Math Review Notes—Calculus

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1 Calculus

These notes include some screenshots from Wikipedia as well as from *Calculus* by Gilbert Strang, available at https://ocw.mit.edu/ans7870/resources/Strang/Edited/Calculus/Calculus.pdf. I also used parts from some other resources which I mention when they arise.

1.1 Differentiation and common derivatives and integrals to know

Theorem 1 (**Clairaut's Theorem**). Let $z = f(x, y)$ be a two variable real-valued function that is defined on a disk $\mathcal{D}$ that contains the point (a, b). Then if $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$ are continuous on $\mathcal{D}$, $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x},\ x > 0$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x},\ x \neq 0$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left(\log_a(x)\right) = \frac{1}{x \ln a},\ x > 0$$

$$\int \tan u \, du = \ln|\sec u| + c$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + c$$

$$\int \frac{1}{a^2+u^2} du = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + c$$

$$\int \frac{1}{\sqrt{a^2-u^2}} du = \sin^{-1}\left(\frac{u}{a}\right) + c$$

$$\int \ln u \, du = u \ln(u) - u + c$$

$$\int \sinh x \, dx = \cosh x + C \qquad \int \cosh x \, dx = \sinh x + C$$

1.2 Matrix Differentiation

Recommended resource: "Matrix Differentiation (and some other stuff)" by Randal J. Barnes (Department of Civil Engineering, University of Minnesota). Available for download at https://atmos.washington.edu/~dennis/MatrixCalculus.pdf.

More information not contained in that pdf (from the appendix of *Convex Optimization* by Stephen Boyd and Lieven Vandenberghe, available for free download at https://web.stanford.edu/~boyd/cvxbook/):

Chain rule. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in \textbf{int dom} f$ and $g : \mathbb{R}^m \to \mathbb{R}^p$ is differentiable at $f(x) \in \textbf{int dom}\ g$. Define the composition $h : \mathbb{R}^n \to \mathbb{R}^p$ by $h(z) = g(f(x))$. Then

$$Dh(x) = Dg(f(x))Df(x)$$

In particular, if $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$,

$$\nabla h(x) = g'(f(x))\nabla f(x)$$

Example with an affine function. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, $A \in \mathbb{R}^{n\times p}$, and $b \in \mathbb{R}^n$. Define $g : \mathbb{R}^p \to \mathbb{R}^m$ as $g(x) = f(Ax + b)$ with **dom** $g = \{x \mid Ax + b \in$ **dom** $f\}$. Then

$$\nabla g(x) = A^T \nabla f(Ax + b)$$

Example 2. Consider $f : \mathbb{R}^n \to \mathbb{R}$ where

$$f(x) = \log \sum_{i=1}^{m} \exp(a_i^T x + b_i) =$$

where $a_1, \ldots, a_m \in \mathbb{R}^n$ and $b_1, \ldots, b_m \in \mathbb{R}$. Note that $f(\cdot)$ can be expressed as a composition of $Ax + b$ (where $A \in \mathbb{R}^{m\times n}$ has rows $a_1^T, \ldots, a_m^T$) and the function $g : \mathbb{R}^m \to \mathbb{R}$ given by $g(y) = \log(\sum_{i=1}^m \exp(y_i)$. We have

$$\nabla g(y) = \left[\sum_{i=1}^{m} e^{y_i}\right]^{-1} \big(\exp(y_1) \ldots \exp(y_m)\big)^T$$

so applying the chain rule yields

$$\nabla f(x) = \left[\sum_{i=1}^{m} \exp(a_i^T x + b_i)\right]^{-1} A^T z$$

where $z_i = \exp(a_i^T x + b_i),\ i = 1, \ldots, m$.

Hessians. The Hessian matrix of $f : \mathbb{R}^n \to \mathbb{R}$ is denoted by $\nabla^2 f(x)$ and is given by

$$\nabla^2 f(x)_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j},\ i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n$$

The quadratic function

$$f(x_0) + \nabla f(x_0)^T (x - x_0) + \frac{1}{2}(x - x_0)^T \nabla^2 f(x_0)(x - x_0)$$

is called the **second-order approximation of f near** x_0.

Chain rule for second derivative. A chain rule for the second derivative is difficult in general. Here are some special cases.

Composition with scalar function. Suppose $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, and $h(x) = g(f(x))$. We have

$$\nabla^2 h(x) = g'(f(x))\nabla^2 f(x) + g''(f(x))\nabla f(x)\nabla f(x)^T$$

Composition with affine function. Suppose $f : \mathbb{R} \to \mathbb{R}$, $a \in \mathbb{R}^m$, $b \in \mathbb{R}$. Define $g : \mathbb{R}^m \to \mathbb{R}$ by $g(x) = f(a^T x + b)$. Then

$$\nabla^2 g(x) = a^T \nabla^2 f(a^T x + b)a$$

More generally, suppose $f : \mathbb{R}^n \to \mathbb{R}$, $A \in \mathbb{R}^{n\times m}$, and $b \in \mathbb{R}^n$. Define $g : \mathbb{R}^m \to \mathbb{R}$ by $g(x) = f(Ax + b)$. Then

$$\nabla^2 g(x) = A^T \nabla^2 f(Ax + b)A$$

1.3 Some theorems in higher dimensions

Proposition 2 (Change of Variables). If U is a "nice" subset of $\mathbb{R}^2$ and ϕ is an injective differentiable function on U, then

$$\int_{\phi(U)} f(u, v)dudv = \int_U f(\phi(x, y))|J\phi(x, y)|dxdy$$

where $J\phi(x, y)$ is the Jacobian of ϕ at (x, y).

Taylor's Theorem (first order). (borrowed from https://www.rose-hulman.edu/~bryan/lottamath/mtaylor.pdf) Consider a function $f : \mathbb{R}^n \to \mathbb{R}$. Let $a \in \mathbb{R}^n$ be a fixed point. Then Taylor's Theorem states:

If $f(x)$ is differentiable on an open ball B around a and $x \in B$ then

$$f(x) = f(a) + \nabla f(b)^T (x - a)$$

for some b on the line segment joining a and x.

This can also be expressed as follows. *Let $x, y \in \mathbb{R}^n$. If $f(x)$ is continuously differentiable, then*

$$f(y) = f(x) + \nabla f(tx + (1 - t)y)^T (y - x)$$

for some $t \in [0, 1]$.

Proof. Consider $g(z) = f(zy + (1 - z)x)$. If f is differentiable then so is g. Then by the Mean Value Theorem, for some $t \in (0, 1)$ we have $g(1) - g(0) = g'(t)$. By the chain rule,

$$g'(t) = \nabla f(x + t(y - x))^T (y - x)$$

Using $g(1) = f(y)$ and $g(0) = f(x)$, we have

$$\iff \nabla f(tx + (1-t)y)^T(y - x) = g(1) - g(0) = f(y) - f(x)$$

$\square$

Taylor's Theorem (second order). Consider a function $f : \mathbb{R}^n \to \mathbb{R}$. Let $a \in \mathbb{R}^n$ be a fixed point. Then Taylor's Theorem states:

If $f(x)$ is twice differentiable on an open ball B around a and $x \in B$ then

$$f(x) = f(a) + (x-a)^T \nabla f(a) + \frac{1}{2}(x-a)^T \nabla^2 f(b)(x-a)$$

for some b on the line segment joining a and x.

This can also be expressed as follows. *Let $x, y \in \mathbb{R}^n$. If $f(x)$ is twice continuously differentiable, then*

$$f(y) = f(x) + \nabla f(x)^T(y-x) + \frac{1}{2}(y-x)^T \nabla^2 f(ty + (1-t)x)(y-x)$$

for some $t \in [0, 1]$.

Proof. Consider $g(z) = f(zy + (1-z)x)$. If f is differentiable then so is g. Then by the second order case of Taylor's Theorem in one dimension, for some $t \in (0, 1)$ we have $g(1) = g(0) + g'(0) + (1/2)g''(t)$. By the chain rule,

$$g''(t) = \frac{\partial}{\partial t} \nabla f(x + t(y-x))^T(y-x) = (y-x)^T \nabla^2 f(x + t(y-x))^T(y-x)$$

Using this result along with $g(1) = f(y)$, $g(0) = f(x)$, and $g'(0) = \nabla f(x)^T(y-x)$, we have

$$f(y) = f(x) + \nabla f(x)^T(y-x) + \frac{1}{2}(y-x)^T \nabla^2 f(x + t(y-x))^T(y-x)$$

$\square$

1.4 Optimizing functions of several variables

Functions of two variables [edit]

Suppose that $f(x, y)$ is a differentiable real function of two variables whose second partial derivatives exist. The Hessian matrix H of f is the 2×2 matrix of partial derivatives of f:

$$H(x,y) = \begin{pmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{pmatrix}.$$

Define $D(x, y)$ to be the determinant

$$D(x,y) = \det(H(x,y)) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2 \ ,$$

of H. Finally, suppose that (a, b) is a critical point of f (that is, $f_x(a, b) = f_y(a, b) = 0$). Then the second partial derivative test asserts the following:[1]

1. If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$ then (a,b) is a local minimum of f.
2. If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$ then (a,b) is a local maximum of f.
3. If $D(a,b) < 0$ then (a,b) is a saddle point of f.
4. If $D(a,b) = 0$ then the second derivative test is inconclusive, and the point (a, b) could be any of a minimum, maximum or saddle point.

Functions of many variables [edit]

For a function f of two or more variables, there is a generalization of the rule above. In this context, instead of examining the determinant of the Hessian matrix, one must look at the eigenvalues of the Hessian matrix at the critical point. The following test can be applied at any critical point $(a, b, \ldots)$ for which the Hessian matrix is invertible:

1. If the Hessian is positive definite (equivalently, has all eigenvalues positive) at $(a, b, \ldots)$, then f attains a local minimum at $(a, b, \ldots)$.
2. If the Hessian is negative definite (equivalently, has all eigenvalues negative) at $(a, b, \ldots)$, then f attains a local maximum at $(a, b, \ldots)$.
3. If the Hessian has both positive and negative eigenvalues then $(a, b, \ldots)$ is a saddle point for f (and in fact this is true even if $(a, b, \ldots)$ is degenerate).

1.5 Lagrange Multipliers

: to flesh out! http://tutorial.math.lamar.edu/Classes/CalcIII/LagrangeMultipliers.aspx

1.6 Line Integrals

(p. 555 of Strang book)

Suppose a force in two-dimensional space is given by $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j}$. Then the work done by this force on a particle moving along a curve C is given by

$$W = \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{R} = \int_C M\mathrm{d}x + N\mathrm{d}y$$

Along a curve in three-dimensional space the work done by a three-dimensional force $\boldsymbol{F} = M\boldsymbol{i} + N\boldsymbol{j} + P\boldsymbol{j}$ is given by

$$W = \int_C \boldsymbol{F} \cdot \boldsymbol{T}\mathrm{d}s = \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{R} = \int_C M\mathrm{d}x + N\mathrm{d}y + P\mathrm{d}z$$

where the tangent vector $\boldsymbol{T}$ is given by

$$\boldsymbol{T} = \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}s}$$

Green's Theorem: Suppose the region R is bounded by the simple closed piecewise smooth curve C. Then an integral over R equals a line integral around C:

$$\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{R} = \oint_C M\mathrm{d}x + N\mathrm{d}y = \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathrm{d}x\ \mathrm{d}y$$

Line integrals chapter! http://tutorial.math.lamar.edu/Classes/CalcIII/LineIntegralsIntro.aspx

Surface inegrals chapter! http://tutorial.math.lamar.edu/Classes/CalcIII/SurfaceIntegralsIntro.aspx

1.7 Miscellaneous

13A The tangent plane at (x_0, y_0, z_0) has the same slopes as the surface $z = f(x, y)$. The equation of the tangent plane (a linear equation) is

$$z - z_0 = \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0). \qquad (1)$$

The normal vector $\mathbf{N}$ to that plane has components $(\partial f/\partial x)_0$, $(\partial f/\partial y)_0$, -1.

13B The tangent plane to the surface $F(x, y, z) = c$ has the linear equation

$$\left(\frac{\partial F}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial F}{\partial y}\right)_0 (y - y_0) + \left(\frac{\partial F}{\partial z}\right)_0 (z - z_0) = 0. \qquad (7)$$

The normal vector is $\mathbf{N} = \left(\frac{\partial F}{\partial x}\right)_0 \mathbf{i} + \left(\frac{\partial F}{\partial y}\right)_0 \mathbf{j} + \left(\frac{\partial F}{\partial z}\right)_0 \mathbf{k}$.

$$dz = (\partial z/\partial x)_0 dx + (\partial z/\partial y)_0 dy \quad \text{or} \quad df = f_x dx + f_y dy. \qquad (10)$$

This is the ***total differential***. All letters dz and df and dw can be used, but ∂z and ∂f are not used. Differentials suggest small movements in x and y; then dz is the resulting movement in z. On the tangent plane, equation (10) holds exactly.

The **directional derivative**, denoted $D_v f(x, y)$, is a derivative of a multivariable function in the direction of a vector $\boldsymbol{v}$. It is the scalar projection of the gradient onto $\boldsymbol{v}$.

$$D_v f(x,y) = \text{comp}_v \nabla f(x,y) = \frac{\nabla f(x,y) \cdot \boldsymbol{v}}{|\boldsymbol{v}|}$$

1.8 Practice Problems

13F The directional derivative is $D_{\mathbf{u}} f = (\text{grad}\, f) \cdot \mathbf{u}$. The level direction is perpendicular to grad f, since $D_{\mathbf{u}} f = 0$. ***The slope $D_{\mathbf{u}} f$ is largest when $\mathbf{u}$ is parallel to*** grad f. That maximum slope is the length $|\text{grad}\, f| = \sqrt{f_x^2 + f_y^2}$:

$$\textit{for} \quad \mathbf{u} = \frac{\text{grad}\, f}{|\text{grad}\, f|} \quad \textit{the slope is} \quad (\text{grad}\, f) \cdot \mathbf{u} = \frac{|\text{grad}\, f|^2}{|\text{grad}\, f|} = |\text{grad}\, f|.$$

$$\int_C g(x, y)\, ds = \text{limit of} \sum_{i=1}^{N} g(x_i, y_i) \Delta s_i \quad \text{as} \quad (\Delta s)_{\max} \to 0.$$

The differential ds becomes $(ds/dt)dt$. Everything changes over to t:

$$\int g(x, y) ds = \int_{t=a}^{t=b} g(x(t), y(t)) \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt.$$

19. Let f and g be twice-differentiable real-valued functions defined on $\mathbb{R}$. If $f'(x) > g'(x)$ for all $x > 0$, which of the following inequalities must be true for all $x > 0$?

(A) $f(x) > g(x)$

(B) $f''(x) > g''(x)$

(C) $f(x) - f(0) > g(x) - g(0)$

(D) $f'(x) - f'(0) > g'(x) - g'(0)$

(E) $f''(x) - f''(0) > g''(x) - g''(0)$

Solution 19. (C) There is no reason that $f(x) > g(x)$, or that $f''(x) > g''(x)$. But we do know that

$$\int_0^x f'(t)\, dt > \int_0^x g'(t)\, dt \implies f(x) - f(0) > g(x) - g(0).$$

This is precisely an answer.

22. What is the volume of the solid in xyz-space bounded by the surfaces $y = x^2$, $y = 2 - x^2$, $z = 0$, and $z = y + 3$?

(A) $\frac{8}{3}$ (B) $\frac{16}{3}$ (C) $\frac{32}{3}$ (D) $\frac{104}{105}$ (E) $\frac{208}{105}$

Solution 22. (C) It looks like our x-coordinates are running over $[-1, 1]$, with y depending on x and z depending on y. To find the volume of the solid, we just need to integrate the constant function 1. We must therefore compute

$$\begin{aligned}\int_{-1}^{1}\int_{x^2}^{2-x^2}\int_{0}^{y+3} 1\,dz\,dy\,dx &= \int_{-1}^{1}\int_{x^2}^{2-x^2} y + 3\,dy\,dx \\ &= \int_{-1}^{1} \left((2-x^2)^2/2 + 3(2-x^2)\right) - \left((x^2)^2/2 + 3(x^2)\right)\,dx \\ &= \int_{-1}^{1} 8 - 8x^2\,dx \\ &= 8x - 8x^3/3\Big|_{-1}^{1} = (8 - 8/3) - (-8 + 8/3) = 32/3.\end{aligned}$$

24. Let h be the function defined by $h(x) = \int_0^{x^2} e^{x+t}\,dt$ for all real numbers x. Then $h'(1) =$

(A) $e - 1$ (B) e^2 (C) $e^2 - e$ (D) $2e^2$ (E) $3e^2 - e$

Solution 24. (E) We can actually just integrate this, and not worry about differentiation under the integral.

$$\int_0^{x^2} e^{x+t}\,dt = e^x \int_0^{x^2} e^t\,dt = e^x(e^{x^2} - 1) = e^{x^2+x} - e^x.$$

Then deriving that,

$$h'(x) = (2x+1)e^{x^2+x} - e^x,$$

whence our result follows immediately.

26. Let $f(x, y) = x^2 - 2xy + y^3$ for all real x and y. Which of the following is true?

(A) f has all of its relative extrema on the line $x = y$.

(B) f has all of its relative extrema on the parabola $x = y^2$.

(C) f has a relative minimum at $(0, 0)$.

(D) f has an absolute minimum at $\left(\frac{2}{3}, \frac{2}{3}\right)$.

(E) f has an absolute minimum at $(1, 1)$.

Solution 26. (A) We are concerned about its extrema, we should find some partial derivatives.

$$f_x = 2x - 2y, \quad f_y = -2x + 3y^2.$$

We would like to know when they are both zero. The first equation gives us $x = y$ and the second gives us $2x = 3y^2$, so that

$$2y = 3y^2 \implies (3y-2)y = 0 \implies y = 0, 2/3.$$

Therefore our solutions are $(0,0)$ and $(2/3, 2/3)$. Indeed, our relative extrema are all on the line $x = y$. To do some more checking (which you should not do on the actual test),

$$f_{xx} = 2, \quad f_{yy} = 6y, \quad f_{xy} = f_{yx} = -2.$$

Then the determinant of the Hessian is $12y - 4$. This shows that $(0,0)$ is a saddle point. There is no reason that $(2/3, 2/3)$ is an absolute minimum without further verification, and $(1,1)$ needn't be an extreme point.

27. Consider the two planes $x + 3y - 2z = 7$ and $2x + y - 3z = 0$ in $\mathbb{R}^3$. Which of the following sets is the intersection of these planes?

(A) $\varnothing$

(B) $\{(0, 3, 1)\}$

(C) $\{(x, y, z) : x = t,\ y = 3t,\ z = 7 - 2t,\ t \in \mathbb{R}\}$

(D) $\{(x, y, z) : x = 7t,\ y = 3 + t,\ z = 1 + 5t,\ t \in \mathbb{R}\}$

(E) $\{(x, y, z) : x - 2y - z = -7\}$

Solution 27. (D) First, we know that the intersection of two planes in $\mathbb{R}^3$ should be either a plane or a line. In our case, the two planes are definitely not the same, so we will obtain a line. The slope of the line can be found by taking the cross product of the normal vectors of the two planes in question.

$$(1, 3, -2) \times (2, 1, -3) = \det \begin{bmatrix} i & j & k \\ 1 & 3 & -2 \\ 2 & 1 & -3 \end{bmatrix} = (-7, -1, -5).$$

The only solution corresponding to this slope is (D), as the coefficients of t in (x, y, z) are $(7, 1, 5)$.

32. $\dfrac{d}{dx}\displaystyle\int_{x^3}^{x^4} e^{t^2}\, dt =$

(A) $e^{x^6}\left(e^{x^8 - x^6} - 1\right)$ (B) $4x^3 e^{x^8}$ (C) $\dfrac{1}{\sqrt{1 - e^{x^2}}}$ (D) $\dfrac{e^{x^2}}{x^2} - 1$ (E) $x^2 e^{x^6}\left(4xe^{x^8 - x^6} - 3\right)$

Solution 32. (E) We can sort this out in two steps and apply the fundamental theorem to each.

$$\frac{d}{dx}\left(\int_{x^3}^{0} e^{t^2}\,dx + \int_0^{x^4} e^{t^2}\,dx\right)$$

For the first,

$$\frac{d}{dx}\int_{x^3}^{0} e^{t^2}\,dx = -\frac{d}{dx}\int_0^{x^3} e^{t^2}\,dx = -3x^2e^{x^6}$$

For the second,

$$\frac{d}{dx}\int_0^{x^4} e^{t^2}\,dx = 4x^3e^{x^8}.$$

All told, our integral is $x^2e^{x^6}(4xe^{x^8-x^6}-3)$.

41. Let ℓ be the line that is the intersection of the planes $x+y+z=3$ and $x-y+z=5$ in $\mathbb{R}^3$. An equation of the plane that contains $(0,0,0)$ and is perpendicular to ℓ is

(A) $x-z=0$

(B) $x+y+z=0$

(C) $x-y-z=0$

(D) $x+z=0$

(E) $x+y-z=0$

Solution 41. (A) The first plane is determined by the normal vector $(1,1,1)$, and the second determined by $(1,-1,1)$. Therefore the slope of ℓ is determined by a vector perpendicular to those, i.e. the cross product.

$$(1,1,1)\times(1,-1,1) = \det\begin{bmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -1 & 1\end{bmatrix} = (2,0,-2).$$

41. Let C be the circle $x^2+y^2=1$ oriented counterclockwise in the xy-plane. What is the value of the line integral $\oint_C (2x-y)\,dx + (x+3y)\,dy$?

(A) 0 (B) 1 (C) $\frac{\pi}{2}$ (D) π (E) 2π

Solution 41. (E) This is a classic Green's theorem problem.

$$\oint_{\partial D} L\,dx + M\,dy = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right)dx\,dy.$$

In our case,

$$\oint_C (2x-y)\,dx + (x+3y)\,dy = \iint_D (1+1)\,dx\,dy = 2A,$$

where A is the area of the unit circle, i.e. π.

So that is the slope of ℓ. We need this to be the normal vector for the plane in question, so it seems that $(1, 0, -1)$ is our best bet (out of the given options).

$$\begin{aligned} y' + xy &= x \\ y(0) &= -1 \end{aligned}$$

44. If y is a real-valued function defined on the real line and satisfying the initial value problem above, then $\lim_{x\to-\infty} y(x) =$

(A) 0 (B) 1 (C) -1 (D) ∞ (E) $-\infty$

Solution 44. (B) Putting it in simpler terms,

$$\frac{dy}{dx} + xy = x \implies \frac{dy}{dx} = x(1-y) \implies \frac{dy}{1-y} = x\,dx.$$

Integrating both sides, we obtain

$$-\log(1-y) = x^2/2 + C' \implies 1 - y = Ce^{-x^2/2} \implies y = 1 - Ce^{-x^2/2}.$$

Solving the initial value problem gives $C = 2$. Furthermore, as $x \to -\infty$, the second term above vanishes so we get 1 in the limit.

48. Let g be the function defined by $g(x, y, z) = 3x^2y + z$ for all real x, y, and z. Which of the following is the best approximation of the directional derivative of g at the point $(0, 0, \pi)$ in the direction of the vector $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$? (Note: $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the standard basis vectors in $\mathbb{R}^3$.)

(A) 0.2 (B) 0.8 (C) 1.4 (D) 2.0 (E) 2.6

Solution 48. (B) It would be good to recall the formula for the directional derivative. We take the gradient of the function then take its scalar product with the normalised vector in the direction we want. To begin,

$$\nabla g = (6xy, 3x^2, 1).$$

At the point $(0, 0, \pi)$, we have $\nabla g = (0, 0, 1)$. That works out pretty well for us. The normalised version of the vector $(1, 2, 3)$ is $(1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14})$. Dotting this with $(0, 0, 1)$ gives $3/\sqrt{14}$, and since $\sqrt{14} = 3.5$ or so our answer should be closer to 0.8 than 0.2.

48. Consider the theorem: If f and f' are both strictly increasing real-valued functions on the interval $(0, \infty)$, then $\lim_{x \to \infty} f(x) = \infty$. The following argument is suggested as a proof of this theorem.

    (1) By the Mean Value Theorem, there is a c_1 in the interval $(1, 2)$ such that

    $$f'(c_1) = \frac{f(2) - f(1)}{2 - 1} = f(2) - f(1) > 0.$$

    (2) For each $x > 2$, there is a c_x in $(2, x)$ such that $\frac{f(x) - f(2)}{x - 2} = f'(c_x)$.

    (3) For each $x > 2$, $\frac{f(x) - f(2)}{x - 2} = f'(c_x) > f'(c_1)$ since f' is strictly increasing.

    (4) For each $x > 2$, $f(x) > f(2) + (x - 2)f'(c_1)$.

    (5) $\lim_{x \to \infty} f(x) = \infty$

    Which of the following statements is true?

    (A) The argument is valid.

    (B) The argument is not valid since the hypotheses of the Mean Value Theorem are not satisfied in (1) and (2).

    (C) The argument is not valid since (3) is not valid.

    (D) The argument is not valid since (4) cannot be deduced from the previous steps.

    (E) The argument is not valid since (4) does not imply (5).

Solution 48. (A) The only issue here seems to be that (4) implies that $f(x)$ gets very large so long as $f'(c_1)$ is positive. But we know that it is, since f is a strictly increasing function. Therefore everything is satisfactory.