate Gaussian with $b(a) - b(s) \sim \mathcal{N}(0, a - s)$.

(iii) For any given realization, $b(a)$ is continuous in a with probability 1.

Other continuous time processes can be generated from the standard Brownian motion. For example, a Brownian motion with variance σ^2 can be obtained as

$$w(a) = \sigma b(a)$$

where $b(a)$ is a standard Brownian motion.

The continuous time process

$$\boldsymbol{w}(a) = \boldsymbol{\Sigma}^{1/2} \boldsymbol{b}(a)$$

is a Brownian motion with covariance matrix $\boldsymbol{\Sigma}$.

Definition 26 (Wiener process). Let $\Delta w(t)$ be the change in $w(t)$ during the time interval dt. Then $w(t)$ is said to follow a Wiener process if

$$\Delta w(t) = \epsilon_t \sqrt{dt}, \quad \epsilon_t \sim IID(0, 1)$$

and $w(t)$ denotes the value of the $w(\cdot)$ at date t. Clearly,

$$\mathbb{E}[\Delta w(t)] = 0, \text{ and } \mathrm{Var}[\Delta w(t)] = dt$$

Theorem 7.64. Donsker's Theorem, Theorem 43, p.335, Section 15.6.3. Let $a \in [0, 1)$, $t \in [0, T]$, and suppose $(J - 1)/T \leq a < J/T, J = 1, 2, \ldots, T$. Define

$$R_T(a) = \frac{1}{\sqrt{T}} s_{[Ta]}$$

where

$$s_{\left[Ta\right]} = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_{\left[Ta\right]}$$

$\left[Ta\right]$ denotes the largest integer part of Ta and $s_{\left[Ta\right]} = 0$ if $\left[Ta\right] = 0$. Then $R_T(a)$ weakly converges to $w(a)$, i.e.,

$$R_T(a) \to w(a)$$

where $w(a)$ is a Wiener process. Note that when $a = 1$, $R_T(1) = 1/\sqrt{T} \cdot S_{\left[T\right]} = 1/\sqrt{T} \cdot (\epsilon_1 + \epsilon_2 + \ldots + \epsilon_T$. Since ϵ_t's are IID, by the central limit theorem, $R_T(1) \to \mathcal{N}(0, 1)$.

Similar (Theorem 2.1 in Phillips and Durlauf [1986]): Let $\{u_t\}$ be a sequence satisfying $\mathbb{E}(u_t) = 0$, $\gamma(0) = \mathbb{E}(T^{-1}S_t^2) \to \sigma^2 < \infty$ as $T \to \infty$, $\{u_t\}$ is square summable, $\sup_t\{\mathbb{E}(|u_t|^\beta)\} < \infty$ for some $2 \leq \beta < \infty$ and all t, $\gamma(h) = \mathbb{E}(T^{-1}(y_t - y_{t-h})^2) \to K_h < \infty$ as $\min\{h, T\} \to \infty$. Then $X_T(t) \implies W(t)$ as $T \to \infty$, where $W(t)$ is a Wiener process.

Theorem 7.65. Continuous Mapping Theorem (Theorem 44 of Pesaran in 15.6.3). Let $a \in [0, 1)$, $i \in [0, n]$, and suppose $(J-1)/n \leq a < J/n, J = 1, 2, \ldots, n$. Define $R_n(a) = n^{-1/2}S_{\left[n \cdot a\right]}$. If $f(\cdot)$ is continuous over $[0, 1)$, then

$$f[R_n(a)] \xrightarrow{d} f[w(a)]$$

8 Asymptotics and Convergence

These notes are based on my notes from chapter 8 of *Time Series and Panel Data Econometrics* (1st edition) by M. Hashem Pesaran [Pesaran, 2015] and coursework for Economics 613: Economic and Financial Time Series I at USC, as well as Math 505A and Math 541A at USC and chapter 7 from *Probability and Random Processes* (Grimmett and Stirzaker) 3rd edition [Grimmett and Stirzaker, 2001].

8.1 Preliminaries (5.9 and 7.1, Grimmett and Stirzaker)

Definition 8.1. Definition 7.1.4, Grimmett and Stirzaker. If for all $x \in [0, 1]$ the sequence $\{f_n(x)\}$ of real numbers satisfies $f_n(x) \to f(x)$ as $n \to \infty$ then we say $f_n \to f$ **pointwise.**

Remark. In practice pointwise convergence is often not useful for functions because a sequence of functions may be continuous while its limit is not. For instance, consider $\{f_n : f_n = x^n \ \forall x \in [0, 1]\}$. Then f_n is continuous for all n but

$$\lim_{n\to\infty} f_n = \begin{cases} 0 & x \leq 1 \\ 1 & x = 1 \end{cases}$$

Instead, the following definition is often more useful.

Definition 8.2. (from class notes.) We say that f_n **uniformly converges to** f **on** $[a, b]$ if for every $\epsilon > 0$ there exists N such that for every $n > N$,

$$\forall x \in [a, b] \ |f_n(x) - f(x)| < \epsilon$$

Definition 8.3. (Definition 7.1.5, Grimmett and Stirzaker.) Let V be a collection of functions mapping $[0, 1]$ into $\mathbb{R}$ and assume V is endowed with a function $\|\cdot\| : V \to \mathbb{R}$ satisfying

(a) $\|f\| \geq 0$ for all $f \in V$

(b) $\|f\| = 0$ if and only if f is the zero function (or equivalent to it)

(c) $\|af\| = |a| \cdot \|f\|$ for all $a \in \mathbb{R}$, $f \in V$

(d) $\|f + g\| \leq \|f\| + \|g\|$ (Triangle Inequality)

The function $\|\cdot\|$ is called a **norm**. If $\{f_n\}$ is a sequence of members of V then we say that $f_n \to f$ **with respect to the norm** $\|\cdot\|$ if $\|f_n - f\| \to 0$ as $n \to \infty$.

Definition 8.4. (Definition 7.16, Grimmett and Stirzaker.) Let $\epsilon > 0$ be prescribed, and define the distance between two functions $g, h : [0, 1] \to \mathbb{R}$ by

$$d_\epsilon(g, h) = \int_E dx$$

where $E = \{u \in [0, 1] : |g(u) - h(u)| > \epsilon\}$. We say that $f_n \to f$ **in measure** if

$$d_\epsilon(f_n, f) \to 0 \text{ as } n \to \infty \text{ for all } \epsilon > 0$$

Theorem 8.1. Inversion Theorem (Theorem 5.9.2, Grimmett and Stirzaker). Let X have distribution function F and characteristic function ϕ. Define $\overline{F} : \mathbb{R} \to [0, 1]$ by

$$\overline{F}(x) = \frac{1}{2}\left[F(x) + \lim_{y \to x^-} F(y)\right]$$

Then

$$\overline{F}(b) - \overline{F}(a) = \lim_{N \to \infty} \int_{-N}^{N} \frac{\exp(-iat) - \exp(-ibt)}{2\pi i t} \cdot \phi(t) dt$$

Proof. See Kingman and Taylor [1966]. □

Corollary 8.1.1. Corollary 5.9.3. Random variables X and Y have the same characteristic function if and only if they have the same distribution function.

Proof. **Available in Grimmett and Stirzaker section 5.9, pp. 189 - 190.** □

Definition 8.5. (Definition 5.9.4, Grimmett and Stirzaker.) We say that the sequence $F_1, F_2, \ldots$ of distribution functions **converges** to the distribution function F (written $F_n \to F$) if $F(x) = \lim_{n \to \infty} F_n(x)$ at each point x where F is continuous.

Theorem 8.2. Continuity theorem (Thereom 5.9.5; in notes from Friday 10/26, Lecture 28). Suppose that $F_1, F_2, \ldots$ is a sequence of distribution functions with corresponding characteristic functions $\phi_1, \phi_2, \ldots$.

(a) If $F_n(x) \to F(x)$ for some distribution function F with characteristic function ϕ (at x where F is continuous), then $\phi_n(t) \to \phi(t)$ for all t.

(b) Conversely, if $\phi(t) = \lim_{n \to \infty} \phi_n(t)$ exists and $\phi(t)$ is continuous at $t = 0$, then ϕ is the characteristic function of some distribution function F, and $F_n \to F$.

Proof. See Kingman and Taylor [1966]. □

8.2 Inequalities (8.6 of Pesaran)

Inequalities

- Probabilities

  –

  Lemma 8.3. Markov's Inequality (Grimmett and Stirzaker p. 311, 319)): Let $X : \Omega \to [-\infty, \infty]$ be a random variable. Then for all $a > 0$,

$$\Pr(|X| \geq a) \leq \frac{\mathbb{E}(|X|)}{a}$$

Proof. Note $t \cdot \mathbf{1}_{\{|X| \geq t\}} \leq |X|$, where $\mathbf{1}$ is the indicator function. Dividing both sides by t and taking expectations, we have

$$\mathbb{E}(\mathbf{1}_{\{|X| \geq t\}}) \leq \frac{\mathbb{E}|X|}{t} \iff \Pr(|X|) \geq t) \leq \frac{\mathbb{E}|X|}{t}, \qquad \forall \, t > 0.$$

□

Corollary 8.3.1. If n is a positive integer, then

$$\Pr(|X| \geq t) \leq \frac{\mathbb{E}(|X|^n)}{t^n} \quad \forall t > 0$$

Proof. By Markov's Inequality (Theorem 8.3),

$$\Pr(|X| \geq t) = \Pr(|X|^n \geq t^n) \leq \frac{\mathbb{E}(|X|^n)}{t^n}$$

□

–

Theorem 8.4. Chebyshev's Inequality: (probability p. 319) Let $X : \Omega \to [-\infty, \infty]$ be an (integrable) random variable with $\mathbb{E}(X^2) < \infty$.Then for any real number $k > 0$

$$\Pr\left(|X - \mathbb{E}(X)| \geq k\sqrt{\mathrm{Var}(X)}\right) \leq \frac{1}{k^2}$$

This can also be written as

$$\Pr(|X - \mathbb{E}(X)| \geq k) \leq \frac{\mathrm{Var}(X)}{k^2}$$

(Can be used to demonstrate consistency of estimators: if we can show that as $T \to \infty$ $\mathrm{Var}(X) = \sigma^2 \to 0$, then this implies $\Pr\left(|X - \mu| \geq k\sigma\right) \to 0$ as $T \to \infty$, showing consistency.)

–

Theorem 8.5. Chernoff For $x \geq 0$, $a > 0$, $\forall \, t > 0$,

$$\Pr(X \geq a) = \Pr(e^{tX} \geq e^{ta}) \leq \frac{\mathbb{E}(e^{tX})}{e^{ta}}$$

- **Moments**

  –

  Theorem 8.6 (Cauchy-Schwarz (and **Bunyakovsky).).** If X and Y are random variables with finite variance then

$$\mathbb{E}(XY)^2 \leq \mathbb{E}(X^2)\mathbb{E}(Y^2)$$

Note that this can is a corollary of Theorem 8.9 with $p = q = 2$. We can also prove this theorem on its own in a different one. We first prove a useful result.

Lemma 8.7. If $\mathrm{Var}(X) = 0$ then X is almost surely constant; that is, $\Pr(X = a) = 1$ for some $a \in \mathbb{R}$.

Proof. Note that because $\text{Var}(X) = 0 < \infty$, we know that $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$ exist. We have

$$\text{Var}(X) = \mathbb{E}\big[(X - \mathbb{E}(X))^2\big] = 0$$

Let $Y = (X - \mathbb{E}(X))^2$. Note that $Y = (X - \mathbb{E}(X))^2 \geq 0$ and that $\mathbb{E}(Y) = \text{Var}(X) = 0$. Therefore $\Pr(Y = 0) = 1$, so $\Pr(Y \neq 0) = 0$. To see why, in the case that X is discrete,

$$\mathbb{E}(Y) = \sum_{k=0}^{\infty} k \cdot \Pr(Y = k) = \text{Var}(X) = 0$$

which is true if and only if $\Pr(Y = k) = 0$ for all $k > 0$. Since we already showed that $Pr(Y < 0) = 0$, it follows that $\Pr(Y = 0) = 1$. In the continuous case,

$$\mathbb{E}(Y) = \int_0^{\infty} y \cdot f_Y(y) dy = \text{Var}(X) = 0$$

which implies that $f_Y(x) = 0$ for all $x > 0$. Again, since $\Pr(Y < 0) = 0$, we have $\Pr(Y \neq 0) = 0$. But $Y = 0 \iff X = \mathbb{E}(X)$ so we have $\Pr(X = \mathbb{E}(X)) = 1$. □

Remark. Note that Lemma 8.7 along with Proposition 6.28 imply that X has variance 0 if and only if it is (almost surely) constant.

We are now ready to prove the Cauchy-Schwarz Inequality.

Proof. if $\mathbb{E}(X^2) = 0$ or $\mathbb{E}(Y^2) = 0$, the Cauchy-Schwarz Inequality follows immediately. To see why, suppose without loss of generality that $\mathbb{E}(X^2) = 0$. Then the right side is 0. Also, $0 \leq \text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = -\mathbb{E}(X)^2$. Since $\mathbb{E}(X)^2 \geq 0$, we must have $E(X)^2 = 0$ and therefore $\text{Var}(X) = 0$. Therefore by Lemma 8.7, X is almost surely constant, which means that $\text{Cov}(X, Y) = 0$.

In the case that $\mathbb{E}(X^2) > 0$ and $\mathbb{E}(Y^2) > 0$, for $a, b \in \mathbb{R}$, let $Z = aX - bY$. Then

$$0 \leq \mathbb{E}(Z^2) = a^2\mathbb{E}(X^2) - 2ab\mathbb{E}(XY) + b^2\mathbb{E}(Y^2) \tag{8.1}$$

The right side of (8.1) is quadratic in a. Because it is greater than or equal to zero, it has at most one real root, which means its discriminant must be non-positive. That is, if $b \neq 0$,

$$(-2b\mathbb{E}(XY))^2 - 4b^2\mathbb{E}(X^2)\mathbb{E}(Y^2) \leq 0 \iff \mathbb{E}(XY)^2 - \mathbb{E}(X^2)\mathbb{E}(Y^2) \leq 0$$

which yields the result. Note that equality holds if and only if $\Pr(aX = bY) = 1$ because the discriminant is zero if and only if the quadratic has a real root, which occurs if and only if

$$\mathbb{E}\big[(aX - bY)^2\big] = 0$$

which is true if and only if $\Pr(aX = bY) = 1$ by by Lemma 8.7 and Proposition 6.28. □

- **Krylov**
-

Definition 8.6. Let $\phi : \mathbb{R} \to \mathbb{R}$. We say that ϕ is **convex** if for any $x, y \in \mathbb{R}$ and for any $t \in [0, 1]$, we have

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$$

Theorem 8.8 (Jensen's Inequality, from Math 541A. Also Grimmett and Stirzaker p.181, 349). Let $X : \Omega \to [-\infty, \infty]$ be a random variable. Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex. If $\mathbb{E}|X| < \infty$ and $\mathbb{E}|\phi(X)| < \infty$, then

$$\phi(\mathbb{E}X) \leq \mathbb{E}\phi(X).$$

(See also Theorem 12.4.)

For the definition of convexity, see Definition 12.1.

Proof. Note that from Theorem 12.2, for any $y \in \mathbb{R}$ there exists a constant a and a function L such that

$$a(x-y) + \phi(y) \leq \phi(x) \quad \forall x \in \mathbb{R}$$

Letting $y = \mathbb{E}(X)$ we have

$$a(X - \mathbb{E}X) + \phi(\mathbb{E}X) \leq \phi(X)$$

Since expectations preserve inequalities,

$$\mathbb{E}[a(X - \mathbb{E}X) + \phi(\mathbb{E}X)] \leq \mathbb{E}\phi(X)$$

But

$$\mathbb{E}[a(X - \mathbb{E}X) + \phi(\mathbb{E}X)] = a(\mathbb{E}X - \mathbb{E}X) + \mathbb{E}(\phi(\mathbb{E}X)) = \phi(\mathbb{E}X)$$

which yields

$$\phi(\mathbb{E}X) \leq \mathbb{E}\phi(X).$$

□

For some corollaries, see section 12.1.

–

Theorem 8.9. [**Hölder (Grimmett and Stirzaker p. p. 143, 319; Theorem 1.99 in Math 541A lecture notes) Generalization of Cauchy-Schwarz**] Let $X, Y : \Omega \to \mathbb{R}$ be random variables. For $p, q \geq 1$ satisfying $1/p + 1/q = 1$ we have

$$\mathbb{E}(|XY|) \leq (\mathbb{E}(|X^p|))^{1/p}(\mathbb{E}(|X^q|))^{1/q} = \|X\|_p \|Y\|_q.$$

The equality case happens only if X is a constant multiple of Y with probability 1. Note that the case $p = q = 2$ recovers the Cauchy-Schwarz Inequality (Theorem 8.6).

Proof. Assume without loss of generality that $\|X\|_p = \|Y\|_q = 1$. Also, the case $p = 1, q = \infty$ follows from the triangle inequality, so we assume $1 < p < \infty$. From concavity of the log function, we have

$$\log\left((x^p)^{1/p}(y^q)^{1/q}\right) = (1/p)\log\left(x^p\right) + (1/q)\log\left(y^q\right)$$

$$\leq \log\left(\frac{1}{p}x^p + \frac{1}{q}y^q\right)$$

$$\implies (x^p)^{1/p}(y^q)^{1/q} \leq \frac{1}{p}x^p + \frac{1}{q}y^q$$

Fixing an $\omega \in \Omega$, we have

$$|X(\omega)Y(\omega)| = (|X(\omega)|^p)^{1/p}(|Y(\omega)|^q)^{1/q} \leq \frac{1}{p}|X(\omega)|^p + \frac{1}{q}|Y(\omega)|^q$$

Integrating we have...

□

Theorem 8.10 (Hölder (vector form)). For any $u, v \in \mathbb{R}^n$,

$$|u^T v| \leq \|u\|_p \|v\|_q$$

for any $p, q \in [0, \infty]$ satisfying $1/p + 1/q = 1$.

–

Theorem 8.11. Minkowski (Grimmett and Stirzaker p. p. 143) For $p \geq 1$,

$$[\mathbb{E}(|X + Y|^p)]^{1/p} \leq (\mathbb{E}|X^p|)^{1/p} + (\mathbb{E}|Y^p|)^{1/p}$$

– Useful for showing lower order moments are finite (e.g. finite variance implies finite mean).

Lemma 8.12. Lyapunov's Inequality (Grimmett and Stirzaker p. 143). For $0 < r \leq s < \infty$,

$$\mathbb{E}(|X|^r)^{1/r} \leq \mathbb{E}(|X|^s)^{1/s}$$

–

Theorem 8.13. Triangle Inequality: Let $X, Y : \Omega \to \mathbb{R}$ be random variables. Let $1 \leq p \leq \infty$. Then

$$\|X + Y\|_p \leq \|X\|_p + \|Y\|_p, 1 \leq p \leq \infty$$

Proof. The case $p = \infty$ follows from the scalar triangle inequality, so assume $1 \leq p < \infty$. By scaling, we may assume $\|X\|_p = 1 - t, \|Y\|_p = t$, for some $t \in (0, 1)$ (zeroes and infinities being trivial). Define $V := X/(1 - t), W := Y/t$. Then by convexity of $x \to |x|^p$ on $\mathbb{R}$,

$$|(1 - t)V(\omega) + t(W(\omega)|^p \leq (1 - t)|V(\omega)|^p + t|W(\omega)|^p$$

Take expectation of both sides:

$$\mathbb{E}|X + Y|^p \leq (a - t)^{1-p}\mathbb{E}(|X|^p) + t^{1-p}\mathbb{E}(|Y|^p)$$

Since $\|X\|_p = t, \|Y\|_p = 1 - t$, we have that the right side is $1 - t + t = 1$. (Note: $\|Y\|_p = t, \mathbb{E}|Y|^p = t^p, \|X\|_p = 1 - t$ Therefore

$$(\mathbb{E}|X + Y|^p)^{1/p} = \|X + Y\|_p \leq 1$$

□

Remark. See also Theorem 5.6 and Corollary 5.6.1.

Theorem 8.14 (Chernoff Bound). Let X be a random variable and let $r > 0$. Define $M_X(t) := \mathbb{E}e^{tX}$ for any $t \in \mathbb{R}$. Then for any $t > 0$,

$$\mathbb{P}(X > r) \le e^{-tr} M_X(t).$$

Proof. Using Markov's Inequality (Theorem 8.3) on e^{tX}, we have

$$\Pr(X \ge r) = \Pr(e^{tX} \ge e^{tr}) \le \frac{\mathbb{E}e^{tX}}{e^{tr}} = e^{-tr} M_X(t), \quad \forall\, t > 0.$$

$\square$

Remark. Consequently, if $X_1, \ldots, X_n$ are independent random variables with the same CDF, and if $r, t > 0$,

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i > r\right) \le e^{-trn}(M_{X_1}(t))^n.$$

For example, if $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter $0 < p < 1$, and if $r, t > 0$,

$$\mathbb{P}\left(\frac{X_1 + \cdots + X_n}{n} - p > r\right) \le e^{-trn}(e^{-tp}[pe^t + (1-p)])^n.$$

And if we choose t appropriately, then the quantity $\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - p) > r\right)$ becomes exponentially small as either n or r become large. That is, $\frac{1}{n}\sum_{i=1}^{n} X_i$ becomes very close to its mean. Importantly, the Chernoff bound is much stronger than either Markov's or Cheyshev's inequality, since they only respectively imply that

$$\mathbb{P}\left(\left|\frac{X_1 + \cdots + X_n}{n} - p\right| > r\right) \le \frac{2p(1-p)}{r}, \quad \mathbb{P}\left(\left|\frac{X_1 + \cdots + X_n}{n} - p\right| > r\right) \le \frac{p(1-p)}{nr^2}.$$

Monotone convergence theorem.

Dominated Convergence Theorem (Theorem 6.73).

8.3 Modes of Convergence (7.2 of Grimmett and Stirzaker, 8.2 and 8.4 of Pesaran)

Let $\{X_n\} = \{X_1, X_2, \ldots\}$ and X be random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 8.7. Convergence in probability. $\{X_n\}$ is said to **converge in probability** to X if

- Grimmett and Strizaker definition:

$$\lim_{n\to\infty} \Pr(|X_n - X| > \epsilon) = 0, \text{ for every } \epsilon > 0$$

- Pesaran definition:

$$\lim_{n\to\infty} \Pr(|X_n - X| < \epsilon) = 1, \text{ for every } \epsilon > 0$$

- More formal (from Math 541A):

$$\forall \epsilon > 0, \ \lim_{n\to\infty} \Pr(\{\omega \in \Omega : |X_n(\omega) - X(\omega)| > \epsilon) = 0$$

Remark. This mode of convergence is also often denoted by $X_n \xrightarrow{p} X$ and when X is a fixed constant it is referred to as the **probability limit of** X_n, written as $Plim(X_n) = x$, as $n \to \infty$.

The above concept is readily extended to multivariate cases where $\{\boldsymbol{X}_n, n = 1, 2, \ldots\}$ denote m-dimensional vectors of random variables. Then the condition is

$$\lim_{n\to\infty} \Pr(\|\boldsymbol{X}_n - \boldsymbol{X}\| < \epsilon) = 1, \text{ for every } \epsilon > 0$$

where $\|\cdot\|$ denotes an appropriate norm (say ℓ_2). Convergence in probability is often referred to as "weak convergence" (in contrast to convergence with probability 1, below).

Definition 8.8. Convergence with probability 1 or almost surely. The sequence of random variables $\{X_n\}$ is said to **converge with probability 1** (or **almost surely**) to X if

- (505A class notes definition)

  $$\Pr\left(\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}\right) = 1$$

  (Note: pointwise convergence can hardly ever be shown here and is not useful.)

- Grimmett and Stirzaker textbook definition:

  $$\Pr\left(\{\omega \in \Omega : X_n(\omega) \to X(\omega) \text{ as } n \to \infty\}\right) = 1$$

- Pesaran textbook definition:

  $$\Pr\left(\lim_{n\to\infty} X_n = X\right) = 1$$

Remark. This is often written as $X_n \xrightarrow{w.p.1} X$ or $X_n \xrightarrow{a.s.} X$. An equivalent condition for convergence with probability 1 is given by

$$\lim_{n\to\infty} \Pr(|X_m - X| < \epsilon, \text{ for all } m \geq= n) = 1, \text{ for every } \epsilon > 0$$

which shows that convergence in probability is a special case of convergence with probability 1 (obtained by setting $m = n$). Convergence with probability 1 is stronger than convergence in probability and is often referred to as "strong convergence."

Definition 8.9. Convergence in r-th mean or **convergence in** ℓ_p. $X_n \to X$ **in rth mean** (or **in** ℓ_p) where $r \geq 1$ (or $0 < p \leq \infty$) if $\mathbb{E}|X_n^r| < \infty$ for all n and

$$\lim_{n\to\infty} \mathbb{E}(|X_n - X|^r) = 0$$

or if $\|X\|_p < \infty$ and

$$\lim_{n\to\infty} \|X_n - X\|_p = 0$$

Remark. Recall that $\|X\|_p := (\mathbb{E}(X)^p)^{1/p}$ if $0 < p < \infty$ and $\|X\|_\infty := \inf\{c > 0 : \Pr(|X| \leq c) = 1\}$. Note that if $p < 1$, $\|\cdot\|_p$ is no longer a norm because it does not satisfy the Triangle Inequality (Corollary **??** and Theorem 8.13), but this property still holds. Convergence in rth mean is often written $X_n \xrightarrow{r} X$.

Definition 8.10. Convergence in Distribution. Let $X_1, X_2, \ldots$ have distribution functions $F_1(\cdot), F_2(\cdot), \ldots$ respectively. Then X_n is said to **converge in distribution to** X if

$$\lim_{n\to\infty} \Pr(X_n \leq u) = \Pr(X \leq u)$$

for all u at which $F_X(x) = \Pr(X \leq x)$ is continuous. This can also be written

$$\lim_{n\to\infty} F_n(u) = F(u)$$

for all u at which F is continuous.

Remark. Convergence in distribution is usually denoted by $X_n \xrightarrow{d} X$, $X_n \xrightarrow{L} X$, or $F_n \implies F$. By the Continuity Theorem (Theorem 8.2, section 8.1), this is equivalent to

$$\lim_{n\to\infty} \phi_{X_n}(t) = \phi_X(t), \quad t \in \mathbb{R}.$$

Note that the random variables are allowed to have different domains.

Definition 8.11. (Convergence in distribution for vector-valued random variables.) We say that random variables $Y^{(1)}, \ldots, : \Omega \to \mathbb{R}^d$ **converge in distribution** to $Y : \Omega \to \mathbb{R}^d$ if for all $v \in \mathbb{R}^d$, $\langle v, Y^{(1)} \rangle, \langle v, Y^{(2)} \rangle, \ldots$ converges in distribution to $\langle v, Y \rangle$.

Theorem 8.15. (Theorem 7.2.3, Grimmett and Stirzaker.) The following implications hold:

- $(X_n \xrightarrow{a.s.} X) \implies (X_n \xrightarrow{p} X)$
- $(X_n \xrightarrow{r} X) \implies (X_n \xrightarrow{p} X)$ for any $r \geq 1$
- $(X_n \xrightarrow{p} X) \implies (X_n \xrightarrow{d} X)$

Also, if $r > s \geq 1$, then $(X_n \xrightarrow{r} X) \implies (X_n \xrightarrow{s} X)$. No other implications hold in general.

Theorem 8.16. Some exceptions (Theorem 7.2.4).

- If $X_n \xrightarrow{d} c$ where c is constant, then $X_n \xrightarrow{p} c$.
- If $X_n \xrightarrow{p} X$ and $\Pr(|X_n| \leq k) = 0$ for all n and some k, then $X_n \xrightarrow{r} X$ for all $r \geq 1$.
- If $P_n(\epsilon) = \Pr(|X_n - X| > \epsilon)$ satisfies $\sum_n P_n(\epsilon) < \infty$ for all $\epsilon > 0$, then $X_n \xrightarrow{a.s.} X$.

Proof. (Part (c).) Let $A_n(\epsilon) = \{|X_n - X| > \epsilon\}$ (so that $P_n(\epsilon) = \Pr[A_n(\epsilon))]$, and let $B_m(\epsilon) = \bigcup_{n \geq m} A_n(\epsilon)$. Then

$$\Pr(B_m(\epsilon)) \leq \sum_{n=m}^{\infty} \Pr(A_n(\epsilon))$$

so $\lim_{m\to\infty} \Pr(B_m(\epsilon)) = 0$ whenever $\sum_n \Pr(A_n(\epsilon)) < \infty$. See also Lemma 8.18 part (b). $\square$

8.4 More on convergence (7.2 of Grimmet and Stirzaker)

Other theorems to include: Fatou's Lemma, Fubini's Theorem, Kolmogorov's Maximal Inequality, Kolmogorov Three-Series Test, Lindeberg Feller Central Limit Theorem, **this and more at beginning of Mike's 505A qual solutions.**

Definition 8.12. Cauchy Convergence. We say that the sequence $\{X_n : n \geq 1\}$ of random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is **almost surely Cauchy convergent** if

$$\Pr\left(\{\omega \in \Omega : X_m(\omega) - X_n(\omega) \to 0 \text{ as } m, n \to \infty\}\right) = 1$$

That is, the set of points ω of the sample space for which the real sequence $\{X_n(\omega) : n \geq 1\}$ is Cauchy convergent is an event having probability 1.

Lemma 8.17. (Lemma 7.2.6 from Grimmett and Stirzaker)

(a) If $r > s \geq 1$ and $X_n \xrightarrow{r} X$, then $X_n \xrightarrow{s} X$.

(b) If $X_n \xrightarrow{1} X$ then $X_n \xrightarrow{p} X$.

The converse assertions fail in general.

Proof. (a) Using Lyapunov's Inequality (Lemma 8.12), if $r > s \geq 1$

$$\left[\mathbb{E}(|X_n - X|^s)\right]^{1/s} \leq \left[\mathbb{E}(|X_n - X|^r)\right]^{1/r}$$

Therefore if $X_n \xrightarrow{r} X$ (meaning $\lim_{n\to\infty} \mathbb{E}(|X_n - X|^r) = 0$), (then $\lim_{n\to\infty} \mathbb{E}(|X_n - X|^s) = 0$, so $X_n \xrightarrow{s} X$. We show the converse fails by counterexample:

$$X_n = \begin{cases} n & \text{with probability } n^{(-1/2)(r+s)} \\ 0 & \text{with probability } 1 - n^{(-1/2)(r+s)} \end{cases}$$

Then $\mathbb{E}|X_n^s| = n^{(1/2)(s-r)} \to 0$ and $\mathbb{E}|X_n^r| = n^{(1/2)(r-s)} \to \infty$.

(b) By Markov's Inequality (Lemma 8.3),

$$\Pr(|X_n - X| > \epsilon) \leq \frac{\mathbb{E}|X_n - X|}{\epsilon} \quad \text{for all } \epsilon > 0$$

Therefore if $X_n \xrightarrow{1} X$; that is, $\lim_{n\to\infty} \mathbb{E}(|X_n - X|) = 0$, then $\lim_{n\to\infty} \Pr(|X_n - X| > \epsilon) = 0$ for every $\epsilon > 0$, so $X_n \xrightarrow{p} X$.

To see the converse fails, define an independent sequence $\{X_n\}$ by

$$X_n = \begin{cases} n^3 & \text{with probability } n^{-2} \\ 0 & \text{with probability } 1 - n^{-2} \end{cases}$$

Then $\Pr(|X| > \epsilon) = n^{-2}$ for all large n, and so $X_n \xrightarrow{p} 0$. However, $\mathbb{E}|X_n| = n \to \infty$.

□

Lemma 8.18. (Lemma 7.2.10, Grimmett and Stirzaker.) Let $A_n(\epsilon) = \{|X_n - X| > \epsilon\}$ and $B_m(\epsilon) = \cup_{n \geq m} A_n(\epsilon)$. Then:

(a) $X_n \xrightarrow{a.s.} X$ if and only if $\Pr(B_m(\epsilon)) \to 0$ as $m \to \infty$ for all $\epsilon > 0$.

(b) $X_n \xrightarrow{a.s.} X$ if $\sum_n \Pr(A_n(\epsilon)) < \infty$ for all $\epsilon > 0$.

(c) If $X_n \xrightarrow{a.s.} X$ then $X_n \xrightarrow{p} X$, but the converse fails in general.

Proof. (a)

(b) As for Theorem 8.16 part (c).

(c) To see the converse fails, define an independent sequence $\{X_n\}$ by

$$X_n = \begin{cases} 1 & \text{with probability } n^{-1} \\ 0 & \text{with probability } 1 - n^{-1} \end{cases}$$

Clearly $X_n \xrightarrow{p} 0$. However, if $0 < \epsilon < 1$,

$$\Pr(B_m(\epsilon)) = 1 - \lim_{r \to \infty} \Pr(X_n = 0 \text{ for all } n \text{ such that } m \leq n \leq r) \text{ (by Lemma 1.3.5)}$$

$$= 1 - \left(1 - \frac{1}{m}\right)\left(1 - \frac{1}{m+1}\right) \cdots \text{ (by independence)}$$

$$= 1 - \lim_{M \to \infty} \left(\frac{m-1}{m} \cdot \frac{m}{m+1} \cdot \frac{m+1}{m+2} \cdots \frac{M}{M+1} \right)$$

$$= 1 - \lim_{M \to \infty} \frac{m-1}{M+1} = 1$$

and so $\{X_n\}$ does not converge almost surely.

□

Lemma 8.19. (Lemma 7.2.12, Grimmett and Stirzaker.) There exist sequences which

(a) converge almost surely but not in mean,

(b) converge in mean but not almost surely.

Proof. (a) As for Lemma 8.17 part (b).

□

Theorem 8.20. (Theorem 7.2.13, Grimmett and Stirzaker.) If $X_n \xrightarrow{p} X$, there exists a non-random increasing sequence of integers $n_1, n_2, \ldots$ such that $X_{n_i} \xrightarrow{a.s.} X$ as $i \to \infty$.

Theorem 8.21. Skorokhod's representation theorem (Theorem 7.2.14, Grimmett and Stirzaker). If $\{X_n\}$ and X with distribution functions $\{F_n\}$ and F are such that $X_n \xrightarrow{d} X$ (or equivalently, $F_n \to F$) as $n \to \infty$, then there exists a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and random variables $\{Y_n\}$ and Y mapping Ω' into $\mathbb{R}$ such that

(a) $\{Y_n\}$ and Y have distribution functions $\{F_n\}$ and F

(b) $Y_n \xrightarrow{a.s.} Y$ as $n \to \infty$

Therefore, although X_n may fail to converge to X in any mode other than in distribution, there exists a sequence $\{Y_n\}$ such that Y_n is distributed identically to X_n for every n, which converges almost surely to a copy of X.

Theorem 8.22. (Theorem 7.2.19, Grimmett and Stirzaker; same as Portmanteau Theorem?) The following three statements are equivalent:

(a) $X_n \xrightarrow{d} X$

(b) $\mathbb{E}[g(X_n)] \to \mathbb{E}[g(X)]$ for all bounded continuous functions g.

(c) $\mathbb{E}[g(X_n)] \to \mathbb{E}[g(X)]$ for all functions g of the form $g(x) = f(x)\mathbf{1}_{[a,b]}(x)$ where f is continuous on $[a, b]$ and a and b are points of continuity of the distribution function of the random variable X.

Theorem 8.23. (Grimmett and Stirzaker Theorem 7.3.9.)

(a) If $X_n \xrightarrow{a.s.} X$ and $Y_n \xrightarrow{a.s.} Y$ then $X_n + Y_n \xrightarrow{a.s.} X + Y$.

(b) If $X_n \xrightarrow{r} X$ and $Y_n \xrightarrow{r} Y$ then $X_n + Y_n \xrightarrow{r} X + Y$.

(c) If $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$ then $X_n + Y_n \xrightarrow{p} X + Y$.

(d) It is not in general true that $X_n + Y_n \xrightarrow{d} X + Y$ whenever $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$.

Theorem 8.24. Borel-Cantelli lemmas (Grimmett and Stirzaker Theorem 7.3.10.) Let $\{A_n\}$ be an infinite sequence of events from some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $A = \bigcap_n \bigcup_{m=n}^\infty A_m = \limsup_{n\to\infty} A_n = \{A_n \text{ i.o.}\}$ be the event that infinitely many of the A_n occur. Then:

(a) $\Pr(A) = 0$ if $\sum_n \Pr(A_n) < \infty$

(b) $\Pr(A) = 1$ if $\sum_n \Pr(A_n) = \infty$ and $A_1, A_2, \ldots$ are independent events.

Proof. (a) We have that $A \subseteq \bigcup_{m=n}^\infty A_n$ for all n, so

$$\Pr(A) \leq \sum_{m=n}^{\infty} \Pr(A_m) \to 0 \text{ as } n \to \infty$$

whenever $\sum_n \Pr(A_n) < \infty$.

(b) One can confirm that

$$A^c = \bigcup_n \bigcap_{m=n}^{\infty} A_m^c$$

But

$$\Pr\left(\bigcap_{m=n}^{\infty} A_m^c\right) = \lim_{r\to\infty} \Pr\left(\bigcap_{m=n}^{r} A_m^c\right) = \prod_{m=n}^{\infty} [1 - \Pr(A_m)] \text{ (by independence) } \leq \prod_{m=n}^{\infty} \exp(-\Pr(A_m))$$

$$= \exp\left(-\sum_{m=n}^{\infty} \Pr(A_m)\right) = 0$$

whenever $\sum_n \Pr(A_n) = \infty$, where the fourth step follows since $1 - x \leq e^{-x}$ if $x \geq 0$. Thus

$$\Pr(A^c) = \lim_{n\to\infty} \Pr\left(\bigcap_{m=n}^{\infty} A_m^c\right) = 0$$

so $\Pr(A) = 1$.

□

Theorem 8.25. Kolmogorov's Two-Series Theorem. Let $X_1, X_2, \ldots$ be independent random variables with $\mathbb{E}(X_n) = \mu_n$ and $\text{Var}(X_n) = \sigma_n^2$ such that $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$. Then $\sum_{n=1}^{\infty} X_n$ converges in $\mathbb{R}$ almost surely.

Proof. Available on wikipedia, https://en.wikipedia.org/wiki/Kolmogorov%27s_two-series_theorem.

□

8.4.1 Slutsky's Convergence Theorems (8.4.1 of Pesaran, 7.3 of Grimmett and Stirzaker)

Theorem 8.26. Theorem 6 of Pesaran, Section 8.4.1, p. 173. Let $\{x_t, y_t\}, t = 1, 2, \ldots$ be a sequence of pairs of random variables with $y_t \xrightarrow{d} y$ and $|y_t - x_t| \xrightarrow{p} 0$. Then $x_t \xrightarrow{d} y$.

Theorem 8.27. Theorem 7 in Pesaran, on p.318 (section 7.3) of Grimmett and Stirzaker. (Section 8.4.1, p. 174) If $x_t \xrightarrow{d} x$ and $y_t \xrightarrow{p} c$ where c is a finite constant, then

(i) $x_t + y_t \xrightarrow{d} x + c$

(ii) $y_t x_t \xrightarrow{d} cx$

(iii) $x_t / y_t \xrightarrow{d} x/c$, if $c \neq 0$.

Theorem 8.28. on p.318 (section 7.3) of Grimmett and Stirzaker. Suppose that $X_n \xrightarrow{d} 0$ and $Y_n \xrightarrow{p} Y$, and let $g : \mathbb{R}^2 \to \mathbb{R}$ be such that $g(x, y)$ is a continuous function of y for all x, and $g(x, y)$ is continuous at $x = 0$ for all y. Then $g(X_n, Y_n) \xrightarrow{p} g(0, Y)$.

Theorem 8.29 (Continuous Mapping Theorem (Theorem 9 of Pesaran, Section 8.4.1, p. 176: convergence properties of transformed sequences.)). Suppose $\{\boldsymbol{x}_j\}$, $\{\boldsymbol{y}_j\}$, $\boldsymbol{x}$, and $\boldsymbol{y}$ are $k \times 1$ vectors of random variables on a probability space, and let $\boldsymbol{g}(\cdot)$ be a continuous vector-valued function. (Alternatively, suppose g has the set of discontinuity points D_g such that $\Pr(X \in D_g) = 0$.) Then

(i) $\boldsymbol{x}_j \xrightarrow{a.s.} x \implies \boldsymbol{g}(\boldsymbol{x}_j) \xrightarrow{a.s.} \boldsymbol{g}(\boldsymbol{x})$

(ii) $\boldsymbol{x}_j \xrightarrow{p} x \implies \boldsymbol{g}(\boldsymbol{x}_j) \xrightarrow{p} \boldsymbol{g}(\boldsymbol{x})$

(iii) $\boldsymbol{x}_j \xrightarrow{d} x \implies \boldsymbol{g}(\boldsymbol{x}_j) \xrightarrow{d} \boldsymbol{g}(\boldsymbol{x})$

(iv) $\boldsymbol{x}_j - \boldsymbol{y}_j \xrightarrow{p} \mathbf{0}$ and $\boldsymbol{y}_j \xrightarrow{d} \boldsymbol{y} \implies \boldsymbol{g}(\boldsymbol{x}_j) - \boldsymbol{g}(\boldsymbol{y}_j) \xrightarrow{d} \mathbf{0}(\boldsymbol{x})$

where $x = (c_1, \ldots, c_k) \in \mathbb{R}^k$.

Proof (part (b), continuous case, one-dimensional codomain). Let $\boldsymbol{x}_j = (M_{j,1}, \ldots, M_{j,k})$. We have that

$$\forall \epsilon_j > 0, \ \lim_{n \to \infty} \Pr(\{\omega \in \Omega : |M_{j,n}(\omega) - c_j| > \epsilon_j) = 0, \qquad \forall j \in \{1, \ldots, k\}.$$

$$\iff \forall \epsilon_j > 0, \ \lim_{n \to \infty} \Pr(\{\omega \in \Omega : |M_{j,n}(\omega) - c_j| < \epsilon_j) = 1, \qquad \forall j \in \{1, \ldots, k\}. \tag{8.2}$$

Because g is continuous, we have that for every $\epsilon^* > 0$ there exists a $\delta^* > 0$ such that

$$0 < \|\big(M_{1,n}(\omega), \ldots, M_{j,n}(\omega)\big)\|_2 < \delta^* \implies |g(M_{1,n}, \ldots, M_{j,n}) - g(c_1, \ldots, c_j)| < \epsilon^*. \tag{8.3}$$

Note that since in $\mathbb{R}$ the L_2 and L_1 norms are equivalent,

$$|M_{j,n}(\omega) - c_j| < \epsilon_j \iff \|M_{j,n}(\omega) - c_j\|_2 < \epsilon_j \implies \sum_{j=1}^k \|M_{j,n}(\omega) - c_j\|_2 < \sum_{j=1}^k \epsilon_j$$

$$\implies \|\big(M_{1,n}(\omega), \ldots, M_{j,n}(\omega)\big)\|_2 < \sum_{j=1}^k \epsilon_j$$

where the last step follows by the Triangle Inequality. Therefore letting $\delta^* = \sum_{j=1}^k \epsilon_j$, we have

$$\Pr(\{\omega \in \Omega : |M_{j,n}(\omega) - c_j| < \epsilon_j) \leq \Pr\left(0 < \|\big(M_{1,n}(\omega), \ldots, M_{j,n}(\omega)\big)\|_2 < \delta^*\right)$$

$$\leq \Pr(\{\omega \in \Omega : |g\big(M_{1,n}(\omega), \ldots, M_{j,n}(\omega)\big) - g(c_1, \ldots, c_j)| < \epsilon^*)$$

where the last step follows from (8.3). So

$$\Pr(\{\omega \in \Omega : |M_{j,n}(\omega) - c_j| < \epsilon_j) \leq \Pr(\{\omega \in \Omega : |g\big(M_{1,n}(\omega), \ldots, M_{j,n}(\omega)\big) - g(c_1, \ldots, c_j)| < \epsilon^*). \qquad (8.4)$$

Taking limits of (8.4) and substituting in (8.2), we have

$$\forall \epsilon^* > 0, \ \lim_{n \to \infty} \Pr(\{\omega \in \Omega : |g\big(M_{1,n}(\omega), \ldots, M_{j,n}(\omega)\big) - g(c_1, \ldots, c_j)| < \epsilon^*) \geq 1$$

$$\iff \forall \epsilon^* > 0, \ \lim_{n \to \infty} \Pr(\{\omega \in \Omega : |g\big(M_{1,n}(\omega), \ldots, M_{j,n}(\omega)\big) - g(c_1, \ldots, c_j)| > \epsilon^*) = 0$$

$$\iff g(M_{1,n}, \ldots, M_{j,n}) \xrightarrow{p} g(c_1, \ldots, c_j).$$

For remaining parts, see Serfling [1980] or Rao [1973]. □

See also:

Theorem 8.30. (Theorem 7.2.18, Grimmett and Stirzaker.) If $X_n \xrightarrow{d} X$ and $g : \mathbb{R} \to \mathbb{R}$ is continuous, then $g(X_n) \xrightarrow{d} g(X)$.

8.5 Stochastic orders $\mathcal{O}_p(\cdot)$ and $o_p(\cdot)$ (Pesaran 8.5)

Definition 8.13 (**Pesaran 8.5 Definition 6.**)**.** Let $\{a_t\}$ be a sequence of positive numbers and $\{x_t\}$ be a sequence of random variables. Then

(i) $x_t = \mathcal{O}_p(a_t)$, or x_t/a_t is bounded in probability, if for every $\epsilon > 0$ there exist real numbers M_ϵ and N_ϵ such that

$$\Pr\left(\frac{|x_t|}{a_t} > M_\epsilon\right) < \epsilon, \quad \text{for } t > N_\epsilon$$

(ii) $x_t = o_p(a_t)$ if

$$\frac{\boldsymbol{x}_t}{a_t} \xrightarrow{p} 0$$

Definition 8.14 (Ross ISE 620 Definition)**.** We say that $f(x)$ is $o(h)$ if $\lim_{h \to 0} f(h)/h = 0$.

8.6 Laws of Large Numbers and Central Limit Theorems (Pesaran 8.6; Grimmett and Stirzaker 7.4, 7.5)

Theorem 8.31. Weak Law of Large Numbers (Khinchine) (Pesaran 8.6 Theorem 10, Grimmett and Stirzaker Theorem 7.4.7, 541A notes Theorem 2.10). Suppose that $\{X_k\}$ is a sequence of (i) independent (ii) identically distributed random variables with (iii) constant means, i.e., $\mathbb{E}(X_k) = \mu < \infty$. Then

$$\overline{X}_k = \frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{p} \mu$$

Theorem 8.32. Weak Law of Large Numbers (Chebyshev) (Pesaran Section 8.6, p. 178, Theorem 11.) Let $\{X_k\}$ be a sequence of random variables. If (i) $\mathbb{E}(X_k) = \mu_k$, (ii) $\text{Var}(X_k) = \sigma_k^2$, and (iii) $\text{Cov}(X_k, X_j) = 0,\ k \neq j$, and (iv)

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \sigma_k^2 < \infty$$

then we have $\overline{X}_n - \overline{\mu}_n \xrightarrow{p} 0$, where $\overline{\mu}_n = n^{-1}\sum_{k=1}^{n} \mu_k$.

Theorem 8.33. Strong Law of Large Numbers (Grimmett and Stirzakker Theorem 7.4.3). Let $\{X_k\}$ be a sequence of (i) independent (ii) identically distributed random variables with (iii) $\mathbb{E}(X_k) = \mu$ and (iv) $\mathbb{E}(X_k^2) < \infty$. Then

$$\frac{1}{n}\sum_{k=1}^{n} X_k \to \mu \text{ almost surely and in mean square.}$$

Theorem 8.34 (**Strong Law of Large Numbers (Grimmett and Stirzakker Theorem 7.5.1, 541A notes Theorem 2.11).**). Let $\{X_k\}$ be a sequence of (i) independent (ii) identically distributed random variables. Then if and only if (iii) $\mathbb{E}|X_k| < \infty$,

$$\frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{a.s.} \mu$$

Theorem 8.35. Strong Law of Large Numbers 1 (Kolmogorov) (Pesaran 8.8 Theorem 12). Let $\{X_k\}$ be a sequence of (i) independent random variables with (ii) $\mathbb{E}(X_k) = \mu_k < \infty$ and (ii) $\text{Var}(X_k) = \sigma_k^2$ such that (iii)

$$\sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^2} < \infty$$

Then $\overline{X}_n - \overline{\mu}_n \xrightarrow{wp1} 0$. If the independence assumption (i) is replaced by a lack of correlation (i.e. $\text{Cov}(X_k, X_j) = 0, k \neq j$), the convergence of $\overline{X}_n - \overline{\mu}_n$ with probability one requires the stronger condition

$$\sum_{k=1}^{\infty} \frac{\sigma_k^2 (\log k)^2}{k^2} < \infty$$

Theorem 8.36. Strong Law of Large Numbers 2 (Pesaran 8.8 Theorem 13) Suppose that $X_1, X_2, \ldots$ are (i) independent random variables, and that (ii) $\mathbb{E}(X_k) = 0$, (iii) $\mathbb{E}(X_k^4) \leq M\ \forall\ k$ where M is an arbitrary positive constant. Then

$$\overline{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{a.s.} 0$$

Theorem 8.37. Central Limit Theorem (Grimmett and Stirzaker theorem 5.10.4.) Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables with finite mean μ and finite non-zero variance σ^2, and let $S_n = \sum_{i=1}^{n} X_i$. Then

$$\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \xrightarrow{d} \mathcal{N}(0,1)$$

Theorem 8.38. (Berry-Esseen Central Limit Theorem.) There exists $c > 0$ such that the following holds. Let $X_1, X_2, \ldots$ be i.i.d. real-valued random variables with mean zero, variance 1, and $\mathbb{E}(|X_1|)^3 < \infty$. Let Z be a standard Gaussian random variable. Then for any $n \geq 1$,

$$\sup_{t\in\mathbb{R}} \left| \Pr\left(\frac{X_1 + \ldots + X_n}{\sqrt{n}} \leq t \right) - \Pr(Z \leq t) \right| \leq c \cdot \frac{\mathbb{E}(|X_1|^3)}{\sqrt{n}}$$

Remark. You can look up what the c is; Heilman doesn't think it's any bigger than around 10.

Theorem 8.39. (Central Limit Theorem in $\mathbb{R}^d$, Heilman notes Theorem 2.33.) Let $X^{(1)}, X^{(2)}, \ldots$ be a sequence of independent identically distributed $\mathbb{R}^d$-valued random variables. (Notation: we write $X^{(1)} = (X_1^{(1)}, \ldots, X_d^{(1)})$.) Assume $\mathbb{E}(X^{(n)} = \boldsymbol{\mu}$ for all $n \geq 1$ and for any $1 \leq i < j \leq d$, all of the covariances

$$a_{ij} = \mathbb{E}\big[(X_i^{(1)} - \mathbb{E}(X_i^{(1)})(X_j^{(1)} - \mathbb{E}(X_j^{(1)})\big]$$

are finite. Let $\boldsymbol{S}_n = \sum_{i=1}^{n} X^{(i)}$. Then as $n \to \infty$,

$$\frac{\boldsymbol{S}_n - n\boldsymbol{\mu}}{\sqrt{n\sigma^2}} \xrightarrow{d} \mathcal{N}(\boldsymbol{\mu}, [a_{ij}])$$

Theorem 8.40. (Grimmett and Stirzaker theorem 5.10.5.) Let $X_1, X_2, \ldots$ be independent random variables satisfying $\mathbb{E}(X_j) = 0$, $\mathrm{Var}(X_j) = \sigma_j^2$, $\mathbb{E}|X_j^3| < \infty$ such that

$$\lim_{n\to\infty} \frac{1}{\sigma(n)^3} \sum_{j=1}^{n} \mathbb{E}|X_j^3| = 0$$

where $\sigma(n)^2 = \mathrm{Var}\big(\sum_{j=1}^{n} X_j\big) = \sum_{j=1}^{n} \sigma_j^2$. Then

$$\frac{1}{\sigma(n)} \sum_{j=1}^{n} X_j \xrightarrow{d} \mathcal{N}(0,1)$$

Proof. See Loeve [1977, p. 287] and Grimmett and Stirzaker Problem 5.12.40. □

Lemma 8.41. Lindeberg's Condition:[Let $\{X_k\}$ be a sequence of independent (not necessarily identically distributed) random variables with expectations μ_k and finite variances σ_k^2. Let $s_n^2 = \sum_{k=1}^{n} \sigma_k^2$. If such a sequence of independent random variables X_k satisfies the condition

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^{n} \mathbb{E}\big[(X_k - \mu_k)^2) \cdot \mathbf{1}_{\{|X_k-\mu_k|>\epsilon s_n\}}\big] = 0$$

for all $\epsilon > 0$ then the central limit theorem holds; that is, the random variables

$$Z_n = \frac{1}{s_n} \sum_{k=1}^{n} (X_k - \mu_k)$$

converge in distrbution to $\mathcal{N}(0,1)$ as $n \to \infty$.

8.7 The case of dependent and heterogeneously distributed observations (Pesaran 8.8)

Theorem 8.42. Central limit theorem for martingale difference sequences (Pesaran 8.8 Theorem 28). Let $\{x_t\}$ be a martingale difference sequence with respect to the information set Ω_t. Let $\overline{\sigma}_T^2 = \mathrm{Var}(\sqrt{T}\overline{x}_T) = T^{-1}\sum_{t=1}^{T} \sigma_t^2$. If $\mathbb{E}(|x_t|^r) < K < \infty$ for any $r > 2$ and for all t, and

$$\frac{1}{T}\sum_{t=1}^{T} x_t^2 - \overline{\sigma}_T^2 \xrightarrow{p} 0$$

then $\sqrt{T}\overline{x}_T/\overline{\sigma}_T \xrightarrow{d} \mathcal{N}(0,1)$.

8.8 Worked Examples from Math 505A Midterm 2

(1) (a) **Fall 2010 Problem 1.** Let X_k, $k \geq 1$, be i.i.d. random variables with mean 1 and variance 1. Show that the limit

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} X_k}{\sum_{k=1}^{n} X_k^2}$$

exists in an appropriate sense, and identify the limit.

(b) **Not included on midterm or final.** Let $(X_j)_{j\geq 1}$ be i.i.d. uniform on $(-1,1)$. Let

$$Y_n = \frac{\sum_{j=1}^{n} X_j}{\sum_{j=1}^{n} X_j^2 + \sum_{j=1}^{n} X_j^3}$$

Prove that $\lim_{n\to\infty} \sqrt{n} Y_n$ exists in an appropriate sense, and identify the limit.

Solution.

(a)

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} X_k}{\sum_{k=1}^{n} X_k^2} = \lim_{n\to\infty} \frac{n^{-1}\sum_{k=1}^{n} X_k}{n^{-1}\sum_{k=1}^{n} X_k^2}$$

Since $X_1, X_2, \ldots$ are i.i.d., $E(X_1^2) = \mathrm{Var}(X_1) + (\mathbb{E}(X_1))^2 = 2 < \infty$, we have

$$n^{-1}\sum_{k=1}^{n} X_k \xrightarrow{a.s.} \mathbb{E}(X_1) = 1 \text{ as } n \to \infty$$

by Theorem 8.33 (Strong Law of Large Numbers). Also, $X_1^2, X_2^2, \ldots$ are clearly identically distributed, and are independent by Theorem 4.2.3 ("If X and Y are independent, then so are $g(X)$ and $g(Y)$."). It is clear also that $\mathbb{E}(|X_1^2|) = \mathbb{E}(X_1^2) = \text{Var}(X_1) + \mathbb{E}(X_1)^2 = 1+1 = 2 < \infty$. Therefore by Theorem 8.34 (Strong Law of Large Numbers),

$$n^{-1} \sum_{k=1}^{n} X_k^2 \xrightarrow{a.s.} \mathbb{E}(X_1^2) = 2 \text{ as } n \to \infty$$

(From here I had two different ways of finishing the problem.)

- Because we have almost sure convergence in the numerator and denominator, by the Continuous Mapping Theorem (Theorem 8.29),

$$\lim_{n \to \infty} \frac{n^{-1} \sum_{k=1}^{n} X_k}{n^{-1} \sum_{k=1}^{n} X_k^2} = \frac{\lim_{n \to \infty} n^{-1} \sum_{k=1}^{n} X_k}{\lim_{n \to \infty} n^{-1} \sum_{k=1}^{n} X_k^2} \xrightarrow{a.s.} \boxed{\frac{1}{2}}$$

- Then, using one of Slutsky's convergence theorems (Theorem 8.27: "If $x_t \xrightarrow{d} x$ and $y_t \xrightarrow{p} c$ where c is a finite constant, then $x_t/y_t \xrightarrow{d} x/c$, if $c \neq 0$."), we have

$$\frac{n^{-1} \sum_{k=1}^{n} X_k}{n^{-1} \sum_{k=1}^{n} X_k^2} \xrightarrow{d} \frac{\mathbb{E}(X_1)}{\mathbb{E}(X_1^2)} = \frac{\mathbb{E}(X_1)}{\text{Var}(X_1) + \mathbb{E}(X_1)^2} = \frac{1}{1+1} = \frac{1}{2}$$

  But then, by Theorem 8.16 (Theorem 7.2.4(a) in Grimmett and Stirzaker: "If $X_n \xrightarrow{d} c$ where c is constant, then $X_n \xrightarrow{p} c$."), we have $\frac{n^{-1} \sum_{k=1}^{n} X_k}{n^{-1} \sum_{k=1}^{n} X_k^2} \xrightarrow{p} 1/2$.

(b) **(Not included on midterm or final.)**

$$Y_n = \frac{\sum_{j=1}^{n} X_j}{\sum_{j=1}^{n} X_j^2 + \sum_{j=1}^{n} X_j^3} = \frac{n^{-1} \sum_{j=1}^{n} X_j}{n^{-1} \sum_{j=1}^{n} X_j^2 + n^{-1} \sum_{j=1}^{n} X_j^3}$$

Note that $\mathbb{E}(X_1) = 0, \mathbb{E}(X_1^2) = \text{Var}(X_1) + \mathbb{E}(X_1)^2 = (1 - -1)^2/12 + 0^2 = 1/3, \mathbb{E}(X_1^3) = (1/2) \int_{-1}^{1} x^3 dx = 0$. (We derived the formulae for the first three moments of a uniform distribution on Homework 4 problem 2(2).)

$$\implies \sqrt{n} Y_n = \frac{\sqrt{1/3} \big(\sum_{j=1}^{n} X_j - n\mathbb{E}(X_1) \big) / \sqrt{n \cdot 1/3}}{n^{-1} \sum_{j=1}^{n} X_j^2 + n^{-1} \sum_{j=1}^{n} X_j^3}$$

By the Central Limit Theorem (Theorem 8.37),

$$\frac{\sum_{j=1}^{n} X_j - n\mathbb{E}(X_1)}{\sqrt{n \cdot 1/3}} \xrightarrow{d} \mathcal{N}(0, 1)$$

By the Law of Large Numbers (Theorem 8.34), since $\mathbb{E}(|X_1^2|) = \mathbb{E}(X_1^2) = 1/3 < \infty$,

$$\frac{1}{n} \sum_{j=1}^{n} X_j^2 \xrightarrow{a.s.} \mathbb{E}(X_1^2) = 1/3$$

By the Law of Large Numbers (Theorem 8.34), since $\mathbb{E}(|X_1^3|) = (1/2) \int_{-1}^{1} |x^3| dx = \int_{0}^{1} x^3 dx = 1/4 < \infty$,

$$\frac{1}{n} \sum_{j=1}^{n} X_j^3 \xrightarrow{a.s.} \mathbb{E}(X_1^3) = 0$$

In the denominator, since we have almost sure convergence, the regular rules of calculus/real analysis apply. That is, using the above results,

$$n^{-1}\sum_{j=1}^{n} X_j^2 + n^{-1}\sum_{j=1}^{n} X_j^3 \xrightarrow{a.s.} 1/3$$

Therefore

$$\sqrt{n}Y_n = \frac{\sqrt{1/3}\big(\sum_{j=1}^{n} X_j - n\mathbb{E}(X_1)\big)/\sqrt{n\cdot 1/3}}{n^{-1}\sum_{j=1}^{n} X_j^2 + n^{-1}\sum_{j=1}^{n} X_j^3} \xrightarrow{d} \frac{\sqrt{1/3}}{1/3}\mathcal{N}(0,1) = \boxed{\mathcal{N}(0,3)}$$

(2) **Fall 2010 Problem 2.** Fix $p \in (0,1)$ and consider independent Poisson random variables $X_k, k \geq 1$ with

$$\mathbb{E}X_k = \frac{p^k}{k}$$

Verify that the sum $\sum_{k=1}^{\infty} kX_k$ converges with probability one and determine the distribution of the random variable $Y = \sum_{k=1}^{\infty} kX_k$.

Solution. Melike's solution (use for midterm): We have $\mathbb{E}[kX_k] = p^k$ and $\sum_{k=1}^{\infty} p^k = p/(1-p) < \infty$, and $\text{Var}(kX_k) = kp^k$ and

$$\sum_{k=1}^{\infty} kp^k = p\sum_{k=1}^{\infty} kp^{k-1} = p\frac{\mathrm{d}}{\mathrm{d}p}\sum_{k=1}^{\infty} p^k = p\frac{\mathrm{d}}{\mathrm{d}p}\frac{p}{1-p} = p\cdot\frac{(1-p)-p(-1)}{(1-p)^2} = \frac{p}{(1-p)^2} < \infty$$

Since the sequence $\{Y_k|\}_{k\geq 1}$ is independent, by Kolmogorov's Two Series Theorem (Theorem 8.25: "Let $X_1, X_2, \ldots$ be independent random variables with $\mathbb{E}(X_n) = \mu_n$ and $\text{Var}(X_n) = \sigma_n^2$ such that $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$. Then $\sum_{n=1}^{\infty} X_n$ converges in $\mathbb{R}$ almost surely."), we conclude that $\sum_{k=1}^{\infty} kX_k$ converges almost surely.

To find the distribution of Y, let X be a Poisson random variable and consider its probability generating function:

$$G_X(s) = \mathbb{E}(s^X) = \sum_{k=0}^{\infty} s^k e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda s)^k}{k!} = e^{-\lambda}e^{\lambda s} = e^{\lambda(s-1)}$$

So $\mathbb{E}(s^{X_k}) = \exp\left(\frac{p^k}{k}(s-1)\right)$ and $\mathbb{E}(s^{kX_k}) = \mathbb{E}[(s^k)^{X_k}] = \exp\left(\frac{p^k}{k}(s^k-1)\right)$. Then define $Y_n = \sum_{k=1}^{n} kX_k$ and consider

$$G_{Y_n}(s) = \mathbb{E}(s^{Y_n}) = \mathbb{E}\bigg(\prod_{k=1}^{n} s^{kX_k}\bigg) = \prod_{k=1}^{n}\mathbb{E}\big(s^{kX_k}\big) = \prod_{k=1}^{n}\exp\left(\frac{p^k}{k}(s^k-1)\right) = \exp\left(\sum_{k=1}^{n}\frac{p^k}{k}(s^k-1)\right)$$

$$= \exp\left(\sum_{k=1}^{n}\frac{(ps)^k}{k} - \sum_{k=1}^{n}\frac{p^k}{k}\right)$$

Now, by taking limits as $n \to \infty$ (since we are allowed to take limit inside of expectation here), we get

$$G_Y(s) = \mathbb{E}(s^Y) = \exp\left(\sum_{k=1}^{\infty}\frac{(ps)^k}{k} - \sum_{k=1}^{\infty}\frac{p^k}{k}\right) = \exp\left(\int\sum_{k=1}^{\infty}(ps)^{k-1}dp - \int\sum_{k=1}^{\infty}p^{k-1}dp\right)$$

$$= \exp\left(\int \frac{1}{1-ps}dp - \int \frac{1}{1-p}dp\right) = \exp(-\log(1-ps) + \log(1-p)), \quad -1 \le ps < 1 \text{ and } -1 \le p < 1$$

$$= \frac{1-p}{1-ps}, \quad -1 \le ps < 1$$

Since we know $\Pr(X = k) = \frac{G_X^{(k)}(0)}{k!}$, we have

$$G_Y(s) = \frac{1-p}{1-sp},\ G'(s) = \frac{p(1-p)}{(1-sp)^2},\ G''(s) = \frac{2p^2(1-p)}{(1-sp)^3},\ G^{(3)}(s) = \frac{3 \cdot 2p^3(1-p)}{(1-sp)^3}, \ldots$$

$$G^{(k)}(s) = \frac{k!p^k(1-p)}{(1-sp)^k} \text{ for } k = 0, 1, 2, \ldots$$

So we have

$$\Pr(Y = k) = (1-p)p^k, \quad k = 0, 1, 2, \ldots$$

$$= \Pr(G_1(1-p) = k+1) = \Pr(G_1(1-p) - 1 = k)$$

which means $Y \sim G_1(1-p) - 1$.

(3) **Spring 2017 Problem 3.**

(a) Consider the sequence $\{X_k, k \ge 1\}$ of random variables such that X_1 is uniform on $(0,1)$ and, given X_k, the distribution of X_{k+1} is uniform on $(0, CX_k)$, where $\sqrt{3} < C < 2$.

(i) For $n \ge 1$, compute the conditional expectation $\mathbb{E}(X_{n+1}^r \mid X_n)$.

(ii) For $n \ge 1$, compute $\mathbb{E}(X_n^r)$.

(iii) Show that $\lim_{x\to\infty} X_n = 0$ in ℓ_1 and with probability one, but not in ℓ_2.

(iv) Investigate the same questions for all other values of $C > 0$.

(b) Let $a > 0$, let $X_n, n \ge 1$ be i.i.d. random variables that are uniform on $(0, a)$, and let $Y_n = \prod_{k=1}^n X_k$. Determine, with a proof, all values of a for which $\lim_{n\to\infty} Y_n = 0$ with probability one.

Solution.

(a) (i) We have that $X_{n+1} \mid X_n \sim U(0, CX_n)$. Therefore

$$\mathbb{E}(X_{n+1}^r \mid X_n) = \frac{1}{CX_n}\int_0^{CX_n} x^r dx = \frac{1}{CX_n} \cdot \frac{x^{r+1}}{r+1}\bigg|_0^{CX_n} = \frac{C^r X_n^r}{r+1}$$

$$\implies \mathbb{E}(X_{n+1}^r) = \mathbb{E}[\mathbb{E}(X_{n+1}^r \mid X_n)] = \frac{C^r}{r+1} \cdot \mathbb{E}(X_n^r)$$

$$\implies \boxed{\mathbb{E}(X_{n+1}^r \mid X_n) = \frac{C^r}{r+1} X_n^r}$$

(ii) Note that $E(X_1^r) = \int_0^1 x^r dr = 1/(r+1)$. Therefore

$$\mathbb{E}(X_{n+1}^r) = \frac{C^r}{r+1} \cdot \mathbb{E}(X_n^r) = \left(\frac{C^r}{r+1}\right)^n \cdot \mathbb{E}(X_1^r) = \boxed{\left(\frac{C^r}{r+1}\right)^n \cdot \frac{1}{r+1}}$$

(iii) We would like to show that $X_n \xrightarrow{w.p.1} 0$ and that $X_n \xrightarrow{1} 0$, but that the same result does not follow for the ℓ_2 norm.

- **Convergence with probability one:** We seek to show that $\Pr\big(\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = 0\}\big) = 1$. By Markov's Inequality (Lemma 8.3), we have

$$\Pr(|X_n| \geq a) \leq \frac{\mathbb{E}(X_n)}{a} \quad \forall\ a > 0$$

$$\iff \Pr(|X_n| \geq a) \leq \left(\frac{C^1}{1+1}\right)^{n-1} \cdot \frac{1}{1+1} \cdot \frac{1}{a} = \left(\frac{C}{2}\right)^{n-1} \cdot \frac{1}{2a} \quad \forall\ a > 0$$

Since $\sqrt{3} < C < 2$, $\sqrt{3}/2 < C/2 < 1$. Since $X_n \in [0, CX_{n-1}]$, $X_n \geq 0$, so $|X_n| = X_n$. Therefore we have

$$\Pr(\lim_{n\to\infty} |X_n| \geq a) = \Pr(\lim_{n\to\infty} X_n \geq a) \leq \lim_{n\to\infty} \left(\frac{C}{2}\right)^{n-1} \cdot \frac{1}{2a} = 0 \quad \forall\ a > 0$$

Since $|X_n| \geq 0$, this implies that $\Pr(\lim_{n\to\infty} X_n = 0) = \Pr\big(\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = 0\}\big) = 1$, so by the Borel-Cantelli Lemma (Theorem 8.24), X_n converges to 0 with probability 1.

- **Convergence in ℓ_1 norm:** We seek to show that $\lim_{n\to\infty} \mathbb{E}(|X_n|) = 0$. Since $X_n \in [0, CX_{n-1}]$, $X_n \geq 0$, so $|X_n| = X_n$. Therefore

$$\lim_{n\to\infty} \mathbb{E}(|X_n|) = \lim_{n\to\infty} \mathbb{E}(X_n) = \lim_{n\to\infty} \left(\frac{C}{2}\right)^{n-1} \cdot \frac{1}{2}$$

Since $\sqrt{3} < C < 2$, $\sqrt{3}/2 < C/2 < 1$, so $C/2 < 1$. Therefore we have

$$\lim_{n\to\infty} \mathbb{E}(|X_n|) = \lim_{n\to\infty} \left(\frac{C}{2}\right)^{n-1} \cdot \frac{1}{2} = 0$$

so X_n converges to 0 in 1st mean.

- **Convergence in ℓ_2 norm:** We seek to show that $\lim_{n\to\infty} \mathbb{E}(|X_n|^2) \neq 0$. We have

$$\lim_{n\to\infty} \mathbb{E}(|X_n|^2) = \lim_{n\to\infty} \mathbb{E}(X_n^2) = \lim_{n\to\infty} \left(\frac{C^2}{3}\right)^{n-1} \cdot \frac{1}{3}$$

Since $\sqrt{3} < C < 2$, $3/3 < C^2/3 < 4/3$, so $C^2/3 > 1$. Therefore we have

$$\lim_{n\to\infty} \mathbb{E}(|X_n|^2) = \lim_{n\to\infty} \left(\frac{C^2}{3}\right)^{n-1} \cdot \frac{1}{3} = \infty \neq 0$$

so X_n does not converge to 0 in 2nd mean.

(iv) From the above, it is clear that for convergence with probability one or in 1st mean we require $0 < C/2 < 1$ and for convergence in second mean we require $0 < C^2/3 < 1$. For $0 < C < \sqrt{3}$, we see that X_n would converge to zero in 2nd mean since this would imply that $0 < C^2/3 < 1$. It would also still converge to 0 in 1st mean (and with probability 1) since we would have $(0 < C/2 < \sqrt{3}/2 < 1$.
For $C = \sqrt{3}$, X_n would still converge to 0 with probability one and in 1st mean for the same reasons. However, it would not converge in 2nd mean because we would have

$$\lim_{n\to\infty} \mathbb{E}(|X_n|^2) = \lim_{n\to\infty} \left(\frac{\sqrt{3}^2}{3}\right)^{n-1} \cdot \frac{1}{3} = \frac{1}{3} \neq 0$$

For $C \geq 2$, it would diverge in all three cases, since in this case $C/2 \geq 2/2 = 1$ and $C^2/3 \geq 4/3 > 1$.

(b) **Probably won't be on midterm.** Note that

$$\lim_{n\to\infty} Y_n = \lim_{n\to\infty} \prod_{k=1}^{n} X_k = 0 \iff \log(Y_n) = \log\left(\prod_{k=1}^{n} X_k\right) = \sum_{k=1}^{n} \log(X_k) \to -\infty$$

Note that

$$\mathbb{E}[\log(Y_n)] = \mathbb{E}\left(\sum_{k=1}^{n} \log(X_k)\right) = \sum_{k=1}^{n} \mathbb{E}[\log(X_k)] = \sum_{k=1}^{n} \mathbb{E}[\log(X_1)] = \sum_{k=1}^{n} \int_0^a (\log(x)/a)dx$$

$$= \sum_{k=1}^{n} \frac{1}{a}\left[x \log x - x\right]_0^a = \sum_{k=1}^{n} \frac{a \log a - a}{a} = \sum_{k=1}^{n} (\log(a) - 1) = n(\log(a) - 1)$$

As $n \to \infty$ we have

$$\mathbb{E}[\log(Y_n)] = \begin{cases} -\infty & a < e \\ 0 & a = e \\ \infty & a > e \end{cases}$$

Since $\mathbb{E}[\log(Y_n)] \to \infty$ for $a < e$, we have $\lim_{n\to\infty} Y_n = 0$ for $a < 3$. Therefore

$$\boxed{\lim_{n\to\infty} Y_n = \lim_{n\to\infty} \prod_{k=1}^{n} X_k = 0 \iff a < e.}$$

8.9 Estimators and Central Limit Theorems (DSO 607)

Lyapunov (?) condition: can prove central limit theorem if we check 3rd moment. Lindeberg's Condition (Lemma 8.41)

9 Mathematical Statistics

These are my notes from taking Math 541A at USC taught by Steven Heilman as well as *Statistical Inference* (2nd edition) by Casella and Berger [Casella and Berger, 2001], Statistics 100B at UCLA taught by Nicolas Christou, ISE 620 at USC taught by Sheldon Ross, Math 505A at USC taught by Sergey Lototsky, and a few other sources I cite within the text.

9.1 Order Statistics

Definition 9.1 (**Order statistics (from Math 541A, more precise)**)**.** Let $X : \Omega \to \mathbb{R}$ be a random variable. Let $X_1, \ldots, X_n$ be a random sample of size n from X. Define $X_{(1)} := \min_{1 \le i \le n} X_i$, and for any $2 \le i \le n$, inductively define

$$X_i := \min \Big\{ \{X_1, \ldots, X_n\} \setminus \{X_{(1)}, \ldots, X_{(i-1)}\} \Big\},$$

so that

$$X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)} = \max_{1 \le i \le n} X_i.$$

The random variables $X_{(1)}, \ldots, X_{(n)}$ are called the **order statistics** of $X_1, \ldots, X_n$.

Definition 9.2 (**Order statistics (from ISE 620, more informal)**)**.** Let $X_1, \ldots X_n \sim iid \enspace F$ with $F' = f$. Define $X_{(1)}$ as the smallest among $X_1, \ldots, X_n$, $X_{(2)}$ as the 2nd smallest, and so on, up to $X_{(n)}$, the largest of the group. We call $X_{(1)}, \ldots, X_{(n)}$ the **order statistics** of $X_1, \ldots, X_n$.

Proposition 9.1 (**Order statistics distribution function; from Math 541A**)**.** Suppose X is a discrete random variable and we can order the values that X takes as $x_1 < x_2 < \cdots$. For any $i \ge 1$, define $p_i := \Pr(X \le x_i)$. Then for any $1 \le i, j \le n$,

$$\Pr(X_{(j)} \le x_i) = \sum_{k=j}^{n} \binom{n}{k} p_i^k (1 - p_i)^{n-k}.$$

Proof. Note that $\{X_{(j)} \le x_i\}$ is equivalent to the event that j or more of the X_i are less than or equal to x_i regardless of order; that is, x_i is the kth smallest observed value. Let A_k be the event that exactly k of the X_i are less than or equal to x_i regardless of order. Then

$$\{X_{(j)} \le x_i\} = \bigcup_{k=j}^{n} A_k.$$

Then since (by definition of p_i)

$$\Pr(A_k) = \binom{n}{k} p_i^k (1 - p_i)^{n-k}$$

and using the fact that the $\{A_k\}$ are disjoint, we have

$$\Pr(\{X_{(j)} \le x_i\}) = \Pr(\bigcup_{k=j}^{n} A_k) = \sum_{k=1}^{n} \Pr(A_k) = \sum_{k=1}^{n} \binom{n}{k} p_i^k (1 - p_i)^{n-k}.$$

□

Corollary 9.1.1. if X is a continuous random variable with density f_X and cumulative distribution function F_X, then for any $1 \leq j \leq n$, $F_{X_{(j)}}$ has density

$$f_{X_{(j)}}(x) := \frac{n!}{(j-1)!(n-j)!} f_X(x)(F_X(x))^{j-1}(1-F_X(x))^{n-j}, \qquad \forall\, x \in \mathbb{R}.$$

Proof. This follows by differentiating the identity from Proposition 9.1 for the cumulative distribution function.

□

Proposition 9.2 (Order statistics joint density function; result from ISE 620). The joint density of the order statistics is

$$f_{X_{(1)},\ldots,X_{(n)}}(x_1, \ldots, x_n) = n! \prod_{i=1}^{n} f(x_i).$$

Proof. Start with $n = 2$. We seek $f_{X_{(1)},X_{(2)}}(x_1, x_2)$. Note that $X_{(1)} = x_1, X_{(2)} = x_2$ if $X_1 = x_1, X_2 = x_2$ or if $X_1 = x_2, X_2 = x_1$. These are mutually exclusive events, so their density is equal to the sums of the two densities. That is,

$$f_{X_{(1)},X_{(2)}}(x_1, x_2) = f_{X_1,X_2}(x_1, x_2) + f_{X_1,X_2}(x_2, x_1) = 2f(x_1)f(x_2)$$

where the last step follows from the i.i.d. distributions. Generalizing, we have

$$f_{X_{(1)},\ldots,X_{(n)}}(x_1, \ldots, x_n) = n! \prod_{i=1}^{n} f(x_i).$$

□

Proposition 9.3 (Distribution of order statistics of uniform random variable; from 541A). Let X be a random variable uniformly distributed in $[0, 1]$. Then for any $1 \leq j \leq n$, $X_{(j)}$ is a beta distributed random variable with parameters j and $n + 1 - j$.

Proof. Note that for a uniform distribution on $[0, 1]$, $f_X(x) = 1, x \in [0, 1]$ and $F_X(x) = x, x \in [0, 1]$. Therefore by Corollary 9.1.1 we have

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}(x)^{j-1}(1-x)^{n-j}, \qquad x \in [0, 1]$$

$$= \frac{\Gamma(n+1)}{\Gamma(j)\Gamma(n+1-j)} x^{j-1}(1-x)^{n-j} = \frac{\Gamma(j+n+1-j)}{\Gamma(j)\Gamma(n+1-j)} x^{j-1}(1-x)^{n+1-j-1}$$

which is the pdf for a beta distribution with parameters j and $n + 1 - j$.

□

Corollary 9.3.1. Let X be a random variable uniformly distributed in $[0,1]$. Then $\mathbb{E}X_{(j)} = \frac{j}{n+1}$

Proof. Follows from Proposition 9.3 since the mean of such a beta distribution is $\frac{j}{n+1}$.

□

Proposition 9.4 (Result from 541A). Let $a, b \in \mathbb{R}$ with $a < b$. Let U be the number of indices $1 \leq j \leq n$ such that $X_j \leq a$. Let V be the number of indices $1 \leq j \leq n$ such that $a < X_j \leq b$. Then the vector $(U, V, n-U-V)$ is a multinomial random variable, so that for any nonnegative integers u, v with $u+v \leq n$, we have

$$\begin{aligned}
&\mathbb{P}(U = u, V = v, n - U - V = n - u - v) \\
&= \frac{n!}{u!v!(n-u-v)!} F_X(a)^u (F_X(b) - F_X(a))^v (1 - F_X(v))^{n-u-v}.
\end{aligned}$$

Consequently, for any $1 \leq i, j \leq n$,

$$\mathbb{P}(X_{(i)} \leq a, X_{(j)} \leq b) = \mathbb{P}(U \geq i, U + V \geq j) = \sum_{k=i}^{j-1} \sum_{m=j-k}^{n-k} \mathbb{P}(U = k, V = m) + \mathbb{P}(U \geq j).$$

So, it is possible to write an explicit formula for the joint distribution of $X_{(i)}$ and $X_{(j)}$.

Proof. We can define a multinomial distribution as follows (from Sheldon Ross *Stochastic Processes*, see Definition 6.26): "Suppose that n independent trials, each of which results in either outcome $1, 2, \ldots, r$ with respective probabilities $p_1, p_2, \ldots, p_r$ (with $\sum_i p_i = 1$), are performed. Let N_i denote the number of trials resulting in outcome i. Then the joint distribution of $N_1, \ldots, N_r$ is called the **multinomial distribution.**" In this case $r = 3$. If we define outcome 1 to be $X_j \leq a$, outcome 2 to be $a < X_j \leq b$, and outcome 3 to be $X_j > b$, then the counts $(U, V, n - U - V)$ meet this definition exactly, with $p_1 = \Pr(X_j \leq a) = F_X(a)$, $p_2 = \Pr(a < X_j \leq b) = F_X(b) - F_X(a)$, $p_3 = \Pr(X_j > b) = 1 - F_X(b)$. Since the pmf of a multinomial distribution with $r = 3$ is

$$\Pr((N_1, N_2, N_3) = (n_1, n_2, n_3)) = \binom{n}{n_1, n_2, n_3} p_1^{n_1} p_2^{n_2} p_3^{n_3} = \frac{n!}{n_1!n_2!n_3!} p_1^{n_1} p_2^{n_2} p_3^{n_3}$$

we have in this case

$$\Pr(U = u, V = v, n - U - V = n - u - v) = \frac{n!}{u!v!(n-u-v)!} F_X(a)^u (F_X(b) - F_X(a))^v (1 - F_X(v))^{n-u-v}$$

as desired.

□

Definition 9.3 (Median; Math 505A defintion). The real number m is called a **median** of a random variable X if

$$\Pr(X \leq m) \geq 1/2, \qquad \Pr(X \geq m) \geq 1/2.$$

Proposition 9.5 (Math 505A homework problem). (a) Every random variable has at least one median.

(b) The set of all medians is a closed interval of the real line.

Proof. (a) Suppose the cdf of X, $F_X : \mathbb{R} \to [0,1]$, is continuous. Because $F_X(x) = \Pr(X \leq x)$ is a cdf, it is also monotonically increasing. By the Intermediate Value Theorem, there exists at least one $m \in \mathbb{R}$ such that $F_X(m) = 1/2$. Because $\Pr(X \leq m) = 1/2 \geq 1/2$ and $\Pr(X \geq m) = 1 - \Pr(X < m) = 1 - 1/2 \geq 1/2$, m is a median.

Suppose F_X is not continuous. If it contains 1/2 in its range, then m such that $F_X(m) = 1/2$ is a median. If there is no $m \in \mathbb{R}$ such that $F_X(m) = 1/2$, then $m = \inf(\{x \mid F_X(x) \geq 1/2\})$ is a median. To see why, note first that $\Pr(X \leq m) = F_X(m) \geq 1/2$. Second,

$$\Pr(X \geq m) = 1 - \Pr(X < m) = 1 - \lim_{x \to m^-} F_X(x) \geq 1 - 1/2 = 1/2$$

because F_X is right continuous. Therefore m is a median of X.

(b) We show that all medians of X must be in one interval by contradiction. Suppose a and b are medians of X but c is not, where $a < c < b$. By the definition of median, $F_X(a) \geq 1/2$ and $F_X(b) \geq 1/2$. Because c is not a median, $F_X(c) < 1/2$. This implies that F_X is decreasing on the interval from a to c, which contradicts the fact that the distribution function F_X monotonically increases.

Finally we prove that all medians of X are in a closed interval. Let $\mathcal{A}$ be the set of all medians of X; that is, $\mathcal{A} = \{x \mid \Pr(X \leq x) \geq 1/2, \Pr(X \geq x) \geq 1/2\} = \{x \mid F_X(x) \geq 1/2, \lim_{y \to x^-} F_X(y) \leq 1/2\}$. We will show that $\mathcal{A}$ contains its infimum and its supremum. The argument above shows that $a = \inf(\{x \mid F_X(x) \geq 1/2\})$ satisfies $\lim_{y \to a^-} F_X(y) \leq 1/2$; that is, $a \in \mathcal{A}$. Since there is no lower value k which satisfies $F_X(k) \geq 1/2$, $a = \inf(\mathcal{A})$, so $\mathcal{A}$ contains its infimum.

Let $b = \sup\{x \mid \lim_{y \to x^-} F_X(y) \leq 1/2\}$. Because b is the supremum of a set containing a, $b \geq a$. Therefore because F_X is nondecreasing, $F_X(b) \geq F_X(a) \geq 1/2$, which shows that $b \in \mathcal{A}$. Since b is the supremum of the set of all values satisfying $\lim_{y \to x^-} F_X(y) \leq 1/2$, b is the supremum of $\mathcal{A}$. Therefore $\mathcal{A}$ contains its infimum and supremum, and the set of all medians of X is closed.

□

Remark. One example of a random variable which has a median of length L: X is a discrete random variable with the following mass function:

$$\Pr(X = 0) = 0.5$$

$$\Pr(X = L) = 0.5$$

Then $m \in [0, L]$ are medians.

9.2 Random Samples

Definition 9.4 (**Random Sample**)**.** Let $n > 0$ be an integer. A **random sample** of size n is a sequence $X_1, \ldots, X_n$ of independent identically distributed random variables.

Definition 9.5 (**Statistic**)**.** Let n, k be positive integers. Let $X_1, \ldots, X_n$ be a random sample of size n. Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be a (measureable) function. A **statistic** is a random variable of the form $Y := f(X_1, \ldots, X_n)$. The distribution of Y is called a **sampling distribution.**

Definition 9.6 (**Sample mean**)**.** The **sample mean** of a random sample $X_1, \ldots, X_n$ of size n, denoted $\overline{X}$, is the following statistic:

$$\overline{X} := \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Proposition 9.6. Suppose we have a random sample of size n from an i.i.d. distribution $X_1, X_2, \ldots, X_n$ with $\mathbb{E}(X_1) = \mu\ in\mathbb{R}$, $\mathrm{Var}(X_1) = \sigma^2 < \infty$. Then

(a) $\mathbb{E}(\overline{X}) = \mathbb{E}(X_1)$.

(b) $\mathrm{Var}(\overline{X}) = \sigma^2/n$.

Proof. (a)

$$\mathbb{E}(\overline{X}) = \mathbb{E}\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} X_i = \frac{1}{n} \cdot n\mu = \mu$$

(b) Using the independence of the X_i,

$$\mathrm{Var}(\overline{X}) = \mathrm{Var}\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} \sum_{i=1}^{n} \mathrm{Var}(X_i) = \frac{1}{n^2} \cdot n\sigma^2 = \boxed{\frac{\sigma^2}{n}}$$

□

Proposition 9.7 (**Stats 100B homework problem**)**.** Suppose that $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ are two samples, with $X \sim \mathcal{N}(\mu_1, \sigma_1)$ and $Y \sim \mathcal{N}(\mu_2, \sigma_2)$. The difference between the sample means, $\bar{X} - \bar{Y}$, is then a linear combination of $m + n$ normal random variables. Then

a. $\mathbb{E}(\bar{X} - \bar{Y}) = \mu_1 - \mu_2$.

b. $\mathrm{Var}(\bar{X} - \bar{Y}) = \frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}$.

c. The distribution of $\bar{X} - \bar{Y}$ is normal with mean and variance equal to the previous results.

Proof. a.

$$\bar{X} = \frac{1}{m} \sum_{i=1}^{m} X_i,\ \bar{Y} = \frac{1}{n} \sum_{j=1}^{n} Y_j$$

$$\mathbb{E}(\bar{X} - \bar{Y}) = \mathbb{E}\left(\frac{1}{m} \sum_{i=1}^{m} X_i - \frac{1}{n} \sum_{j=1}^{n} Y_j\right) = \frac{1}{m} \mathbb{E}\left(\sum_{i=1}^{m} X_i\right) - \frac{1}{n} \mathbb{E}\left(\sum_{j=1}^{n} Y_j\right)$$

$$= \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}(X_i) - \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}(Y_j) = \frac{1}{m}\sum_{i=1}^{m}\mu_1 - \frac{1}{n}\sum_{j=1}^{n}\mu_2 = \frac{1}{m}m\cdot\mu_1 - \frac{1}{n}n\cdot\mu_2$$

$$\implies \mathbb{E}(\bar{X}-\bar{Y}) = \mu_1 - \mu_2$$

b. Since X and Y are independent,

$$\text{Var}(\bar{X}-\bar{Y}) = \text{Var}(\bar{X}) + \text{Var}(\bar{Y})$$

$$= \mathbb{E}\big[(\bar{X}-\mathbb{E}[\bar{X}])^2\big] + \mathbb{E}\big[(\bar{Y}-\mathbb{E}[\bar{Y}])^2\big]$$

$$= \mathbb{E}\left(\frac{1}{m}\sum_{i=1}^{m}X_i - \mu_1\right)^2 + \mathbb{E}\left(\frac{1}{n}\sum_{j=1}^{n}Y_j - \mu_2\right)^2$$

$$= \mathbb{E}\left(\frac{1}{m}\sum_{i=1}^{m}\left(X_i - m\frac{1}{m}\mu_1\right)\right)^2 + \mathbb{E}\left(\frac{1}{n}\sum_{j=1}^{n}\left(Y_j - n\frac{1}{n}\mu_2\right)\right)^2$$

$$= \frac{1}{m^2}\mathbb{E}\left(\sum_{i=1}^{m}\left(X_i-\mu_1\right)\right)^2 + \frac{1}{n^2}\mathbb{E}\left(\sum_{j=1}^{n}\left(Y_j-\mu_2\right)\right)^2$$

Since X_i and X_j are independent for $i \neq j$ (and likewise for Y), $\text{Cov}(X_i, X_j) = 0$ for $i \neq j$, so

$$\mathbb{E}[(X_i-\mu_1)(X_j-\mu_1)] = 0$$

for $i \neq j$ (and likewise for Y). Therefore the above equation can be written as

$$\frac{1}{m^2}\mathbb{E}\left(\sum_{i=1}^{m}\left(X_i-\mu_1\right)^2\right) + \frac{1}{n^2}\mathbb{E}\left(\sum_{j=1}^{n}\left(Y_j-\mu_2\right)^2\right)$$

$$\frac{1}{m^2}\sum_{i=1}^{m}\mathbb{E}\left(X_i-\mu_1\right)^2 + \frac{1}{n^2}\sum_{j=1}^{n}\mathbb{E}\left(Y_j-\mu_2\right)^2$$

$$= \frac{1}{m^2}\left(\sum_{i=1}^{m}\sigma_1^2\right) + \frac{1}{n^2}\left(\sum_{j=1}^{n}\sigma_2^2\right) = \frac{1}{m^2}m\cdot\sigma_1^2 + \frac{1}{n^2}n\cdot\sigma_2^2$$

$$\implies \text{Var}(\bar{X}-\bar{Y}) = \frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}$$

c.

$$M_{X_i}(t) = \exp\left(\mu_1 t + \frac{t^2\sigma_1^2}{2}\right),\ M_{Y_i}(t) = \exp\left(\mu_2 t + \frac{t^2\sigma_2^2}{2}\right)$$

Since individual observations from X and Y are independent,

$$M_{\bar{X}}(t) = \prod_{i=1}^{m}M_{X_i}\left(\frac{1}{m}t\right),\ M_{\bar{Y}}(t) = \prod_{j=1}^{n}M_{Y_j}\left(\frac{1}{n}t\right)$$

and

$$M_{\bar{X}-\bar{Y}}(t) = M_{\bar{X}}(t)M_{-\bar{Y}}(t) = M_{\bar{X}}(t)M_{\bar{Y}}(-t) = \prod_{i=1}^{m} M_{X_i}\left(\frac{1}{m}t\right)\prod_{j=1}^{n} M_{Y_j}\left(\frac{-1}{n}t\right)$$

$$= \left[M_{X_i}\left(\frac{t}{m}\right)\right]^m \left[M_{Y_j}\left(\frac{-t}{n}\right)\right]^n = \left[\exp\left(\frac{\mu_1 t}{m} + \frac{t^2\sigma_1^2}{2m^2}\right)\right]^m \left[\exp\left(\frac{-\mu_2 t}{n} + \frac{(-t)^2\sigma_2^2}{2n^2}\right)\right]^n$$

$$= \exp\left(\frac{m\mu_1 t}{m} + \frac{mt^2\sigma_1^2}{2m^2}\right)\exp\left(\frac{-n\mu_2 t}{n} + \frac{nt^2\sigma_2^2}{2n^2}\right)$$

$$\implies \boxed{M_{\bar{X}-\bar{Y}}(t) = \exp\left[(\mu_1-\mu_2)t + \frac{1}{2}t^2\left(\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}\right)\right]}$$

This is the moment generating function of a normal distribution with mean $\mu_1 - \mu_2$ and variance $\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}$, consistent with the results from parts (a) and (b).

□

Definition 9.7 (Sample variance). Let $n > 1$. The **sample variance** of a random sample $X_1, \ldots, X_n$ of size n, denoted S^2, is the following statistic:

$$S^2 := \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

The **sample standard deviation** of a random sample of size n is $\sqrt{S^2}$.

Proposition 9.8 (Unbiasedness of sample variance). Suppose we have a random sample of size n from an i.i.d. distribution $X_1, X_2, \ldots, X_n$ with $\mathbb{E}(X_1) = \mu\ in\mathbb{R}$, $\mathrm{Var}(X_1) = \sigma^2 < \infty$. Then $\mathbb{E}(S^2) = \sigma^2$. Further, S^2 is a consistent estimator of σ^2.

Proof. We have

$$\mathbb{E}(S^2) = \mathbb{E}\left(\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2\right) = \frac{1}{n-1}\mathbb{E}\left(\sum_{i=1}^{n}X_i^2 - 2X_i\overline{X} + \overline{X}^2\right)$$

$$= \frac{1}{n-1}\left(\sum_{i=1}^{n}\mathbb{E}(X_i^2) - 2\mathbb{E}\left(\overline{X}\sum_{i=1}^{n}X_i\right) + n\mathbb{E}\overline{X}^2\right) = \frac{1}{n-1}\left(n\mathbb{E}(X_i^2) - 2n\mathbb{E}\overline{X}^2 + n\mathbb{E}\overline{X}^2\right)$$

$$= \frac{n}{n-1}\left(\mathbb{E}(X_i^2) - \mathbb{E}\overline{X}^2\right) = \frac{n}{n-1}\left(\mathrm{Var}(X_i) + \mathbb{E}(X_i)^2 - [\mathrm{Var}(\overline{X}) + \mathbb{E}(\overline{X})^2]\right)$$

Using the results from Proposition 9.6, we have

$$\mathbb{E}(S^2) = \frac{n}{n-1}\left(\sigma^2 + \mu^2 - [\sigma^2/n + \mu^2]\right) = \frac{n}{n-1}\cdot\frac{(n-1)\sigma^2}{n} = \boxed{\sigma^2}$$

□

Alternative proof from Stats 100B homework.

$$\mathbb{E}\bigg(\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2\bigg) = \frac{1}{n}\mathbb{E}\bigg(\sum_{i=1}^{n}(x_i-\mu)^2\bigg)$$

Assuming independence of samples, this can be written as

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}((x_i-\mu)^2) = \frac{1}{n}n\sigma^2 = \boxed{\sigma^2}$$

Since S^2 is unbiased, it is a consistent estimator if we can show $\text{Var}(S^2) \to 0$ as $n \to \infty$. We have

$$\text{Var}(\frac{(n-1)S^2}{\sigma^2}) = 2(n-1)$$

$$\frac{(n-1)^2}{\sigma^4}\text{Var}(S^2) = 2(n-1)$$

$$\text{Var}(S^2) = \frac{2\sigma^4}{n-1}$$

$$\implies \lim_{n\to\infty}\text{Var}(S^2) = \lim_{n\to\infty}\frac{2\sigma^4}{n-1} = \boxed{0}$$

Therefore S^2 is a consistent estimator of σ^2.

□

Lemma 9.9. Let $X := (X_1, \ldots, X_n)$ be i.i.d. mean zero, variance 1 Gaussian random variables. Let $v_1, \ldots, v_n \in \mathbb{R}^n$. Then $\langle X, v_1\rangle, \ldots, \langle X, v_n\rangle$ are independent if and only if $v_1, \ldots, v_m$ are pairwise orthogonal; that is, $\langle v_i, v_j\rangle = 0 \ \forall\ 1 \leq i < j \leq m$.

Proof. By Theorem 6.56, we have that for any $v \in \mathbb{R}^n$, $\langle X, v\rangle$ is a mean zero Gaussian with variance $\langle v, v\rangle$. For notational convenience, let $\langle X, v_k\rangle = A_k$. Because all the A_k are Gaussian random variables by Theorem 6.56, the A_k are uncorrelated if and only if they are independent. That is, we would like to show that their covariances

$$\mathbb{E}[(A_k - \mathbb{E}A_k)(A_\ell - \mathbb{E}A_\ell)]$$

equal zero for all $\{(k,\ell) : k,\ell \in \{1,2,\ldots,m\}, k \neq \ell\}$ if and only if the vectors $v_1, \ldots, v_m$ are pairwise orthogonal; that is, $\langle v_k, v_\ell\rangle = 0$ for all $\{(k,\ell) : k,\ell \in \{1,2,\ldots,m\}, k \neq \ell\}$. Note that since $A_k = \sum_{i=1}^n X_i v_{ki}$, $\mathbb{E}(A_k) = \sum_{i=1}^n v_{ki}\mathbb{E}(X_i)$. So for any $\{(k,\ell) : k,\ell \in \{1,2,\ldots,m\}, k \neq \ell\}$ we have

$$\mathbb{E}[(A_k - \mathbb{E}A_k)(A_\ell - \mathbb{E}A_\ell)] = \mathbb{E}\bigg[\bigg(\sum_{i=1}^{n}X_i v_{ki} - \sum_{i=1}^{n}v_{ki}\mathbb{E}(X_i)\bigg)\bigg(\sum_{i=1}^{n}X_i v_{\ell i} - \sum_{i=1}^{n}v_{\ell i}\mathbb{E}(X_i)\bigg)\bigg]$$

$$= \mathbb{E}\Bigg[\bigg(\sum_{i=1}^n X_i v_{ki}\bigg)\bigg(\sum_{i=1}^n X_i v_{\ell i}\bigg) - \bigg(\sum_{i=1}^n X_i v_{ki}\bigg)\bigg(\sum_{i=1}^n v_{\ell i}\mathbb{E}(X_i)\bigg)$$

$$-\bigg(\sum_{i=1}^n v_{ki}\mathbb{E}(X_i)\bigg)\bigg(\sum_{i=1}^n X_i v_{\ell i}\bigg) + \bigg(\sum_{i=1}^n v_{ki}\mathbb{E}(X_i)\bigg)\bigg(\sum_{i=1}^n v_{\ell i}\mathbb{E}(X_i)\bigg)\Bigg]$$

$$= \mathbb{E}\bigg(\sum_{i=1}^n X_i^2 v_{ki}v_{\ell i} + \sum_{\{a,b\in\{1,\ldots,n\},a\neq b\}} X_a X_b v_{ka}v_{\ell b}\bigg) - 2\mathbb{E}\bigg(\sum_{i=1}^n X_i\mathbb{E}(X_i)v_{ki}v_{\ell i} + \sum_{\{a,b\in\{1,\ldots,n\},a\neq b\}} X_a\mathbb{E}(X_b)v_{ka}v_{\ell b}\bigg)$$

$$+\mathbb{E}\bigg(\sum_{i=1}^n \mathbb{E}(X_i)^2 v_{ki}v_{\ell i} + \sum_{\{a,b\in\{1,\ldots,n\},a\neq b\}} \mathbb{E}(X_a)\mathbb{E}(X_b)v_{ka}v_{\ell b}\bigg)$$

Recall that $\mathbb{E}(X_i) = 0$ for all i. Also, due to independence of the X_i, all of the terms that involve $\mathbb{E}(X_a X_b),\ a \neq b$ disappear. This leaves only

$$= \mathbb{E}\bigg(\sum_{i=1}^n X_i^2 v_{ki}v_{\ell i}\bigg) = \sum_{i=1}^n \mathbb{E}(X_i^2)v_{ki}v_{\ell i} = \mathbb{E}(X_1^2)\sum_{i=1}^n v_{ki}v_{\ell i} \tag{9.1}$$

where the last step follows from the i.i.d. distributions of X_i. Recall

$$\langle v_k, v_\ell\rangle = 0 \iff \sum_{i=1}^n v_{ki}v_{\ell i} = 0.$$

Since $\mathbb{E}(X_i^2) \neq 0$, (9.1) equals 0 for all $\{(k,\ell) : k,\ell \in \{1,2,\ldots,m\}, k \neq \ell\}$ if and only if $\langle v_k, v_\ell\rangle = 0$ for all $\{(k,\ell) : k,\ell \in \{1,2,\ldots,m\}, k \neq \ell\}$. Therefore the random variables $\langle X, v_1\rangle, \ldots, \langle X, v_m\rangle$ are independent if and only if the vectors $v_1, \ldots, v_m$ are pairwise orthogonal.

$\square$

Proposition 9.10 (Proposition 4.7 in 541A notes). Let $n \geq 2$ be an integer. Let $X_1, \ldots, X_n$ be a random sample from the Gaussian distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$. Let $\overline{X}$ be the sample mean and let S be the sample standard deviation. Then

(i) $\overline{X}$ and S are independent random variables.

(ii) $\overline{X}$ is a Gaussian random variable with mean μ and variance σ^2/n.

(iii) $(n-1)S^2/\sigma^2$ is a χ^2-distributed random variable with $n-1$ degrees of freedom.

Proof. (i) Replace $X_1, \ldots, X_n$ with $X_1 - \mu, \ldots, X_n - \mu$ so that $\mu = 0$. Also divide by σ so that $\sigma = 1$. Note that $\overline{X}$ is independent of all random variables $X_2 - \overline{X}, \ldots, X_n - \overline{X}$ by Lemma 9.9 because for example

$$X_2 - \overline{X} = \langle X_2, e_2 - \frac{1}{n}(1,1,\ldots,1)\rangle$$

where the second vector in the inner product is orthogonal to $(1,1,\ldots,1)$ (in fact, $(1,\ldots,1)$ is orthogonal to anything in the span of these vectors). Likewise for all the remaining vectors you could use to construct X_i. (Note that the other random variables [e.g. $X_2 - \overline{X}$ and $X_3 - \overline{X}$] are not independent.)

So the proof will be complete if we can write S as a function of $X_2 - \overline{X}, \ldots, X_n - \overline{X}$. Observe

$$(n-1)S^2 = \sum_{i=1}^n (X_i - \overline{X})^2 = (X_1 - \overline{X})^2 + \sum_{i=2}^n (X_i - \overline{X})^2 = \left(n\overline{X} - \left[\sum_{i=2}^n X_i \right] - \overline{X} \right)^2 + \sum_{i=2}^n \left(X_i - \overline{X} \right)^2$$

$$= \left(\sum_{i=2}^n (X_i - \overline{X} \right)^2 + \sum_{i=2}^n (X_i - \overline{X})^2$$

(ii) Follows from Proposition 1.45, Example 1.108, and Exercise 1.58 in 541A notes (condense later?)

(iii) Like above, replace $X_1, \ldots, X_n$ with $X_1 - \mu, \ldots, X_n - \mu$ so that $\mu = 0$. Also divide by σ so that $\sigma = 1$. We will prove by induction. Let $\overline{X}_n := \frac{1}{n}\sum_{i=1}^n X_i$ and let $S_n^2 := \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X}_n)^2$. In the case $n = 2$ we have

$$S_2^2 = \left(X_1 - \frac{1}{2}(X_1 + X_2) \right)^2 + \left(X_2 - \frac{1}{2}(X_1 + X_2) \right)^2 = \frac{1}{4}(X_2 - X_1)^2 + \frac{1}{4}(X_2 - X_1)^2 = \frac{1}{2}(X_2 - X_1)^2$$

$$= \left(\frac{1}{\sqrt{2}}(X - 2 - X_1) \right)^2$$

Note that $1/\sqrt{2}(X_2 - X_1)$ is a mean zero Gaussian random variable with variance 1 (see example 1.108 in 541A notes for details). So S_2^2 is χ_1^2 by Definition 1.33 in 541A notes.

We now induct on n. From Lemma 4.8 in 541A notes (will prove later),

$$nS_{n+1}^2 = (n-1)S_n^2 + \frac{n}{n+1}(X_{n+1} - \overline{X}_n)^2, \quad \forall n \geq 2$$

From the first item, S_n is independent of $\overline{X}_n$. Also, X_{n+1} is independent of S_n by Proposition 1.61 in Math 541A notes, since S_n is a function of $X_1, \ldots, X_n$, which are independent of X_{n+1}. So S_n is independent of $(X_{n+1} - \overline{X}_n)^2$. By the inductive hypothesis, $(n-1)S_n^2$ is a χ_{n-1}^2 random variable. From Example 1.108 in Math 541A notes, $X_{n+1} - \overline{X}_n$ is a Gaussian random variable with mean zero and variance $1 + 1/n = (n+1)/n$ so that $\sqrt{n/(n+1)}(X_{n+1} - \overline{X}_n)$ is a mean zero Gaussian with variance 1, implying $n/(n+1)(X_{n+1} - \overline{X}_n)^2$ is χ^2. Definition 1.33 in 541A notes then implies that nS_{n+1} is a χ_n^2 random variable, completing the inductive step.

□

Lemma 9.11 (Lemma 4.8 in 541A notes.).

Let $X_1, X_2, \ldots$ be random variables. For any $n \geq 2$, let $\overline{X}_n := (1/n)\sum_{i=1}^n X_i$ and let $S_n^2 := 1/(n-1)\sum_{i=1}^n (X_i - \overline{X}_n)^2$. Then

$$nS_{n+1}^2 - (n-1)S_n^2 = \frac{n}{n+1}(X_{n+1} - \overline{X}_n)^2.$$

Proof.

$$nS_{n+1}^2 - (n-1)S_n^2 = \sum_{i=1}^{n+1}(X_i - \overline{X}_{n+1})^2 - \sum_{i=1}^{n}(X_i - \overline{X}_n)^2$$

Note:

$$(a-b)^2 - (a-c)^2 = a^2 - 2ab + b^2 - a^2 - c^2 + 2ac = b^2 - c^2 + 2a(c-b)$$

$$= (b-c)[(b+c) - 2a] = (b-c)(b+c-2a)$$

for all real a, b, c. Using $a = X_n, b = \overline{X}_{n+1}, c = \overline{X_n}$ we have

$$= (X_{n+1} - \overline{X}_{n+1})^2 + \sum_{i=1}^{n}(\overline{X}_{n+1} - \overline{X}_n)(\overline{X}_{n+1} + \overline{X}_n - 2X_i)$$

$$= (X_{n+1} - \overline{X}_{n+1})^2 + (\overline{X}_{n+1} - \overline{X}_n)\sum_{i=1}^{n}(\overline{X}_{n+1} + \overline{X}_n - 2X_i)$$

$$= (X_{n+1} - \overline{X}_{n+1})^2 + (\overline{X}_{n+1} - \overline{X}_n) \cdot n(\overline{X}_{n+1} + \overline{X}_n - 2\overline{X}_n)$$

$$= (X_{n+1}(1 - 1/(n+1)) - \frac{n}{n++1}\overline{X}_n)^2 + n(\overline{X}_{n+1} - \overline{X}_n)^2$$

$$= \frac{n^2}{(n+1)^2}(X_{n+1} - \overline{X}_n)^2 + n(\frac{X_{n+1}}{n+1} + (\frac{1}{n+1} - \frac{1}{n})\sum_{i=1}^{n} X_i)^2$$

Algebra: $1/(n+1) - 1/n = \frac{n-(n+1)}{n(n+1)} = -\frac{1}{n(n+1)}$. So we have

$$= \frac{n^2}{(n+1)^2}(X_{n+1} - \overline{X}_n)^2 + \frac{n}{(n+1)^2}(X_{n+1} - \frac{1}{n}\sum_{i=1}^{n} X_i)^2$$

$$= \frac{n^2}{(n+1)^2}(X_{n+1} - \overline{X}_n)^2 + \frac{n}{(n+1)^2}(X_{n+1} - \overline{X}_n)^2$$

$$= \frac{n^2+n}{(n+1)^2}(X_{n+1} - \overline{X}_n)^2 = \frac{n}{n+1}(X_{n+1} - \overline{X}_n)^2$$

□

Proposition 9.12 (**Proposition 4.9 in 541A notes**). Let X be a standard Gaussian random variable. Let Y be a χ_p^2 random variable. Assume that X and Y are independent. Then $X/\sqrt{Y/p}$ has the following density, known as **Student's t-distribution** with $p =$ degrees of freedom: $(p = n + 1?)$

$$f_{X/(Y/\sqrt{p})}(t) := \frac{\Gamma((p+1)/2)}{\sqrt{p}\sqrt{\pi}\Gamma(p/2)}\left(1 + \frac{t^2}{p}\right)^{-(p+1)/2}, \quad \forall t \in \mathbb{R}$$

(should have $p + 1$ in a bunch of the expressions above? that's what was written on board, not in notes.)

Proof. Let $Z := \sqrt{Y/p}$. We find the density of Z as follows. Let $t > 0$. Then

$$f_Z(y) = \frac{\mathrm{d}}{\mathrm{d}y}\Big|_{y=0} \Pr(Z \leq y) = \frac{\mathrm{d}}{\mathrm{d}y}\Big|_{y=0} \Pr(Y \leq y^2 p)$$

$$= \frac{\mathrm{d}}{\mathrm{d}y}\Big|_{y=0} \int_0^{y^2 p} \frac{x^{(p/2)-1}e^{-x/2}}{2^{p/2}\Gamma(p/2)} dx = 2yp \cdot p^{(p/2)-1} y^{p-2} e^{-y^2 p/2} \cdot \frac{1}{2^{p/2}\Gamma(p/2)}$$

$$= p^{p/2} y^{p-1} e^{-y^2 p/2} \cdot \frac{1}{2^{p/2-1}\Gamma(p/2)}$$

$\vdots$ skipped this stuff in class proof

$$\Pr(X/Z \leq t) = \Pr(X \leq tZ)$$

$$= \text{ (by definition of joint density) } \int \int_{\{(x,y) \in \mathbb{R}^2 : x \leq ty\}} f_X(x) f_Z(y) dx dy$$

We use the change of variables formula:

$$\int \int_{\phi(U)} f(x,y) dx dy = \int \int_U f(\phi(a,b)) |\operatorname{Jac} \phi(a,b)| da db$$

$$\phi : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\phi(a,b) = (ab, a)$$

$$\phi^{-1}(x,y) = (y, x/y)$$

We chose x/y as the second variable so that an upper limit of the variable will end up being t after the transformation. We need the Jacobian of ϕ:'

$$|\operatorname{Jac}\phi(a,b)| = \left|\det\begin{pmatrix} b & a \\ 1 & 0 \end{pmatrix}\right| = |a|$$

By the change in variables formula,

$$\int\int_{\phi(U)} f(x,y)dxdy = \int\int_U f(\phi(a,b))\left|\operatorname{Jac}\phi(a,b)\right| dadb$$

$$= \int\int_{\{(a,b)\in\mathbb{R}^2 : a\geq 0, b\leq t\}} f_X(ab)f_Z(a)\,|a|\,dadb$$

$$\implies \Pr(X/Z \leq t) = \int_{-\infty}^{t}\int_0^\infty |a|f_X(ab)f_Z(a)dadb$$

By the Fundamental Theorem of Calculus,

$$f_{X/Z}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Pr(X/Z\leq t) = \int_0^\infty |a|f_X(at)f_Z(a))da = \int_0^\infty af_X(at)f_Z(a))da$$

By the definitions of X and Z,

$$= \frac{1}{2^{-/2-1}\Gamma(p/2)}\int_0^\infty a\cdot\frac{1}{\sqrt{2\pi}}e^{-(a^2t^2)/2}\cdot p^{p/2}a^{p-1}e^{-a^2p/2}da$$

$$= \frac{p^{p/2}}{2^{-/2-1}\Gamma(p/2)\sqrt{2\pi}}\int_0^\infty e^{-[a^2(t^2+p)]/2}\cdot a^p da$$

Change of variables: let $x = a^2, dx = 2ada, da = \frac{1}{2a}dx = 1/(2\sqrt{x})dx$. Then this integral is

$$= c\int_0^\infty e^{-[x(t^2+p)]/2}\cdot x^{p/2-1/2}da, \quad \text{where } c = \frac{p^{p/2}}{2^{p/2}\sqrt{2\pi}\Gamma(p/2)}$$

So the integrand is a Gamma density function with parameters α, β: $\alpha - 1 = p/2 - 1/2 \iff \alpha = p/2 + 1/2$, $\beta = 2/(t^2+p)$. So if we multiply and divide $\beta^\alpha\Gamma(\alpha)$

So

$$f_{X/Z}(t) = \frac{p^{p/2}}{2^{p/2}\sqrt{2\pi}\Gamma(p/2)}\cdot\beta^\alpha\Gamma(\alpha)\cdot 1 = \frac{p^{p/2}\Gamma((p_1)/2)}{2^{p/2}\sqrt{2\pi}\Gamma(p/2)}\cdot\left(\frac{2}{t^2+p}\right)^{(p-1)/2}$$

$$= \frac{p^{p/2}\Gamma((p+1)/2)}{\sqrt{\pi}\Gamma(p/2)}\cdot(t^2+p)^{-(p+1)/2} = \frac{\Gamma((p+1)/2)}{\sqrt{\pi p}\Gamma(p/2)}\cdot(1+t^2/p)^{-(p+1)/2}$$

□

Remark (Remark 4.10 in 541A notes)**.** If $X_1, \ldots, X_n$ is a random sample from a Gaussian distribution with mean $\mu \in \mathbb{R}$, standard deviation $\sigma < 0$, then

$$\frac{\overline{X} - \mu}{S/\sqrt{n}}$$

also has Student's t distribution. ($\overline{X} := n^{-1} \sum_{i=1}^n X_i, S = \sqrt{(n-1)^{n-1} \sum_{i=1}^n (X_i - \overline{X})^2}$.)

Proposition 9.13 (**Stats 100B homework 3 problem**)**.** Let X_1, X_2 be a random sample from a normal distribution with a mean μ and standard deviation σ. Then $(n-1)s^2/\sigma^2$ has a χ_1^2 distribution.

Proof.

$$s^2 = \frac{1}{2-1} \sum_{i=1}^{2} (X_i - \bar{X})^2 = (X_1 - \bar{X})^2 + (X_2 - \bar{X})^2 = (X_1 - \frac{X_1 + X_2}{2})^2 + (X_2 - \frac{X_1 + X_2}{2})^2$$

$$= X_1^2 - 2X_1(\frac{X_1 + X_2}{2}) + (\frac{X_1 + X_2}{2})^2 + X_2^2 - 2X_2(\frac{X_1 + X_2}{2}) + (\frac{X_1 + X_2}{2})^2$$

$$= X_1^2 + X_2^2 - X_1(X_1 + X_2) - X_2(X_1 + X_2) + 2(\frac{X_1 + X_2}{2})^2$$

$$= X_1^2 + X_2^2 - (X_1 + X_2)(X_1 + X_2) + \frac{(X_1 + X_2)^2}{2}$$

$$= \frac{1}{2}(2X_1^2 + 2X_2^2) - \frac{1}{2}(X_1^2 + 2X_1X_2 + X_2^2)$$

$$= \frac{1}{2}(X_1^2 - 2X_1X_2 + X_2^2)$$

$$\boxed{s^2 = \frac{1}{2}(X_1 - X_2)^2}$$

$$\implies \frac{(n-1)s^2}{\sigma^2} = (2-1)\frac{1}{2\sigma^2}(X_1 - X_2)^2 = \left(\frac{X_1 - X_2}{\sigma\sqrt{2}}\right)^2$$

Since X_1 and X_2 are normal,

$$X_1 - X_2 \sim \mathcal{N}(\mu - \mu, \sqrt{\sigma^2 + \sigma^2}) = \mathcal{N}(0, \sigma\sqrt{2}) \implies \frac{X_1 - X_2}{\sigma\sqrt{2}} \sim \mathcal{N}(0, 1)$$

$$\implies \left(\frac{X_1 - X_2}{\sigma\sqrt{2}}\right)^2 = \boxed{\frac{(n-1)s^2}{\sigma^2} \sim \chi_1^2}$$

$\square$

Proposition 9.14 (Stats 100B homework problem). Suppose two independent random samples of n_1 and n_2 observations are selected from two normal populations. Further, assume that the populations possess a common variance σ^2 which is unknown. Let the sample variances be S_1^2 and S_2^2 and assume they are unbiased. Then the pooled estimator for σ^2

$$S^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

is unbiased and has variance $\frac{2\sigma^4}{n_1+n_2-2}$.

Proof. First we show S^2 is unbiased.

$$\mathbb{E}(S^2) = \mathbb{E}\left(\frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}\right) = \frac{n_1 - 1}{n_1 + n_2 - 2}\mathbb{E}(S_1^2) + \frac{n_2 - 1}{n_1 + n_2 - 2}\mathbb{E}(S_2^2)$$

$$= \frac{n_1 - 1}{n_1 + n_2 - 2}\sigma^2 + \frac{n_2 - 1}{n_1 + n_2 - 2}\sigma^2 = \frac{(n_1 + n_2 - 2)\sigma^2}{n_1 + n_2 - 2} = \boxed{\sigma^2}$$

Now we derive its variance.

$$\text{Var}(S^2) = \text{Var}\left(\frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}\right)$$

Since S_1 and S_2 are independent, this can be written as

$$\frac{1}{(n_1 + n_2 - 2)^2}\left(\text{Var}[(n_1 - 1)S_1^2] + \text{Var}[(n_2 - 1)S_2^2]\right)$$

Since the populations are normal, we know

$$\frac{(n_i - 1)S_i^2}{\sigma^2} \sim \chi^2_{n_i - 1} \implies \text{Var}\left(\frac{(n_i - 1)S_i^2}{\sigma^2}\right) = 2(n_i - 1)$$

$$\text{Var}(S^2) = \frac{\sigma^4}{(n_1 + n_2 - 2)^2}\left(\text{Var}\left[\frac{(n_1 - 1)S_1^2}{\sigma^2}\right] + \text{Var}\left[\frac{(n_2 - 1)S_2^2}{\sigma^2}\right]\right)$$

$$\frac{\sigma^4}{(n_1 + n_2 - 2)^2}(2(n_1 - 1) + 2(n_2 - 1)) = \sigma^4 \frac{2(n_1 + n_2 - 2)}{(n_1 + n_2 - 2)^2}$$

$$= \frac{2\sigma^4}{n_1 + n_2 - 2}$$

□

Proposition 9.15 (**Stats 100B Homework problem**). Suppose that $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ are two samples, with $X \sim \mathcal{N}(\mu_1, \sigma_1)$ and $Y \sim \mathcal{N}(\mu_2, \sigma_2)$. The difference between the sample means, $\bar{X} - \bar{Y}$, is then a linear combination of $m + n$ normal random variables.

a. $\mathbb{E}(\bar{X} - \bar{Y})$.

b. $\text{Var}(\bar{X} - \bar{Y})$

c. The distribution of $\bar{X} - \bar{Y}$ is normal.

Proof. a.

$$\bar{X} = \frac{1}{m}\sum_{i=1}^{m} X_i, \ \bar{Y} = \frac{1}{n}\sum_{j=1}^{n} Y_j$$

$$\mathbb{E}(\bar{X} - \bar{Y}) = \mathbb{E}\bigg(\frac{1}{m}\sum_{i=1}^{m} X_i - \frac{1}{n}\sum_{j=1}^{n} Y_j\bigg) = \frac{1}{m}\mathbb{E}\bigg(\sum_{i=1}^{m} X_i\bigg) - \frac{1}{n}\mathbb{E}\bigg(\sum_{j=1}^{n} Y_j\bigg)$$

$$= \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}(X_i) - \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}(Y_j) = \frac{1}{m}\sum_{i=1}^{m}\mu_1 - \frac{1}{n}\sum_{j=1}^{n}\mu_2 = \frac{1}{m}m \cdot \mu_1 - \frac{1}{n}n \cdot \mu_2$$

$$\boxed{\mathbb{E}(\bar{X} - \bar{Y}) = \mu_1 - \mu_2}$$

b. Since X and Y are independent,

$$\text{Var}(\bar{X} - \bar{Y}) = \text{Var}(\bar{X}) + \text{Var}(\bar{Y})$$

$$= \mathbb{E}\big[(\bar{X} - \mathbb{E}[\bar{X}])^2\big] + \mathbb{E}\big[(\bar{Y} - \mathbb{E}[\bar{Y}])^2\big]$$

$$= \mathbb{E}\bigg(\frac{1}{m}\sum_{i=1}^{m} X_i - \mu_1\bigg)^2 + \mathbb{E}\bigg(\frac{1}{n}\sum_{j=1}^{n} Y_j - \mu_2\bigg)^2$$

$$= \mathbb{E}\bigg(\frac{1}{m}\sum_{i=1}^{m}\big(X_i - m\frac{1}{m}\mu_1\big)\bigg)^2 + \mathbb{E}\bigg(\frac{1}{n}\sum_{j=1}^{n}\big(Y_j - n\frac{1}{n}\mu_2\big)\bigg)^2$$

$$= \frac{1}{m^2}\mathbb{E}\bigg(\sum_{i=1}^{m}\big(X_i - \mu_1\big)\bigg)^2 + \frac{1}{n^2}\mathbb{E}\bigg(\sum_{j=1}^{n}\big(Y_j - \mu_2\big)\bigg)^2$$

Since X_i and X_j are independent for $i \neq j$ (and likewise for Y), $\text{Cov}(X_i, X_j) = 0$ for $i \neq j$, so

$$\mathbb{E}[(X_i - \mu_1)(X_j - \mu_1)] = 0$$

for $i \neq j$ (and likewise for Y). Therefore the above equation can be written as

$$\frac{1}{m^2}\mathbb{E}\bigg(\sum_{i=1}^{m}\big(X_i - \mu_1\big)^2\bigg) + \frac{1}{n^2}\mathbb{E}\bigg(\sum_{j=1}^{n}\big(Y_j - \mu_2\big)^2\bigg)$$

$$\frac{1}{m^2}\sum_{i=1}^{m}\mathbb{E}\big(X_i-\mu_1\big)^2+\frac{1}{n^2}\sum_{j=1}^{n}\mathbb{E}\big(Y_j-\mu_2\big)^2$$

$$=\frac{1}{m^2}\left(\sum_{i=1}^{m}\sigma_1^2\right)+\frac{1}{n^2}\left(\sum_{j=1}^{n}\sigma_2^2\right)=\frac{1}{m^2}m\cdot\sigma_1^2+\frac{1}{n^2}n\cdot\sigma_2^2$$

$$\boxed{\operatorname{Var}(\bar{X}-\bar{Y})=\frac{\sigma_1^2}{m}+\frac{\sigma_2^2}{n}}$$

c.

$$M_{X_i}(t)=\exp\left(\mu_1 t+\frac{t^2\sigma_1^2}{2}\right),\ M_{Y_i}(t)=\exp\left(\mu_2 t+\frac{t^2\sigma_2^2}{2}\right)$$

Since individual observations from X and Y are independent,

$$M_{\bar{X}}(t)=\prod_{i=1}^{m}M_{X_i}\left(\frac{1}{m}t\right),\ M_{\bar{Y}}(t)=\prod_{j=1}^{n}M_{Y_j}\left(\frac{1}{n}t\right)$$

and

$$M_{\bar{X}-\bar{Y}}(t)=M_{\bar{X}}(t)M_{-\bar{Y}}(t)=M_{\bar{X}}(t)M_{\bar{Y}}(-t)=\prod_{i=1}^{m}M_{X_i}\left(\frac{1}{m}t\right)\prod_{j=1}^{n}M_{Y_j}\left(\frac{-1}{n}t\right)$$

$$=\left[M_{X_i}\left(\frac{t}{m}\right)\right]^m\left[M_{Y_j}\left(\frac{-t}{n}\right)\right]^n=\left[\exp\left(\frac{\mu_1 t}{m}+\frac{t^2\sigma_1^2}{2m^2}\right)\right]^m\left[\exp\left(\frac{-\mu_2 t}{n}+\frac{(-t)^2\sigma_2^2}{2n^2}\right)\right]^n$$

$$=\exp\left(\frac{m\mu_1 t}{m}+\frac{mt^2\sigma_1^2}{2m^2}\right)\exp\left(\frac{-n\mu_2 t}{n}+\frac{nt^2\sigma_2^2}{2n^2}\right)$$

$$\implies\boxed{M_{\bar{X}-\bar{Y}}(t)=\exp\left[(\mu_1-\mu_2)t+\frac{1}{2}t^2\left(\frac{\sigma_1^2}{m}+\frac{\sigma_2^2}{n}\right)\right]}$$

This is the moment generating function of a normal distribution with mean $\mu_1-\mu_2$ and variance $\frac{\sigma_1^2}{m}+\frac{\sigma_2^2}{n}$, consistent with the results from parts (a) and (b).

□

9.2.1 The Delta Method

Theorem 9.16 (Delta Method, Theorem 4.14 in 541A notes, 5.5.24 in Casella and Berger). Let $\theta\in\mathbb{R}$. Let $Y_1,Y_2,\ldots$ be random variables such that $\sqrt{n}(Y_n-\theta)$ converges in distribution to a mean zero Gaussian random variable with variance $\sigma^2>0$. Let $f:\mathbb{R}\to\mathbb{R}$. Assume that f' exists and is continuous, and $f'(\theta)\neq 0$. Then

$$\sqrt{n}(f(Y_n)-f(\theta))\xrightarrow{d}\mathcal{N}(0,\sigma^2(f'(\theta))^2).$$

Proof from new class notes. Since $f'(\theta)$ exists, $\lim_{y\to\theta} \frac{f(y)-f(\theta)}{y-\theta}$ exists. That is, there exists $h : \mathbb{R} \to \mathbb{R}$ such that $\lim_{z\to 0} \frac{h(z)}{z} = 0$ and for all $y \in \mathbb{R}$,

$$f'(\theta) = \frac{f(y) - f(\theta)}{(y-\theta)} + h(y-\theta)$$

$$\iff f(y) = f(\theta) + f'(\theta)(y-\theta) + h(y-\theta).$$

In particular,

$$\sqrt{n}[f(Y_n) - f(\theta)] = \underbrace{f'(\theta)}_{\text{(constant)}} \underbrace{\sqrt{n}(Y_n - \theta)}_{\implies \mathcal{N}(0,\sigma^2)} + \underbrace{\sqrt{n}h(Y_n - \theta)}_{?}. \tag{9.2}$$

where we note that $\sqrt{n}(Y_n - \theta) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$ by assumption. Since it is multiplied by $f'(\theta) \in \mathbb{R}$, the product of these two terms converges to $\mathcal{N}(0, \sigma^2[f'(\theta)]^2)$ by Slutsky's Theorem (Theorem 8.27(b)). We seek to show what happens to the third term of (9.2) as $n \to \infty$ (the result follows if the term converges in probability to 0). Note that for any $n \geq 1$ and for any $t > 0$,

$$\Pr(\sqrt{n}|h(Y_n-\theta)| > t) = \Pr\left(\sqrt{n}|h(Y_n-\theta)| > t \cap |Y_n-\theta| > \frac{t}{\sqrt{n}}\right) + \Pr\left(\sqrt{n}|h(Y_n-\theta)| > t \cap |Y_n-\theta| \leq \frac{t}{\sqrt{n}}\right)$$

$$\iff \Pr(\sqrt{n}|h(Y_n - \theta)| > t) \leq \Pr(|Y_n - \theta| > t/\sqrt{n}) + \Pr(\sqrt{n}|h(Y_n - \theta)| > t \cap |Y_n - \theta| \leq t/\sqrt{n}). \tag{9.3}$$

Since we already have by assumption $\sqrt{n}(Y_n - \theta) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$, it follows that $|Y_n - \theta| \xrightarrow{p} 0$. (For completeness, a detailed argument is included in the below lemma.) It then follows that the second term converges in probability to 0 if $\lim_{n\to\infty} \Pr(|Y_n - \theta| > t/\sqrt{n}) = 0$ because $\lim_{z\to 0} h(z)/z = 0$. Therefore for any $t > 0$,

$$\lim_{n\to\infty} \Pr(\sqrt{n}|h(Y_n - \theta)| > t) = 0 \iff \sqrt{n}|h(Y_n - \theta)| \xrightarrow{p} 0$$

which yields the result by (9.2).

□

Lemma 9.17. Under the same assumptions and notation as in Theorem 9.16,

$$\lim_{n\to\infty} \Pr(|Y_n - \theta| > t/\sqrt{n}) = 0$$

Proof. We will examine the behavior of the right side of (9.3) as $n \to \infty$ by looking at the first term and showing that $Y_n - \theta$ converges in probability to 0. If $t > 0$, then $\Pr(|Y_n - \theta| > t) = \Pr(\sqrt{n}|Y_n - \theta| > t\sqrt{n})$, and if $c > 0$ is a constant, then for sufficiently large n, the last quantity is at most $\Pr(\sqrt{n}|Y_n - \theta| > c)$. So we have

$$\Pr(|Y_n - \theta| > t) = \Pr(\sqrt{n}|Y_n - \theta| > t\sqrt{n}) \leq \Pr(\sqrt{n}|Y_n - \theta| > c)$$

But as $n \to \infty$, c can be any constant (arbitrarily large). So

$$\lim_{n\to\infty} \Pr(\sqrt{n}|Y_n - \theta| > t) \leq \int_c^\infty e^{-y^2/2} \frac{1}{\sqrt{2\pi}} dy.$$

Therefore

$$\lim_{n\to\infty} \Pr(|Y_n - \theta| > t/\sqrt{n}) = 0.$$

□

Theorem 9.18 (Convergence Theorem with Bounded Moment, Theorem 4.16 in 541A notes.). Let $X_1, X_2, \ldots$ be random variables that converge in distribution to a random variable X. Assume $\exists\ \epsilon > 0, c < \infty$ such that $\mathbb{E}(|X_n|^{1+\epsilon}) \leq c, \quad \forall\ n \geq 1$. Then

$$\mathbb{E}(X) = \lim_{n\to\infty} \mathbb{E}(X_n).$$

Proof. In Heilman's Graduate Probability Notes, Theorem 1.59 and Exercise 3.8(iii).

□

If $f'(\theta) = 0$ in the Delta Method, we can instead use a second order Taylor expansion as follows.

Theorem 9.19 (Second Order Delta Method, Theorem 4.17 in Math 541A Notes.). Let $\theta \in \mathbb{R}$. Let $Y_1, Y_2, \ldots$ be random variables such that $\sqrt{n}(Y_n - \theta)$ converges in distribution to a mean zero Gaussian random variable with variance $\sigma^2 > 0$. Let $f : \mathbb{R} \to \mathbb{R}$. Assume that f'' exists and is continuous, $f'(\theta) = 0$ and $f''(\theta) \neq 0$. Then

$$n(f(Y_n) - f(\theta))$$

converges in distribution to a χ_1^2 random variable multiplied by $\sigma^2 \frac{1}{2}|f''(\theta)|$ as $n \to \infty$.

Proof. Using a second order Taylor expansion of f, there exists a random Z_n between θ and Y_n such that

$$f(Y_n) = f(\theta) + f'(\theta)(Y_n - \theta) + \frac{1}{2} f''(Z_n)(Y_n - \theta)^2 = f(\theta) + \frac{1}{2} f''(Z_n)(Y_n - \theta)^2 \tag{9.4}$$

where the second equality follows because $f'(\theta) = 0$. As in the proof of Theorem 9.16, $Z_n \xrightarrow{p} \theta$. Since f'' is continuous, $f''(Z_n)$ converges in probability to $f''(\theta)$ by Proposition 2.36 in the Math 541A notes (Theorem 8.29, continuous functions conserve convergence in probability). Therefore using (9.4),

$$n(f(Y_n) - f(\theta)) = \frac{1}{2} f''(Z_n) \cdot n(Y_n - \theta)^2$$

Note that $\sqrt{n}(Y_n - \theta)$ converges in distribution to a mean zero Gaussian random variable by assumption, so $n(Y_n - \theta)^2$ converges in distribution to a χ_1^2 random variable by Proposition 2.36 in the Math 541A notes (Theorem 8.29). So since $f''(Z_n)$ converges in probability to a constant, by Proposition 2.36 in the Math 541A notes (Slutsky's Theorem, Theorem 8.27), the right side converges in probability to $\frac{1}{2}f''(\theta)\sigma$ multiplied by a χ_1^2 random variable.

□

9.2.2 Simulation of Random Variables

Proposition 9.20. If $X : \Omega \to \mathbb{R}$ is an arbitrary random variable with cumulative distribution function $F : \mathbb{R} \to [0, 1]$, then the function F^{-1} (if it exists) is a random variable on $[0, 1]$ with the uniform probability law on $(0, 1)$ that is equal in distribution to X.

Proof. Starting with the cdf of $F^{-1}(u)$,

$$\Pr(s \in [0,1] : F^{-1}(s) \leq t) = \Pr(s \in [0,1] : F(t) > s) = F(t) = \Pr(\omega \in \Omega : X(\omega) \leq t)$$

where the third equality uses the definition of a uniform probability law on $(0, 1)$.

□

Remark. If F^{-1} does not exist, it can still work if you construct a generalized inverse of F as follows:

Proposition 9.21 (Exercise 4.20 in Math 541A notes). Let $X : \Omega \to \mathbb{R}$ be a random variable on a sample space Ω equipped with a probability law $\mathbb{P}$. For any $t \in \mathbb{R}$ let $F(t) := \mathbb{P}(X \leq t)$. For any $s \in (0, 1)$ define

$$Y(s) := \sup\{t \in \mathbb{R} : F(t) < s\}.$$

So Y is a random variable on $(0, 1)$ with the uniform probability law on $(0, 1)$. Then X and Y are equal in distribution. That is, $\mathbb{P}(Y \leq t) = F(t)$ for all $t \in \mathbb{R}$.

Proof. Note that since F is a cumulative distribution function, F is nondecreasing and F is right-continuous. So we have

$$\sup\{t \in \mathbb{R} : F(t) < s\} = \begin{cases} F^{-1}(s) & \text{if } F \text{ is strictly increasing (i.e. invertible) near } s \\ \inf\{x : F(x) = F(s)\} & \text{if } F \text{ is constant near } s \end{cases}$$

That is, the only time this quantity is different from $F^{-1}(s)$ is when $F^{-1}(\cdot)$ is undefined because F is constant on some interval around s. But if that is the case, $F(\sup\{t \in \mathbb{R} : F(t) < s\}) = F(\inf\{x : F(x) = F(s)\}) = s$ anyway. With that in mind we proceed:

$$\mathbb{P}(Y \leq t) = \mathbb{P}(s \in (0,1) : Y(s) \leq t) = \mathbb{P}(s \in (0,1) : \sup\{t' \in \mathbb{R} : F(t') < s\} \leq t)$$

$$= \mathbb{P}(s \in (0,1) : F\big(\sup\{t' \in \mathbb{R} : F(t') < s\}\big) \le F(t)\big) = \mathbb{P}\big(s \in (0,1) : s \le F(t)\big)$$

$$= \mathbb{P}(s \in (0,1) : F(t) > s) = F(t) = \Pr(\omega \in \Omega : X(\omega) \le t).$$

□

Example 9.1 (Example 4.22 in Math 541A notes). Let X be an exponential random variable with parameter 1.

$$\Pr(X \le t) = \int_0^t e^{-x} dx = [-e^{-x}]_0^t = 1 - e^{-t} = F(t)$$

We seek $F^{-1}(t)$:

$$1 - e^{-y} = t \iff e^{-y} = 1 - t \iff -y = \log(1-t) \iff y = -\log(1-t) \implies F^{-1}(t) = -\log(1-t)$$

So to simulate an exponential random variable with parameter 1, sample $-\log(1-U)$ where $U \sim \mathrm{U}(0,1)$.

Remark. What if the cdf is hard to compute? For example, in a Gaussian distribution:

$$F(t) = \int_{-\infty}^t (2\pi)^{-1/2} \exp(-x^2/2) dx.$$

F^{-1} cannot be described using elementary formulas, so $F^{-1}(u)$ is not the best way to simulate a Gaussian random variable. When using the Central Limit Theorem approach (see 541A notes for details), Edgeworth expansion says: if we replace $U_1, \ldots, U_n$ with i.i.d. $X_1, \ldots, X_n$ and the first m moments of X_1 agree with the first m moments of Gaussian random variables, then the error in the CLT approximation to a Gaussian is $n^{-(m-1)/2}$. (See https://en.wikipedia.org/wiki/Edgeworth_series.) But this is still inefficient, because one Gaussian sample requires n uniform samples.

Proposition 9.22 (Box-Muller Algorithm). Let U_1, U_2 be independent random variable distributed in $(0,1)$. Define

$$R := \sqrt{-2\log(U_1)}$$

t̂his density is something like $e^{-x^2/2}$

$$\Psi := 2\pi U_2$$

$$X := R\cos(\Psi), \quad Y := R\sin(\Psi)$$

Then X, Y are independent standard Gaussian random variables.

Proof. Homework problem.

□

9.3 Data Reduction

Suppose we have some data and an exponential family. We would like to find the parameter θ among the exponential family that fits the data well. Suppose we have a large data set, maybe so large that you can't store all the data in RAM at once. What is the "least memory" or "most efficient" method for finding θ? The answer: try to find a statistic that captures all the relevant information about θ. For example, to find the mean of a Gaussian sample, use the sample mean. You don't have to store all the raw data, you can just store the sample mean. The following is a generalization of this concept:

9.3.1 Sufficient Statistics

Definition 9.8 (Sufficient Statistic; definition 5.1 in Math 541A notes). Suppose $X_1, \ldots, X_n$ is a sample of size n from a distribution f where $f \in \{f_\theta : \theta \in \Theta\}$ is a family of distributions (such as an exponential family). Let $g : \mathbb{R}^n \to \mathbb{R}^k$ so that $Y := g(X_1, \ldots, X_n)$ is a statistic. We say that Y is a **sufficient statistic** for θ if for every $y \in \mathbb{R}^k$ and for every $\theta \in \Theta$, the conditional distribution of $(X_1, \ldots, X_n)$ given $Y = y$ (with respect to probabilities given by f_θ) does not depend on θ. That is, Y provides sufficient information to determine θ from $X_1, \ldots, X_n$.

Remark. Based on a comment Heilman made on class, this definition assumes independence of the random variables? Basically everything in this class does?

Goldstein lecture: Suppose we have a model $\{f_\theta : \theta \in \Theta\}$ which we interpret as a set of densities or mass functions. We have $\Theta \in \mathbb{R}^p$, and we know the model up to p parameters. Example; we have $X_1, X_2, \ldots, X_n \sim$ i.i.d. f_θ where $\theta \in (\mu, \sigma^2), \mu \in \mathbb{R}$, where $f_\theta \sim \mathcal{N}(\mu, \sigma^2)$, then

$$\Theta = \{(\mu, \sigma^2) : \mu \in \mathbb{R}, \sigma^2 > 0\}.$$

Example 9.2 (Example 5.2 in 541A notes). Let $X_1, \ldots, X_n$ be a random sample of size n from a Bernoulli distribution with parameter $0 < \theta < 1$. Then $Y := X_1 = \ldots + X_n$ is sufficient for θ.

Proposition 9.23 (Example 5.2 in 541A notes). Let $X_1, \ldots, X_n$ be a random sample of size n from a Bernoulli distribution with parameter $0 < \theta < 1$. Let $Y := X_1 = \ldots + X_n$. Then

$$\mathbb{P}_\theta(X = x \mid Y = y) = \begin{cases} 0 & y \neq \sum_i x_i \\ \Big/ \binom{n}{y} & \sum_i x_i = y \end{cases}$$

Remark. If a statistic is sufficient for θ, then we can use that sufficient statistic to re-create the data (or re-create an equivalent data set with the same statistical properties as far we are concerned with estimating the parameter of interest).

Proof. Let $x_1, \ldots, x_n \in [0, 1]$. Let $0 \leq y \leq n$ be an integer. Then Y is binomial with parameters n and θ. We may assume $y = x_1 + \ldots + x_n$, otherwise there is nothing to show. Using the definition of conditional probability,

$$\Pr\big((X_1,\ldots,X_n)=(x_1,\ldots,x_n)\mid Y=y\big)=\frac{1}{\Pr(Y=y)}\cdot\Pr\big((X_1,\ldots,X_n)=(x_1,\ldots,x_n)\cap Y=y\big)$$

$$=\frac{1}{\Pr(Y=y)}\cdot\Pr\big((X_1,\ldots,X_n)=(x_1,\ldots,x_n)\big)$$

Using independence and the definition of a binomial distribution, we have

$$=\frac{1}{\binom{n}{y}\theta^y(1-\theta)^{n-y}}\cdot\prod_{i=1}^n\Pr\big(X_i=x_i\big)=\frac{1}{\binom{n}{y}\theta^y(1-\theta)^{n-y}}\cdot\prod_{i=1}^n\theta^{x_i}(1-\theta)^{1-x_i}$$

$$=\frac{1}{\binom{n}{y}\theta^y(1-\theta)^{n-y}}\cdot\theta^y(1-\theta)^{n-y}=\frac{1}{\binom{n}{y}}.$$

Since this expression does not depend on θ, Y is sufficient for θ.

□

Example 9.3 (Example 5.3 in 541A notes). Let $X_1,\ldots,X_n$ be a sample of size n from a Gaussian distribution with known variance $\sigma^2>0$ and unknown mean $\mu\in\mathbb{R}$. Then $Y:=(X_1,\ldots,X_n)/n$ is a sufficient statistic for μ.

Proof. Note that Y is a Gaussian random variable with mean μ and variance σ^2/n. Let $x_1,\ldots,x_n\in\mathbb{R}$ and let $y=(x_1+\ldots+x_n)/n$. Then

$$f_{X_1,\ldots,X_n|Y}(x_1,\ldots,x_n\mid y)=\frac{1}{f_Y(y)}\cdot f_{X_1,\ldots,X_n,Y}(x_1,\ldots,x_n,y)=\frac{1}{f_Y(y)}\cdot f_{X_1,\ldots,X_n}(x_1,\ldots,x_n,y)$$

Since

$$f_X(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(\frac{-x^2-\mu^2+2\mu x}{2\sigma^2}\right)$$

we have

$$=\frac{1}{f_Y(y)}\cdot\prod_{i=1}^n f_{X_i}(x_i)=\frac{1}{f_Y(y)}\cdot\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n\cdot\exp\left(-\frac{1}{2\sigma^2}(x_1^2+\ldots+x_n^2)-\frac{n\mu^2}{2\sigma^2}+\frac{\mu}{\sigma^2}\sum_{i=1}^n x_i\right)$$

$$=\frac{\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n\cdot\exp\left(-\frac{1}{2\sigma^2}(x_1^2+\ldots+x_n^2)-\frac{n\mu^2}{2\sigma^2}+\frac{\mu}{\sigma^2}\sum_{i=1}^n x_i\right)}{n^{1/2}(\sigma^2 2\pi)^{-1/2}\exp\left(-\frac{n}{2\sigma^2}y^2-\frac{n}{2\sigma^2}\mu^2+\frac{n\mu}{\sigma^2}y\right)}$$

$$=\frac{\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n\cdot\exp\left(-\frac{1}{2\sigma^2}(x_1^2+\ldots+x_n^2)\right)}{n^{1/2}(\sigma^2 2\pi)^{-1/2}\exp\left(-\frac{n}{2\sigma^2}y^2\right)}$$

Because μ does not appear in this expression, Y is sufficient for μ.

$\square$

Theorem 9.24 (Theorem 6.2.2 in Casella and Berger; not in 541A lecture notes). If $p(x \mid \theta)$ is the joint pdf or pmf of a random sample $\boldsymbol{X} = X_1, \ldots, X_n$ and $q(t \mid \theta)$ is the pdf or pmf of the statistic $T(\boldsymbol{X})$, then $T(\boldsymbol{X})$ is a sufficient statistic for θ if, for every $\boldsymbol{x} = x_1, \ldots, x_n$ in the sample space, the ratio $p(\boldsymbol{x} \mid \theta)/q(T(\boldsymbol{x} \mid \theta)$ is constant as a function of θ.

Theorem 9.25 (Factorization Theorem, Theorem 5.4 in 541A notes). Suppose $X = (X_1, \ldots, X_n)$ is a random sample of size n from a distribution f where $f \in \{f_\theta : \theta \in \Theta\}$ is a family of probability density functions or probability mass functions. Let $t : \mathbb{R}^n \to \mathbb{R}^k$, so $Y := t(X_1, \ldots, X_n)$ is a statistic. Then Y is sufficient for θ if and only if there exists a nonnegative $\{g_\theta : \theta \in \Theta\}$, $h : \mathbb{R}^n \to \mathbb{R}$, $g_\theta : \mathbb{R}^k \to \mathbb{R}$ such that

$$f_\theta(x) = g_\theta(t(x))h(x), \quad \forall\, x \in \mathbb{R}^n, \ \ \forall \theta \in \Theta. \tag{9.5}$$

Proof. We will prove only the discrete case to avoid measure theory. For a general case, see Keener Section 6.4.

Suppose Y is sufficient. Let $x \in \mathbb{R}^n$. Note that by definition and using $Y = t(X)$,

$$f_\theta(x) = \mathbb{P}_\theta(X = x) = \mathbb{P}_\theta(X = x \cap t(X) = t(x)) = \mathbb{P}_\theta(Y = t(x))\mathbb{P}_\theta(X = x \mid Y = t(x))$$

By sufficiency, $\mathbb{P}_\theta(X = x \mid Y = t(x))$ does not depend on θ. Therefore we can satisfy (9.5) with $g_\theta(t(x)) = \mathbb{P}_\theta(Y = t(x))$, $h(x) = \mathbb{P}_\theta(X = x \mid Y = t(x))$, so the factorization holds.

Now suppose there exists a nonnegative $\{g_\theta : \theta \in \Theta\}$, $h : \mathbb{R}^n \to \mathbb{R}$, $g_\theta : \mathbb{R}^k \to \mathbb{R}$ such that

$$f_\theta(x) = g_\theta(t(x))h(x), \quad \forall\, x \in \mathbb{R}^n, \ \ \forall \theta \in \Theta.$$

Define $r_\theta(z) := \mathbb{P}_\theta(t(X) = z) \ \ \forall\, z \in \mathbb{R}^k$ (the probability mass function for $t(X)$). Also define $t^{-1}t(x) := \{y \in \mathbb{R}^n; t(y) = t(x)\} \ \ \forall x \in \mathbb{R}^n$. To show sufficiency, we need to show that $\mathbb{P}_\theta(X = x \mid Y = t(x))$ does not depend on θ. Note that

$$\mathbb{P}_\theta(X = x \mid Y = t(x)) = \frac{f_\theta(x)}{f_Y(t(x))} = \frac{f_\theta(x)}{r_\theta(t(x))}$$

Using our assumption and the Total Probability Theorem, we have

$$= \frac{g_\theta(t(x)h(x))}{\mathbb{P}_\theta(t(X) = t(x))} = \frac{g_\theta(t(x))h(x)}{\sum_{z \in t^{-1}t(x)} \mathbb{P}_\theta(X = z)} = \frac{g_\theta(t(x))h(x)}{\sum_{z \in t^{-1}t(x)} f_\theta(z)} = \frac{g_\theta(t(x))h(x)}{\sum_{z \in t^{-1}t(x)} g_\theta(t(z))h(z)}$$

By definition of $t^{-1}t(z)$, we we can write this as

$$= \frac{g_\theta(t(x))h(x)}{\sum_{z\in t^{-1}t(x)} g_\theta(t(x))h(z)} = \frac{g_\theta(t(x))h(x)}{g_\theta(t(x))\sum_{z\in t^{-1}t(x)} h(z)} = \frac{h(x)}{\sum_{z\in t^{-1}t(x)} h(z)}$$

where the second-to-last step follows since $t(x)$ is constant for all $z \in t^{-1}t(x)$. Since this expression does not contain θ, Y is sufficient for θ.

□

Remark. Intuition: data only cares about θ through $t(x)$.

To use the Factorization Theorem (Theorem 9.5) to find a sufficient statistic, we factor the joint pdf of the sample into two parts, with one part not depending on θ. The other part, the one that depends on θ, usually depends on the sample only through the function $t(x)$, and this function is a sufficient statistic for θ.

Exercise 16. Suppose $X_1, X_2, \ldots, X_n \sim$ i.i.d. $\mathcal{N}(0, 1)$. So density is

$$\frac{1}{\sqrt{2\pi}} e^{-1/2(x-\theta)^2}$$

Show that

$$e^{-1/2(x^2-2x\theta+\theta^2)} =$$

$$f_\theta(x) = \left(\frac{1}{2\pi}\right)^{n/2} e^{2/12\sum_i X_i^2} e^{\theta \sum X_i - n\theta^2/2}$$

so if $t(x) = \sum_{i=1}^n X_i$, $h(x) = \left(\frac{1}{2\pi}\right)^{n/2} e^{2/12\sum_i X_i^2}$, $g_\theta(t(x)) = e^{\theta\sum X_i - n\theta^2/2}$, then by the Factorization Theorem (Theorem 9.5) this $(\overline{x})$ is a sufficient statistic.

Remark. In this case, if we deleted the original data we could recreate the original data by sampling from a $\mathcal{N}(0, 1)$ distribution, then add the difference between the mean we get and the original sample mean to get an equivalent data set to the original one.

Remark. Suppose we define $t(x) := x,\ \forall\ x \in \mathbb{R}^n$. Then $Y = t(X_1, \ldots, X_n) = (X_1, \ldots, X_n)$ is (trivially) sufficient for θ. In general there will be infinitely many sufficient statistics for θ. For instance, in Example 9.23, $(X_1 + \ldots + X_n)^2$ is also sufficient. So is $(X_1 + \ldots + X_n)^3$, etc. More generally, any invertible function of any sufficient statistic is itself sufficient.

We can see that $(X_1k \ldots, X_n)$ is sufficient for θ if $(t(x_1, \ldots, x_n) = (x_1, \ldots, x_n)$, $g_\theta = f_\theta, h = 1$. But this is not really helpful. We see we are interested in sufficient statistics that are smaller—reduce the data (in some sense) as much as possible.

9.3.2 Minimal Sufficient Statistics

Proposition 9.26. Suppose $X = (X_1, \ldots, X_n)$ is a random sample of size n from a distribution f where $f \in \{f_\theta : \theta \in \Theta\}$ is a family of probability density functions or probability mass functions. Let $t : \mathbb{R}^n \to \mathbb{R}^k$. Let $Y := t(X_1, \ldots, X_n)$. Assume Y is sufficient of θ. Let $a : \mathbb{R}^n \to \mathbb{R}^m$, let $Z := u(X_1, \ldots, X_n)$. suppose there exists $r : \mathbb{R}^m \to \mathbb{R}^k$ such that $r(u(x)) = t(x)$ for all $x \in \mathbb{R}^n$. That is, suppose $Y = r(Z)$. Then Z is sufficient for θ.

Proof.

$$f_\theta(x) = g_\theta(t(x))h(x) = g_\theta(r(u(x))h(x)$$

there exists $g_\theta : \mathbb{R}^k \to [0, \infty)$. Y is sufficient.

Define

$$\tilde{g}_\theta(y) := g_\theta(r(y)) \ \ \forall\, y \in \mathbb{R}^m$$

So

$f_\theta(x) = \tilde{g}_\theta(u(x))h(x) \ \ \forall x \in \mathbb{R}^n$. So Z is sufficient for θ by the Factorization Theorem (Theorem 9.5).

□

Definition 9.9 (Minimal sufficient statistic). Suppose $X = (X_1, \ldots, X_n)$ is a random sample of size n from a distribution f where $f \in \{f_\theta : \theta \in \Theta\}$ is a family of probability density functions or probability mass functions. Let $t : \mathbb{R}^n \to \mathbb{R}^k$. Let $Y := t(X_1, \ldots, X_n)$. Assume Y is sufficient of θ. Then Y is a **minimal sufficient statistic** for θ if for every statistic $Z : \Omega \to \mathbb{R}^m$ that is sufficient for θ there exists a function $\mathbb{R}^m \to \mathbb{R}^k$ such that $Y = r(Z)$.

Remark. Minimal sufficient statistics are not in general unique (because if you take any one-to-one function you get another one), but they are unique up to invertible transformations. (This is true because if Y and Z are both minimal sufficient, $Y = r(Z)$ and $Z = s(Y)$, so $Y = r(s(Y)), Z = s(r(Z))$). They exist under mild assumptions (for a family of densities or probability mass functions).

Proposition 9.27 (Proposition Larry Goldstein gave in class; Proposition 5.12 in notes). Suppose $X_1, \ldots, X_n$ is a random sample of size n from a distribution f where $f \in \{f_\theta : \theta \in \Theta\}$, is a family of probability density functions or probability mass functions ($\Theta \in \mathbb{R}^n$). (In the case of probability mass functions, we also assume that the set $\cup_{\theta \in \Theta}\{x \in \mathbb{R}^n : f_\theta(x) > 0\}$ is countable.) Then there exists a statistic Y that is minimal sufficient for θ.

Proof where θ is countable. By relabeling, let $\Theta = \{1, 2, \ldots\}$. We say for x, y sequences, we define the equivalence relation $x \sim y$ if $\exists\ \alpha \in \mathbb{R}$ such that $x = \alpha y$. Finite

$$t : \mathbb{R}^n \to \mathbb{R}^m / \sim, \qquad \Theta = \{1, \ldots, m\}$$

$$t(x) = (f_1(x), f_2(x), \ldots, f_m(x))$$

these likelihood are multiples of each other where α is a constant. The likelihood ratio is a constant not depending on θ. If they have the same $t(x)$ then we have that.

$\square$

Theorem 9.28 (Theorem 5.8 in 541A notes). Let $\{f_\theta : \theta \in \Theta\}$ be a family of probability density functions or probability mass functions. Let $X_1, \ldots, X_n$ be a random sample from a member of the family. Let $t : \mathbb{R}^n \to \mathbb{R}^m$ and define $Y := t(X_1, \ldots, X_n)$. Assume that Y is sufficient for θ. Y is minimal sufficient if and only if the following condition holds for every $x, y \in \mathbb{R}^n$:

$$\boxed{\begin{array}{c}\text{There exists } c(x,y) \in \mathbb{R} \text{ that does not depend on } \theta \text{ such that } f_\theta(x) = c(x,y) f_\theta(y) \quad \forall\, \theta \in \Theta \\ \\ \text{if and only if} \\ \\ t(x) = t(y).\end{array}}$$

Proof. We are only considering probability mass functions to make things easier. We first prove sufficiency. We will show that the condition holding implies that Y is minimal sufficient.

Recall the likelihood ratio:

$$\frac{f_\theta(x)}{f_\theta(y)}$$

Note that the condition is equivalent to the likelihood ratio not depending on θ if and only if $t(x) = t(y)$. Consider the range $R = \{t(x) : x \in \mathbb{R}^n\}$ jnd then for $t \in R$ let $S_t = \{y : S(y) = t\}$. If t is in R, then there must be some z so that $t(z)$ is that z. This ensure that S_t is nonempty (there is at least one z so that $t(z) = t$. Let $t(x) \in R$, then $S_{t(x)}$ is nonempty (in particular it contains x). Pick any y you like in $t(x)$: $y \in S$. S depends on $t(x)$ so we can index it by $t(x)$: $y_{t(x)} \in S_{t(x)}$. let $y_t \in S_t$. Note that

$$t(y_{t(x)}) = t(x)$$

But now by the assumption, we have

$$\text{There exists } c(x, y_{t(x)}) \in \mathbb{R} \text{ that does not depend on } \theta \text{ such that } f_\theta(x) = c(x, y_{t(x)}) f_\theta(y_{t(x)}) \quad \forall\, \theta \in \Theta$$

Then note that if $h(x) = c(x, y_{t(x)})$, $g_\theta(t) = f_\theta(y_t) \iff g_\theta(t(x)) = f_\theta(y_{t(x)})$, we meet the conditions for the Factorization Theorem (Theorem 9.5). So using the Factorization Theorem, Y is sufficient.

$$\vdots$$

Part we did in class on Friday 02/15: evidently (according to Goldstein) this shows that the statistic is minimal but not necessarily sufficient. Let $Z = u(X_1, \ldots, X_n)$ be any other sufficient statistic. We need to eventually show that Y is a function of Z. By the Factorization Theorem (Theorem 9.5), there exists $h : \mathbb{R}^m \to \mathbb{R}, g_\theta; \mathbb{R}^n \to \mathbb{R}$ such that for all $\theta \in \Theta$,

$$f_\theta(x) = g'_\theta(u(x))h'(x), \quad \forall\, x \in \mathbb{R}^n.$$

Let $y \in \mathbb{R}^n$. If $h'(y) = 0$, then $f_\theta(y) = 0$ for all $\theta \in \Theta$. So, $\mathbb{P}_\theta(y \in \mathbb{R}^n : h'(y) = 0) = 0$ for all $\theta \in \Theta$. So we can ignore this possibility since it's a probability 0 event and assume $h'(y) > 0, \forall y \in \mathbb{R}^n$.

Now let $x, y \in \mathbb{R}^n$ such that $u(x) = u(y)$. **By an exercise we're going to do later**, if $t(x) = t(y)$ then t is a function of u, so we will be done if we can show that $t(x) = t(y)$. Note that since $u(x) = u(y)$, for any $\theta \in \Theta$

$$f_\theta(x) = g'_\theta(u(x))h'(x) = \frac{g'_\theta(u(y))h'(x) = g'_\theta(u(y))h'(y)}{f_\theta(y)}\frac{h'(x)}{h'(y)} = f_\theta(y)\frac{h'(x)}{h'(y)}, \quad forall \theta \in \Theta$$

So define $c(x, y) = h'(x)/h'(y)$, we have

$$f_\theta(x) = f_\theta(y)c(x, y), \quad \forall \theta \in \Theta$$

Therefore $t(x) = t(y)$, so we're done showing that if the condition holds then Y is minimal sufficient.

Then next thing to show is that if Y is minimal sufficient then the condition holds.

$$\vdots$$

For any $z \in \{t(x) : x \in \mathbb{R}^n\}$, let x_z be any element of $t^{-1}(z)$

$\square$

Proposition 9.29 (Exercise 5.10 in Math 541A notes). Let $\{f_\theta : \theta \in \Theta\}$ be a k-parameter exponential family $\{f_\theta : \theta \in \Theta, a(w(\theta)) < \infty\}$ of probability density functions or probability mass functions, where

$$f_\theta(x) := h(x) \exp\Big(\sum_{i=1}^k w_i(\theta)t_i(x) - a(w(\theta))\Big), \qquad \forall\, x \in \mathbb{R}.$$

For any $\theta \in \Theta$, let $w(\theta); = (w_1(\theta), \ldots, w_k(\theta))$. Assume that the following subset of $\mathbb{R}^k$ is k-dimensional:

$$\{(w_1(\theta), \ldots, w_k(\theta)) \in \mathbb{R}^k : \theta \in \Theta\}.$$

That is, if $x \in \mathbb{R}^k$ satisfies $\langle x, y \rangle = 0$ for all y in this set, then $x = 0$.

Let $X = (X_1, \ldots, X_n)$ be a random sample of size n from f_θ. Define $t : \mathbb{R}^n \to \mathbb{R}^n$ by

$$t(X) := \sum_{j=1}^{n} (t_1(X_j), \ldots, t_n(X_j)) .$$

Then $t(X)$ is minimal sufficient for θ.

Proof. First note that $t(X)$ is sufficient by the Factorization Theorem (Theorem 9.5) because we have for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$

$$f_\theta(x) = \prod_{j=1}^{n} \left[h(x_j) \exp\left(\sum_{i=1}^{k} w_i(\theta) t_i(x_j) - a(w(\theta)) \right) \right] = \exp\left(\sum_{j=1}^{n} \sum_{i=1}^{k} w_i(\theta) t_i(x_j) - n \cdot a(w(\theta)) \right) \prod_{j=1}^{n} h(x_j)$$

$$= g_\theta(t(x)) h(x), \qquad \forall\, x \in \mathbb{R}^n$$

where

$$h(x) = \prod_{j=1}^{n} h(x_j), \qquad g_\theta(t(x)) = \exp\left(\sum_{j=1}^{n} \sum_{i=1}^{k} w_i(\theta) t_i(x_j) \right).$$

Now we will show minimal sufficiency using Theorem 5.8 from the lecture notes (Theorem 9.28). We seek to show that for every $x, y \in \mathbb{R}^n$, the likelihood ratio $\frac{f_\theta(x)}{f_\theta(y)}$ is constant (the constant may depend on x and y) if and only if $t(x) = t(y)$. Let $x, y \in \mathbb{R}^n$. Suppose there is some constant $c(x, y) > 0$ that may depend on x and y but not θ such that

$$\exp\left(\sum_{j=1}^{n} \sum_{i=1}^{k} w_i(\theta) t_i(x_j) - n \cdot a(w(\theta)) \right) \prod_{j=1}^{n} h(x_j)$$

$$= c(x, y) \exp\left(\sum_{j=1}^{n} \sum_{i=1}^{k} w_i(\theta) t_i(y_j) - n \cdot a(w(\theta)) \right) \prod_{j=1}^{n} h(y_j)$$

$$\iff \exp\left(\sum_{j=1}^{n} \sum_{i=1}^{k} w_i(\theta) t_i(x_j) \right) = c_1(x, y) \exp\left(\sum_{j=1}^{n} \sum_{i=1}^{k} w_i(\theta) t_i(y_j) \right)$$

(where cancellation of the h functions is permissible since they depend only on x and y, so we can let $c_1(x, y) = c(x, y) \cdot \prod_{j=1}^{n} h(y_j) / \prod_{j=1}^{n} h(x_j) > 0$)

$$\iff \sum_{j=1}^{n} \sum_{i=1}^{k} w_i(\theta) t_i(x_j) = c_2(x, y) + \sum_{j=1}^{n} \sum_{i=1}^{k} w_i(\theta) t_i(y_j)$$

where $c_2(x, y) = \log\big(c_1(x, y)\big)$. Then if θ_0, θ_1 are any two points in Θ,

$$\sum_{j=1}^{n}\sum_{i=1}^{k} w_i(\theta_0)t_i(x_j) - \sum_{j=1}^{n}\sum_{i=1}^{k} w_i(\theta_1)t_i(x_j) = \sum_{j=1}^{n}\sum_{i=1}^{k} w_i(\theta_0)t_i(y_j) - \sum_{j=1}^{n}\sum_{i=1}^{k} w_i(\theta_1)t_i(y_j)$$

(where the $c_2(x, y)$ terms cancel since (x, y) is held fixed.)

$$\iff \sum_{j=1}^{n}\sum_{i=1}^{k}[w_i(\theta_0) - w_i(\theta_1)]t_i(x_j) = \sum_{j=1}^{n}\sum_{i=1}^{k}[w_i(\theta_0) - w_i(\theta_1)]t_i(y_j)$$

$$\iff \sum_{j=1}^{n}\sum_{i=1}^{k}[w_i(\theta_0) - w_i(\theta_1)][t_i(x_j) - t_i(y_j)] = 0.$$

This equation holds for all $\theta \in \Theta$ if and only if $t_i(x_j) - t_i(y_j) = 0, i = 1, \ldots, k, j = 1, \ldots, n$; that is, it holds if and only if $t(x) = t(y)$. Therefore we have shown that $t(\cdot)$ is minimal sufficient by Theorem 9.28.

□

Remark. Note that the assumption of the exercise is always satisfied for an exponential family in canonical form. From this proposition we can conclude that if we sample from a Gaussian with unknown mean μ and variance $\sigma^2 > 0$, then $\overline{X}$ is minimal sufficient for θ and $(\overline{X}, S)$ is minimal sufficient for (μ, σ^2).

Proposition 9.30 (Exercise 5.13 in Math 541A notes). Let $\mathbb{P}_1, \mathbb{P}_2$ be two probability laws on the sample space $\Omega = \mathbb{R}$. Suppose these laws have densities $f_1, f_2 : \mathbb{R} \to [0, \infty)$ so that

$$\mathbb{P}_i(A) = \int_A f_i(x)dx, \qquad \forall\, i = 1, 2, \qquad \forall\, A \subseteq \mathbb{R}.$$

Then

(a)

$$\sup_{A \subseteq \mathbb{R}} |\mathbb{P}_1(A) - \mathbb{P}_2(A)| = \frac{1}{2}\int_{\mathbb{R}} |f_1(x) - f_2(x)|dx.$$

(b) If $\mathbb{P}_1, \mathbb{P}_2$ are probability laws on $\Omega = \mathbb{Z}$

$$\sup_{A \subseteq \mathbb{Z}} |\mathbb{P}_1(A) - \mathbb{P}_2(A)| = \frac{1}{2}\sum_{z \in \mathbb{Z}} |\mathbb{P}_1(z) - \mathbb{P}_2(z)|.$$

Proof. (a) Note that $\sup_{A \subseteq \mathbb{R}} |\mathbb{P}_1(A) - \mathbb{P}_2(A)|$ returns the difference in areas under f_1 and f_2 in the region $A \subset \mathbb{R}$ where that difference is positive. Suppose without loss of generality that

$$\sup_{A \subseteq \mathbb{R}} |\mathbb{P}_1(A) - \mathbb{P}_2(A)| = \sup_{A \subseteq \mathbb{R}} \{\mathbb{P}_1(A) - \mathbb{P}_2(A)\}. \tag{9.6}$$

(There is no loss of generality because in the case that $\sup_{A \subseteq \mathbb{R}} |\mathbb{P}_1(A) - \mathbb{P}_2(A)| = \sup_{A \subseteq \mathbb{R}} \{\mathbb{P}_2(A) - \mathbb{P}_1(A)\}$, we can simply switch the names of $\mathbb{P}_1$ and $\mathbb{P}_2$ to get the desired result.) Then the region A which maximizes the quantity on the right side of (9.6) is the suggested region, $A := \{x \in \mathbb{R} : f_1(x) > f_2(x)\}$. That is,

$$\sup_{A \subseteq \mathbb{R}} |\mathbb{P}_1(A) - \mathbb{P}_2(A)| = \int_A (f_1(x) - f_2(x))dx.$$

Note that

$$\int_{\mathbb{R}} |f_1(x) - f_2(x)|dx = \int_A |f_1(x) - f_2(x)|dx + \int_{\mathbb{R}\setminus A} |f_1(x) - f_2(x)|dx$$

$$\iff \int_{\mathbb{R}} |f_1(x) - f_2(x)|dx = \int_A (f_1(x) - f_2(x))dx + \int_{\mathbb{R}\setminus A} (f_2(x) - f_1(x))dx. \tag{9.7}$$

Since we already have $\int_A (f_1(x)-f_2(x))dx = \sup_{A\subseteq\mathbb{R}} |\mathbb{P}_1(A)-\mathbb{P}_2(A)|$, if it is also true that $\int_{\mathbb{R}\setminus A}(f_2(x)-f_1(x))dx = \sup_{A\subseteq\mathbb{R}} |\mathbb{P}_1(A) - \mathbb{P}_2(A)|$ we are done by (9.7), so this is what we will show next. Let $\int_A f_2(x)dx = a_2$ and let $\int_A f_1(x)dx = a_1$, so that

$$\sup_{A \subseteq \mathbb{R}} |\mathbb{P}_1(A) - \mathbb{P}_2(A)| = \int_A (f_1(x) - f_2(x))dx = a_1 - a_2.$$

Note that

$$1 = \int_{\mathbb{R}} f_2(x)dx = \int_A f_2(x)dx + \int_{\mathbb{R}\setminus A} f_2(x)dx = a_2 + \int_{\mathbb{R}\setminus A} f_2(x)dx \iff \int_{\mathbb{R}\setminus A} f_2(x)dx = 1 - a_2,$$

and similarly $\int_{\mathbb{R}\setminus A} f_1(x)dx = 1 - a_1$. Therefore

$$\int_{\mathbb{R}\setminus A} (f_2(x)-f_1(x))dx = \int_{\mathbb{R}\setminus A} f_2(x)dx - \int_{\mathbb{R}\setminus A} f_1(x)dx = 1-a_2-(1-a_1) = a_1-a_2 = \sup_{A\subseteq\mathbb{R}} |\mathbb{P}_1(A)-\mathbb{P}_2(A)|.$$

So by (9.7), we have

$$\int_{\mathbb{R}} |f_1(x) - f_2(x)|dx = 2 \sup_{A\subseteq\mathbb{R}} |\mathbb{P}_1(A) - \mathbb{P}_2(A)| \iff \sup_{A\subset\mathbb{R}} |\P_1(A) - \P_2(A)| = \frac{1}{2}\int_{\mathbb{R}} |f_1(x) - f_2(x)|dx.$$

(b) Analogous to the proof of (a). Note that $\sup_{A\subseteq\mathbb{Z}} |\mathbb{P}_1(A) - \mathbb{P}_2(A)|$ returns the difference in probabilities for those numbers in the set $A \subset \mathbb{Z}$ where that difference is positive. Suppose without loss of generality that

$$\sup_{A\subseteq\mathbb{Z}} |\mathbb{P}_1(A) - \mathbb{P}_2(A)| = \sup_{A\subseteq\mathbb{Z}} \{\mathbb{P}_1(A) - \mathbb{P}_2(A)\}. \tag{9.8}$$

(There is no loss of generality because in the case that $\sup_{A\subseteq\mathbb{Z}} |\mathbb{P}_1(A) - \mathbb{P}_2(A)| = \sup_{A\subseteq\mathbb{Z}}\{\mathbb{P}_2(A) - \mathbb{P}_1(A)\}$, we can simply switch the names of $\mathbb{P}_1$ and $\mathbb{P}_2$ to get the desired result.) Then the set A which maximizes the quantity on the right side of (9.8) can be defined as (similarly to part (a)) $A := \{z \in \mathbb{Z} : \mathbb{P}_1(z) > \mathbb{P}_2(z)\}$. That is,

$$\sup_{A\subseteq\mathbb{Z}} |\mathbb{P}_1(A) - \mathbb{P}_2(A)| = \sum_{z\in A} \mathbb{P}_1(z) - \mathbb{P}_2(z).$$

Note that

$$\sum_{z\in\mathbb{Z}} |\mathbb{P}_1(z) - \mathbb{P}_2(z)| = \sum_{z\in A} |\mathbb{P}_1(z) - \mathbb{P}_2(z)| + \sum_{z\in\{\mathbb{Z}\setminus A\}} |\mathbb{P}_1(z) - \mathbb{P}_2(z)|$$

$$\iff \sum_{z\in\mathbb{Z}} |\mathbb{P}_1(z) - \mathbb{P}_2(z)| = \sum_{z\in A} \mathbb{P}_1(z) - \mathbb{P}_2(z) + \sum_{z\in\{\mathbb{Z}\setminus A\}} (\mathbb{P}_2(z) - \mathbb{P}_1(z)). \tag{9.9}$$

Since we already have $\sum_{z\in A}\mathbb{P}_1(z)-\mathbb{P}_2(z) = \sup_{A\subseteq\mathbb{Z}} |\mathbb{P}_1(A)-\mathbb{P}_2(A)|$, if it is also true that $\sum_{z\in\{\mathbb{Z}\setminus A\}}(\mathbb{P}_2(z)-\mathbb{P}_1(z)) = \sup_{A\subseteq\mathbb{Z}} |\mathbb{P}_1(A)-\mathbb{P}_2(A)|$ we are done by (9.9), so this is what we will show next. Let $\sum_{z\in A}\mathbb{P}_2(z) = a_2$ and let $\sum_{z\in A}\mathbb{P}_1(z) = a_1$, so that

$$\sup_{A\subseteq\mathbb{Z}} |\mathbb{P}_1(A) - \mathbb{P}_2(A)| = \sum_{z\in A} \mathbb{P}_1(z) - \mathbb{P}_2(z) = a_1 - a_2.$$

Note that

$$1 = \sum_{z\in Z} \mathbb{P}_2(z) = \sum_{z\in A} \mathbb{P}_2(z) + \sum_{z\in\{\mathbb{Z}\setminus A\}} \mathbb{P}_2(z) = a_2 + \sum_{z\in\{\mathbb{Z}\setminus A\}} \mathbb{P}_2(z) \iff \sum_{z\in\{\mathbb{Z}\setminus A\}} \mathbb{P}_2(z) = 1 - a_2,$$

and similarly $\sum_{z\in\{\mathbb{Z}\setminus A\}}\mathbb{P}_1(z) = 1 - a_1$. Therefore

$$\sum_{z\in\{\mathbb{Z}\setminus A\}} (\mathbb{P}_2(z) - \mathbb{P}_1(z)) = \sum_{z\in\{\mathbb{Z}\setminus A\}} \mathbb{P}_2(z) - \sum_{z\in\{\mathbb{Z}\setminus A\}} \mathbb{P}_1(z) = 1 - a_2 - (1 - a_1) = a_1 - a_2 = \sup_{A\subseteq\mathbb{Z}} |\mathbb{P}_1(A) - \mathbb{P}_2(A)|.$$

So by (9.9), we have

$$\sum_{z\in\mathbb{Z}} |\mathbb{P}_1(z) - \mathbb{P}_2(z)| = 2\sup_{A\subseteq\mathbb{Z}} |\mathbb{P}_1(A) - \mathbb{P}_2(A)| \iff \sup_{A\subset\mathbb{Z}} |\P_1(A) - \P_2(A)| = \frac{1}{2}\sum_{z\in\mathbb{Z}} |\mathbb{P}_1(z) - \mathbb{P}_2(z)|.$$

□

9.3.3 Ancillary Statistics

Definition 9.10 (Ancillary Statistic). Suppose $X_1, \ldots, X_n$ is a random sample of size n from a distribution f where $f \in \{f_\theta : \theta \in \Theta\}$ is a family of distributions. A statistic $Y = t(X_1, \ldots, X_n), t : \mathbb{R}^n \to \mathbb{R}^m$ is **ancillary** for θ if the distribution of Y does not depend on θ.

Example 9.4 (Example 5.15 from 541A notes). Let $X_1, \ldots, X_n$ be random sample of size n from the location family for the Cauchy distribution:

$$f_\theta(x) := \prod_{i=1}^n \frac{1}{\pi}\frac{1}{1 + (x_i - \theta)^2}, \qquad \forall\ x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad \forall\theta \in \mathbb{R}.$$

Then the order statistics $X_{(1)} \leq \ldots \leq X_{(n)}$ are minimal sufficient for θ.

Proof. Sufficiency follows by the Factorization Theorem (Theorem 9.5, Theorem 5.4 in Math 541A notes) (**usually the easiest way to prove sufficiency**) since if $t(x) := (x_{(1)}, \ldots, x_{(n)})$, then $f_\theta(t(x)) = f_\theta(x)$ (because $t(x)$ is just a permutation so the product won't change).

To get minimal sufficiency, apply Theorem 9.28 (Theorem 5.8 in 541A notes). Recall we have minimal sufficiency if the following condition holds for every $x, y \in \mathbb{R}^n$:

There exists $c(x, y) \in \mathbb{R}$ that does not depend on θ such that $f_\theta(x) = c(x, y) f_\theta(y) \quad \forall\, \theta \in \Theta$

if and only if

$$t(x) = t(y).$$

Let's try to show it.

$$\frac{f_\theta(x)}{f_\theta(y)} = \prod_{i=1}^{n} \frac{1}{\pi} \frac{1}{1 + (x_i - \theta)^2} \Bigg/ \prod_{i=1}^{n} \frac{1}{\pi} \frac{1}{1 + (y_i - \theta)^2} = \prod_{i=1}^{n} [1 + (y_i - \theta)^2] \Bigg/ \prod_{i=1}^{n} [1 + (x_i - \theta)^2] \tag{9.10}$$

Keep x, y fixed, $\theta \in \mathbb{R}$ variable. Then the likelihood ratio (9.10) does not depend on θ if and only if the roots in θ on top are equal to the roots on bottom. Roots on top: $\theta = y_i \pm \sqrt{-i}, 1 \leq i \leq n$. Roots on bottom: $\theta = x_i \pm \sqrt{-i}, 1 \leq i \leq n$. So we can see this is true if and only if the vector $(x_1, \ldots, x_n)$ is a permutation of $(y_1, \ldots, y_n)$, which is exactly the case if $t(x) = t(y)$.

□

However, this statistic has ancillary information. Specifically, $X_{(n)} - X_{(1)}$ is ancillary (its distribution does not depend on θ).

Proof. Let $Z_1, \ldots, Z_n$ be independent centered Cauchy random variables; that is, they all have density $\frac{1}{\pi} \frac{1}{1+a^2}, a \in \mathbb{R}$. Then $X_i = Z_i + \theta$, $\forall 1 \leq i \leq n, \forall \theta \in \mathbb{R}$. Also, $X_{(i)} = Z_{(i)} + \theta$. So $X_{(n)} - X_{(1)} = Z_{(n)} - Z_{(1)}$ does not depend on $\theta \in \mathbb{R}$. So, $X_{(n)} - X_{(1)}$ is ancillary for θ. That is, there exists a constant c that does not depend on θ such that $\mathbb{E}_\theta[X_{(n)} - X_{(1)} - c] = 0$ for all $\theta \in \mathbb{R}$.

Let $f : \mathbb{R}^n \to \mathbb{R}$, $f(x_1, \ldots, x_n) = x_n - x_1 - c \;\forall\, (x_1, \ldots, x_n) \in \mathbb{R}^n$. Then $\mathbb{E}_\theta f(Y) = 0$, $\forall \theta \in \Theta, Y = (X_{(1)}, \ldots, X_{(n)})$. Note that $f \neq 0$ (in fact $f(Y) \neq 0$ with probability 1).

□

9.3.4 Complete Statistics

Definition 9.11 (Complete statistic; definition 5.16 in 541A notes). Suppose $X_1, \ldots, X_n$ is a random sample of size n from a distribution f where $f \in \{f_\theta : \theta \in \Theta\}$ is a family of distributions. A statistic $Y = t(X_1, \ldots, X_n), t : \mathbb{R}^n \to \mathbb{R}^k$ is **complete** for θ if the following holds:

$$\text{For any } f : \mathbb{R}^k \to \mathbb{R} \text{ such that } \mathbb{E}_\theta f(Y) = 0 \ \forall \ \theta \in \Theta, \text{ it holds that } f(Y) = 0.$$

Intuition: Y has no "excess information."

Remark. If a statistic has ancillary information, then it is not complete. Therefore if a statistic is complete, it is not ancillary. Minimal sufficient statistics pretty much always exist, but a complete sufficient statistic might not exist (**see example from homework 4**).

Remark. A complete statistic may not be sufficient (example: a constant).

Example 9.5 (**Exercise 5.19 in Math 541A notes**)**.** The following is an example of a statistic Y that is complete and nonconstant, but not sufficient. Suppose $X_1, \ldots, X_n$ is a random sample of known size n from a Bernoulli distribution with unknown probability parameter $\theta \in (0, 1)$. Let

$$Y = t(X_1, \ldots, X_n) = \sum_{i=1}^{n-1} X_i.$$

Then Y is complete for μ because for any $f : \mathbb{R}^m \to \mathbb{R}$, suppose

$$0 = \mathbb{E}_\theta f(Y) = \sum_{j=0}^{n-1} f(j) \Pr(Y = j \mid \theta) = \sum_{j=0}^{n-1} f(j) \binom{n-1}{j} \theta^j (1-\theta)^{n-1-j}, \qquad \forall \theta \in (0, 1).$$

Divide by $(1-\theta)^{n-1}$ and let $\alpha = \theta/(1-\theta)$ for notational ease. (Note that since $\theta \in (0, 1)$, $\alpha > 0$.)

$$0 = \sum_{j=0}^{n-1} f(j) \binom{n-1}{j} \theta^j \frac{(1-\theta)^{n-1-j}}{(1-\theta)^{n-1}} = \sum_{j=0}^{n-1} f(j) \binom{n-1}{j} \left(\frac{\theta}{1-\theta} \right)^j = \sum_{j=0}^{n-1} f(j) \binom{n-1}{j} \alpha^j, \qquad \forall \alpha > 0.$$

The sum on the right side is a polynomial in $\alpha > 0$. That means the sum on the right can only equal 0 if every coefficient on the polynomial equals zero. $\binom{n}{j}$ is of course nonzero for all $j \in 0, \ldots, n-1$. Therefore for all $\alpha > 0$ we have that $\mathbb{E}_\theta f(Y) = 0$ only if $f(Y) = 0$, so Y is complete.

However, Y is not sufficient for μ. Using the definition of conditional probability,

$$\Pr\big((X_1, \ldots, X_n) = (x_1, \ldots, x_n) \mid Y = y\big) = \frac{1}{\Pr(Y = y)} \cdot \Pr\big((X_1, \ldots, X_n) = (x_1, \ldots, x_n) \cap Y = y\big)$$

$$= \frac{1}{\Pr(Y = y)} \cdot \Pr\big((X_1, \ldots, X_n) = (x_1, \ldots, x_n)\big)$$

Using independence and the definition of a binomial distribution, we have

$$= \frac{1}{\binom{n-1}{y} \theta^y (1-\theta)^{n-1-y}} \cdot \prod_{i=1}^{n} \Pr\left(X_i = x_i\right) = \frac{1}{\binom{n-1}{y} \theta^y (1-\theta)^{n-1-y}} \cdot \prod_{i=1}^{n} \theta^{x_i} (1-\theta)^{1-x_i}$$

$$= \frac{1}{\binom{n-1}{y}\theta^y(1-\theta)^{n-1-y}} \cdot \theta^y(1-\theta)^{n-y} = \frac{1-\theta}{\binom{n-1}{y}}.$$

Because $\Pr\big((X_1, \ldots, X_n) = (x_1, \ldots, x_n) \mid Y = y\big) = (1-\theta)/\binom{n-1}{y}$ depends on θ, Y is not sufficient for θ.

Exercise 17 (Exercise 5.20 in Math 541A notes). This exercise shows that a complete sufficient statistic might not exist.

Let $X_1, \ldots, X_n$ be a random sample of size n from the uniform distribution on the three points $\{\theta, \theta + 1, \theta + 2\}$, where $\theta \in \mathbb{Z}$.

(a) Show that the vector $Y := (X_{(1)}, X_{(n)})$ is minimal sufficient for θ.

(b) Show that Y is not complete by considering $X_{(n)} - X_{(1)}$.

(c) Using minimal sufficiency, conclude that any sufficient statistic for θ is not complete.

Proof. (a) First we need to show that Y is sufficient for θ. Informally, it makes sense that this would be the case because there are three possibilities:

(1) If θ and $\theta + 2$ appear in the data set, we can identify θ with certainty as the smallest of them. Simply observing $x_{(1)}$ and $x_{(n)}$ would show us the smallest and largest observations, which would be 2 units apart. Then we would known that the these observations are θ and $\theta + 2$, and we could identify θ with certainty. (Note that it doesn't matter in this case whether we observe $\theta + 1$.)

(2) If only the values $\{\theta, \theta + 1\}$ or $\{\theta + 1, \theta + 2\}$ appear in the data set, we have to guess which one of these pairs we observed. If we guess that we have observed $\theta + 1$ and $\theta + 2$ and subtract one from the smallest observation to estimate θ, or we can guess that we have observed θ and $\theta + 1$ and use the smallest value as our estimate for θ. (Or we can hedge and take the mean of these values.) In any case, simply observing $x_{(1)}$ and $x_{(n)}$ would show us the smallest and largest observations, which would be 1 apart, leaving us in the same position as if we had all the data.

(3) If only one value appears, we have to guess if it is θ, $\theta + 1$, or $\theta + 2$ in a way similar to if we only observe two values. But if we observe that the smallest and largest values are equal, we are observing the same information and we are in the same position for estimating θ as if we had all of the data.

We can formally show that Y is sufficient using the Factorization Theorem (Theorem 9.5). Note that because on each trial we observe $\theta, \theta + 1$, or $\theta + 2$ with equal probability, the mass function for the unordered observations $f_{u,\theta} : \mathbb{Z}^n \to \mathbb{R}$ is the same as that of a multinomial distribution with three outcomes with equal probabilities. That is, if $n_0 = \sum_{i=1}^n \mathbf{1}_{\{x_i = \theta\}}$ (where $\mathbf{1}_{\{x_i=\theta\}}$ is an indicator variable for the ith observation having value θ), $n_1 = \sum_{i=1}^n \mathbf{1}_{\{x_i=\theta=1\}}$, and $n_2 = \sum_{i=1}^n \mathbf{1}_{\{x_i=\theta+2\}}$, we have

$$f_{u,\theta}(x) = \binom{n}{n_0, n_1, n_2} \left(\frac{1}{3}\right)^n = \frac{n!}{n_0! n_1! n_2!} \left(\frac{1}{3}\right)^n.$$

But taking into account the order in which we observe the samples, the probability of observing any one sample of size n ($x = (x_1, \ldots, x_n)$) is simply

$$f_\theta(x) = \begin{cases} \left(\frac{1}{3}\right)^n & x_1 \in \{\theta, \theta+1, \theta+2\}, \ldots, x_n \in \{\theta, \theta+1, \theta+2\} \\ 0 & \text{otherwise.} \end{cases}$$

We have $t : \mathbb{Z}^n \to \mathbb{Z}^2$ is given by

$$t(X_1, \ldots, X_n) = (X_{(1)}, X_{(n)}).$$

Choose

$$h(x) = (1/3)^n, \qquad g_\theta(t(x)) = \begin{cases} 1 & t(x) \in \{\theta, \theta+1, \theta+2\} \times \{\theta, \theta+1, \theta+2\} \\ 0 & \text{otherwise.} \end{cases} \tag{9.11}$$

Then we have $f_\theta(x) = g_\theta(t(x))h(x)$, as desired. Now we will use Theorem 9.28 (Theorem 5.8 from the lecture notes) to show that Y is not only sufficient, but is also minimal sufficient. Let $x, z \in \mathbb{Z}^n$, and let $y_x = (x_{(1)}, x_{(n)})$, $y_z = (z_{(1)}, z_{(n)})$. We seek to show that for every $x, z \in \mathbb{Z}^n$, the likelihood ratio $\frac{f_\theta(x)}{f_\theta(z)}$ is constant (the constant may depend on x and z) if and only if $y_x = y_z$. (We only need to consider $x, z \in \mathbb{Z}^n$ rather than all of $\mathbb{R}^n$ because since $\theta \in \mathbb{Z}$, $X_i \in \mathbb{Z}\ \forall i \in \{1, \ldots, n\},\ \forall \theta \in \Theta$.) Using the expressions in (9.11), we can write the equation in (9.12) as

$$f_\theta(x) = c(x,y) f_\theta(z) \iff g_\theta(t(x)) \left(\frac{1}{3}\right)^n = c(x,z) g_\theta(t(z)) \left(\frac{1}{3}\right)^n \iff g_\theta(t(x)) = c(x,z) g_\theta(t(z)) \tag{9.12}$$

I will argue that the equality in (9.12) only holds for some $c(x, z) \in \mathbb{R}$ if $t(x) = t(z)$. Suppose we have observed data x and z from a distribution with a specific θ_0. There are three cases to consider:

(1) $t(x) = \{\theta_0, \theta_0 + 2\}$ **(full information):** Then the only θ for which $g_\theta(t(x)) \neq 0$ is $\theta = \theta_0$. (This corresponds to situation (1) above.)

(2) $t(x) = \{\theta_0, \theta_0 + 1\}$ **or** $\{\theta_0, \theta_0 + 1\}$**:** Then $g_\theta(t(x)) \neq 0$ for two values of θ. In the first case, those two values will be $\theta_0 - 1$ and θ_0. In the second case, those two values will be θ_0 and $\theta_0 + 1$. (This corresponds to situation (2) above.)

(3) $t(x) = \{\theta_0, \theta_0\}$**,** $\{\theta_0 + 1, \theta_0 + 1\}$**, or** $\{\theta_0 + 2, \theta_0 + 2\}$**:** Then $g_\theta(t(x)) \neq 0$ for three values of θ. In the first case, those three values will be $\theta_0 - 2$, $\theta_0 - 1$, and θ_0. In the second case, those three values will be $\theta_0 - 1$, θ_0, and $\theta_0 + 1$. In the last case, those three values will be θ_0, $\theta_0 + 1$, and $\theta_0 + 2$. (This corresponds to situation (3) above.)

I have enumerated all possible values of $t(x)$ or $t(z)$ for a given true $\theta = \theta_0$, and note that there is no overlap among any of the possibilities for what values of θ will yield identical values of $g_\theta(t(x))$ and $g_\theta(t(z))$ for all θ. That is, that is true (and (9.12) holds for all $\theta \in \Theta = \mathbb{Z}$) if and only if $t(x) = t(z)$, which is what we were trying to show. So Y is minimal sufficient.

(b) Recall the definition of a complete statistic:

Definition 9.12 (Complete statistic; definition 5.16 in 541A notes). Suppose $X_1, \ldots, X_n$ is a random sample of size n from a distribution f where $f \in \{f_\theta : \theta \in \Theta\}$ is a family of distributions. A statistic $Y = t(X_1, \ldots, X_n), t : \mathbb{R}^n \to \mathbb{R}^k$ is **complete** for θ if the following holds:

$$\text{For any } f : \mathbb{R}^k \to \mathbb{R} \text{ such that } \mathbb{E}_\theta f(Y) = 0\ \forall\ \theta \in \Theta, \text{ it holds that } f(Y) = 0.$$

We will show that Y is not complete by showing that Y contains ancillary information. Specifically, we will show that $\mathbb{E}_\theta[f(Y)] \neq 0$ where $f(Y) = X_{(n)} - X_{(1)} - c$ for some $c \in \mathbb{Z}$.

Let $Z_1, \ldots, Z_n$ be a random sample of size n from the uniform distribution on the three points $\{0, 1, 2\}$. Then $X_i = Z_i + \theta,\ \forall\ 1 \leq i \leq n, \forall\ \theta \in \mathbb{Z}$. Also, $X_{(i)} = Z_{(i)} + \theta$. So $X_{(n)} - X_{(1)} = Z_{(n)} - Z_{(1)}$ does not depend on $\theta \in \mathbb{Z}$. So, $X_{(n)} - X_{(1)}$ is ancillary for θ. That is, there exists a constant $c \in \mathbb{Z}$ that does not depend on θ such that $\mathbb{E}_\theta[X_{(n)} - X_{(1)} - c] = 0$ for all $\theta \in \mathbb{Z}$.

Define this c and let $f : \mathbb{Z}^2 \to \mathbb{Z}$ be $f(x_1, x_n) := x_n - x_1 - c\ \forall\ (x_1, x_n) \in \mathbb{R}^n$. Then $\mathbb{E}_\theta f(Y) = 0,\ \forall \theta \in \Theta, Y = (X_{(1)}, X_{(n)})$.

(c) Let S be any sufficient statistic for θ. Since Y is minimal sufficient, there exists a function ϕ such that $Y = \phi(\theta)$. Therefore S is not complete because $\mathbb{E}_\theta(f(\phi(S)) = \mathbb{E}_\theta(f(Y)) = 0$ for all $\theta \in \mathbb{Z}$. So any sufficient statistic for θ is not complete.

□

Example 9.6 (Discrete RV example, Example 5.21 in Math 541A notes; return to Example 9.23). Suppose we take a sample of size n from a Bernoulli distribution with parameter $0 < \theta < 1$. We already showed $Y := X_1 + \ldots + X_n$ is sufficient for θ. Now we show Y is complete.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ with $\mathbb{E}_\theta f(Y) = 0\ \ \forall \theta \in \Theta$. Since Y is binomial,

$$0 = \mathbb{E}_\theta f(Y) = \sum_{j=0}^{n} f(j) \binom{n}{j} \theta^j (1-\theta)^{n-j}, \qquad \forall \theta \in (0, 1)$$

Let $\alpha = \theta/(1-\theta)$ and divide by $(1-\theta)^n$;

$$0 = \sum_{j=0}^{n} f(j) \binom{n}{j} \alpha^j, \qquad \forall \alpha > 0.$$

The sum on the right side is a polynomial in $\alpha > 0$. That means the sum on the right can only equal 0 if every coefficient on the polynomial equals zero. $\binom{n}{j}$ is of course nonzero for all $j \in 0, \ldots, n-1$. Therefore for all $\alpha > 0$ we have that $\mathbb{E}_\theta f(Y) = 0$ only if we have $f(j) = 0$ for all $0 \leq j \leq n$, so $f(Y) = 0$ so Y is complete.

□

Example 9.7 (Continuous RV example; return to Example 9.3). For a random sample from a Gaussian distribution with known variance $\sigma^2 > 0$ and unknown $\mu \in \mathbb{R}$, we showed that $Y = (X_1 + \ldots + X_n)/n$ is sufficient for μ. Now we will show it is complete.

Proof. **Found this proof confusing** For simplicity, let $\sigma = 1, n = 1$. Let $f : \mathbb{R} \to \mathbb{R}$ with $\mathbb{E}_\mu |f(Y)| < \infty$ for all $\mu \in \mathbb{R}$. Then

$$0 = \mathbb{E}_\mu(f(Y)) = \int_{-\infty}^{\infty} f(y) \exp\bigg(-\frac{(y-\mu)^2}{2}\bigg) \frac{1}{\sqrt{2\pi}} dy, \qquad \forall \mu \in \mathbb{R}$$

Multiplying both sides by $e^{\mu^2/2}\sqrt{2\pi}$ yields

$$0 = \int_{-\infty}^{\infty} f(y) e^{-y^2/2} e^{y\mu} dy, \qquad \forall \mu \in \mathbb{R} \tag{9.13}$$

If $f(y) \geq 0$, we are done since (9.13) is the moment generating function of a random variable with density

$$\frac{f(y)e^{-y^2/2}}{\int_{\mathbb{R}} f(x) e^{-x^2/2} dx}$$

Then by Theorem 9.2 from the appendix of the Math 541A notes (uniqueness of moment-generating functions), this describes a unique random variable. But this is a contradiction because we can't divide by 0. **(???)**

In the case that f is positive and negative at different points, we write $f = f_+ - f_-$ where $f_+(x) := \max\{f(x), 0\}$ and $f_-(x) := \max\{-f(x), 0\}$. use (9.13) for any μ and divide by case $\mu = 0$,

$$\int_{-\infty}^{\infty} f_-(y) e^{-y^2/2} e^{y\mu} dy, \qquad \int_{-\infty}^{\infty} f_-(y) e^{-y^2/2} dy$$

which yields

$$\int_{-\infty}^{\infty} f_+(y) e^{-y^2/2} e^{y\mu} dy \Big/ \int_{-\infty}^{\infty} f_-(y) e^{-y^2/2} dy$$

so we are done again by Theorem 9.2. So $f_y = f_-$, $f = f_+ - f_- = 0$.

So basically we started by assuming that the expression equals zero and concluded that f must equal 0; therefore the statistic is complete.

$\square$

Remark. Later we will show that complete and sufficient statistics are minimal sufficient.

Exercise 18 (Conditional expectation exercise relevant to proof of Bahadur's Theorem). Let $X, Y : \Omega \to \mathbb{R}$ both be discrete or both continuous. For all y in the range of Y, define $g(y) := \mathbb{E}(X \mid Y = y)$. Define the conditional expectation of X given Y, denoted $\mathbb{E}(X \mid Y)$, as the random variable $g(Y)$.

Solution: Theorem 6.6

Theorem 9.31 (Bahadur's Theorem; Theorem 5.25 in Math 541A notes). If Y is a complete sufficient statistic for a family $\{f_\theta : \theta \in \Theta\}$ of probability densities or probability mass functions, then Y is a minimal sufficient statistic for θ.

Remark. By Remark 5.11 in Math 541A notes, a complete sufficient statistic is unique up to an invertible map. Also by Example 5.15 in Math 541A notes, the converse of Bahadur's Theorem is false.

Proof. By Proposition 9.27 (Proposition 5.12 in Math 541A notes), there exists a minimal sufficient statistic Z for θ. To show that Y is minimal sufficient, it suffices to find a function r such that $Y = r(Z)$. Define $r(Z) = \mathbb{E}_\theta(Y \mid Z)$. Since Z is minimal sufficient and Y is sufficient by assumption, there exists a function u such that $Z = u(Y)$. By conditioning on Y we have by Exercise 18 (Exercise 5.24 in the Math 541A notes)

$$\mathbb{E}_\theta(r(u(Y)) = \mathbb{E}_\theta(r(Z)) = \mathbb{E}_\theta[\mathbb{E}_\theta(r(Z) \mid Y)] = \mathbb{E}_\theta[\mathbb{E}_\theta(\mathbb{E}_\theta(Y \mid Z) \mid Y)] = \mathbb{E}_\theta[\mathbb{E}_\theta(\mathbb{E}_\theta(Y \mid u(Y)) \mid Y)]$$

$$= \mathbb{E}_\theta[\mathbb{E}_\theta(Y \mid u(Y))] = \mathbb{E}_\theta(Y).$$

That is, $\mathbb{E}_\theta(r(u(Y) - Y) = 0$ for all $\theta \in \Theta$. Since Y is complete, we conclude that $r(u(Y)) = Y$, and since $r(u(Y)) = r(Z)$, we have $r(Z) = Y$, as desired.

□

Basu's theorem tells us that a complete sufficient statistic implies independence from any ancillary statistic. So complete sufficient statistics have no ancillary information, unlike minimal sufficient statistics.

Theorem 9.32 (Basu's Theorem, Theorem 5.27 in Math 541A notes). Let $Y : \Omega \to \mathbb{R}^k$ and $Z : \Omega \to \mathbb{R}^m$ be statistics. If Y is a complete sufficient statistic for $\{f_\theta : \theta \in \Theta\}$ and Z is ancillary for θ, then for all $\theta \in \Theta$, Y and Z are independent with respect to f_θ.

Proof. Let $A \subseteq \mathbb{R}^k$ and $B \subseteq \mathbb{R}^m$. We need to show that

$$\mathbb{P}_\theta(Y \in A, Z \in B) = \mathbb{P}_\theta(Y \in A)\mathbb{P}_\theta(Z \in B), \qquad \forall \theta \in \Theta.$$

Note that

$$\mathbb{P}_\theta(Y \in A, Z \in B) = \mathbb{E}_\theta \mathbf{1}_{\{Y \in A\}} \mathbf{1}_{\{Z \in B\}} = \mathbb{E}_\theta \mathbb{E}_\theta(\mathbf{1}_{\{Y \in A\}} \mathbf{1}_{\{Z \in B\}} \mid Y) = \mathbb{E}_\theta[\mathbf{1}_{\{Y \in A\}} \mathbb{E}_\theta(\mathbf{1}_{\{Z \in B\}} \mid Y)].$$

Let $g(Y) := \mathbb{E}_\theta(\mathbf{1}_{\{Z \in B\}} \mid Y)$. Then

$$\mathbb{E}_\theta(g(Y)) = \mathbb{E}_\theta(\mathbb{E}_\theta(\mathbf{1}_{\{Z \in B\}}) = \mathbb{E}_\theta(\mathbf{1}_{\{Z \in B\}}) = \mathbb{P}_\theta(Z \in B). \tag{9.14}$$

Let $c := \mathbb{P}_\theta(Z \in B) = \mathbb{E}_\theta(g(Y)) = \mathbb{E}_\theta[\mathbb{E}_\theta(\mathbf{1}_{\{Z \in B\}} \mid Y)]$. Then c does not depend on θ since Z is ancillary by assumption. Then $\mathbb{E}_\theta(g(Y) - c) = 0, \forall\ \theta \in \Theta$ for all $\theta \in \Theta$. Note that $g(Y) := \mathbb{E}_\theta(\mathbf{1}_{\{Z \in B\}} \mid Y)$ does not depend on θ since Y is sufficient. Since Y is complete, $g(Y) - c - 0 \iff g(Y) = c$, so Y is constant. Therefore by (9.14)

$$c = \mathbb{E}_\theta(c) = \mathbb{E}_\theta(g(Y) = \mathbb{P}_\theta(Z \in B),$$

so we have

$$\mathbb{P}_\theta(Y \in A, Z \in B) = \mathbb{P}_\theta(Y \in A)g(Y) = \mathbb{P}_\theta(Y \in A)c = \mathbb{P}_\theta(Y \in A)\mathbb{P}_\theta(Z \in B), \qquad \forall \theta \in \Theta.$$

as desired.

□

9.4 Point Estimation

Definition 9.13 (Point estimator). Let $X_1, \ldots, X_n$ be a random sample of size n from a family of distribution $\{f_\theta : \theta \in \Theta\}$. If Y is a statistic that is used to estimate the parameter θ that fits the data at hand, we then refer to Y as a **point estimator** or **estimator**.

9.4.1 Heuristic Principles for Finding Good Estimators

Definition 9.14 (Likelihood, Definition 6.1 in Math 541A notes). Let $X_1, \ldots, X_n$ be a random sample of size n from a family of distributions $\{f_\theta : \theta \in \Theta\}$. If we have data $x \in \mathbb{R}^n$, then the function $L : \Theta \to [0, \infty)$ defined by $L(\theta) := f_\theta(x)$ is called the **likelihood function**.

- **Likelihood principle:** All data relevant to estimating the parameter θ is contained in the likelihood function.
- **Sufficiency principle:** If $Y = t(X_1, \ldots, X_n)$ is a sufficient statistic and if we have two results $x, y \in \mathbb{R}^n$ from an experiment with the same statistics $t(x) = t(y)$, then our estimate of the parameter θ should be the same for either experimental result.
- **Equivariance principle:** If the family of distributions $\{f_\theta : \theta \in \Theta\}$ is invariant under some symmetry, then the estimator of θ should respect the same symmetry. (For example, a location family is invariant under translation, so an estimator for the location parameter should commute with translations.)

9.4.2 Evaluating Estimators

We can enumerate several desirable properties for estimators.

Definition 9.15 (Unbiasedness, Definition 6.2 in Math 541A notes). Let $X_1, \ldots, X_n$ be a random sample of size n from a family of distributions $\{f_\theta : \theta \in \Theta\}$. Let $t : \mathbb{R}^n \to \mathbb{R}^k$ and let $Y := t(X_1, \ldots, X_n$ be an estimator for $g(\theta)$. Let $g : \Theta \to \mathbb{R}^k$. We say that Y is **unbiased** for $g(\theta)$ if $\mathbb{E}_\theta Y = g(\theta)$ for all $\theta \in \Theta$.

One common way to check the quality of an estimator is the mean squared error, or squared L_2 norm, of the estimator minus θ, $\mathbb{E}_\theta(Y - g(\theta))^2$. If the estimator is unbiased, this quantity is equal to the variance of Y.

Definition 9.16 (UMVU, sometimes called MVUE (minimum variance unbiased estimator); Definition 6.3 in Math 541A Notes). Let $X_1, \ldots, X_n$ be a random sample of size n from a family of distributions $\{f_\theta : \theta \in \Theta\}$. Let $g : \Theta \to \mathbb{R}$. Let $t : \mathbb{R}^n \to \mathbb{R}$ and let $Y := t(X_1, \ldots, X_n)$ be an unbiased estimator for $g(\theta)$. We say that Y is **uniformly minimum variance unbiased (UMVU)** if, for any other unbiased estimator Z for $g(\theta)$, we have $\mathrm{Var}_\theta(Y) \leq \mathrm{Var}_\theta(Z)$ for all $\theta \in \Theta$.

Remark. The "uniform" property has to do with the fact that this inequality must hold for every $\theta \in \Theta$ (as opposed to for a particular θ, or averaged over all $\theta \in \Theta$, or something like that).

More generally, given a family of distributions $\{f_{\tilde{\theta}} : \tilde{\theta} \in \Theta\}$, we could be given a **loss function** $L(\theta, y) : \Theta \times \mathbb{R}^k \to \mathbb{R}$ and be asked to minimize the **risk function** $r(\theta, Y) := \mathbb{E}_{\tilde{\theta}}\big(\ell(\theta, Y)\big)$ over all possible estimators Y. In the case of mean squared error loss, we have $L(\theta, y) := (y - g(\theta))^2$ for all $y, \theta \in \mathbb{R}$.

The Rao-Blackwell Theorem says that if $L(\theta, y)$ is convex in y then we can create an optimal estimator for $g(\theta)$ from a sufficient statistic and any estimator for $g(\theta)$ (we can lower the risk of an estimator Y by conditioning on a sufficient statistic Z.).

Theorem 9.33 (Rao-Blackwell; Theorem 6.4 in Math 541A notes). Let Z be a sufficient statistic for $\{f_\theta : \theta \in \Theta\}$ and let Y be an estimator for $g(\theta)$. Define $W := \mathbb{E}_\theta(Y \mid Z)$. Let $\theta \in \Theta$. Then

$$\mathrm{Var}_\theta(W) \leq \mathrm{Var}_\theta(Y).$$

Further, let $r(\theta, y) < \infty$ and such that $\ell(\theta, y)$ is convex in y. Then

$$r(\theta, W) \leq r(\theta, Y).$$

Proof. Note that since Z is sufficient, W does not depend on θ. By the Conditional Jensen's Inequality (Theorem 12.5) and using the convexity of $\ell(\theta, y)$ in y,

$$\ell(\theta, w) = \ell(\theta, \mathbb{E}_{\tilde{\theta}}(Y \mid Z)) \leq \mathbb{E}_{\tilde{\theta}}[\ell(\theta, Y) \mid Z].$$

Take expectations of both sides to get

$$\mathbb{E}_{\tilde{\theta}}\ell(\theta, w) = r(\theta, W) \leq \mathbb{E}_{\tilde{\theta}}\mathbb{E}_{\tilde{\theta}}[\ell(\theta, Y) \mid Z] = \mathbb{E}_{\tilde{\theta}}\ell(\theta, Y) = r(\theta, Y).$$

$\square$

Definition 9.17 (Definition 6.4 in 541A notes; MISSED SOME NOTES TODAY). We say Y is **uniformly minimum risk unbiased** (UMRU) if for any other unbiased estimator Z for $g(\theta)$,

$$r(\theta, Y) \leq r(\theta, z), \qquad \forall\, \theta \in \Theta$$

Remark. Unfortunately, UMRU or UMVU may not exist. More fundamentally, an unbiased estimator for $g(\theta)$ may not exist. For example, let X be a binomial random variable with known n, unknwon $0 < \theta < 1$, and $g(\theta) = \theta/(1-\theta)$. Then no unbiased estimator exists for $g(\theta)$. Why?

$$\mathbb{E}_\theta t(X) = \sum_{j=0}^{n} t(j) \binom{n}{j} \theta^j (1-\theta)^{n-j}, \qquad \forall \theta \in \Theta, \text{ by definition of } X.$$

where the summation is a polynomial of degree at most n in θ. Then it is impossible to have $\mathbb{E}_\theta t(x) = g(\theta)$ when $g(\theta) = \theta/(1-\theta)$.

Recall the definition of strict convexity (Definition 12.2).

Theorem 9.34 (**Rao-Blackwell restated; Theorem 6.7 in Math 541A notes**)**.** Let Z be a sufficient statistic for $\{f_\theta : \theta \in \Theta\}$ and let Y be an estimator for $g(\theta)$. Define $W := \mathbb{E}_\theta(Y \mid Z)$. Let $\theta \in \Theta$ with $r(\theta, y) < \infty$ and such that $\ell(\theta, y)$ is convex in y. Then

$$r(\theta, W) \leq r(\theta, Y).$$

Further, if $\ell(\theta, y)$ is strictly convex in $y \in \mathbb{R}$, then $r(\theta, W) < r(\theta, Y)$ unless $W = Y$ (that is, there is a unique minimizer of the risk).

So Z makes the estimator better. Question: can we construct $\mathbb{E}_\theta(Y \mid Z)$ to be UMRU?

Remark (**Remark 6.9 in Math 541A notes**)**.** $\mathbb{E}_\theta W = \mathbb{E}_\theta \mathbb{E}_\theta(Y \mid Z) = \mathbb{E}_\theta Y$. So if Y is unbiased for $g(\theta)$, then so is W.

Remark. What happens if Z is constant in Rao-Blackwell? Then in general Z will not be sufficient, so W might depend on θ which is not allowed. Put another way, if Z has insufficient information, then W gets messed up **(???)**.

Example 9.8. Let $X_1, \ldots, X_n$ be a random sample with unknown mean $\mu \in \mathbb{R}$. We want to construct an estimator for μ using Rao-Blackwell. Let $t : \mathbb{R}^n \to \mathbb{R}$ so that $t(x_1, \ldots, x_n) = x_1$ for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Let $Y = t(X_1, \ldots, X_n) = X_1$. Note that Y is unbiased. First use of Rao-Blackwell: use $Z = (X_1, \ldots, X_n)$. Then by Exercise 5.24,

$$W := \mathbb{E}_\mu(X_1 \mid (X_1, \ldots, X_n)) = \mathbb{E}(X_1 \mid X_1) = X_1.$$

We can think of this as failing to improve the estimator because we used "too much" information. Second try: use $Z = \sum_{i=1}^n X_i$. Note that in the Gaussian case Z is sufficient for μ and unbiased for $n\mu$. Since $X_1, \ldots, X_n$ are i.i.d. for all $1 \leq k \leq \ell \leq n$ the joint distribution of $(X_k, \sum_{i=1}^n X_i)$ is the same as the joint distribution of $(X_\ell, \sum_{i=1}^n X_i)$. So

$$\mathbb{E}(X_k \mid \sum_{i=1}^n X_i) = \mathbb{E}(X_\ell \mid \sum_{i=1}^n X_i).$$

So we have

$$W := \mathbb{E}_\mu\left(X_1 \mid \sum_{i=1}^n X_i\right) = \frac{1}{n}\sum_{j=1}^n \mathbb{E}_\mu\left(X_j \mid \sum_{i=1}^n X_i\right) = \frac{1}{n}\mathbb{E}_\mu\left(\sum_{j=1}^n X_j \mid \sum_{i=1}^n X_i\right) = \frac{1}{n}\sum_{i=1}^n X_i.$$

So we started with a trivial estimator X_1 and ended up with the sample mean using Rao-Blackwell.

Theorem 9.35 (**Lehmann-Scheffe, Theorem 6.13 in Math 541A notes**)**.** Let Z be a complete sufficient statistic for a family of distributions $\{f_\theta : \theta \in \Theta\}$. Let Y be an unbiased estimator for $g(\theta)$. Define

$W := \mathbb{E}_\theta(Y \mid Z)$. (Since Z is sufficient, W does not depend on θ.) Then W is UMRU for $g(\theta)$. Further, if $\ell(\theta, y)$ is strictly convex in y for all $\theta \in \Theta$, then W is unique. In particular, W is the unique UMVU for $g(\theta)$.

Proof. W is unbiased by Remark 6.10 in math 541A notes ("$\mathbb{E}_\theta W = \mathbb{E}_\theta \mathbb{E}_\theta(Y \mid Z) = \mathbb{E}_\theta Y$. So if Y is unbiased for $g(\theta)$, then so is W.") We first show W does not depend on Y. Let Y' be an unbiased estimator for $g(\theta)$. We show that $\mathbb{E}_\theta(Y \mid Z) = \mathbb{E}_\theta(Y' \mid Z)$ for all $\theta \in \Theta$. Note that

$$\mathbb{E}_\theta(\mathbb{E}_\theta(Y \mid Z) - \mathbb{E}_\theta(Y' \mid Z)) = \mathbb{E}_\theta(Y - Y') = g(\theta) - g(\theta) = 0, \ \forall \ \theta \in \Theta$$

Note that $\mathbb{E}_\theta(Y \mid Z)$ and $\mathbb{E}_\theta(Y' \mid Z)$ are functions of Z. Therefore since Z is complete, $\mathbb{E}_\theta(Y \mid Z) = \mathbb{E}_\theta(Y' \mid Z)$ for all $\theta \in \Theta$.

Next, by Rao Blackwell,

$$r(\theta, Y') = r(\theta, \mathbb{E}_\theta(Y' \mid Z)) = r(\theta, \mathbb{E}_\theta(Y \mid Z)) = r(\theta, W), \ \forall \theta \in \Theta.$$

$\square$

Remark (Remark 6.14 in Math 541A notes, Theorem 7.3.23 in Casella and Berger [2001, p. 347]). Let $Z : \Omega \to \mathbb{R}^k$ be a complete sufficient statistic for $\{f_\theta : \theta \in \Theta\}$ and let $h : \mathbb{R}^k \to \mathbb{R}^m$. Let $g(\theta) := \mathbb{E}_\theta h(Z)$ for all $\theta \in \Theta$. Then $h(Z)$ is unbiased for $g(\theta)$, since $\mathbb{E}_\theta h(Z) = g(\theta) = \mathbb{E}_\theta(g(\theta))$. Applying Theorem 9.35, we have

$$W := \mathbb{E}_\theta(h(Z) \mid Z) = \mathbb{E}_\theta(\mathbb{E}_\theta[h(Z) \mid h(Z)] \mid Z) = \mathbb{E}_\theta[h(Z) \mid h(Z)] = h(Z).$$

Therefore by Theorem 9.35, $h(Z)$ is UMVU for $g(\theta)$. That is, any function of a complete sufficient statistic is UMVU for its expected value. So one way to find a UMVU is to come up with a function of a complete sufficient statistic that is unbiased for a given function $g(\theta)$.

Summary of methods for finding UMVU (given a complete sufficient statistic Z, want to estimate $g(\theta)$)

(1) **(Condition method/Rao-Blackwell):** Follow Theorem 9.35: find an unbiased Y and let $W := \mathbb{E}_\theta(Y \mid Z)$. (problem: can be hard to find an unbiased Y.)

(2) Solve for $h : \mathbb{R}^k \to \mathbb{R}$ satisfying

$$\mathbb{E}_\theta h(Z) = g(\theta) \tag{9.15}$$

by the above remark, (9.15) will also give you the UMVU. Then $h(Z)$ is UMVU for $g(\theta)$. By "solve", consider that we have g and Z and somehow solve for the h satisfying (9.15). For example if Z is binomial the left side of (9.15) will be the sum of a bunch of numbers. Find the h values that satisfy (9.15), if possible.

(3) **(Luck method):** Somehow guess the h such that (9.15) is satisfied.

Example 9.9. Suppose we are sampling from a Gaussian distribution with unknown mean and variance. By the Factorization Theorem and Exercise 5.23 (**on homework 4**), we know $(\overline{X}, S^2)$ is complete sufficient of (μ, σ^2) (sufficiency follows by the Factorization Theorem (Theorem 9.5), completeness follows by the exercise). For example, using method (2) above, $\overline{X}$ is UMVU for μ (with finite σ) by method (3) above, since $\overline{X}$ is a function of $(\overline{X}, S^2)$, $h(x,y) := x$, $g(\mu, \sigma^2) := \mu$ (then (9.15) is satisfied). Similarly, S^2 is UMVU for σ^2 be (3) using $h(x,y) := y$, $g(\mu, \sigma^2) := \sigma^2$ (then (9.15) is satisfied).

Suppose we want a UMVU for μ^2. Try guessing $(\overline{X})^2$ as an estimator. Note that

$$\mathbb{E}[(\overline{X})^2] = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^n X_i\right)^2 = \frac{1}{n^2}\left(\sum_{i=1}^n \mathbb{E}X_i^2 + 2\sum_{1 \le i < j \le n} \mathbb{E}X_i \mathbb{E}X_j\right) = \ldots = \mu^2 + \sigma^2/n$$

So,

$$\mathbb{E}\left(\overline{X}^2 - S^2/n\right) = \mu^2$$

which means that $\overline{X}^2 - S^2/n$ is UMVU since it is a function of $(\overline{X}, S^2)$.

Example 9.10. Try Method (2). Let X be a binomial random variable with known parameter n and unknown $0 < \theta < 1$. Suppose we want to estimate $g(\theta) := \theta(1-\theta)$. Solve (9.15): find the h satisfying

$$\theta(1-\theta) = \mathbb{E}_\theta h(X)$$

$$\iff \theta(1-\theta) = \sum_{j=0}^n h(j)\binom{n}{j}\theta^j(1-\theta)^{n-j}, \qquad \forall \theta \in (0,1)$$

For convenience, let $a := \theta/(1-\theta)$, so that $a(1-\theta) = \theta \iff \theta = a/(1+a)$, $1-\theta = 1/(1+a)$. Then we have

$$(1-\theta)^{-n} = \sum_{j=0}^n h(j)\binom{n}{j}a^j = \theta(1-\theta)^{1-n} = \frac{a}{1+a}\left(\frac{1}{1+a}\right)^{1-n} = a(1+a)^{n-2} \tag{9.16}$$

So we want to solve for $h(j)$ so that for all $a > 0$,

$$\sum_{j=0}^n h(j)\binom{n}{j}a^j = a(1+a)^{n-2} = \text{ (RHS of (9.16), by binomial theorem) } a\sum_{j=0}^{n-2}\binom{n-2}{j}a^j = \sum_{j=1}^{n-1}\binom{n-2}{j}-1a^j$$

We need the coefficients to match up one by one. So $h(0) = h(n) = 0$, and then it works if $h(j)\binom{n}{j} = \binom{n-2}{j-1}$ for all $j \in 1, \ldots, n-1$. So

$$h(j) = \frac{\binom{n-2}{j-1}}{\binom{n}{j}} = \frac{(n-2)!}{n!}\frac{(n-j)!j!}{|}(n-j-1)!(j-1)! = \frac{(n-j)j}{n(n-1)}$$

So in fact,

$$h(j) = \frac{(n-j)j}{n(n-1)}, \qquad \forall\ 0 \leq j \leq n$$

Therefore the UMVU for $\theta(1-\theta)$ is

$$\frac{X(n-X)}{n(n-1)}.$$

Example 9.11. Try Method (1). Suppose we have n independent samples $X_1, \ldots, X_n$ from a Bernoulli distribution with unknown $\theta \in (0,1)$. From Example 6.7 (Example 3.15 in Math 541A notes) and Exercise 5.23 in Math 541A notes, a complete sufficient statistic is $Z := \sum_{i=1}^n X_i$ is complete and sufficient for θ. Also, $(1/n)\sum_{i=1}^n X_i$ is unbiased for θ. So, $(1/n)\sum_{i=1}^n X - i$ is UMVU for θ. Suppose we want to estimate θ^2. We need an unbiased estimator Y for θ^2. Let $Y = X_1X_2$. Then $\mathbb{E}(Y) = \mathbb{E}(X_1)\mathbb{E}(X_2) = \theta^2$. By Theorem 9.35, $W := \mathbb{E}_\theta(Y \mid Z)$ is UMVU for σ^2. Note, $Y = 1$ when $X_1 = X_2 = 1$ and 0 otherwise. So

$$\mathbb{E}_\theta(Y \mid Z = z) = \mathbb{E}_\theta(\mathbf{1}_{\{X_1=X_2=1\}} \mid Z = z) = \mathbb{P}_\theta(X_1 = X_2 = 1 \mid Z = z) = \mathbb{P}_\theta(X_1 = X_2 = 1 \mid \sum_{i=1}^n X_i = z)$$

$$= \frac{1}{\mathbb{P}_\theta\Big(\sum_{i=1}^n X_i = z\Big)} \cdot \mathbb{P}_\theta\Big(X_1 = X_2 = 1 \cap \sum_{i=1}^n X_i = z\Big)$$

$$= \frac{1}{\mathbb{P}_\theta\Big(\sum_{i=1}^n X_i = z\Big)} \cdot \mathbb{P}_\theta\Big(X_1 = X_2 = 1 \cap \sum_{i=3}^n X_i = z-2\Big)$$

$$= \frac{\theta^2\binom{n-2}{z-2}\theta^{z-2}(1-\theta)^{n-z}}{\binom{n}{z}\theta^z(1-\theta)^{n-z}} = \frac{\binom{n-z}{z-2}}{\binom{n}{z}} = \frac{(n-z)!(n-z)!z!}{n!(n-z)!(z-2)!} = \frac{z(z-1)}{n(n-1)}$$

So we have shown that for all $0 \leq Z \leq n$,

$$\mathbb{E}_\theta(Y \mid Z = z) = \frac{z(z-1)}{n(n-1)}.$$

So, by Theorem 9.35,

$$W := \mathbb{E}_\theta(Y \mid Z) = \frac{Z(Z-1)}{n(n-1)}$$

is UMVU for θ^2.

Question: If W_1 is UMVU for $g_1(\theta)$ and W_2 is UMVU for $g_2(\theta)$, is $W_1 + W_2$ UMVU for $g_1(\theta) + g_2(\theta)$? If there is a complete sufficient statistic, then by Lehmann-Scheffe (Theorem 9.35), $W_1 = \mathbb{E}_\theta(Y_1 \mid Z)$, $W_2 = \mathbb{E}(Y_2 \mid Z)$ where Y_1 is unbiased for g_1, Y_2 is unbiased for g_2. Then

$$W_1 + W_2 = \mathbb{E}_\theta(Y_1 + Y_2 \mid Z).$$

Note: $\mathbb{E}_\theta(Y-1+Y_2) = \mathbb{E}_\theta Y_1 + \mathbb{E}_\theta Y_2 = g_1(\theta) = g_Z 2(\theta)$. So $W_1 + W_2$ is UMVU by Lehmann-Scheffe (Theorem 9.35) for $g_1(\theta) + g_2(\theta)$. But is it true if we don't have a complete sufficient statistic (and this argument doesn't apply)? **yes, by the theorem below; this condition will clearly hold across sums.**

Theorem 9.36 (**Alternate Characterization of UMVU; Theorem 6.18 in Math 541A notes**). Let $f \in \{f_\theta : \theta \in \Theta\}$ be a family of distributions and let $g : \Theta \to \mathbb{R}$. Let W be an unbiased estimator for $g(\theta)$ (note that the existence of an unbiased estimator is a nontrival assumption). Let $L_2(\Omega)$ be the set of statistics with finite second moment. Then $W \in L_2(\Omega)$ is UMVU for $g(\theta)$ if and only if for any $\theta \in \Theta$,

$$\mathbb{E}_\theta(WU) = 0, \qquad \forall U \in L_2(\Omega) \text{ that are unbiased estimators of } 0$$

Thinking of this as an inner product, we have to be orthogonal to all such U.

Proof. Assume W is UMVU for $g(\theta)$. Let U be an unbiased estimator of 0. Let $s \in \mathbb{R}$, consider $W + sU$. Note that $W + sU$ is also unbiased for $g(\theta)$. Since W is UMVU,

$$\text{Var}_\theta(W) \leq \text{Var}_\theta(W + sU) = \text{Var}_\theta(W) + s^2\text{Var}_\theta(U) + 2s\text{Cov}_\theta(W, U)$$

$$= \text{Var}_\theta(W) + s^2\text{Var}(U) + 2s\mathbb{E}_\theta[(W - \mathbb{E}_\theta(W))U], \qquad \forall \theta \in \Theta.$$

Note that we have equality when $s = 0$. Also, the derivative of the right side with respect to s must be 0 when $s = 0$ or else the inequality does not hold (the minimum value occurs at $s = 0$ if and only if the derivative of the right side in s is 0 at $s = 0$). Note that the derivative of the right side is

$$0 = 2\mathbb{E}_\theta[(W - \mathbb{E}_\theta W)U] = 2\mathbb{E}_\theta(WU).$$

The converse is also true because this reasoning can be reversed, since if Y is any unbiased estimator for $g(\theta)$, then $U := W - Y$ is an unbiased estimator for 0, and $Y = W + sU$ with $s = 1$. We have

$$\text{Var}_\theta(Y) = \text{Var}_\theta(W - U) = \text{Var}_\theta W + \text{Var}_\theta U + 2\text{Cov}_\theta(W, U) = \text{Var}_\theta W + \text{Var}_\theta U + 2\mathbb{E}_\theta(WU)$$

So $\text{Var}_\theta Y \geq \text{Var}_\theta W$ for all $\theta \in \Theta$.

□

Remark. If we have a complete sufficient statistic, better to use the earlier methods in general (unless it is really complicated to work with). If we don't have a complete sufficient statistic, use this theorem.

9.4.3 Efficiency of an Estimator

Another desirable property of an estimator is high efficiency—"good" with a small number of samples. One way to quantify this notion is to define a notion of "information" and try to maximize the information content of the estimator.

Definition 9.18 (Fisher Information, Definition 6.19 in Math 541A notes). Let $f \in \{f_\theta : \theta \in \Theta\}$ be a family of multivariate probability densities or probability mass functions. Assume $\Theta \subseteq \mathbb{R}$ (this is a one-parameter situation). Let X be a random variable with distribution f_θ. Define the **Fisher information** of the family to be

$$I(\theta) = I_X(\theta) := \mathbb{E}_\theta \left(\frac{\mathrm{d}}{\mathrm{d}\theta} \log f_\theta(X) \right)^2, \qquad \forall \theta \in \Theta$$

if this quantity exists and is finite.

Remark. Note that if X is continuous,

$$\mathbb{E}_\theta \left(\frac{\mathrm{d}}{\mathrm{d}\theta} \log f_\theta(X) \right) = \int_{\mathbb{R}^n} \frac{1}{f_\theta(x)} \frac{\mathrm{d}}{\mathrm{d}\theta} f_\theta(x) \cdot f_\theta(x) dx = \int_{\mathbb{R}^n} \frac{\mathrm{d}}{\mathrm{d}\theta} f_\theta(x) dx = \frac{\mathrm{d}}{\mathrm{d}\theta} \int_{\mathbb{R}^n} f_\theta(x) dx = \frac{\mathrm{d}}{\mathrm{d}\theta} 1 = 0.$$

So we could have equivalently defined the Fisher information as

$$I_X(\theta) = \mathrm{Var}_\theta \left(\frac{\mathrm{d}}{\mathrm{d}\theta} \log f_\theta(X) \right)$$

Example 9.12 (Example 6.20 in Math 541A notes). Let $\theta > 0$, let

$$f_\theta(x) := \frac{1}{\sigma\sqrt{2\pi}} \exp\left(- \frac{(x-\theta)^2}{2\sigma^2} \right), \qquad \forall \theta \in \mathbb{R} = \Theta, \ \forall x \in \mathbb{R}.$$

Then

$$I(\theta) = \mathrm{Var}_\theta \left(\frac{\mathrm{d}}{\mathrm{d}\theta} - \frac{(x-\theta)^2}{2\sigma^2} \right) = \frac{1}{\sigma^4} \mathrm{Var}_\theta(x-\theta) = \frac{\sigma^2}{\sigma^4} = \frac{1}{\sigma^2}.$$

Observe that a small σ means large $I(\theta)$ in this case.

Proposition 9.37 (Proposition 6.21 in Math 541A notes). Let X be a random variable with distribution from $\{f_\theta : \theta \in \Theta\}$ (densities or mass functions). Let Y be a random variable with distribution from $\{g_\theta : \theta \in \Theta\}$ (densities or mass functions. Assume $\Theta \subseteq \mathbb{R}$ (one parameter in θ). If X and Y are independent, then

$$I_{(X,Y)}(\theta) = I_X(\theta) I_Y(\theta).$$

Proof. This proof will just be for the case of densities (continuous random variables; the case for probability mass functions is similar). Since X and Y are independent, (X, Y) has a distribution with density $f_\theta(x)g_\theta(y)$, for all $x, y \in \mathbb{R}$. Also, $\frac{\mathrm{d}}{\mathrm{d}\theta} \log f_\theta(X)$ and $\frac{\mathrm{d}}{\mathrm{d}\theta} \log g_\theta(Y)$ are independent for all $\theta \in \Theta$. So,

$$I_{(X,Y)}(\theta) = \mathrm{Var}_\theta\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\log[f_\theta(X)g_\theta(Y)]\right) = \mathrm{Var}_\theta\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\log[f_\theta(X)] + \frac{\mathrm{d}}{\mathrm{d}\theta}\log[g_\theta(Y)]\right)$$

By independence we can write

$$= \mathrm{Var}_\theta\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\log[f_\theta(X)]\right) + \mathrm{Var}_\theta\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\log[g_\theta(Y)]\right) = I_X(\theta) + I_Y(\theta).$$

□

Remark. This is consistent with a notion of "information" since if variables are independent, the information is the sum of the information of each variable. This proof also shows the main reason why the logarithm is in the definition of the Fisher information—it brings a product to a sum.

Proposition 9.38 (Exercise 6.22 in Math 541A notes). Let X be a random variable with distribution from $\{f_\theta : \theta \in \Theta\}$ (densities or mass functions). Let Y be a random variable with distribution from $\{g_\theta : \theta \in \Theta\}$ (densities or mass functions). Then

$$I_{(X,Y)}(\theta) = I_X(\theta) + I_{Y|X=x}(\theta), \qquad \forall \theta \in \Theta, x \in \mathbb{R}.$$

Proof. Recall that $Y \mid X$ has density $f_{X,Y}(x,y)/f_X(x)$ for any fixed x. And if X, Y are discrete random variables, recall that $Y \mid X$ has mass function $\mathbb{P}(X = x, Y = y)/\mathbb{P}(Y = y)$.

$$I_{(X,Y)}(\theta) = \mathrm{Var}_\theta\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\log[f_{\theta(X,Y)}(x,y)]\right) = \mathrm{Var}_\theta\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\log[f_\theta(x)f_{\theta(Y|X=x)}(y)]\right)$$

$$= \mathrm{Var}_\theta\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\log[f_\theta(X)] + \frac{\mathrm{d}}{\mathrm{d}\theta}\log[f_{\theta(Y|X=x)}(y)]\right)$$

Note that $Y \mid X = x$ is independent of X (because we have conditioned out the dependence). Therefore we have

$$= \mathrm{Var}_\theta\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\log[f_\theta(X)]\right) + \mathrm{Var}_\theta\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\log[f_{\theta(Y|X=x)}(y)]\right) = I_X(\theta) + I_{Y|X=x}(\theta).$$

□

Theorem 9.39 (Cramer-Rao/Information Inequality, Theorem 6.23 in Math 541A Notes). Let $X : \Omega \to \mathbb{R}^n$ be a random variable with distribution from a family of multivariable probability densities or probability mass functions $\{f_\theta : \theta \in \Theta\}$ with $\Theta \subseteq \mathbb{R}$. Let $t : \mathbb{R}^n \to \mathbb{R}$ and let $Y := t(X)$ be a statistic. For any $\theta \in \Theta$ let $g(\theta) := \mathbb{E}_\theta Y$. Then

$$\mathrm{Var}_\theta(Y) \geq \frac{|g'(\theta)|^2}{I_X(\theta)}, \qquad \forall \theta \in \Theta.$$

In particular, if Y is unbiased for θ, then $g(\theta) = \theta$, so,

$$\mathrm{Var}_\theta(Y) \geq \frac{1}{I_X(\theta)}, \qquad \forall \theta \in \Theta.$$

Equality occurs for some $\theta \in \Theta$ only when $\frac{\mathrm{d}}{\mathrm{d}\theta} \log f_\theta(x)$ and $Y - \mathbb{E}_\theta Y$ are multiples of each other.

Remark. For a one-parameter family of distributions, the equality case of Theorem 9.39 gives a new way to find a UMVU that avoids any discussion of complete sufficient statistics. This is another way to find a UMVU ($\frac{\mathrm{d}}{\mathrm{d}\theta} \log f_\theta(X)$) that sidesteps the need for a complete sufficient statistic. That is, to find a UMVU. we look for affine functions of $\frac{\mathrm{d}}{\mathrm{d}\theta} \log f_\theta(X)$.

Proof.

$$|g'(\theta)| = \left| \frac{\mathrm{d}}{\mathrm{d}\theta} \int_{\mathbb{R}} f_\theta(x) t(x) dx \right| = \left| \int_{\mathbb{R}} \big(\frac{\mathrm{d}}{\mathrm{d}\theta} \log f_\theta(x)\big) t(x) f_\theta(x) dx \right| = \left| \mathbb{E}_\theta \frac{\mathrm{d}}{\mathrm{d}\theta} \log f_\theta(X) t(X) \right|$$

Note that $\mathbb{E}_\theta \frac{\mathrm{d}}{\mathrm{d}\theta} \log f_\theta(X) = 0$, so this can be written as

$$= \left| \mathrm{Cov}_\theta \left(\frac{\mathrm{d}}{\mathrm{d}\theta} \log f_\theta(X), t(X) \right) \right|$$

Then by Remark 1.63 in math 541A notes, by the Cauchy-Schwarz inequality,

$$\mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)] \leq \sqrt{\mathrm{Var}_\theta(X)\mathrm{Var}_\theta(Y)},$$

so we have

$$\left| \mathrm{Cov}_\theta \left(\frac{\mathrm{d}}{\mathrm{d}\theta} \log f_\theta(X), t(X) \right) \right| \leq \sqrt{\mathrm{Var}_\theta \left(\frac{\mathrm{d}}{\mathrm{d}\theta} \log f_\theta(X) \right) \mathrm{Var}_\theta(Y)} = \sqrt{I_X(\theta)} \sqrt{\mathrm{Var}_\theta(Y)}$$

$$\iff |g'(\theta)|^2 \leq I_X(\theta) \mathrm{Var}_\theta(Y) \iff \mathrm{Var}_\theta(Y) \geq \frac{|g'(\theta)|^2}{I_X(\theta)}, \qquad \forall \theta \in \Theta.$$

Recall that equality occurs in the Cauchy-Schwarz inequality if and only if $\frac{\mathrm{d}}{\mathrm{d}\theta} \log f_\theta(x)$ is a constant multiple of $Y - \mathbb{E}_\theta Y$ with probability 1. (See also Corollary 7.3.15 in Casella and Berger [2001, p. 341].)

□

Example 9.13 (Example 6.24). Suppose $f_\theta(x) := \theta x^{\theta-1} \mathbf{1}_{0<x<1}$ for all $x \in \mathbb{R}, \theta > 0$. (This is a beta distribution with $\beta = 1$.) We have

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \log f_\theta(x) = \frac{1}{\theta} + \log x, \qquad \forall 0 < x < 1.$$

A vector $X = (X_1, \ldots, X_n)$ of n independent samples from f_θ is distributed according to the product $\prod_{i=1}^n f_\theta(x_i)$, so that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\log\prod_{i=1}^{n} f_\theta(x_i) = \frac{\mathrm{d}}{\mathrm{d}\theta}\sum_{i=1}^{n}\log f_\theta(x_i) = \sum_{i=1}^{n}\left(\frac{1}{\theta}+\log x_i\right) = n\left(\frac{1}{\theta}+\frac{1}{n}\log\prod_{i=1}^{n} x_i\right), \qquad \forall 0 < x_i < 1, 1 \le i \le n.$$

Then by Theorem 9.39 (Theorem 6.23 in Math 541A notes), any function of $\frac{\mathrm{d}}{\mathrm{d}\theta}\log\prod_{i=1}^{n} f_\theta(X_i)$ (plus a constant) is UMVU of its expectation. So, e.g.

$$Y := -\frac{1}{n}\log\prod_{i=1}^{n} X_i$$

is UMVU of its expectation, and $\mathbb{E}_\theta Y = \theta^{-1}$ since $\mathbb{E}_\theta \frac{\mathrm{d}}{\mathrm{d}\theta}\log\prod_{i=1}^{n} f_\theta(X_i) = \mathbb{E}n\left(\frac{1}{\theta}+\frac{1}{n}\log\prod_{i=1}^{n} x_i\right) = 0$.

Definition 9.19 (**Efficiency, Defintion 6.25 in Math 541A notes**). Let $X : \Omega \to \mathbb{R}$ be a random variable with distribution from a family of multivariable probability densities or probability mass functions $\{f_\theta : \theta \in \Theta\}$ with $\Theta \in \mathbb{R}$. Let $t : \mathbb{R}^n \to \mathbb{R}$ and let $Y := t(X)$ be a statistic. Define the **efficiency** of Y to be

$$\frac{1}{I_X(\theta)\mathrm{Var}_\theta(Y)}, \qquad \forall \theta \in \Theta$$

if this quantity exists and is finite. If Z is another statistic, we define the **relative efficiency** of Y to Z to be

$$\frac{I_X(\theta)\mathrm{Var}_\theta(Z)}{I_X(\theta)\mathrm{Var}_\theta(Y)} = \frac{\mathrm{Var}_\theta(Z)}{\mathrm{Var}_\theta(Y)}, \qquad \forall \theta \in \Theta.$$

9.4.4 Bayes Estimation

In Bayes estimation, the parameter $\theta \in \Theta$ is regarded as a random variable Ψ. The distribution of Ψ reflects our prior knowledge about the probable values of Ψ. Then, given that $\Psi = \theta$, the conditional distribution of $X \mid \Psi = \theta$ is assumed to be $\{f_\theta : \theta \in \Theta\}$, where $f_\theta : \mathbb{R}^n \to [0, \infty)$. Suppose $t : \mathbb{R}^n \to \mathbb{R}^k$ and we have a statistic $Y := t(X)$ and a loss function $\ell : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$. Let $g : \Theta \to \mathbb{R}^k$.

Definition 9.20 (**Bayes estimator, Defintion 6.26 in Math 541A notes**). A **Bayes estimator** Y for $g(\theta)$ with respect to Ψ is defined such that

$$\mathbb{E}\ell(g(\Psi), Y) \le \mathbb{E}\ell(g(\Psi), Z)$$

for all estimators Z. Here the expectation is with respect to both Ψ and Y. Note that we have not made any assumptions about bias for Y or Z. To find a Bayes estimator, it is sufficient to minimize the conditional risk.

Remark. $t(X)$ can depend on Ψ.

Proposition 9.40 (Proposition 6.27 in Math 541A notes). Suppose there exists $t : \mathbb{R}^k \to \mathbb{R}$ such that for almost every $x \in \mathbb{R}^n$, $Y := t(X)$ minimizes

$$\mathbb{E}(\ell(g(\Psi), Z) \mid X = x)$$

over all estimators Z. Then $t(X)$ is a Bayes estimator for $g(\theta)$ with respect to Ψ.

Proof. By assumption,

$$\mathbb{E}(\ell(g(\Psi), t(X)) \mid X = x) \leq \mathbb{E}(\ell(g(\Psi), Y) \mid X = x)$$

for any estimator Y and for almost every x. Taking expected values of both sides, we get

$$\mathbb{E}\ell(g(\Psi, t(X)) \leq \mathbb{E}\ell(g(\Psi, Y).$$

□

Example 9.14 (Example 6.29 in Math 541A notes). Suppose $n = 1, g(\theta) = \theta$, and $\ell(\Psi, Y) = (\Psi - Y)^2$. The minimum value of

$$\mathbb{E}[(\Psi - Y(X))^2 \mid X = x] = \mathbb{E}(\Psi^2 - 2\Psi t(X) + (t(X))^2 \mid X = x)$$

$$= \mathbb{E}(\Psi^2 \mid X = x) - 2t(X)\mathbb{E}(\Psi \mid X = x) + (t(X))^2$$

(where we remove expressions with x from the expectation because we take x to be fixed) occurs when $t(x) = \mathbb{E}(\Psi \mid X = x)$. So in this specific case the Bayes estimator is $Y = t(X) = \mathbb{E}(\Psi \mid X)$.

Given that $\Psi = \theta < 0$, suppose X is uniform on the interval $[0, \theta]$. (Suppose X is a single sample $n = 1$ from this distribution.) Also assume that Ψ has the gamma distribution with parameters $\alpha = 2$ and $\beta = 1$, so that Ψ has density $\theta e^{-\theta}\mathbf{1}_{\{\theta>0\}}$. The joint distribution of X and Ψ is then

$$f_{\Psi,X}(\theta, x) := \frac{1}{\theta}\mathbf{1}_{\{0<x<\theta\}}\theta e^{-\theta}\mathbf{1}_{\{\theta>0\}} = \mathbf{1}_{\{0<x<\theta\}}e^{-\theta}.$$

The marginal distribution of X is then

$$f_X(x) = \mathbf{1}_{\{x>0\}}\int_{-\infty}^{\infty} f_{\Psi,X}(\theta, x)d\theta = \mathbf{1}_{\{x>0\}}\int_{x}^{\infty} e^{-\theta}d\theta = \mathbf{1}_{\{x>0\}}e^{-x}.$$

So the conditional distribution of Ψ given X is

$$f_{\Psi|X=x}(\theta \mid x) = \frac{f_{\Psi,X}(\theta, x)}{f_X(x)} = \frac{e^{-\theta}\mathbf{1}_{\{0<x<\theta\}}}{e^{-x}\mathbf{1}_{\{x>0\}}} = e^{x-\theta}\mathbf{1}_{\{0<x<\theta\}}.$$

So,

$$\mathbb{E}(\Psi \mid X = x) = \int_{-\infty}^{\infty} \theta f_{\Psi|X=x}(\theta \mid x) d\theta = e^x \int_x^{\infty} \theta e^{-\theta} d\theta = e^x((x+1)e^{-x}) = x + 1.$$

So the Bayes estimator is $Y = t(X) = \mathbb{E}(\Psi \mid X) = X + 1$. This estimator minimizes $\mathbb{E}(Y - Z)^2$ over all estimators Z. $(\ell(a, b) = (a - b)^2, g(\theta) = \theta, \mathbb{E}\ell(g(\Psi), Z))$.

In contrast, the UMVU for one sample is $2X$ by Theorem 9.35, since $2X$ is complete sufficient and unbiased for θ and $\mathbb{E}_\theta(2X \mid 2X) = 2X$. ($X$ uniform on $[0, \theta]$, θ unknown, $\mathbb{E}_\theta X = \theta/2, \forall\, \theta < 0$.)

(Not obvious) For n samples, $(1 + n^{-1})X_{(n)}$ is the UMVU for θ. Note that $2X$ is recovered when $n = 1$. Remarks: this estimator seems to be sufficient because you could factorize it as in the Factorization Theorem (Theorem 9.5).

Exercise 19 (2018 DSO Statistics Group In-Class Screening Exam, Question 3). Suppose that given the vector μ, the random vector X has a normal distribution in $\mathbb{R}^n$ with mean μ and identity covariance matrix. We want to make inference about $\|\mu\|^2$.

(a) Find an unbiased estimate of $\|\mu\|^2$. Call this estimator $\hat{\delta}_{\text{unbiased}}$.

(b) Suppose that a Bayesian has a proper prior distribution for μ that is Gaussian with mean vector 0 and covariance kI, where k is any fixed positive real number and I is the identity matrix. He wants to minimize mean squared error (MSE). The estimator minimizing the MSE is the posterior mean of $\|\mu\|^2$, i.e., $\mathbb{E}(\|\mu\|^2 \mid X)$. Find this estimator. Call this estimator $\hat{\delta}_{\text{proper}}$.

(c) Suppose now the Bayesian uses the uniform prior (which is also called a "flat" or "noninformative" prior) for μ. Report $\mathbb{E}(\|\mu\|^2 \mid X)$ in this case. Call it $\hat{\delta}_{\text{flat}}$. Report $\hat{\delta}_{\text{flat}} - \hat{\delta}_{\text{unbiased}}$.

(d) Now, if the true distribution of μ is indeed Gaussian with mean vector 0 and covariance kI, then show that with respect to the unconditional (i.e. marginal) distribution of X, the Bayes estimator $\hat{\delta}_{\text{proper}}$ is closer in Euclidean distance to $\hat{\delta}_{\text{unbiased}}$ than it is to $\hat{\delta}_{\text{flat}}$ when n is large. That is, show

$$\mathbb{E}\left(\hat{\delta}_{\text{proper}} - \hat{\delta}_{\text{unbiased}}\right)^2 < \mathbb{E}\left(\hat{\delta}_{\text{flat}} - \hat{\delta}_{\text{unbiased}}\right)^2$$

for large n, where the expectation is over the unconditional distribution of X, which is

$$\int_{\mathbb{R}^n} f(x \mid \mu)\pi(\mu)\; d\mu$$

with $f(x \mid \mu) = \mathcal{N}_n(\mu, I$ and $\pi(\mu) = \mathcal{N}_n(0, kI)$. (Hint: let $\hat{D} = \hat{\delta}_{\text{proper}} - \hat{\delta}_{\text{unbiased}}$. Compute the mean and variance of $\hat{D}$ under the unconditional distribution of X.)

Solution.

(a) We have

$$X \mid \mu \sim \mathcal{N}(\mu, \boldsymbol{I}_n)$$

Let $X = (X_1, \ldots, X_n)^T$ and let $\mu = (\mu_1, \ldots, \mu_n)^T$. Notice that

$$\mathbb{E}(\boldsymbol{X}^T\boldsymbol{X}) = \mathbb{E}[\mathbb{E}(\boldsymbol{X}^T\boldsymbol{X} \mid \mu)] = \mathbb{E}\left[\mathbb{E}\left(X_1^2 + X_2^2 + \ldots + X_n^2 \mid \mu\right)\right] = \mathbb{E}\left[\sum_{i=1}^n \mathbb{E}(X_i^2 \mid \mu)\right]$$

$$= \mathbb{E}\left[\sum_{i=1}^n \text{Var}(X_i \mid \mu) + \mathbb{E}(X_i \mid \mu)^2\right] = \mathbb{E}\left[\sum_{i=1}^n 1 + \mu_i^2\right] = n + \mathbb{E}\|\mu\|_2^2$$

$$\implies \mathbb{E}(\boldsymbol{X}^T\boldsymbol{X} - n) = \mathbb{E}\|\mu\|_2^2$$

Therefore $\boxed{\hat{\delta}_{\text{unbiased}} = \boldsymbol{X}^T\boldsymbol{X} - n}$ is unbiased for $\mathbb{E}\|\mu\|_2^2$ (and given μ it is unbiased for $\|\mu\|_2^2$).

(b) We will begin by finding the posterior distribution of μ. The prior distribution of μ is

$$f(\boldsymbol{\mu}) = (2\pi)^{-n/2}|k\boldsymbol{I}_n|^{-1/2} \cdot \exp\left(-\frac{1}{2}\boldsymbol{\mu}^T(k\boldsymbol{I}_n)^{-1}\boldsymbol{\mu}\right) = \frac{1}{\sqrt{(2\pi k)^n}} \exp\left(-\frac{1}{2k}\boldsymbol{\mu}^T\boldsymbol{\mu}\right).$$

The likelihood is

$$f_{\boldsymbol{X}|\boldsymbol{\mu}}(\boldsymbol{x} \mid \boldsymbol{\mu}) = (2\pi)^{-n/2}|\boldsymbol{I}_n|^{-1/2} \cdot \exp\left(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})^T(\boldsymbol{I}_n)^{-1}(\boldsymbol{x} - \boldsymbol{\mu})\right)$$

$$= (2\pi)^{-n/2} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})^T(\boldsymbol{x} - \boldsymbol{\mu})\right).$$

So the unconditional distribution of $\boldsymbol{X}$ is

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \int_{\mathbb{R}^n} f_{\boldsymbol{X}|\boldsymbol{\mu}}(\boldsymbol{x} \mid \boldsymbol{\mu}) f(\boldsymbol{\mu}) \, d\boldsymbol{\mu}$$

$$= \int_{\mathbb{R}^n} (2\pi)^{-n/2} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})^T(\boldsymbol{x} - \boldsymbol{\mu})\right) \cdot \frac{1}{\sqrt{(2\pi k)^n}} \exp\left(-\frac{1}{2k}\boldsymbol{\mu}^T\boldsymbol{\mu}\right) \, d\boldsymbol{\mu}$$

$$= \frac{1}{(2\pi\sqrt{k})^n} \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\left[\boldsymbol{\mu}^T\boldsymbol{\mu} + \frac{1}{k}\boldsymbol{\mu}^T\boldsymbol{\mu} - 2\boldsymbol{x}^T\boldsymbol{\mu} + \boldsymbol{x}^T\boldsymbol{x}\right]\right) \, d\boldsymbol{\mu}$$

$$= \frac{1}{(2\pi\sqrt{k})^n} \int_{\mathbb{R}^n} \exp\left(-\frac{k+1}{2k}\left[\boldsymbol{\mu}^T\boldsymbol{\mu} - \frac{2k}{k+1}\boldsymbol{x}^T\boldsymbol{\mu}\right] - \frac{1}{2}\boldsymbol{x}^T\boldsymbol{x}\right) \, d\boldsymbol{\mu}$$

$$= \frac{1}{(2\pi\sqrt{k})^n} \int_{\mathbb{R}^n} \exp\left(-\frac{k+1}{2k}\left[\boldsymbol{\mu}^T\boldsymbol{\mu} - \frac{2k}{k+1}\boldsymbol{x}^T\boldsymbol{\mu} + \left(\frac{k}{k+1}\right)^2\boldsymbol{x}^T\boldsymbol{x}\right] - \left(-\frac{k+1}{2k}\right)\left(\frac{k}{k+1}\right)^2\boldsymbol{x}^T\boldsymbol{x} - \frac{1}{2}\boldsymbol{x}^T\boldsymbol{x}\right) \, d\boldsymbol{\mu}$$

$$= \frac{1}{(2\pi\sqrt{k})^n} \int_{\mathbb{R}^n} \exp\left(-\frac{k+1}{2k} \left[\left(\boldsymbol{\mu} - \frac{k}{k+1}\boldsymbol{x} \right)^T \left(\boldsymbol{\mu} - \frac{k}{k+1}\boldsymbol{x} \right) \right] - \frac{1}{2} \left[- \left(\frac{k}{k+1} \right) \boldsymbol{x}^T \boldsymbol{x} + \frac{k+1}{k+1} \boldsymbol{x}^T \boldsymbol{x} \right] \right) \, d\boldsymbol{\mu}$$

$$= \frac{1}{(2\pi\sqrt{k})^n} \int_{\mathbb{R}^n} \exp\left(-\frac{k+1}{2k} \left[\left(\boldsymbol{\mu} - \frac{k}{k+1}\boldsymbol{x} \right)^T \left(\boldsymbol{\mu} - \frac{k}{k+1}\boldsymbol{x} \right) \right] \right) \exp\left(-\frac{1}{2} \frac{1}{k+1} \boldsymbol{x}^T \boldsymbol{x} \right) \, d\boldsymbol{\mu}$$

$$= \frac{1}{\sqrt{(2\pi k)^n}} \exp\left(-\frac{1}{2} \frac{1}{k+1} \boldsymbol{x}^T \boldsymbol{x} \right) \left(\frac{k}{k+1} \right)^{n/2}$$

$$\cdot \int_{\mathbb{R}^n} \frac{1}{\sqrt{(2\pi)^n}} \cdot \left(\frac{k}{k+1} \right)^{-n/2} \exp\left(-\frac{1}{2} \left(\boldsymbol{\mu} - \frac{k}{k+1}\boldsymbol{x} \right)^T \left(\frac{k}{k+1} \boldsymbol{I}_n \right)^{-1} \left(\boldsymbol{\mu} - \frac{k}{k+1}\boldsymbol{x} \right) \right) \, d\boldsymbol{\mu}$$

The second row is the integral over $\mathbb{R}^n$ of an n-dimensional multivariate Gaussian distribution with mean $k/(k+1)\boldsymbol{x}$ and covariance $k/(k+1)\boldsymbol{I}_n$, so it equals 1. Then we are left with

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^n}} \frac{1}{k^{n/2}} \exp\left(-\frac{1}{2} \frac{1}{k+1} \boldsymbol{x}^T \boldsymbol{x} \right) \left(\frac{k}{k+1} \right)^{n/2}$$

$$= \frac{1}{\sqrt{(2\pi)^n}} \left[(k+1)^n \right]^{-1/2} \exp\left(-\frac{1}{2} \boldsymbol{x}^T \left((k+1) \boldsymbol{I}_n \right)^{-1} \boldsymbol{x} \right) \tag{9.17}$$

which is the density of an n-dimensional multivariate Gaussian random variable with mean $\mathbf{0}$ and covariance $(k+1)\boldsymbol{I}_n$. Therefore the posterior distribution of $\boldsymbol{\mu}$ is

$$f_{\boldsymbol{\mu}|\boldsymbol{X}}(\boldsymbol{\mu} \mid \boldsymbol{x}) = \frac{f_{\boldsymbol{X}|\boldsymbol{\mu}}(\boldsymbol{x} \mid \boldsymbol{\mu}) f(\boldsymbol{\mu})}{f_{\boldsymbol{X}}(\boldsymbol{x})}$$

$$= \left[\frac{1}{(2\pi\sqrt{k})^n} \exp\left(-\frac{1}{2} \left[\boldsymbol{\mu}^T \boldsymbol{\mu} + \frac{1}{k} \boldsymbol{\mu}^T \boldsymbol{\mu} - 2\boldsymbol{x}^T \boldsymbol{\mu} + \boldsymbol{x}^T \boldsymbol{x} \right] \right) \right] \Big/ \left[\frac{1}{\sqrt{(2\pi)^n}} \left[(k+1)^n \right]^{-1/2} \exp\left(-\frac{1}{2} \frac{1}{k+1} \boldsymbol{x}^T \boldsymbol{x} \right) \right]$$

$$= \frac{1}{\sqrt{(2\pi)^n}} \left(\frac{k+1}{k} \right)^{n/2} \exp\left(-\frac{k+1}{2k} \left[\left(\boldsymbol{\mu} - \frac{k}{k+1}\boldsymbol{x} \right)^T \left(\boldsymbol{\mu} - \frac{k}{k+1}\boldsymbol{x} \right) \right] - \frac{1}{2} \frac{1}{k+1} \boldsymbol{x}^T \boldsymbol{x} + \frac{1}{2} \frac{1}{k+1} \boldsymbol{x}^T \boldsymbol{x} \right)$$

$$= \frac{1}{\sqrt{(2\pi)^n}} \left(\frac{k+1}{k} \right)^{n/2} \exp\left(-\frac{k+1}{2k} \left(\boldsymbol{\mu} - \frac{k}{k+1}\boldsymbol{x} \right)^T \left(\boldsymbol{\mu} - \frac{k}{k+1}\boldsymbol{x} \right) \right)$$

$$= \frac{1}{\sqrt{(2\pi)^n}} \cdot \left(\frac{k}{k+1} \right)^{-n/2} \exp\left(-\frac{1}{2} \left(\boldsymbol{\mu} - \frac{k}{k+1}\boldsymbol{x} \right)^T \left(\frac{k}{k+1} \boldsymbol{I}_n \right)^{-1} \left(\boldsymbol{\mu} - \frac{k}{k+1}\boldsymbol{x} \right) \right)$$

which is an n-dimensional multivariate Gaussian distribution with mean $k/(k+1)\boldsymbol{x}$ and covariance $k/(k+1)\boldsymbol{I}_n$. That is, conditional on $\boldsymbol{X}$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$, where

$$\mu_i \mid \boldsymbol{X} \overset{i.i.d.}{\sim} \mathcal{N}\left(\frac{k}{k+1} X_i, \frac{k}{k+1} \right).$$

$$\iff \left(\mu_i - \frac{k}{k+1}X_i\right)\frac{k+1}{k} \mid \boldsymbol{X} \overset{i.i.d.}{\sim} \mathcal{N}(0,1) \iff \frac{k+1}{k}\mu_i - X_i \mid \boldsymbol{X} \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$$

$$\implies \mathbb{E}\left(\left[\frac{k+1}{k}\mu_i - X_i\right]^2 \mid \boldsymbol{X}\right) = 1 \iff \mathbb{E}\left(\left[\frac{k+1}{k}\right]^2 \mu_i^2 + X_i^2 - 2\frac{k+1}{k}\mu_i X_i \mid \boldsymbol{X}\right) = 1 \tag{9.18}$$

So,

$$\hat{\delta}_{\text{proper}} = \mathbb{E}\left(\|\boldsymbol{\mu}\|_2^2 \mid \boldsymbol{X}\right) = \mathbb{E}\left(\sum_{i=1}^n \mu_i^2 \mid \boldsymbol{X}\right) = \left[\frac{k}{k+1}\right]^2 \mathbb{E}\left(\sum_{i=1}^n \left[\frac{k+1}{k}\right]^2 \mu_i^2 \mid \boldsymbol{X}\right)$$

$$= \left[\frac{k}{k+1}\right]^2 \mathbb{E}\left(\sum_{i=1}^n \left[\frac{k+1}{k}\right]^2 \mu_i^2 + X_i^2 - 2\frac{k+1}{k}\mu_i X_i \mid \boldsymbol{X}\right) - \left[\frac{k}{k+1}\right]^2 \mathbb{E}\left(\sum_{i=1}^n X_i^2 - 2\frac{k+1}{k}\mu_i X_i \mid \boldsymbol{X}\right)$$

$$= \left[\frac{k}{k+1}\right]^2 - \left[\frac{k}{k+1}\right]^2 \sum_{i=1}^n X_i^2 + 2\frac{k+1}{k}\left[\frac{k}{k+1}\right]^2 \sum_{i=1}^n X_i \mathbb{E}\left(\mu_i \mid \boldsymbol{X}\right)$$

$$= \left[\frac{k}{k+1}\right]^2 - \left[\frac{k}{k+1}\right]^2 \sum_{i=1}^n X_i^2 + 2\frac{k}{k+1}\sum_{i=1}^n X_i \cdot \frac{k}{k+1}X_i$$

$$= \left[\frac{k}{k+1}\right]^2 + 2\left[\frac{k}{k+1}\right]^2 \sum_{i=1}^n X_i^2 - \left[\frac{k}{k+1}\right]^2 \sum_{i=1}^n X_i^2 = \left[\frac{k}{k+1}\right]^2 \left(1 + \boldsymbol{X}^T\boldsymbol{X}\right).$$

(c) We will again begin by finding the posterior distribution of μ. The (improper) prior distribution of μ is constant; that is, for some $c \in \mathbb{R}$,

$$f(\boldsymbol{\mu}) = c, \qquad \forall \boldsymbol{\mu} \in \mathbb{R}^n.$$

The likelihood is

$$f_{\boldsymbol{X}|\boldsymbol{\mu}}(\boldsymbol{x} \mid \boldsymbol{\mu}) = (2\pi)^{-n/2}|\boldsymbol{I}_n|^{-1/2} \cdot \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T(\boldsymbol{I}_n)^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right)$$

$$= (2\pi)^{-n/2}\exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T(\boldsymbol{x}-\boldsymbol{\mu})\right).$$

So the unconditional distribution of $\boldsymbol{X}$ is

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \int_{\mathbb{R}^n} f_{\boldsymbol{X}|\boldsymbol{\mu}}(\boldsymbol{x} \mid \boldsymbol{\mu}) f(\boldsymbol{\mu})\, d\boldsymbol{\mu} = c\int_{\mathbb{R}^n} (2\pi)^{-n/2}\exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T(\boldsymbol{x}-\boldsymbol{\mu})\right)\, d\boldsymbol{\mu}$$

$$= c\int_{\mathbb{R}^n} (2\pi)^{-n/2}\exp\left(-\frac{1}{2}(\boldsymbol{\mu}-\boldsymbol{x})^T(\boldsymbol{\mu}-\boldsymbol{x})\right)\, d\boldsymbol{\mu}$$

The expression inside the integral is the density of a Gaussian random variable with mean $\boldsymbol{x}$ and covariance $\boldsymbol{I}_n$, so the integral evaluates to 1. Therefore the unconditional distribution of $\boldsymbol{X}$ is also flat. Therefore the posterior distribution of $\boldsymbol{\mu}$ is the same as the likelihood:

$$f_{\boldsymbol{\mu}|\boldsymbol{X}}(\boldsymbol{\mu} \mid \boldsymbol{x}) = \frac{f_{\boldsymbol{X}|\boldsymbol{\mu}}(\boldsymbol{x} \mid \boldsymbol{\mu}) f(\boldsymbol{\mu})}{f_{\boldsymbol{X}}(\boldsymbol{x})} = (2\pi)^{-n/2} \exp\left(-\frac{1}{2}(\boldsymbol{\mu} - \boldsymbol{x})^T(\boldsymbol{\mu} - \boldsymbol{x})\right);$$

that is, conditional on $\boldsymbol{X}$, $\boldsymbol{\mu}$ is normally distributed with mean $\boldsymbol{X}$ and covariance $\boldsymbol{I}_n$. So conditional on $\boldsymbol{X}$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$, where

$$\mu_i \mid \boldsymbol{X} \overset{i.i.d.}{\sim} \mathcal{N}(X_i, 1).$$

$$\iff \mu_i - X_i \mid \boldsymbol{X} \overset{i.i.d.}{\sim} \mathcal{N}(0, 1) \implies \mathbb{E}\left([\mu_i - X_i]^2 \mid \boldsymbol{X}\right) = 1 \iff \mathbb{E}\left(\mu_i^2 + X_i^2 - 2\mu_i X_i \mid \boldsymbol{X}\right) = 1 \tag{9.19}$$

So,

$$\hat{\delta}_{\text{flat}} = \mathbb{E}\left(\|\boldsymbol{\mu}\|_2^2 \mid \boldsymbol{X}\right) = \mathbb{E}\left(\sum_{i=1}^n \mu_i^2 \mid \boldsymbol{X}\right) = \mathbb{E}\left(\sum_{i=1}^n \mu_i^2 + X_i^2 - 2\mu_i X_i \mid \boldsymbol{X}\right) - \mathbb{E}\left(\sum_{i=1}^n X_i^2 - 2\mu_i X_i \mid \boldsymbol{X}\right)$$

$$= 1 - \sum_{i=1}^n X_i^2 + 2\sum_{i=1}^n X_i \mathbb{E}(\mu_i \mid \boldsymbol{X}) = 1 - \sum_{i=1}^n X_i^2 + 2\sum_{i=1}^n X_i^2 = 1 + \boldsymbol{X}^T\boldsymbol{X}.$$

So,

$$\hat{\delta}_{\text{flat}} - \hat{\delta}_{\text{unbiased}} = 1 + \boldsymbol{X}^T\boldsymbol{X} - \left(\boldsymbol{X}^T\boldsymbol{X} - n\right) = 1 + n. \tag{9.20}$$

(d) If the true distribution of $\boldsymbol{\mu}$ is the prior from part (b), then the marginal distribution of $\boldsymbol{X}$ is (9.17):

$$= \frac{1}{\sqrt{(2\pi)^n}} \left[(k+1)^n\right]^{-1/2} \exp\left(-\frac{1}{2}\boldsymbol{x}^T\left((k+1)\boldsymbol{I}_n\right)^{-1}\boldsymbol{x}\right);$$

that is, $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{0}, (k+1)\boldsymbol{I}_n)$. (Note that this means $(k+1)^{-1}X_i$ is standard Gaussian and i.i.d. for all $i \in \{1, \ldots, n\}$.) Per the suggestion, let

$$\hat{D} = \hat{\delta}_{\text{proper}} - \hat{\delta}_{\text{unbiased}} = \left[\frac{k}{k+1}\right]^2\left(1 + \boldsymbol{X}^T\boldsymbol{X}\right) - \left(\boldsymbol{X}^T\boldsymbol{X} - n\right) = \frac{k^2 - (k^2 + 2k + 1)}{(k+1)^2}\boldsymbol{X}^T\boldsymbol{X} + \left[\frac{k}{k+1}\right]^2 + n$$

$$= -\frac{2k+1}{(k+1)^2}\boldsymbol{X}^T\boldsymbol{X} + \left[\frac{k}{k+1}\right]^2 + n \tag{9.21}$$

Since from (9.20) we have

$$\mathbb{E}\left(\hat{\delta}_{\text{proper}} - \hat{\delta}_{\text{unbiased}}\right)^2 = n^2 + 2n + 1,$$

we seek

$$\mathbb{E}\left(\hat{\delta}_{\text{flat}} - \hat{\delta}_{\text{unbiased}}\right)^2 = \mathbb{E}(\hat{D}^2) = \text{Var}(\hat{D}) + \left[\mathbb{E}(\hat{D})\right]^2.$$

Note that since $(k+1)^{-1} X_i \overset{i.i.d.}{\sim} \mathcal{N}(0,1)$ for all $i \in \{1, \ldots, n\}$,

$$\sum_{i=1}^n \left((k+1)^{-1} X_i\right)^2 \sim \chi_n^2,$$

so

$$\mathbb{E}\left[\sum_{i=1}^n \left(\frac{1}{k+1} X_i\right)^2\right] = n \iff \frac{1}{(k+1)^2}\mathbb{E}\left[\sum_{i=1}^n X_i^2\right] = n \iff \mathbb{E}\left[\sum_{i=1}^n X_i^2\right] = n(k+1)^2,$$

and

$$\text{Var}\left[\sum_{i=1}^n \left(\frac{1}{k+1} X_i\right)^2\right] = 2n \iff \frac{1}{(k+1)^4}\text{Var}\left[\sum_{i=1}^n X_i^2\right] = 2n \iff \text{Var}\left[\sum_{i=1}^n X_i^2\right] = 2n(k+1)^4.$$

Under the unconditional distribution of $\boldsymbol{X}$,

$$\mathbb{E}\left(\hat{D}\right) = \mathbb{E}\left(-\frac{2k+1}{(k+1)^2}\boldsymbol{X}^T\boldsymbol{X} + \left[\frac{k}{k+1}\right]^2 + n\right) = -\frac{2k+1}{(k+1)^2}\mathbb{E}\left(\boldsymbol{X}^T\boldsymbol{X}\right) + \left[\frac{k}{k+1}\right]^2 + n$$

$$= -\frac{2k+1}{(k+1)^2}\mathbb{E}\left(\sum_{i=1}^n X_i^2\right) + \left[\frac{k}{k+1}\right]^2 + n = -\frac{2k+1}{(k+1)^2} n(k+1)^2 + \left[\frac{k}{k+1}\right]^2 + n$$

$$= -n\frac{(2k+1)(k^2+2k+1)+k^2}{(k+1)^2} + n = n\left[\frac{k^2+2k+1}{(k+1)^2} - \frac{2k^3+4k^2+2k+k^2+2k+1+k^2}{(k+1)^2}\right]$$

$$= n\left[\frac{-2k^3-5k^2-2k}{(k+1)^2}\right] = -n\left[\frac{2k^3+5k^2+2k}{(k+1)^2}\right]$$

Next,

$$\text{Var}(\hat{D}) = \text{Var}\left(-\frac{2k+1}{(k+1)^2}\boldsymbol{X}^T\boldsymbol{X} + \left[\frac{k}{k+1}\right]^2 + n\right) = \frac{(2k+1)^2}{(k+1)^4}\text{Var}\left(\sum_{i=1}^n X_i^2\right)$$

$$= \frac{(2k+1)^2}{(k+1)^4} 2n(k+1)^4 = 2n(2k+1)^2$$

So

$$\mathbb{E}\left(\hat{\delta}_{\text{flat}} - \hat{\delta}_{\text{unbiased}}\right)^2 = 2n(2k+1)^2 + \left(-n\left[\frac{2k^3+5k^2+2k}{(k+1)^2}\right]\right)^2 = n^2\left[\frac{2k^3+5k^2+2k}{(k+1)^2}\right]^2 + n\cdot 2(2k+1)^2$$

$$\approx 4k^2n^2 + 8k^2 n,$$

so in general when $k \geq 1$ and n is large,, $\mathbb{E}\left(\hat{\delta}_{\text{flat}} - \hat{\delta}_{\text{unbiased}}\right)^2 > \mathbb{E}\left(\hat{\delta}_{\text{proper}} - \hat{\delta}_{\text{unbiased}}\right)^2$; that is, the flat prior Bayes estimator is further from the unbiased estimator than the proper prior Bayes estimator.

9.4.5 Method of Moments

Definition 9.21 (Consistency, Definition 6.30 in Math 541A notes). Let $\{f_\theta : \theta \in \Theta\}$ be a family of distributions. Let $Y_1, Y_2, \ldots$ be a sequence of estimators of $g(\theta)$. We say that $Y_1, Y_2, \ldots$ is **consistent** for $g(\theta)$ if for any $\theta \in \Theta$, $Y_1, Y_2, \ldots$ converges in probability to the constant value $g(\theta)$ with respect to the probability distribution f_θ. That is, $Y_n \xrightarrow{p} g(\theta)$. (Typically we will take Y_n to be a function of a random sample of size n for all $n \geq 1$.)

Example 9.15 (Example 6.31 in Math 541A notes). Let $X_1, \ldots, X_n$ be a random sample of size n with distribution f_θ. The Weak Law of Large Numbers (Theorem 8.31) says that the sample mean is consistent when $\mathbb{E}_\theta |X_1| M\infty$ for all $\theta \in \Theta$. More generally, if $j \geq 1$ is a positive integer such that $\mathbb{E}_\theta |X_1|^j < \infty$ for all $\theta \in \Theta$, then the jth sample moment

$$M_j = M_{j,n}(\theta) := \frac{1}{n} \sum_{i=1}^n X_i^j$$

is a consistent estimator for $\mu_j(\theta) := \mathbb{E} X_1^j$.

Definition 9.22 (Method of Moments, Definition 6.32 in Math 541A notes). Let $g : \Theta \to \mathbb{R}^k$. Suppose we want to estimate $g(\theta)$ for any $\theta \in \Theta$. Suppose there exists $h : \mathbb{R}^j \to \mathbb{R}^k$ such that $g(\theta) = h(\mu_1, \ldots, \mu_j)$. Then the estimator

$$h(M_1, \ldots, M_j)$$

is a **method of moments** estimator for $g(\theta)$, where M_j is the jth sample moment

$$M_j = M_{j,n}(\theta) := \frac{1}{n} \sum_{i=1}^n X_i^j$$

Example 9.16 (Example 6.33 in Math 541A notes). Recall that the standard deviation is

$$\sqrt{\operatorname{Var}(X)} = \sqrt{\mathbb{E}(X^2) - [\mathbb{E}(X)]^2}.$$

To estimate the standard deviation, we can use $\Theta = \mathbb{R} \times (0, \infty) = \{(\mu_1, \mu_2) : \mu_1 \in \mathbb{R}, \mu_2 > 0\}$, $j = 2$, and $h(\mu_1, \mu_2) = \sqrt{\mu_2 - \mu_1^2}$, so that the method of moments estimator of the standard deviation is $\sqrt{M_2 - M_1^2}$.

Remark. The method of moments estimator is not necessarily unbiased.

9.4.6 Maximum likelihood estimator

Remark. Under some reasonable assumptions, the maximum likelihood estimator is consistent. See Keener for details.

Proposition 9.41 (Math 541A Proposition 6.40). For all $i \in 1, \ldots, n$, if $\Theta \to f_\theta(x_i)$ is strictly log-concave, then $\ell(\theta)$ has at most one maximum value.

Remark. One example of a function whose log likelihood has no maximum value is $\exp(-e^{-\theta})$ (as in an extreme value distribution).

Remark. Intuition of Lemma 6.50 in Math 541A: if distribution follows θ then it is more likely to match distribution function of f_θ than f_ω.

Note on Theorem 6.53: exponential family is an example of a family that satisfies condition (a).

Note from proof:

$$\sqrt{n}\ell_n(\theta) = \frac{1}{\sqrt{n}}\left(\sum_{i=1}^n \log f_\theta(x_i) - \mathbb{E}(\log f_\theta(x_i))\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{I_{X_1}(\theta)}\right)$$

by the Central Limit Theorem (and Assumption 3 for the variance) since $\mathbb{E}(\log f_\theta(x_i)) = 0$. Also,

$$\ell_n''(\theta') = \frac{1}{n}\sum_{i=1}^n \frac{\mathrm{d}^2}{\mathrm{d}[\theta']^2} \log f_{\theta'}(x_i)$$

which explains the applicability of the Weak Law of Large Numbers.

Proposition 9.42 (Stats 100B homework problem). Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a Bernoulli(p) distribution. Let $X = \sum_{i=1}^n X_i$. Then

(a) The maximum likelihood estimator of p is $\hat{p} = X/n$.

(b) The maximum likelihood estimator attains the Cramer-Rao lower bound.

(c) The maximum likelihood estimator is a consistent estimator for p.

(d) $\frac{\hat{p}(1-\hat{p})}{n-1}$ is an unbiased estimator for $\mathrm{Var}(\hat{p}) = p(1-p)/n$.

Proof. a. Bernoulli random variable:

$$P(X_i = x) = p^x(1-p)^{1-x}$$

Assuming independent samples,

$$L = \prod_{i=1}^n p^{X_i}(1-p)^{1-X_i} = p^{\sum_{i=1}^n X_i}(1-p)^{\sum_{i=1}^n 1-X_i}$$

$$\log(L) = \sum_{i=1}^n X_i \log(p) + \left(\sum_{i=1}^n 1 - X_i\right)\log(1-p)$$

$$\frac{\mathrm{d}\log(L)}{\mathrm{d}p} = \frac{1}{p}\sum_{i=1}^n X_i - \frac{1}{1-p}\sum_{i=1}^n (1-X_i) = 0$$

$$\frac{1}{\hat{p}}\sum_{i=1}^n X_i = \frac{1}{1-\hat{p}}\sum_{i=1}^n (1-X_i)$$

$$(1-\hat{p})\sum_{i=1}^n X_i = \hat{p}\sum_{i=1}^n (1-X_i)$$

$$\sum_{i=1}^{n} X_i = \hat{p} \sum_{i=1}^{n} (X_i + 1 - X_i)$$

$$\sum_{i=1}^{n} X_i = n\hat{p}$$

$$\boxed{\hat{p} = \frac{1}{n} \sum_{i=1}^{n} X_i}$$

b.

$$\text{Var}(\hat{p}) = \frac{1}{n^2} \text{Var}\left(\sum_{i=1}^{n} X_i \right)$$

Since X_i are independent, we can write this as

$$\frac{1}{n^2} \sum_{i=1}^{n} \text{Var}(X_i)$$

And since X_i is Bernoulli, $\text{Var}(X_i) = p(1-p)$.

$$= \frac{1}{n^2} \sum_{i=1}^{n} p(1-p) = \frac{1}{n^2} np(1-p) = \boxed{\frac{p(1-p)}{n}}$$

Cramer-Rao lower bound:

$$\text{Var}(\hat{\theta}) \geq 1/\left(-n\mathbb{E}\left[\frac{\partial^2 \log(f(X;\theta))}{\partial \theta^2} \right] \right)$$

$$\frac{\partial}{\partial p} \log(p^x(1-p)^{1-x}) = \frac{\partial}{\partial p}(x\log(p) + (1-x)\log(1-p)) = \frac{x}{p} - \frac{1-x}{1-p}$$

$$\frac{\partial^2 \log(f(X;\theta))}{\partial \theta^2} = \frac{\partial}{\partial p}(\frac{x}{p} - \frac{1-x}{1-p}) = -\frac{x}{p^2} - \frac{1-x}{(1-p)^2}$$

$$\mathbb{E}\left[\frac{\partial^2 \log(f(X;\theta^2))}{\partial \theta} \right] = \mathbb{E}(-\frac{x}{p^2} - \frac{1-x}{(1-p)^2}) = -\frac{1}{p^2}\mathbb{E}(x) - \frac{1}{(1-p)^2}\mathbb{E}(1-x) = -\frac{1}{p^2}p - \frac{1}{(1-p)^2}(1-p)$$

$$= -\frac{1}{p} - \frac{1}{1-p} = -\frac{1-p}{p(1-p)} - \frac{p}{p(1-p)} = \frac{-1}{p(1-p)}$$

$$\implies \text{Var}(\hat{p}) \geq 1/\left(-n(\frac{-1}{p(1-p)}) \right) = \frac{p(1-p)}{n} = \text{Var}(\hat{p})$$

c. (1) **Unbiased:**

$$E\left(\frac{X}{n} \right) = \frac{np}{p} = p$$

(2) $\text{Var}(\hat{\theta}) \to 0$ **as** $n \to \infty$ **:**

$$\lim_{n\to\infty} \text{Var}\left(\frac{X}{n}\right) = \lim_{n\to\infty} \frac{1}{n^2}\sum_{i=1}^{n} \text{Var}(X_i) = \lim_{n\to\infty} \frac{1}{n^2} \cdot np(1-p) = \lim_{n\to\infty} \frac{p(1-p)}{n} = \boxed{0}$$

Therefore $\frac{X}{n}$ is a consistent estimator of p.

d.

$$\mathbb{E}(\hat{\sigma^2}) = \mathbb{E}\left[\frac{1}{n}\left(\frac{X}{n}\big(1-\frac{X}{n}\big)\right)\right] = \mathbb{E}\left[\frac{X(n-X)}{n^3}\right] = \frac{1}{n^3}\mathbb{E}[nX-(X)^2] = \frac{1}{n^2}\mathbb{E}(X) - \frac{1}{n^3}\mathbb{E}(X^2)$$

$$= \frac{1}{n^2}\cdot np - \frac{1}{n^3}\big(\text{Var}(X) + (E(X))^2\big) = \frac{p}{n} - \frac{np(1-p)}{n^3} - \frac{p^2n^2}{n^3} = \frac{pn - p + p^2 - p^2 n}{n^2}$$

$$= \frac{p(n-1+p-pn)}{n^2} = \frac{p(n-1)(1-p)}{n^2}$$

This is a biased estimator since $\text{Var}(X) = \frac{p(1-p)}{n}$ (since X is bionomial).

$$c \cdot \frac{p(n-1)(1-p)}{n^2} = \frac{p(1-p)}{n} \implies \boxed{c = \frac{n}{n-1}}$$

□

Proposition 9.43 (Stats 100B homework problem). Suppose that X follows a geometric distribution and we take an i.i.d. sample of size n. Then the maximum likelihood estimator of p is

$$\hat{p} = \frac{n}{\sum_{i=1}^n X_i} = \frac{1}{\bar{X}}.$$

Proof. Since sample is i.i.d.:

$$L = \prod_{i=1}^{n} p(1-p)^{X_i - 1} = p^n(1-p)^{-n+\sum_{i=1}^n X_i}$$

$$\log(L) = n\log(p) + \left(-n + \sum_{i=1}^{n} X_i\right)\log(1-p)$$

$$\frac{\mathrm{d}\log(L)}{\mathrm{d}p} = \frac{n}{p} - \frac{1}{1-p}\left(-n + \sum_{i=1}^{n} X_i\right) = 0$$

$$\frac{n}{\hat{p}} = \frac{1}{1-\hat{p}}\left(-n + \sum_{i=1}^{n} X_i\right)$$

$$(1-\hat{p})\hat{p} = -n\hat{p} + \hat{p}\sum_{i=1}^{n} X_i$$

$$n = \hat{p} \sum_{i=1}^{n} X_i$$

$$\hat{p} = \frac{n}{\sum_{i=1}^{n} X_i} = \frac{1}{\bar{X}}$$

□

Proposition 9.44 (Stats 100B homework problem). Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a Poisson(λ) distribution. Then

(a) The maximum likelihood estimator of λ is

$$\hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

(b) The variance of the maximum likelihood estimator is

$$\mathrm{Var}(\hat{\lambda}) = \frac{\lambda}{n}$$

(c) The maximum likelihood estimator is a minimum variance unbiased estimator.

(d) The maximum likelihood estimator is consistent.

Proof. (a)

$$f(X_i; \lambda) = \frac{\lambda^{X_i} e^{-\lambda}}{X_i!}$$

Assuming the samples are independent,

$$L = \prod_{i=1}^{n} \frac{\lambda^{X_i} e^{-\lambda}}{X_i!} = \left(e^{-n\lambda} \lambda^{\sum_{i=1}^{n} X_i} \right) / \prod_{i=1}^{n} X_i!$$

$$\log(L) = -n\lambda + \left(\sum_{i=1}^{n} X_i \right) \log(\lambda) - \sum_{i=1}^{n} \log(X_i!)$$

$$\frac{\mathrm{d} \log(L)}{\mathrm{d}\lambda} = -n + \frac{1}{\lambda} \sum_{i=1}^{n} X_i = 0$$

$$\implies \hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} X_i = \bar{x}$$

(b)

$$\mathrm{Var}(\hat{\lambda}) = \mathrm{Var}(\frac{1}{n} \sum_{i=1}^{n} X_i) = \frac{1}{n^2} \mathrm{Var}(\sum_{i=1}^{n} X_i)$$

Since X_i are i.i.d. this can be written as

$$= \frac{1}{n^2} \sum_{i=1}^{n} \mathrm{Var}(X_i) = \frac{1}{n^2} \sum_{i=1}^{n} \lambda = \frac{\lambda}{n}$$

(c) Cramer-Rao lower bound:

$$\mathrm{Var}(\hat{\lambda}) \geq 1/\Bigg(-n\mathbb{E}\Bigg[\frac{\partial^2 \log(f(X;\lambda))}{\partial \lambda^2}\Bigg]\Bigg)$$

$$\log(f(X;\lambda)) = \log\left(\frac{\lambda^{X_i}e^{-\lambda}}{X_i!}\right) = X_i \log(\lambda) - \lambda - \log(X_i!)$$

$$\frac{\partial}{\partial \lambda}\log(f(X;\lambda)) = \frac{1}{\lambda}X_i - 1$$

$$\frac{\partial^2 \log(f(X;\lambda))}{\partial \lambda^2} = -\frac{1}{\lambda^2}X_i$$

$$\mathbb{E}\Bigg[\frac{\partial^2 \log(f(X;\lambda^2))}{\partial \lambda}\Bigg] = -\frac{1}{\lambda^2}\mathbb{E}(X_i) = -\frac{1}{\lambda^2}\lambda = -\frac{1}{\lambda}$$

$$\implies \mathrm{Var}(\hat{\lambda}) \geq 1/\Bigg(-n\mathbb{E}\Bigg[\frac{\partial^2 \log(f(X;\lambda))}{\partial \lambda^2}\Bigg]\Bigg) = \frac{1}{n/\lambda} = \boxed{\frac{\lambda}{n} = \mathrm{Var}(\hat{\lambda})}$$

Since $\mathrm{Var}(\hat{\lambda})$ equals the Cramer-Rao lower bound, $\hat{\lambda}$ is a MVUE.

(d) We already know the MLE is unbiased. To show consistency, we show $\mathrm{Var}(\hat{\theta}) \to 0$ as $n \to \infty$.

$$\lim_{n\to\infty} \mathrm{Var}(\hat{\lambda}) = \lim_{n\to\infty} \frac{\lambda}{n} = \boxed{0}$$

Therefore $\hat{\lambda}$ is a consistent estimator of λ.

□

Proposition 9.45 (Stats 100B homework problem, similar to Math 541A example 6.47). Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a Exponential(λ) distribution. Then the maximum likelihood estimator of λ is

$$\hat{\lambda} = n/\sum_{i=1}^n X_i = \frac{1}{\overline{X}}.$$

Proof.

$$f(X_i;\lambda) = \lambda e^{-\lambda X_i}$$

Assuming the samples are independent,

$$L = \prod_{i=1}^n \lambda e^{-\lambda X_i} = \lambda^n \exp(-\lambda \sum_{i=1}^n X_i)$$

$$\log(L) = n\log(\lambda) + -\lambda \sum_{i=1}^n X_i$$

$$\frac{\mathrm{d}\log(L)}{\mathrm{d}\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} X_i = 0$$

$$\implies \hat{\lambda} = n / \sum_{i=1}^{n} X_i = \frac{1}{\overline{X}}$$

$\square$

Proposition 9.46 (Stats 100B homework problem; similar to Math 541A Example 6.45). Let $X_1, X_2, \ldots, X_n$ be an i.i.d. random sample from a normal population with mean zero and unknown variance σ^2. Then

(a) The maximum likelihood estimator of σ^2 is

$$\hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n} X_i^2.$$

(b) The maximum likelihood estimator of σ^2 is biased, but asymptotically unbiased.

(c) The maximum likelihood estimator of σ^2 has variance

$$\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \frac{2\sigma^4}{n}$$

and is consistent.

(d) The variance of the maximum likelihood estimator of σ^2 reaches the Cramer-Rao lower bound.

(e) The maximum likelihood estimator of μ is

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and it is unbiased and UMVU/MVUE.

Proof. a. Since sample is i.i.d. $\mathcal{N}(0, \sigma^2)$:

$$L = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left[\frac{X_i - \mu}{\sigma}\right]^2\right) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2\right)$$

$$\log(L) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2 = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - (\sigma^2)^{-1}\frac{1}{2}\sum_{i=1}^{n}(X_i - \mu)^2$$

$$\frac{\partial \log(L)}{\partial \sigma^2} = -\frac{n}{2}\frac{1}{\sigma^2} + (\sigma^2)^{-2}\frac{1}{2}\sum_{i=1}^{n}(X_i - \mu)^2 = 0$$

$$\frac{\sum_{i=1}^{n}(X_i - \mu)^2}{(\hat{\sigma^2})^2} = \frac{n}{\hat{\sigma^2}}$$

$$\hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$$

b. **something wrong with this part**

$$\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \frac{1}{n}\mathbb{E}\left(\sum_{i=1}^{n} X_i^2\right)$$

Since the sample is i.i.d., this can be written as

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(X_i^2)$$

Since $X_i \sim \mathcal{N}(0, \sigma^2)$, $X_i^2/\sigma^2 \sim \chi_1^2$. So we have

$$\mathbb{E}\left(\frac{X_i^2}{\sigma^2}\right) = 1$$

$$\frac{1}{\sigma^2}\mathbb{E}(X_i^2) = 1$$

$$\mathbb{E}(X_i^2) = \sigma^2$$

Therefore

$$\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \frac{1}{n}\sum_{i=1}^{n}\ \sigma^2 = \frac{1}{n}n\sigma^2 = \boxed{\sigma^2}$$

However it is asymptotically biased: that is,

$$\lim_{n\to\infty}\frac{\mathbb{E}(\hat{\sigma}^2)}{\sigma^2} = 1$$

c.

$$\text{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \frac{1}{n^2}\text{Var}\left(\sum_{i=1}^{n} X_i^2\right)$$

Since X_i is i.i.d. this can be written as

$$\frac{1}{n^2}\sum_{i=1}^{n}\text{Var}(X_i^2)$$

Again, , $X_i^2/\sigma^2 \sim \chi_1^2$, so we have

$$\text{Var}\left(\frac{X_i^2}{\sigma^2}\right) = 2$$

$$\frac{1}{\sigma^4}\text{Var}(X_i^2) = 2$$

$$\text{Var}(X_i^2) = 2\sigma^4$$

Therefore

$$\text{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \frac{1}{n^2}\sum_{i=1}^{n} 2\sigma^4 = \frac{2n\sigma^4}{n^2} = \boxed{\frac{2\sigma^4}{n}}$$

Test for consistency (already known that estimate is unbiased):

$$\lim_{n\to\infty} \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \lim_{n\to\infty} \frac{2\sigma^4}{n} = \boxed{0}$$

So this is a consistent estimator of σ^2.

d. Cramer-Rao lower bound:

$$\mathrm{Var}(\hat{\theta}) \geq 1/\left(-n\mathbb{E}\left[\frac{\partial^2 \log(f(X;\theta))}{\partial\theta^2}\right]\right)$$

$$\log(f(X;\theta)) = \log\left[\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left[\frac{X_i}{\sigma}\right]^2\right)\right] = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log(\sigma^2) - \frac{1}{2}X_i^2(\sigma^2)^{-1}$$

$$\frac{\partial}{\partial\sigma^2}\left(-\frac{1}{2}\log(2\pi) - \frac{1}{2}\log(\sigma^2) - \frac{1}{2}X_i^2(\sigma^2)^{-1}\right) = -\frac{1}{2\sigma^2} + \frac{1}{2}(X_i)^2(\sigma^2)^{-2}$$

$$\frac{\partial^2 \log(f(X;\theta))}{\partial(\sigma^2)^2} = \frac{1}{2}(\sigma^2)^{-2} - X_i^2(\sigma^2)^{-3}$$

$$\mathbb{E}\left[\frac{\partial^2 \log(f(X;\theta^2))}{\partial\theta}\right] = \mathbb{E}\left[\frac{1}{2}(\sigma^2)^{-2} - X_i^2(\sigma^2)^{-3}\right] = \frac{1}{2\theta^4} - \frac{1}{\theta^6}\mathbb{E}(X_i^2) = \frac{1}{2\theta^4} - \frac{\theta^2}{\theta^6} = -\frac{1}{2\theta^4}$$

$$\implies \mathrm{Var}(\hat{\sigma^2}) \geq 1/\left(-n\mathbb{E}\left[\frac{\partial^2 \log(f(X;\theta))}{\partial\theta^2}\right]\right) = \frac{1}{n/(2\theta^4)} = \boxed{\frac{2\theta^4}{n}}$$

Therefore the variance of this estimator is equal to the Cramer-Rao lower bound.

Alternative solution (Math 541A):

$$I_X(\sigma) = I_{(X_1,\ldots,X_n)}(\sigma) = \text{(by Proposition 9.37 (Proposition 6.21 in Math 541A notes))}\ nI_{X_1}(\sigma)$$

$$= n\mathrm{Var}_\sigma\left(\frac{\mathrm{d}}{\mathrm{d}\sigma}\log\left(\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(X_1-\mu)^2}{2\sigma^2}\right)\right.\right. \text{(by definition of Fisher information)}$$

$$= n\mathrm{Var}_\sigma\left[-\frac{1}{\sigma} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{(-X_1-\mu)^2}{2\sigma^2}\right)\right] = n\sigma^{-6}\mathrm{Var}_\sigma\big((X_1-\mu)^2\big)$$

$$= n\sigma^{-6}\big(\mathbb{E}(X_1-\mu)^4 - \big[\mathbb{E}(X_1-\mu)^2\big]^2\big) = n\sigma^{-6}\sigma^4(3-1) = \frac{2n}{\sigma^2}.$$

By Cramer-Rao,

$$\mathrm{Var}_\sigma(z) \geq \frac{|g'(0)|^2}{I_X(0)}$$

Note that $g(\sigma) = \mathbb{E}Y = \frac{n-1}{n}\sigma^2$ and that $\mathbb{E}\sum_{j=1}^n (X_j - \overline{X})^2 = \sigma^2(n-1)$. If Z is unbiased for σ^2,

$$g'(\sigma) = \frac{2\sigma(n-1)}{n}|g'(\sigma)|^2 = \frac{4\sigma^2(n-1)^2}{n^2}$$

So,

$$\mathrm{Var}_\sigma(z) \geq \frac{4\sigma^2(n-1)^2}{n^2 2n\sigma^{-2}} = \frac{2(n-1)^2\sigma^4}{n^3}.$$

Note that

$$fracn - 1\sigma^2 \sum_{j=1}^{n} (X_j - \overline{X})^2 \sim \chi^2_{n-1}$$

$$\vdots$$

Note that

$$\frac{(n-1)S_n^2}{\sigma^2} \sim \chi_n^2 \implies \mathrm{Var}(\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \overline{X})^2) = (n-1) \cdot 2$$

$$\implies \mathrm{Var}(\hat{\sigma}^2) = \mathrm{Var}_\sigma(\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2) = \frac{1}{n^2}\sigma^4 \mathrm{Var}_\sigma(\sigma^{-2}\sum_{i=1}^{n}(X_i - \overline{X})^2) = \frac{2\sigma^4(n-1)}{n^2}$$

e.

$$\log(L) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - (\sigma^2)^{-1}\frac{1}{2}\sum_{i=1}^{n} X_i^2$$

$$\frac{\partial \log(L)}{\partial \mu} = \sum_{i=1}^{n} \frac{x_i - \mu}{\sigma^2} = 0 \iff n\mu = \sum_{i=1}^{n} x_i \iff \mu = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\implies \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Also note that

$$\mathbb{E}(\hat{\mu}) = \mathbb{E}\frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{n} \cdot n\mu = \mu.$$

□

Example 9.17 (Example 6.46, Math 541A notes). Alternative solution at https://math.stackexchange.com/questions/49543/maximum-estimator-method-more-known-as-mle-of-a-uniform-distribution

Proposition 9.47 (Functional Equivariance of the MLE; Proposition 6.49 from Lecture Notes, Theorem 7.2.10 in Casella and Berger). Let $g : \Theta \to \Theta'$ be a bijection. Suppose Y is the MLE of θ. Then $g(Y)$ is the MLE of $g(\theta)$.

Proof (case when g is invertible). Note that if $\ell(\theta)$ is the likelihood function for θ, then the likelihood function for $g(\theta)$ can be expressed as

$$\ell(g(\theta)) = \prod_{i=1}^{n} f_\theta(x_i \mid g^{-1}(g(\theta))) = \ell(g^{-1}(g(\theta))) = \ell(g^{-1}(\theta'))$$

where $\theta' = g(\theta)$. By definition of the MLE, $Y = t(X_1, \ldots, X_n)$ achieves the maximum value of $\theta \mapsto \ell(\theta)$. Therefore we can equivalently say $g(Y) = g(t(X_1, \ldots, X_n))$ achieves the maximum value of $\theta' \mapsto \ell(g^{-1}(\theta'))$.

(For a proof when g is not invertible, see Theorem 7.2.10 in Casella and Berger (p.320).)

□

Proposition 9.48 (Math 541A Homework Problem). Let $X_1, \ldots, X_n$ be a random sample of size n, so that X_1 has the Laplace density $\frac{1}{2}e^{-|x-\theta|}$ for all $x \in \mathbb{R}$, where $\theta \in \mathbb{R}$ is unknown. Then the MLE of θ is the mdiean.

Proof.

$$f(x_i; \theta) = \frac{1}{2}e^{-|x_i-\theta|}$$

$$\implies L = \prod_{i=1}^{n} \frac{1}{2}e^{-|x_i-\theta|} = 2^{-n} \exp\left(-\sum_{i=1}^{n} |x_i - \theta|\right)$$

$$\implies \log(L) = -n\log(2) - \sum_{i=1}^{n} |x_i - \theta| \implies \frac{\mathrm{d}\log(L)}{\mathrm{d}\theta} = -\sum_{i=1}^{n} \frac{\mathrm{d}}{\mathrm{d}\theta}|x_i - \theta| = -\sum_{i=1}^{n} \operatorname{sgn}(x_i - \theta)$$

since $\frac{\mathrm{d}|x|}{\mathrm{d}x} = \operatorname{sgn}(x)$. Next set this equal to 0 and solve:

$$-\sum_{i=1}^{n} \operatorname{sgn}(x_i - \hat{\theta}_{MLE}) = 0$$

Notice that if n is even then the median set as $\hat{\theta}_{MLE}$ satisfies the above equation. If n is odd, the median is still the best we can do. So the MLE is the median.

□

9.4.7 Bayes estimator

9.4.8 EM Algorithm

Remark (Correction to Remark 6.57). If Y is constant, the algorithm just outputs θ_0 in one step by the Likelihood Inequality (Lemma 6.50 in lecture notes):

$$\mathbb{E}_\theta \log\left(\frac{f_\theta(X)}{f_\omega(X)}\right) \geq 0 \iff \mathbb{E}_\theta \log f_\theta(X) - \mathbb{E}_\theta \log f_\omega(X) \geq 0$$

has equality only when $\omega = \theta$ (if $\mathbb{P}_\theta \neq \mathbb{P}_\omega \ \forall \theta \neq \omega$). So

$$\mathbb{E}_\theta \log f_\theta(X) \geq \mathbb{E}_\theta \log f_\omega(X).$$

Remark (note on proof of Lemma 6.58).

$$Y = t(X), \qquad f_{X,Y}(x, .y) = f_X(x)\mathbf{1}_{y=t(x)}.$$

9.4.9 Comparison of estimators

9.5 Resampling and Bias Reduction

9.5.1 Jackknife Resampling

$$Z_n := Y_n + (n-1)\bigg(Y_n - \frac{1}{n}\sum_{i=1}^{n} t_{n-1}(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)\bigg)$$

9.5.2 Bootstrapping

9.6 Some Concentration of Measure

9.6.1 Concentration for Independent Sums

Can generate similar results for other random variables—just need different bound on moment-generating function (generally is fine as long as values of random variable are bounded between two real numbers, but more work to prove). However, doesn't work when values aren't bounded (e.g. for Gaussian random variables).

- What about other unbounded random variables?
- What about dependent random variables?

General question: **how far is a random variable from its mean?** Will first address functions of independent Gaussian random variables.

Definition 9.23 (Lipschitz functions). A real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is called **Lipschitz continuous** or **L-Liptschitz** if there exists a positive real constant L such that for all $x_1, x_2 \in \mathbb{R}^n$,

$$|f(x_1) - f(x_2)| \leq L\|x_1 - x_2\|_2.$$

Theorem 9.49 (Theorem 8.5 in 541A notes). Note from proof: use the fact that since f is 1-Lipschitz, $\|\frac{\mathrm{d}f(X)}{\mathrm{d}x_i}\|_2^2 \leq 1$.

Theorem 9.50 (Theorem 8.6 in 541A notes). Notes from proof: change bounds on integral to $8a/\pi$. Then since $u \geq 8a/\pi \iff -a \geq -\pi u/8$, we have

$$\|\Pi(X)\| > u \implies \|\Pi(X)\| - a > u - a \geq u - \pi/8u > u/2.$$

Therefore

$$\Pr(\|\Pi(X)\| > u) = \Pr(\|\Pi(X)\| > u) \geq \Pr(|\|\Pi(x)\| - a| > u/2).$$

$$\vdots$$

Loose bound: $a^2 e^{-a^2} \leq 10$ for all $a > 0$. $(k > 1)$.

$$\vdots$$

$$\mathbb{E}\|\Pi(X)\|^4 \leq 2^{12}a^4 + 10^6 k^2$$

then by Jensen's Inequality,

$$a^4 = (\mathbb{E}\|\Pi(X)\|)^4 \leq (\mathbb{E}\|\Pi(X)\|^2)^2$$

So $2^{12}a^4 \leq 2^{12}(\mathbb{E}\|\Pi(X)\|^2)^2$. Then we chose 10^{10} to make things easy and say

$$2^{12}a^4 + 10^6 k^2 \leq 10^{10}(\mathbb{E}\|\Pi(X)\|^2)^2.$$

Summary:

$$Z := \|\Pi(X)\|^2, \qquad \mathbb{E}(Z^2) = k, \qquad \mathbb{E}(Z^4) \leq 10^{10}(\mathbb{E}(Z^2))^2.$$

$$\vdots$$

Union bound (Boole's Inequality)

9.7 Hypothesis Testing

Proposition 9.51 (Stats 100B Homework problem). Let $Y_1, Y_2, \ldots, Y_n$ be the outcomes of n independent Bernoulli trials. Then by the Neyman-Pearson lemma, the best critical region for testing

$$H_0 : p = p_0 \qquad H_a : p > p_0$$

is

$$\frac{y}{n} = \frac{1}{n}\sum Y_i > \frac{\log(K) + n\log\left(\frac{1-p_a}{1-p_0}\right)}{n\log\left(\frac{p_0(1-p_a)}{p_a(1-p_0)}\right)}.$$

Proof.

$$\Pr(\sum Y_i = y) = \binom{n}{y} p^y (1-p)^{n-y}$$

Using the Neyman-Pearson lemma (let p_a be some particular value of $p > p_0$):

$$\frac{L(p_0)}{L(p_a)} = \frac{\binom{n}{y} p_0^y (1-p_0)^{n-y}}{\binom{n}{y} p_a^y (1-p_a)^{n-y}} < K$$

$$\left(\frac{p_0}{p_a}\right)^y \left(\frac{1-p_0}{1-p_a}\right)^n \left(\frac{1-p_0}{1-p_a}\right)^{-y} < K$$

$$\left(\frac{p_0(1-p_a)}{p_a(1-p_0)}\right)^y < K \left(\frac{1-p_a}{1-p_0}\right)^n$$

$$y \log\left(\frac{p_0(1-p_a)}{p_a(1-p_0)}\right) < \log(K) + n \log\left(\frac{1-p_a}{1-p_0}\right)$$

Aside:

$$\frac{p_0(1-p_a)}{p_a(1-p_0)} = \frac{p_0 - p_0 p_a}{p_a - p_0 p_a} < 1$$

since by assumption $p_a > p_0$. Therefore $\log\left(\frac{p_0(1-p_a)}{p_a(1-p_0)}\right) < 0$. So we have

$$\frac{y}{n} = \frac{1}{n} \sum Y_i > \frac{\log(K) + n \log\left(\frac{1-p_a}{1-p_0}\right)}{n \log\left(\frac{p_0(1-p_a)}{p_a(1-p_0)}\right)}$$

as the form for our critical region.

$\square$

10 Linear Regression

These notes are based on my notes from *Time Series and Panel Data Econometrics* (1st edition) by M. Hashem Pesaran [Pesaran, 2015] and coursework for Economics 613: Economic and Financial Time Series I at USC taught by M. Hashem Pesaran, DSO 607 at USC taught by Jinchi Lv, Statistics 100B at UCLA taught by Nicolas Christou, and the Coursera MOOC "Econometrics: Methods and Applications" from Erasmus University Rotterdam. I also borrowed from some other sources which I mention when I use them.

10.1 Chapter 1: Linear Regression

10.1.1 Preliminaries

Suppose the true model is $y_i = \alpha + \beta x_i + \epsilon_i$. Classical assumptions:

(i) $\mathbb{E}(\epsilon_i) = 0$

(ii) $\text{Var}(\epsilon_i \mid x_i = \sigma^2$ (constant)

(iii) $\text{Cov}(\epsilon_i, \epsilon_j) = 0$ if $i \neq j$

(iv) ϵ_i is uncorrelated to x_i, or $\mathbb{E}(\epsilon_i \mid x_j) = 0$ for all i, j.

10.1.2 Estimation

$$\hat{\beta} = \frac{n\sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \sum_{i=1}^n y_i}{n\sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i\right)^2} = \frac{\sum_{i=1}^n x_i y_i - n\overline{xy}}{\sum_{i=1}^n x_i^2 - n\overline{x}^2}$$

$$\hat{\alpha} = \overline{y} - \hat{\beta}\overline{x}$$

or

$$\hat{\beta} = \frac{\sum_{i=1}^n (x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^n (x_i - \overline{x})^2} = \frac{S_{XY}}{S_{XX}}$$

or

$$\hat{\beta} = r\frac{S_{YY}}{S_{XX}}$$

where r is the correlation coefficient.

Let

$$w_i = \frac{x_i - \overline{x}}{\sum_{i=1}^n (x_i - \overline{x})^2}$$

so that

$$\hat{\beta} = \sum_{i=1}^{n} w_i(y_i - \overline{y}) = \sum_{i=1}^{n} w_i y_i - \overline{y}\frac{\sum_{i=1}^{n} x_i - \overline{x}}{\sum_{i=1}^{n}(x_i - \overline{x})^2} = \sum_{i=1}^{n} w_i y_i$$

since $\sum_{i=1}^{n} x_i - \overline{x} = 0$. Then a simple expression for $\text{Var}(\hat{\beta})$ is

$$\text{Var}(\hat{\beta}) = \sum_{i=1}^{n} w_i^2 \text{Var}(y_i \mid x_i) = \sum_{i=1}^{n} w_i^2 \text{Var}(\epsilon \mid x_i) = \sigma^2 \sum_{i=1}^{n} w_i^2 = \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \overline{x})^2} = \frac{\sigma^2}{S_{XX}}$$

We can estimate these quantities as follows:

$$\hat{\sigma}^2 = \frac{1}{n-2} \cdot \sum_{i=1}^{n}(y_i - \hat{\alpha} - \hat{\beta}x_i)^2$$

Note that

$$\hat{\sigma}^2 = \frac{1}{n-2}\sum_{t=1}^{T}(y_t - \hat{\alpha} - \hat{\beta}x_t)^2 = \frac{1}{n-2}\sum_{t=1}^{T}\left[(y_t - (\overline{y} - \hat{\beta}\overline{x}) - \hat{\beta}x_t)^2\right] = \frac{1}{n-2}\sum_{t=1}^{T}(y_t - \overline{y} - \hat{\beta}(x_t - \overline{x}))^2$$

$$= \frac{1}{n-2}\sum_{t=1}^{T}(y_t - \overline{y})^2 - 2\hat{\beta}(x_t - \overline{x})(y_t - \overline{y}) + \hat{\beta}^2(x_t - \overline{x})^2$$

In the case where there is no intercept, we have

$$\hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=1}^{T}(y_t - \hat{\beta}x_t)^2 = \frac{1}{T-1}\sum_{t=1}^{T}\left(y_t^2 - 2r\frac{S_{YY}}{S_{XX}}x_t y_t + r^2\frac{S_{YY}^2}{S_{XX}^2}x_t^2\right)$$

Also,

$$\widehat{\text{Var}}(\hat{\beta}) = \frac{\hat{\sigma}^2}{S_{XX}} = \frac{1}{n-2} \cdot \frac{\sum_{i=1}^{n}(y_i - \hat{\alpha} - \hat{\beta}x_i)^2}{\sum_{i=1}^{n}(x_i - \overline{x})^2}$$

Correlation coefficient:

$$r^2 = \frac{\left(\sum_{t=1}^{T} x_t y_t\right)^2}{\sum_{t=1}^{T} x_t^2 \sum_{t=1}^{T} y_t^2}$$

$$r = \frac{1}{T-1}\frac{S_{XY}}{\sqrt{S_{XX}S_{YY}}}$$

Remark. The formulas for the coefficients in univariate OLS can also be derived by considering (x, y) as a bivariate normal distribution and calculating the conditional expectation of y given x. (See Proposition (6.69).)

Proposition 10.1 (Stats 100B homework problem). Consider the regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$ with x_i fixed and $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$, ϵ_i i.i.d. Let $e_i = y_i - \hat{y}_i$ be the residuals.

(a)

$$\sum_{i=1}^{n} e_i = 0$$

(b) $\text{Cov}(\overline{Y}, \hat{\beta}_1) = 0$ where $\overline{Y}$ is the sample mean of the y values.

(c)

$$\text{Cov}(e_i, e_j) = \sigma^2 \left(-\frac{1}{n} - \frac{(x_i - \bar{x})(x_j - \bar{x})}{\sum_{k=1}^{n}(x_k - \bar{x})^2} \right)$$

(d) We can construct a confidence interval for σ^2 as

$$\Pr\left(\frac{\sum_{i=1}^{n} e_i^2}{\chi^2_{1-\frac{\alpha}{2};n-2}} \leq \sigma^2 \leq \frac{\sum_{i=1}^{n} e_i^2}{\chi^2_{\frac{\alpha}{2};n-2}} \right) = 1 - \alpha$$

Proof. (a)

$$\sum_{i=1}^{n} e_i = \sum_{i=1}^{n}(y_i - \hat{y}_i) = \sum_{i=1}^{n}(y_i - [\bar{y} + \hat{\beta}_1(x_i - \bar{x})])$$

$$= \sum_{i=1}^{n} \left(y_i - \bar{y} - \frac{\sum(x_i - \bar{x})y_i}{\sum(x_i - \bar{x})^2}(x_i - \bar{x}) \right) = \sum_{i=1}^{n} y_i - n\bar{y} - \frac{\sum(x_i - \bar{x})y_i}{\sum(x_i - \bar{x})^2} \sum_{i=1}^{n}(x_i - \bar{x})$$

$$= \sum_{i=1}^{n} y_i - n\frac{1}{n}\sum_{i=1}^{n} y_i - \left(\frac{\sum(x_i - \bar{x})y_i}{\sum(x_i - \bar{x})^2} \right) \left[\sum_{i=1}^{n} \left(x_i - \frac{1}{n}\sum_{i=1}^{n} x_i \right) \right]$$

$$= \sum_{i=1}^{n}(y_i - y_i) - \left(\frac{\sum(x_i - \bar{x})y_i}{\sum(x_i - \bar{x})^2} \right) \left[\sum_{i=1}^{n} x_i - \frac{1}{n} \cdot n \sum_{i=1}^{n} x_i \right] = 0 - 0 = \boxed{0}$$

Or:

$$\sum_{i=1}^{n} e_i = \sum_{i=1}^{n}(y_i - \hat{y}_i) = \sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = \sum_{i=1}^{n}(y_i - (\bar{y} - \hat{\beta}_1\bar{x}) - \hat{\beta}_1 x_i)$$

$$= \sum_{i=1}^{n}(y_i - \bar{y}) - \hat{\beta}_1 \sum_{i=1}^{n}(x_i - \bar{x}) = 0$$

(b)

$$\text{Cov}(\bar{Y}, \hat{\beta}_1) = \text{Cov}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i, \frac{\sum_{i=1}^{n}(x_i - \bar{x})Y_i}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \right) = \frac{1}{n\sum_{i=1}^{n}(x_i - \bar{x})^2} \text{Cov}\left(\sum_{i=1}^{n} Y_i, \sum_{i=1}^{n}(x_i - \bar{x})Y_i \right)$$

x_i is fixed, $\text{Cov}(Y_i, Y_j) = 0$ for $i \neq j$ by assumption of the model, $\text{Var}(Y_i) = \sigma^2$ by assumption of the model.

$$= \frac{1}{n\sum_{i=1}^{n}(x_i - \bar{x})^2} \sum_{i=1}^{n}[(x_i - \bar{x})\text{Var}(Y_i)] = \frac{\sigma^2}{n\sum_{i=1}^{n}(x_i - \bar{x})^2} \sum_{i=1}^{n}(x_i - \bar{x}) = \boxed{0}$$

(c)
$$\text{Cov}(e_i, e_j) = \text{Cov}(y_i - \bar{y} - \hat{\beta}_1(x_i - \bar{x}), y_j - \bar{y} - \hat{\beta}_1(x_j - \bar{x}))$$

$$= \text{Cov}(y_i, y_j) - \text{Cov}(y_i, \bar{y}) - \text{Cov}(y_i, \hat{\beta}_1(x_j - \bar{x})) - \text{Cov}(\bar{y}, y_j) + \text{Cov}(\bar{y}, \bar{y}) + \text{Cov}(\bar{y}, \hat{\beta}_1(x_j - \bar{x})) - \text{Cov}(\hat{\beta}_1(x_i - \bar{x}), y_j)$$

$$+ \text{Cov}(\hat{\beta}_1(x_i - \bar{x}), \bar{y}) + \text{Cov}(\hat{\beta}_1(x_i - \bar{x}), \hat{\beta}_1(x_j - \bar{x}))$$

By assumption of the model, $\text{Cov}(y_i, y_j) = 0$.

$$= 0 - \text{Cov}(y_i, \bar{y}) - (x_j - \bar{x})\text{Cov}(y_i, \hat{\beta}_1) - \text{Cov}(\bar{y}, y_j) + \text{Var}(\bar{y}) + (x_j - \bar{x})\text{Cov}(\bar{y}, \hat{\beta}_1) - (x_i - \bar{x})\text{Cov}(\hat{\beta}_1, y_j)$$

$$+ (x_i - \bar{x})\text{Cov}(\hat{\beta}_1, \bar{y}) + (x_i - \bar{x})(x_j - \bar{x})\text{Cov}(\hat{\beta}_1, \hat{\beta}_1)$$

In part 7(b) we showed $\text{Cov}(\bar{y}, \hat{\beta}_1) = 0$. $\text{Var}(\bar{y}) = \sigma^2/n$. $\text{Cov}(\hat{\beta}_1, \hat{\beta}_1) = \text{Var}(\hat{\beta}_1) = \sigma^2 / \sum(x_k - \bar{x})^2$. So this simplifies to

$$= -\text{Cov}(y_i, \bar{y}) - (x_j - \bar{x})\text{Cov}(y_i, \hat{\beta}_1) - \text{Cov}(y_j, \bar{y}) + \frac{\sigma^2}{n} + 0 - (x_i - \bar{x})\text{Cov}(y_j, \hat{\beta}_1) + 0 + (x_i - \bar{x})(x_j - \bar{x})\frac{\sigma^2}{\sum_{k=1}^n (x_k - \bar{x})^2}$$

$$= -\text{Cov}(y_i, \bar{y}) - (x_j - \bar{x})\text{Cov}(y_i, \hat{\beta}_1) - \text{Cov}(y_j, \bar{y}) + \frac{\sigma^2}{n} - (x_i - \bar{x})\text{Cov}(y_j, \hat{\beta}_1) + (x_i - \bar{x})(x_j - \bar{x})\frac{\sigma^2}{\sum_{k=1}^n (x_k - \bar{x})^2} \tag{10.1}$$

Find $\text{Cov}(y_i, \bar{y}), \text{Cov}(y_j, \bar{y}), \text{Cov}(y_i, \hat{\beta}_1)$, and $\text{Cov}(y_j, \hat{\beta}_1)$:

Using that x_i is fixed, $\text{Cov}(Y_i, Y_j) = 0$ for $i \neq j$ by assumption of the model, $\text{Var}(Y_i) = \sigma^2$ by assumption of the model:

$$\text{Cov}(y_i, \bar{y}) = \text{Cov}\left(y_i, \frac{1}{n}\sum_{k=1}^n y_k\right) = \frac{1}{n}\text{Cov}(y_i, y_i) = \frac{\sigma^2}{n}$$

Similarly,

$$\text{Cov}(y_j, \bar{y}) = \frac{\sigma^2}{n}$$

$$\text{Cov}(y_i, \hat{\beta}_1) = \text{Cov}\left(y_i, \frac{\sum_{k=1}^n (x_k - \bar{x})y_i}{\sum_{k=1}^n (x_k - \bar{x})^2}\right) = \frac{1}{\sum_{k=1}^n (x_k - \bar{x})^2}\text{Cov}\left(y_i, \sum_{k=1}^n (x_k - \bar{x})y_i\right)$$

$$= \frac{1}{\sum_{k=1}^n (x_k - \bar{x})^2}\text{Cov}(y_i, (x_i - \bar{x})y_i) = \frac{x_i - \bar{x}}{\sum_{k=1}^n (x_k - \bar{x})^2}\text{Var}(y_i) = \frac{x_i - \bar{x}}{\sum_{k=1}^n (x_k - \bar{x})^2}\sigma^2$$

Similarly,

$$\text{Cov}(y_j, \hat{\beta}_1) = \frac{x_j - \bar{x}}{\sum_{k=1}^n (x_k - \bar{x})^2}\sigma^2$$

Plugging these in to equation (10.1) yields

$$\mathrm{Cov}(e_i, e_j) = -\frac{\sigma^2}{n} - (x_j - \bar{x})\frac{(x_i - \bar{x})\sigma^2}{\sum_{k=1}^n (x_k - \bar{x})^2} - \frac{\sigma^2}{n} + \frac{\sigma^2}{n} - (x_i - \bar{x})\frac{(x_j - \bar{x})\sigma^2}{\sum_{k=1}^n (x_k - \bar{x})^2}$$

$$+(x_i - \bar{x})(x_j - \bar{x})\frac{\sigma^2}{\sum_{k=1}^n (x_k - \bar{x})^2}$$

$$= \frac{-\sigma^2}{n} - \sigma^2 \frac{(x_i - \bar{x})(x_j - \bar{x})}{\sum_{k=1}^n (x_k - \bar{x})^2}$$

$$\mathrm{Cov}(e_i, e_j) = \sigma^2\left(-\frac{1}{n} - \frac{(x_i - \bar{x})(x_j - \bar{x})}{\sum_{k=1}^n (x_k - \bar{x})^2}\right)$$

(d) From class notes 08/29:

$$\frac{(n-2)S_e^2}{\sigma^2} \sim \chi^2_{n-2}$$

$$\implies \Pr\left(\chi^2_{\frac{\alpha}{2};n-2} \le \frac{(n-2)S_e^2}{\sigma^2} \le \chi^2_{1-\frac{\alpha}{2};n-2}\right) = 1 - \alpha$$

$$\implies \boxed{\Pr\left(\frac{(n-2)S_e^2}{\chi^2_{1-\frac{\alpha}{2};n-2}} \le \sigma^2 \le \frac{(n-2)S_e^2}{\chi^2_{\frac{\alpha}{2};n-2}}\right) = 1 - \alpha}$$

Since

$$S_e^2 = \frac{1}{n-2}\sum_{i=1}^n e_i^2$$

this interval can be expressed as

$$\Pr\left(\frac{\sum_{i=1}^n e_i^2}{\chi^2_{1-\frac{\alpha}{2};n-2}} \le \sigma^2 \le \frac{\sum_{i=1}^n e_i^2}{\chi^2_{\frac{\alpha}{2};n-2}}\right) = 1 - \alpha$$

□

Proposition 10.2 (Stats 100B homework problem). Suppose $Y_i = \beta_1 x_i + \epsilon_i$ (no intercept). Suppose x_i is fixed and $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$.

(a) The maximum likelihood estimator of β_1 is

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2}$$

which is unbiased. Its variance is $\frac{\sigma^2}{\sum_{i=1}^n x_i^2}$ and it is normally distributed.

(b) The maximum likelihood estimator of σ^2 is

$$\hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n (y_i - \beta_1 x_i)^2.$$

Proof. (a) First we find the likelihood function to find the MLE. Assuming the n observations are independent,

$$L = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\bigg(- \frac{1}{2\sigma^2}(y_i - \beta_1 x_i)^2 \bigg)$$

$$= (2\sigma^2\pi)^{-n/2} \exp\bigg(- \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta_1 x_i)^2 \bigg)$$

Next,

$$\log(L) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta_1 x_i)^2$$

$$\frac{\mathrm{d}\log(L)}{\mathrm{d}\beta_1} = \frac{\mathrm{d}}{\mathrm{d}\beta_1}\bigg(- \frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta_1 x_i)^2 \bigg)$$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i(y_i - \beta_1 x_i) = 0$$

$$\sum_{i=1}^{n} x_i y_i - \hat{\beta}_1 \sum_{i=1}^{n} x_i^2 = 0$$

$$\implies \hat{\beta}_1 = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

Next we show that this estimator is unbiased.

$$\mathbb{E}(\hat{\beta}_1) = \mathbb{E}\bigg(\frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2} \bigg) = \frac{1}{\sum_{i=1}^{n} x_i^2}\mathbb{E}\bigg(\sum_{i=1}^{n} x_i(\beta_1 x_i + \epsilon_i) \bigg) = \frac{1}{\sum_{i=1}^{n} x_i^2}\bigg[\mathbb{E}\bigg(\sum_{i=1}^{n} x_i^2 \beta_1 \bigg) + E\bigg(\sum_{i=1}^{n} x_i \epsilon_i \bigg) \bigg]$$

Since x_i and β_1 are non-random and ϵ_i are independent, this can be written as

$$\frac{1}{\sum_{i=1}^{n} x_i^2}\bigg[\sum_{i=1}^{n} x_i^2 \beta_1 + \sum_{i=1}^{n} x_i \mathbb{E}(\epsilon_i) \bigg] = \frac{1}{\sum_{i=1}^{n} x_i^2}\beta_1 \sum_{i=1}^{n} x_i^2 = \beta_1$$

Next we find the variance.

$$\mathrm{Var}(\hat{\beta}_1) = \mathrm{Var}\bigg(\frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2} \bigg) = \frac{1}{(\sum_{i=1}^{n} x_i^2)^2}\mathrm{Var}\bigg(\sum_{i=1}^{n} x_i(\beta_1 x_i + \epsilon_i) \bigg)$$

$$= \frac{1}{(\sum_{i=1}^{n} x_i^2)^2}\bigg[\mathrm{Var}\bigg(\sum_{i=1}^{n} x_i^2 \beta_1 \bigg) + \mathrm{Var}\bigg(\sum_{i=1}^{n} x_i \epsilon_i \bigg) \bigg]$$

Since x_i and β_1 are non-random and ϵ_i are independent, this can be written as

$$\frac{1}{(\sum_{i=1}^{n} x_i^2)^2}\bigg[0 + \sum_{i=1}^{n} x_i^2 \mathrm{Var}(\epsilon_i) \bigg] = \frac{1}{(\sum_{i=1}^{n} x_i^2)^2}\sigma^2 \sum_{i=1}^{n} x_i^2 = \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}$$

β_1 is a linear combination of y_i which is normally distributed, therefore β_1 is normally distributed.

$$\implies \beta_1 \sim \mathcal{N}\left(\beta_1, \frac{\sigma}{\sqrt{\sum_{i=1}^n x_i^2}}\right)$$

(b)

$$\frac{\mathrm{d}\log(L)}{\mathrm{d}\sigma^2} = \frac{\mathrm{d}}{\mathrm{d}\sigma^2}\left(-\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^n (y_i - \beta_1 x_i)^2\right)$$

$$= -\frac{n}{2}\frac{1}{2\pi\sigma^2}2\pi - \frac{1}{2}\left(-\frac{1}{(\sigma^2)^2}\right)\sum_{i=1}^n (y_i - \beta_1 x_i)^2 = -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^n (y_i - \beta_1 x_i)^2 = 0$$

$$\frac{1}{2(\hat{\sigma^2})^2}\sum_{i=1}^n (y_i - \beta_1 x_i)^2 = \frac{n}{2\hat{\sigma^2}}$$

$$\hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n (y_i - \beta_1 x_i)^2$$

□

Remark. More details on this problem available in Math 541A Homework 7.

10.2 Chapter 2: Multiple Regression

General OLS:

$$\hat{\beta} = (\boldsymbol{X}'X)^{-1}X'y = (X'X)^{-1}X'(X\beta + u) = (X'X)^{-1}X'X\beta + (X'X)^{-1}X'u = \beta + (X'X)^{-1}X'u$$

$$\mathrm{Var}(\hat{\beta}) = \mathrm{Var}(\beta + (X'X)^{-1}X'u) = \mathrm{Var}(\beta) + \mathrm{Var}((X'X)^{-1}X'u) = 0 + \mathbb{E}[(X'X)^{-1}X'uu'X(X'X)^{-1}]$$

$$= \mathbb{E}\big[(X'X)^{-1}X'\mathbb{E}(uu' \mid X)X(X'X)^{-1}\big] = \sigma^2\mathbb{E}\big[(X'X)^{-1}X'I_T X(X'X)^{-1}\big] = \sigma^2\mathbb{E}\big[(X'X)^{-1}\big]$$

$$= \sigma^2(X'X)^{-1}$$

$$\hat{\sigma}^2 = \frac{\hat{\boldsymbol{u}}'\hat{\boldsymbol{u}}}{T-k}$$

10.3 Chapter 3: Hypothesis testing in regression

In this section, I borrow from C. Flinn's notes "Asymptotic Results for the Linear Regression Model," available online at http://www.econ.nyu.edu/user/flinnc/notes1.pdf.

Lemma 10.3.

$$\frac{1}{n} \cdot X'\epsilon \xrightarrow{p} 0$$

Proof. Note that $\mathbb{E}\frac{1}{n} \cdot X'\epsilon = 0$ for any n. Then we have

$$\text{Var}\left(\frac{1}{n} \cdot X'\epsilon\right) = \mathbb{E}\left(\frac{1}{n} \cdot X'\epsilon\right)^2 = n^{-2}\mathbb{E}(X'\epsilon\epsilon'X) = n^{-2}\mathbb{E}(\epsilon\epsilon')X'X = \frac{\sigma^2}{n}\frac{X'X}{n}$$

implying that $\lim_{n\to\infty} \text{Var}\left(\frac{1}{n} \cdot X'\epsilon\right) = 0$. Therefore the result follows from Chebyshev's Inequality (Theorem 8.4). □

Lemma 10.4. If ϵ is i.i.d. with $E(\epsilon_i) = 0$ and $\mathbb{E}(\epsilon_i^2) = \sigma^2$ for all i, the elements of the matrix X are uniformly bounded so that $|X_{ij}| < U$ for all i and j and for U finite, and $\lim_{n\to\infty} X'X/n = Q$ is finite and nonsingular, then

$$\frac{1}{\sqrt{n}}X'\epsilon \xrightarrow{d} \mathcal{N}(0, \sigma^2 Q)$$

Proof. If we have one regressor, then $n^{-1/2}\sum_{i=1}^n X_i\epsilon_i$ is a scalar. Let G_i be the cdf of $X_i\epsilon_i$. Let

$$S_n^2 = \sum_{i=1}^n \text{Var}(X_i\epsilon_i) = \sigma^2 \sum_{i=1}^n X_i^2$$

In this scalar case, $Q = \lim_{n\to\infty} n^{-1}\sum_i X_i^2$. By the Lindberg-Feller Theorem, a necessary and sufficient condition for $Z_n \to \mathcal{N}(0\sigma^2 Q)$ is

$$\lim_{n\to\infty} \frac{1}{S_n^2}\sum_{i=1}^n \int_{|\omega|>\nu S_n} \omega^2 dG_i(\omega) = 0$$

for all $\nu > 0$. Now $G_i(\omega) = F(\omega/|X_i|)$. Then rewrite the above equation as

$$\lim_{n\to\infty} \frac{n}{S_n^2}\sum_{i=1}^n \frac{X_i^2}{n}\int_{|\omega/X_i|>\nu S_n/|X_i|} \left(\frac{\omega}{X_i}\right)^2 dF(\omega/|X_i|) = 0$$

Since $\lim_{n\to\infty} S_n^2 = \lim_{n\to\infty} n\sigma^2\sum_{i=1}^n X_i^2/n = n\sigma^2 Q$, we have $\lim_{n\to\infty} n/S_n^2 = (\sigma^2 Q)^{-1}$, which is a finite and nonzero scalar. Then we need to show

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n X_i^2 \delta_{i,n} = 0$$

where

$$\delta_{i,n} = \int_{|\omega/X_i|>\nu S_n/|X_i|} \left(\frac{\omega}{X_i}\right)^2 dF(\omega/|X_i|)$$

But $\lim_{n\to\infty} \delta_{i,n} = 0$ for all i and any fixed ν since $|X_i|$ is bounded while $\lim_{n\to\infty} X_n = \infty$, so the measure of the set $\{|\omega/X_i| > \nu S_n/|X_i|\}$ goes to 0 asymptotically. Since $\lim_{n\to\infty} n^{-1}\sum_i X_i^2$ is finite and $\lim_{n\to\infty} \delta_{i,n} = 0$ for all i, $\lim_{n\to\infty} n^{-1}\sum_i X_i^2 \delta_{i,n} = 0$, so $\frac{1}{n} \cdot X'\epsilon \xrightarrow{p} 0$.

□

Theorem 10.5. Under the conditions of Lemma 10.4 (ϵ is i.i.d. with $E(\epsilon_i) = 0$ and $\mathbb{E}(\epsilon_i^2) = \sigma^2$ for all i, the elements of the matrix X are uniformly bounded so that $|X_{ij}| < U$ for all i and j and for U finite, and $\lim_{n\to\infty} X'X/n = Q$ is finite and nonsingular),

$$\sqrt{n}(\hat\beta - \beta) \xrightarrow{d} \mathcal{N}(0, \sigma^2 Q^{-1})$$

Proof.

$$\sqrt{n}(\hat\beta - \beta) = \left(\frac{X'X}{n}\right)^{-1} \frac{1}{\sqrt{n}} X'\epsilon$$

Since $\lim_{n\to\infty}(X'X/n)^{-1} = Q^{-1}$ and by Lemma 10.4

$$\frac{1}{\sqrt{n}} X'\epsilon \xrightarrow{d} \mathcal{N}(0, \sigma^2 Q)$$

then

$$\sqrt{n}(\hat\beta - \beta) \xrightarrow{d} \mathcal{N}(0, \sigma^2 Q^{-1} Q Q^{-1}) = \mathcal{N}(0, \sigma^2 Q^{-1})$$

□

t-test statistic:

$$t = \frac{\hat\beta - 0}{s.e.(\hat\beta)}$$

F-test statistic:

$$F = \left(\frac{T-k-1}{r}\right)\left(\frac{SSR_R - SSR_U}{SSR_U}\right)$$

Since

$$R^2 = \frac{\sum_t (y_t - \overline{y})^2 - \sum_t (y_t - \hat y_t)^2}{\sum_t (y_t - \overline{y})^2} = \frac{\sum_t (y_t - \overline{y})^2 - SSR_U}{\sum_t (y_t - \overline{y})^2}$$

we have

$$SSR_U = \sum_t (y_t - \overline{y})^2 - R^2 \sum_t (y_t - \overline{y})^2 = (1 - R^2) \sum_t (y_t - \overline{y})^2$$

yielding

$$F = \left(\frac{T - k - 1}{r} \right) \left(\frac{\sum_t (y_t - \overline{y})^2 - (1 - R^2) \sum_t (y_t - \overline{y})^2}{(1 - R^2) \sum_t (y_t - \overline{y})^2} \right) = \left(\frac{T - k - 1}{r} \right) \left(\frac{R^2}{1 - R^2} \right)$$

Confidence interval for sums of coefficients. (Two coefficient case.) Suppose we want to test $H_0 : \beta_1 + \beta_2 = k$. Let $\delta = \beta_1 + \beta_2 - k$, $\hat{\delta} = \hat{\beta}_1 + \hat{\beta}_2 - k$. Note that under the null hypothesis $\delta = 0$. We can construct a t-statistic

$$t_{\hat{\delta}} = \frac{\hat{\delta} - 0}{\sqrt{\hat{\text{Var}}(\hat{\delta})}} = \frac{\hat{\beta}_1 + \hat{\beta}_2 - k}{\sqrt{\hat{\text{Var}}(\hat{\delta})}}$$

where

$$\hat{\text{Var}}(\hat{\delta}) = \hat{\text{Var}}(\hat{\beta}_1) + \hat{\text{Var}}(\hat{\beta}_2) + 2\hat{\text{Cov}}(\hat{\beta}_1, \hat{\beta}_2)$$

This means that a 95% confidence interval for δ can be constructed in the following way:

$$\hat{\delta} \pm t^* \sqrt{\hat{\text{Var}}(\hat{\delta})}$$

where t^* is the 95% critical value for the t-distribution.

10.4 Chapter 4: Heteroskedasticity

Under heteroskedasticity, the OLS estimator $\hat{\beta} = (X'X)^{-1}X'y$ is unbiased, but the true covariance matrix of $\hat{\beta}$ no longer matches the OLS formula. For instance, suppose we have

$$y_t = \sum_{i=1}^{K} \beta_i x_{ti} + u_t$$

where $\text{Var}(u_t) = \sigma^2 z_t^2$.

$$\hat{\beta} = (X'X)^{-1}X'y = (X'X)^{-1}X'X\beta + (X'X)^{-1}X'u = \beta + (X'X)^{-1}X'u$$

$$\implies \mathbb{E}(\hat{\beta}) = \mathbb{E}[\beta] + (X'X)^{-1}X'\mathbb{E}[u] = \beta$$

since $\mathbb{E}(u)$ is still 0. However,

$$\text{Var}(\hat{\beta}) = \mathbb{E}\big[\big(\hat{\beta} - \mathbb{E}(\hat{\beta})\big)\big(\hat{\beta} - \mathbb{E}(\hat{\beta})\big)'\big] = \mathbb{E}\big[\big(\beta + (X'X)^{-1}X'u - \beta\big)\big(\beta + (X'X)^{-1}X'u - \beta\big)'\big]$$

$$= \mathbb{E}\big[\big((X'X)^{-1}X'u\big)\big((X'X)^{-1}X'u\big)'\big] = \mathbb{E}\big[(X'X)^{-1}X'uu'X\big((X'X)^{-1}\big)'\big]$$

$$= (X'X)^{-1}X'\mathbb{E}[uu' \mid X]X(X'X)^{-1}$$

$$= (X'X)^{-1}X' \begin{bmatrix} \sigma^2 z_1^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma^2 z_2^2 & 0 & \dots & 0 \\ 0 & 0 & \sigma^2 z_3^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sigma^2 z_T^2 \end{bmatrix} X(X'X)^{-1}$$

$$= \sigma^2 (X'X)^{-1}X' \begin{bmatrix} z_1^2 & 0 & 0 & \dots & 0 \\ 0 & z_2^2 & 0 & \dots & 0 \\ 0 & 0 & z_3^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & z_T^2 \end{bmatrix} X(X'X)^{-1}$$

which is different from the OLS estimator of the covariance matrix $\sigma^2(X'X)^{-1}$. Therefore the estimate of the variances of $\hat{\beta}$ will be biased if the OLS formulas are used, and the usual t and F tests for $\hat{\beta}$ will be invalid.

10.5 Chapter 5: Autocorrelated disturbances

Generalized least squares model:

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{u}$$

where

$$\mathbb{E}(\boldsymbol{u} \mid \boldsymbol{X}) = 0 \ \ \forall\, t$$

$$\mathbb{E}(\boldsymbol{uu}' \mid \boldsymbol{X}) = \boldsymbol{\Sigma}$$

where $\boldsymbol{\Sigma}$ is a positive definite matrix.

$$\hat{\beta}_{GLS} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y$$

$$\text{Var}(\hat{\beta}_{GLS}) = (X'\Sigma^{-1}X)^{-1}$$

10.6 DSO 607

Generalized linear models:

$$f_n(z, \beta) = \prod_{i=1}^{n} \exp\left[\theta_i z_i - b(\theta_i) h(z_i)\right], \quad z = (z_1, \ldots, z_n)^T$$

Natural parameter θ_i: $\theta_i = x_i^T \beta$, $x_i = \{x_{ij} : j \in \mathscr{M}\}$

$h(z_i)$: normalization constant

linear regression: $b(\theta) = \frac{1}{2}\theta^2$

other: $b(\theta) = \log(1 + e^{\theta})$

If $Y = (Y_1, \ldots, Y_n)^T \sim F_n(\cdot, \beta)$, then $\mathbb{E}(Y) = (b'(\theta_1), \ldots, b'(\theta_n))^T = \mu(\theta)$ and

$\mathrm{Cov}(Y) = \mathrm{diag}\{b''(\theta_1), \ldots, b''(\theta_n)\} = \Sigma(\theta)$ where $\theta = X\beta$ and $X = (x_1, \ldots, x_n)^T$ is the $n \times d$ design matrix.

Quasi-log-likelihood ("quasi" because error may be misspecified):

$$\ell_n(y, \beta) = y^T X \beta - \mathbf{1}^T b(X\beta) + \mathbf{1}^T h(y)$$

Like MLE, maximizing $\ell_n(y, \beta)$ with respect to β gives the quasi-MLE $\hat{\beta}_n$. Solution exists and is unique due to strict convexity of b, solves the score equation

$$\frac{\partial \ell_n(y, \beta)}{\partial \beta} = x^T[y - \mu(X\beta)] = \mathbf{0}$$

(Intuition of score equation: the columns of X are all orthogonal to the errors (uncorrelated if X is random).

10.6.1 Akaike Information Criterion (AIC)

AIC: proposed by Akaike [1973] to choose a model by minimizing the Kullback-Leibler (KL) divergence of the fitted model from the true model (or equivalently, maximize the expected log-likelihood). Recall the KL Divergence

$$I(\theta; \theta_0) := 2\mathbb{E}_{\theta_0}\left[\log\left(f(X \mid \theta_0)\right)\right] - 2\mathbb{E}_{\theta_0}\left[\log\left(f(X \mid \theta)\right)\right].$$

We will try to maximizing the KL Divergence by estimating θ_0 as best as we can by maximizing the **probabilistic negentropy**

$$\mathbb{E}_Z I(\theta; \hat{\theta}_0(Z)) := 2\mathbb{E}_{\theta_0}\left[\log\left(f(X \mid \theta_0)\right] - 2\mathbb{E}_{\theta_0, Z}\left[\log\left(f\left(X \mid \hat{\theta}_0(Z)\right)\right)\right].$$

Because the true model θ_0 is unknown we cannot carry out this maximization directly. Note that as the number of independent observations increases, the **mean log-likelihood ratio**

$$\hat{I}(\theta; \theta_0) := \frac{2}{n} \sum_{i=1}^{n} \log \frac{f(x_i \mid \theta_0)}{f(x_i \mid \theta)} \xrightarrow{p} I(\theta; \theta_0).$$

Because of this, Akaike reasons that maximizing the mean log-likelihood ratio over θ_0 (i.e. computing the maximum likelihood estimate) tend to maximize the entropy. So the maximum likelihood estimate $\hat{\theta}_0(Z)$ is substituted for the unknown θ_0.

Way we wrote KL Divergence in DSO 607: density f from density g:

$$I(g_n; f_n(\cdot, \beta)) = \int [\log g(z)] g(z) dz - \int [\log f(z)] g(z) dz$$

Akaike [1973] found that up to an additive constant, the KL divergence of the fitted model from the true model can be asymptotically expanded as

$$-\ell_n(\hat{\theta}) + \lambda \dim(\hat{\theta}) = -\ell_n(\hat{\theta}) + \lambda \sum_{j=1}^{p} \mathbf{1}_{\{\hat{\theta}_j \neq 0\}}$$

where $\ell_n(\theta)$ is the log-likelihood function and $\lambda = 1$. This leads to the Akaike information criterion (AIC) for comparing models:

$$AIC\left(\hat{\theta}_k(Z)\right) := n\hat{I}\left(\hat{\theta}_k(Z; \hat{\theta}_0(Z)\right) + 2\|\hat{\theta}_k(Z)\|_0 = 2\sum_{i=1}^{n} \log \frac{f(x_i \mid \hat{\theta}_0(Z))}{f(x_i \mid \hat{\theta}_k(Z))} + 2\|\hat{\theta}_k(Z)\|_0$$

Way we wrote this is DSO 607:

$$AIC(\hat{\theta}) := -2\ell_n(\hat{\theta}) + 2\|\hat{\theta}\|_0$$

Intuition: $\log g(x)$ is the log likelihood. Penalty term can be interpreted as penalty, or as a bias correction since you are doing training and feature selection simultaneously on the same data.

$$I(g_n; f_n(\cdot, \beta)) = \sum_{i=1}^{n} \left[\int \right]$$

To minimize the KL divergence

$$\frac{\partial I(g_n; f_n(\cdot, \beta))}{\partial \beta} = -X^T[\mathbb{E}(Y) = \mu(X\beta)] = 0$$

the inverse of the Fisher information matrix is the covariance of the MLE (?).

$$\vdots$$

(For more information on KL Divergence, see Section 6.5). For AIC, we minimize the KL divergence. For BIC, we maximize the Bayes factor (posterior probability for the model).

10.6.2 Bayesian Information Criterion (BIC)

A typical Bayesian model selection procedure is to first give nonzero prior probability α_M om each model M and then prescribe a prior distribution μ_M for the parameter vector in the corresponding model. The Bayesian principle of model selection is to choose the most probable model *a posteriori*; that is, to choose a model that maximizes the log-marginal likelihood (or the Bayes factor)

$$\log \int \alpha_M \exp[\ell_n(\theta)] d\mu_m(\theta).$$

Schwarz [1978] took a Bayesian approach with prior distributions that have nonzero prior probabilities on some lower dimensional subspaces of $\mathbb{R}^p$ and showed that the negative log-marginal likelihood can be asymptotically expanded as

$$-\ell_n(\hat{\theta}) + \lambda \|\hat{\theta}\|_0$$

where $\lambda = (\log n)/2$. This asymptotic expansion leads to the Bayesian information criterion (BIC) for comparing models:

$$BIC(\hat{\theta}) := -2\log\left(f(x \mid \hat{\theta}; \hat{\theta}_{MLE})\right) + (\log n)\|\hat{\theta}\|_0.$$

where f is the density function parameterized by $\hat{\theta}_{MLE}$, the maximum likelihood estimate for the density given the data x.

Way we wrote this in DSO 607:

$$BIC(\hat{\theta}) := -2\ell_n(\hat{\theta}) + (\log n)\|\hat{\theta}\|_0.$$

$$\vdots$$

$$B_n^{1/2} A_n(\hat{\beta}_n - \beta_{n,0}) = W_n \xrightarrow{D} \mathcal{N}(0, I_d)$$

$$\hat{\beta}_n - \beta_{n,0} = A_n^{-1} B_n^{1/2} W_n \implies \text{Cov}(\hat{\beta}_n) = \text{Cov}(\hat{\beta} - n - \beta_{n,0})$$

$$= \text{Cov}(A_n^{-1}B^{1/2}W_n) = A_n^{-1}B_n^{1/2}\text{Cov}(W_n)B_n^{1/2}A_n^{-1} = A_n^{-1}B_n^{1/2}I_dB_n^{1/2}A_n^{-1} = \boxed{A_n^{-1}B_nA_n^{-1}}$$

Note that if the model is correct, $A_n = B_n$ so this reduces to conventional asymptotic MLE theory ($\text{Cov}(\hat{\beta}_n) = A_n^{-1}$).

$\vdots$

A_n from working model, B_n from true model (unknown).

GBIC in misspecified models: $H_n = A_n^{-1}B_n$ (covariance contrast matrix). Note that when model is specified, $H_n = I_d$ so the log of its determinant is 0 so it vanishes. If not, then it is a misspecification penalty.

$\vdots$

Note: $\log(y, \hat{\beta}_n) > \log(y, \beta_{n,0})$ because $\hat{\beta}_n$ is by definition the MLE on the observed data. But $\mathbb{E}(\log(\tilde{y}, \beta_{n,0}) > \mathbb{E}(\log(\tilde{y}, \hat{\beta}_n)$ because $\beta_{n,0}$ is the true parameter. We have a systematic upward bias when we use the empirical estimate. (p.18 of week 2-2 slides)

Proposition 10.6 (**Result from *"Econometrics: Methods and Applications"* homework**). Consider the usual linear model, where $y = X\beta + \epsilon$. Suppose we compare two regressions, which differ in how many variables are included in the matrix X. In the full (unrestricted) model p_1 regressors are included. In the restricted model only a subset of $p_0 < p_1$ regressors are included. Then for large n, selection based on AIC corresponds to an F-test with a critical value of approximately 2.

Proof. Let e_R be the vector of residuals for the restricted model with p_0 parameters and e_U the vector of residuals for the full unrestricted model with p_1 parameters. Then we have the sample standard deviations

$$s_0^2 = \frac{1}{n - p_0}{e_R}'e_R, s_1^2 = \frac{1}{n - p_1}{e_U}'e_U \tag{10.2}$$

Recall the AIC:

$$\log(s^2) + \frac{2k}{n}$$

where k is the number of regressors included in the model.
For the small model, we have

$$AIC_0 = \log(s_0^2) + \frac{2p_0}{n}.$$

For the big model, we have

$$AIC_1 = \log(s_1^2) + \frac{2p_1}{n}.$$

Therefore the smallest model is preferred according to the AIC if

$$AIC_0 < AIC_1$$

$$\iff \log(s_0^2) + \frac{2p_0}{n} < \log(s_1^2) + \frac{2p_1}{n} \iff \log(s_0^2) - \log(s_1^2) < \frac{2p_1}{n} - \frac{2p_0}{n} \iff \log(\frac{s_0^2}{s_1^2}) < \frac{2}{n}(p_1 - p_0)$$

$$\iff \frac{s_0^2}{s_1^2} < e^{\frac{2}{n}(p_1 - p_0)} \tag{10.3}$$

If n is very large, $\frac{2}{n}(p_1 - p_0)$ is small. Therefore, using the first order Taylor approximation $e^x \approx 1 + x$ we can approximate that

$$e^{\frac{2}{n}(p_1 - p_0)} \approx 1 + \frac{2}{n}(p_1 - p_0)$$

(if n is very large.) Substituting this expression into the right side of (10.3) yields

$$\frac{s_0^2}{s_1^2} < 1 + \frac{2}{n}(p_1 - p_0) \iff \frac{s_0^2}{s_1^2} - 1 < \frac{2}{n}(p_1 - p_0) \iff \frac{s_0^2}{s_1^2} - \frac{s_1^2}{s_1^2} < \frac{2}{n}(p_1 - p_0)$$

$$\iff \frac{s_0^2 - s_1^2}{s_1^2} < \frac{2}{n}(p_1 - p_0)$$

for n very large. Plugging in the expressions from (10.2), we have

$$\frac{\frac{1}{n-p_0} e_R{}' e_R - \frac{1}{n-p_1} e_U{}' e_U}{\frac{1}{n-p_1} e_U{}' e_U} < \frac{2}{n}(p_1 - p_0).$$

For large values of n, $n - p_0 \approx n - p_1 \approx n$. This yields

$$\frac{\frac{1}{n} e_R{}' e_R - \frac{1}{n} e_U{}' e_U}{\frac{1}{n} e_U{}' e_U} < \frac{2}{n}(p_1 - p_0)$$

$$= \frac{e_R{}' e_R - e_U{}' e_U}{e_U{}' e_U} < \frac{2}{n}(p_1 - p_0) \tag{10.4}$$

Now recall the F statistic:

$$F = \frac{(e_R{}' e_R - e_U{}' e_U)/g}{e_U{}' e_U/(n-k)} \tag{10.5}$$

where k is the number of explanatory factors in the unrestricted model, and g is the number of explanatory factors removed from the unrestricted model to create the restricted model. Under this test, we believe there is significant evidence to suggest that $\beta \neq 0$ (so the unrestricted model is preferred) if $F > F_{critical}$. Therefore a larger model is preferred if $F > F_{critical}$, and we stay with (prefer) a smaller model if $F < F_{critical}$.

Let $F_{critical} = 2$. Then a smaller model is preferred if $F < 2$:

$$\frac{(e_R{}'e_R - e_U{}'e_U)/g}{e_U{}'e_U/(n-k)} < 2$$

In this case, with p_1 factors in the unrestricted model and p_0 in the restricted model, we get

$$\frac{(e_R{}'e_R - e_U{}'e_U)/(p_1 - p_0)}{e_U{}'e_U/(n-p_1)} < 2$$

$$\frac{(e_R{}'e_R - e_U{}'e_U)}{e_U{}'e_U} < \frac{2(p_1 - p_0)}{n - p_1}$$

If n is very large, $n - p_1 \approx n$. Substituting this in yields

$$\frac{(e_R{}'e_R - e_U{}'e_U)}{e_U{}'e_U} < \frac{2(p_1 - p_0)}{n} \tag{10.6}$$

which equals (10.4). Our condition for preferring a restricted model when doing an F-test with $F_{critical} = 2$ (and when n is very large) is approximately the same as our condition for preferring a restricted model when using the AIC (when n is very large).

□

10.7 Ridge Regression

Suppose $\boldsymbol{\beta} \in \mathbb{R}^p$ is an unknown vector, and for all $1 \leq i \leq n$, there are known vectors $x^{(1)}, \ldots, x^{(n)} \in \mathbb{R}^p$. Our observed data are $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Let $\boldsymbol{X}$ be the $n \times p$ matrix so that the i^{th} row of $\boldsymbol{X}$ is the row vector $x^{(i)}$. Assume that $p \leq n$ and the matrix $\boldsymbol{X}$ has full rank. Let $\lambda > 0$ and consider the quantity

$$\sum_{i=1}^n \left(y_i - {x^{(i)}}^T \boldsymbol{\beta}\right)^2 + \lambda \|\boldsymbol{\beta}\|_2^2 = \|\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta}\|_2^2 + \lambda \|\boldsymbol{\beta}\|_2^2 \tag{10.7}$$

The term $\|\boldsymbol{\beta}\|_2^2$ penalizes $\boldsymbol{\beta}$ from having large entries. By Lagrange Multipliers, a critical point $\boldsymbol{\beta}$ of the constrained minimization problem

$$\text{minimize } \sum_{i=1}^n (\boldsymbol{y}_i - \langle x^{(i)}, \boldsymbol{\beta}\rangle)^2 \quad \text{subject to } \|\boldsymbol{\beta}\|_2^2 \leq 1$$

is equivalent to the existence of a $\lambda \in \mathbb{R}$ such that $\boldsymbol{\beta}$ is a critical point of (10.7). We call the $\hat{\beta}$ that minimizes (10.7) the **ridge regression** estimator for β.

Proposition 10.7 (Math 541A Homework Problem). The value of $\hat{\boldsymbol{\beta}} \in \mathbb{R}^p$ that minimizes (10.7) is $\hat{\boldsymbol{\beta}}_{\text{ridge}} = (\boldsymbol{X}^T\boldsymbol{X} + \lambda \boldsymbol{I}_p)^{-1}\boldsymbol{X}^T\boldsymbol{y}$.

Proof.

$$\sum_{i=1}^n \left(y_i - {x^{(i)}}^T \boldsymbol{\beta}\right)^2 + \lambda \|\boldsymbol{\beta}\|^2 = (\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})^T(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta}) + \lambda \boldsymbol{\beta}^T\boldsymbol{\beta}$$

$$= \boldsymbol{y}^T\boldsymbol{y} - \boldsymbol{y}^T\boldsymbol{X}\boldsymbol{\beta} - \boldsymbol{\beta}^T\boldsymbol{X}^T\boldsymbol{y} + \boldsymbol{\beta}^T\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{\beta} + \lambda\boldsymbol{\beta}^T\boldsymbol{\beta} = \boldsymbol{y}^T\boldsymbol{y} - 2\boldsymbol{y}^T\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\beta}^T\boldsymbol{X}^T\boldsymbol{X}\boldsymbol{\beta} + \lambda\boldsymbol{\beta}^T\boldsymbol{\beta}$$

where $\boldsymbol{y}^T\boldsymbol{X}\boldsymbol{\beta} = \boldsymbol{\beta}^T\boldsymbol{X}^T\boldsymbol{y}$ because a scalar equals its transpose. Differentiating with respect to $\boldsymbol{\beta}$ yields

$$-2\boldsymbol{y}^T\boldsymbol{X} + 2\boldsymbol{\beta}^T\boldsymbol{X}^T\boldsymbol{X} + 2\lambda\boldsymbol{\beta}^T = 0 \iff \boldsymbol{\beta}^T(2\boldsymbol{X}^T\boldsymbol{X} + 2\lambda\boldsymbol{I}_p) = 2\boldsymbol{y}^T\boldsymbol{X}$$

$$\iff (\boldsymbol{X}^T\boldsymbol{X} + \lambda\boldsymbol{I}_p)\boldsymbol{\beta} = \boldsymbol{X}^T\boldsymbol{y} \iff \hat{\boldsymbol{\beta}}_{\text{ridge}} = (\boldsymbol{X}^T\boldsymbol{X} + \lambda\boldsymbol{I}_p)^{-1}\boldsymbol{X}^T\boldsymbol{y}$$

where $\boldsymbol{X}^T\boldsymbol{X} + \lambda\boldsymbol{I}_p$ is invertible by the following argument. $\boldsymbol{X}^T\boldsymbol{X}$ must be positive semidefinite. In fact, it is positive definite because $\boldsymbol{X} \in \mathbb{R}^{n\times p}$ has full rank; that is, $\text{rank}(\boldsymbol{X}) = p$, so $\boldsymbol{X}^T\boldsymbol{X} \in \mathbb{R}^{p\times p}$ has rank p (full rank) and is invertible. So $\boldsymbol{X}^T\boldsymbol{X}$ is positive definite (all positive eigenvalues). Then since $\text{Tr}(\boldsymbol{X}^T\boldsymbol{X} + \lambda\boldsymbol{I}_p) > \text{Tr}(\boldsymbol{X}^T\boldsymbol{X})$, the eigenvalues of $\boldsymbol{X}^T\boldsymbol{X} + \lambda\boldsymbol{I}_p$ are also all positive, which means the determinant of $\boldsymbol{X}^T\boldsymbol{X} + \lambda\boldsymbol{I}_p$ is nonzero, which means it is invertible.

□

Proposition 10.8 (**DSO 607 Homework Problem**)**.** Suppose $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ where $\boldsymbol{\epsilon} \sim \mathcal{N}(\boldsymbol{0}, \sigma^2\boldsymbol{I}_n)$.

(a) The asymptotic behavior of the ridge estimator is as follows: as $\lambda \to \infty$, $\hat{\boldsymbol{\beta}}_{\text{ridge}} \to \mathbf{0}$, and as $\lambda \to 0$, $\hat{\boldsymbol{\beta}}_{\text{ridge}} \to X^\dagger(X\beta + \epsilon)$.

(b) For any fixed $\lambda > 0$, the probability that each component of the ridge estimator $\hat{\boldsymbol{\beta}}_{\text{ridge}}$ equals 0 is 0.

Proof. (a) Since X is fixed, as $\lambda \to \infty$ we have

$$\hat{\beta}_{\text{ridge}} = (X^TX + \lambda I_p)^{-1}X^Ty \to (\lambda I_p)^{-1}X^Ty = \frac{1}{\lambda}I_pX^T(X\beta + \epsilon) = \frac{1}{\lambda}(X^TX\beta + X^T\epsilon) \to \mathbf{0}$$

where $\mathbf{0}$ is a p-dimensional vector of zeroes. As $\lambda \to 0^+$ we have

$$\hat{\beta}_{\text{ridge}} = (X^TX + \lambda I_p)^{-1}X^Ty \to X^\dagger y = X^\dagger(X\beta + \epsilon)$$

where we substitute the pseudoinverse instead of the inverse because since X^TX is rank deficient, $(X^TX)^{-1}$ does not exist.

(b) We have

$$\hat{\beta}_{\text{ridge}} = (X^TX+\lambda I_p)^{-1}X^Ty = (X^TX+\lambda I_p)^{-1}X^T(X\beta+\epsilon) = (X^TX+\lambda I_p)^{-1}X^TX\beta+(X^TX+\lambda I_p)^{-1}X^T\epsilon$$

Let e_i be a selection vector, with the ith entry equal to 1 and all other entries equal to 0. Let the ith entry of $\hat{\beta}_{\text{ridge}}$ be $\hat{\beta}_{\text{ridge}}^{(i)} = e_i^T\hat{\beta}_{\text{ridge}}$. We have

$$\Pr(\hat{\beta}_{\text{ridge}}^{(i)} = 0) = \Pr\left(e_i^T[(X^TX + \lambda I_p)^{-1}X^TX\beta + (X^TX + \lambda I_p)^{-1}X^T\epsilon] = 0\right)$$

$$= \Pr\left(e_i^T(X^TX + \lambda I_p)^{-1}X^T\epsilon = -e_i^T(X^TX + \lambda I_p)^{-1}X^TX\beta\right)$$

Since every entry of ϵ is distributed continuously, the probability of it equaling a particular value is 0. Therefore the probability that each component of the ridge estimator equals 0 is 0. (For an intuitive argument as to why this is, see Figure 2.)

□

10.8 Lasso

From KKT theory, the correlation between all selected features and residual will be λ (see the remark in Section 10.8.4 for an explanation why).

Consider the linear regression model $y = X\beta + \epsilon$. If we assume the errors ϵ have a multivariate Gaussian distribution, that is,

$$f_\epsilon(t) = \left(\frac{1}{\sqrt{2\pi\sigma^2}} \right)^n \exp\left(-\frac{t^T t}{2\sigma^2} \right), \quad t = (t_1, \ldots, t_n)^T$$

then the log likelihood is

$$\log(f(t)) = n \log[(2\pi\sigma^2)^{-1/2}] - t^T t/(2\sigma^2)$$

Suppose we want the MLE estimator. When we maximize the log likelihood, we can disregard the first term which does not include t (it is constant). So we seek

$$\arg\max_{\beta \in \mathbb{R}^p} \left\{ -t^T t/(2\sigma^2) \right\} = \arg\max_{\beta \in \mathbb{R}^p} \left\{ -\|y - X\beta\|_2^2/(2\sigma^2) \right\}$$

which is the same as

$$\arg\min_{\beta \in \mathbb{R}^p} \left\{ \|y - X\beta\|_2^2/(2\sigma^2) \right\}$$

We commonly scale this with an n in the denominator to match the empirical risk; note that this does not affect the arguments which minimize the quantity. When the design matrix X multiplied by $n^{-1/2}$ is orthonormal ($X^T X = nI_p$), the penalized least squares reduces to the minimization of

$$\min_{\beta \in \mathbb{R}^p} \left\{ \frac{1}{2n} \|y - X\hat{\beta}\|_2^2 + \frac{1}{2} \|\hat{\beta} - \beta\|_2^2 + \sum_{j=1}^p p_\lambda(|\beta_j|) \right\}$$

where $\hat{\beta} = (X^T X)^{-1} X^T y = nX^T y$ is the OLS estimator. Disregarding the first term which does not contain β, we have a **separable** loss function (we can solve for one parameter at a time):

$$\min_{\beta \in \mathbb{R}^p} \left\{ \frac{1}{2} \|\hat{\beta} - \beta\|_2^2 + \sum_{j=1}^{p} p_\lambda(|\beta_j|) \right\}.$$

So we can consider the univariate penalized least squares function

$$\hat{\theta}(z) = \arg\min_{\theta \in \mathbb{R}} \left\{ \frac{1}{2}(z - \theta)^2 + p_\lambda(|\theta|)) \right\}.$$

Antoniadis and Fan [2001] showed that the PLS estimator $\hat{\theta}$ possesses the following properties:

- *sparsity* if $\min_{t \geq 0}\{t + p'_\lambda(t)\} > 0$;
- *approximate unbiasedness* if $p'_\lambda(t) = 0$ for large t;
- *continuity* if and only if $\arg\min_{t \geq 0}\{t + p'_\lambda(t)\} = 0$. Intuition: if you perturb data a little, the solution should remain similar.

In general, the singularity of the penalty function at the origin (i.e., $p'_\lambda(0+) < 0$) is needed for generating sparsity in variable selection and the concavity is needed to reduce the bias.

To recap: constrained version:

$$\begin{aligned} \hat{\beta}_{\text{lasso}} = & \arg\min_{\beta \in \mathbb{R}^p} && \frac{1}{2n} \|y - X\beta\|_2^2 \\ & \text{subject to} && \|\beta\|_1 \leq t \end{aligned}$$

Unconstrained version:

$$\hat{\beta}_{\text{lasso}} = \arg\min \left\{ \frac{1}{2n} \|y - X\beta\|_2^2 + \lambda \|\beta\|_1 \right\}$$

Use $1/n$ to rescale RSS due to $\|1\| - 2 = \sqrt{n}$.

Proposition 10.9 (**Math 541A Homework Problem**)**.** Suppose $\boldsymbol{\beta} \in \mathbb{R}^p$ is an unknown vector, and for all $1 \leq i \leq n$, there are known vectors $x^{(1)}, \ldots, x^{(n)} \in \mathbb{R}^p$. Our observed data are $\boldsymbol{y} = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$. Let $\boldsymbol{X}$ be the $n \times p$ matrix so that the i^{th} row of $\boldsymbol{X}$ is the row vector $x^{(i)}$. Assume that $p \leq n$ and the matrix $\boldsymbol{X}$ has full rank. Let $\lambda > 0$ and consider the quantity

$$\sum_{i=1}^{n} \left(\boldsymbol{y}_i - {x^{(i)}}^T \boldsymbol{\beta} \right)^2 + \lambda \sum_{i=1}^{p} |\boldsymbol{\beta}_i| \tag{10.8}$$

Then there exists a $\hat{\boldsymbol{\beta}} \in \mathbb{R}^p$ that minimizes this quantity (this $\hat{\boldsymbol{\beta}}$ is known as the LASSO, or least absolute shrinkage and selection operator).

Proof. We can write (10.8) as

$$\sum_{i=1}^{n} \left(\boldsymbol{y}_i - {x^{(i)}}^T \boldsymbol{\beta} \right)^2 + \lambda \sum_{i=1}^{p} |\boldsymbol{\beta}_i| = \|\boldsymbol{y} - \boldsymbol{X\beta}\|_2^2 + \lambda \|\boldsymbol{\beta}\|_1$$

$$= \|\boldsymbol{y} - \boldsymbol{X\beta}\|_2^2 + \lambda \|\boldsymbol{\beta}\|_1 \tag{10.9}$$

By Proposition 12.11, $\|\boldsymbol{y} - \boldsymbol{X\beta}\|_2^2$ is convex, and by Proposition 12.12, $\lambda\|\boldsymbol{\beta}\|_1$ is convex. Therefore by Proposition 12.8, (10.9) is convex. Differentiating and setting equal to 0 yields

$$-2\boldsymbol{y}^T \boldsymbol{X} + 2\boldsymbol{\beta}^T \boldsymbol{X}^T \boldsymbol{X} + \lambda \left[\operatorname{sgn}(\boldsymbol{\beta}_i) \right] = 0 \tag{10.10}$$

where $\left[\operatorname{sgn}(\boldsymbol{\beta}_i) \right]$ is vector resulting from the sgn function being applied elementwise to $\boldsymbol{\beta}$. Since (10.10) is linear in $\boldsymbol{\beta}$, it has one solution. Since (10.9) is convex, any solution to (10.10) minimizes (10.8).

□

Remark. The L_1 penalization term in (10.8) is better at penalizing large entries of $\boldsymbol{\beta}$ (a similar observation applies in the compressed sensing literature). Unfortunately, there is no closed form solution to (10.8) in general. The constrained minimization problem

$$\text{minimize } \sum_{i=1}^{n} (\boldsymbol{y}_i - \langle x^{(i)}, \boldsymbol{\beta} \rangle)^2 \quad \text{subject to } \sum_{i=1}^{n} |\boldsymbol{\beta}_i| \leq 1$$

is morally equivalent to (10.8), but technically Lagrange Multipliers does not apply since the constraint is not differentiable everywhere.

10.8.1 Soft Thresholding

Classical ideas of nonparametric models: kernels (locally constant/linear), splines (smooth basis functions). But wavelets are non-smooth. Why is this beneficial? Some real life functions are non-smooth. (example; image data with noise. There will be non-smooth edges to objects.) Also, the wavelet basis functions are orthonormal (which is closely related to the assumption we made above about the orthonormal design matrix). So when working with wavelets, we have a separable optimization problem. Soft thresholding is something like the lasso idea for wavelets (but before the lasso was developed).

Suppose we wish to recover an unknown function f on $[0, 1]$ from noisy data

$$d_i = f(t_i) + \sigma z_i, \qquad i = 0, \ldots, n-1$$

where $t_i = i/n$ and $z_i \sim \mathcal{N}(0, 1)$. The term de-noising is to optimize the mean squared error $n^{-1} \, E\|\hat{f} - f\|_2^2$. Donoho and Johnstone [1994] proposed a soft-thresholding estimator

$$\hat{\beta}_j = \operatorname{sgn}(\hat{\beta}_j^0)(|\hat{\beta}_j^0| - \gamma)_+$$

where γ is some small number. (So estimator gets shrunk by γ, and if γ is bigger than the original estimator, we set it equal to 0.) They applied this estimator to the coefficients of a wavelet transform of a function measured with noise, then back-transformed to obtain a smooth estimate of the function.

Example 10.1. Suppose we have an image in data in the form of $X \in \mathbb{R}^n$. We have a wavelet basis $W \in \mathbb{R}^{n \times n}$ where W is orthonormal. We transform the image into the frequency domain by

$$Wx \to \tilde{x}$$

where $\tilde{x}$ is the frequency domain representation. Then we apply soft-thresholding to $\tilde{x}$ to yield $\tilde{x}^*$, which we hope is de-noised. Finally, we bring the image back into the original domain according to

$$\hat{x} = W^{-1}\tilde{x}^* = W^T\tilde{x}^*.$$

The asymptotic risk of this estimator is

$$[2(\log p) + 1](\sigma^2 + R_{DP})$$

Note that the $2 \log p$ term is related to the result (described informally) below:

Proposition 10.10. if we have n i.i.d. $\mathcal{N}(0, 1)$ random variables, the maximum of them is near $\sqrt{2 \log n}$ if n is large. (The order is this large with high probability)

Remark. In the language of wavelets, sometimes ℓ_0 penalization is called "hard-thresholding."

10.8.2 Lasso theory

Drawbacks of previous techniques that lasso helps with: subset selection is interpretable but computationally intensive and not stable because it is a discrete process (small changes in the data can result in very different models being selected). Ridge regression is a continuous process and more stable, but it does not set any coefficients equal to 0 and hence does not give an easily interpretable model.

In the orthonormal design case $X^TX = nI_p$, the lasso solution can be shown to be the same as soft thresholding:

$$\hat{\beta}_j = \text{sgn}(\hat{\beta}_j^0)(|\hat{\beta}_j^0| - \gamma)_+$$

where $\gamma \geq 0$ is determined by the condition $\sum_{j=1}^p |\beta_j| = t$.

Geometry: the criterion $\sum_{i=1}^n (y_i - \sum_{j=1}^p \beta_j x_{ij})^2$ equals the quadratic function (plus a constant)

$$(\beta - \hat{\beta}^0)^T X^T X(\beta - \hat{\beta}^0).$$

Proof.

$$\sum_{i=1}^{n}\left(y_i - \sum_{j=1}^{p}\beta_j x_{ij}\right)^2 = \sum_{i=1}^{n}\left(y_i - X_i\hat{\beta}\right)^2 = (\boldsymbol{y} - \boldsymbol{X}\hat{\beta})^T(\boldsymbol{y} - \boldsymbol{X}\hat{\beta}) = [\boldsymbol{X}(\beta^0 - \hat{\beta})]^T[\boldsymbol{X}(\beta^0 - \hat{\beta})]$$

$$= (\beta^0 - \hat{\beta})^T\boldsymbol{X}^T\boldsymbol{X}(\beta^0 - \hat{\beta})$$

□

The contours (level sets) are therefore elliptical and centered at the OLS estimates. If the constraint region does not have corners, as in ridge regression, zero solutions result with probability zero (see Proposition 10.8 and Figure 2).

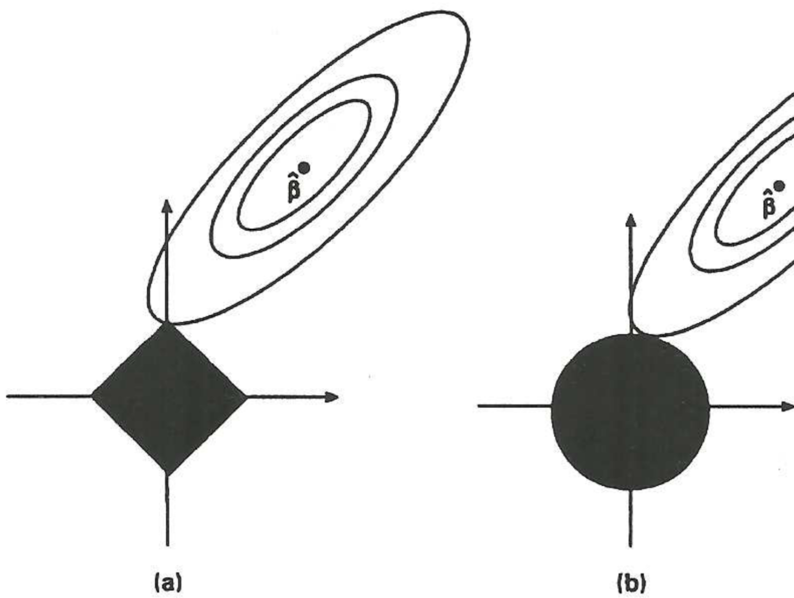


Figure 2: Level sets of least squares loss function with feasible sets for (a) lasso and (b) ridge regression in the case of $\beta \in \mathbb{R}^2$.

Proposition 10.11 (2018 DSO Statistics Group In-Class Screening Exam, Question 5). Consider the optimization problem

$$\underset{\beta \in \mathbb{R}^p}{\text{minimize}} \quad \frac{1}{2}\|y - X\beta\|_2^2 + \lambda\|\beta\|_1 \tag{10.11}$$

where $y \in \mathbb{R}^n, X \in \mathbb{R}^{n \times p}$, and $\lambda > 0$.

(a) The following problem is a dual of (10.11):

$$\underset{u \in \mathbb{R}^n}{\text{maximize}} \quad \frac{1}{2}\|y\|_2^2 - \frac{1}{2}\|y - u\|_2^2 \quad \text{subject to} \quad \|X^T u\|_\infty \leq \lambda.$$

Also, $\hat{u} = y - X\hat{\beta}$, where $\hat{\beta}$ is a solution of (10.11) and $\hat{u}$ is a solution of the dual.

(b) $\hat{\beta}$ is not necessarily unique, but $\hat{u}$, $\|y - X\hat{\beta}\|_2^2$, and $\|\hat{\beta}\|_1$ are.

(c) Suppose $y = X\beta^* + \epsilon$, and suppose the tuning parameter λ is chosen to satisfy $\lambda \geq \|X^T\epsilon\|_\infty$. Then

(i)
$$\frac{1}{2}\|y - X\hat{\beta}\|_2^2 + \lambda\|\hat{\beta}\|_1 \leq \frac{1}{2}\|\epsilon\|_2^2 + \lambda\|\beta^*\|_1.$$

(ii)
$$\frac{1}{2}\|y - X\hat{\beta}\|_2^2 + \lambda\|\hat{\beta}\|_1 \geq \frac{1}{2}\|y\|_2^2 - \frac{1}{2}\|X\beta^*\|_2^2.$$

(iii)
$$\frac{1}{2}\|y - X\hat{\beta}\|_2^2 + \lambda\|\hat{\beta}\|_1 \geq \frac{1}{2}\|\epsilon\|_2^2 - \lambda\|\beta^*\|_1.$$

Remark. We can express the original optimization problem (10.11) as

$$\begin{aligned} \underset{\beta \in \mathbb{R}^p, z \in \mathbb{R}^n}{\text{minimize}} \quad & \frac{1}{2}\|y - z\|_2^2 + \lambda\|\beta\|_1 \\ \text{subject to} \quad & z = X\beta. \end{aligned} \tag{10.12}$$

We will also refer to another expression of the lasso optimization problem,

$$\begin{aligned} \underset{\beta \in \mathbb{R}^p}{\text{minimize}} \quad & \frac{1}{2}\|y - X\beta\|_2^2 \\ \text{subject to} \quad & \|\beta\|_1 \leq t \end{aligned} \tag{10.13}$$

for some $t > 0$.

Before proving the main results, we will show a few simpler results. Whenever $\lambda > 0$, the lasso objective function (10.11) is the Lagrangian of (10.13). We will prove a useful lemma about the relationship between these functions.

Lemma 10.12. For a given $\lambda > 0$, let $\hat{\beta}$ minimize (10.11). Then there is exactly one $t = \|\hat{\beta}\|_1$ such that any $\hat{\beta}$ minimizing (10.11) also minimizes (10.13).

Proof. This must be true by contradiction. First of all, since the objective function of (10.13) is continuous and the feasible region $\|\beta\|_1 \leq t$ is compact, a minimum of (10.13) is guaranteed to exist. Now suppose $\hat{\beta}$ minimizes (10.11) for a fixed λ, with $\|\hat{\beta}\|_1 = t$, but there is a different solution $\hat{\beta}^*$ that is feasible for (10.13) and achieves a lower value. That is,

$$\frac{1}{2}\|y - X\hat{\beta}^*\|_2^2 < \frac{1}{2}\|y - X\hat{\beta}\|_2^2.$$

and $\|\hat{\beta}^*\|_1 \leq \|\hat{\beta}\|_1 = t$. Since $\lambda > 0$, $\|\hat{\beta}\|_1 < \|\hat{\beta}_{global}\|_1$, where $\hat{\beta}_{global}$ is a global minimum for $\frac{1}{2}\|y - X\hat{\beta}\|_2^2$. Since (10.13) is convex and all global minima lie outside the feasible region, $\hat{\beta}^*$ lies on the boundary; that is, $\|\hat{\beta}^*\|_1 = \|\hat{\beta}\|_1 = t$. But then

$$\frac{1}{2}\|y - X\hat{\beta}^*\|_2^2 < \frac{1}{2}\|y - X\hat{\beta}\|_2^2 \iff \frac{1}{2}\|y - X\hat{\beta}^*\|_2^2 + \lambda\|\hat{\beta}^*\|_1 < \frac{1}{2}\|y - X\hat{\beta}\|_2^2 + \lambda\|\hat{\beta}\|_1$$

which contradicts the fact that $\hat{\beta}$ minimizes (10.11).

□

Another useful result follows in a simple way from Lemma 10.12.

Proposition 10.13. Let $\mathcal{B}$ be the set of all $\hat{\beta}$ that minimize (10.11) for some fixed $\lambda > 0$. Then for any two $\hat{\beta}_1, \hat{\beta}_2 \in \mathcal{B}$, $\|\hat{\beta}_1\|_1 = \|\hat{\beta}_2\|_1$. That is, $\|\hat{\beta}\|_1$ is unique.

Proof. Suppose $\hat{\beta}_1$ and $\hat{\beta}_2$ both minimize (10.11), and (without loss of generality) $\|\hat{\beta}_1\|_1 < \|\hat{\beta}_2\|_1$. By Lemma 10.12, these values both minimize (10.13) with $t = \|\hat{\beta}_2\|_1$ (we cannot choose $t = \|\hat{\beta}_1\|_1$ because $\hat{\beta}_1$ is not feasible for that problem). Because the global minimum of (10.13) lies outside the feasible region and (10.13) is convex, all solutions to (10.13) lie on the boundary of the feasible region. But $\|\hat{\beta}_1\|_1 < \|\hat{\beta}_2\|_1$, so $\hat{\beta}_1$ is not on the boundary of the feasible region, contradiction. Therefore $\|\hat{\beta}_1\|_1 = \|\hat{\beta}_2\|_1$ for all solutions $\hat{\beta}_1$, $\hat{\beta}_2$ to (10.11); that is, $\|\hat{\beta}\|_1$ is unique. (See Osborne et al. [2000] for more details.)

□

Now we are ready to prove Proposition 10.11.

Proof of Proposition 10.11. (a) The Lagrangian of (10.12) is

$$\mathcal{L}(\beta, z, u) = \frac{1}{2}\|y - z\|_2^2 + \lambda\|\beta\|_1 + u^T(z - X\beta),$$

so the Lagrange dual function is

$$\inf_{\beta,z} \{\mathcal{L}(x, u)\} = \inf_{\beta,z} \left\{ \frac{1}{2}\|y - z\|_2^2 + \lambda\|\beta\|_1 + u^T(z - X\beta) \right\}$$

$$= \inf_{\beta,z} \left\{ \frac{1}{2}(y - z)^T(y - z) + u^T z + \lambda\|\beta\|_1 - u^T X\beta \right\}$$

This minimization is separable:

$$= \inf_{z} \left\{ \frac{1}{2}\left(y^T y - 2y^T z + z^T z\right) + u^T z \right\} + \inf_{\beta} \left\{ \lambda\|\beta\|_1 - u^T X\beta \right\} \tag{10.14}$$

We will handle each part of (10.14) separately. First, the left side:

$$\inf_{z} \left\{ \frac{1}{2}\left(y^T y - 2y^T z + z^T z\right) + u^T z \right\} = \inf_{z} \left\{ \frac{1}{2}z^T z + (u - y)^T z + \frac{1}{2}y^T y \right\}$$

Since this is a convex quadratic form, differentiate with respect to z and set equal to zero:

$$z + (u - y) = 0 \implies z = y - u \tag{10.15}$$

$$\implies \inf_{z} \left\{ \frac{1}{2}z^T z + (u - y)^T z + \frac{1}{2}y^T y \right\} = \frac{1}{2}(y - u)^T(y - u) + (u - y)^T(y - u) + \frac{1}{2}y^T y$$

$$= \frac{1}{2}\left(y^T y - 2u^T y + u^T u\right) + 2u^T y - y^T y - u^T u + \frac{1}{2} y^T y = -\frac{1}{2} u^T u + u^T y = \frac{1}{2} y^T y - \frac{1}{2} y^T y + u^T y - \frac{1}{2} u^T u$$

$$= \frac{1}{2} y^T y - \frac{1}{2}(y^T y - 2u^T y + u^T u) = \frac{1}{2} y^T y - \frac{1}{2}(y-u)^T (y-u) = \frac{1}{2}\|y\|_2^2 - \frac{1}{2}\|y-u\|_2^2$$

Next we will minimize the right side of (10.14):

$$\inf_\beta \left\{\lambda \|\beta\|_1 - u^T X \beta\right\} = \inf_\beta \left\{\lambda \sum_{i=1}^p |\beta_i| - \sum_{i=1}^p \left[u^T X\right]_i \beta_i \right\} = \inf_\beta \left\{ \sum_{i=1}^p \left(\lambda |\beta_i| - \left[u^T X\right]_i \beta_i \right) \right\}$$

$$= \inf_\beta \left\{ \sum_{i=1}^p \left(\operatorname{sgn}(\beta_i)\lambda - \left[u^T X\right]_i \right) \beta_i \right\} = \sum_{i=1}^p \inf_{\beta_i} \left\{ \left(\operatorname{sgn}(\beta_i)\lambda - \left[u^T X\right]_i \right) \beta_i \right\}.$$

Notice that when β_i is negative, if $\left(\operatorname{sgn}(\beta_i)\lambda - \left[u^T X\right]_i\right) = -\left(\lambda + \left[u^T X\right]_i\right)$ is positive there is no lower bound on the quantity we are minimizing; otherwise, when β_i is negative the infimum is 0. When β_i is positive, if $\left(\operatorname{sgn}(\beta_i)\lambda - \left[u^T X\right]_i\right) = \left(\lambda - \left[u^T X\right]_i\right)$ is negative there is no lower bound on the quantity we are minimizing; otherwise, when β_i is negative the infimum is 0. That is, the only dual feasible points satisfy for all i

$$-\left(\lambda + \left[u^T X\right]_i\right) \leq 0, \qquad \lambda - \left[u^T X\right]_i \geq 0 \iff \left[u^T X\right]_i \geq -\lambda, \qquad \left[u^T X\right]_i \leq \lambda$$

which is equivalent to the condition

$$\|u^T X\|_\infty \leq \lambda.$$

Therefore the Lagrange dual function is

$$\inf_{\beta, z} \{\mathcal{L}(x, u)\} = \frac{1}{2}\|y\|_2^2 - \frac{1}{2}\|y-u\|_2^2 \tag{10.16}$$

subject to the constraint $\|u^T X\|_\infty \leq \lambda$. This quantity represents a lower bound on the minimum value of the original optimization problem for all $u \in \mathbb{R}^p$. The dual problem is to find the best lower bound by maximizing over u; that is, the dual problem is

$$\begin{aligned} \underset{u \in \mathbb{R}^p}{\text{maximize}} \quad & \frac{1}{2}\|y\|_2^2 - \frac{1}{2}\|y-u\|_2^2 \\ \text{subject to} \quad & \|u^T X\|_\infty \leq \lambda. \end{aligned} \tag{10.17}$$

Lastly, suppose $\hat{\beta}$ and $\hat{u}$ satisfy

$$\hat{\beta} = \underset{\beta \in \mathbb{R}^p}{\arg\min} \quad \frac{1}{2}\|y - X\beta\|_2^2 + \lambda \|\beta\|_1 \quad ,$$

$$\hat{u} = \begin{array}{ll} \underset{u \in \mathbb{R}^p}{\arg\max} & \frac{1}{2}\|y\|_2^2 - \frac{1}{2}\|y-u\|_2^2 \\ \text{subject to} & \|u^T X\|_\infty \leq \lambda \end{array} = \begin{array}{ll} \underset{u \in \mathbb{R}^p}{\arg\min} & -\frac{1}{2}\|y\|_2^2 + \frac{1}{2}\|y-u\|_2^2 \\ \text{subject to} & \|u^T X\|_\infty \leq \lambda \end{array}$$

Then since (10.15) is a requirement for dual feasibility of u and strong duality applies, we have $\hat{u} = y - X\hat{\beta}$.

(b) (i) **Not necessarily unique.** Per Tibshirani [2013], if $\text{rank}(X) < p$, the lasso solution is not necessarily unique. Intuitively, this is because the columns of X are linearly dependent, so there may exist more than one linear combination of the columns that minimizes (10.11). **Jacob's suggestion: counterexample. X is two columns that are equal; then convex combinations of two solutions are equal as long as same sign (can't be opposite sign because then ℓ_1 could be smaller by setting one equal to 0.**

(ii) **Necessarily unique.** The dual problem (10.17) is strictly concave, so the value $\hat{u}$ that maximizes it is unique.

(iii) **Necessarily unique** (except in the trivial case $\lambda = 0$). Per part 5(b)(iv), $\|\hat{\beta}\|_1$ is unique. (10.11) is convex, so the minimum $\frac{1}{2}\|y - X\hat{\beta}\|_2^2 + \lambda\|\hat{\beta}\|_1$ is unique. Therefore $\|y - X\hat{\beta}\|_2^2$ must be unique. **Jacob's solution:** Since $\hat{u}$ is unique and by (10.15) $\hat{u} = y - X\hat{\beta}$, we must have that $\|\hat{u}\| = \|y - X\hat{\beta}\|$ is unique.

(iv) **Necessarily unique** (except in the trivial case $\lambda = 0$). This is immediate from Proposition 10.13.

(c) (i) Since β^* is clearly feasible for (10.11) and $\hat{\beta}$ achieves the minimum, we have

$$\frac{1}{2}\|y - X\beta^*\|_2^2 + \lambda\|\beta^*\|_1 \geq \frac{1}{2}\|y - X\hat{\beta}\|_2^2 + \lambda\|\hat{\beta}\|_1 \iff \frac{1}{2}\|y - X\hat{\beta}\|_2^2 + \lambda\|\hat{\beta}\|_1 \leq \frac{1}{2}\|\epsilon\|_2^2 + \lambda\|\beta^*\|_1$$

(ii) We know that the expression in the dual problem (10.17) is a lower bound for the solution of the primal problem (10.11) for any u feasible for (10.17) (that is, any u satisfying $\|u^T X\|_\infty \leq \lambda$). Therefore we have

$$\frac{1}{2}\|y - X\hat{\beta}\|_2^2 + \lambda\|\beta\|_1 \geq \frac{1}{2}\|y\|_2^2 - \frac{1}{2}\|y - u\|_2^2.$$

Since by assumption $\lambda \geq \|X^T\epsilon\|_\infty$, ϵ is feasible for (10.17). Therefore we have

$$\frac{1}{2}\|y - X\hat{\beta}\|_2^2 + \lambda\|\beta\|_1 \geq \frac{1}{2}\|y\|_2^2 - \frac{1}{2}\|y - \epsilon\|_2^2 = \frac{1}{2}\|y\|_2^2 - \frac{1}{2}\|X\beta^*\|_2^2 \tag{10.18}$$

as desired.

(iii) We can rewrite the right side of (10.18) as

$$\frac{1}{2}\|y\|_2^2 - \frac{1}{2}\|X\beta^*\|_2^2 = \frac{1}{2}\|X\beta^*\|_2^2 + \frac{1}{2}\|\epsilon\|_2^2 + \epsilon^T X\beta^* - \frac{1}{2}\|X\beta^*\|_2^2 = \frac{1}{2}\|\epsilon\|_2^2 + \epsilon^T X\beta^*. \tag{10.19}$$

By assumption, we have

$$\lambda \geq \|X^T\epsilon\|_\infty \iff \lambda\mathbf{1} - X^T\epsilon \succeq 0 \implies \lambda\mathbf{1}\beta^* - X^T\epsilon\beta^* \succeq 0$$

$$\iff -\lambda\|\beta^*\|_1 \leq \epsilon^T X\beta^* \leq \lambda\|\beta^*\|_1.$$

By Hölder's Inequality, we have for any two vectors $u, v \in \mathbb{R}^n$, $|u^T v| \leq \|u\|_\infty\|v\|_1$. Therefore

$$|\epsilon^T X\beta^*| = |(X^T\epsilon)^T\beta^*| \leq \|X^T\epsilon\|_\infty\|\beta^*\|_1 \leq \lambda\|\beta^*\|_1$$

where the last step used the assumption $\|X^T\epsilon\|_\infty \leq \lambda$. So we have

$$\frac{1}{2}\|\epsilon\|_2^2 + \lambda\|\beta^*\|_1 \leq \frac{1}{2}\|\epsilon\|_2^2 + \epsilon^T X\beta^*.$$

Substituting in to (10.18), using the identity in (10.19), and using the result from part 5(c)(iii) yields

$$\frac{1}{2}\|\epsilon\|_2^2 + \lambda\|\beta^*\|_1 \leq \frac{1}{2}\|\epsilon\|_2^2 + \epsilon^T X\beta^* = \frac{1}{2}\|y\|_2^2 - \frac{1}{2}\|X\beta^*\|_2^2 \leq \frac{1}{2}\|y - X\hat{\beta}\|_2^2 + \lambda\|\beta\|_1$$

as desired.

(iv) We see from parts (i) and (iii) that

$$\frac{1}{2}\|y - X\hat{\beta}\|_2^2 + \lambda\|\beta\|_1 - \lambda\|\beta^*\|_1 \leq \frac{1}{2}\|\epsilon\|_2^2 \leq \frac{1}{2}\|y - X\hat{\beta}\|_2^2 + \lambda\|\beta\|_1 + \lambda\|\beta^*\|_1$$

$$\iff \frac{1}{n}\|y - X\hat{\beta}\|_2^2 + \frac{2}{n}\lambda\|\beta\|_1 - \frac{2}{n}\lambda\|\beta^*\|_1 \leq \frac{1}{n}\|\epsilon\|_2^2 \leq \frac{1}{n}\|y - X\hat{\beta}\|_2^2 + \frac{2}{n}\lambda\|\beta\|_1 + \frac{2}{n}\lambda\|\beta^*\|_1$$

that is, we can lower bound and upper bound $\frac{1}{n}\|\epsilon\|_2^2$ by taking the quantity $\frac{1}{n}\|y - X\hat{\beta}\|_2^2 + \frac{2}{n}\lambda\|\beta\|_1$ and adding or subtracting $\frac{2}{n}\lambda\|\beta^*\|_1$. Therefore it seems that the quantity in the middle of this interval, $\frac{1}{n}\|y - X\hat{\beta}\|_2^2 + \frac{2}{n}\lambda\|\beta\|_1$, is a reasonable estimator for $\sigma^2 = \mathbb{E}\left[n^{-1}\|\epsilon\|_2^2\right]$.

□

10.8.3 Non-Negative Garotte

This idea inspired the lasso. Proposed by Breiman [1995]. It minimizes

$$\sum_{i=1}^{n}\left(y_i - \alpha - \sum_{j=1}^{p} c_j \hat{\beta}_j^o x_{ij}\right)^2 \text{ subject to } c_j \geq 0, \sum_{j=1}^{p} c_j \leq t$$

It starts with OLS estimates and shrinks them by non-negative factors whose sum is constrained. It depends on both the sign and magnitude of OLS estimates. In contrast, lasso avoids the explicit use of OLS estimates.

10.8.4 LARS—Preliminaries and Intuition

Intuition: the algorithm takes steps from a model where all coefficients are 0 to the biggest model (the unpenalized OLS model). Covariates are considered from the highest correlation with y to the least. (The variable most highly correlated with y is the one at the "least angle" from y.) Recall the original definition of the lasso estimator:

$$\hat{\beta}_{lasso} = \underset{\beta \in \mathbb{R}^p}{\arg\min}\left\{\frac{1}{2n}\|y - X\beta\|_2^2\right\} \text{ subject to } \|\beta\|_1 \leq t \tag{10.20}$$

The more common version now:

$$\hat{\beta}_{lasso} = \underset{\beta \in \mathbb{R}^p}{\arg\min}\left\{\frac{1}{2n}\|y - X\beta\|_2^2 + \lambda\|\beta\|_1\right\} \tag{10.21}$$

One form can be changed to the other by applying Lagrangians[1]. Have to be careful because this is a convex program (quadratic with "linear" constraint—use a slack variable).

Taking the gradient of the loss function in (10.21) yields

$$\nabla\left(\frac{1}{2n}\|y - X\beta\|_2^2 + \lambda\|\beta\|_1\right) = \nabla\left(\frac{1}{2n}\|y - X\beta\|_2^2\right) + \lambda\nabla\big(\|\beta\|_1\big)$$

$$= -\frac{1}{n}X^T(y - X\beta) + \lambda\nabla\big(\|\beta\|_1\big) \tag{10.22}$$

We set this equal to zero. If the first term equals 0, the residual has to equal 0. For the second part to equal zero, we have to account for the fact that the gradient doesn't exist at 0. In the one-dimensional case $g(t) = |t|$, we have

$$g'(t) = \begin{cases} -1 & t < 0 \\ 1 & t > 0 \end{cases}$$

but it doesn't exist at 0. Instead of using the gradient, we will use ∂, the subdifferential, which is the set of all subgradients. We have a solution if 0 is in the subdifferential. We can rewrite (10.22) using the subdifferential instead of the gradient:

$$\partial\left(\frac{1}{2n}\|y - X\beta\|_2^2 + \lambda\|\beta\|_1\right) = \nabla\left(\frac{1}{2n}\|y - X\beta\|_2^2\right) + \lambda\partial\big(\|\beta\|_1\big) = -\frac{1}{n}X^T(y - X\beta) + \lambda\partial\big(\|\beta\|_1\big)$$

Then rather than setting the gradient equal to 0, our condition is

$$0 \in -\frac{1}{n}X^T(y - X\beta) + \lambda\partial\big(\|\beta\|_1\big)$$

Note that

$$\partial g(t) = \begin{cases} -1 & t < 0 \\ [-1, 1] & t = 0 \\ 1 & t > 0 \end{cases} = \begin{cases} \operatorname{sgn}(t) & t \neq 0 \\ [-1, 1] & t = 0 \end{cases}$$

so we have

$$0 \in -\frac{1}{n}X^T(y - X\beta) + \lambda \cdot \left[\begin{cases} \operatorname{sgn}(\beta_j) & t \neq 0 \\ [-1, 1] & \beta_j = 0 \end{cases}\right] \tag{10.23}$$

where

[1] However, the correspondence between t and λ is **not** one-to-one. Because with $t = \infty$, $\lambda = 0$. But a slightly smaller t would result in the same solution.

$$\left[\begin{cases}\operatorname{sgn}(\beta_j) & t \neq 0 \\ [-1,1] & \beta_j = 0\end{cases}\right] \in \mathbb{R}^p$$

is a vector with each entry as specified.

Remark. (1) Examining the jth component of the separable equation (10.23), if $\beta_j \neq 0$, we have

$$0 = -\frac{1}{n}X_j^T(y - X\beta) + \lambda \cdot \operatorname{sgn}(\beta_j) \iff \frac{1}{n}X_j^T(y - X\beta) = \lambda \cdot \operatorname{sgn}(\beta_j)$$

Note that the left side contains the correlation between X_j and $e = y - X\beta$, the residual vector. **So if lasso chooses k variables, all k of them will have the same correlation with the residual (λ).**

(2) If $\beta_j = 0$, we have

$$0 \in -\frac{1}{n}X^T(y - X\beta) + \lambda \cdot [-1,1] \iff \left|\frac{1}{n}X^T(y - X\beta)\right| \leq \lambda$$

So for unselected features, the (absolute) correlation should be bounded by λ.

These two conditions relate to the KKT conditions (first order conditions).

So if we start with λ very large and gradually decrease it, we will let in as the first feature the one that is most highly correlated with y—that is, the feature with the *least angle* between it and y.

10.8.5 LARS

In Figure 3, note that we choose feature X_1 first because it has the highest correlation with y. As the coefficient on X_1 increases, the correlation between X_1 and the residual with y decreases, while the correlation between X_2 and the residual remains constant (**increases?**). When the correlation between X_1 and the residual becomes equal to the correlation between X_2 and the residual, X_2 enters the lasso path.

Remark. Just like in lasso, in LARS the correlation between all included features and the residual are equal (see the remark in Section 10.8.4). However, LARS is a stepwise procedure—once we add a feature, it stays in the model. In the lasso, features can be dropped later in the path after they are selected—whenever β_j becomes 0, it is dropped from the current active set. A feature's sign cannot change in lasso—it is not possible. If we modify the LARS algorithm to have this property ("lasso modification"), then the result is the lasso estimator.

The LARS algorithm for lasso has order $\mathcal{O}(np \cdot \min\{n, p\})$. In particular, if $p > n$ it has order $\mathcal{O}(n^2 p)$.

10.9 Quadratic Loss

Theorem 10.14. Let $X : \Omega \to \mathbb{R}$ be a random variable with $\mathbb{E}X^2 < \infty$. Then $\mathbb{E}(X - t)^2$ is minimized for $t \in \mathbb{R}$ uniquely when $t = \mathbb{E}X$.

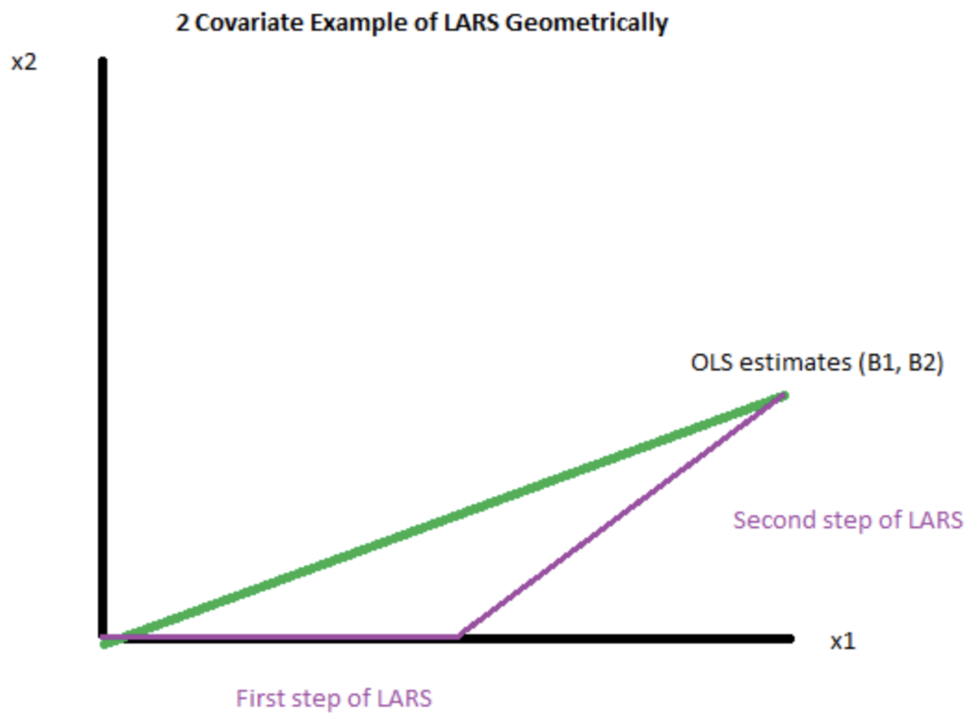


Figure 3: LARS figure in 2d case.

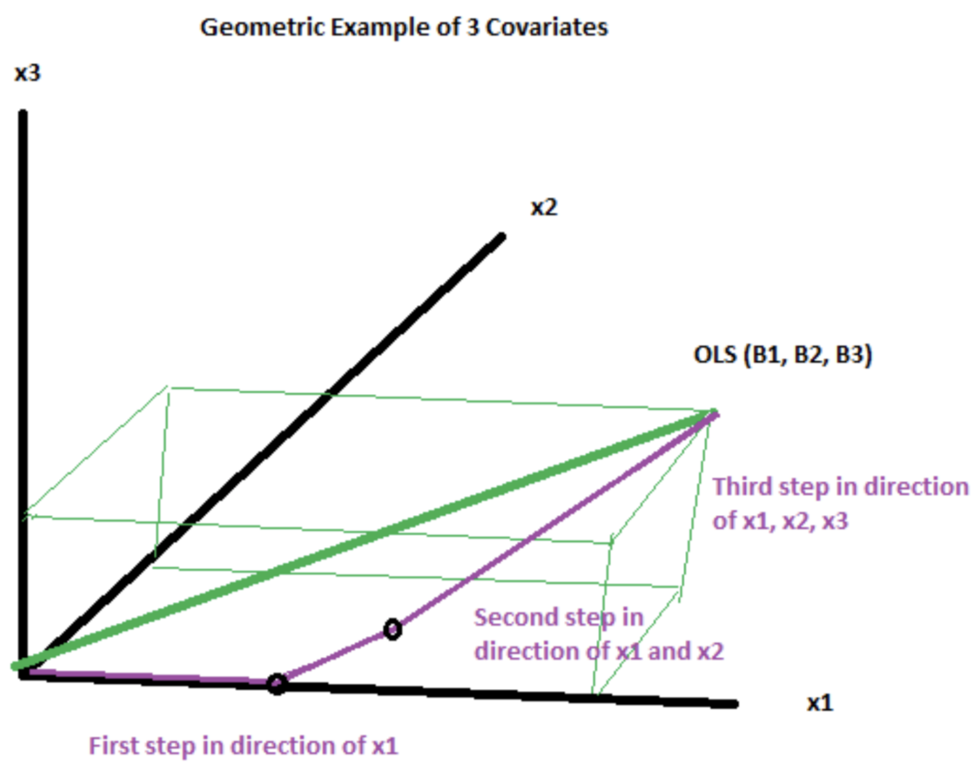


Figure 4: LARS figure in 3d case.

Proof. We seek

$$\arg\min_t \mathbb{E}(X-t)^2 = \arg\min_t \left[\mathbb{E}(X^2) - 2t\mathbb{E}(X) + t^2\right] = \arg\min_t \left[t^2 - 2t\mathbb{E}(X)\right]$$

where the last step follows because $\mathbb{E}(X^2)$ is independent of t. This expression is quadratic in t. Differentiating with respect to t and setting equal to 0, we have

$$2t - 2\mathbb{E}(X) = 0 \implies \boxed{\arg\min_t \mathbb{E}(X-t)^2 = \mathbb{E}(X)}$$

□

10.9.1 Feature Selection properties

Model selection consistency: $\Pr(\text{supp}(\hat{\beta}) = \text{supp}(\beta_0)) \to 1$.

Oracle property: model selection consistency, asymptotic efficiency as efficient as if true model were known ("efficiency" having to do with the variance given n).

Definition 10.1 (Oracle property). Let β^0 denote the true parameter vector for data generated from a linear model. Let S_0 be the true support; that is, $S_0 = \{j : \beta_j^0 \neq 0, j = 1, \ldots, p\}$. Denote $\hat{\beta}(\delta)$ the coefficient estimator for fitting procedure δ. We call δ an **oracle procedure** if $\hat{\beta}(\delta)$ asymptotically has the following properties:

- Identifies the right subset model (consistency): $\{j : \hat{\beta}_j \neq 0\} = S_0$.
- Has the optimal estimation rate: $\sqrt{n}(\hat{\beta}(\delta)_{S_0} - \beta^0_{S_0}) \xrightarrow{d} \mathcal{N}(0, \Sigma_0)$ where Σ_0 is the covariance matrix knowing the true subset model.

The lasso problem is convex but not necessarily strictly convex if $p > n$. That is, there is some flat region, so the minimizer may not be unique. Consider the KKT conditions from convex optimization:

$$g(\beta) = \arg\min \left\{ \frac{1}{2n}\|y - X\beta\|_2^2 + \lambda\|\beta\|_1 \right\} = \arg\min \left\{ f_1(\beta) + f_2(\beta) \right\}$$

Then $\hat{\beta}$ is a lasso solution if and only if 0 is in the subdifferential of $g(\hat{\beta})$. Note that

$$\partial g(\hat{\beta}) = \nabla f_1 + \partial f_2 = \frac{1}{n}X^T(X\beta - y) + \lambda \begin{bmatrix} \vdots \\ \partial|\beta_j| \\ \vdots \end{bmatrix} = \frac{1}{n}X^T(X\beta - y) + \lambda \begin{bmatrix} \vdots \\ \begin{cases} \text{sgn}(\beta_j) & \beta_j \neq 0 \\ [-1, 1] & \beta_j = 0 \end{cases} \\ \vdots \end{bmatrix}$$

Now assume $\text{supp}(\hat{\beta}) = \text{supp}(\beta_0)$ (that is, assume lasso recovers the correct support). Suppose the first s features are nonzero and consider one of them (so we know that we should have $\hat{\beta}_j \neq 0$):

$$0 \in \partial g(\hat{\beta}) \implies 0 \in \partial_j g(\hat{\beta}) = \left[\frac{1}{n} X^T (X\beta - y)\right]_j + \lambda \operatorname{sgn}(\hat{\beta}_j)$$

Therefore

$$\frac{1}{n} X_A^T (X\hat{\beta} - y) + \lambda \operatorname{sgn}(\hat{\beta}_j) = 0 \tag{10.24}$$

where X_A is a submatrix of X containing the columns corresponding to the features in the true support, is our first condition. Next, consider what happens for $j > s$ (features not in the true support). We have

$$0 \in \partial g(\hat{\beta}) \implies 0 \in \partial_j g(\hat{\beta}) = \left[\frac{1}{n} X^T (X\beta - y)\right] + \lambda[-1, 1]$$

$$\implies \left\| \frac{1}{n} X_{A^C}^T (X\hat{\beta} - y) \right\|_\infty \leq \lambda \tag{10.25}$$

where X_{A^C} is a submatrix of X containing the columns corresponding to the features not in the true support, is our boundary condition. Recall the true model

$$y = X\beta_0 + \epsilon$$

and consider the case $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$ where X_1 are the features in the true model and X_2 are noise features; that is, $\beta_0 = \begin{bmatrix} \beta_1 \\ 0 \end{bmatrix}$. Then we are assuming

$$\hat{\beta}_{\text{lasso}} = \begin{bmatrix} \hat{\beta}_1 \\ 0 \end{bmatrix}.$$

We have from (10.24)

$$0 = \frac{1}{n} X_1^T (X\hat{\beta} - y) + \lambda \operatorname{sgn}(\hat{\beta}_1) = \frac{1}{n} X_1^T (X_1 \hat{\beta}_1 - X_1 \beta_1 - \epsilon) + \lambda \operatorname{sgn}(\hat{\beta}_1)$$

$$\iff \frac{1}{n} X_1^T X_1 (\hat{\beta}_1 - \beta_1) = \frac{1}{n} X_1^T \epsilon - \lambda \operatorname{sgn}(\hat{\beta}_1)$$

Let's assume that $\operatorname{sgn}(\hat{\beta}) = \operatorname{sgn}(\beta_0)$ (sign consistency).

$$\iff \frac{1}{n} X_1^T X_1 (\hat{\beta}_1 - \beta_1) = \frac{1}{n} X_1^T \epsilon - \lambda \operatorname{sgn}(\beta_1)$$

which is linear in $\hat{\beta}$. Solving, we have

$$\iff \hat{\beta}_1 - \beta_1 = (X_1^T X_1)^{-1} \big(X_1^T \epsilon - n\lambda \operatorname{sgn}(\beta_1)\big) \iff \hat{\beta}_1 = \beta_1 + (n^{-1} X_1^T X_1)^{-1} \big(n^{-1} X_1^T \epsilon - \lambda \operatorname{sgn}(\beta_1)\big) \tag{10.26}$$

Looking at the second (boundary) condition (10.25), we have

$$\left\| \frac{1}{n} X_2^T (X\hat{\beta} - y) \right\|_\infty \leq \lambda. \tag{10.27}$$

Consider that

$$X\hat{\beta} - y = X_1\hat{\beta}_1 - X_1\beta_1 - \epsilon = X_1(\hat{\beta}_1 - \beta_1) - \epsilon$$

Substituting in the result from (10.26) yields

$$X\hat{\beta} - y = X_1\big[(n^{-1}X_1^T X_1)^{-1}\big(n^{-1}X_1^T\epsilon - \lambda\,\mathrm{sgn}(\beta_1)\big)\big] - \epsilon$$

which when we plug into (10.27) yields

$$\left\| \frac{1}{n} X_2^T \left[X_1(n^{-1}X_1^T X_1)^{-1}\big(n^{-1}X_1^T\epsilon - \lambda\,\mathrm{sgn}(\beta_1)\big) - \epsilon\right] \right\|_\infty \leq \lambda.$$

$$\iff \left\| \frac{1}{n} X_2^T X_1(n^{-1}X_1^T X_1)^{-1}\big(n^{-1}X_1^T\epsilon - \lambda\,\mathrm{sgn}(\beta_1)\big) - \frac{1}{n}X_2^T\epsilon \right\|_\infty \leq \lambda.$$

Using the Triangle Inequality, we have

$$\left\| \frac{1}{n} X_2^T X_1(n^{-1}X_1^T X_1)^{-1}\big(n^{-1}X_1^T\epsilon - \lambda\,\mathrm{sgn}(\beta_1)\big) - \frac{1}{n}X_2^T\epsilon \right\|_\infty$$

$$\leq \left\| \frac{1}{n} X_2^T X_1(n^{-1}X_1^T X_1)^{-1}\big(n^{-1}X_1^T\epsilon - \lambda\,\mathrm{sgn}(\beta_1)\big) \right\|_\infty + \left\| \frac{1}{n}X_2^T\epsilon \right\|_\infty$$

$$\leq \left\| \frac{1}{n} X_2^T X_1(n^{-1}X_1^T X_1)^{-1} \right\|_\infty \cdot \left\| n^{-1}X_1^T\epsilon - \lambda\,\mathrm{sgn}(\beta_1) \right\|_\infty + \left\| \frac{1}{n}X_2^T\epsilon \right\|_\infty \tag{10.28}$$

Assume that the jth column of X has L_2 norm $n^{1/2}$ (as it would if all entries equaled 1). We have

$$\left\| n^{-1}X_1^T\epsilon \right\|_\infty \leq \lambda/2, \qquad \left\| n^{-1}X_2^T\epsilon \right\|_\infty \leq \lambda/2$$

$$\left\| n^{-1}X^T\epsilon \right\|_\infty \leq \lambda/2 \text{ with large probability}$$

Recall that $\lambda = \sigma\sqrt{\frac{c\log p}{n}}$ for some $c > 2$. Then we have (continuing from (10.28)), and using $\left\| n^{-1}X_2^T\epsilon \right\| \leq \lambda/2$,

$$\leq \left\| n^{-1}X_1^T\epsilon \right\|_\infty + \left\| \lambda\,\mathrm{sgn}(\beta_1) \right\|_\infty$$

$$\left\|n^{-1}X_2^TX_1(n^{-1}X_1^TX_1)^{-1}\right\|_\infty \cdot \underbrace{\|\cdot\|_\infty}_{3/2\lambda} + \underbrace{\|\cdot\|_\infty}_{\lambda/2} \leq \lambda$$

$$\left\| \underbrace{n^{-1}X_2^TX_1}_{\text{corr. between noise and true}} (\underbrace{n^{-1}X_1^TX_1}_{\text{sample covariance matrix}})^{-1} \right\|_\infty \leq 1/3 \tag{10.29}$$

It turns out we're fine as long as it's less than or equal to 1. This is known as the **irrepresentable condition**. Note that the sample covariance matrix is the same as the sample correlation since the columns are standardized. So this is the correlation between the true variables. Note that this matrix has dimension $(p-s)\times s$ where s is the dimension of the true support. Note that

$$n^{-1}X_2^TX_1(n^{-1}X_1^TX_1)^{-1} = (X_2^TX_1(X_1^TX_1)^{-1}]^T = (X_1^TX_1)^TX_1^TX_2$$

which is ordinary least squares for regressing X_2 on X_1. In the end, the irrepresentable condition says the correlation between the noise and true variables can't be too high.

10.10 Dantzig Selector

Dantzig selector:

$$\begin{aligned} \hat{\beta}_{\text{Dantzig}} = & \arg\min_{\beta\in\mathbb{R}^p} && \|\beta\|_1 \\ & \text{subject to} && \|n^{-1}X^T(y-X\beta)\|_\infty \leq \lambda \end{aligned}$$

Can be recast as a linear program:

$$\begin{aligned} \hat{\beta}_{\text{Dantzig}} = & \arg\min_{u\in\mathbb{R}^p} && \sum_{i=1}^p u_i \\ & \text{subject to} && -u \leq \beta \leq u \\ & && -\lambda_p\sigma\mathbf{1} \leq n^{-1}X^T(y-X\beta) \leq \lambda_p\sigma\mathbf{1} \end{aligned} \tag{10.30}$$

where $|u|$ denotes the absolute value of u componentwise. (This is a benefit because linear programming is easy to use and very popular in industry and other applications.) Note that $n^{-1}X^t(y-X\beta)$ corresponds to the correlations between the residuals and the design matrix. Recall that in OLS this correlation is 0—the design matrix is orthogonal to the residuals. In the Dantzig selector we relax this, bounding the L_∞ norm by λ. Recall that the gradient of the log-likelihood is the **score function**, in this case $n^{-1}X6t(y-X\beta)$. For example, the score equation in linear regression is $n^{-1}X^Ty = n^{-1}X^TX\beta$. Note:

$$\nabla\left(\frac{1}{2n}\|y-X\beta\|_2^2\right) = \frac{1}{n}X^T(X\beta-y)$$

Note for Theorem 1: in original paper, assumed columns had L_2 norm 1, resulting in $\lambda_p = \sqrt{2\log p}$. We are instead assuming each column has L_2 norm $\sqrt{n}$, which results in $\lambda = \sigma \cdot \sqrt{\frac{c\log p}{n}}$. Intuition of $\log p$ term:

By a theorem in James et al. [2009], the lasso and Dantzig selector estimates equal each other under certain conditions:

Theorem 10.15. Let I_l be the support of the lasso estimate $\hat{\beta}_{\text{lasso}}$. Let $\boldsymbol{X}_L$ be the $n \times |I_L|$ matrix constructed by taking $\boldsymbol{X}_{I_L}$ and multiplying its columns by the signs of the corresponding coefficients in $\hat{\beta}_{\text{lasso}}$. Suppose that $\lambda_{\text{lasso}} = \lambda_{\text{Dantzig}}$. Then $\hat{\beta}_{\text{lasso}} = \hat{\beta}_{\text{Dantzig}}$ if $\boldsymbol{X}_L$ has full rank and

$$\boldsymbol{u} = \left(\boldsymbol{X}_L^T \boldsymbol{X}_L\right)^{-1}\mathbf{1} \succeq 0 \text{ and } \|\boldsymbol{X}^T\boldsymbol{X}_L\boldsymbol{u}\|_\infty \leq 1$$

where $\mathbf{1}$ is an $|I_L|$-vector of ones and the vector inequality is understood componentwise.

Corollary 10.15.1. If $\boldsymbol{X}$ is orthonormal ($\boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{I}_p$), then the entire lasso and Dantzig selector coefficient paths are identical.

Proof. For each index set $\boldsymbol{I}$, $\boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{I}_{|\boldsymbol{I}|}$, so clearly both of the conditions of Theorem 10.15 are satisfied. ☐

The entire paths can be identical under another condition presented in the same paper.

Theorem 10.16. Suppose that all pairwise correlations between the columns of $\boldsymbol{X}$ are equal to the same value ρ where $0 \leq \rho < 1$. Then the entire Lasso and Dantzig selector coefficient paths are identical. In addition, when $p = 2$, the same holds for every $\rho \in (-1, 1)$.

10.11 Coordinate Descent

Start with β_1 varying and all other βs fixed. Optimize β_1. Then cycle through each β_j, run until convergence.

10.12 Total Variational Distance

11 Time Series

These notes are based on my notes from *Time Series and Panel Data Econometrics* (1st edition) by M. Hashem Pesaran [Pesaran, 2015] as well as coursework for Economics 613: Economic and Financial Time Series I at USC.

11.1 Chapter 6: ARDL Models

In an ARDL model, if the error are serially correlated, then the coefficient estimates are biased (even as $T \to \infty$).

11.2 Chapters 12 and 13: Intro to Stochastic Processes and Spectral Analysis

Stationarity conditions: $\{X_t\}$ is **strictly stationary** if the joint distribution functions of $\{X_{t_1}, X_{t_2}, \ldots, X_{t_k}\}$ and $\{X_{t_1+h}, X_{t_2+h}, \ldots, X_{t_k+h}\}$ are identical for all values of $t_1, t_2, \ldots, t_k$ and h and all positive integers k.

Definition 11.1. X_t is **weakly (or covariance) stationary** if it has a constant mean and variance and its covariance function $\gamma(t_1, t_2)$ depends only on the absolute difference $|t_1 - t_2|$, namely $\gamma(t_1, t_2) = \gamma(|t_1 - t_2|)$.

Definition 11.2. X_t is said to be **trend stationary** if $y_t = X_t - d_t$ is covariance stationary, where d_t is the perfectly predictable component of X_t.

The process $\{\epsilon_t\}$ is said to be a **white noise process** if it has mean zero, a constant variance, and ϵ_t and ϵ_s are uncorrelated for all $s \neq t$.

Autocovariance generating function: The autocovariance generating function for the general linear stationary process $y_t = \sum_{i=0}^{\infty} a_i \epsilon_{t-i}$ is given by:

$$G(z) = \sigma^2 a(z) a(z^{-1})$$

where $a(z) = \sum_{i=0}^{\infty} a_i z^i$.

Wold's Decomposition (Theorem 42, p. 275, Section 12.5) Any trend-stationary process $\{y_t\}$ can be represented in the form of $y_t = d_t + \sum_{i=0}^{\infty} \alpha_i \epsilon_{t-i}$ where $\alpha_0 = 1$ and $\sum_{i=0}^{\infty} \alpha_i^2 < K < \infty$. The term d_t is a deterministic component, while $\{\epsilon_t\}$ is a serially uncorrelated process: $\epsilon_t = y_t - \mathbb{E}(y_t \mid y_{t-1}, y_{t-2}, \ldots)$.

Stationarity conditions for an ARMA(p, q) process: Consider the ARMA(p, q) process

$$y_t = \sum_{i=1}^{p} \phi_i y_{t-i} + \sum_{i=0}^{q} \theta_i \epsilon_{t-i}, \quad \theta_0 = 1$$

The MA part is stationary for any finite q. The AR part is stationary if the roots of the characteristic equation

$$\lambda^t = \sum_{i=1}^{p} \phi_i \lambda^{t-i}$$

lie strictly inside the unit circle. Alternatively, in terms of $z = \lambda^{-1}$, the process is stationary if the roots of

$$1 - \sum_{i=1}^{p} \phi_i z^i = 0$$

lie outside the unit circle. The ARMA process is **invertible** (so that y_t can be solved uniquely in terms of its past values) if all the roots of

$$1 - \sum_{i=1}^{p} \theta_i z^i = 0$$

fall outside the unit circle.

Spectral Density Function: Definition (Equation 13.3):

$$f(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h) e^{ih\omega}, \omega \in (-\pi, \pi)$$

Equation (13.5):

$$f(\omega) = \frac{1}{2\pi} \left[\gamma(0) + 2 \sum_{h=1}^{\infty} \gamma(h) \cos(h\omega) \right], \quad \omega \in [0, \pi]$$

Can also be found using the autocovariance generating function. We have (Equation 13.6, section 13.3.1)

$$f(\omega) = \frac{1}{2\pi} G(e^{i\omega}) = \frac{\sigma^2}{2\pi} a(e^{i\omega}) a(e^{-i\omega})$$

Properties of spectral density function:

(1) $f(\omega)$ always exists and is bounded if $\gamma(h)$ is absolutely summable.

(2) $f(\omega)$ is symmetric.

(3) The spectrum of a stationary process is finite at zero frequency; that is, $f(0) < \infty$.

Linear (time-domain) processes don't have to be stationary, but to write something as a frequency-domain process, it must be stationary.

11.2.1 Worked Examples

Midterm Problem 2 part (1) (chapter 12 exercise 6)

Midterm Problem 2 part (2) (exercise 7 in chapter 12; similar to exercise 1 in chapter 14. Suppose $\{y_t\}$ has the following general linear process

$$y_t = \mu + \alpha(L)\epsilon_t, \qquad \epsilon_t \sim i.i.d.\ (0, \sigma^2)$$

where $\alpha(L) = \alpha_0 + \alpha_1 L + \alpha_2 L^2 + \ldots;\ \alpha_0 = 1$. Let

$$\overline{y}_T = \frac{1}{T}\sum_{t=1}^{T} y_t$$

$$\gamma(h) = \mathbb{E}[(y_t - \mu)(y_{t-h} - \mu)]$$

$$\hat{\gamma}(h) = \frac{1}{T}\sum_{t=h+1}^{T}(y_t - \overline{y}_T)(y_{t-h} - \overline{y}_T)$$

Derive the conditions under which

(a) $\overline{y}_T$ is a consistent estimator of μ as $T \to \infty$

(b) For fixed h, $\hat{\gamma}(h)$ is a consistent estimator of $\gamma(h)$ as $T \to \infty$.

Solution.

(a) This is an MA(∞) process. By Chebyshev's Inequality (Theorem 8.4), $\overline{y}_T$ is a consistent estimator of μ as $T \to \infty$ if $\lim_{T\to\infty} \mathbb{E}(\overline{y}_T) = \mathbb{E}(y_T) = \mu$ and $\lim_{T\to\infty} \mathrm{Var}(\overline{y}_T) = 0$. In this case in particular (MA(∞) process), we can write

$$\overline{y}_T = \frac{1}{T}\sum_{t=1}^{T}\big(\mu + \alpha(L)\epsilon_t\big) = \frac{1}{T}\cdot T\mu + \frac{1}{T}\sum_{t=1}^{T}\alpha(L)\epsilon_t = \mu + \frac{1}{T}\sum_{t=1}^{T}\alpha(L)\epsilon_t$$

Then we have

$$\mathbb{E}(\overline{y}_T) = \mu + \frac{1}{T}\mathbb{E}\left(\sum_{t=1}^{T}\alpha(L)\epsilon_t\right) = \mu + \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}(\alpha(L)\epsilon_t) = \mu$$

$$\mathrm{Var}(\overline{y}_T) = 0 + \frac{1}{T^2}\mathrm{Var}\left(\sum_{t=1}^{T}\alpha(L)\epsilon_t\right) = \frac{1}{T^2}\sum_{t=1}^{T}\mathrm{Var}[\alpha(L)\epsilon_t] = \frac{1}{T^2}\sum_{t=1}^{T}\mathbb{E}[\alpha(L)\epsilon_t]^2 = \frac{1}{T}\alpha(1)^2\mathbb{E}[\epsilon_t]^2$$

$$= \frac{\sigma^2}{T}\alpha(1)^2$$

Therefore a sufficient condition for consistency is

$$\lim_{T\to\infty} \frac{\sigma^2}{T}\alpha(1)^2 = 0 \iff \alpha(1)^2 < \infty \impliedby \boxed{\sum_{i=0}^{\infty}\alpha_i = 0}$$

(b) See section 11.4.2.

11.3 Some time series and their properties

11.3.1 White noise process:

$x_t = \epsilon_t$, $\epsilon_t \sim IID(0, \sigma^2)$

- Autocovariances:

$$\gamma(0) = \sigma^2$$

$$\gamma(h) = 0, \quad \forall h \neq 0$$

- Spectral density function:

$$f_x(\omega) = \frac{1}{2\pi}\cdot\sigma^2 = \frac{\sigma^2}{2\pi} \text{ (flat spectrum)}$$

11.3.2 MA(1) process:

$x_t = \epsilon_t + \theta\epsilon_{t-1}$ with $\epsilon_t \sim iid(0, \sigma^2)$, $|\rho| < 1$.

- Autocovariances: By Equation (12.2), the autocovariance function is

$$\text{Cov}(u_t, u_{t-h}) = \gamma(h) = \sigma^2 \sum_{i=0}^{1-|h|} a_i a_{i+|h|} \text{ if } 0 \leq |h| \leq 1$$

$$\implies \mathbb{E}(x_t^2) = \gamma(0) = (1+\theta^2)\sigma^2$$

$$\mathbb{E}(x_t x_{t-1}) = \gamma(1) = \theta\sigma^2$$

$$\gamma(h) = 0 \;\; \forall \; |h| > 1$$

So the covariance matrix is

$$\begin{pmatrix} \sigma^2(1+\theta^2) & \sigma^2\theta & 0 & 0 & \cdots & 0 \\ \sigma^2\theta & \sigma^2(1+\theta^2) & \sigma^2\theta & 0 & \cdots & 0 \\ 0 & \sigma^2\theta & \sigma^2(1+\theta^2) & \sigma^2\theta & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma^2\theta & \sigma^2(1+\theta^2) & \sigma^2\theta \\ 0 & 0 & \cdots & 0 & \sigma^2\theta & \sigma^2(1+\theta^2) \end{pmatrix}$$

$$= \sigma^2(1+\theta^2)I_T + \sigma^2\theta A$$

where A is defined as in section 14.3.2 (p. 304).

- Spectral density function:

$$f(\omega) = \frac{\sigma^2}{2\pi}\left[1 + 2\theta\cos(\omega) + \rho^2\right], \quad \omega \in [0, \pi]$$

11.3.3 MA(∞) process:

This process is covariance stationary.

- Autocovariances:

11.3.4 AR(1) process:

$x_t = \phi x_{t-1} + \epsilon_t$, $|\phi| < 1$, $\epsilon_t \sim IID(0, \sigma^2)$.

- Yule-Walker Equations:

$$\mathbb{E}[x_t x_{t-h}] = \mathbb{E}[\phi x_{t-1} x_{t-h}] + \mathbb{E}[\epsilon x_{t-h}]$$

$$\gamma_h = \phi\gamma_{h-1} + \mathbb{E}[\epsilon x_{t-h}]$$

$$\implies \gamma_0 = \phi\gamma_1 + \sigma^2, \quad \gamma_h = \phi\gamma_{h-1} \ \forall h \geq 1$$

- Autocovariances:

$$\gamma(0) = \frac{\sigma^2}{1-\phi^2}$$

$$\gamma_h = \frac{\sigma^2\phi^h}{1-\rho^2} = \phi^h\gamma(0) \ \ \forall \ h \geq 1$$

$$\implies \operatorname{Cov}(x) =$$

$$\begin{pmatrix} \sigma^2/(1-\phi^2) & \sigma^2\phi/(1-\phi^2) & \sigma^2\phi^2/(1-\phi^2) & \sigma^2\phi^3/(1-\phi^2) & \cdots & \sigma^2\phi^{T-1}/(1-\phi^{)} \\ \sigma^2\phi^{/}(1-\phi) & \sigma^2/(1-\phi^2) & \sigma^2\phi/(1-\phi^2) & \sigma^2\phi^2/(1-\phi^2) & \cdots & \sigma^2\phi^{T-2}/(1-\phi^2) \\ \sigma^2\phi^2/(1-\phi^2) & \sigma^2\phi/(1-\phi^2) & \sigma^2/(1-\phi^2) & \sigma^2\phi/(1-\phi^2) & \cdots & \sigma^2\phi^{T-3}/(1-\phi^2) \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \sigma^2\phi^{T-2}/(1-\phi^2) & \sigma^2\phi^{T-3}/(1-\phi^2) & \cdots & \sigma^2\phi/(1-\phi^2) & \sigma^2/(1-\phi^2) & \sigma^2\phi/(1-\phi^2) \\ \sigma^2\phi^{T-1}/(1-\phi^2) & \sigma^2\phi^{T-2}/(1-\phi^2) & \cdots & \sigma^2\phi^2/(1-\phi^2) & \sigma^2\phi/(1-\phi^2) & \sigma^2/(1-\phi^2) \end{pmatrix}$$

- If stationary, can be written as an infinite MA process with absolutely summable coefficients

$$x_t = \sum_{i=0}^{\infty} \phi^i \epsilon_{t-i} = \left(\frac{1}{1-\phi L}\right)\epsilon_t$$

- Autocovariance generating function:

$$G(z) = \left(\frac{\sigma^2}{1-\phi^2}\right)\left(1 + \sum_{h=1}^{\infty} \phi^h (z^h + z^{-h})\right)$$

- Spectral density function:

$$f(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \frac{\sigma^2 \phi^{|h|}}{(1-\phi^2)} (e^{i\omega})^h = \frac{1}{2\pi} \frac{\sigma^2}{(1-\phi e^{i\omega})(1-\phi e^{-i\omega})} = \frac{1}{2\pi} \frac{\sigma^2}{1-2\phi\cos(\omega)+\phi^2}$$

11.3.5 AR(2) process:

$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \epsilon$, $|\phi_1| < 1$, $|\phi_2| < 1$, $\epsilon_t \sim IID(0, \sigma^2)$.

Can be written as

$$x_t = \frac{1}{1-\phi L}\epsilon_t = \epsilon_t + \phi\epsilon_{t-1} + \phi^2\epsilon_{t-2} + \dots$$

- Yule-Walker equations:

$$\mathbb{E}[x_t x_{t-h}] = \mathbb{E}[\phi_1 x_{t-1} x_{t-h}] + \mathbb{E}[\phi_2 x_{t-2} x_{t-h}] + \mathbb{E}[\epsilon x_{t-h}]$$

$$\gamma_h = \phi_1 \gamma_{h-1} + \phi_2 \gamma_{h-2} + \mathbb{E}[\epsilon x_{t-h}]$$

$$\implies \boxed{\gamma_0 = \phi_1\gamma_1 + \phi_2\gamma_2 + \sigma^2, \quad \gamma_1 = \phi_1\gamma_0 + \phi_2\gamma_1, \quad \gamma_2 = \phi_1\gamma_1 + \phi_2\gamma_0}$$

- Autocovariances:

11.3.6 AR(p) process:

$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \ldots + \phi_p x_{t-p} + \epsilon$, $|\phi_i| < 1$, $\epsilon_t \sim IID(0, \sigma^2)$.

- Stationary if the eigenvalues of $\mathbf{\Phi}$ lie inside the unit circle, which is equivalent to all the roots of

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \ldots - \phi_p z^p = 0$$

  being strictly larger than unity. Under this condition the AR process has the infinite-order MA representation'

$$x_t = \sum_{i=0}^{\infty} \alpha_i \epsilon_{t-i}$$

  where $\alpha_i = \phi_1 \alpha_{i-1} + \ldots + \phi_p \alpha_{i-p}$.

- Autocovariance generating function:

$$G(z) = \frac{\sigma^2}{\phi(z)\phi(z^{-1})}$$

11.3.7 ARMA(1, 1) process:

$x_t = \phi x_{t-1} + \epsilon_t + \theta \epsilon_{t-1}$, with $|\phi| < 1$ (implying stationarity), $\mathbb{E}(\epsilon_t^2) = \sigma^2$, $\mathbb{E}(\epsilon_t \epsilon_s) = 0$ for $t \neq s$.

- Yule-Walker Equations:

$$\gamma(0) = \phi\gamma(1) + \sigma^2(1 + \theta^2)$$

$$\gamma(1) = \phi\gamma(0) + \sigma^2\phi^2$$

$$\gamma(h) = \phi\gamma(h-1) \quad \forall\ h \geq 2$$

- Autocovariances:

$$\gamma(0) = \sigma^2\left(1_{\frac{(\phi+\theta)^2}{1-\phi^2}}\right)$$

$$\gamma(1) = \sigma^2\left(\phi + \theta + \frac{(\phi+\theta)^2\phi}{1-\phi^2}\right)$$

$$\gamma(2) = \phi^{h-1}\gamma(1)\ \forall\ h \geq 2$$

- Autocorrelation function:

$$\rho(h) = \begin{cases} 1 & h = 0 \\ \frac{(\phi+\theta)(1+\phi\theta)}{1+2\phi\theta+\theta^2} & h = 1 \\ \phi^{h-1}\rho(1) & h \geq 2 \end{cases}$$

- Autocovariance generating function: the autocovariance function of an ARMA(p, q) process $\phi(L)y_t = \theta(L)\epsilon_t$ is given by

$$f(\omega) = \sigma^2 \frac{\theta(z)\theta(z^{-1})}{\phi(z)\phi(z^{-1})}$$

Plugging in for the ARMA(1,1) case yields (**double-check**)

$$f(\omega) = \sigma^2 \frac{(1+\theta)^2}{(1-\rho)^2}$$

- Spectral Density Function: the spectral density function of an ARMA(p, q) process $\phi(L)y_t = \theta(L)\epsilon_t$ is given by

$$f(\omega) = \frac{\sigma^2}{2\pi} \frac{\theta(e^{i\omega})\theta(e^{-i\omega})}{\phi(e^{i\omega})\phi(e^{-i\omega})}, \quad \omega \in [0, 2\pi]$$

Plugging in for the ARMA(1,1) case yields

$$f_x(\omega) = \frac{\sigma^2}{2\pi} \frac{(e^{i\omega} - \theta e^{i\omega})(e^{-i\omega} - \theta e^{-i\omega})}{(e^{i\omega} - \phi e^{i\omega})(e^{-i\omega} - \phi e^{-i\omega})} = \frac{\sigma^2}{2\pi} \frac{1 - 2\theta + \theta^2}{1 - 2\phi + \phi^2}$$

- If $\phi = \theta$, the ARMA(1,1) process becomes a white noise process. We can see this two ways. The ARMA(1, 1) process can be represented in the following way:

$$(1 - \phi L)y_t = (1 - \theta L)\epsilon_t$$

Therefore $\phi(L) = \theta(L)$ yields $y_t = \epsilon_t$.

We can also see that when $\phi = \theta$, an ARMA(1,1) process is equivalent to a white noise process as follows. Plugging in $\phi = \theta$ to the spectral density function, we have

$$f_x(\omega) = \frac{\sigma^2}{2\pi} \frac{1 - 2\theta + \theta^2}{1 - 2\theta + \theta^2} = \frac{\sigma^2}{2\pi}$$

showing that if $\theta = \phi$, the spectral density function is constant and independent of θ and ϕ. We can see that it in fact is a white noise process. Since a white noise process has the following covariances:

$$\gamma(0) = \sigma^2$$

$$\gamma(h) = 0, \quad \forall h \neq 0$$

for a white noise process we have

$$f_x(\omega) = \frac{1}{2\pi} \cdot \sigma^2 = \frac{\sigma^2}{2\pi}$$

11.4 Chapter 14: Estimation of Stationary Time Series Processes

11.4.1 Sufficient conditions for ergodicity of mean. (Book section 14.2.1)

By Chebyshev's Inequality (see section 8.2), $\overline{y}_T$ is a consistent estimator of μ as $T \to \infty$ if $\lim_{T\to\infty} \mathbb{E}(\overline{y}_T) = \mathbb{E}(y_T) = \mu$ and $\lim_{T\to\infty} \text{Var}(\overline{y}_T) = 0$. We have

$$\mathbb{E}(\overline{y}_T) = \frac{1}{T}\mathbb{E}\bigg(\sum_{t=1}^{T} y_t\bigg) = \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}(y_t) = \mu$$

$$\text{Var}(\overline{y}_T) = \frac{1}{T^2}\text{Var}\bigg(\sum_{t=1}^{T} y_t\bigg) = \frac{1}{T^2}\bigg(\sum_{t=1}^{T}\text{Var}(y_t) + 2\sum_{0\leq i<j\leq T}\text{Cov}(y_i, y_j)\bigg)$$

$$= \frac{1}{T^2}\bigg(\sum_{t=1}^{T}\gamma(0) + 2\sum_{0\leq i<j\leq T}\gamma(j-i)\bigg) = \frac{1}{T^2}\bigg(T\gamma(0) + 2\sum_{h=1}^{T-1}(T-h)\gamma(h)\bigg)$$

$$= \frac{1}{T}\bigg[\gamma(0) + 2\sum_{h=1}^{T-1}\bigg(1 - \frac{h}{T}\bigg)\gamma(h)\bigg] = \frac{1}{T^2}\mathbf{1}'\text{Var}(\boldsymbol{y})\mathbf{1}$$

where $\mathbf{1}$ is a vector of ones and

$$\text{Var}(\boldsymbol{y}) = \begin{pmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(T-2) & \gamma(T-1) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(T-3) & \gamma(T-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma(T-2) & \gamma(T-3) & \cdots & \gamma(0) & \gamma(1) \\ \gamma(T-1) & \gamma(T-2) & \cdots & \gamma(1) & \gamma(0) \end{pmatrix}$$

Notice that

$$\bigg|\gamma(0) + 2\sum_{h=1}^{T-1}\bigg(1 - \frac{h}{T}\bigg)\gamma(h)\bigg| < \bigg|2\sum_{h=0}^{T-1}\gamma(h)\bigg| \leq 2\sum_{h=0}^{T-1}|\gamma(h)|$$

Therefore

$$\sum_{h=0}^{\infty}|\gamma(h)| < \infty$$

is a sufficient condition for

$$\lim_{T\to\infty}\text{Var}(\overline{y}_T) = \lim_{T\to\infty}\frac{1}{T}\bigg[\gamma(0) + 2\sum_{h=1}^{T-1}\bigg(1 - \frac{h}{T}\bigg)\gamma(h)\bigg] = 0$$

11.4.2 Estimation of autocovariances (Book section 14.2.2).

A moment estimator of $\gamma(h) = \mathbb{E}[(y_t - \mu)(y_{t-h} - \mu)]$ is

$$\hat{\gamma}(h) = \frac{1}{T} \sum_{t=h+1}^{T} (y_t - \overline{y}_T)(y_{t-h} - \overline{y}_T)$$

By Chebyshev's Inequality (Theorem 8.4), $\hat{\gamma}(h)$ is a consistent estimator of $\gamma(h)$ as $T \to \infty$ if $\lim_{T\to\infty} \mathbb{E}(\hat{\gamma}(h)) = \gamma(h)$ and $\lim_{T\to\infty} \mathrm{Var}(\hat{\gamma}(h)) = 0$.

$$\hat{\gamma}(h) = \frac{1}{T} \sum_{t=h+1}^{T} (y_t - \overline{y}_T)(y_{t-h} - \overline{y}_T) = \frac{1}{T} \sum_{t=h+1}^{T} (y_t - \mu + \mu - \overline{y}_T)(y_{t-h} - \mu + \mu - \overline{y}_T)$$

$$= \frac{1}{T} \sum_{t=h+1}^{T} (y_t - \mu)(y_{t-h} - \mu) + (y_t - \mu)(\mu - \overline{y}_T) + (\mu - \overline{y}_T)(y_{t-h} - \mu) + (\mu - \overline{y}_T)^2$$

$$= \frac{1}{T} \sum_{t=h+1}^{T} (y_t - \mu)(y_{t-h} - \mu) + (\mu - \overline{y}_T)\frac{1}{T} \sum_{t=h+1}^{T} (y_t - \mu) + (\mu - \overline{y}_T)\frac{1}{T} \sum_{t=h+1}^{T} (y_{t-h} - \mu) + \frac{1}{T}(T-h)(\mu - \overline{y}_T)^2$$

$$\vdots$$

Because **where does this line come from? on page 300 of book/331 of pdf.**

$$\overline{y}_T = \mu + \mathcal{O}_p(T^{-1/2})$$

and for any fixed h

$$T^{-1/2} \sum_{t=h+1}^{T} (y_t - \mu) = \mathcal{O}_p(1)$$

it follows that

$$(\mu - \overline{y}_T)\frac{1}{T} \sum_{t=h+1}^{T} (y_t - \mu) = \frac{\mu}{T} \sum_{t=h+1}^{T} (y_t - \mu) - \frac{\overline{y}_T}{\sqrt{T}} \cdot \frac{1}{\sqrt{T}} \sum_{t=h+1}^{T} (y_t - \mu) = \mathcal{O}_p(T^{-1})$$

$$(\mu - \overline{y}_T)\frac{1}{T} \sum_{t=h+1}^{T} (y_{t-h} - \mu) = \mathcal{O}_p(T^{-1})$$

$$\frac{1}{T}(T-h)(\mu - \overline{y}_T)^2 = (\mu - \overline{y}_T)^2 - \frac{h}{T}(\mu - \overline{y}_T)^2 = \mathcal{O}_p(T^{-1})$$

$$\implies \hat{\gamma}(h) = \frac{1}{T}\sum_{t=h+1}^{T}(y_t - \mu)(y_{t-h} - \mu) + \mathcal{O}_p(T^{-1})$$

For $\hat{\gamma}(h)$ to be consistent, we need

$$\frac{1}{T}\sum_{t=h_1}^{T}(y_t - \mu)(y_{t-h} - \mu) \xrightarrow{p} \gamma(h)$$

First we show that $(y_t - \mu)(y_{t-h} - \mu)$ is a martingale difference process:

$$\mathbb{E}[(y_t - \mu)(y_{t-h} - \mu) \mid F_{t-h}] = (y_{t-h} - \mu)\mathbb{E}[y_t - \mu \mid F_{t-h}] = 0$$

We need to show that

$$\mathbb{E}[(y_t - \mu)^2(y_{t-h} - \mu)^2] = \mathbb{E}\left[\left(\sum_{i=0}^{\infty}\alpha_i\epsilon_{t-i}\right)^2\left(\sum_{j=0}^{\infty}\alpha_j\epsilon_{t-h-j}\right)^2\right] < \infty$$

By the Cauchy-Schwarz Inequality (Theorem 8.6), we have

$$\mathbb{E}\left|\left[\left(\sum_{i=0}^{\infty}\alpha_i\epsilon_{t-i}\right)^2\left(\sum_{j=0}^{\infty}\alpha_j\epsilon_{t-h-j}\right)^2\right]\right|^2 \leq \mathbb{E}\left[\left(\sum_{i=0}^{\infty}\alpha_i\epsilon_{t-i}\right)^2\right]^2\mathbb{E}\left[\left(\sum_{j=0}^{\infty}\alpha_j\epsilon_{t-h-j}\right)^2\right]^2$$

$$< \infty \iff \mathbb{E}\left[\sum_{i=0}^{\infty}\alpha_i\epsilon_{t-i}\right]^4 < \infty, \quad \mathbb{E}\left[\sum_{j=0}^{\infty}\alpha_j\epsilon_{t-h-j}\right]^4 < \infty$$

These conditions hold if $\mathbb{E}(\epsilon_t^4) < \infty$ and $\sum_{i=0}^{\infty}|\alpha_i| < \infty$. Then $\mathbb{E}[(y_t - \mu)^2(y_{t-h} - \mu)^2] < \infty$ holds and

$$\hat{\gamma} \xrightarrow{p} \gamma(h)$$

11.4.3 Worked examples

Midterm Problem 3 parts (3) and (4) (similar to 14.7 and 14.8 material. Consider the following ARMA(1, 1) model

$$y_t = \phi y_{t-1} + u_t + \theta u_{t-1}, \text{ for } t = -\infty, \ldots, -1, 0, 1, \ldots$$

where $|\theta| < 1$, $|\phi| < 1$, and u_t is i.i.d. with mean zero and variance σ_u^2, $\mathbb{E}(u_t^4) < \infty$.

(1) Suppose that we have the data $\{y_t : t = 0, 1, \ldots, T\}$. Consider the following estimator of ϕ:

$$\hat{\phi}_T = \frac{\sum_{t=2}^T y_t y_{t-2}}{\sum_{t=2}^T y_{t-1} y_{t-2}}$$

Show that $\hat{\phi}$ is a consistent estimator of ϕ and derive the asymptotic distribution of $\sqrt{T}(\hat{\phi}_T - \phi)$. Comment on the case where $\theta = \phi$.

(2) Suppose that $\sigma_u^2 = 1$ is known. Show that θ can be consistently estimated by

$$\hat{\theta}_T = \frac{1}{T}\sum_{t=1}^T y_t y_{t-1} - \frac{\hat{\phi}_T}{T}\sum_{t=1}^T y_{t-1}^2$$

Solution.

(1) From the results in Question 2 part 2(b) (in section 11.2.1), since $\mathbb{E}(y_t) = \mathbb{E}(y_{t-1}) = \mathbb{E}(y_{t-2}) = 0$, we know that

$$\hat{\phi}_T = \frac{\sum_{t=2}^T y_t y_{t-2}}{\sum_{t=2}^T y_{t-1} y_{t-2}} = \frac{T^{-1}\sum_{t=2}^T y_t y_{t-2}}{T^{-1}\sum_{t=2}^T y_{t-1} y_{t-2}} \xrightarrow{p} \frac{\gamma(2)}{\gamma(1)}$$

By the result from Question 3 part (2), we have $\gamma(h) = \phi\gamma(h-1)$ for $h \geq 2$. Therefore $\gamma(2)/\gamma(1) = \phi$, so $\hat{\phi}_T$ is a consistent estimator for ϕ. To obtain the asymptotic distribution, note that

$$\sqrt{T}(\hat{\phi}_T - \phi) = \sqrt{T}\left(\frac{T^{-1}\sum_{t=2}^T y_t y_{t-2}}{T^{-1}\sum_{t=2}^T y_{t-1} y_{t-2}} - \phi\right)$$

$$= \frac{T^{-1/2}\sum_{t=2}^T (\phi y_{t-1} + u_t + \theta u_{t-1}) y_{t-2}}{T^{-1}\sum_{t=2}^T y_{t-1} y_{t-2}} - \frac{\phi T^{-1/2}\sum_{t=2}^T y_{t-1} y_{t-2}}{T^{-1}\sum_{t=2}^T y_{t-1} y_{t-2}}$$

$$= \frac{T^{-1/2}\sum_{t=2}^T (u_t + \theta u_{t-1}) y_{t-2}}{T^{-1}\sum_{t=2}^T y_{t-1} y_{t-2}}$$

In Question 2 part 2(b) (in section 11.2.1), we showed that

$$\frac{1}{T}\sum_{t=h_1}^T (y_t - \mu)(y_{t-h} - \mu) \xrightarrow{p} \gamma(h)$$

Therefore in the denominator, since $\mathbb{E}(y_{t-1}) = \mathbb{E}(y_{t-h})$=0, we have

$$T^{-1}\sum_{t=2}^T y_{t-1} y_{t-2} \xrightarrow{p} \gamma(1)$$

In the numerator,

$$T^{-1/2}\sum_{t=2}^T (u_t + \theta u_{t-1}) y_{t-2} = \frac{1}{\sqrt{T}}\sum_{t=2}^T \left[u_t y_{t-2} + \theta u_{t-1} y_{t-2}\right]$$

$$= \frac{1}{\sqrt{T}}\sum_{t=2}^T u_t y_{t-2} + \frac{1}{\sqrt{T}}\sum_{t=2}^T \theta u_{t-1} y_{t-2} = \frac{1}{\sqrt{T}}\left(\sum_{t=2}^{T-1} u_t y_{t-2} + u_T y_{T-2}\right) + \frac{1}{\sqrt{T}}\sum_{t'=1}^{T-1} \theta u_{t'} y_{t'-1}$$

$$= \frac{1}{\sqrt{T}}\left(\sum_{t=2}^{T-1} u_t y_{t-2} + u_T y_{T-2}\right) + \frac{1}{\sqrt{T}}\left(\theta u_1 y_0 + \sum_{t=2}^{T-1} \theta u_t y_{t-1}\right) = \frac{1}{\sqrt{T}}\left(\sum_{t=2}^{T-1} u_t(y_{t-2} + \theta y_{t-1}) + \theta u_1 y_0 + u_T y_{T-2}\right)$$

Since $\mathbb{E}(u_t(y_{t-2} + \theta y_{t-1}) \mid F_{t-1}) = 0$. Further, $T^{-1/2}(\theta u_1 y_0 + u_{T-1} y_{T-2}) = o_p(1)$. Then by the Central Limit Theorem in martingale difference processes (Theorem (8.42)):

Theorem 28 (Central limit theorem for martingale difference sequences). Let $\{x_t\}$ be a martingale difference sequence with respect to the information set Ω_t. Let $\overline{\sigma}_T^2 = \text{Var}(\sqrt{T}\overline{x}_T) = T^{-1}\sum_{t=1}^T \sigma_t^2$. If $\mathbb{E}(|x_t|^r) < K < \infty$, $r > 2$ and for all t, and

$$\frac{1}{T}\sum_{t=1}^T x_t^2 - \overline{\sigma}_T^2 \xrightarrow{p} 0$$

then

$$\sqrt{T} \cdot \frac{\overline{x}_T}{\overline{\sigma}_T} \xrightarrow{d} \mathcal{N}(0,1)$$

we have

$$\sqrt{T} \cdot \frac{\overline{x}_T}{T^{-1/2}\sqrt{\sum_{t=1}^T \sigma_t^2}} \xrightarrow{d} \mathcal{N}(0,1)$$

$$\vdots$$

$$\frac{1}{\sigma^2}\frac{\gamma(1)^2}{(1+\theta)^2)\gamma(0) + 2\theta\gamma(1)}\sqrt{T}(\hat{\phi}_T - \phi) \xrightarrow{d} \mathcal{N}(0,1)$$

$$\iff \sqrt{T}(\hat{\phi}_T - \phi) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\frac{(1+\theta)^2)\gamma(0) + 2\theta\gamma(1)}{\gamma(1)^2}\right)$$

(2) From the results of Question 2 part 2(b) (in section 11.2.1), where we showed that

$$\frac{1}{T}\sum_{t=h_1}^T (y_t - \mu)(y_{t-h} - \mu) \xrightarrow{p} \gamma(h)$$

(and since $\mathbb{E}(y_{t-1}) = \mathbb{E}(y_{t-h})$=0,)

$$T^{-1}\sum_{t=2}^T y_t y_{t-1} \xrightarrow{p} \gamma(1), \quad T^{-1}\sum_{t=2}^T y_{t-1}^2 \xrightarrow{p} \gamma(0)$$

and by the Weak Law of Large Numbers (Theorem 8.31) we have

$$\hat{\theta}_T = \frac{1}{T}\sum_{t=1}^T y_t y_{t-1} - \frac{\hat{\phi}_T}{T}\sum_{t=1}^T y_{t-1}^2 \xrightarrow{p} \gamma(1) - \phi\gamma(0) = \phi\gamma(0) + \theta\sigma^2 - \phi\gamma(0) = \theta$$

Chapter 14 Question 3. The time series $\{y_t\}$ and $\{x_t\}$ are independently generated according to the following schemes:

$$y_t = \lambda y_{t-1} + \epsilon_{1t}, \quad |\lambda| < 1$$

$$x_t = \rho x_{t-1} + \epsilon_{2t}, \quad |\rho| < 1$$

for $t = 1, 2, \ldots, T$, where ϵ_{1t} and ϵ_{2t} are non-autocorrelated and distributed independently of each other with zero means and variances equal to σ_1^2 and σ_2^2 respectively. An investigator estimates the simple regression

$$y_t = \beta x_t + u_t \qquad t = 1, 2, \ldots T$$

by the OLS method. Show that

(a) $\hat{\beta} \xrightarrow{p} 0$ as $T \to \infty$

(b)
$$t_{\hat{\beta}}^2 = \frac{\hat{\beta}^2}{\widehat{\mathrm{Var}}(\hat{\beta})} = \frac{(T-1)r^2}{1-r^2}$$

(c)
$$Tr^2 \xrightarrow{p} \frac{1+\lambda\rho}{1-\lambda\rho} \text{ as } T \to \infty$$

where $\hat{\beta}$ is the OLS estimator of β, $\widehat{\mathrm{Var}}(\hat{\beta})$ is the estimated variance of $\hat{\beta}$, and r is the sample correlation coefficient between x and y, i.e.

$$r^2 = \left(\sum_{t=1}^{T} x_t y_t \right)^2 \Bigg/ \left(\sum_{t=1}^{T} x_t^2 \sum_{t=1}^{T} y_t^2 \right)$$

What are the implications of these results for problems of spurious correlation in economic time series analysis?

Solution.

(a)
$$\hat{\beta} = \frac{\sum_{t=1}^{T} x_t y_t}{\sum_{t=1}^{T} x_t^2} = \frac{T^{-1}\sum_{t=1}^{T} x_t y_t}{T^{-1}\sum_{t=1}^{T} x_t^2}$$

By the Weak Law of Large Numbers (Theorem 8.31), we have

$$T^{-1}\sum_{t=1}^{T} x_t y_t = T^{-1}\sum_{t=1}^{T} x_t y_t \xrightarrow{p} \mathbb{E}(x_t y_t) = \mathrm{Cov}(x_t, y_t) = 0$$

because of the i.i.d. distributions of ϵ_1 and ϵ_{2t}. By the Law of Large Numbers (Theorem 8.34),

$$T^{-1}\sum_{t=1}^T x_t^2 \xrightarrow{a.s.} \mathbb{E}(x_t^2) = \gamma_x(0) > 0$$

Therefore

$$\hat{\beta} \xrightarrow{p} \frac{0}{\gamma_x(0)} = 0$$

(b)

$$\widehat{\text{Var}}(\hat{\beta}) = \frac{\hat{\sigma}^2}{S_{XX}}$$

where

$$\hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=1}^T (y_t - \hat{\beta}x_t)^2 = \frac{1}{T-1}\sum_{t=1}^T \left(y_t^2 - 2y_t\hat{\beta}x_t + \hat{\beta}^2 x_t^2\right)$$

$$= \frac{1}{T-1}\sum_{t=1}^T y_t^2 - \frac{1}{T-1}\sum_{t=1}^T \left(2y_t x_t \frac{\sum_{t=1}^T x_t y_t}{\sum_{t=1}^T x_t^2}\right) + \frac{1}{T-1}\sum_{t=1}^T \left(\left[\frac{\sum_{t=1}^T x_t y_t}{\sum_{t=1}^T x_t^2}\right]^2 x_t^2\right)$$

$$= \frac{1}{T-1}\sum_{t=1}^T y_t^2 - 2\cdot\frac{1}{T-1}\frac{\left(\sum_{t=1}^T x_t y_t\right)^2}{\sum_{t=1}^T x_t^2} + \frac{1}{T-1}\frac{\left(\sum_{t=1}^T x_t y_t\right)^2}{\sum_{t=1}^T x_t^2}$$

$$= \frac{1}{T-1}\left(\sum_{t=1}^T y_t^2 - \frac{\left(\sum_{t=1}^T x_t y_t\right)^2}{\sum_{t=1}^T x_t^2}\right)$$

$$\implies \hat{t}^2 = \frac{\left(\sum_{t=1}^T x_t y_t\right)^2}{\left(\sum_{t=1}^T x_t^2\right)^2}\cdot\sum_{t=1}^T x_t^2\cdot(T-1)\Bigg/\left(\sum_{t=1}^T y_t^2 - \frac{\left(\sum_{t=1}^T x_t y_t\right)^2}{\sum_{t=1}^T x_t^2}\right)$$

$$= \frac{\left(\sum_{t=1}^T x_t y_t\right)^2}{\sum_{t=1}^T x_t^2}\cdot(T-1)\Bigg/\left(\sum_{t=1}^T y_t^2 - \frac{\left(\sum_{t=1}^T x_t y_t\right)^2}{\sum_{t=1}^T x_t^2}\right)$$

$$= \left(\sum_{t=1}^T x_t y_t\right)^2\cdot(T-1)\Bigg/\left(\sum_{t=1}^T y_t^2\sum_{t=1}^T x_t^2 - \left(\sum_{t=1}^T x_t y_t\right)^2\right)$$

Note that

$$r^2 = \left(\sum_{t=1}^T x_t y_t\right)^2\Bigg/\left(\sum_{t=1}^T x_t^2\sum_{t=1}^T y_t^2\right), \quad 1 - r^2 = \left[\sum_{t=1}^T x_t^2\sum_{t=1}^T y_t^2 - \left(\sum_{t=1}^T x_t y_t\right)^2\right]\Bigg/\left(\sum_{t=1}^T x_t^2\sum_{t=1}^T y_t^2\right)$$

$$\implies \frac{r^2}{1-r^2} = \left(\sum_{t=1}^T x_t y_t\right)^2\Bigg/\left[\sum_{t=1}^T x_t^2\sum_{t=1}^T y_t^2 - \left(\sum_{t=1}^T x_t y_t\right)^2\right]$$

Therefore

$$\hat{t}^2 = \left(\sum_{t=1}^T x_t y_t\right)^2\cdot(T-1)\Bigg/\left(\sum_{t=1}^T y_t^2\sum_{t=1}^T x_t^2 - \left(\sum_{t=1}^T x_t y_t\right)^2\right) = \boxed{\frac{(T-1)r^2}{1-r^2}}$$

(c)

$$Tr^2 = T \cdot \frac{\left(T^{-1}\sum_{t=1}^T x_t y_t\right)^2}{T^{-1}\sum_{t=1}^T x_t^2 \cdot T^{-1}\sum_{t=1}^T y_t^2} = \frac{T^{-1}\left(2\sum_{1\leq i<j\leq T} x_i x_j y_i y_j + \sum_{t=1}^T x_t^2 y_t^2\right)}{T^{-1}\sum_{t=1}^T x_t^2 \cdot T^{-1}\sum_{t=1}^T y_t^2}$$

$$= \frac{2T^{-1}\sum_{1\leq i<j\leq T} x_i x_j y_i y_j + T^{-1}\sum_{t=1}^T x_t^2 y_t^2}{T^{-1}\sum_{t=1}^T x_t^2 \cdot T^{-1}\sum_{t=1}^T y_t^2}$$

Note that

$$2T^{-1}\sum_{1\leq i<j\leq T}(x_i x_j)(y_i y_j) = \sum_{h=1}^{T-1} 2T^{-1}(T-h)(x_t x_{t-h})(y_t y_{t-h})$$

By the Weak Law of Large Numbers (Theorem 8.31) and the independence of x_t and y_t,

$$\sum_{h=1}^{T-1} 2T^{-1}(T-h)(x_t x_{t-h})(y_t y_{t-h}) \approx 2T^{-1}\sum_{h=1}^{T-1}(x_t x_{t-h})(y_t y_{t-h}) \xrightarrow{p} 2\sum_{h=1}^{\infty}\gamma_x(h)\gamma_y(h)$$

Recall that for an AR(1) process with coefficient ϕ, $\gamma(h) = \phi^h \gamma(0)$. (See section 11.3.4).

$$= 2\sum_{h=1}^{\infty}\rho^h\gamma_x(0)\lambda^h\gamma_y(0) = 2\gamma_x(0)\gamma_y(0)\sum_{h=1}^{\infty}(\rho\lambda)^h$$

By the Weak Law of Large Numbers (Theorem 8.31) and the independence of x_t and y_t,

$$T^{-1}\sum_{t=1}^T x_t^2 y_t^2 \xrightarrow{a.s.} \mathbb{E}(x_t^2 y_t^2) = \mathbb{E}(x_t^2)\mathbb{E}(y_t^2) = \mathrm{Var}(x_t)\mathrm{Var}(y_t) = \sigma_1^2\sigma_2^2 = \gamma_x(0)\gamma_y(0)$$

Therefore in the numerator, we have

$$2T^{-1}\sum_{1\leq i<j\leq T} x_i x_j y_i y_j + T^{-1}\sum_{t=1}^T x_t^2 y_t^2 \xrightarrow{p} 2\gamma_x(0)\gamma_y(0)\sum_{h=1}^{\infty}(\rho\lambda)^h + \gamma_x(0)\gamma_y(0)$$

$$= \gamma_x(0)\gamma_y(0)\left(2\frac{\rho\lambda}{1-\rho\lambda} + 1\right) = \gamma_x(0)\gamma_y(0)\left(\frac{1+\rho\lambda}{1-\rho\lambda}\right)$$

Next we examine the denominator. By the Law of Large Numbers (Theorem 8.34),

$$T^{-1}\sum_{t=1}^T x_t^2 \xrightarrow{a.s.} \mathbb{E}(x_t^2) = \mathrm{Var}(x_t) = \gamma_x(0) = \sigma_2^2$$

By the Law of Large Numbers (Theorem 8.34),

$$T^{-1}\sum_{t=1}^T y_t^2 \xrightarrow{a.s.} \mathbb{E}(y_t^2) = \gamma_y(0) = \sigma_1^2$$

Therefore in the denominator, we have

$$T^{-1}\sum_{t=1}^T x_t^2 \cdot T^{-1}\sum_{t=1}^T y_t^2 \xrightarrow{a.s.} \gamma_x(0)\gamma_y(0) = \sigma_1^2\sigma_2^2$$

This yields

$$\boxed{Tr^2 \xrightarrow{p} \frac{1+\rho\lambda}{1-\rho\lambda}}$$

Lidan's explanation: Because as $T \to \infty$

$$t_{\hat{\beta}} \to \sqrt{Tr^2} = \sqrt{\frac{1+\lambda\rho}{1-\lambda\rho}}$$

so if $\lambda\rho \approx 1$, at very high probability we will reject the null $\hat{\beta} = 0$ when in fact $\hat{\beta} \xrightarrow{p} 0$.

My original explanation: Because $\hat{\beta}$ converges in probability to 0 and $t^2_{\hat{\beta}}$ is proportional to r^2 (which we would expect to be close to 0), this suggests that a regression of uncorrelated variables should result in an insignificant $\hat{\beta}$, but if r^2 is high due to a spurious correlation, the $\hat{\beta}$ could be found to be statistically significant even if there is no meaningful relationship between x and y.

11.5 Chapter 15: Unit Root Processes

Need to review concepts in this chapter.

11.5.1 Worked Problems

Problem 3. Suppose that a time series of interest can be decomposed into a deterministic trend, a random walk component, and stationary errors:

$$y_t = \alpha + \delta t + \gamma_t + v_t \tag{11.1}$$

$$\gamma_t = \gamma_{t-1} + u_t$$

with

$$v_t \sim iid\, \mathcal{N}(0, \sigma_v^2), \quad u_t \sim iid\, \mathcal{N}(0, \sigma_u^2), \quad u_t \perp\!\!\!\perp v_t$$

Let $\lambda = \sigma_u^2/\sigma_v^2$.

(a) Show that under $\lambda = 0$, y_t reduces to a trend stationary process.

(b) Alternatively, suppose that y_t follows and ARIMA(0,1,1) process of the form

$$y_t = \delta + y_{t-1} + w_t \tag{11.2}$$

$$w_t = \epsilon_t + \theta \epsilon_{t-1}$$

where ϵ_t are iid $\mathcal{N}(0, \sigma_\epsilon^2)$. In this case show that under $\theta = -1$, y_t is a trend stationary process.

(c) Derive a relation between λ and the MA(1) parameter θ, and hence or otherwise show that a test of $\theta = -1$ in (11.2) is equivalent to a test of $\lambda = 0$ in (11.1).

(d) Show that (11.2) as a characterization of (11.1) implies $\theta < 0$.

Solution.

(a) $\lambda = 0 \implies \sigma_u^2 = 0 \implies u_t = 0$ (constant)

$$\implies \gamma_t = \gamma_{t-1} + 0 \iff \gamma_t = \gamma_0$$

$$\implies y_t = \alpha + \delta t + \gamma_0 + v_t = (\alpha + \gamma_0) + \delta t + v_t$$

which is trend stationary because $d_t = (\alpha + \gamma_0) + \delta t$ is perfectly predictable, and $y_t - d_t = v_t$ is covariance stationary.

Also note that $\text{Var}(y_t - \delta t) = \sigma_v^2$, $\text{Cov}\big([y_t - \delta t][y_{t-h} - \delta(t-h)]\big) = 0$, which implies trend stationarity of y_t. (Recall Definition 11.2: "X_t is said to be **trend stationary** if $y_t = X_t - d_t$ is covariance stationary, where d_t is the perfectly predictable component of X_t.")

(b) $\theta = -1 \implies w_t = \epsilon_t - \epsilon_{t-1} = (1 - L)\epsilon_t$

$$\implies (1 - L)y_t = \delta + (1 - L)\epsilon_t \iff y_t = (1 - L)^{-1}\delta + \epsilon_t$$

which is trend stationary because $d_t = (1-L)^{-1}\delta$ is perfectly predictable, and $y_t - d_t = \epsilon_t$ is covariance stationary.

(c) **Lidan's solution:** From (11.1), let

$$z_t = y_t - y_{t-1} = \alpha + \delta t + \gamma_{t-1} + u_t + v_t - (\alpha + \delta(t-1) + \gamma_{t-1} + v_{t-1})$$

$$= \delta + u_t + v_t - v_{t-1}$$

From (11.2), let

$$a_t = y_t - y_{t-1} = \delta + y_{t-1} + w_t - y_{t-1} = \delta + \epsilon_t + \theta \epsilon_{t-1}$$

Calculate the autocovariances for each and set them equal (using the serial independence of u_t, v_t, and ϵ_t as well as the independence of u_t, v_t, and ϵ_t for all t):

- $$\gamma(0) : \text{Var}(z_t) = \text{Var}(a_t) \iff \text{Var}(u_t + v_t - v_{t-1}) = \text{Var}(\epsilon_t + \theta\epsilon_{t-1})$$

$$\iff \sigma_u^2 + 2\sigma_v^2 = \sigma_\epsilon^2(1 + \theta^2) \tag{11.3}$$

- $$\gamma(1) : \text{Cov}(z_t, z_{t-1}) = \text{Cov}(a_t, a_{t-1}) \iff \text{Cov}(-v_{t-1}, v_{t-1}) = \text{Cov}(\theta\epsilon_{t-1}, \epsilon_{t-1})$$

$$\iff -\sigma_v^2 = \theta\sigma_\epsilon^2 \tag{11.4}$$

- $$\gamma(h)\ (h \geq 2) : 0 = 0$$

Plugging (11.4) into (11.3) and using $\lambda = \sigma_u^2/\sigma_v^2$ we have

$$\sigma_u^2 + 2\sigma_v^2 = -\frac{\sigma_v^2}{\theta}(1 + \theta^2) \iff \sigma_u^2 = \sigma_v^2(-1/\theta - \theta - 2) \iff \lambda = -1/\theta - \theta - 2$$

$$\iff \theta^2 + (2 + \lambda)\theta + 1 = 0 \iff \theta = \frac{-(2 + \lambda) \pm \sqrt{(2 + \lambda)^2 - 4}}{2}$$

$$\iff \boxed{\theta = \frac{-(2 + \lambda) \pm \sqrt{\lambda^2 + 4\lambda}}{2}}$$

Clearly if $\lambda = 0$ then $\theta = -1$. The reverse is also true:

$$\theta = -1 \implies \lambda = -1/(-1) - (-1) - 2 = 1 + 1 - 2 = 0$$

Therefore a test of $\theta = -1$ in (11.2) is equivalent to a test of $\lambda = 0$ in (11.1).

original solution:

$$y_t = \alpha + \delta t + \gamma_{t-1} + u_t + v_t$$

$$y_{t-1} = \alpha + \delta(t - 1) + \gamma_{t-1} + v_{t-1}$$

$$\implies y_t = y_{t-1} + \delta + u_t - v_{t-1} + v_t$$

Comparing this to the second expression, $y_t = \delta + y_{t-1} + \epsilon_t + \theta\epsilon_{t-1}$, they match if $u_t - v_{t-1} + v_t = \epsilon_t + \theta\epsilon_{t-1}$. Since the distributions of ϵ_t and v_t are both i.i.d. normal, this is the case if $\lambda = 0$ (so that $\sigma_u^2 = 0$ and $u_t = 0\ \forall\ t$), $\theta = -1$, and $\sigma_\epsilon^2 = \sigma_v^2$.

(d) **Lidan's Solution:** By (11.4) (which follows from (11.2)),

$$-\sigma_v^2 = \theta\sigma_\epsilon^2 \implies \theta < 0$$

My solution: Again consider (2)

$$y_t = \delta + y_{t-1} + \epsilon_t + \theta\epsilon_{t-1}$$

which is clearly a unit root process with a drift, compared to a re-written version of (1)

$$y_t = \delta + y_{t-1} + u_t + v_t - v_{t-1}$$

which has a strong resemblance to a unit root process with a drift. These match up if $\epsilon_t = u_t + v_t$ and $\theta < 0$.

Problem 4

Homework 4 Problem 3. Let $\{u_t\}$ be an i.i.d. sequence with mean zero and variance σ^2, and let

$$y_t = u_1 + u_2 + \ldots + u_t$$

with $y_0 = 0$.

(a) Show that

$$T^{-3/2}\sum_{t=1}^{T} y_{t-1} = T^{-1/2}\sum_{t=1}^{T-1} u_t - T^{-3/2}\sum_{t=1}^{T-1} tu_t$$

(b) Show that

$$\begin{bmatrix} T^{-1/2}\sum_{t=1}^{T} u_t \\ T^{-3/2}\sum_{t=1}^{T} y_{t-1} \end{bmatrix} \xrightarrow{d} \mathcal{N}\left(0, \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}\right)$$

(c) Use the functional central limit theorem (Donsker's Theorem, Theorem 7.64) to show that

$$T^{-3/2}\sum_{t=1}^{T} y_{t-1} \xrightarrow{d} \sigma \int_0^1 W(r)dr$$

where $W(\cdot)$ is a standard Brownian motion process.

(d) Use parts (a) - (c) to show that

$$T^{-3/2}\sum_{t=1}^{T} tu_t \xrightarrow{d} \sigma \cdot W(a) - \sigma \int_0^1 W(r)dr$$

Solution.

$$y_t = u_1 + u_2 + \ldots + u_t = \sum_{i=1}^{t} u_i = \sum_{i=0}^{t} u_i$$

where the last equality follows because $y_0 = 0 \iff u_0 = 0$.

$$u_t \sim iid\ (0, \sigma^2), \quad y_0 = 0$$

(a)

$$y_T = \sum_{t=1}^{T} u_t$$

$$T^{-3/2} \sum_{t=1}^{T} y_{t-1} = T^{-3/2} \sum_{t=1}^{T} \left(\sum_{i=0}^{t-1} u_i \right) = T^{-3/2} \sum_{t=1}^{T} (T - (t-1)) u_{t-1} = T^{-3/2} \sum_{t'=0}^{T-1} (T - t') u_{t'}$$

$$= T^{-3/2} \cdot T \sum_{t'=0}^{T-1} u_{t'} - T^{-3/2} \sum_{t'=0}^{T-1} t' u_{t'} = T^{-1/2} \sum_{t'=1}^{T-1} u_{t'} - T^{-3/2} \sum_{t'=1}^{T-1} t' u_{t'}$$

$$\implies \boxed{T^{-3/2} \sum_{t=1}^{T} y_{t-1} = T^{-1/2} \sum_{t'=1}^{T-1} u_{t'} - T^{-3/2} \sum_{t'=1}^{T-1} t' u_{t'}} \tag{11.5}$$

(b) By the Central Limit Theorem (Theorem 8.37), since $u_t \sim (0, \sigma^2)$, we have

$$\frac{1}{\sqrt{T\sigma^2}} \sum_{t=1}^{T} u_t \xrightarrow{d} \mathcal{N}(0, 1)$$

which implies

$$\boxed{\frac{1}{\sqrt{T}} \sum_{t=1}^{T} u_t \xrightarrow{d} \mathcal{N}(0, \sigma^2)} \tag{11.6}$$

The distribution of $T^{-3/2} \sum_{t=1}^{T} y_{t-1}$ is trickier. Note that from (11.5),

$$T^{-3/2} \sum_{t=1}^{T} y_{t-1} = T^{-1/2} \left(\sum_{t=0}^{T-1} u_t - \sum_{t=0}^{T-1} \frac{t u_t}{T} \right)$$

Note that $x_t = (t u_t)/T$ is a martingale difference process because

$$\mathbb{E}\left(\frac{t u_t}{T} \mid u_{t-1}, u_{t-2}, \ldots \right) = 0$$

and $\text{Var}(x_t) = \frac{1}{T^2} \text{Var}(t u_t) = \frac{t^2}{T^2} \sigma^2 < \infty$. By the Central Limit Theorem in martingale difference processes (Theorem (8.42)):

Theorem 28 (Central limit theorem for martingale difference sequences). Let $\{x_t\}$ be a martingale difference sequence with respect to the information set Ω_t. Let $\overline{\sigma}_T^2 = \text{Var}(\sqrt{T}\overline{x}_T) = T^{-1}\sum_{t=1}^T \sigma_t^2$. If $\mathbb{E}(|x_t|^r) < K < \infty$ for any $r > 2$ and for all t, and

$$\frac{1}{T}\sum_{t=1}^T x_t^2 - \overline{\sigma}_T^2 \xrightarrow{p} 0$$

then

$$\sqrt{T} \cdot \frac{\overline{x}_T}{\overline{\sigma}_T} \xrightarrow{d} \mathcal{N}(0,1)$$

since

$$T^{-1}\sum_{t=1}^T \sigma_t^2 = T^{-1}\sum_{t=1}^T \frac{t^2}{T^2}\sigma^2 = \frac{\sigma^2}{T^3} \cdot \frac{T(T+1)(2T+1)}{6} = \frac{\sigma^2}{6} \cdot \frac{2T^2+3T+1}{T^2},$$

$$\mathbb{E}(|x_t|^r) = \mathbb{E}\left(\left| \frac{tu_t}{T} \right|^r \right) = \frac{t^r}{T^r}\mathbb{E}(|u_t|^r)$$

which is finite for $r = 4$ if u_t has finite fourth moment, and since by the Weak Law of Large Numbers (Theorem 8.31)

$$T^{-1}\sum_{t=1}^T x_t^2 - T^{-1}\sum_{t=1}^T \frac{t^2}{T^2}\sigma^2 = T^{-1}\sum_{t=1}^T \frac{t^2}{T^2}\left(u_t^2 - \sigma^2\right)$$

$$\left| T^{-1}\sum_{t=1}^T \frac{t^2}{T^2}\left(u_t^2-\sigma^2\right) \right| \leq \left| T^{-1}\sum_{t=1}^T \left(u_t^2-\sigma^2\right) \right| \leq T^{-1}\sum_{t=1}^T \left|u_t^2-\sigma^2\right| \xrightarrow{p} 0$$

$$\implies T^{-1}\sum_{t=1}^T x_t^2 - T^{-1}\sum_{t=1}^T \frac{t^2}{T^2}\sigma^2 \xrightarrow{p} 0$$

we have (if u_t has finite fourth moment)

$$\sqrt{T} \cdot \frac{1}{T}\left(\sum_{t=1}^T \frac{tu_t}{T} \right) \Bigg/ \sqrt{\frac{\sigma^2}{6} \cdot \frac{2T^2+3T+1}{T^2}} \xrightarrow{d} \mathcal{N}(0,1)$$

$$\iff T^{-3/2}\left(\sum_{t=1}^T tu_t \right) \cdot \sqrt{\frac{6}{\sigma^2} \cdot \frac{1}{2T^2+3T+1}} \xrightarrow{d} \mathcal{N}(0,1) \implies \sqrt{T}\left(\sum_{t=1}^T tu_t \right) \cdot \sqrt{\frac{6}{\sigma^2} \cdot \frac{1}{2T^2}} \xrightarrow{d} \mathcal{N}(0,1)$$

$$\iff T^{-3/2}\left(\sum_{t=1}^T \frac{tu_t}{T} \right) \cdot \sqrt{\frac{3}{\sigma^2}} \xrightarrow{d} \mathcal{N}(0,1) \iff T^{-1/2}\frac{\sqrt{3}}{\sigma}\sum_{t=0}^{T-1} \frac{tu_t}{T} \xrightarrow{d} \mathcal{N}\left(0,1\right)$$

$$\iff T^{-1/2}\sum_{t=0}^{T-1} \frac{tu_t}{T} \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2}{3}\right) \tag{11.7}$$

To get the covariance between these distributions, we have (using the serial independence of u_t)

$$\text{Cov}\left(T^{-1/2}\sum_{t=0}^{T-1} u_t, T^{-3/2}\sum_{t=0}^{T-1} tu_t\right) = \mathbb{E}\left[T^{-1/2}\sum_{t=0}^{T-1} u_t \cdot T^{-3/2}\sum_{t=0}^{T-1} tu_t\right] = T^{-2}\cdot\mathbb{E}\left[\sum_{t=0}^{T-1} u_t \cdot \sum_{t=0}^{T-1} tu_t\right]$$

$$= T^{-2}\cdot\mathbb{E}\left[\sum_{t=0}^{T-1} tu_t^2\right] = T^{-2}\sum_{t=0}^{T-1} t\mathbb{E}(u_t^2) = \frac{\sigma^2}{T^2}\cdot\frac{T(T-1)}{2} \to \boxed{\frac{\sigma^2}{2}} \tag{11.8}$$

Putting (11.6), (11.7), and (11.8) together, we have $T^{-3/2}\sum_{t=1}^{T} y_{t-1} \xrightarrow{d} \mathcal{N}(0, \sigma^2 + \sigma^2/3 - 2\cdot\sigma^2/2) = \boxed{\mathcal{N}(0, \sigma^2/3)}$.

Lastly, to get the covariance between these distributions, we have (again using the serial independence of u_t)

$$\text{Cov}\left(T^{-1/2}\sum_{t=1}^{T-1} u_t, T^{-3/2}\sum_{t=1}^{T} y_{t-1}\right) = T^{-2}\text{Cov}\left(\sum_{t=1}^{T-1} u_t, \sum_{t=1}^{T}(T-(t-1))u_{t-1}\right)$$

$$= T^{-2}\text{Cov}\left(\sum_{t=1}^{T-1} u_t, \sum_{t'=0}^{T-1}(T-t')u_{t'}\right) = T^{-2}\mathbb{E}\left(\sum_{t=1}^{T-1}(T-t)u_t^2\right) = \frac{(T-1)T - T(T-1)/2}{T^2}\sigma^2$$

$$= \frac{(T-1)T}{2T^2}\sigma^2 \to \boxed{\sigma^2/2} \tag{11.9}$$

which yields the result.

(c) Let $r \in [0,1)$, $t \in [0,T]$. Define

$$R_T(r) = \frac{1}{\sigma\sqrt{T}} y_{\left[rT\right]}$$

where $\left[rT\right]$ denotes the largest integer part of rT and $y_{\left[rT\right]} = 0$ if $\left[rT\right] = 0$. That is,

$$R_T(r) = \begin{cases} 0 & 0 \le r < 1/T \\ \frac{y_1}{\sigma\sqrt{T}} & 1/T \le r < 2/T \\ \frac{y_2}{\sigma\sqrt{T}} & 2/T \le r < 3/T \\ \vdots & \vdots \\ \frac{y_{T-1}}{\sigma\sqrt{T}} & (T-1)/T \le r < 1 \end{cases}$$

We have

$$T^{-3/2}\sum_{t=1}^{T} y_{t-1} = T^{-3/2}\sum_{t'=0}^{T-1} y_{t'} = T^{-3/2}\sum_{t'=0}^{T-1}(y_{t'-1} + u_{t'}) = \sigma\sum_{t'=0}^{T-1}\int_{t'/T}^{(t'+1)/T} R_T(r)dr + o_p(1)$$

$$= \sigma\int_0^1 R_T(r)dr + o_p(1) \implies \sigma\int_0^1 W(r)dr \text{ as } T\to\infty$$

where the last step follows from Donsker's Theorem (Theorem 7.64, the functional central limit theorem) and the continuous mapping theorem (Theorem 7.65) and $W(r)$ is a standard Weiner process.

Donsker's Theorem, Theorem 43, p.335, Section 15.6.3. Let $a \in [0,1)$, $t \in [0,T]$, and suppose $(J-1)/T \leq a < J/T, J = 1,2,\ldots,T$. Define

$$R_T(a) = \frac{1}{\sqrt{T}} s_{\left[Ta\right]}$$

where

$$s_{\left[Ta\right]} = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_{\left[Ta\right]}$$

$\left[Ta\right]$ denotes the largest integer part of Ta and $s_{\left[Ta\right]} = 0$ if $\left[Ta\right] = 0$. Then $R_T(a)$ weakly converges to $w(a)$, i.e.,

$$R_T(a) \to w(a)$$

where $w(a)$ is a Wiener process. Note that when $a = 1$, $R_T(1) = 1/\sqrt{T} \cdot S_{\left[T\right]} = 1/\sqrt{T} \cdot (\epsilon_1 + \epsilon_2 + \ldots + \epsilon_T$. Since ϵ_t's are IID, by the central limit theorem, $R_T(1) \to \mathcal{N}(0,1)$.

Continuous Mapping Theorem (Theorem 44 of Pesaran in 15.6.3). Let $a \in [0,1)$, $i \in [0,n]$, and suppose $(J-1)/n \leq a < J/n, J = 1,2,\ldots,n$. Define $R_n(a) = n^{-1/2} S_{\left[n \cdot a\right]}$. If $f(\cdot)$ is continuous over $[0,1)$, then

$$f[R_n(a)] \xrightarrow{d} f[w(a)]$$

(d) We have $T^{-1/2} \sum_{t=1}^T u_t = \sigma \cdot R_T(1) \implies \sigma \cdot W(1)$ by Donsker's Theorem. Using that and the result from (c), we have

$$T^{-3/2} \sum_{t=1}^{T-1} t u_t = T^{-1/2} \sum_{t=1}^{T-1} u_t - T^{-3/2} \sum_{t=1}^{T} y_{t-1} \xrightarrow{d} \sigma \cdot W(1) - \sigma \int_0^1 W(r) dr$$

11.6 Chapter 17: Introduction to Forecasting

Feel pretty good on concepts

11.6.1 17.7: Iterated and direct multi-step AR methods

Suppose y_t follows the AR(1) model:

$$y_t = a + \phi y_{t-1} + \epsilon_t, \quad |\phi| < 1, \epsilon_t \sim iid(0, \sigma_\epsilon^2) \tag{11.10}$$

$$\iff y_t = \frac{a}{1-\phi} + \sum_{i=0}^{\infty} \phi^i \epsilon_{t-i}$$

$$\iff y_t = a\left(\frac{1-\phi^h}{1-\phi}\right) + \phi^h y_{t-h} + \sum_{j=0}^{h-1} \phi^j \epsilon_{t-j} \tag{11.11}$$

We have two methods for forecasting y_{t+h} $h > 1$ steps ahead.

(1) **Iterated method:** In this method, we first calculate the OLS estimates of $\hat{a}_T$ and $\hat{\phi}_T$ in Equation (11.12) using all available data Ω_T. Then we use the form of Equation (11.11):

$$\hat{y}^*_{T+h|T} = \hat{a}_T \left(\frac{1 - \hat{\phi}_T^h}{1 - \hat{\phi}_T} \right) + \hat{\phi}_T^h y_T$$

(2) **Direct method:** We directly calculate OLS estimates of the parameters in Equation (11.11) using all available data Ω_T:

$$\tilde{y}^*_{T+h|T} = \tilde{a}_{h,T} + \tilde{\phi}_{h,T} y_T$$

Proposition 11.1. (Pesaran Chapter 17 Proposition 45.) Suppose data is generated by Equation (11.12). If $u_t = \sum_{i=0}^{\infty} \phi^i \epsilon_{t-i}$ and $v_t = \sum_{j=0}^{h-1} \phi^j \epsilon_{t-j}$ are symmetrically distributed around zero and have finite second moments, and if $\mathbb{E}(\hat{\phi}_T$ and $\mathbb{E}(\tilde{\phi}_{h,T})$ exist, then for any finite T and h we have

$$\mathbb{E}(\hat{y}^*_{T+h|T} - y_{T+h}) = \mathbb{E}(\tilde{y}^*_{T+h|T} - y_{T+h}) = 0$$

11.6.2 Worked Problems

Problem 4 (Homework 5 Question 3—fine on all but part (c), which even Lidan is hazy on.

Consider the AR(1) model

$$y_t = \phi y_{t-1} + u_t, \qquad u_t \sim iid(0, \sigma^2) \tag{11.12}$$

(a) Derive iterated and direct forecasts of y_{T+2} condition on y_T, and show that they can be estimated as

$$\text{Iterated: } \hat{y}^{(it)}_{T+2|T} = \hat{\phi}^2 y_T$$

$$\text{Direct: } \hat{y}^{(d)}_{T+2|T} = \hat{\phi}_2 y_T$$

where $\hat{\phi}$ and $\hat{\phi}_2$ are OLS coefficients in the regressions of y_t on y_{t-1} and y_{t-2}, respectively, using the M observations $y_{T-M+1}, y_{T-M+2}, \ldots, y_T$.

(b) Show that conditional on y_t,

$$\mathbb{E}(y_{T+2} - \hat{y}^{(it)}_{T+2|T})^2 = \mathbb{E}(\phi^2 - \hat{\phi}^2)^2 y_T^2 + (1 + \phi^2)\sigma^2$$

$$\mathbb{E}(y_{T+2} - \hat{y}^{(d)}_{T+2|T})^2 = \mathbb{E}(\phi^2 - \hat{\phi}_2)^2 y_T^2 + (1 + \phi^2)\sigma^2$$

(c) Hence, or otherwise, show that

$$\lim_{M\to\infty} \mathbb{E}(d_{T+2}) = 0$$

where d_{T+2} is the loss differential of the two forecasting methods, defined by

$$d_{T+2} = (y_{T+2} - \hat{y}^{(it)}_{T+2|T})^2 + (y_{T+2} - \hat{y}^{(d)}_{T+2|T})^2$$

Solution.

(a) Note that we can write

$$y_t = \phi\big[\phi y_{t-2} + u_{t-1}\big] + u_t = \phi^2 y_{t-2} + u_t + \phi u_{t-1}$$

$$\implies y_{T+2} = \phi^2 y_T + u_{T+2} + \phi u_{T+1}$$

$$\implies \mathbb{E}(y_{T+2} \mid y_{T-M+1}, y_{T-M+2}, \ldots, y_T) = \phi^2 y_T + \mathbb{E}(u_{T+2} \mid y_{T-M+1}, \ldots, y_T) + \phi\mathbb{E}(u_{T+1} \mid y_{T-M+1}, \ldots, y_T)$$

$$= \phi^2 y_T$$

$$\iff \mathbb{E}(y_{T+2} \mid y_{T-M+1}, y_{T-M+2}, \ldots, y_T) = \phi^2 y_T \tag{11.13}$$

If we calculate the OLS estimate $\hat{\phi}$ in Equation (11.12), we can substitute that into Equation (11.13) to obtain the iterated estimate of y_{T+2}:

$$\hat{y}^{(it)}_{T+2|T} = \hat{\phi}^2 y_T$$

Since we have no intercept term, the OLS estimate would be simply

$$\hat{\phi} = (x'x)^{-1}x'y = \frac{\sum_{t=T-M+2}^{T} y_t y_{t-1}}{\sum_{t=T-M+1}^{T-1} y_t^2}$$

Alternatively, we could directly calculate the OLS estimate $\hat{\phi}_2$ of ϕ^2 in Equation (11.13)

$$\hat{y}^{(d)}_{T+2|T} = \hat{\phi}_2 y_T$$

The OLS estimate would be simply

$$\hat{\phi}_2 = (x'x)^{-1}x'y = \frac{\sum_{t=T-M+3}^{T} y_t y_{t-2}}{\sum_{t=T-M+1}^{T-2} y_t^2}$$

(b) We have

$$\mathbb{E}(y_{T+2} - \hat{y}^{(it)}_{T+2|T})^2 = \mathbb{E}\big(y^2_{T+2}\big) + \mathbb{E}\big((\hat{y}^{(it)}_{T+2|T})^2\big) - 2\mathbb{E}\big(y_{T+2} \cdot \hat{y}^{(it)}_{T+2|T}\big)$$

$$= \mathbb{E}((\phi^2 y_T + u_{T+2} + \phi u_{T+1})^2) + \mathbb{E}\big((\hat{\phi}^2 y_T)^2\big) - 2\mathbb{E}\big((\phi^2 y_T + u_{T+2} + \phi u_{T+1}) \cdot \hat{\phi}^2 y_T\big)$$

$$= \mathbb{E}(\phi^4 y_T^2 + (u_{T+2} + \phi u_{T+1})^2 + 2\phi^2 y_T(u_{T+2} + \phi u_{T+1})) + y_T^2\mathbb{E}\big(\hat{\phi}^2\big) - 2\phi^2 y_T \mathbb{E}\big(\hat{\phi}^2 y_T\big)$$

$$= \mathbb{E}(\phi^4)y_T^2 + \mathbb{E}(u^2_{T+2} + \phi^2 u^2_{T+1}) + y_T^2\mathbb{E}\big(\hat{\phi}^2\big) - 2\phi^2 y_T^2\mathbb{E}\big(\hat{\phi}^2\big) = \mathbb{E}(\phi^4)y_T^2 + (1+\phi^2)\sigma^2 + y_T^2\mathbb{E}\big(\hat{\phi}^2\big) - 2\phi^2 y_T^2\mathbb{E}\big(\hat{\phi}^2\big)$$

$$= y_T^2\big[\mathbb{E}\big(\phi^4 + \hat{\phi}^2 - 2\phi^2\hat{\phi}^2\big)\big] + (1+\phi^2)\sigma^2 = \boxed{\mathbb{E}(\phi^2 - \hat{\phi}^2)^2 y_T^2 + (1+\phi^2)\sigma^2}$$

$$\mathbb{E}(y_{T+2} - \hat{y}^{(d)}_{T+2|T})^2 = \mathbb{E}\big(y^2_{T+2}\big) + \mathbb{E}\big((\hat{y}^{(d)}_{T+2|T})^2\big) - 2\mathbb{E}\big(y_{T+2} \cdot \hat{y}^{(d)}_{T+2|T}\big)$$

$$= \mathbb{E}((\phi^2 y_T + u_{T+2} + \phi u_{T+1})^2) + \mathbb{E}\big((\hat{\phi}_2 y_T)^2\big) - 2\mathbb{E}\big((\phi^2 y_T + u_{T+2} + \phi u_{T+1}) \cdot \hat{\phi}_2 y_T\big)$$

$$= \mathbb{E}(\phi^4 y_T^2 + (u_{T+2} + \phi u_{T+1})^2 + 2\phi^2 y_T(u_{T+2} + \phi u_{T+1})) + y_T^2\mathbb{E}\big(\hat{\phi}_2\big) - 2\phi^2 y_T \mathbb{E}\big(\hat{\phi}_2 y_T\big)$$

$$= \mathbb{E}(\phi^4)y_T^2 + \mathbb{E}(u^2_{T+2} + \phi^2 u^2_{T+1}) + y_T^2\mathbb{E}\big(\hat{\phi}_2\big) - 2\phi^2 y_T^2\mathbb{E}\big(\hat{\phi}_2\big) = \mathbb{E}(\phi^4)y_T^2 + (1+\phi^2)\sigma^2 + y_T^2\mathbb{E}\big(\hat{\phi}_2\big) - 2\phi^2 y_T^2\mathbb{E}\big(\hat{\phi}_2\big)$$

$$= y_T^2\big[\mathbb{E}\big(\phi^4 + \hat{\phi}_2 - 2\phi^2\hat{\phi}_2\big)\big] + (1+\phi^2)\sigma^2 = \boxed{\mathbb{E}(\phi^2 - \hat{\phi}_2)^2 y_T^2 + (1+\phi^2)\sigma^2}$$

(c) Begin by expanding $\hat{\phi}^2$ in a first order Taylor series about ϕ:

$$\hat{\phi} = \phi + \mathcal{O}_p(M^{-1/2}) \implies \hat{\phi}^2 = \phi^2 + 2\phi(\hat{\phi} - \phi) + \mathcal{O}_p(M^{-1})$$

Then

$$\mathbb{E}\big[(\hat{\phi}^2 - \phi^2)^2 \mid y_T\big] = \mathbb{E}\big[\big(2\phi(\hat{\phi} - \phi) + \mathcal{O}_p(M^{-1})\big)^2 \mid y_T\big] = 4\phi^2\mathbb{E}\big[\big((\hat{\phi} - \phi) + \mathcal{O}_p(M^{-1})\big)^2 \mid y_T\big]$$

$$= 4\phi^2\mathbb{E}\big[\big((\hat{\phi} - \phi) + \mathcal{O}_p(M^{-1})\big)^2 \mid y_T\big]$$

11.7 Chapter 18: Measurement and Modeling of Volatility

Maybe review a little, but feel pretty okay on concepts

GARCH(1, 1) model (Pesaran Equation 18.5):

$$h_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \phi_1 h_{t-1}^2, \quad \alpha_0 > 0 \tag{11.14}$$

- This process is unconditionally stationary if $|\alpha_1 + \phi_1| < 1$.
- The unconditional variance exists and is fixed if $|\alpha_1 + \phi_1| < 1$.
- The case where $\alpha_1 + \phi_1 = 1$ is known as the Integrated GARCH(1,1), or IGARCH(1,1) for short. The RiskMetrics exponentially weighted formulation of h_t^2 for large H is a special case of the IGARCH(1,1) model where α_0 is set to 0. RiskMetrics formulation avoids the variance non-existence problem by focusing on H fixed.

11.7.1 Higher order GARCH models (Pesaran Section 18.4.2)

The various members of the GARCH and GARCH-M class of models can be written compactly as

$$y_t = \boldsymbol{\beta}' \boldsymbol{x}_{t-1} + \gamma h_t^2 + \epsilon_t \tag{11.15}$$

where

$$h_t^2 = \text{Var}(\epsilon_t \mid \Omega_{t-1}) = \mathbb{E}(\epsilon_t^2 \mid \Omega_{t-1}) = \alpha_0 + \sum_{i=1}^q \alpha_i \epsilon_{t-i}^2 + \sum_{i=1}^p \phi_i h_{t-i}^2 \tag{11.16}$$

and Ω_{t-1} is the information set at time $t-1$ containing at least $(\boldsymbol{x}_{t-1}, \boldsymbol{x}_{t-2}, \ldots, y_{t-1}, y_{t-2}, \ldots)$. The unconditional variance of ϵ_t is determined by

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^q \alpha_i \sigma_{t-i}^2 + \sum_{i=1}^p \phi_i \sigma_{t-i}^2$$

and yields a stationary outcome if all the roots of

$$1 - \sum_{i=1}^q \alpha_i \lambda^i + \sum_{i=1}^p \phi_i \lambda^i = 0$$

like outside the unit circle. In that case

$$\text{Var}(\epsilon_t) = \sigma^2 = \frac{\alpha_0}{1 - \sum_{i=1}^q \alpha_i - \sum_{i=1}^p \phi_i} > 0 \tag{11.17}$$

Clearly the necessary condition for (11.16) to be covariance stationary is given by

$$\sum_{i=1}^{q} \alpha_i + \sum_{i=1}^{p} \phi_i < 1$$

11.7.2 Testing for GARCH effects (Pesaran Section 18.5.1)

If we consider y_t as periodic data defined by $y_t = r_t - \overline{r}$ with r_t representing, say, asset return and $\overline{r}$ representing the unconditional mean, then we have the GARCH(1,1) representation of volatility:

$$\text{Var}(y_t \mid \Omega_{t-1}) = h_t^2 = \overline{\sigma}^2(1 - \alpha - \beta) + \alpha y_{t-1}^2 + \beta h_{t-1}^2$$

Then the test for GARCH effects would test

$$H_0 : \alpha = 0$$

against

$$H_1 : \alpha \neq 0$$

GARCH(1,1) can be approximated by ARCH(q):

$$\text{Var}(y_t \mid \Omega_{t-1}) = \frac{\overline{\sigma}^2(1 - \alpha - \beta)}{1 - \beta} + \alpha y_{t-1}^2 + \alpha\beta y_{t-2}^2 + \ldots + \alpha\beta^{q-1} y_{t-q}^2$$

$$= \tilde{\alpha}_0 + \tilde{\alpha}_1 y_{t-1}^2 + \tilde{\alpha}_2 z_{t-2}^2 + \ldots + \tilde{\alpha}_q z_{t-q}^2$$

which means that we can approximate this hypothesis test by instead using the Lagrange multiplier test proposed by Engle:

$$H_0 : \tilde{\alpha}_1 = \tilde{\alpha}_2 = \ldots = \tilde{\alpha}_q$$

against

$$H_1 : \tilde{\alpha}_1 \neq 0, \tilde{\alpha}_2 \neq 0, \ldots, \tilde{\alpha}_q \neq 0$$

11.7.3 Worked Problems

Problem 1. Consider the generalized autoregressive heteroskedastic model

$$y_t = h_t z_t$$

where

$$z_t \mid \Omega_{t-1} \sim IID\mathcal{N}(0,1) \tag{11.18}$$

$$h_t^2 = \text{Var}(y_t \mid \Omega_{t-1}) = \mathbb{E}(y_t^2 \mid \Omega_{t-1}) = \overline{\sigma}^2(1-\alpha-\beta) + \alpha y_{t-1}^2 + \beta h_{t-1}^2 \tag{11.19}$$

and Ω_t is the information set that contains at least y_t and its lagged values.

(a) Derive the conditions under which $\{y_t\}$ is a stationary process.

(b) Are the observations $\{y_t\}$ serially independent and/or serially uncorrelated?

(c) Develop a test of the GARCH effect and discuss the estimation of the above model by the maximum likelihood method.

(d) Discuss the relevance of GARCH models for the analysis of financial time series data.

Solution.

(a) Note that (using the fact that y_t and h_t are conditionally independent given Ω_{t-1})

$$\mathbb{E}(y_t) = \mathbb{E}(h_t z_t) = \mathbb{E}\big[\mathbb{E}(h_t z_t \mid \Omega_{t-1})\big] = \mathbb{E}\big[\mathbb{E}(h_t \mid \Omega_{t-1})\mathbb{E}(z_t \mid \Omega_{t-1})\big] = \mathbb{E}\big[\mathbb{E}(h_t \mid \Omega_{t-1}) \cdot 0\big] = 0$$

We have

$$\text{Var}(y_t) = \mathbb{E}\big[\big(y_t - \mathbb{E}(y_t)\big)^2\big] = \mathbb{E}\big[y_t^2\big] = \mathbb{E}\big[\mathbb{E}(y_t^2 \mid \Omega_{t-1})\big] = \mathbb{E}(h_t^2) = \mathbb{E}(\overline{\sigma}^2(1-\alpha-\beta) + \alpha y_{t-1}^2 + \beta h_{t-1}^2)$$

$$= \mathbb{E}\big(\overline{\sigma}^2(1-\alpha-\beta)\big) + \alpha\mathbb{E}(y_{t-1}^2) + \beta\mathbb{E}(h_{t-1}^2) = \overline{\sigma}^2(1-\alpha-\beta) + (\alpha+\beta)\text{Var}(y_{t-1})$$

Therefore in order for this to be a stationary process, we require $\boxed{|\alpha+\beta| < 1}$.

more stuff I did on original homework

By Definition 11.1 $\{y_t\}$ is a stationary process if it has constant mean and its covariance function depends only on the absolute difference $|t_1 - t_2|$; that is,

$$\text{Cov}(y_{t_1}, y_{t_2}) = \gamma(t_1, t_2) = \gamma(|t_1 - t_2|) \text{ for all } t_1, t_2$$

$$\vdots$$

In order for this to be true, we must first show that h_t is finite for all t. It is sufficient to find the conditions that make h_t stationary. Using Equation (11.16)

$$h_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{i=1}^{p} \phi_i h_{t-i}^2$$

From equation (11.19) we have

$$h_t^2 = \overline{\sigma}^2(1 - \alpha - \beta) + \alpha y_{t-1}^2 + \beta h_{t-1}^2$$

Therefore we note that $p = q = 1$ and we have a GARCH(1,1) model. From Equation (11.14):

$$h_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \phi_1 h_{t-1}^2, \quad \alpha_0 > 0$$

is unconditionally stationary if $|\alpha_1 + \phi_1| < 1$. In this case, we require $\boxed{|\alpha + \beta| < 1}$ for stationarity of h_t.

$$\vdots$$

Assume without loss of generality that $t_2 \geq t_1$. (Note that this implies $z_{t_2} \perp\!\!\!\perp z_{t_1}, h_{t_2}, h_{t_1} \mid \Omega_{t_2 - 1}$.)

$$\text{Cov}(y_{t_1}, y_{t_2}) = \mathbb{E}\big[\big(y_{t_1} - \mathbb{E}(y_{t_1})\big)\big(y_{t_2} - \mathbb{E}(y_{t_2})\big)\big] = \mathbb{E}\big[y_{t_1} y_{t_2}\big] = \mathbb{E}\big[h_{t_1} h_{t_2} z_{t_1} z_{t_2}\big]$$

$$= \mathbb{E}\big[\mathbb{E}(h_{t_1} h_{t_2} z_{t_1} z_{t_2} \mid \Omega_{t_2-1})\big] = \mathbb{E}\big[\mathbb{E}(h_{t_1} h_{t_2} z_{t_1} \mid \Omega_{t_2-1})\mathbb{E}(z_{t_2} \mid \Omega_{t_2-1})\big]$$

$$= \mathbb{E}\big[\mathbb{E}(h_{t_1} h_{t_2} z_{t_1} \mid \Omega_{t_2-1}) \cdot 0\big] = \boxed{0}$$

Therefore given that h_t is finite (that is, given $|\alpha + \beta| < 1$), we have that $\mathbb{E}(y_t) = 0$, $\text{Cov}(y_{t_1}, y_{t_2}) = 0 \;\; \forall \; t_1, t_2$ which implies that under these conditions y_t is stationary.

(b) y_t is serially uncorrelated because $\text{Cov}(y_{t_1}, y_{t_2}) = 0 \;\; \forall \; t_1, t_2$. However, it is clear that y_t is not serially independent since past values of y_t affect $h_t^2 = \text{Var}(y_t)$. Observe that

$$\Pr(y_t \leq y \mid y_{t-1}) = \Pr(h_t z_t \leq y \mid y_{t-1}) = \Pr\left(z_t \sqrt{\overline{\sigma}^2(1 - \alpha - \beta) + \alpha y_{t-1}^2 + \beta h_{t-1}^2} \leq y \mid y_{t-1}\right)$$

In other words, even though the mean of y_t remains constant, because y_{t-1} affects the variance of y_t, it affects the heaviness of the tails of y_t, changing the probability distribution of y_t. Therefore the conditional cumulative distribution function of y_t given y_{t-1} is not equal to the unconditional cdf, so y_t and y_{t-1} are not independent.

Lidan's Explanation: From (a) we know that $\{y_t\}$ is serially uncorrelated, but they are not necessarily independent because from Equation (11.19) we know that h_t is not independent from h_{t-1}.

(c) See section 11.7.2.

(d) **Lidan:** Usually financial time series data are fat-tailed. So the series may not look stationary, and therefore the local variance would be clustered in some very low and very high values. To capture this serial correlation and heterogeneity in volatility, we need to use a GARCH model.

Book: In financial econometrics, ARCH and GARCH are fundamental tools for analyzing the time-variation of conditional variance. In many applications in finance, the assumption that the conditional variance of the disturbances is constant over time is not valid. GARCH models allow for time variation in volatility, relating (unobserved) volatility to squares of past innovations in price changes. However, this approach only partly overcomes the deficiency of the historical measure and continues to respond very slowly when volatility undergoes rapid changes.

11.8 Chapter 21: Vector Autoregressive Models

Feel ok on concepts.

11.8.1 Worked Problems

Problem 1. Consider the bivariate autoregressive model:

$$Y_t = \mathbf{\Phi} Y_{t-1} + U_t, \qquad U_t \sim IID\mathcal{N}(0, \Sigma) \tag{11.20}$$

where

$$Y_t = (y_{1t}, y_{2t})',\ Y_{t-1} = (y_{1,t-1}, y_{2,t-1})',\ U_t = (u_{1t}, u_{2t})'$$

and

$$\mathbf{\Phi} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \succ 0$$

(a) Derive the conditional mean and variance of y_{1t} with respect to y_{2t} and lagged values of y_{1t} and y_{2t}.

(b) Show that the univariate representation of y_{1t} is an ARMA(2,1) process.

Solution

(a) Note that

$$y_{1t} = \phi_{11} y_{1,t-1} + \phi_{12} y_{2,t-1} + u_{1t} \tag{11.21}$$

$$y_{2t} = \phi_{21} y_{1,t-1} + \phi_{22} y_{2,t-1} + u_{2t} \tag{11.22}$$

First,

$$\mathbb{E}(y_{1t} \mid y_{2t}, y_{2,t-1}, y_{2,t-2}, \ldots, y_{1,t-1}, y_{1,t-2}, \ldots) = \mathbb{E}(y_{1t} \mid y_{2t}, \Omega_{t-1})$$

$$= \phi_{11}y_{1,t-1} + \phi_{12}y_{2,t-1} + \mathbb{E}(u_{1t} \mid y_{2t}) = \phi_{11}y_{1,t-1} + \phi_{12}y_{2,t-1} + \mathbb{E}(u_{1t} \mid \phi_{21}y_{1,t-1} + \phi_{22}y_{2,t-1} + u_{2t})$$

$$= \phi_{11}y_{1,t-1} + \phi_{12}y_{2,t-1} + \mathbb{E}(u_{1t} \mid u_{2t})$$

Recall Proposition 6.69: for a bivariate normal distribution with mean 0, the conditional distribution of u_{1t} given u_{2t} is

$$u_{1t} \mid u_{2t} \sim \mathcal{N}\left(\rho\frac{\sqrt{\sigma_{11}}}{\sqrt{\sigma_{22}}}u_{2t}, (1-\rho^2)\sigma_{11}\right)$$

where $\rho = \sigma_{12}/\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}$.

$$\iff u_{1t} \mid u_{2t} \sim \mathcal{N}\left(\frac{\sigma_{12}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}} \cdot \frac{\sqrt{\sigma_{11}}}{\sqrt{\sigma_{22}}}u_{2t}, \left[1 - \left(\frac{\sigma_{12}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}}\right)^2\right]\sigma_{11}\right)$$

$$\iff u_{1t} \mid u_{2t} \sim \mathcal{N}\left(\frac{\sigma_{12}}{\sigma_{22}}u_{2t}, \sigma_{11} - \frac{\sigma_{12}^2}{\sigma_{22}}\right)$$

$$\implies \mathbb{E}(u_{1t} \mid u_{2t}) = \frac{\sigma_{12}}{\sigma_{22}}u_{2t}, \ \operatorname{Var}(u_{1t} \mid u_{2t}) = \sigma_{11} - \frac{\sigma_{12}^2}{\sigma_{22}}$$

$$\implies \mathbb{E}(y_{1t} \mid y_{2t}, \Omega_{t-1}) = \phi_{11}y_{1,t-1} + \phi_{12}y_{2,t-1} + \frac{\sigma_{12}}{\sigma_{22}}u_{2t}$$

$$= \phi_{11}y_{1,t-1} + \phi_{12}y_{2,t-1} + \frac{\sigma_{12}}{\sigma_{22}}(y_{2t} - \phi_{21}y_{1,t-1} - \phi_{22}y_{2,t-1})$$

$$= \boxed{\left(\phi_{11} - \frac{\sigma_{12}}{\sigma_{22}}\phi_{21}\right)y_{1,t-1} + \left(\phi_{12} - \frac{\sigma_{12}}{\sigma_{22}}\phi_{22}\right)y_{2,t-1} + \frac{\sigma_{12}}{\sigma_{22}}y_{2t}}$$

Second,

$$\operatorname{Var}(y_{1t} \mid y_{2t}, \Omega_{t-1}) = \phi_{11}^2\operatorname{Var}(y_{1,t-1} \mid y_{2t}, \Omega_{t-1}) + \phi_{12}^2\operatorname{Var}(y_{2,t-1} \mid y_{2t}, \Omega_{t-1}) + \operatorname{Var}(u_{1t} \mid y_{2t}, \Omega_{t-1})$$

$$= \operatorname{Var}(u_{1t} \mid u_{2t}) = \boxed{\sigma_{11} - \frac{\sigma_{12}^2}{\sigma_{22}}}$$

(b) Again, using equations (11.21) and (11.22),

$$y_{1t} = \phi_{11}y_{1,t-1} + \phi_{12}y_{2,t-1} + u_{1t}, \quad y_{2t} = \phi_{21}y_{1,t-1} + \phi_{22}y_{2,t-1} + u_{2t}$$

$$\implies (1 - \phi_{11}L)y_{1t} = \phi_{12}Ly_{2t} + u_{1t} \tag{11.23}$$

$$(1 - \phi_{22}L)y_{2t} = \phi_{21}Ly_{1t} + u_{2t} \tag{11.24}$$

We can multiply $(1 - \phi_{22}L)$ on both sides of (11.23) to yield

$$(1-\phi_{22}L)(1-\phi_{11}L)y_{1t} = \phi_{12}L(1-\phi_{22}L)y_{2t} + (1-\phi_{22}L)u_{1t}$$

Using (11.24), we have

$$(1-(\phi_{22}+\phi_{11})L + \phi_{11}\phi_{22}L^2)y_{1t} = \phi_{12}L\big[\phi_{21}Ly_{1t} + u_{2t}\big] + (1-\phi_{22}L)u_{1t}$$

$$\iff (1-(\phi_{22}+\phi_{11})L + (\phi_{11}\phi_{22}-\phi_{12}\phi_{21})L^2)y_{1t} = \phi_{12}Lu_{2t} + (1-\phi_{22}L)u_{1t} \tag{11.25}$$

If we can show that the left side of (11.25) is an AR(2) process and the right side is an MA(1) process, we are done. **Lidan: "It is obvious that the left hand side of (11.25) is an AR(2) process."** The left side is a stationary AR(2) process if the absolute values of all the roots of

$$1-(\phi_{22}+\phi_{11})z + (\phi_{11}\phi_{22}-\phi_{12}\phi_{21})z^2 = 0$$

are greater than 1. The roots are

$$z = \frac{\phi_{22}+\phi_{11} \pm \sqrt{(\phi_{22}+\phi_{11})^2 - 4(\phi_{11}\phi_{22}-\phi_{12}\phi_{21})}}{2(\phi_{11}\phi_{22}-\phi_{12}\phi_{21})} = \frac{\phi_{22}+\phi_{11} \pm \sqrt{\phi_{22}^2+\phi_{11}^2 - 2\phi_{22}\phi_{11} + 4\phi_{12}\phi_{21}}}{2(\phi_{11}\phi_{22}-\phi_{12}\phi_{21})}$$

$$= \frac{\phi_{22}+\phi_{11} \pm \sqrt{(\phi_{22}-\phi_{11})^2 + 4\phi_{12}\phi_{21}}}{2(\phi_{11}\phi_{22}-\phi_{12}\phi_{21})}$$

$$\vdots$$

To check if the right side of (11.25) is MA(1), we will write $x_t = \phi_{12}Lu_{2t} + (1-\phi_{22}L)u_{1t}$ as an MA(1) process; that is,

$$x_t = \phi_{12}u_{2,t-1} + u_{1t} - \phi_{22}u_{1,t-1} = \xi_t + \theta\xi_{t-1}$$

with $\xi_t \sim iid(0,\sigma_\xi^2)$, $|\theta|<1$. If x_t is an MA(1) process, it must satisfy

$$\gamma(0) = \mathbb{E}(x_t^2) = (1+\theta^2)\sigma_\xi^2, \quad \gamma(1) = \mathbb{E}(x_tx_{t-1}) = \theta\sigma_\xi^2$$

We have (using the serial independence of the u_{1t}, u_{2t})

$$\mathbb{E}(x_t^2) = \mathbb{E}\big(\big[\phi_{12}u_{2,t-1} + u_{1t} - \phi_{22}u_{1,t-1}\big]^2\big) = \mathbb{E}\big(\phi_{12}^2u_{2,t-1}^2 - 2\phi_{12}\phi_{22}u_{2,t-1}u_{1,t-1} + \phi_{22}^2u_{1,t-1}^2 + u_{1t}^2\big)$$

$$= \phi_{12}^2\mathbb{E}\big(u_{2,t-1}^2\big) - 2\phi_{12}\phi_{22}\mathbb{E}\big(u_{2,t-1}u_{1,t-1}\big) + \phi_{22}^2\mathbb{E}\big(u_{1,t-1}^2\big) + \mathbb{E}\big(u_{1t}^2\big)$$

$$= \phi_{12}^2\sigma_{22} - 2\phi_{12}\phi_{22}\sigma_{12} + \phi_{22}^2\sigma_{11} + \sigma_{11} = (1+\phi_{22}^2)\sigma_{11} - 2\phi_{12}\phi_{22}\sigma_{12} + \phi_{12}^2\sigma_{22}$$

$$\mathbb{E}(x_tx_{t-1}) = \mathbb{E}\big(\big[\phi_{12}u_{2,t-1} + u_{1t} - \phi_{22}u_{1,t-1}\big]\big[\phi_{12}u_{2,t-2} + u_{1,t-1} - \phi_{22}u_{1,t-2}\big]\big)$$

$$= \mathbb{E}\big(\phi_{12}u_{2,t-1}u_{1,t-1} - \phi_{22}u_{1,t-1}^2\big) = \phi_{12}\sigma_{12} - \phi_{22}\sigma_{11}$$

Therefore we require

$$(1+\theta^2)\sigma_\xi^2 = (1+\phi_{22}^2)\sigma_{11} - 2\phi_{12}\phi_{22}\sigma_{12} + \phi_{12}^2\sigma_{22} \tag{11.26}$$

$$\theta\sigma_\xi^2 = \phi_{12}\sigma_{12} - \phi_{22}\sigma_{11} \tag{11.27}$$

Dividing (11.26) by (11.27) we have

$$\frac{1+\theta^2}{\theta} = \frac{(1+\phi_{22}^2)\sigma_{11} - 2\phi_{12}\phi_{22}\sigma_{12} + \phi_{12}^2\sigma_{22}}{\phi_{12}\sigma_{12} - \phi_{22}\sigma_{11}} \tag{11.28}$$

For simplicity, let the right side of (11.28) be $A \in \mathbb{R}$. Then we have

$$\theta^2 - A\theta + 1 = 0 \iff \boxed{\theta_1 = \frac{1}{2}\big(A + \sqrt{A^2-4}\big), \theta_2 = \frac{1}{2}\big(A + \sqrt{A^2-4}\big)}$$

Then the corresponding σ_ξ^2 are

$$\boxed{\sigma_{\xi,1}^2 = \sigma_{\xi,2}^2 = \frac{\phi_{12}\sigma_{12} - \phi_{22}\sigma_{11}}{\theta_1}}$$

As a double check,

$$\mathbb{E}(x_t x_{t-2}) = \mathbb{E}\big(\big[\phi_{12}u_{2,t-1} + u_{1t} - \phi_{22}u_{1,t-1}\big]\big[\phi_{12}u_{2,t-3} + u_{1,t-2} - \phi_{22}u_{1,t-3}\big]\big) = 0$$

as expected for an MA(1) process. Therefore the right side of (11.25) is an MA(1) process (provided that $0 < |\theta_1| < 1$ and/or $0 < |\theta_2| < 1$). Since the left side is an AR(2) process, this proves that the univariate representation (11.25) of y_{1t} is an ARMA(2,1) process.

Problem 2. Consider the VAR(2) model in the m-dimensional vector Y_t:

$$Y_t = \mu + \Phi_1 Y_{t-1} + \Phi_2 Y_{t-2} + U_t, \quad U_t \sim (0, \Sigma) \tag{11.29}$$

where μ is an m-vector of fixed constants.

(a) Derive the conditions under which the VAR(2) model defined in (11.29) is stationary.

(b) Derive the error correction form of (11.29) and discuss what is meant by the process Y_t being cointegrated.

(c) Suppose now that one or more elements of Y_t is I(1). Derive suitable restrictions on the intercepts μ such that despite the I(1) nature of the variables in (11.29), Y_t has a fixed mean. Discuss the importance of such restrictions for the analysis of cointegration.

Solution.

(a) Let

$$Y_t^* = Y_t - (I - \Phi_1 - \Phi_2)^{-1}\mu. \tag{11.30}$$

Then

$$Y_t^* = \mu + \Phi_1 Y_{t-1} + \Phi_2 Y_{t-2} + U_t - (I - \Phi_1 - \Phi_2)^{-1}\mu$$

$$= \mu + \Phi_1 Y_{t-1} + \Phi_2 Y_{t-2} + U_t - [(I - \Phi_1 - \Phi_2) + \Phi_1 + \Phi_2](I - \Phi_1 - \Phi_2)^{-1}\mu$$

$$= \mu - (I - \Phi_1 - \Phi_2)(I - \Phi_1 - \Phi_2)^{-1}\mu + \Phi_1(Y_{t-1} - (I - \Phi_1 - \Phi_2)^{-1}\mu) + \Phi_2(Y_{t-2} - (I - \Phi_1 - \Phi_2)^{-1}\mu) + U_t$$

$$= \Phi_1 Y_{t-1}^* + \Phi_2 Y_{t-2}^* + U_t \tag{11.31}$$

Equation (11.31) can be rewritten in the companion form as follows:

$$\begin{bmatrix} Y_t^* \\ Y_{t-1}^* \end{bmatrix} = \begin{bmatrix} \Phi_1 & \Phi_2 \\ I & 0 \end{bmatrix} \begin{bmatrix} Y_{t-1}^* \\ Y_{t-2}^* \end{bmatrix} + \begin{bmatrix} U_t \\ 0 \end{bmatrix} \tag{11.32}$$

which is a VAR(1) model. If Y_{-M+1}, Y_{-M+2} are given, equation (11.32) can be solved iteratively from $t = -M + 2$ to obtain

$$\begin{bmatrix} Y_t^* \\ Y_{t-1}^* \end{bmatrix} = \begin{bmatrix} \Phi_1 & \Phi_2 \\ I & 0 \end{bmatrix}^{t+M-2} \begin{bmatrix} Y_{-M+2}^* \\ Y_{-M+1}^* \end{bmatrix} + \sum_{j=0}^{t+M-3} \begin{bmatrix} \Phi_1 & \Phi_2 \\ I & 0 \end{bmatrix}^j \begin{bmatrix} U_{t-j} \\ 0 \end{bmatrix} \tag{11.33}$$

Then the condition for (11.33) to be covariance stationary is for all the eigenvalues of

$$\Phi = \begin{bmatrix} \Phi_1 & \Phi_2 \\ I & 0 \end{bmatrix}$$

to lie inside the unit circle; that is, the solutions of

$$|\Phi - \lambda I_4| = 0$$

must satisfy $|\lambda| < 1$. Equivalently, the stability condition can be written in terms of the roots of the determinantal equation

$$|I_2 - \Phi_1 z - \Phi_2 z^2| = 0$$

in which case the process Y_t^* will be stationary if all the roots lie outside the unit circle ($|z| > 1$). Then if Y_t^* is stationary, so is Y_t, so either of these conditions (plus the invertibility of $(I - \Phi_1 - \Phi_2)$) ensure stationarity of Y_t.

(b) From (11.31) we have

$$Y_t^* = \Phi_1 Y_{t-1}^* + \Phi_2 Y_{t-2}^* + U_t$$

Note that (letting $\Delta Y_t^* = Y_t^* - Y_{t-1}^*$)

$$Y_t^* - Y_{t-1}^* + Y_{t-1}^* = \Phi_1 Y_{t-1}^* + \Phi_2\big(Y_{t-2}^* - Y_{t-1}^* + Y_{t-1}^*\big) + U_t$$

$$\iff \Delta Y_t^* + Y_{t-1}^* = \Phi_1 Y_{t-1}^* + \Phi_2\big(Y_{t-1}^* - \Delta Y_{t-1}^*\big) + U_t$$

$$\iff \Delta Y_t^* = -(I - \Phi_1 - \Phi_2) Y_{t-1}^* - \Phi_2 \Delta Y_{t-1}^* + U_t \iff \boxed{\Delta Y_t^* = -\Pi Y_{t-1}^* + \Gamma \Delta Y_{t-1}^* + U_t}$$

where $\Pi = (I - \Phi_1 - \Phi_2)$ and $\Gamma = -\sum_{i=2}^{2} \Phi_i = -\Phi_2$. Because

$$\Delta Y_t^* = Y_t^* - Y_{t-1}^* = Y_t - (I - \Phi_1 - \Phi_2)^{-1}\mu - \big[Y_{t-1} - (I - \Phi_1 - \Phi_2)^{-1}\mu\big] = Y_t - Y_{t-1} = \Delta Y_t,$$

this can also be written as

$$\Delta Y_t = -\Pi\big[Y_{t-1} - (I - \Phi_1 - \Phi_2)^{-1}\mu\big] + \Gamma \Delta Y_{t-1} + U_t$$

$$= -\Pi\big[Y_{t-1} - \Pi^{-1}\mu\big] + \Gamma \Delta Y_{t-1} + U_t = \Pi \cdot \Pi^{-1}\mu - \Pi Y_{t-1} + \Gamma \Delta Y_{t-1} + U_t$$

$$\implies \boxed{\Delta Y_t = \mu - \Pi Y_{t-1} + \Gamma \Delta Y_{t-1} + U_t} \tag{11.34}$$

For the definition of cointegration, see Definition 11.3 and Section 11.9.1. In this particular case, if $Y_{t-1}^* \sim I(1)$ and the linear combinations ΠY_{t-1}^* of Y_{t-1}^* are covariance stationary (that is, $\Pi Y_{t-1}^* \sim I(0)$), we say Y_t^* is cointegrated (and therefore so is Y_t).

(c) Since (11.34) is I(0), for Y_t to have fixed mean, take expectations on both sides of (11.34):

$$\mu - (I_m - \Phi_1 - \Phi_2)\mathbb{E}(Y_{t-1}) = 0$$

If this restriction is violated, then (11.34) becomes a stationary process with a drift, which implies (11.29) has a time trend.

11.9 Chapter 22: Cointegration Analysis

Feel pretty good except for long run effects, examples we went over in class, the restrictions, and the 5 cases. Don't need to understand SURE.

11.9.1 22.4 Cointegrating VAR: multiple cointegrating relations and 22.5: Identification of long-run effects

Definition 11.3. We say that the m variables in Y_t are *cointegrated* if they are individually integrated (or have a random walk component) but there exist linear combinations of them which are stationary. That is, $y_{it} \sim I(1)$ for $i = 1, 2, \ldots, m$, but there exists an $m \times r$ matrix β such that $\beta' Y_t = \xi_t \sim I(0)$.

- In this case r denotes the number of cointegrating vectors, also known as the dimension of the cointegration space.
- The cointegrating relations summarized in the $r \times 1$ vector $\beta' Y_t$ are also known as long-run relations.
- $r = \textbf{rank}(\Pi)$ is the dimension of the cointegration space.
- Cointegration is present if Π is rank-deficient; that is, $r < m$.

When $\textbf{rank}(\Pi) = r < m$, we can write Π as

$$\Pi = \alpha\beta' \tag{11.35}$$

where α and β are $m \times r$ matrices of full column rank. Then

$$\Pi y_{t-1} = \alpha\beta' y_{t-1} \sim I(0)$$

and the VECM can be written as

$$\Delta y_t = -\alpha\beta' y_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta y_{t-j} + u_t \tag{11.36}$$

Since α is full rank, we have

$$\beta' y_{t-1} \sim I(0)$$

where $\beta' y_t$ is the r-vector of cointegrating relations, also known as the long-run relations.

However, β as defined above is not uniquely determined. Consider a linear transformation of β by a non-singular $r \times r$ matrix Q: $\tilde{\beta} = \beta Q$. Then since $\textbf{rank}(\Pi) = r < m$, Π can be expressed as $\Pi = \alpha\beta'$ where α is a rank r $m \times r$ matrix. But consider that

$$\Pi = \alpha\beta = (\alpha {Q'}^{-1})(Q'\beta') = \tilde{\alpha}\tilde{\beta}'$$

with $\tilde{\alpha} = \alpha {Q'}^{-1}$. Therefore β is not uniquely determined without r^2 exact- or just-identifying restrictions, r restrictions on each of the r cointegrating relations.

11.9.2 Worked Problems

Problem 2: See Chapter 21 Problem 2.

11.10 Chapter 23: VARX Modeling

Lidan says will not be on final.

11.10.1 Worked Problems

11.11 Chapter 24: Impulse Response Analysis

Feel ok on concepts, should review and do an example problem.

11.11.1 Worked Problems

Chapter 24 Problem 1. Consider the VAR(2) model

$$x_t = \Phi_1 x_{t-1} + \Phi_2 x_{t-2} + \epsilon_t, \quad \epsilon_t \sim IID(0, \Sigma)$$

in the $m \times 1$ vectof of random variables x_t and Σ is the covariance matrix of the errors with typical element σ_{ij}.

(a) Derive the conditions under which this process is stationary, and show that it has the following moving average representation:

$$x_t = \sum_{j=0}^{\infty} A_j \epsilon_{t-j} \tag{11.37}$$

(b) Derive the coefficient matrices A_j in terms of Φ_1 and Φ_2.

(c) Using the above result, write down the orthogonalized (OIR) and generalize impulse (GIR) response functions of one standard error shock (i.e. $\sqrt{\sigma_{ii}}$ to the error of the ith equation, $\epsilon_{it} = s_i' \epsilon_t$, where s_i is am $m \times 1$ selection vector.

(d) What are the main differences betwen OIR and GIR functions?

Chapter 24 Problem 1 Solution.

(a) For stationarity conditions, see Ch. 21 Problem 2 in Section 11.8.1. To get the MA representation

$$x_t = \sum_{j=0}^{\infty} A_j \epsilon_{t-j} \tag{11.38}$$

of (11.37) note that

$$(I - \Phi_1 L - \Phi_2 L^2)x_t = \epsilon_t$$

$$\implies x_t = (I - \Phi_1 L - \Phi_2 L^2)^{-1}\epsilon_t = \sum_{j=0}^{\infty} A_j \epsilon_{t-j}$$

for some $\{A_j\}$.

(b) Now we seek to evaluate $(I - \Phi_1 L - \Phi_2 L^2)^{-1}$. To do this, let $(I - \Phi_1 L - \Phi_2 L^2)^{-1} = A_0 + A_1 L + A_2 L^2 + A_3 L^3 + \ldots$ and note that

$$(I - \Phi_1 L - \Phi_2 L^2)(I - \Phi_1 L - \Phi_2 L)^{-1} = I$$

$$\iff (I - \Phi_1 L - \Phi_2 L^2)(A_0 + A_1 L + A_2 L^2 + A_3 L^3 + \ldots + A_j L^j + \ldots) = I$$

$$\iff A_0 + (A_1 - \Phi_1)L + (A_2 - \Phi_1 A_1 - \Phi_2)L^2 + (A_3 - \Phi_1 A_2 - \Phi_2 A_1)L^3 + \ldots$$

$$+(A_j - \Phi_1 A_{j-1} + \Phi_2 A_{j-2})L^j + \ldots = I$$

In order for this equation to hold, all the lag terms must equal zero and the constant matrix A_0 must equal I.

$$\implies \boxed{A_0 = I}$$

$$A_1 - \Phi_1 = 0 \iff \boxed{A_1 = \Phi_1}$$

$$A_2 - \Phi_1 A_1 - \Phi_2 = 0 \iff A_2 - \Phi_1^2 - \Phi_2 = 0 \iff \boxed{A_2 = \Phi_1^2 + \Phi_2}$$

$$A_3 - \Phi_1 A_2 - \Phi_2 A_1 = 0 \iff A_3 - \Phi_1^3 - \Phi_1\Phi_2 - \Phi_2\Phi_1 = 0 \iff \boxed{A_3 = \Phi_1^3 + \Phi_1\Phi_2 + \Phi_2\Phi_1}$$

and, in general,

$$\boxed{A_j = \Phi_1 A_{j-1} + \Phi_2 A_{j-2}}$$

(c)
- **OIR:** We employ the Cholesky decomposition of Σ:

$$\Sigma = PP' \tag{11.39}$$

where P is a lower-triangular matrix. Then the MA representation (11.38) can be written as

$$x_t = \sum_{j=0}^{\infty}(A_j P)(P^{-1}u_{t-j}) = \sum_{j=0}^{\infty} B_j \eta_{t-j} \tag{11.40}$$

where $B_j = A_j P$, $\eta_t = P^{-1}u_t$, so we have

$$\mathbb{E}(\eta_t\eta_t') = P^{-1}\mathbb{E}(u_t u_t')(P^{-1})' = P^{-1}\Sigma(P^{-1})' = P^{-1}PP'(P^{-1})' = I_m$$

so the new errors $\eta_{1t}, \eta_{2t}, \ldots, \eta_{mt}$ are contemporaneously uncorrelated. Then the orthogonalized impact of a unit shock at time t to the ith equation on y at time $t+n$ is given by

$$B_n e_i, n = 0, 1, \ldots \tag{11.41}$$

where e_i is an $m \times 1$ selection vector. Written more compactly, the orthogonalized impulse response function of a unit (one standar error) shock to the ith variable on the jth variable is given by

$$OI_{ij,n} = e_j' A_n P e_i, \quad i, j = 1, 2, \ldots, m \tag{11.42}$$

These orthogonalized impulse responses are not unique and depend on the particular ordering of the variables in the VAR. The orthogonalized responses are invariant to the ordering of the variables only if Σ is diagonal.

- **GIR:** If the VAR model is perturbed by a shock of size $\delta_i = \sqrt{\sigma_{ii}}$ to its ith equation at time t, by the definition of the generalized IR function we have

$$GI_y(n, \delta_i, \Omega_{t-1}^0) = \mathbb{E}(y_t \mid u_{it} = \delta_i, \Omega_{t-1}^0) - \mathbb{E}(y_t \mid \Omega_{t-1}^0) \tag{11.43}$$

Once again using the MA(∞) representation (11.38 we obtain

$$GI_y(n, \delta_i, \Omega_{t-1}^0) = A_n \mathbb{E}(u_t \mid u_{it} = \delta_i) \tag{11.44}$$

which is history invariant (i.e. does not depend on $\Omega_{t=1}^0$. The computation of the conditional expectations $\mathbb{E}(u_t \mid u_{it} = \delta_i)$ depends on the nature of the multivariate distribution assumed for the disturbances u_t. In the case where $u_t \sim IID\mathcal{N}(0, \Sigma)$, we hae

$$\mathbb{E}(u_t \mid u_{it} = \delta_i) = \begin{bmatrix} \sigma_{1i}/\sigma_{ii} \\ \sigma_{2i}/\sigma_{ii} \\ \vdots \\ \sigma_{mi}/\sigma_{ii} \end{bmatrix} \delta_i \tag{11.45}$$

where as before $\Sigma = [\sigma_{ij}]$. Hence for a unit shock $\delta_i = \sqrt{\sigma_{ii}}$ we have

$$GI_y(n, \delta_i = \sqrt{\sigma_{ii}}, \Omega_{t-1}^0) = \frac{A_n \Sigma e_i}{\sqrt{\sigma_{ii}}}, \quad i, j = 1, 2, \ldots, m \tag{11.46}$$

The GIRF of a unit shock to the ith equation in the VAR(p) model

$$y_t = \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \ldots + \Phi_p y_{t-p} + u_t, \quad u_t \sim IID(0, \Sigma) \tag{11.47}$$

on the jth variable at horizon n is given by the jth element of (11.46), expressed more compactly by

$$GI_y(n, \delta_i = \sqrt{\sigma_{ii}}, \Omega_{t-1}^0) = \frac{e_j' A_n \Sigma e_i}{\sqrt{\sigma_{ii}}}, \quad i, j = 1, 2, \ldots, m \tag{11.48}$$

(d) The GIRF circumvents the problem of the dependence of the orthogonalized impulse responses to the ordering of the variables in the VAR. Unlike the OIR responses in (11.42), the GIR responses in (11.48) are invariant to the ordering of the variables in the VAR. The two responses coincide only for the first variable in the VAR or when Σ is diagonal.

Chapter 24 Problem 4.

Chapter 24 Problem 4 Solution.

11.12 Chapter 33: Theory and Practice of GVAR Modeling

Lidan says will not be on final.

12 Convex Optimization

These are my notes from taking EE 588 at USC taught by Mahdi Soltanolkotabi and the textbook *Convex Optimization* (Boyd and Vandenberghe) 7th printing [Boyd et al., 2004], as well as Math 541A at USC taught by Steven Heilman.

Need to cover:

- Update rules for optimization problems (e.g. gradient descent, be able to write down gradient, etc.)
- Know which algorithms are useful in which settings
- Homework-like problems from first part of class (no proofs though) (Boyd homework is good practice)
- Understand how to derive algorithms
- Understand how to calculate gradients, proximal functions, etc.
- Understand examples, how to run algorithms
- Only conceptual thing: duality question (write down dual)
- Formulate problems as convex optimization problems

Do not need to cover:

- ADMM
- Proofs from 2nd half of class (rates of convergence, etc.)
- Coding

12.1 Convex Functions

Definition 12.1 (Math 541A definition). Let $\phi : \mathbb{R} \to \mathbb{R}$. We say that ϕ is **convex** if, for any $x, y \in \mathbb{R}$ and for any $t \in [0, 1]$, we have

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y).$$

Definition 12.2 (Strict convexity, Math 541A notes definition 6.6). Let $\phi : \mathbb{R} \to \mathbb{R}$. We say that ϕ is **strictly convex** if, for any $x, y \in \mathbb{R}, x \neq y$ and for any $t \in (0, 1)$, we have

$$\phi(tx + (1-t)y) < t\phi(x) + (1-t)\phi(y).$$

Definition 12.3 (Convex function in $\mathbb{R}^n$, Math 541A Definition). Let $\phi : \mathbb{R}^n \to \mathbb{R}$. We say that ϕ is **convex** if, for any $x, y \in \mathbb{R}^n$ and for any $t \in [0, 1]$, we have

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y). \tag{12.1}$$

Lemma 12.1 (Result from Math 541A Homework 2)**.** The slope of a convex function is nondecreasing. More formally, let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function. For any $x \in \mathbb{R}$, let

$$M_R := \left\{ \frac{\phi(c) - \phi(x)}{c - x} : c > x \right\}, \quad M_L := \left\{ \frac{\phi(x) - \phi(b)}{x - b} : b > x \right\}$$

be the slopes of the secant lines through ϕ using points to the right and left of x, respectively. Then for any $m \in M_R$, $p \in M_L$ we have $m \geq p$.

Proof. Fix $x \in \mathbb{R}$. Let $m \in M_R, p \in M_L$. By definition, there exist $b < x < c$ such that

$$m = \frac{\phi(c) - \phi(x)}{c - x}, \quad p = \frac{\phi(x) - \phi(b)}{x - b}.$$

Let $t \in (0, 1)$ such that

$$tb + (1 - t)c = x. \tag{12.2}$$

Then we have

$$m \geq p \iff \frac{\phi(c) - \phi(x)}{c - x} \geq \frac{\phi(x) - \phi(b)}{x - b} \iff (x - b)(\phi(c) - \phi(x)) \geq (c - x)(\phi(x) - \phi(b))$$

$$\iff (x - b)\phi(c) + b\phi(x) \geq c\phi(x) - (c - x)\phi(b)$$

From (12.2), we have $x - b = tb + (1 - t)c - b = (t - 1)b + (1 - t)c = (1 - t)(c - b)$ and $t(b - c) = x - c \iff t(c - b) = c - x$. Therefore

$$(x - b)\phi(c) + b\phi(x) \geq c\phi(x) - (c - x)\phi(b) \iff (1 - t)(c - b)\phi(c) + b\phi(x) \geq c\phi(x) - t(c - b)\phi(b)$$

$$\iff (1 - t)(c - b)\phi(c) \geq (c - b)\phi(x) - t(c - b)\phi(b) \iff (1 - t)\phi(c) + t\phi(b) \geq \phi(x)$$

But $t\phi(b) + (1 - t)\phi(c) \geq \phi(x)$ since ϕ is convex. Therefore $m \geq p$.

□

Theorem 12.2 (**Result from 541A Homework 2; equivalent conditions for convexity**)**.** Let $\phi : \mathbb{R} \to \mathbb{R}$. Then ϕ is convex if and only if: for any $y \in \mathbb{R}$, there exists a constant a and there exists a function $L : \mathbb{R} \to \mathbb{R}$ defined by $L(x) = a(x - y) + \phi(y)$, $x \in \mathbb{R}$, such that $L(y) = \phi(y)$ and such that $L(x) \leq \phi(x)$ for all $x \in \mathbb{R}$. (In the case that ϕ is differentiable, the latter condition says that ϕ lies above all of its tangent lines.)

Proof. $\implies$: As in Lemma 12.1, let

$$M_R := \{\frac{\phi(c) - \phi(y)}{c - y} : c > y\}, \quad M_L := \{\frac{\phi(y) - \phi(b)}{y - b} : b > y\}$$

be the slopes of the secant lines through ϕ using points to the right and left of y, respectively. Then by Lemma 12.1, for any $m \in M_R$, $p \in M_L$ we have $m \geq p$, so we can choose some $a_0 \in \mathbb{R}$ such that $p \leq a_0 \leq m$ for all $p \in M_L, m \in M_R$. Then let $L : \mathbb{R} \to \mathbb{R}$ be defined by $L(x) = a_0(x - y) + \phi(y)$, $x \in \mathbb{R}$. Note that $L(y) = \phi(y)$.

We argue that $L(x) \leq \phi(x)$ for all $x \in \mathbb{R}$ by contradiction. Suppose there is some $z \in \mathbb{R}$ with $L(z) > \phi(z)$. Note that $z \neq y$ because we have already shown that $L(y) = \phi(y)$. Then we have

$$L(z) > \phi(z) \iff a_0(z - y) + \phi(y) > \phi(z) \tag{12.3}$$

If $z > y$, then we can solve (12.3) for a_0 as follows:

$$a_0 > \frac{\phi(z) - \phi(y)}{z - y}$$

But $z \in M_R$, so we have $\frac{\phi(z)-\phi(y)}{z-y} > a_0$. Contradiction. If $z < y$, we solve (12.3) for a_0 as follows:

$$a_0 < \frac{\phi(z) - \phi(y)}{z - y} = \frac{\phi(y) - \phi(z)}{y - z}$$

But $z \in M_L$, so we have $\frac{\phi(y)-\phi(z)}{y-z} < a_0$. Contradiction. Therefore for every $z \in \mathbb{R}$ we have $L(z) \leq \phi(z)$ as desired.

$\Longleftarrow$: Now suppose that for any $y \in \mathbb{R}$ there exists a constant a and a function $L : \mathbb{R} \to \mathbb{R}$ defined by $L(x) = a(x - y) + \phi(y)$, $x \in \mathbb{R}$, such that $L(y) = \phi(y)$ and such that $L(x) \leq \phi(x)$ for all $x \in \mathbb{R}$.

Fix $b, c \in \mathbb{R}$ and let $t \in (0, 1)$. Set $y := tb + (1 - t)c$. Then by assumption we can write

$$a(b - y) + \phi(y) \leq \phi(b), \quad a(c - y) + \phi(y) \leq \phi(c)$$

Multiply by $t > 0$ and $(1 - t) > 0$ respectively to yield

$$ta(b - y) + t\phi(y) \leq t\phi(b), \quad (1 - t)a(c - y) + (1 - t)\phi(y) \leq (1 - t)\phi(c) \tag{12.4}$$

Note that

$$ta(b - y) + (1 - t)a(c - y) = a(tb - ty + (c - y - ct + yt)) = a(tb + c - y - ct)$$

$$= a(tb + c(1-t) - tb - (1-t)c) = 0$$

So adding the inequalities in (12.4) yields

$$t\phi(y) + (1-t)\phi(y) \leq t\phi(b) + (1-t)\phi(c) \iff \phi(y) \leq t\phi(b) + (1-t)\phi(c)$$

$$\iff \phi(tb + (1-t)c) \leq t\phi(b) + (1-t)\phi(c).$$

$\square$

My original proof from submitted homework. If ϕ is differentiable at y, let $a = \phi'(y)$. If not, let a be any subgradient of ϕ at y. Then $L(x)$ is (a) tangent line to ϕ at y, which should be lesser than or equal to ϕ for all $x \in \mathbb{R}$ if ϕ is convex. If and only if this is true at every $y \in \mathbb{R}$ (the tangent line is a global underestimator at y for every $y \in \mathbb{R}$), then ϕ must be convex. We proceed to show this formally:

$\implies$: We will show that if ϕ is convex; that is, if for any $x, y \in \mathbb{R}$ and for any $t \in [0, 1]$, we have

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y) \tag{12.5}$$

then the inequality

$$\phi'(y)(x - y) + \phi(y) \leq \phi(x) \quad \forall x \in \mathbb{R} \tag{12.6}$$

holds. Starting from (12.5) note that

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y) \implies \phi(tx + (1-t)y) - \phi(x) \leq (1-t)(\phi(y) - \phi(x))$$

Suppose $y > x$. Then $tx + (1-t)y - x = (1-t)(y-x) > 0$, so we can divide by it on both sides:

$$\implies \frac{\phi(tx + (1-t)y) - \phi(x)}{tx + (1-t)y - x} \leq \frac{(1-t)(\phi(y) - \phi(x))}{(1-t)(y-x)} \implies \frac{\phi(tx + (1-t)y) - \phi(x)}{tx + (1-t)y - x} \leq \frac{\phi(y) - \phi(x)}{y - x}$$

Taking the limit as $t \to 1$ yields

$$\phi'(x) \leq \frac{\phi(y) - \phi(x)}{y - x}$$

if ϕ is differentiable, which is (equivalent to) what we hoped to prove. The case where $x > y$ is analogous.

$\impliedby$: We will show that if (12.6) holds then ϕ is convex; that is, (12.5) holds for any $x, y \in \mathbb{R}$ and for any $t \in [0, 1]$. Starting from (12.6) note that

$$\phi'(y)(x-y)+\phi(y) \le \phi(x) \iff \phi'(y) \le \frac{\phi(x)-\phi(y)}{x-y} \quad \forall\, x, y \in \mathbb{R}$$

□

Theorem 12.3 (Global minimum of convex functions; Math 541A Homework problem). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Let $x \in \mathbb{R}^n$ be a local minimum of f. Then

(a) x is a global minimum of f.

(b) If f is strictly convex, then there is at most one global minimum of f.

(c) If f is a C^1 function (all derivatives of f exist and are continuous), and $x \in \mathbb{R}^n$ satisfies $\nabla f(x) = 0$, then x is a global minimum of f.

Proof. (a) Since x is a local minimum, we have that there exists $\epsilon > 0$ such that $f(x) \le f(y)\ \forall\ y \in B(x, \epsilon)$ where $B(x, \epsilon) \subseteq \mathbb{R}^n$ is an n-dimensional L_2 ball of radius ϵ centered at x. Suppose there exists some $z \in \mathbb{R}^n$ such that $f(z) < f(x)$. Then by convexity of f, for $t \in [0, 1]$,

$$f(tx + (1-t)z) \le tf(x) + (1-t)f(z) < tf(x) + (1-t)f(x) = f(x)$$

which when $t = 1$ leads to the contradiction $f(x) < f(x)$. (Also, for $t = 1 - \delta$ with δ sufficiently small, we get $f(x') \le tf(x) + (1-t)f(z) < f(x)$ where $x' = tx + (1-t)z$ such that $x' \in B(x, \epsilon)$, contradicting the fact that $f(x)$ is a local minimum.) Therefore there is no $z \in \mathbb{R}^n$ such that $f(z) < f(x)$, so x is a global minimum.

(b) If f is strictly convex, for any $z \in \{\mathbb{R}^n \setminus x\}$ we have

$$f(tx + (1-t)z) < tf(x) + (1-t)f(z), \qquad \forall x, z \in \mathbb{R}^n, x \ne z \tag{12.7}$$

We have already shown that there exists no $z \in \{\mathbb{R}^n \setminus x\}$ such that $f(z) < f(x)$. Suppose there is more than one global minimum of f; that is, there exists $z \in \{\mathbb{R}^n \setminus x\}$ such that $f(z) = f(x)$. That is, for all $y \in \{\mathbb{R}^n \setminus \{x, z\}\}$,

$$f(x) = f(z) \le f(y). \tag{12.8}$$

But then by strict convexity,

$$f\left(\frac{x+z}{2}\right) < \frac{1}{2}f(x) + \frac{1}{2}f(z) = \frac{1}{2}f(x) + \frac{1}{2}f(x) = f(x)$$

which contradicts (12.8) if $y = (x+z)/2$. Therefore the global minimum of f is unique.

(c) Recall from Exercise 4 in Homework 2 that f is convex if and only if for any $x \in \mathbb{R}^n$ there exists a constant $a \in \mathbb{R}^n$ and a function $L : \mathbb{R}^n \to \mathbb{R}$ defined by $L(y) = a^T(y - x) + f(x), y \in \mathbb{R}^n$ such that $L(x) = f(x)$ and $L(y) \le f(y)$ for all $y \in \mathbb{R}^n$. Further, if f is a C^1 function then this function exists for $a = \nabla f(x)$. That is,

$$f(y) \ge f(x) + \nabla f^T(x)(y - x),\ \forall y \in \mathbb{R}^n.$$

Since $\nabla f(x) = 0$, if we plug in $y = x$ we get

$$f(y) \geq f(x), \quad \forall y \in \mathbb{R}^n.$$

□

Theorem 12.4 (Jensen's Inequality, from Math 541A). Let $X : \Omega \to [-\infty, \infty]$ be a random variable. Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex. Assume that $\mathbb{E}|X| < \infty$ and $\mathbb{E}|\phi(X)| < \infty$. Then

$$\phi(\mathbb{E}X) \leq \mathbb{E}\phi(X).$$

Proof. Note that from Theorem 12.2, for any $y \in \mathbb{R}$ there exists a constant a and a function L such that

$$a(x - y) + \phi(y) \leq \phi(x) \quad \forall x \in \mathbb{R}$$

Letting $y = \mathbb{E}(X)$ we have

$$a(X - \mathbb{E}X) + \phi(\mathbb{E}X) \leq \phi(X)$$

Since expectations preserve inequalities,

$$\mathbb{E}[a(X - \mathbb{E}X) + \phi(\mathbb{E}X)] \leq \mathbb{E}\phi(X)$$

But

$$\mathbb{E}[a(X - \mathbb{E}X) + \phi(\mathbb{E}X)] = a(\mathbb{E}X - \mathbb{E}X) + \mathbb{E}(\phi(\mathbb{E}X)) = \phi(\mathbb{E}X)$$

which yields

$$\phi(\mathbb{E}X) \leq \mathbb{E}\phi(X).$$

□

Corollary 12.4.1 (Jensen's Inequality: EE 588 Formulation). f is convex if and only if

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a) + f(b)}{2}$$

for all $a, b \in \mathbf{dom}(f)$.

Proof. Follows from Theorem 12.4 if X is a discrete random variable that equals a or b each with probability 1/2 and $\phi(X) = f(X)$. Note that $\phi(X)$ is convex.

□

Corollary 12.4.2 (Triangle Inequality). Let $X : \Omega \to [-\infty, \infty]$ be a random variable with $\mathbb{E}|X| < \infty$. Then

$$\|EX| \leq \mathbb{E}|X|.$$

Proof. Note that $\phi(x) = |x|$ is convex by the definition of convexity: for any $x, y \in \mathbb{R}$ and for any $t \in (0, 1)$, we have

$$\phi(tx + (1-t)y) = |tx + (1-t)y| \leq \ldots = t|x| + (1-t)|y| = t\phi(x) + (1-t)\phi(y).$$

Then the result follows immediately from Jensen's Inequality (Theorem 12.4) using $\phi(X) = |X|$:

$$|\mathbb{E}X| \leq \mathbb{E}|X|$$

□

Theorem 12.5 (Conditional Jensen Inequality). Let $X, Y : \Omega \to \mathbb{R}$ be random variables that are either both discrete or both continuous. Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex. Then

$$\phi(\mathbb{E}(X|Y)) \leq \mathbb{E}(\phi(X)|Y).$$

If ϕ is strictly convex, then equality holds only if X is constant on any set where Y is constant. That is, (by an Exercise from the previous homework) equality holds only if X is a function of Y.

Proof. Recall that from Exercise 4 in Homework 2 that since ϕ is convex, for any $y \in \mathbb{R}$ there exists a constant a and a function L such that

$$a(x - y) + \phi(y) \leq \phi(x) \quad \forall x \in \mathbb{R}$$

Letting $x = X$ and $y = \mathbb{E}(X \mid Y)$ we have

$$a(X - \mathbb{E}(X \mid Y)) + \phi(\mathbb{E}(X \mid Y)) \leq \phi(X)$$

Since by Lemma 6.7 conditional expectations preserve inequalities,

$$\mathbb{E}[a(X - \mathbb{E}[X \mid Y]) + \phi(\mathbb{E}[X \mid Y]) \mid Y] \leq \mathbb{E}(\phi(X) \mid Y)$$

But

$$\mathbb{E}[a(X - \mathbb{E}[X \mid Y]) + \phi(\mathbb{E}[X \mid Y]) \mid Y] = a(\mathbb{E}[X \mid Y] - \mathbb{E}[\mathbb{E}(X \mid Y) \mid Y]) + \mathbb{E}[\phi(\mathbb{E}[X \mid Y]) \mid Y].$$

By Corollary 6.6.1 (letting $h(Y) = \phi(\mathbb{E}[X \mid Y])$), $\mathbb{E}[\phi(\mathbb{E}[X \mid Y]) \mid Y] = \phi(\mathbb{E}[X \mid Y])$. By Corollary 6.6.2, $\mathbb{E}[\mathbb{E}(X \mid Y)) \mid Y] = \mathbb{E}(X \mid Y)$. Therefore we have

$$= a(\mathbb{E}[X \mid Y] - \mathbb{E}[X \mid Y]) + \phi(\mathbb{E}[X \mid Y]) = \phi(\mathbb{E}(X \mid Y))$$

which yields

$$\phi(\mathbb{E}(X \mid Y)) \leq \mathbb{E}(\phi(X) \mid Y).$$

$\square$

Proposition 12.6 (**Convexity of affine functions**). Let $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$ be fixed. Let $x \in \mathbb{R}^n$. Then the function $f(x) = c^T x + d$ is convex.

Proof. We will show that f satisfies (12.1) for any $x, y \in \mathbb{R}^n$ and any $t \in [0, 1]$:

$$f(tx + (1-t)y) = c^T(tx + (1-t)y) + d = c^T(tx + (1-t)y) + d = tc^T x + td + (1-t)c^T y + (1-t)d$$

$$= t[c^T x + d] + (1-t)[c^T y + d] = tf(x) + (1-t)f(y).$$

In particular, (12.1) is satisfied with equality.

$\square$

Proposition 12.7 (**Convexity of quadratic forms**). Let $A \in \mathbb{R}^{m \times n}$ and $k > 0$ be fixed. Let $x \in \mathbb{R}^n$. Then the function $f(x) = kx^T A^T Ax$ is convex.

Proof. We will show that f satisfies the definition of convexity. Plugging $\phi(x) = kx^T A^T Ax$ into the left side of (12.1), we have

$$k(tx + (1-t)y)^T A^T A(tx + (1-t)y) = k(tx^T + (1-t)y^T)A^T A(tx + (1-t)y)$$

$$= k\big[tx^T A^T Atx + tx^T A^T A(1-t)y + (1-t)y^T A^T Atx + (1-t)y^T A^T A(1-t)y\big]$$

$$= k\big[t^2 x^T A^T Ax + t(1-t)x^T A^T Ay + t(1-t)y^T A^T Ax + (1-t)^2 y^T A^T Ay\big]$$

Note that $x^T A^T Ay \in \mathbb{R} = [x^T A^T Ay]^T = y^T A^T Ax$, so we have

$$= k\big[t^2 x^T A^T Ax + 2t(1-t)x^T A^T Ay + (1-t)^2 y^T A^T Ay\big] \tag{12.9}$$

Plugging $\phi(x) = kx^T A^T Ax$ into the right side of (12.1) yields

$$ktx^T A^T Ax + k(1-t)y^T A^T Ay \tag{12.10}$$

We can verify the inequality in (12.1) by subtracting (12.10) from (12.9) to see if a negative number results:

$$kt^2x^TA^TAx + 2kt(1-t)x^TA^TAy + k(1-t)^2y^TA^TAy - ktx^TA^TAx - k(1-t)y^TA^TAy$$

$$= kt(t-1)x^TA^TAx + 2kt(1-t)x^TA^TAy + k(1-t)[1-t-1]y^TA^TAy$$

$$= -kt(1-t)[x^TA^TAx - 2x^TA^TAy + y^TA^TAy] = -kt(1-t)[x^TA^T - y^TA^T][Ax - Ay]$$

$$= -kt(1-t)[A(x-y)]^TA(x-y) \leq 0$$

for all $x, y \in \mathbb{R}^n$ and any $t \in [0,1]$ since $-kt(1-t) \leq 0$ (with equality only when $t = 0$ or $t = 1$) and $[A(x-y)]^TA(x-y) \geq 0$ (with equality only when $x = y$). This verifies the inequality in (12.1), which proves that kx^TA^TAx is convex.

□

Proposition 12.8 (**Sum of convex functions is convex**). Let $f_1, \ldots, f_n : \mathbb{R}^n \to \mathbb{R}$ be (strictly) convex functions. Then the function $g(x) := \sum_{i=1}^n f_i(x)$ is (strictly) convex.

Proof. Since f_i is convex for all $i \in \{1, \ldots, n\}$, f_i satisfies

$$f_i(tx + (1-t)y) \leq tf_i(x) + (1-t)f_i(y), \qquad \forall i \in \{1, \ldots, n\}.$$

We make use of these inequalities to show that g satisfies (12.1) for any $x, y \in \mathbb{R}^n$ and any $t \in [0,1]$:

$$\begin{aligned} g(tx + (1-t)y) = \sum_{i=1}^n f_i(tx + (1-t)y) &\leq \sum_{i=1}^n [tf_i(x) + (1-t)f_i(y)] \\ = t\sum_{i=1}^n f_i(x) + (1-t)\sum_{i=1}^n f_i(y) &= tg(x) + (1-t)g(y) \end{aligned}$$

which proves the result. (Note that if the initial inequality is strict then strict convexity follows.)

□

Proposition 12.9 (**Exercise 6.43 in Math 541A Lecture Notes**). Let $f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}$ be n strictly convex functions. Define $g : \mathbb{R}^n \to \mathbb{R}$ by

$$g(x_1, \ldots, x_n) := \sum_{i=1}^n f_i(x_i), \qquad \forall (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Then $g : \mathbb{R}^n \to \mathbb{R}$ is convex.

Proof. Since f_i is strictly convex for all $i \in \{1, \ldots, n\}$, we have that for any $x_i, y_i \in \mathbb{R}$, for all $t \in (0,1)$

$$f_i(tx_i + (1-t)y_i) < tf_i(x_i) + (1-t)f_i(y_i).$$

Therefore for any $x, y \in \mathbb{R}^n$ (where $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$), for all $t \in (0,1)$

$$g(tx + (1-t)y) = \sum_{i=1}^n f_i(tx_i + (1-t)y_i) < \sum_{i=1}^n [tf_i(x_i) + (1-t)f_i(y_i)] = t\sum_{i=1}^n f_i(x_i) + (1-t)\sum_{i=1}^n f_i(y_i)$$
$$= tg(x) + (1-t)g(y).$$

□

Proposition 12.10 (Exercise 6.44 from Math 541A lecture notes). Let $f : \mathbb{R}^n \to \mathbb{R}$. Suppose that for any fixed $i \in \{1, \ldots, n\}$ and for any $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$, the function

$$x_i \mapsto f(x_1, \ldots, x_n)$$

is strictly convex. Then f has at most one global minimum.

Proof. An equivalent statement to our assumption is that for any i, f is strictly convex in x_i keeping $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ fixed. That is, if we let $h_i : \mathbb{R} \to \mathbb{R}$ be defined for all $i \in \{1, \ldots, n\}$ by

$$h_i\big(x_i \mid (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\big) := f\big((x_1, \ldots, x_n)\big) \qquad \forall i \in \{1, \ldots, n\},$$

then h_i is strictly convex for all $i \in \{1, \ldots, n\}$. That is, for any $x_i, y_i \in \mathbb{R}$, for all $t \in (0,1)$

$$h_i\big(tx_i + (1-t)y_i \mid (tx_1 + (1-t)y_1, \ldots, tx_{i-1} + (1-t)y_{i-1}, tx_{i+1} + (1-t)y_{i+1}, \ldots, tx_n + (1-t)y_n)\big)$$

$$< th_i\big(x_i \mid (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\big) + (1-t)h_i\big(y_i \mid (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)\big). \tag{12.11}$$

By Theorem 12.3(b), there is at most one global minimum of f if it is strictly convex, so all we need to show is that f is strictly convex. That is, we must show that for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$, for all $t \in (0,1)$,

$$f\big(t(x_1, \ldots, x_n) + (1-t)(y_1, \ldots, y_n)\big) < tf\big((x_1, \ldots, x_n)\big) + (1-t)f\big((y_1, \ldots, y_n)\big). \tag{12.12}$$

We will argue by contradiction. Suppose that for some $(x_1^*, \ldots, x_n^*) \in \mathbb{R}^n$ and $(y_1^*, \ldots, y_n^*) \in \mathbb{R}^n$, (12.12) does not hold. That is,

$$f\big(t^*(x_1^*, \ldots, x_n^*) + (1-t^*)(y_1^*, \ldots, y_n^*)\big) \geq t^* f\big((x_1^*, \ldots, x_n^*)\big) + (1-t^*)f\big((y_1^*, \ldots, y_n^*)\big).$$

for some $t^* \in (0,1)$. This is equivalent to

$$h_1\big(t^* x_1^* + (1-t^*)y_1^* \mid (t^* x_2^* + (1-t^*)y_2^*, \ldots, t^* x_n^* + (1-t^*)y_n^*)\big)$$
$$\geq t^* h_1\big(x_1^* \mid (x_2^*, \ldots, x_n^*)\big) + (1-t^*) h_1\big(y_1^* \mid (y_2^*, \ldots, y_n^*)\big).$$

But this contradicts (12.11). Therefore (12.12) holds for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$, for all $t \in (0,1)$, so f is strictly convex, which means (by Theorem 12.3(b)) that f has at most one global minimum.

□

Proposition 12.11. Let A be a real $m \times n$ matrix. Let $x \in \mathbb{R}^n$ and let $b \in \mathbb{R}^m$. Then the function $f \colon \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) = \frac{1}{2}\|Ax - b\|^2$ is convex.

Proof. We have

$$f(x) = \frac{1}{2}\|Ax-b\|^2 = \frac{1}{2}(Ax-b)^T(Ax-b) = \frac{1}{2}(x^TA^T - b^T)(Ax-b)$$

$$= \frac{1}{2}(x^TA^TAx - b^TAx - x^TA^Tb + b^Tb) = \frac{1}{2}x^TA^TAx - b^TAx + \frac{1}{2}b^Tb$$

where the last step follows because $b^TAx \in \mathbb{R} = (b^TAx)^T = x^TA^Tb$ since a real number equals its transpose. The affine function $-b^TAx + \frac{1}{2}b^Tb$ is convex by Proposition 12.6, and the quadratic form $\frac{1}{2}x^TA^TAx$ is convex by Proposition 12.7. Since the sum of convex functions is convex by Proposition 12.8, the result follows.

□

Remark. Moreover,

$$\nabla f(x) = A^T(Ax - b), \qquad D^2 f(x) = A^TA.$$

(Here $D^2 f$ denotes the matrix of second derivatives of f.)

So, if $\nabla f(x) = 0$, i.e. if $A^TAx = A^Tb$, then x is the global minimum of f. And if A has full rank, then A^TA is invertible, so that $x = (A^TA)^{-1}A^Tb$ is the global minimum of f.

Proposition 12.12 (Convexity of norms). Every norm on $\mathbb{R}^n$ is convex.

Proof. Suppose we have a generic norm $\|\cdot\|_*$ in $\mathbb{R}^n$. Because $\|\cdot\|_*$ is a norm, it satisfies the triangle inequality; that is, for all $x, y \in \mathbb{R}^n$,$\|x+y\|_* \leq \|x\|_* + \|y\|_*$. Further, for any $t \in [0,1]$, we have

$$\|tx + (1-t)y\|_* \leq \|tx\|_* + \|(1-t)y\|_* = t\|x\|_* + (1-t)\|y\|_*$$

where the last step also follows from a property of all norms.

□

Proposition 12.13 (Mentioned in in-class 541A review; might have been on HW?). If ϕ is strictly convex and $\mathbb{E}(\phi(X)) = \phi(\mathbb{E}(X))$ then X is almost surely constant.

Proposition 12.14 (2018 DSO Statistics Group In-Class Screening Exam, Question 5). The function $f(\theta) = (\|\theta\|_1)^2$ is convex.

Proof. Note that

$$\|tx + (1-t)y\|_1 \leq \|tx\|_1 + \|(1-t)y\|_1 = t\|x\|_1 + (1-t)\|y\|_1 \tag{12.13}$$

where the first step follows by the Triangle Inequality (which all norms satisfy, including the ℓ_1 norm) and the second step follows by the homogeneity property of norms. Therefore $\|\theta\|_1$ is convex. Next, by (12.13) and the monotonicity of $g(\theta) = \theta^2$ when $\theta \geq 0$,

$$f(tx + (1-t)y) = \left(\|tx + (1-t)y\|_1\right)^2 \leq \left(t\|x\|_1 + (1-t)\|y\|_1\right)^2$$

$$= t^2\|x\|_1^2 + (1-t)^2\|y\|_1^2 + 2t(1-t)\|x\|_1\|y\|_1$$

and

$$tf(x) + (1-t)f(y) = t\|x\|_1^2 + (1-t)\|y\|_1^2$$

Taking the difference of these yields

$$tf(x) + (1-t)f(y) - f(tx + (1-t)y) \geq t\|x\|_1^2 + (1-t)\|y\|_1^2 - \left(t^2\|x\|_1^2 + (1-t)^2\|y\|_1^2 + 2t(1-t)\|x\|_1\|y\|_1\right)$$

$$= (t - t^2)\|x\|_1^2 + [(1-t) - (1-t)^2]\|y\|_1^2 - 2t(1-t)\|x\|_1\|y\|_1$$

$$= (t - t^2)\left(\|x\|_1^2 + \|y\|_1^2 - 2\|x\|_1\|y\|_1\right) = t(1-t)(\|x\|_1 - \|y\|_1)^2 \geq 0$$

$$\iff tf(x) + (1-t)f(y) \geq f(tx + (1-t)y)$$

which proves convexity.

$\square$

12.2 Schur Complement Trick

12.2.1 Definition

For a matrix $X \in \boldsymbol{S}^n$ partitioned as

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

the Schur complement is (if $\mathbf{det}(A) \neq 0$)

$$S = C - B^T A^{-1} B$$

The Schur complement has two useful properties in convex analysis.

Theorem 12.15. (a) $X \succ 0$ if and only if $A \succ 0$ and $S \succ 0$.

(b) If $A \succ 0$, then $X \succeq 0$ if and only if $S \succeq 0$.

12.2.2 The Trick

Suppose we are trying to express a problem as a semidefinite program (SDP); that is, in the form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 F_1 + \ldots + x_n F_n + G \preceq 0 \\ & Ax = b \end{array}$$

where $G, F_1, \ldots, F_n \in \boldsymbol{S}^k$ and $A \in \mathbb{R}^{p \times n}$. If we have a constraint of the form $c^T F(x)^{-1} c \leq t$ where $F(x)$ is symmetric and positive definite and $t \in \mathbb{R}$, by Theorem 12.15(b) we can write

$$c^T F(x)^{-1} c \leq t \iff \begin{bmatrix} F(x) & c \\ c^T & t \end{bmatrix} \succeq 0$$

in order to get our constraint in the form required for an SDP.

12.2.3 Example 1: Last Year's Final, Question 2(b)

Suppose we have the constraints

$$\begin{gathered} Ax + b \geq 0 \\ \frac{(c^T x)^2}{d^T x} \leq t \end{gathered}$$

which we would like to express in an SDP. By Theorem 12.15(b) we can write

$$\frac{(c^T x)^2}{d^T x} \leq t \iff d^T x - (c^T x)^T t^{-1} c^T x \geq 0 \iff \begin{bmatrix} t & c^T x \\ c^T x & d^T x \end{bmatrix} \succeq 0$$

Since

$$Ax + b \geq 0 \iff \mathbf{diag}(Ax + b) \succeq 0$$

we can finally write our constraints as

$$\begin{bmatrix} \mathbf{diag}(Ax + b) & 0 & 0 \\ 0 & t & c^T x \\ 0 & c^T x & d^T x \end{bmatrix} \succeq 0$$

12.2.4 Example 2: Last Year's Final, Question 4(b)

Suppose we have the constraints

$$\begin{aligned} &Ax + b \geq 0 \\ &\frac{(c^T x)^2}{d^T x} \leq t \end{aligned}$$

which we would like to express in an SDP. By Theorem 12.15(b) we can write

$$\frac{(c^T x)^2}{d^T x} \leq t \iff d^T x - (c^T x)^T t^{-1} c^T x \geq 0 \iff \begin{bmatrix} t & c^T x \\ c^T x & d^T x \end{bmatrix} \succeq 0$$

Since

$$Ax + b \geq 0 \iff \mathbf{diag}(Ax + b) \succeq 0$$

we can finally write our constraints as

$$\begin{bmatrix} \mathbf{diag}(Ax + b) & 0 & 0 \\ 0 & t & c^T x \\ 0 & c^T x & d^T x \end{bmatrix} \succeq 0$$

12.3 Duality

Theorem 12.16. Slater's condition/constraint qualification: Strong duality holds for a convex problem

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, i = 1, \ldots, m \\ & Ax = b \end{aligned}$$

if it is strictly feasible, i.e., there exists at least one x in the domain of f_0 such that $f_i(x) < 0,\ i = 1, 2, \ldots, m$, $Ax = b$.

12.4 MLE estimates

For linear estimates with iid noise

$$y_i = a_i^T x + v_i, i = 1, \ldots, m$$

where a is observed and $x \in \mathbb{R}^n$ are the parameters to be estimated, the likelihood function is

$$p_x(y) = \prod_{i=1}^{m} \Pr(v_i = y_i - a_i^T x \mid x)$$

Therefore the log likelihood function is:

$$\ell_x(y) = \sum_{i=1}^{m} \log[\Pr(v_i = y_i - a_i^T x \mid x)]$$

12.5 Practice Final (2017 Final)

(1) (a) Strictly convex. Multiply by x/x (allowed in this case since $x > 0$) to get $\frac{x^2}{x+1}$ which is a quadratic over linear, which is convex in $\mathbb{R}^{++}$ according to CVX rules.

(b) Not convex, it is convex for $x \geq -1$, but there is a boundary problem at $x = -1$. Note that Jensen's inequality (Theorem 12.4.1)

$$\frac{f(a) + f(b)}{2} \geq f\left(\frac{a+b}{2}\right)$$

is violated because

$$\frac{f(-1.3) + f(-0.9)}{2} = \frac{2.3 + 0}{2} = 1.15 \leq 2.2 = f(-1.1) = f\left(\frac{-1.3 + -0.9}{2}\right)$$

(c)

(d)

$$f(x) = \sup \log\left(\frac{p(t)}{q(t)}\right) = \sup\{\log p(t) - \log q(t)\} = \sup\{\log\left(\sum_{i=1}^{n} \exp(x_i \sin(it))\right) - \sum_{i=1}^{n} x_i \sin(it)\}$$

(e) The proximal mapping is

$$\text{prox}_{\mathcal{R}}(z) = \arg\min_y \frac{1}{2}\|z - y\|_2^2 + \mathcal{R}(y) = \arg\min_y \frac{1}{2}\sum_{i=1}^{n}(z_i - y_i)^2 + \sum_{i=1}^{n} w_i |y_i|$$

$$= \arg\min_y \frac{1}{2}\sum_{i=1}^{n}\left[(z_i - y_i)^2 + w_i|y_i|\right]$$

Taking the gradient of the inside quantity with respect to y, we have

$$\nabla(y) = \begin{pmatrix} \frac{1}{2} \cdot 2(z_1 - y_1) + \mathbf{sign}(y_1)w_1 \\ \frac{1}{2} \cdot 2(z_2 - y_2) + \mathbf{sign}(y_2)w_2 \\ \vdots \\ \frac{1}{2} \cdot 2(z_n - y_n) + \mathbf{sign}(y_n)w_n \end{pmatrix} = \begin{pmatrix} z_1 - y_1 + \mathbf{sign}(y_1)w_1 \\ z_2 - y_2 + \mathbf{sign}(y_2)w_2 \\ \vdots \\ z_n - y_n + \mathbf{sign}(y_n)w_n \end{pmatrix}$$

Setting equal to 0, we have

$$y = \begin{pmatrix} z_1 \pm w_1 \\ z_2 \pm w_2 \\ \vdots \\ z_n \pm w_n \end{pmatrix}$$

(2) (a) The constraint is convex (affine). The denominator is affine. Since $c^T x = x^T c$, the numerator

$$(c^T x)^2 = (c^T x)(c^T x) = x^T c c^T x = x^T (cc^T) x$$

is convex since cc^T is positive semidefinite.

(b) We start by using the epigraph trick to transform the problem:

$$\begin{aligned} \text{minimize} \quad & t \\ \text{subject to} \quad & \frac{(c^T x)^2}{d^T x} \leq t \\ & Ax + b \geq 0 \end{aligned}$$

We are trying to express this problem as a semidefinite program (SDP); that is, in the form

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & x_1 F_1 + \ldots + x_n F_n + G \preceq 0 \\ & Ax = b \end{aligned}$$

where $G, F_1, \ldots, F_n \in \boldsymbol{S}^k$ and $A \in \mathbb{R}^{p \times n}$. The first constraint

$$\frac{(c^T x)^2}{d^T x} \leq t$$

can be expressed in the form

$$(c^T x)^2 \leq t d^T x \iff (c^T x c^T - t d^T) x \leq 0$$

We have a constraint

$$Ax + b \geq 0$$

which can be expressed in the form

$$Ax \geq -b$$

$$c^T F(x)^{-1} c \leq t$$

where $F(x)$ is symmetric and positive definite and $t \in \mathbb{R}$, by Theorem 12.15(b) we can write

$$c^T F(x)^{-1} c \le t \iff \begin{bmatrix} F(x) & c \\ c^T & t \end{bmatrix} \succeq 0$$

in order to get our constraint in the form required for an SDP.

(3) (a) Yes, g is convex over $\mathcal{X}$ since it is quadratic over linear.

(b) The only points satisfying the constraint have $x_1 = 0$. Therefore the primal optimal value (the only feasible value) is $e^0 = \boxed{1}$.

(c) Lagrangian:

$$L(x, \lambda) = e^{-x_1} + \lambda(x_1^2/x_2)$$

The Lagrangian obtains its minimum value of 0 when $x_2 = x_1^3$ and $x_1 \to \infty$. Thus, its dual function ($g(\lambda) = \min_x L(x, \lambda)$) is

$$g(\lambda) = 0$$

The dual problem is then

$$\boxed{\begin{array}{ll} \text{maximize} & 0 \\ \text{subject to} & \lambda \ge 0 \end{array}}$$

(d) The optimal value of the dual problem is 0. Strong duality does not hold since the optimum of the dual problem is less than the optimum of the primal problem. We can also tell this because Slater's Condition (Theorem 12.16) is violated; that is, there is no (x_1, x_2) that is strictly feasible since x_1 must equal 0, which is on the boundary of the feasible region.

(e) Now for the primal problem, instead of $x_1 = 0$, we have

$$\frac{x_1^2}{x_2} \le u \iff x_1^2 \le ux_2 \implies -\sqrt{ux_2} \le x_1 \le \sqrt{ux_2}$$

Since e^{-x_1} is minimized as $x_1 \to \infty$, our optimal solution is $x_2 \to \infty, x_1 = \sqrt{ux_2} \to \infty$ yielding a primal optimal value of $\boxed{0}$. For the dual problem, we have

$$L(x, \lambda) = e^{-x_1} + \lambda\left(\frac{x_1^2}{x_2} - u\right)$$

Dual function ($g(\lambda) = \min_x L(x, \lambda)$):

$$\frac{x_1^2}{x_2} - u = 0 \implies x_2 = \frac{x_1^2}{u}$$

and let $x_1 \to -\infty$ to yield

$$g(\lambda) = 0$$

The dual problem is then

$$\boxed{\text{maximize} \quad 0}$$

with optimal value 0, so there is no longer a duality gap. We can also tell this because Slater's Condition (Theorem 12.16) is satisfied; that is, there exists an (x_1, x_2) which is strictly feasible (say $(x_1, x_2) = (\sqrt{u}, 10)$).

(4) (a) Yes, the set is convex. If $(u_i, v_i) = \boldsymbol{u}_i$, each

$$\sqrt{(x-u_i)^2 + (y-v_i)^2} = \|\boldsymbol{x} - \boldsymbol{u}_i\|_2$$

is convex in $\boldsymbol{x}$. Therefore the function

$$\sum_{i=1}^{k} \|\boldsymbol{x} - \boldsymbol{u}_i\|_2$$

is convex. For any fixed d, this set is a sublevel set of this function, which is convex since the function is convex.

(b) This is a feasibility problem:

$$\begin{array}{ll} \text{find} & \boldsymbol{x} \\ \text{subject to} & \displaystyle\sum_{i=1}^{k} \|\boldsymbol{x} - \boldsymbol{u}_i\| \le d \\ & \displaystyle\sum_{i=1}^{j} \|\boldsymbol{x} - \boldsymbol{v}_i\| \le e \end{array}$$

or

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & \displaystyle\sum_{i=1}^{k} \|\boldsymbol{x} - \boldsymbol{u}_i\| \le d \\ & \displaystyle\sum_{i=1}^{j} \|\boldsymbol{x} - \boldsymbol{v}_i\| \le e \end{array}$$

for two sets of points in $\mathbb{R}^2$ $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_j$. We would like to express these constraints as matrix inequalities in order to have an SDP. To do this, first rewrite the problem as

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & \|\boldsymbol{x} - \boldsymbol{u}_i\| \le t_i, i = 1, \ldots, k \\ & \|\boldsymbol{x} - \boldsymbol{v}_i\| \le s_i, s = 1, \ldots, j \\ & \mathbf{1}^T t \le d \\ & \mathbf{1}^T s \le e \end{array}$$

Then note that we can use the Schur trick:

$$(\boldsymbol{x} - \boldsymbol{u}_i)^T I (\boldsymbol{x} - \boldsymbol{u}_i) \le t_i \iff \begin{bmatrix} I & \boldsymbol{x} - \boldsymbol{u}_i \\ (\boldsymbol{x} - \boldsymbol{u}_i)^T & t_i \end{bmatrix} \succeq 0$$

and write the optimization problem as an SDP:

$$\boxed{\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & \begin{bmatrix} I & \boldsymbol{x}-\boldsymbol{u}_i \\ (\boldsymbol{x}-\boldsymbol{u}_i)^T & t_i \end{bmatrix} \succeq 0, i=1,\ldots,k \\ & \begin{bmatrix} I & \boldsymbol{x}-\boldsymbol{v}_i \\ (\boldsymbol{x}-\boldsymbol{v}_i)^T & s_i \end{bmatrix} \succeq 0, s=1,\ldots,j \\ & \mathbf{1}^T t \leq d \\ & \mathbf{1}^T s \leq e \end{array}}$$

(5) (a) To minimize the MSE:

$$\mathcal{L}(z) = \sum_r (y_r - |a_r^T x|^2)^2$$

For MLE estimate:

$$p_x(y) = \prod_{r=1}^{m} \Pr(w_r = y_r - (a_r^T x)^2 \mid x) = \frac{1}{(y_r - (a_r^T x)^2)!} \cdot \exp\big(-(a_r^T x)^2\big) \cdot (a_r^T x)^{2[y_r - (a_r^T x)^2]}$$

Therefore the log likelihood function is:

$$\ell_x(y) = \sum_{i=1}^{m} \log[\Pr(y_i - a_i^T x \mid x)] = \sum_{i=1}^{m} \log\left[\frac{1}{(y_r - (a_r^T x)^2)!} \cdot \exp\big(-(a_r^T x)^2\big) \cdot (a_r^T x)^{2[y_r - (a_r^T x)^2]}\right]$$

$$= \sum_{i=1}^{m} \log\left[\frac{1}{(y_r - (a_r^T x)^2)!}\right] - (a_r^T x)^2 + 2[y_r - (a_r^T x)^2] \cdot \log\big[(a_r^T x)\big]$$

(b) b

(c) c

(d) d

(e) e

13 Abstract Algebra

These are my notes from reading *Elementary Abstract Algebra* by W. Edwin Clark, available for free download on his website: http://shell.cas.usf.edu/~wclark/#ELEMENTARY_ABSTRACT_ALGEBRA

13.1 Chapter 1: Binary Operations

Definition 1.1 *A* **binary operation** $*$ *on a set S is a function from $S \times S$ to S. If $(a, b) \in S \times S$ then we write $a * b$ to indicate the image of the element (a, b) under the function $*$.*

The following lemma explains in more detail exactly what this definition means.

Lemma 1.1 *A binary operation $*$ on a set S is a rule for combining two elements of S to produce a third element of S. This rule must satisfy the following conditions:*

(a) $a \in S$ *and* $b \in S \Longrightarrow a * b \in S$. [$S$ is closed under $*$.]

(b) *For all a, b, c, d in S*
$a = c$ *and* $b = d \Longrightarrow a * b = c * d$. [Substitution is permissible.]

(c) *For all a, b, c, d in S*
$a = b \Longrightarrow a * c = b * c$.

(d) *For all a, b, c, d in S*
$c = d \Longrightarrow a * c = a * d$.

Definition: A **function** f from the set A to the set B is a rule which assigns to each element $a \in A$ an element $f(a) \in B$ in such a way that the following condition holds for all $x, y \in A$:

$$x = y \implies f(x) = f(y)$$

To indicate that f is a function from A to B we write $f : A \to B$. The set A is called the **domain** of f and the set B is called the **codomain** of f.

A function $f : A \to B$ is said to be **one-to-one** or **injective** if the following condition holds for all $x, y \in A$:

$$f(x) = f(y) \implies x = y$$

A function $f : A \to B$ is said to be **onto** or **surjective** if the following condition holds:

$$\forall \ b \in B \ \exists \ a \in A \mid f(a) = b$$

A function $f : A \to B$ is said to be **bijective** if it is both one-to-one and onto. Then f is sometimes said to be a **bijection** or a **one-to-one correspondence** between A and B.

15. Let S, T, and U be nonempty sets, and let $f: S \to T$ and $g: T \to U$ be functions such that the function $g \circ f: S \to U$ is one-to-one (injective). Which of the following must be true?

(A) f is one-to-one.

(B) f is onto.

(C) g is one-to-one.

(D) g is onto.

(E) $g \circ f$ is onto.

Solution 15. (A) For a composition of functions, if the first function isn't one-to-one, there's no way the composite is. It's worth mentioning here that the opposite is true for onto: the second function had better be onto.

Let S be a set. The **power set** $\mathcal{P}(S)$ of S is the set of all subsets of S (including S itself).

Definition 1.2 *Assume that $*$ is a binary operation on the set S.*

1. *We say that $*$ is* **associative** *if*

$$x * (y * z) = (x * y) * z \quad \text{for all } x, y, z \in S.$$

2. *We say that an element e in S is an* **identity** *with respect to $*$ if*

$$x * e = x \text{ and } e * x = x \quad \text{for all } x \text{ in } S.$$

3. *Let $e \in S$ be an identity with respect to $*$. Given $x \in S$ we say that an element $y \in S$ is an* **inverse** *of x if both*

$$x * y = e \text{ and } y * x = e.$$

4. *We say that $*$ is* **commutative** *if*

$$x * y = y * x \quad \text{for all } x, y \in S.$$

5. *We say that an element a of S is* **idempotent** *with respect to $*$ if*

$$a * a = a.$$

6. *We say that an element z of S is a* **zero** *with respect to $*$ if*

$$z * x = z \text{ and } x * z = z \quad \text{for all } x \in S.$$

For each integer $n \geq 2$ define the set

$$\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$$

For all $a, b \in \mathbb{Z}_n$ let

$$a + b = \text{remainder when the ordinary sum of a and b is divided by n}$$

and

$$a \cdot b = \text{remainder when the ordinary product of a and b is divided by n.}$$

These binary operations are referred to as **addition modulo** n and **multiplication modulo** n. The integer n in $\mathbb{Z}_n$ is called the **modulus**. The plural of modulus is **moduli**.

Let K denote any one of the following: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}_n$.

$$M_n(K)$$

is the set of all $n \times n$ matrices containing elements of K.

$$GL(n, K)$$

is the set of all matrices in $M_n(K)$ with non-zero determinant. $(GL(n, k), \cdot)$ is called the **general linear group of degree n over K**. It is non-abelian.

$$SL(n, K) = \{A \in GL(n, K) \mid \det(A) = 1\}$$

$SL(n, K)$ is called the **Special Linear Group of degree** n **over** K.

13.2 Chapter 2: Groups

Definition A **group** is an ordered pair $(G, *)$ where G is a set and $*$ is a binary operation on G satisfying the following properties:

1. **The binary operation is associative on G:** $\forall\ x, y, z \in G$,

$$x * (y * z) = (x * y) * z$$

2. **The binary operation contains a (unique) identity in G:** $\exists\ e \in G \mid \forall\ x \in G$

$$e * x = x,\ x * e = x$$

3. **Every element in G has a (unique) inverse on $*$ in G:** $\forall\ x \in G\ \exists\ y \in G \mid$

$$x * y = e, y * x = e$$

A group $(G, *)$ is said to be **abelian** if $\forall\ x, y \in G,\ x * y = y * x$. A group is said to be **non-abelian** if it is not abelian.

Theorem 2.2 *Let $(G, *)$ be a group with identity e. Then the following hold for all elements a, b, c, d in G:*

1. *If $a * c = a * b$, then $c = b$.* [Left cancellation law for groups.]
2. *If $c * a = b * a$, then $c = b$.* [Right cancellation law for groups.]
3. *Given a and b in G there is a* unique *element x in G such that $a * x = b$.*
4. *Given a and b in G there is a* unique *element x in G such that $x * a = b$.*
5. *If $a * b = e$ then $a = b^{-1}$ and $b = a^{-1}$.* [Characterization of the inverse of an element.]
6. *If $a * b = a$ for just one a, then $b = e$.*
7. *If $b * a = a$ for just one a, then $b = e$.*
8. *If $a * a = a$, then $a = e$.* [The only idempotent in a group is the identity.]
9. *$(a^{-1})^{-1} = a$.*
10. *$(a * b)^{-1} = b^{-1} * a^{-1}$.*

13.3 Chapter 3: The Symmetric Groups

If n is a positive integer,

$$[n] = \{1, 2, \ldots, n\}$$

A **permutation** of $[n]$ is a one-to-one, onto function from $[n]$ to $[n]$, and

$$S_n$$

is the set of all permutations of $[n]$.

The identity of S_n is the so-called **identity function**

$$\iota : [n] \to [n]$$

which is defined by the rule

$$\iota(x) = x, \quad \forall\ x \in [n]$$

The inverse of an element $\sigma \in S_n$: Suppose $\sigma \in S_n$. Since σ is by definition one-to-one and onto, the rule

$$\sigma^{-1}(y) = x \iff \sigma(x) = y$$

defines a function $\sigma^{-1} : [n] \to [n]$. This function σ^{-1} is also one-to-one and onto and satisfies

$$\sigma\sigma^{-1} = \iota \text{ and } \sigma^{-1}\sigma = \iota$$

so it is the inverse of σ in the group sense also.

Since the binary operation of composition on S_n is associative $[(\gamma\beta)\alpha = \gamma(\beta\alpha)]$, S_n under the binary operation of composition is a group (it is associative, it has an inverse, and it has an identity).

Definition 3.2 *Let $i_1, i_2, \ldots, i_k$ be a list of k distinct elements from $[n]$. Define a permuation σ in S_n as follows:*

$$\begin{aligned} \sigma(i_1) &= i_2 \\ \sigma(i_2) &= i_3 \\ \sigma(i_3) &= i_4 \\ \vdots \quad &\vdots \quad \vdots \\ \sigma(i_{k-1}) &= i_k \\ \sigma(i_k) &= i_1 \end{aligned}$$

and if $x \notin \{i_1, i_2, \ldots, i_k\}$ then

$$\sigma(x) = x$$

Such a permutation is called a **cycle** *or a* k-**cycle** *and is denoted by*

$$(i_1 \ i_2 \ \cdots \ i_k).$$

If $k = 1$ then the cycle $\sigma = (i_1)$ is just the identity function, i.e., $\sigma = \iota$.

Two cycles $(i_1 \ i_2 \ \ldots \ i_k)$ and $(j_1 \ j_2 \ \ldots \ j_l)$ are said to be **disjoint** if the sets $\{i_1, i_2, \ldots, i_k\}$ and $\{j_1, j_2, \ldots, j_l\}$ are disjoint.

So for example, the cycles (1 2 3) and (4 5 8) are disjoint, but the cycles (1 2 3) and (4 2 8) are not disjoint.

If σ and τ are disjoint cycles, then $\sigma\tau = \tau\sigma$.

Theorem 3.4 *Every element $\sigma \in S_n$, $n \geq 2$, can be written as a product*

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_m \tag{3.1}$$

where $\sigma_1, \sigma_2, \ldots, \sigma_m$ are pairwise disjoint cycles, that is, for $i \neq j$, σ_i and σ_j are disjoint. If all 1-cycles of σ are included, the factors are unique except for the order. ■

The factorization (3.1) is called the **disjoint cycle decomposition of** σ.

An element of S_n is called a **transposition** if and only if it is a 2-cycle.

Every element of S_n can be written as a product of transpositions. The factors of such a product are not unique. However, if $\sigma \in S_n$ can be written as a product of k transpositions and if the same σ can also be written as a product of l transpositions, then k and l have the same parity.

A permutation is **even** if it is a product of an even number of transpositions and **odd** if it is a product of an odd number of transpositions. We define the function sign : $S_n \to \{1, -1\}$ by

$$\text{sign}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

If $n = 1$ then there are no transpositions. In this case, to be complete we define the identity permutation ι to be even.

If σ is a k-cycle, then $\text{sign}(\sigma) = 1$ if k is odd and $\text{sign}(\sigma) = -1$ if k is even.

Remark. Let $A = [a_{ij}]$ be an $n \times n$ matrix. The determinant of A may be defined by the sum

$$\det(A) = \sum_{\sigma \in S_n} \text{sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

For example, if $n = 2$ we have only two permutations ι and $(1\ 2)$. Since $\text{sign}(\iota) = 1$ and $\text{sign}((1\ 2)) = -1$ we obtain

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

Definition: If $(G, *)$ is a group, the number of elements in G is called the **order** of G. We use $|G|$ to denote the order of G. Note that $|G|$ may be finite or infinite.

Let

$$A_n$$

be the set of all even permutations in the group S_n. A_n is called the **alternating group of degree** n.

13.4 Chapter 4: Subgroups

Definition: Let G be a group. A **subgroup** of G is a subset H of G which satisfies the following three conditions:

1. $e \in H$
2. $a, b \in H \implies ab \in H$
3. $a \in H \implies a^{-1} \in H$

If H is a subgroup of G, we write $H \leq G$. The subgroups $\{e\}$ and G are said to be **trivial** subgroups of G.

Every finite subgroup may be thought of as a subgroup of one of the groups S_n.

Let A_n be the set of all even permutations in the group S_n. A_n is then a subgroup of S_n. A_n is called the **alternating group of degree** n.

Let a be an element of the group G. If $\exists\ n \in \mathbb{N} \mid a^n = e$ we say that a has **finite order** and we define

$$\mathrm{o}(a) = \min\{n \in \mathbb{N} \mid a^n = e\}$$

If $a^n \neq e\ \forall\ n \in \mathbb{N}$ we say that a has **infinite order** and we define

$$\mathrm{o}(a) = \infty$$

In either case we call $\mathrm{o}(a)$ the **order** of a. Note carefully the difference between the order of a group and the order of an element of a group. Note also that $a = e \iff \mathrm{o}(a) = 1$. So every element of a group other than e has order $n \geq 2$ or ∞.

Let a be an element of group G. Define

$$\langle a \rangle = \{a^i : i \in \mathbb{Z}\}$$

We call $\langle a \rangle$ the **subgroup of** G **generated by** a. Note that $e = a^0$ and a^{-1} are in $\langle a \rangle$.

Theorem. For each $a \in G$, $\langle a \rangle$ is a subgroup of G. $\langle a \rangle$ contains a and is the smallest subgroup of G containing a.

Proof of second statement. If H is any subgroup of G containing a, $\langle a \rangle \subseteq H$ since H is closed under taking products and inverses. That is, every subgroup of G containing a also contains $\langle a \rangle$. This implies that $\langle a \rangle$ is the smallest subgroup of G containing a.

Theorem. Let G be a group and let $a \in G$. If $\mathrm{o}(a) = 1$, then $\langle a \rangle = \{e\}$. If $\mathrm{o}(a) = n$ where $n \geq 2$, then

$$\langle a \rangle = \{e, a, a^2, \ldots, a^{n-1}\}$$

and the elements $e, a, a^2, \ldots, a^{n-1}$ are distinct; that is,

$$\mathrm{o}(a) = |\langle a \rangle|$$

Proof Assume that $\mathrm{o}(a) = n$. The case $n = 1$ is left to the reader. Suppose $n \geq 2$. We must prove two things.

1. If $i \in \mathbb{Z}$ then $a^i \in \{e, a, a^2, \ldots, a^{n-1}\}$.
2. The elements $e, a, a^2, \ldots, a^{n-1}$ are distinct.

To establish 1 we note that if i is any integer we can write it in the form $i = nq + r$ where $r \in \{0, 1, \ldots, n-1\}$. Here q is the quotient and r is the remainder when i is divided by n. Now using Theorem 2.4 we have

$$a^i = a^{nq+r} = a^{nq}a^r = (a^n)^q a^r = e^q a^r = ea^r = a^r.$$

This proves 1. To prove 2, assume that $a^i = a^j$ where $0 \leq i < j \leq n-1$. It follows that

$$a^{j-i} = a^{j+(-i)} = a^j a^{-i} = a^i a^{-i} = a^0 = e.$$

But $j - i$ is a positive integer less than n, so $a^{j-i} = e$ contradicts the fact that $\mathrm{o}(a) = n$. So the assumption that $a^i = a^j$ where $0 \leq i < j \leq n-1$ is false. This implies that 2 holds. It follows that $\langle a \rangle$ contains exactly n elements, that is, $\mathrm{o}(a) = |\langle a \rangle|$.

Theorem. If G is a finite group, then every element of G has finite order.

49. What is the largest order of an element in the group of permutations of 5 objects?

(A) 5 (B) 6 (C) 12 (D) 15 (E) 120

Solution 49. (B) The greatest order is given by the product of a 2-cycle and a 3-cycle acting on disjoint elements. That gives order 6.

13.5 Chapter 5: The Group of Units of $\mathbb{Z}_n$

Let $n \geq 2$. An element $a \in \mathbb{Z}_n$ is said to be a **unit** if $\exists\, b \in \mathbb{Z}_n \mid ab = 1$ (where the product is multiplication modulo n).

The set of all units in $\mathbb{Z}_n$ is denoted by

$$U_n$$

and is a group under multiplication modulo n called the **group of units of** $\mathbb{Z}_n$.

Theorem. For $n \geq 2$, $U_n = \{a \in \mathbb{Z}_n : \gcd(a, n) = 1\}$

Theorem. p is a prime $\implies \exists\, a \in U_p \mid U_p = \langle a \rangle$

Theorem. If $n \geq 2$ then U_n contains an element a satisfying $U_n = \langle a \rangle$ if and only if a has one of the following forms: $2,\ 4,\ p^k$, or $2p^k$ where p is an odd prime and $k \in \mathbb{N}$.

13.6 Chapter 6: Direct Products of Groups

If $G_1, G_2, \ldots, G_n$ is a list of n groups we make the Cartesian product $G_1 \times G_2 \times \cdots \times G_n$ into a group by defining the binary operation

$$(a_1, a_2, \ldots, a_n) \cdot (b_1, b_2, \ldots, b_n) = (a_1 \cdot b_1, a_2 \cdot b_2, \ldots, a_n \cdot b_n)$$

Here for each $i \in \{1, 2, \ldots, n\}$ the product $a_i \cdot b_i$ is the product of a_i and b_i in the group G_i. We call this group the **direct product** of the groups $G_1, G_2, \ldots, G_n$.

The direct product contains an identity and an inverse, and is associative (since it is composed of groups which must themselves be associative), so it is a group per below:

Theorem. If $G_1, G_2, \ldots, G_n$ is a list of n groups, the direct product $G = G_1 \times G_2 \times \cdots \times G_n$ as defined above is a group. Moreover, if for each i, e_i is the identity of G_i, then $e_1, e_2, \ldots, e_n$ is the identity of G, and if

$$\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in G$$

then the inverse of $\boldsymbol{a}$ is given by

$$\boldsymbol{a}^{-1} = (a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1})$$

where a_i^{-1} is the inverse of a_i in the group G_i.

13.7 Chapter 7: Isomorphism of Groups

Let $G = \{g_1, g_2, \ldots, g_n\}$. Let $\text{o}(g_i) = k_i$ for $i = 1, 2, \ldots, n$. We say that the sequence $(k_1, k_2, \ldots, k_m)$ is the **order sequence** of the group G. To make the sequence unique we assume the elements are ordered so that $k_1 \leq k_2 \leq \ldots \leq k_n$.

Let $(G, *)$ and $(H, \bullet)$ be groups. A function $f : G \to H$ is said to be a **homomorphism** from G to H if

$$f(a * b) = f(a) \bullet f(b)$$

for all $a, b \in G$. If in addition f is one-to-one and onto, f is said to be an **isomorphism** from G to H.

We say that G and H are **isomorphic** if and only if there is an isomorphism from G to H. We write $G \cong H$ to indicate that G is isomorphic to H.

Isomorphism is an equivalence relation: If G, H, and K are groups then

1. $G \cong G$
2. If $G \cong H$ then $H \cong G$, and
3. If $G \cong H$ and $H \cong K$, then $G \cong K$.

Theorem. Let $(G, *)$ and $(H, \bullet)$ be groups and let $f : G \to H$ be a homomorphism. Let e_G denote the identity of G, and let e_H denote the identity of H. Then

1. $f(e_G) = e_H$

   Proof: Let $x_G \in G$ and let $f(x_G) = x_H \in H$. Then
   $x_H = f(x_G) = f(e_G * x_G) = f(e_G) \bullet f(x_G) = f(e_G) \bullet x_H = e_H \bullet x_H$.

2. $f(a^{-1}) = f(a)^{-1}$

   Proof: $f(a)^{-1} \bullet f(a) = e_H = f(e_G) = f(a^{-1} * a) = f(a^{-1}) \bullet f(a)$

3. $f(a^n) = f(a)^n \ \forall \ n \in \mathbb{Z}$

   Proof by induction.

Theorem. Let $(G, *)$ and $(H, \bullet)$ be groups and let $f : G \to H$ be an isomorphism. Then $\text{o}(a) = \text{o}(f(a)) \ \forall \ a \in G$. It follows that G and H have the same number of elements of each possible order.

Theorem. If G and H are isomorphic groups, and G is abelian, then so is H.

Proof: Let $a_G, b_G \in G$ and let $f(a_G) = a_H \in H, f(b_G) = b_H \in H$.
$a_H \bullet b_H = f(a_G) \bullet f(b_G) = f(a_G * b_G) = f(b_G * a_G) = f(b_G) \bullet f(a_G) = b_H \bullet a_H$.

A group G is **cyclic** if there is an element $a \in G \mid \langle a \rangle = G$. If $\langle a \rangle = G$ then we say that a is a **generator** for G.

Theorem. If G and H are isomorphic groups and G is cyclic then H is cyclic.

Theorem. Let a be an element of group G.

1. $\mathrm{o}(a) = \infty \implies \langle a \rangle \cong \mathbb{Z}$.
2. $\mathrm{o}(a) = n \in \mathbb{N} \implies \langle a \rangle \cong \mathbb{Z}_n$

Cayley's Theorem. If G is a finite group of order n, then there is a subgroup H of S_n such that $G \cong H$.

66. Let $\mathbb{Z}_{17}$ be the ring of integers modulo 17, and let $\mathbb{Z}_{17}{}^{\times}$ be the group of units of $\mathbb{Z}_{17}$ under multiplication. Which of the following are generators of $\mathbb{Z}_{17}{}^{\times}$?

I. 5

II. 8

III. 16

(A) None (B) I only (C) II only (D) III only (E) I, II, and III

Solution 66. (B) We need to pick elements of order 16 in $\mathbb{Z}/17^{\times}$. It is easy to rule out $16 \equiv -1$, since -1 has order 2. We see that $5^2 = 25 \equiv 8$, so there's no way that 8 can be a generator. We just need to verify that the order of 5 is more than 8, so we can check 5^8:

$$5^4 = 8^2 = 64 \equiv -4, \quad 5^8 = (-4)^2 = 16 \neq 1.$$

That makes 5 a generator.

13.8 Chapter 8: Cosets and Lagrange's Theorem

Let G be a group and let H be subgroup of G. For each element a of G we define

$$aH = \{ah \mid h \in H\}$$

We call aH the **coset of H in G generated by** a.

Let $a, b \in G$. Then

1. $a \in aH$ (since H must contain an identity; specifically, the identity of G)
2. $|aH| = |H|$ (since ah is unique)
3. $aH \cap bH \neq \emptyset \implies aH = bH$

Lagrange's Theorem. If G is a finite group and $H \leq G$ then $|H|$ divides $|G|$.

Any group of prime order is cyclic; therefore, there is only one such group up to isomorphism.

Exercise 3. Use Lagrange's theorem to prove that any group of prime order is cyclic.

Proof. Let G be a group whose order is a prime p. Since $p > 1$, there is an element $a \in G$ such that $a \neq e$. The group $\langle a \rangle$ generated by a is a subgroup of G. By Lagrange's theorem, the order of $\langle a \rangle$ divides $|G|$. But the only divisors of $|G| = p$ are 1 and p. Since $a \neq e$ we have $|\langle a \rangle| > 1$, so $|\langle a \rangle| = p$. Hence $\langle a \rangle = G$ and G is cyclic. $\square$

We say that there are k **isomorphism classes of groups of order** n if there are k groups $G_1, G_2, \ldots, G_k$ such that

1. if $i \neq j$ then G_i and G_j are not isomorphic, and
2. Every group of order n is isomorphic to G_i for some $i \in \{1, 2, \ldots, k\}$.

This is sometimes expressed by saying that "there are k groups of order n up to isomorphism" or that "there are k non-isomorphic groups of order n."

12. For which integers n such that $3 \leq n \leq 11$ is there only one group of order n (up to isomorphism) ?

(A) For no such integer n

(B) For 3, 5, 7, and 11 only

(C) For 3, 5, 7, 9, and 11 only

(D) For 4, 6, 8, and 10 only

(E) For all such integers n

Solution 12. (B) Any group of prime order is necessarily cyclic, and hence there is only one up to isomorphism. This limits are choices to (B), (C), and (E). But there are two groups of order 9 (at least): $\mathbb{Z}/3 \times \mathbb{Z}/3$ and $\mathbb{Z}/9$. This makes (B) our only option.

In more advanced courses in algebra, it is shown that the number of isomorphism classes of groups of order n for $n \leq 17$ is given by the following table:

Order :	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Number :	1	1	1	2	1	2	1	5	2	2	1	5	1	2	1	14	1

This table means, for example, that one may find 14 groups of order 16 such that every group of order 16 is isomorphic to one and only one of these 14 groups.

There is only one isomorphism class of groups of order n if n is prime. But there are some non-primes that have this property; for example, 15.

The Fundamental Theorem of Finite Abelian Groups. If G is a finite abelian group of order at least 2, then

$$G \cong \mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \cdots \times \mathbb{Z}_{p_s^{n_s}}$$

where for each i, p_i is a prime and n_i is a positive integer. Moreover, the prime powers $p_i^{n_i}$ are unique except for the order of the factors.

If the group G in the above theorem has order n then

$$n = p_1^{n_1} p_2^{n_2} \cdots p_s^{n_2}$$

So the p_i may be obtained from the prime factorization of the order of the group G. These primes are not necessarily distinct, so we cannot say what the n_i are. However, we can find all possible choices for the n_i. For example, if G is an abelian group of order $72 = 3^2 \cdot 2^3$ then G is isomorphic to one and only one of the following groups. Note that each corresponds to a way of factoring 72 as a product of prime powers.

$$\begin{array}{ll}
\mathbb{Z}_9 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 & 72 = 9 \cdot 2 \cdot 2 \cdot 2 \\
\mathbb{Z}_9 \times \mathbb{Z}_4 \times \mathbb{Z}_2 & 72 = 9 \cdot 4 \cdot 2 \\
\mathbb{Z}_9 \times \mathbb{Z}_8 & 72 = 9 \cdot 8 \\
\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 & 72 = 3 \cdot 3 \cdot 2 \cdot 2 \cdot 2 \\
\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_2 & 72 = 3 \cdot 3 \cdot 4 \cdot 2 \\
\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_8 & 72 = 3 \cdot 3 \cdot 8
\end{array}$$

Thus there are exactly 6 non-isomorphic abelian groups of order 72.

Corollary. For $n \geq 2$, the number of isomorphism classes of abelian groups of order n is equal to the number of ways to factor n as a product of prime powers (where the order of the factors does not count).

13.9 Chapter 9: Introduction to Ring Theory

Definition 9.1 *A **ring** is an ordered triple $(R, +, \cdot)$ where R is a set and $+$ and $\cdot$ are binary operations on R satisfying the following properties:*

A1 $a + (b + c) = (a + b) + c$ *for all* a, b, c *in* R.

A2 $a + b = b + a$ *for all* a, b *in* R.

A3 *There is an element* $0 \in R$ *satisfying* $a + 0 = a$ *for all* a *in* R.

A4 *For every* $a \in R$ *there is an element* $b \in R$ *such that* $a + b = 0$.

M1 $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ *for all* a, b, c *in* R.

D1 $a \cdot (b + c) = a \cdot b + a \cdot c$ *for all* a, b, c *in* R.

D2 $(b + c) \cdot a = b \cdot a + c \cdot a$ *for all* a, b, c *in* R.

Terminology If $(R, +, \cdot)$ is a ring, the binary operation $+$ is called *addition* and the binary operation $\cdot$ is called *multiplication*. *In the future we will usually write* ab *instead of* $a \cdot b$. The element 0 mentioned in A3 is called the **zero** of the ring. Note that we have not assumed that 0 behaves like a *zero*, that is, we have not assumed that $0 \cdot a = a \cdot 0 = 0$ for all $a \in R$. What A3 says is that 0 is an identity with respect to addition. Note that *negative* (as the opposite of *positive*) has no meaning for most rings. We do not assume that multiplication is commutative and we have not assumed that there is an identity for multiplication, much less that elements have inverses with respect to multiplication.

23. Let $(\mathbb{Z}_{10}, +, \cdot)$ be the ring of integers modulo 10, and let S be the subset of $\mathbb{Z}_{10}$ represented by $\{0, 2, 4, 6, 8\}$. Which of the following statements is FALSE?

(A) $(S, +, \cdot)$ is closed under addition modulo 10.

(B) $(S, +, \cdot)$ is closed under multiplication modulo 10.

(C) $(S, +, \cdot)$ has an identity under addition modulo 10.

(D) $(S, +, \cdot)$ has no identity under multiplication modulo 10.

(E) $(S, +, \cdot)$ is commutative under addition modulo 10.

Solution 23. (D) Examining the choices, we see $S \subset \mathbb{Z}/10$ is a subgroup of an abelian group. Therefore it still have an additive identity and the operation is commutative. It is also closed under addition and multiplication. While S does not contain the multiplicative identity of $\mathbb{Z}/10$, it does have a multiplicative identity. $6 \in S$ is such an identity, as

$$6x = (5+1)x = 5x + x.$$

Since $x \in S$ are all even, $5x = 0$, so $6x = x$.

50. Let R be a ring and let U and V be (two-sided) ideals of R. Which of the following must also be ideals of R ?

I. $U + V = \{u + v : u \in U \text{ and } v \in V\}$

II. $U \cdot V = \{uv : u \in U \text{ and } v \in V\}$

III. $U \cap V$

(A) II only (B) III only (C) I and II only (D) I and III only (E) I, II, and III

Solution 50. (D) The sum of the ideals is still an ideal: it is clearly closed under addition (using commutativity of addition), and still under left and right multiplication due to the distributive property. The intersection of ideals is still an ideal, which is not too hard to work out. The product of ideals, however, need not be closed under addition. Consider, for example, $R = \mathbb{Z}[X]$, $U = (2, X)$, and $V = (3, X)$ (the ideals generated by two elements). Then we know that $-2X \in U \cdot V$ and $3X \in U \cdot V$, and hence we should expect $3X - 2X = X \in U \cdot V$. However, there is no way to get X as the product of an element of U and an element of V.

18. Let V be the real vector space of all real 2×3 matrices, and let W be the real vector space of all real 4×1 column vectors. If T is a linear transformation from V <u>onto</u> W, what is the dimension of the subspace $\{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}$?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Solution 18. (A) We see that $\dim V = 6$ and $\dim W = 4$. Since $\dim \operatorname{im} T = \dim W = 4$, we must have $\dim \ker T = 6 - 4 = 2$.

14 Miscellaneous

14.1 Set Theory

Proposition 14.1 (Math 541A Homework Problem). Let A, B, Ω be sets. Let $u : \Omega \to A$ and let $t : \Omega \to B$. Assume that, for every $x, y \in \Omega$, if $u(x) = u(y)$, then $t(x) = t(y)$. Show that there exists a function $s : A \to B$ such that

$$t = s(u).$$

Proof. Let $B' \subseteq B$ be the image of Ω under t (so that t is surjective onto B'). Fix $x \in \Omega$ and let $y \in \Omega$ range over $\{\Omega \setminus x\}$. Let $Y \subseteq \Omega$ be the set of values such that $t(y) = t(x)$ for all $y \in Y$. Then let s map $u(y) \in A$ to $t(y) = t(x) \in B$ for every $y \in Y$.

(Note that if x is not the only value in Ω that t maps to $t(x)$, then Y contains elements other than x; otherwise, $Y = \{x\}$. In either case this mapping is fine. Note that $u(y_1)$ does not necessarily equal $u(y_2)$ for every $y_1 \neq y_2 \in Y$, but again this does not pose any difficulties for the mapping.)

Since this argument is true for every $x \in \Omega$, we can argue by contradiction that s is surjective onto B'. Let $A' \subseteq A$ be the image of Ω under u (so that u is surjective onto A'). Suppose there is some $b \in B'$ such that there is no unused $a \in A'$ to correspond to it. That is, there are some $y, z \in \Omega$ such that $t(z) \neq t(y)$ but $u(z) = u(y)$. In that case the mapping s would map $u(z)$ and $u(y)$ both to the same value in B', so one of the values $t(z)$ or $t(y)$ would necessarily be missed. But by assumption there is no $z \in \Omega$ such that $t(z) \neq t(y)$ but $u(z) = u(y)$. (Note the contrapositive of the assumption: "for every $x, y \in \Omega$, if $t(x) \neq t(y)$, then $u(x) \neq u(y)$.")

$\square$

Remark. By showing this mapping exists, we have shown that the cardinality of B' is less than or equal to the cardinality of A'.

14.2 Other

6. **Which of the following circles has the greatest number of points of intersection with the parabola $x^2 = y + 4$?**

   (A) $x^2 + y^2 = 1$

   (B) $x^2 + y^2 = 2$

   (C) $x^2 + y^2 = 9$

   (D) $x^2 + y^2 = 16$

   (E) $x^2 + y^2 = 25$

Solution 6. (C) We can try to do this algebraically, but non-algebraically is simpler. Graphing $y = x^2 - 4$ shows that the graph crosses the x-axis at ± 2. Therefore a circle of radius 1 or $\sqrt{2}$ will not intersect the parabola at all. A circle of radius 3 will intersect four times – twice above and twice below the x-axis. A circle of radius 4 will only intersect at one point below the x-axis (and twice above), and a circle of radius 5 will only intersect at the two points above.

19. If z is a complex variable and $\overline{z}$ denotes the complex conjugate of z, what is $\lim\limits_{z \to 0} \frac{(\overline{z})^2}{z^2}$?

(A) 0 (B) 1 (C) i (D) ∞ (E) The limit does not exist.

Solution 19. (E) Let us represent $z = a + bi$. Then our limit becomes

$$\lim_{(a,b)\to 0} \frac{(a-bi)^2}{(a+bi)^2} = \lim_{(a,b)\to 0} \frac{a^2 - b^2 - 2abi}{a^2 - b^2 + 2abi}.$$

If we let $a = 0$ (for instance), it is easy to see that the limit is equal to 1. However, if we let $a = b$, then our limit becomes

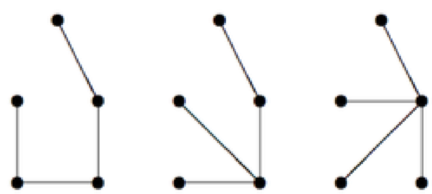
$$\lim_{a\to 0} \frac{-2a^2 i}{2a^2 i} = -1.$$

Therefore the limit does not exist.

29. A tree is a connected graph with no cycles. How many nonisomorphic trees with 5 vertices exist?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution 29. (C) It's probably easiest to draw this out for yourself. The maximum degree of any vertex is 2, 3, or 4. If there is a vertex of degree 4, then our tree looks like a star. If the maximum degree of any vertex is 2, then we have a straight line. In the middle case, we obtain a 3-pointed star to which we attach one more vertex – the choice of branch yields isomorphic graphs. See Figure 1.



38. The maximum number of acute angles in a convex 10-gon in the Euclidean plane is

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution 38. (C) The total angle measure of a 10-gon is $180 \cdot 8 = 1440°$. If the polygon is to be convex, all angles must be less than $180°$. If we have 5 acute angles, then the remaining 5 angles would have to make up for $> 1440 - 5 \cdot 90 = 990$ degrees. This is impossible to do and remain convex. If we have 4 acute angles, the remaining 6 angles need to make up for $> 1440 - 4 \cdot 90 = 1080$ degrees. This is our edge case, so the answer must be 3 acute angles.

45. How many positive numbers x satisfy the equation $\cos(97x) = x$?

(A) 1 (B) 15 (C) 31 (D) 49 (E) 96

Solution 45. (C) Certainly our solutions are concentrated in $[0, 1]$. We know that every $2\pi/97$ units in x, we get another period of $\cos(97x)$, and each period must meet $y = x$ twice. Therefore there are

$$\frac{1}{2\pi/97} = \frac{97}{2\pi} \approx \frac{97}{6.3} \approx 15$$

periods in $[0, 1]$ and about 30 meetings. There's only one answer in that range, so we'll stick with it.

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