DSO Screening Exam: 2018 Take-Home Exam

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Exercise 1. (a) Let S_D be the event that the server wins after the game reaches deuce. Then the server wins if she wins two points in a row. If she wins the first point and loses the second point, then they return to deuce, so the server wins with probability $\Pr(S_D)$. Lastly, if the server's opponent wins the first point of deuce, the server needs to win the next point, then the server is back at deuce and again has probability $\Pr(S_D)$ of ultimately winning. Therefore we have

$$\Pr(S_D) = p^2 + p(1-p)\Pr(S_D) + (1-p)p\Pr(S_D) \iff \Pr(S_D)(1 - 2p(1-p)) = p^2$$

$$\iff \boxed{Pr(S_D) = \frac{p^2}{1 - 2p(1-p)}.}$$

If $p = 0.7$, we have $Pr(S_D) = \frac{0.7^2}{1-1.4(0.3)} = \boxed{\frac{49}{58}.}$

(b) The server can win by either winning without reaching deuce (that is, winning 4 points while her opponent wins two or fewer points, or the 4th point won by the server happens before the 7th point) or by reaching deuce and then winning deuce. Let S be the event that the server wins the game, let S_W be the event that the server wins without reaching deuce and let D be the event that the game reaches deuce. Then $\Pr(S_W)$ is the probability that a negative binomial random variable with parameters 4 and p is less than or equal to 6; that is,

$$\Pr(S_W) = \binom{4-1}{3}p^4(1-p)^{4-4} + \binom{5-1}{3}p^4(1-p)^{5-4} + \binom{6-1}{3}p^4(1-p)^{6-4}$$

$$= p^4 + 4p^4(1-p) + 10p^4(1-p)^2$$

Next $\Pr(D)$ is equal to the probability that each player wins exactly three of the first six games; that is,

$$\Pr(D) = \binom{6}{3}p^3(1-p)^3 = 20p^3(1-p)^3$$

So we have

$$\Pr(S) = \Pr(S_W) + \Pr(D)\Pr(S_D) = p^4 + 4p^4(1-p) + 10p^4(1-p)^2 + 20p^3(1-p)^3 \cdot \frac{p^2}{1-2p(1-p)}$$

$$= \boxed{p^4 + 4p^4(1-p) + 10p^4(1-p)^2 + \frac{20p^5(1-p)^3}{1-2p(1-p)}}$$

For $p = 0.7$, this yields $\boxed{\Pr(S) = 0.900789 = \frac{33,567}{37,264}.}$

Exercise 2. (a)

(b)

Exercise 3. (a)

(b)

Exercise 4. (a)

(b)

Exercise 5. (a)

(b)