Math Review Notes—Asymptotics and Convergence

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1 Asymptotics and Convergence

These notes are based on my notes from chapter 8 of *Time Series and Panel Data Econometrics* (1st edition) by M. Hashem Pesaran [Pesaran, 2015] and coursework for Economics 613: Economic and Financial Time Series I at USC, as well as Math 505A and Math 541A at USC and chapter 7 from *Probability and Random Processes* (Grimmett and Stirzaker) 3rd edition [Grimmett and Stirzaker, 2001].

1.1 Preliminaries (5.9 and 7.1, Grimmett and Stirzaker)

Definition 1.1. Definition 7.1.4, Grimmett and Stirzaker. If for all $x \in [0,1]$ the sequence $\{f_n(x)\}$ of real numbers satisfies $f_n(x) \to f(x)$ as $n \to \infty$ then we say $f_n \to f$ **pointwise.**

Remark. In practice pointwise convergence is often not useful for functions because a sequence of functions may be continuous while its limit is not. For instance, consider $\{f_n : f_n = x^n \ \forall x \in [0,1]\}$. Then f_n is continuous for all n but

$$\lim_{n\to\infty} f_n = \begin{cases} 0 & x \leq 1 \\ 1 & x = 1 \end{cases}$$

Instead, the following definition is often more useful.

Definition 1.2. (from class notes.) We say that f_n **uniformly converges to** f **on** $[a,b]$ if for every $\epsilon > 0$ there exists N such that for every $n > N$,

$$\forall x \in [a,b] \ |f_n(x) - f(x)| < \epsilon$$

Definition 1.3. (Definition 7.1.5, Grimmett and Stirzaker.) Let V be a collection of functions mapping $[0,1]$ into $\mathbb{R}$ and assume V is endowed with a function $\|\cdot\| : V \to \mathbb{R}$ satisfying

(a) $\|f\| \geq 0$ for all $f \in V$

(b) $\|f\| = 0$ if and only if f is the zero function (or equivalent to it)

(c) $\|af\| = |a| \cdot \|f\|$ for all $a \in \mathbb{R}$, $f \in V$

(d) $\|f + g\| \leq \|f\| + \|g\|$ (Triangle Inequality)

The function $\|\cdot\|$ is called a **norm**. If $\{f_n\}$ is a sequence of members of V then we say that $f_n \to f$ **with respect to the norm** $\|\cdot\|$ if $\|f_n - f\| \to 0$ as $n \to \infty$.

Definition 1.4. (Definition 7.16, Grimmett and Stirzaker.) Let $\epsilon > 0$ be prescribed, and define the distance between two functions $g, h : [0,1] \to \mathbb{R}$ by

$$d_\epsilon(g,h) = \int_E dx$$

where $E = \{u \in [0,1] : |g(u) - h(u)| > \epsilon\}$. We say that $f_n \to f$ **in measure** if

$$d_\epsilon(f_n, f) \to 0 \text{ as } n \to \infty \text{ for all } \epsilon > 0$$

Theorem 1.1.1. Inversion Theorem (Theorem 5.9.2, Grimmett and Stirzaker). Let X have distribution function F and characteristic function ϕ. Define $\overline{F} : \mathbb{R} \to [0,1]$ by

$$\overline{F}(x) = \frac{1}{2}\left[F(x) + \lim_{y \to x^-} F(y)\right]$$

Then

$$\overline{F}(b) - \overline{F}(a) = \lim_{N \to \infty} \int_{-N}^{N} \frac{\exp(-iat) - \exp(-ibt)}{2\pi i t} \cdot \phi(t) dt$$

Proof. See Kingman and Taylor [1966]. □

Corollary 1.1.1.1. Corollary 5.9.3. Random variables X and Y have the same characteristic function if and only if they have the same distribution function.

Proof. **Available in Grimmett and Stirzaker section 5.9, pp. 189 - 190.** □

Definition 1.5. (Definition 5.9.4, Grimmett and Stirzaker.) We say that the sequence $F_1, F_2, \ldots$ of distribution functions **converges** to the distribution function F (written $F_n \to F$) if $F(x) = \lim_{n\to\infty} F_n(x)$ at each point x where F is continuous.

Theorem 1.1.2. Continuity theorem (Thereom 5.9.5; in notes from Friday 10/26, Lecture 28). Suppose that $F_1, F_2, \ldots$ is a sequence of distribution functions with corresponding characteristic functions $\phi_1, \phi_2, \ldots$.

(a) If $F_n(x) \to F(x)$ for some distribution function F with characteristic function ϕ (at x where F is continuous), then $\phi_n(t) \to \phi(t)$ for all t.

(b) Conversely, if $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ exists and $\phi(t)$ is continuous at $t = 0$, then ϕ is the characteristic function of some distribution function F, and $F_n \to F$.

Proof. See Kingman and Taylor [1966]. □

1.2 Inequalities (8.6 of Pesaran)

Inequalities

- Probabilities
  -

    Lemma 1.2.1. Markov's Inequality (Grimmett and Stirzaker p. 311, 319)): Let $X : \Omega \to [-\infty, \infty]$ be a random variable. Then for all $a > 0$,

    $$\Pr(|X| \geq a) \leq \frac{\mathbb{E}(|X|)}{a}$$

Proof. Note $t \cdot \mathbf{1}_{\{|X| \geq t\}} \leq |X|$, where $\mathbf{1}$ is the indicator function. Dividing both sides by t and taking expectations, we have

$$\mathbb{E}(\mathbf{1}_{\{|X|\geq t\}}) \leq \frac{\mathbb{E}|X|}{t} \iff \Pr(|X|) \geq t) \leq \frac{\mathbb{E}|X|}{t}, \qquad \forall\, t > 0.$$

□

Corollary 1.2.1.1. If n is a positive integer, then

$$\Pr(|X| \geq t) \leq \frac{\mathbb{E}(|X|^n)}{t^n} \quad \forall t > 0$$

Proof. By Markov's Inequality (Theorem 1.2.1),

$$\Pr(|X| \geq t) = \Pr(|X|^n \geq t^n) \leq \frac{\mathbb{E}(|X|^n)}{t^n}$$

□

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Theorem 1.2.2. Chebyshev's Inequality: (probability p. 319) Let $X : \Omega \to [-\infty, \infty]$ be an (integrable) random variable with $\mathbb{E}(X^2) < \infty$.Then for any real number $k > 0$

$$\Pr\left(|X - \mathbb{E}(X)| \geq k\sqrt{\text{Var}(X)}\right) \leq \frac{1}{k^2}$$

This can also be written as

$$\Pr(|X - \mathbb{E}(X)| \geq k) \leq \frac{\text{Var}(X)}{k^2}$$

(Can be used to demonstrate consistency of estimators: if we can show that as $T \to \infty$ $\text{Var}(X) = \sigma^2 \to 0$, then this implies $\Pr\left(|X - \mu| \geq k\sigma\right) \to 0$ as $T \to \infty$, showing consistency.)

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Theorem 1.2.3. Chernoff For $x \geq 0$, $a > 0$, $\forall\ t > 0$,

$$\Pr(X \geq a) = \Pr(e^{tX} \geq e^{ta}) \leq \frac{\mathbb{E}(e^{tX})}{e^{ta}}$$

- **Moments**

  –

  Theorem 1.2.4 (Cauchy-Schwarz (and **Bunyakovsky).).** If X and Y are random variables with finite variance then

  $$\mathbb{E}(XY)^2 \leq \mathbb{E}(X^2)\mathbb{E}(Y^2)$$

  Note that this can is a corollary of Theorem 1.2.7 with $p = q = 2$. We can also prove this theorem on its own in a different one. We first prove a useful result.

  Lemma 1.2.5. If $\text{Var}(X) = 0$ then X is almost surely constant; that is, $\Pr(X = a) = 1$ for some $a \in \mathbb{R}$.

Proof. Note that because $\text{Var}(X) = 0 < \infty$, we know that $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$ exist. We have

$$\text{Var}(X) = \mathbb{E}\big[(X - \mathbb{E}(X))^2\big] = 0$$

Let $Y = (X - \mathbb{E}(X))^2$. Note that $Y = (X - \mathbb{E}(X))^2 \geq 0$ and that $\mathbb{E}(Y) = \text{Var}(X) = 0$. Therefore $\Pr(Y = 0) = 1$, so $\Pr(Y \neq 0) = 0$. To see why, in the case that X is discrete,

$$\mathbb{E}(Y) = \sum_{k=0}^{\infty} k \cdot \Pr(Y = k) = \text{Var}(X) = 0$$

which is true if and only if $\Pr(Y = k) = 0$ for all $k > 0$. Since we already showed that $Pr(Y < 0) = 0$, it follows that $\Pr(Y = 0) = 1$. In the continuous case,

$$\mathbb{E}(Y) = \int_0^{\infty} y \cdot f_Y(y) dy = \text{Var}(X) = 0$$

which implies that $f_Y(x) = 0$ for all $x > 0$. Again, since $\Pr(Y < 0) = 0$, we have $\Pr(Y \neq 0) = 0$. But $Y = 0 \iff X = \mathbb{E}(X)$ so we have $\Pr(X = \mathbb{E}(X)) = 1$. □

Remark. Note that Lemma 1.2.5 along with Proposition **??** imply that X has variance 0 if and only if it is (almost surely) constant.

We are now ready to prove the Cauchy-Schwarz Inequality.

Proof. if $\mathbb{E}(X^2) = 0$ or $\mathbb{E}(Y^2) = 0$, the Cauchy-Schwarz Inequality follows immediately. To see why, suppose without loss of generality that $\mathbb{E}(X^2) = 0$. Then the right side is 0. Also, $0 \leq \text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = -\mathbb{E}(X)^2$. Since $\mathbb{E}(X)^2 \geq 0$, we must have $E(X)^2 = 0$ and therefore $\text{Var}(X) = 0$. Therefore by Lemma 1.2.5, X is almost surely constant, which means that $\text{Cov}(X, Y) = 0$.

In the case that $\mathbb{E}(X^2) > 0$ and $\mathbb{E}(Y^2) > 0$, for $a, b \in \mathbb{R}$, let $Z = aX - bY$. Then

$$0 \leq \mathbb{E}(Z^2) = a^2\mathbb{E}(X^2) - 2ab\mathbb{E}(XY) + b^2\mathbb{E}(Y^2) \tag{1}$$

The right side of (1) is quadratic in a. Because it is greater than or equal to zero, it has at most one real root, which means its discriminant must be non-positive. That is, if $b \neq 0$,

$$(-2b\mathbb{E}(XY))^2 - 4b^2\mathbb{E}(X^2)\mathbb{E}(Y^2) \leq 0 \iff \mathbb{E}(XY)^2 - \mathbb{E}(X^2)\mathbb{E}(Y^2) \leq 0$$

which yields the result. Note that equality holds if and only if $\Pr(aX = bY) = 1$ because the discriminant is zero if and only if the quadratic has a real root, which occurs if and only if

$$\mathbb{E}\big[(aX - bY)^2\big] = 0$$

which is true if and only if $\Pr(aX = bY) = 1$ by by Lemma 1.2.5 and Proposition **??**. □

- **Krylov**
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Definition 1.6. Let $\phi : \mathbb{R} \to \mathbb{R}$. We say that ϕ is **convex** if for any $x, y \in \mathbb{R}$ and for any $t \in [0, 1]$, we have

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$$

Theorem 1.2.6 (Jensen's Inequality, from Math 541A. Also Grimmett and Stirzaker p.181, 349). Let $X : \Omega \to [-\infty, \infty]$ be a random variable. Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex. If $\mathbb{E}|X| < \infty$ and $\mathbb{E}|\phi(X)| < \infty$, then

$$\phi(\mathbb{E}X) \leq \mathbb{E}\phi(X).$$

(See also Theorem **??**.)

For the definition of convexity, see Definition **??**.

Proof. Note that from Theorem **??**, for any $y \in \mathbb{R}$ there exists a constant a and a function L such that

$$a(x - y) + \phi(y) \leq \phi(x) \quad \forall x \in \mathbb{R}$$

Letting $y = \mathbb{E}(X)$ we have

$$a(X - \mathbb{E}X) + \phi(\mathbb{E}X) \leq \phi(X)$$

Since expectations preserve inequalities,

$$\mathbb{E}[a(X - \mathbb{E}X) + \phi(\mathbb{E}X)] \leq \mathbb{E}\phi(X)$$

But

$$\mathbb{E}[a(X - \mathbb{E}X) + \phi(\mathbb{E}X)] = a(\mathbb{E}X - \mathbb{E}X) + \mathbb{E}(\phi(\mathbb{E}X)) = \phi(\mathbb{E}X)$$

which yields

$$\phi(\mathbb{E}X) \leq \mathbb{E}\phi(X).$$

□

For some corollaries, see section **??**.

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Theorem 1.2.7. [Hölder (Grimmett and Stirzaker p. p. 143, 319; Theorem 1.99 in Math 541A lecture notes) Generalization of Cauchy-Schwarz] Let $X, Y : \Omega \to \mathbb{R}$ be random variables. For $p, q \geq 1$ satisfying $1/p + 1/q = 1$ we have

$$\mathbb{E}(|XY|) \leq (\mathbb{E}(|X^p|))^{1/p}(\mathbb{E}(|X^q|))^{1/q} = \|X\|_p \|Y\|_q.$$

The equality case happens only if X is a constant multiple of Y with probability 1. Note that the case $p = q = 2$ recovers the Cauchy-Schwarz Inequality (Theorem 1.2.4).

Proof. Assume without loss of generality that $\|X\|_p = \|Y\|_q = 1$. Also, the case $p = 1, q = \infty$ follows from the triangle inequality, so we assume $1 < p < \infty$. From concavity of the log function, we have

$$\log\left((x^p)^{1/p}(y^q)^{1/q}\right) = (1/p)\log\left(x^p\right) + (1/q)\log\left(y^q\right)$$

$$\leq \log\left(\frac{1}{p}x^p + \frac{1}{q}y^q\right)$$

$$\implies (x^p)^{1/p}(y^q)^{1/q} \leq \frac{1}{p}x^p + \frac{1}{q}y^q$$

Fixing an $\omega \in \Omega$, we have

$$|X(\omega)Y(\omega)| = (|X(\omega)|^p)^{1/p}(|Y(\omega)|^q)^{1/q} \leq \frac{1}{p}|X(\omega)|^p + \frac{1}{q}|Y(\omega)|^q$$

Integrating we have...

□

Theorem 1.2.8 (Hölder (vector form)). For any $u, v \in \mathbb{R}^n$,

$$|u^T v| \leq \|u\|_p \|v\|_q$$

for any $p, q \in [0, \infty]$ satisfying $1/p + 1/q = 1$.

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Theorem 1.2.9. Minkowski (Grimmett and Stirzaker p. p. 143) For $p \geq 1$,

$$[\mathbb{E}(|X+Y|^p)]^{1/p} \leq (\mathbb{E}|X^p|)^{1/p} + (\mathbb{E}|Y^p|)^{1/p}$$

– Useful for showing lower order moments are finite (e.g. finite variance implies finite mean).

Lemma 1.2.10. Lyapunov's Inequality (Grimmett and Stirzaker p. 143). For $0 < r \leq s < \infty$,

$$\mathbb{E}(|X|^r)^{1/r} \leq \mathbb{E}(|X|^s)^{1/s}$$

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Theorem 1.2.11. Triangle Inequality: Let $X, Y : \Omega \to \mathbb{R}$ be random variables. Let $1 \leq p \leq \infty$. Then

$$\|X+Y\|_p \leq \|X\|_p + \|Y\|_p, 1 \leq p \leq \infty$$

Proof. The case $p = \infty$ follows from the scalar triangle inequality, so assume $1 \leq p < \infty$. By scaling, we may assume $\|X\|_p = 1 - t, \|Y\|_p = t$, for some $t \in (0, 1)$ (zeroes and infinities being trivial). Define $V := X/(1-t), W := Y/t$. Then by convexity of $x \to |x|^p$ on $\mathbb{R}$,

$$|(1-t)V(\omega) + t(W(\omega)|^p \leq (1-t)|V(\omega)|^p + t|W(\omega)|^p$$

Take expectation of both sides:

$$\mathbb{E}|X+Y|^p \leq (a-t)^{1-p}\mathbb{E}(|X|^p) + t^{1-p}\mathbb{E}(|Y|^p)$$

Since $\|X\|_p = t, \|Y\|_p = 1 - t$, we have that the right side is $1 - t + t = 1$. (Note: $\|Y\|_p = t, \mathbb{E}|Y|^p = t^p, \|X\|_p = 1 - t$ Therefore

$$(\mathbb{E}|X+Y|^p)^{1/p} = \|X+Y\|_p \leq 1$$

□

Remark. See also Theorem **??** and Corollary **??**.

Theorem 1.2.12 (Chernoff Bound). Let X be a random variable and let $r > 0$. Define $M_X(t) := \mathbb{E}e^{tX}$ for any $t \in \mathbb{R}$. Then for any $t > 0$,

$$\mathbb{P}(X > r) \leq e^{-tr} M_X(t).$$

Proof. Using Markov's Inequality (Theorem 1.2.1) on e^{tX}, we have

$$\Pr(X \geq r) = \Pr(e^{tX} \geq e^{tr}) \leq \frac{\mathbb{E}e^{tX}}{e^{tr}} = e^{-tr} M_X(t), \quad \forall\, t > 0.$$

□

Remark. Consequently, if $X_1, \ldots, X_n$ are independent random variables with the same CDF, and if $r, t > 0$,

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i > r\right) \leq e^{-trn}(M_{X_1}(t))^n.$$

For example, if $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter $0 < p < 1$, and if $r, t > 0$,

$$\mathbb{P}\left(\frac{X_1 + \cdots + X_n}{n} - p > r\right) \leq e^{-trn}(e^{-tp}[pe^t + (1-p)])^n.$$

And if we choose t appropriately, then the quantity $\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - p) > r\right)$ becomes exponentially small as either n or r become large. That is, $\frac{1}{n}\sum_{i=1}^{n} X_i$ becomes very close to its mean. Importantly, the Chernoff bound is much stronger than either Markov's or Cheyshev's inequality, since they only respectively imply that

$$\mathbb{P}\left(\left|\frac{X_1 + \cdots + X_n}{n} - p\right| > r\right) \leq \frac{2p(1-p)}{r}, \quad \mathbb{P}\left(\left|\frac{X_1 + \cdots + X_n}{n} - p\right| > r\right) \leq \frac{p(1-p)}{nr^2}.$$

Monotone convergence theorem.

Dominated Convergence Theorem (Theorem **??**).

1.3 Modes of Convergence (7.2 of Grimmett and Stirzaker, 8.2 and 8.4 of Pesaran)

Let $\{X_n\} = \{X_1, X_2, \ldots\}$ and X be random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 1.7. Convergence in probability. $\{X_n\}$ is said to **converge in probability** to X if

- Grimmett and Strizaker definition:

$$\lim_{n\to\infty} \Pr(|X_n - X| > \epsilon) = 0, \text{ for every } \epsilon > 0$$

- Pesaran definition:

$$\lim_{n\to\infty} \Pr(|X_n - X| < \epsilon) = 1, \text{ for every } \epsilon > 0$$

- More formal (from Math 541A):

$$\forall \epsilon > 0, \ \lim_{n\to\infty} \Pr(\{\omega \in \Omega : |X_n(\omega) - X(\omega)| > \epsilon) = 0$$

Remark. This mode of convergence is also often denoted by $X_n \xrightarrow{p} X$ and when X is a fixed constant it is referred to as the **probability limit of** X_n, written as $Plim(X_n) = x$, as $n \to \infty$.

The above concept is readily extended to multivariate cases where $\{\boldsymbol{X}_n, n = 1, 2, \ldots\}$ denote m-dimensional vectors of random variables. Then the condition is

$$\lim_{n\to\infty} \Pr(\|\boldsymbol{X}_n - \boldsymbol{X}\| < \epsilon) = 1, \text{ for every } \epsilon > 0$$

where $\|\cdot\|$ denotes an appropriate norm (say ℓ_2). Convergence in probability is often referred to as "weak convergence" (in contrast to convergence with probability 1, below).

Definition 1.8. Convergence with probability 1 or almost surely. The sequence of random variables $\{X_n\}$ is said to **converge with probability 1** (or **almost surely**) to X if

- (505A class notes definition)

  $$\Pr\left(\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}\right) = 1$$

  (Note: pointwise convergence can hardly ever be shown here and is not useful.)

- Grimmett and Stirzaker textbook definition:

  $$\Pr\left(\{\omega \in \Omega : X_n(\omega) \to X(\omega) \text{ as } n \to \infty\}\right) = 1$$

- Pesaran textbook definition:

  $$\Pr\left(\lim_{n\to\infty} X_n = X\right) = 1$$

Remark. This is often written as $X_n \xrightarrow{w.p.1} X$ or $X_n \xrightarrow{a.s.} X$. An equivalent condition for convergence with probability 1 is given by

$$\lim_{n\to\infty} \Pr(|X_m - X| < \epsilon, \text{ for all } m \geq= n) = 1, \text{ for every } \epsilon > 0$$

which shows that convergence in probability is a special case of convergence with probability 1 (obtained by setting $m = n$). Convergence with probability 1 is stronger than convergence in probability and is often referred to as "strong convergence."

Definition 1.9. Convergence in r-th mean or **convergence in** ℓ_p. $X_n \to X$ **in rth mean** (or **in** ℓ_p) where $r \geq 1$ (or $0 < p \leq \infty$) if $\mathbb{E}|X_n^r| < \infty$ for all n and

$$\lim_{n\to\infty} \mathbb{E}(|X_n - X|^r) = 0$$

or if $\|X\|_p < \infty$ and

$$\lim_{n\to\infty} \|X_n - X\|_p = 0$$

Remark. Recall that $\|X\|_p := (\mathbb{E}(X)^p)^{1/p}$ if $0 < p < \infty$ and $\|X\|_\infty := \inf\{c > 0 : \Pr(|X| \leq c) = 1\}$. Note that if $p < 1$, $\|\cdot\|_p$ is no longer a norm because it does not satisfy the Triangle Inequality (Corollary **??** and Theorem 1.2.11), but this property still holds. Convergence in rth mean is often written $X_n \xrightarrow{r} X$.

Definition 1.10. Convergence in Distribution. Let $X_1, X_2, \ldots$ have distribution functions $F_1(\cdot), F_2(\cdot), \ldots$ respectively. Then X_n is said to **converge in distribution to** X if

$$\lim_{n\to\infty} \Pr(X_n \leq u) = \Pr(X \leq u)$$

for all u at which $F_X(x) = \Pr(X \leq x)$ is continuous. This can also be written

$$\lim_{n\to\infty} F_n(u) = F(u)$$

for all u at which F is continuous.

Remark. Convergence in distribution is usually denoted by $X_n \xrightarrow{d} X$, $X_n \xrightarrow{L} X$, or $F_n \implies F$. By the Continuity Theorem (Theorem 1.1.2, section 1.1), this is equivalent to

$$\lim_{n\to\infty} \phi_{X_n}(t) = \phi_X(t), \quad t \in \mathbb{R}.$$

Note that the random variables are allowed to have different domains.

Definition 1.11. (Convergence in distribution for vector-valued random variables.) We say that random variables $Y^{(1)}, \ldots, : \Omega \to \mathbb{R}^d$ **converge in distribution** to $Y : \Omega \to \mathbb{R}^d$ if for all $v \in \mathbb{R}^d$, $\langle v, Y^{(1)} \rangle, \langle v, Y^{(2)} \rangle, \ldots$ converges in distribution to $\langle v, Y \rangle$.

Theorem 1.3.1 (**(Theorem 7.2.3, Grimmett and Stirzaker.)**). The following implications hold:

(a) $(X_n \xrightarrow{a.s.} X) \implies (X_n \xrightarrow{p} X)$

(b) $(X_n \xrightarrow{r} X) \implies (X_n \xrightarrow{p} X)$ for any $r \geq 1$

(c) $(X_n \xrightarrow{p} X) \implies (X_n \xrightarrow{d} X)$

Also, if $r > s \geq 1$, then $(X_n \xrightarrow{r} X) \implies (X_n \xrightarrow{s} X)$. No other implications hold in general.

Proof. (a) By Markov's Inequality (Lemma 1.2.1),

$$\Pr(|X_n - X| > \epsilon) \leq \frac{\mathbb{E}|X_n - X|}{\epsilon} \quad \text{for all } \epsilon > 0$$

Therefore if $X_n \xrightarrow{1} X$; that is, $\lim_{n\to\infty} \mathbb{E}(|X_n - X|) = 0$, then $\lim_{n\to\infty} \Pr(|X_n - X| > \epsilon) = 0$ for every $\epsilon > 0$, so $X_n \xrightarrow{p} X$.

(b) Let $\epsilon > 0$ and let $0 < r < \infty$. From Markov's Inequality,

$$\Pr(|X_n - X| > \epsilon) = \Pr(|X_n - X|^r > \epsilon^r) \leq \frac{\mathbb{E}|X_n - X|^r}{\epsilon^r}$$

The right side converges to zero by assumption. Therefore $X_1, X_2, \ldots$ converges to X in probability. The case $r = \infty$ follows from any case $r < \infty$, since for example

$$(|\mathbb{E}(X_n) - X|)^2 \leq \mathbb{E}(|X_n - X|^2) \leq \|X_n - X\|_\infty$$

which shows that L_∞ convergence implies L_2 convergence. So since L_2 convergence implies convergence in probability, L_∞ convergence does too. We can also see this by simply examining the definition of the L_∞ norm:

$$\|X_n - X\|_\infty := \inf\{c > 0 : \Pr(|X_n - X| \leq c) = 1\}.$$

Clearly if this number goes to 0, then $X_n \xrightarrow{p} X$.

(c)

$\square$

Remark. Here are counterexamples showing that the converses are not always true:

(a) To see the converse fails, define an independent sequence $\{X_n\}$ by

$$X_n = \begin{cases} n^3 & \text{with probability } n^{-2} \\ 0 & \text{with probability } 1 - n^{-2} \end{cases}$$

Then $\Pr(|X| > \epsilon) = n^{-2}$ for all large n, and so $X_n \xrightarrow{p} 0$. However, $\mathbb{E}|X_n| = n \to \infty$.

(b) To see why the converse is false, fix $0 < r < \infty$, let $\Omega := [0, 1]$ with $\mathbb{P}$ uniform on Ω and consider $X_n := n^{1/r}\mathbf{1}_{\{[0,1/n]\}}$. Then $X_1, X_2, \ldots$ converges in probability to 0 since if $1 > \epsilon > 0$, then $\mathbb{P}(|X_n - 0| > \epsilon) \leq \mathbb{P}([0, 1/n]) = 1/n \to 0$ as $n \to \infty$. However, $X_1, X_2, \ldots$ does not converge in L_r to 0 since $\mathbb{E}|X_n - 0|^r = n/n = 1$ for all $n \geq 1$.

(c)

Theorem 1.3.2. Some exceptions (Theorem 7.2.4).

(a) If $X_n \xrightarrow{d} c$ where c is constant, then $X_n \xrightarrow{p} c$.

(b) If $X_n \xrightarrow{p} X$ and $\Pr(|X_n| \leq k) = 0$ for all n and some k, then $X_n \xrightarrow{r} X$ for all $r \geq 1$.

(c) If $P_n(\epsilon) = \Pr(|X_n - X| > \epsilon)$ satisfies $\sum_n P_n(\epsilon) < \infty$ for all $\epsilon > 0$, then $X_n \xrightarrow{a.s.} X$.

Proof. (Part (c).) Let $A_n(\epsilon) = \{|X_n - X| > \epsilon\}$ (so that $P_n(\epsilon) = \Pr[A_n(\epsilon))]$, and let $B_m(\epsilon) = \bigcup_{n \geq m} A_n(\epsilon)$. Then

$$\Pr(B_m(\epsilon)) \leq \sum_{n=m}^{\infty} \Pr(A_n(\epsilon))$$

so $\lim_{m\to\infty} \Pr(B_m(\epsilon)) = 0$ whenever $\sum_n \Pr(A_n(\epsilon)) < \infty$. See also Lemma 1.4.1 part (b). $\square$

Lemma 1.3.3. (Lemma 7.2.6 from Grimmett and Stirzaker)

(a) If $r > s \geq 1$ and $X_n \xrightarrow{r} X$, then $X_n \xrightarrow{s} X$.

(b) If $X_n \xrightarrow{1} X$ then $X_n \xrightarrow{p} X$.

The converse assertions fail in general.

Proof. (a) Using Lyapunov's Inequality (Lemma 1.2.10), if $r > s \geq 1$

$$\left[\mathbb{E}(|X_n - X|^s)\right]^{1/s} \leq \left[\mathbb{E}(|X_n - X|^r)\right]^{1/r}$$

Therefore if $X_n \xrightarrow{r} X$ (meaning $\lim_{n\to\infty} \mathbb{E}(|X_n - X|^r) = 0$), (then $\lim_{n\to\infty} \mathbb{E}(|X_n - X|^s) = 0$, so $X_n \xrightarrow{s} X$. We show the converse fails by counterexample:

$$X_n = \begin{cases} n & \text{with probability } n^{(-1/2)(r+s)} \\ 0 & \text{with probability } 1 - n^{(-1/2)(r+s)} \end{cases}$$

Then $\mathbb{E}|X_n^s| = n^{(1/2)(s-r)} \to 0$ and $\mathbb{E}|X_n^r| = n^{(1/2)(r-s)} \to \infty$.

(b) See proof of Theorem 1.3.1(a).

□

1.4 More on convergence (7.2 of Grimmet and Stirzaker)

Other theorems to include: Fatou's Lemma, Fubini's Theorem, Kolmogorov's Maximal Inequality, Kolmogorov Three-Series Test, Lindeberg Feller Central Limit Theorem, **this and more at beginning of Mike's 505A qual solutions.**

Definition 1.12. Cauchy Convergence. We say that the sequence $\{X_n : n \geq 1\}$ of random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is **almost surely Cauchy convergent** if

$$\Pr\left(\{\omega \in \Omega : X_m(\omega) - X_n(\omega) \to 0 \text{ as } m, n \to \infty\}\right) = 1$$

That is, the set of points ω of the sample space for which the real sequence $\{X_n(\omega) : n \geq 1\}$ is Cauchy convergent is an event having probability 1.

Lemma 1.4.1. (Lemma 7.2.10, Grimmett and Stirzaker.) Let $A_n(\epsilon) = \{|X_n - X| > \epsilon\}$ and $B_m(\epsilon) = \cup_{n\geq m} A_n(\epsilon)$. Then:

(a) $X_n \xrightarrow{a.s.} X$ if and only if $\Pr(B_m(\epsilon)) \to 0$ as $m \to \infty$ for all $\epsilon > 0$.

(b) $X_n \xrightarrow{a.s.} X$ if $\sum_n \Pr(A_n(\epsilon)) < \infty$ for all $\epsilon > 0$.

(c) If $X_n \xrightarrow{a.s.} X$ then $X_n \xrightarrow{p} X$, but the converse fails in general.

Proof. (a)

(b) As for Theorem 1.3.2 part (c).

(c) To see the converse fails, define an independent sequence $\{X_n\}$ by

$$X_n = \begin{cases} 1 & \text{with probability } n^{-1} \\ 0 & \text{with probability } 1 - n^{-1} \end{cases}$$

Clearly $X_n \xrightarrow{p} 0$. However, if $0 < \epsilon < 1$,

$$\Pr(B_m(\epsilon)) = 1 - \lim_{r \to \infty} \Pr(X_n = 0 \text{ for all } n \text{ such that } m \leq n \leq r) \text{ (by Lemma 1.3.5)}$$

$$= 1 - \left(1 - \frac{1}{m}\right)\left(1 - \frac{1}{m+1}\right) \cdots \text{ (by independence)}$$

$$= 1 - \lim_{M \to \infty} \left(\frac{m-1}{m} \cdot \frac{m}{m+1} \cdot \frac{m+1}{m+2} \cdots \frac{M}{M+1} \right)$$

$$= 1 - \lim_{M \to \infty} \frac{m-1}{M+1} = 1$$

and so $\{X_n\}$ does not converge almost surely.

□

Lemma 1.4.2. (Lemma 7.2.12, Grimmett and Stirzaker.) There exist sequences which

(a) converge almost surely but not in mean,

(b) converge in mean but not almost surely.

Proof. (a) As for Lemma 1.3.3 part (b).

□

Theorem 1.4.3. (Theorem 7.2.13, Grimmett and Stirzaker.) If $X_n \xrightarrow{p} X$, there exists a non-random increasing sequence of integers $n_1, n_2, \ldots$ such that $X_{n_i} \xrightarrow{a.s.} X$ as $i \to \infty$.

Theorem 1.4.4. Skorokhod's representation theorem (Theorem 7.2.14, Grimmett and Stirzaker). If $\{X_n\}$ and X with distribution functions $\{F_n\}$ and F are such that $X_n \xrightarrow{d} X$ (or equivalently, $F_n \to F$) as $n \to \infty$, then there exists a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and random variables $\{Y_n\}$ and Y mapping Ω' into $\mathbb{R}$ such that

(a) $\{Y_n\}$ and Y have distribution functions $\{F_n\}$ and F

(b) $Y_n \xrightarrow{a.s.} Y$ as $n \to \infty$

Therefore, although X_n may fail to converge to X in any mode other than in distribution, there exists a sequence $\{Y_n\}$ such that Y_n is distributed identically to X_n for every n, which converges almost surely to a copy of X.

Theorem 1.4.5. (Theorem 7.2.19, Grimmett and Stirzaker; same as Portmanteau Theorem?) The following three statements are equivalent:

(a) $X_n \xrightarrow{d} X$

(b) $\mathbb{E}[g(X_n)] \to \mathbb{E}[g(X)]$ for all bounded continuous functions g.

(c) $\mathbb{E}[g(X_n)] \to \mathbb{E}[g(X)]$ for all functions g of the form $g(x) = f(x)\mathbf{1}_{[a,b]}(x)$ where f is continuous on $[a, b]$ and a and b are points of continuity of the distribution function of the random variable X.

Theorem 1.4.6. (Grimmett and Stirzaker Theorem 7.3.9.)

(a) If $X_n \xrightarrow{a.s.} X$ and $Y_n \xrightarrow{a.s.} Y$ then $X_n + Y_n \xrightarrow{a.s.} X + Y$.

(b) If $X_n \xrightarrow{r} X$ and $Y_n \xrightarrow{r} Y$ then $X_n + Y_n \xrightarrow{r} X + Y$.

(c) If $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$ then $X_n + Y_n \xrightarrow{p} X + Y$.

(d) It is not in general true that $X_n + Y_n \xrightarrow{d} X + Y$ whenever $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$.

Proposition 1.4.7. Almost sure convergence does not imply convergence in L_2, and convergence in L_2 does not imply almost sure convergence. That is, find random variables that converge in L_2 but not almost surely. Then, find random variables that converge almost surely but not in L_2.

Proof. (a) **Almost sure convergence does not imply ℓ_2 convergence:** Let $Z_n := n\mathbf{1}_{[0,1/n]}$. Then $Z_1, Z_2, \ldots$ converges almost surely to 0 since $\lim_{n\to\infty} Z_n = 0$ for all $\omega \in (0, 1]$, but it does not converge in L_2 since $\mathbb{E}(Z_n - 0)^2 = \mathbb{E}Z_n^2 = n^2\mathbb{E}\mathbf{1}_{[0,1/n]} = n \to \infty$ as $n \to \infty$.

(b) **ℓ_2 convergence does not imply almost sure convergence:** Define an independent sequence $\{X_n\}$ by

$$X_n = \begin{cases} 1 & \text{with probability } n^{-1} \\ 0 & \text{with probability } 1 - n^{-1} \end{cases}$$

and let $B_m(\epsilon) = \cup_{n\geq m}\{|X_n - X| > \epsilon\}$. $X_n \xrightarrow{2} 0$ because

$$\lim_{n\to\infty} \|X_n - 0\|_2 = 0$$

However, if $0 < \epsilon < 1$,

$$\begin{aligned} \Pr(B_m(\epsilon)) &= 1 - \lim_{r\to\infty} \Pr(X_n = 0 \text{ for all } n \text{ such that } m \leq n \leq r) \\ &= 1 - \left(1 - \frac{1}{m}\right)\left(1 - \frac{1}{m+1}\right)\cdots \text{ (by independence)} \\ &= 1 - \lim_{M\to\infty}\left(\frac{m-1}{m} \cdot \frac{m}{m+1} \cdot \frac{m+1}{m+2} \cdots \frac{M}{M+1}\right) \\ &= 1 - \lim_{M\to\infty} \frac{m-1}{M+1} = 1 \end{aligned}$$

and so $\{X_n\}$ does not converge almost surely.

□

Theorem 1.4.8. Borel-Cantelli lemmas (Grimmett and Stirzaker Theorem 7.3.10.) Let $\{A_n\}$ be an infinite sequence of events from some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $A = \bigcap_n \bigcup_{m=n}^{\infty} A_m = \limsup_{n \to \infty} A_n = \{A_n \text{ i.o.}\}$ be the event that infinitely many of the A_n occur. Then:

(a) $\Pr(A) = 0$ if $\sum_n \Pr(A_n) < \infty$. (That is, with probability 1 only finitely many of the events occur. If each event is an indicator, the sum of them is finite.)

(b) $\Pr(A) = 1$ if $\sum_n \Pr(A_n) = \infty$ and $A_1, A_2, \ldots$ are independent events.

Proof. (a) We have that $A \subseteq \bigcup_{m=n}^{\infty} A_n$ for all n, so

$$\Pr(A) \leq \sum_{m=n}^{\infty} \Pr(A_m) \to 0 \text{ as } n \to \infty$$

whenever $\sum_n \Pr(A_n) < \infty$.

Proof from 541B: Let

$$I_j(\omega) = \begin{cases} 1, & \omega \in A_j \\ 0, & \omega \notin A_j. \end{cases}$$

Then we need to show that $\sum_{j=1}^{\infty} I_j(\omega) < \infty$ with probability 1. Observe that

$$\mathbb{E}\left(\sum_{j=1}^{\infty} I_j\right) = \sum_{j=1}^{\infty} \mathbb{E}\left(I_j\right) = \sum_{j=1}^{\infty} \mathbb{P}(A_j)$$

where the last sum is finite by assumption. (Switching of the sum and expectation is allowed by monotone convergence theorem or Fatou's lemma because of monotonicity: $g_k = \sum_{j=1}^{k} I_j$ is increasing in k.)

(b) One can confirm that

$$A^c = \bigcup_n \bigcap_{m=n}^{\infty} A_m^c$$

But

$$\Pr\left(\bigcap_{m=n}^{\infty} A_m^c\right) = \lim_{r \to \infty} \Pr\left(\bigcap_{m=n}^{r} A_m^c\right) = \prod_{m=n}^{\infty} [1 - \Pr(A_m)] \text{ (by independence) } \leq \prod_{m=n}^{\infty} \exp(-\Pr(A_m))$$

$$= \exp\left(-\sum_{m=n}^{\infty} \Pr(A_m)\right) = 0$$

whenever $\sum_n \Pr(A_n) = \infty$, where the fourth step follows since $1 - x \leq e^{-x}$ if $x \geq 0$. Thus

$$\Pr(A^c) = \lim_{n \to \infty} \Pr\left(\bigcap_{m=n}^{\infty} A_m^c\right) = 0$$

so $\Pr(A) = 1$.

$\square$

Theorem 1.4.9. Kolmogorov's Two-Series Theorem. Let $X_1, X_2, \ldots$ be independent random variables with $\mathbb{E}(X_n) = \mu_n$ and $\text{Var}(X_n) = \sigma_n^2$ such that $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$. Then $\sum_{n=1}^{\infty} X_n$ converges in $\mathbb{R}$ almost surely.

Proof. Available on wikipedia, https://en.wikipedia.org/wiki/Kolmogorov%27s_two-series_theorem. □

1.4.1 Slutsky's Convergence Theorems (8.4.1 of Pesaran, 7.3 of Grimmett and Stirzaker)

Theorem 1.4.10. Theorem 6 of Pesaran, Section 8.4.1, p. 173. Let $\{x_t, y_t\}, t = 1, 2, \ldots$ be a sequence of pairs of random variables with $y_t \xrightarrow{d} y$ and $|y_t - x_t| \xrightarrow{p} 0$. Then $x_t \xrightarrow{d} y$.

Theorem 1.4.11. Theorem 7 in Pesaran, on p.318 (section 7.3) of Grimmett and Stirzaker. (Section 8.4.1, p. 174) If $x_t \xrightarrow{d} x$ and $y_t \xrightarrow{p} c$ where c is a finite constant, then

(i) $x_t + y_t \xrightarrow{d} x + c$

(ii) $y_t x_t \xrightarrow{d} cx$

(iii) $x_t / y_t \xrightarrow{d} x/c$, if $c \neq 0$.

Theorem 1.4.12. on p.318 (section 7.3) of Grimmett and Stirzaker. Suppose that $X_n \xrightarrow{d} 0$ and $Y_n \xrightarrow{p} Y$, and let $g : \mathbb{R}^2 \to \mathbb{R}$ be such that $g(x, y)$ is a continuous function of y for all x, and $g(x, y)$ is continuous at $x = 0$ for all y. Then $g(X_n, Y_n) \xrightarrow{p} g(0, Y)$.

Theorem 1.4.13 (**Continuous Mapping Theorem (Theorem 9 of Pesaran, Section 8.4.1, p. 176: convergence properties of transformed sequences.)**)**.** Suppose $\{\boldsymbol{x}_j\}$, $\{\boldsymbol{y}_j\}$, $\boldsymbol{x}$, and $\boldsymbol{y}$ are $k \times 1$ vectors of random variables on a probability space, and let $\boldsymbol{g}(\cdot)$ be a continuous vector-valued function. (Alternatively, suppose g has the set of discontinuity points D_g such that $\Pr(X \in D_g) = 0$.) Then

(i) $\boldsymbol{x}_j \xrightarrow{a.s.} x \implies \boldsymbol{g}(\boldsymbol{x}_j) \xrightarrow{a.s.} \boldsymbol{g}(\boldsymbol{x})$

(ii) $\boldsymbol{x}_j \xrightarrow{p} x \implies \boldsymbol{g}(\boldsymbol{x}_j) \xrightarrow{p} \boldsymbol{g}(\boldsymbol{x})$

(iii) $\boldsymbol{x}_j \xrightarrow{d} x \implies \boldsymbol{g}(\boldsymbol{x}_j) \xrightarrow{d} \boldsymbol{g}(\boldsymbol{x})$

(iv) $\boldsymbol{x}_j - \boldsymbol{y}_j \xrightarrow{p} \mathbf{0}$ and $\boldsymbol{y}_j \xrightarrow{d} \boldsymbol{y} \implies \boldsymbol{g}(\boldsymbol{x}_j) - \boldsymbol{g}(\boldsymbol{y}_j) \xrightarrow{d} \mathbf{0}(\boldsymbol{x})$

where $x = (c_1, \ldots, c_k) \in \mathbb{R}^k$.

Proof (part (b), continuous case, one-dimensional codomain). Let $\boldsymbol{x}_j = (M_{j,1}, \ldots, M_{j,k})$. We have that

$$\forall \epsilon_j > 0, \lim_{n \to \infty} \Pr(\{\omega \in \Omega : |M_{j,n}(\omega) - c_j| > \epsilon_j) = 0, \qquad \forall j \in \{1, \ldots, k\}.$$

$$\iff \forall \epsilon_j > 0, \lim_{n \to \infty} \Pr(\{\omega \in \Omega : |M_{j,n}(\omega) - c_j| < \epsilon_j) = 1, \qquad \forall j \in \{1, \ldots, k\}. \tag{2}$$

Because g is continuous, we have that for every $\epsilon^* > 0$ there exists a $\delta^* > 0$ such that

$$0 < \|\big(M_{1,n}(\omega), \ldots, M_{j,n}(\omega)\big)\|_2 < \delta^* \implies |g(M_{1,n}, \ldots, M_{j,n}) - g(c_1, \ldots, c_j)| < \epsilon^*. \tag{3}$$

Note that since in $\mathbb{R}$ the L_2 and L_1 norms are equivalent,

$$|M_{j,n}(\omega) - c_j| < \epsilon_j \iff \|M_{j,n}(\omega) - c_j\|_2 < \epsilon_j \implies \sum_{j=1}^{k} \|M_{j,n}(\omega) - c_j\|_2 < \sum_{j=1}^{k} \epsilon_j$$

$$\implies \|\big(M_{1,n}(\omega), \ldots, M_{j,n}(\omega)\big)\|_2 < \sum_{j=1}^{k} \epsilon_j$$

where the last step follows by the Triangle Inequality. Therefore letting $\delta^* = \sum_{j=1}^{k} \epsilon_j$, we have

$$\Pr(\{\omega \in \Omega : |M_{j,n}(\omega) - c_j| < \epsilon_j) \leq \Pr\left(0 < \|\big(M_{1,n}(\omega), \ldots, M_{j,n}(\omega)\big)\|_2 < \delta^*\right)$$

$$\leq \Pr(\{\omega \in \Omega : |g\big(M_{1,n}(\omega), \ldots, M_{j,n}(\omega)\big) - g(c_1, \ldots, c_j)| < \epsilon^*)$$

where the last step follows from (3). So

$$\Pr(\{\omega \in \Omega : |M_{j,n}(\omega) - c_j| < \epsilon_j) \leq \Pr(\{\omega \in \Omega : |g\big(M_{1,n}(\omega), \ldots, M_{j,n}(\omega)\big) - g(c_1, \ldots, c_j)| < \epsilon^*). \tag{4}$$

Taking limits of (4) and substituting in (2), we have

$$\forall \epsilon^* > 0, \ \lim_{n\to\infty} \Pr(\{\omega \in \Omega : |g\big(M_{1,n}(\omega), \ldots, M_{j,n}(\omega)\big) - g(c_1, \ldots, c_j)| < \epsilon^*) \geq 1$$

$$\iff \forall \epsilon^* > 0, \ \lim_{n\to\infty} \Pr(\{\omega \in \Omega : |g\big(M_{1,n}(\omega), \ldots, M_{j,n}(\omega)\big) - g(c_1, \ldots, c_j)| > \epsilon^*) = 0$$

$$\iff g(M_{1,n}, \ldots, M_{j,n}) \xrightarrow{p} g(c_1, \ldots, c_j).$$

For remaining parts, see Serfling [1980] or Rao [1973]. □

See also:

Theorem 1.4.14. (Theorem 7.2.18, Grimmett and Stirzaker.) If $X_n \xrightarrow{d} X$ and $g : \mathbb{R} \to \mathbb{R}$ is continuous, then $g(X_n) \xrightarrow{d} g(X)$.

1.5 Stochastic orders $\mathcal{O}_p(\cdot)$ and $o_p(\cdot)$ (Pesaran 8.5)

Definition 1.13 (Pesaran 8.5 Definition 6.). Let $\{a_t\}$ be a sequence of positive numbers and $\{x_t\}$ be a sequence of random variables. Then

(i) $x_t = \mathcal{O}_p(a_t)$, or x_t/a_t is bounded in probability, if for every $\epsilon > 0$ there exist real numbers M_ϵ and N_ϵ such that

$$\Pr\left(\frac{|x_t|}{a_t} > M_\epsilon\right) < \epsilon, \quad \text{for } t > N_\epsilon$$

(ii) $x_t = o_p(a_t)$ if

$$\frac{\boldsymbol{x}_t}{a_t} \xrightarrow{p} 0$$

Definition 1.14 (Ross ISE 620 Definition)**.** We say that $f(x)$ is $o(h)$ if $\lim_{h\to 0} f(h)/h = 0$.

1.6 Laws of Large Numbers and Central Limit Theorems (Pesaran 8.6; Grimmett and Stirzaker 7.4, 7.5)

Theorem 1.6.1. Weak Law of Large Numbers (Khinchine) (Pesaran 8.6 Theorem 10, Grimmett and Stirzaker Theorem 7.4.7, 541A notes Theorem 2.10). Suppose that $\{X_k\}$ is a sequence of (i) independent (ii) identically distributed random variables with (iii) constant means, i.e., $\mathbb{E}(X_k) = \mu < \infty$. Then

$$\overline{X}_k = \frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{p} \mu$$

Theorem 1.6.2. Weak Law of Large Numbers (Chebyshev) (Pesaran Section 8.6, p. 178, Theorem 11.) Let $\{X_k\}$ be a sequence of random variables. If (i) $\mathbb{E}(X_k) = \mu_k$, (ii) $\mathrm{Var}(X_k) = \sigma_k^2$, and (iii) $\mathrm{Cov}(X_k, X_j) = 0,\ k \neq j$, and (iv)

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \sigma_k^2 < \infty$$

then we have $\overline{X}_n - \overline{\mu}_n \xrightarrow{p} 0$, where $\overline{\mu}_n = n^{-1}\sum_{k=1}^{n} \mu_k$.

Theorem 1.6.3. Strong Law of Large Numbers (Grimmett and Stirzakker Theorem 7.4.3). Let $\{X_k\}$ be a sequence of (i) independent (ii) identically distributed random variables with (iii) $\mathbb{E}(X_k) = \mu$ and (iv) $\mathbb{E}(X_k^2) < \infty$. Then

$$\frac{1}{n}\sum_{k=1}^{n} X_k \to \mu \text{ almost surely and in mean square.}$$

Theorem 1.6.4 (Strong Law of Large Numbers (Grimmett and Stirzakker Theorem 7.5.1, 541A notes Theorem 2.11).). Let $\{X_k\}$ be a sequence of (i) independent (ii) identically distributed random variables. Then if and only if (iii) $\mathbb{E}|X_k| < \infty$,

$$\frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{a.s.} \mu$$

Theorem 1.6.5. Strong Law of Large Numbers 1 (Kolmogorov) (Pesaran 8.8 Theorem 12). Let $\{X_k\}$ be a sequence of (i) independent random variables with (ii) $\mathbb{E}(X_k) = \mu_k < \infty$ and (ii) $\text{Var}(X_k) = \sigma_k^2$ such that (iii)

$$\sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^2} < \infty$$

Then $\overline{X}_n - \overline{\mu}_n \xrightarrow{wp1} 0$. If the independence assumption (i) is replaced by a lack of correlation (i.e. $\text{Cov}(X_k, X_j) = 0, k \neq j$), the convergence of $\overline{X}_n - \overline{\mu}_n$ with probability one requires the stronger condition

$$\sum_{k=1}^{\infty} \frac{\sigma_k^2 (\log k)^2}{k^2} < \infty$$

Theorem 1.6.6. Strong Law of Large Numbers 2 (Pesaran 8.8 Theorem 13) Suppose that $X_1, X_2, \ldots$ are (i) independent random variables, and that (ii) $\mathbb{E}(X_k) = 0$, (iii) $\mathbb{E}(X_k^4) \leq M \; \forall \; k$ where M is an arbitrary positive constant. Then

$$\overline{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{a.s.} 0$$

Theorem 1.6.7. Central Limit Theorem (Grimmett and Stirzaker theorem 5.10.4.) Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables with finite mean μ and finite non-zero variance σ^2, and let $S_n = \sum_{i=1}^{n} X_i$. Then

$$\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \xrightarrow{d} \mathcal{N}(0, 1)$$

Theorem 1.6.8. (Berry-Esseen Central Limit Theorem.) There exists $c > 0$ such that the following holds. Let $X_1, X_2, \ldots$ be i.i.d. real-valued random variables with mean zero, variance 1, and $\mathbb{E}(|X_1|)^3 < \infty$. Let Z be a standard Gaussian random variable. Then for any $n \geq 1$,

$$\sup_{t \in \mathbb{R}} \left| \Pr\left(\frac{X_1 + \ldots + X_n}{\sqrt{n}} \leq t \right) - \Pr(Z \leq t) \right| \leq c \cdot \frac{\mathbb{E}(|X_1|^3)}{\sqrt{n}}$$

Remark. You can look up what the c is; Heilman doesn't think it's any bigger than around 10.

Theorem 1.6.9. (Central Limit Theorem in $\mathbb{R}^d$, Heilman notes Theorem 2.33.) Let $X^{(1)}, X^{(2)}, \ldots$ be a sequence of independent identically distributed $\mathbb{R}^d$-valued random variables. (Notation: we write $X^{(1)} = (X_1^{(1)}, \ldots, X_d^{(1)})$.) Assume $\mathbb{E}(X^{(n)} = \boldsymbol{\mu}$ for all $n \geq 1$ and for any $1 \leq i < j \leq d$, all of the covariances

$$a_{ij} = \mathbb{E}\big[(X_i^{(1)} - \mathbb{E}(X_i^{(1)})(X_j^{(1)} - \mathbb{E}(X_j^{(1)})\big]$$

are finite. Let $\boldsymbol{S}_n = \sum_{i=1}^n X^{(i)}$. Then as $n \to \infty$,

$$\frac{\boldsymbol{S}_n - n\boldsymbol{\mu}}{\sqrt{n\sigma^2}} \xrightarrow{d} \mathcal{N}(\boldsymbol{\mu}, [a_{ij}])$$

Theorem 1.6.10. (Grimmett and Stirzaker theorem 5.10.5.) Let $X_1, X_2, \ldots$ be independent random variables satisfying $\mathbb{E}(X_j) = 0$, $\text{Var}(X_j) = \sigma_j^2$, $\mathbb{E}|X_j^3| < \infty$ such that

$$\lim_{n\to\infty} \frac{1}{\sigma(n)^3} \sum_{j=1}^n \mathbb{E}|X_j^3| = 0$$

where $\sigma(n)^2 = \text{Var}\big(\sum_{j=1}^n X_j\big) = \sum_{j=1}^n \sigma_j^2$. Then

$$\frac{1}{\sigma(n)} \sum_{j=1}^n X_j \xrightarrow{d} \mathcal{N}(0,1)$$

Proof. See Loeve [1977, p. 287] and Grimmett and Stirzaker Problem 5.12.40. □

Lemma 1.6.11. Lindeberg's Condition:[Let $\{X_k\}$ be a sequence of independent (not necessarily identically distributed) random variables with expectations μ_k and finite variances σ_k^2. Let $s_n^2 = \sum_{k=1}^n \sigma_k^2$. If such a sequence of independent random variables X_k satisfies the condition

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^n \mathbb{E}\big[(X_k - \mu_k)^2) \cdot \mathbf{1}_{\{|X_k - \mu_k| > \epsilon s_n\}}\big] = 0$$

for all $\epsilon > 0$ then the central limit theorem holds; that is, the random variables

$$Z_n = \frac{1}{s_n} \sum_{k=1}^n (X_k - \mu_k)$$

converge in distrbution to $\mathcal{N}(0,1)$ as $n \to \infty$.

1.7 The case of dependent and heterogeneously distributed observations (Pesaran 8.8)

Theorem 1.7.1. Central limit theorem for martingale difference sequences (Pesaran 8.8 Theorem 28). Let $\{x_t\}$ be a martingale difference sequence with respect to the information set Ω_t. Let $\overline{\sigma}_T^2 = \text{Var}(\sqrt{T}\overline{x}_T) = T^{-1}\sum_{t=1}^T \sigma_t^2$. If $\mathbb{E}(|x_t|^r) < K < \infty$ for any $r > 2$ and for all t, and

$$\frac{1}{T}\sum_{t=1}^T x_t^2 - \overline{\sigma}_T^2 \xrightarrow{p} 0$$

then $\sqrt{T}\overline{x}_T/\overline{\sigma}_T \xrightarrow{d} \mathcal{N}(0,1)$.

1.8 Worked Examples from Math 505A Midterm 2

(1) (a) **Fall 2010 Problem 1.** Let X_k, $k \geq 1$, be i.i.d. random variables with mean 1 and variance 1. Show that the limit

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n X_k}{\sum_{k=1}^n X_k^2}$$

exists in an appropriate sense, and identify the limit.

(b) **Not included on midterm or final.** Let $(X_j)_{j\geq 1}$ be i.i.d. uniform on $(-1,1)$. Let

$$Y_n = \frac{\sum_{j=1}^n X_j}{\sum_{j=1}^n X_j^2 + \sum_{j=1}^n X_j^3}$$

Prove that $\lim_{n\to\infty} \sqrt{n} Y_n$ exists in an appropriate sense, and identify the limit.

Solution.

(a)

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n X_k}{\sum_{k=1}^n X_k^2} = \lim_{n\to\infty} \frac{n^{-1}\sum_{k=1}^n X_k}{n^{-1}\sum_{k=1}^n X_k^2}$$

Since $X_1, X_2, \ldots$ are i.i.d., $E(X_1^2) = \text{Var}(X_1) + (\mathbb{E}(X_1))^2 = 2 < \infty$, we have

$$n^{-1}\sum_{k=1}^n X_k \xrightarrow{a.s.} \mathbb{E}(X_1) = 1 \text{ as } n \to \infty$$

by Theorem 1.6.3 (Strong Law of Large Numbers). Also, $X_1^2, X_2^2, \ldots$ are clearly identically distributed, and are independent by Theorem 4.2.3 ("If X and Y are independent, then so are $g(X)$ and $g(Y)$."). It is clear also that $\mathbb{E}(|X_1^2|) = \mathbb{E}(X_1^2) = \text{Var}(X_1) + \mathbb{E}(X_1)^2 = 1+1 = 2 < \infty$. Therefore by Theorem 1.6.4 (Strong Law of Large Numbers),

$$n^{-1}\sum_{k=1}^n X_k^2 \xrightarrow{a.s.} \mathbb{E}(X_1^2) = 2 \text{ as } n \to \infty$$

(From here I had two different ways of finishing the problem.)

- Because we have almost sure convergence in the numerator and denominator, by the Continuous Mapping Theorem (Theorem 1.4.13),

$$\lim_{n\to\infty} \frac{n^{-1}\sum_{k=1}^n X_k}{n^{-1}\sum_{k=1}^n X_k^2} = \frac{\lim_{n\to\infty} n^{-1}\sum_{k=1}^n X_k}{\lim_{n\to\infty} n^{-1}\sum_{k=1}^n X_k^2} \xrightarrow{a.s.} \boxed{\frac{1}{2}}$$

- Then, using one of Slutsky's convergence theorems (Theorem 1.4.11: "If $x_t \xrightarrow{d} x$ and $y_t \xrightarrow{p} c$ where c is a finite constant, then $x_t/y_t \xrightarrow{d} x/c$, if $c \neq 0$."), we have

$$\frac{n^{-1}\sum_{k=1}^n X_k}{n^{-1}\sum_{k=1}^n X_k^2} \xrightarrow{d} \frac{\mathbb{E}(X_1)}{\mathbb{E}(X_1^2)} = \frac{\mathbb{E}(X_1)}{\text{Var}(X_1) + \mathbb{E}(X_1)^2} = \frac{1}{1+1} = \frac{1}{2}$$

But then, by Theorem 1.3.2 (Theorem 7.2.4(a) in Grimmett and Stirzaker: "If $X_n \xrightarrow{d} c$ where c is constant, then $X_n \xrightarrow{p} c$."), we have $\frac{n^{-1}\sum_{k=1}^n X_k}{n^{-1}\sum_{k=1}^n X_k^2} \xrightarrow{p} 1/2$.

(b) **(Not included on midterm or final.)**

$$Y_n = \frac{\sum_{j=1}^n X_j}{\sum_{j=1}^n X_j^2 + \sum_{j=1}^n X_j^3} = \frac{n^{-1}\sum_{j=1}^n X_j}{n^{-1}\sum_{j=1}^n X_j^2 + n^{-1}\sum_{j=1}^n X_j^3}$$

Note that $\mathbb{E}(X_1) = 0, \mathbb{E}(X_1^2) = \text{Var}(X_1) + \mathbb{E}(X_1)^2 = (1 - -1)^2/12 + 0^2 = 1/3, \mathbb{E}(X_1^3) = (1/2)\int_{-1}^{1} x^3 dx = 0$. (We derived the formulae for the first three moments of a uniform distribution on Homework 4 problem 2(2).)

$$\implies \sqrt{n} Y_n = \frac{\sqrt{1/3}\big(\sum_{j=1}^n X_j - n\mathbb{E}(X_1)\big)/\sqrt{n \cdot 1/3}}{n^{-1}\sum_{j=1}^n X_j^2 + n^{-1}\sum_{j=1}^n X_j^3}$$

By the Central Limit Theorem (Theorem 1.6.7),

$$\frac{\sum_{j=1}^n X_j - n\mathbb{E}(X_1)}{\sqrt{n \cdot 1/3}} \xrightarrow{d} \mathcal{N}(0,1)$$

By the Law of Large Numbers (Theorem 1.6.4), since $\mathbb{E}(|X_1^2|) = \mathbb{E}(X_1^2) = 1/3 < \infty$,

$$\frac{1}{n}\sum_{j=1}^n X_j^2 \xrightarrow{a.s.} \mathbb{E}(X_1^2) = 1/3$$

By the Law of Large Numbers (Theorem 1.6.4), since $\mathbb{E}(|X_1^3|) = (1/2)\int_{-1}^{1} |x^3| dx = \int_0^1 x^3 dx = 1/4 < \infty$,

$$\frac{1}{n}\sum_{j=1}^n X_j^3 \xrightarrow{a.s.} \mathbb{E}(X_1^3) = 0$$

In the denominator, since we have almost sure convergence, the regular rules of calculus/real analysis apply. That is, using the above results,

$$n^{-1}\sum_{j=1}^n X_j^2 + n^{-1}\sum_{j=1}^n X_j^3 \xrightarrow{a.s.} 1/3$$

Therefore

$$\sqrt{n} Y_n = \frac{\sqrt{1/3}\big(\sum_{j=1}^n X_j - n\mathbb{E}(X_1)\big)/\sqrt{n \cdot 1/3}}{n^{-1}\sum_{j=1}^n X_j^2 + n^{-1}\sum_{j=1}^n X_j^3} \xrightarrow{d} \frac{\sqrt{1/3}}{1/3}\mathcal{N}(0,1) = \boxed{\mathcal{N}(0,3)}$$

(2) **Fall 2010 Problem 2.** Fix $p \in (0,1)$ and consider independent Poisson random variables $X_k, k \geq 1$ with

$$\mathbb{E}X_k = \frac{p^k}{k}$$

Verify that the sum $\sum_{k=1}^{\infty} kX_k$ converges with probability one and determine the distribution of the random variable $Y = \sum_{k=1}^{\infty} kX_k$.

Solution. Melike's solution (use for midterm): We have $\mathbb{E}[kX_k] = p^k$ and $\sum_{k=1}^{\infty} p^k = p/(1-p) < \infty$, and $\text{Var}(kX_k) = kp^k$ and

$$\sum_{k=1}^{\infty} kp^k = p\sum_{k=1}^{\infty} kp^{k-1} = p\frac{\mathrm{d}}{\mathrm{d}p}\sum_{k=1}^{\infty} p^k = p\frac{\mathrm{d}}{\mathrm{d}p}\frac{p}{1-p} = p \cdot \frac{(1-p) - p(-1)}{(1-p)^2} = \frac{p}{(1-p)^2} < \infty$$

Since the sequence $\{Y_k|\}_{k\geq 1}$ is independent, by Kolmogorov's Two Series Theorem (Theorem 1.4.9: "Let $X_1, X_2, \ldots$ be independent random variables with $\mathbb{E}(X_n) = \mu_n$ and $\text{Var}(X_n) = \sigma_n^2$ such that $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$. Then $\sum_{n=1}^{\infty} X_n$ converges in $\mathbb{R}$ almost surely."), we conclude that $\sum_{k=1}^{\infty} kX_k$ converges almost surely.

To find the distribution of Y, let X be a Poisson random variable and consider its probability generating function:

$$G_X(s) = \mathbb{E}(s^X) = \sum_{k=0}^{\infty} s^k e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda s)^k}{k!} = e^{-\lambda} e^{\lambda s} = e^{\lambda(s-1)}$$

So $\mathbb{E}(s^{X_k}) = \exp\left(\frac{p^k}{k}(s-1)\right)$ and $\mathbb{E}(s^{kX_k}) = \mathbb{E}[(s^k)^{X_k}] = \exp\left(\frac{p^k}{k}(s^k-1)\right)$. Then define $Y_n = \sum_{k=1}^{n} kX_k$ and consider

$$G_{Y_n}(s) = \mathbb{E}(s^{Y_n}) = \mathbb{E}\left(\prod_{k=1}^{n} s^{kX_k}\right) = \prod_{k=1}^{n} \mathbb{E}\left(s^{kX_k}\right) = \prod_{k=1}^{n} \exp\left(\frac{p^k}{k}(s^k-1)\right) = \exp\left(\sum_{k=1}^{n} \frac{p^k}{k}(s^k-1)\right)$$

$$= \exp\left(\sum_{k=1}^{n} \frac{(ps)^k}{k} - \sum_{k=1}^{n} \frac{p^k}{k}\right)$$

Now, by taking limits as $n \to \infty$ (since we are allowed to take limit inside of expectation here), we get

$$G_Y(s) = \mathbb{E}(s^Y) = \exp\left(\sum_{k=1}^{\infty} \frac{(ps)^k}{k} - \sum_{k=1}^{\infty} \frac{p^k}{k}\right) = \exp\left(\int \sum_{k=1}^{\infty} (ps)^{k-1} dp - \int \sum_{k=1}^{\infty} p^{k-1} dp\right)$$

$$= \exp\left(\int \frac{1}{1-ps} dp - \int \frac{1}{1-p} dp\right) = \exp(-\log(1-ps) + \log(1-p)), \quad -1 \leq ps < 1 \text{ and } -1 \leq p < 1$$

$$= \frac{1-p}{1-ps}, \quad -1 \leq ps < 1$$

Since we know $\Pr(X = k) = \frac{G_X^{(k)}(0)}{k!}$, we have

$$G_Y(s) = \frac{1-p}{1-sp},\ G'(s) = \frac{p(1-p)}{(1-sp)^2},\ G''(s) = \frac{2p^2(1-p)}{(1-sp)^3},\ G^{(3)}(s) = \frac{3 \cdot 2p^3(1-p)}{(1-sp)^3}, \ldots$$

$$G^{(k)}(s) = \frac{k!p^k(1-p)}{(1-sp)^k} \text{ for } k = 0, 1, 2, \ldots$$

So we have

$$\Pr(Y = k) = (1-p)p^k, \quad k = 0, 1, 2, \ldots$$

$$= \Pr(G_1(1-p) = k+1) = \Pr(G_1(1-p) - 1 = k)$$

which means $Y \sim G_1(1-p) - 1$.

(3) **Spring 2017 Problem 3.**

(a) Consider the sequence $\{X_k, k \geq 1\}$ of random variables such that X_1 is uniform on $(0, 1)$ and, given X_k, the distribution of X_{k+1} is uniform on $(0, CX_k)$, where $\sqrt{3} < C < 2$.

(i) For $n \geq 1$, compute the conditional expectation $\mathbb{E}(X_{n+1}^r \mid X_n)$.

(ii) For $n \geq 1$, compute $\mathbb{E}(X_n^r)$.

(iii) Show that $\lim_{x\to\infty} X_n = 0$ in ℓ_1 and with probability one, but not in ℓ_2.

(iv) Investigate the same questions for all other values of $C > 0$.

(b) Let $a > 0$, let $X_n, n \geq 1$ be i.i.d. random variables that are uniform on $(0, a)$, and let $Y_n = \prod_{k=1}^n X_k$. Determine, with a proof, all values of a for which $\lim_{n\to\infty} Y_n = 0$ with probability one.

Solution.

(a) (i) We have that $X_{n+1} \mid X_n \sim U(0, CX_n)$. Therefore

$$\mathbb{E}(X_{n+1}^r \mid X_n) = \frac{1}{CX_n}\int_0^{CX_n} x^r dx = \frac{1}{CX_n} \cdot \frac{x^{r+1}}{r+1}\bigg|_0^{CX_n} = \frac{C^r X_n^r}{r+1}$$

$$\implies \mathbb{E}(X_{n+1}^r) = \mathbb{E}[\mathbb{E}(X_{n+1}^r \mid X_n)] = \frac{C^r}{r+1} \cdot \mathbb{E}(X_n^r)$$

$$\implies \boxed{\mathbb{E}(X_{n+1}^r \mid X_n) = \frac{C^r}{r+1} X_n^r}$$

(ii) Note that $E(X_1^r) = \int_0^1 x^r dr = 1/(r+1)$. Therefore

$$\mathbb{E}(X_{n+1}^r) = \frac{C^r}{r+1} \cdot \mathbb{E}(X_n^r) = \left(\frac{C^r}{r+1}\right)^n \cdot \mathbb{E}(X_1^r) = \boxed{\left(\frac{C^r}{r+1}\right)^n \cdot \frac{1}{r+1}}$$

(iii) We would like to show that $X_n \xrightarrow{w.p.1} 0$ and that $X_n \xrightarrow{1} 0$, but that the same result does not follow for the ℓ_2 norm.

- **Convergence with probability one:** We seek to show that $\Pr\big(\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = 0\}\big) = 1$. By Markov's Inequality (Lemma 1.2.1), we have

$$\Pr(|X_n| \geq a) \leq \frac{\mathbb{E}(X_n)}{a} \quad \forall\ a > 0$$

$$\iff \Pr(|X_n| \geq a) \leq \left(\frac{C^1}{1+1}\right)^{n-1} \cdot \frac{1}{1+1} \cdot \frac{1}{a} = \left(\frac{C}{2}\right)^{n-1} \cdot \frac{1}{2a} \quad \forall\ a > 0$$

Since $\sqrt{3} < C < 2$, $\sqrt{3}/2 < C/2 < 1$. Since $X_n \in [0, CX_{n-1}]$, $X_n \geq 0$, so $|X_n| = X_n$. Therefore we have

$$\Pr(\lim_{n\to\infty} |X_n| \geq a) = \Pr(\lim_{n\to\infty} X_n \geq a) \leq \lim_{n\to\infty} \left(\frac{C}{2}\right)^{n-1} \cdot \frac{1}{2a} = 0 \quad \forall\ a > 0$$

Since $|X_n| \geq 0$, this implies that $\Pr(\lim_{n\to\infty} X_n = 0) = \Pr\big(\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = 0\}\big) = 1$, so by the Borel-Cantelli Lemma (Theorem 1.4.8), X_n converges to 0 with probability 1.

- **Convergence in ℓ_1 norm:** We seek to show that $\lim_{n\to\infty} \mathbb{E}(|X_n|) = 0$. Since $X_n \in [0, CX_{n-1}]$, $X_n \geq 0$, so $|X_n| = X_n$. Therefore

$$\lim_{n\to\infty} \mathbb{E}(|X_n|) = \lim_{n\to\infty} \mathbb{E}(X_n) = \lim_{n\to\infty} \left(\frac{C}{2}\right)^{n-1} \cdot \frac{1}{2}$$

Since $\sqrt{3} < C < 2$, $\sqrt{3}/2 < C/2 < 1$, so $C/2 < 1$. Therefore we have

$$\lim_{n\to\infty} \mathbb{E}(|X_n|) = \lim_{n\to\infty} \left(\frac{C}{2}\right)^{n-1} \cdot \frac{1}{2} = 0$$

so X_n converges to 0 in 1st mean.

- **Convergence in ℓ_2 norm:** We seek to show that $\lim_{n\to\infty} \mathbb{E}(|X_n|^2) \neq 0$. We have

$$\lim_{n\to\infty} \mathbb{E}(|X_n|^2) = \lim_{n\to\infty} \mathbb{E}(X_n^2) = \lim_{n\to\infty} \left(\frac{C^2}{3}\right)^{n-1} \cdot \frac{1}{3}$$

Since $\sqrt{3} < C < 2$, $3/3 < C^2/3 < 4/3$, so $C^2/3 > 1$. Therefore we have

$$\lim_{n\to\infty} \mathbb{E}(|X_n|^2) = \lim_{n\to\infty} \left(\frac{C^2}{3}\right)^{n-1} \cdot \frac{1}{3} = \infty \neq 0$$

so X_n does not converge to 0 in 2nd mean.

(iv) From the above, it is clear that for convergence with probability one or in 1st mean we require $0 < C/2 < 1$ and for convergence in second mean we require $0 < C^2/3 < 1$. For $0 < C < \sqrt{3}$, we see that X_n would converge to zero in 2nd mean since this would imply that $0 < C^2/3 < 1$. It would also still converge to 0 in 1st mean (and with probability 1) since we would have $(0 < C/2 < \sqrt{3}/2 < 1$.
For $C = \sqrt{3}$, X_n would still converge to 0 with probability one and in 1st mean for the same reasons. However, it would not converge in 2nd mean because we would have

$$\lim_{n\to\infty} \mathbb{E}(|X_n|^2) = \lim_{n\to\infty} \left(\frac{\sqrt{3}^2}{3}\right)^{n-1} \cdot \frac{1}{3} = \frac{1}{3} \neq 0$$

For $C \geq 2$, it would diverge in all three cases, since in this case $C/2 \geq 2/2 = 1$ and $C^2/3 \geq 4/3 > 1$.

(b) **Probably won't be on midterm.** Note that

$$\lim_{n\to\infty} Y_n = \lim_{n\to\infty} \prod_{k=1}^n X_k = 0 \iff \log(Y_n) = \log\left(\prod_{k=1}^n X_k\right) = \sum_{k=1}^n \log(X_k) \to -\infty$$

Note that

$$\mathbb{E}[\log(Y_n)] = \mathbb{E}\left(\sum_{k=1}^n \log(X_k)\right) = \sum_{k=1}^n \mathbb{E}[\log(X_k)] = \sum_{k=1}^n \mathbb{E}[\log(X_1)] = \sum_{k=1}^n \int_0^a (\log(x)/a)dx$$
$$= \sum_{k=1}^n \frac{1}{a}\left[x\log x - x\right]_0^a = \sum_{k=1}^n \frac{a\log a - a}{a} = \sum_{k=1}^n (\log(a) - 1) = n(\log(a) - 1)$$

As $n \to \infty$ we have

$$\mathbb{E}[\log(Y_n)] = \begin{cases} -\infty & a < e \\ 0 & a = e \\ \infty & a > e \end{cases}$$

Since $\mathbb{E}[\log(Y_n)] \to \infty$ for $a < e$, we have $\lim_{n\to\infty} Y_n = 0$ for $a < 3$. Therefore

$$\boxed{\lim_{n\to\infty} Y_n = \lim_{n\to\infty} \prod_{k=1}^{n} X_k = 0 \iff a < e.}$$

1.9 Estimators and Central Limit Theorems (DSO 607)

Lyapunov (?) condition: can prove central limit theorem if we check 3rd moment. Lindeberg's Condition (Lemma 1.6.11)

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