Math Review Notes—Stochastic Processes

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1 Stochastic Processes

These notes are based on my notes from *Time Series and Panel Data Econometrics* (1st edition) by M. Hashem Pesaran and coursework for Economics 613: Economic and Financial Time Series I at USC, as well as notes from *Probability and Random Processes* by Grimmett and Stirzaker.

1.1 Poisson Processes

Definition 1.1. A **Poisson process with intensity** λ is a process $N = \{N(t) : t \geq 0\}$ taking values in $S = \{0, 1, 2, \ldots\}$ such that

(a) $N(0) = 0$; if $s < t$ then $N(s) \leq N(t)$.

(b) $\Pr(N(t+h) = n + m \mid N(t) = n) = \begin{cases} \lambda h + o(h) & \text{if } m = 1, \\ o(h) & \text{if } m > 1; \\ 1 - \lambda h + o(h) & \text{if } m = 0 \end{cases}$

(c) If $s < t$, the number $N(t) - N(s)$ of emissions in the interval $(s, t]$ is independent of the times of emissions during $[0, s]$.

Remark. λ can be interpreted as the average or long-run frequency of the Poisson process.

Theorem 1. (Grimmett and Stirzaker theorem 6.8.2.) Let $N(t)$ be a Poisson process with intensity λ. Then $N(t)$ has the Poisson distribution with parameter λt; that is,

$$\Pr(N(t) = j) = \frac{(\lambda t)^j \exp(-\lambda t)}{j!}, \quad j = 0, 1, 2, \ldots$$

Proof. See Grimmett and Stirzaker section 6.8.2, page 247. □

Definition 1.2. Let $N(t)$ be a Poisson process with intensity λ. Let $T_0, T_1, \ldots$ be given by

$$T_0 = 0, T_n = \inf\{t : N(t) = n\} \tag{1}$$

so that T_n is the time of the nth arrival. The **interarrival times** are the random variables $X_1, X_2, \ldots$ given by

$$X_n = T_n - T_{n-1}. \tag{2}$$

Remark. From knowledge of N, we can find the values of $X_1, X_2, \ldots$ by (1) and (2). Conversely, we can construct N from knowledge of the X_i by

$$T_n = \sum_{i=1}^{n} X_i, \quad N(t) = \max\{n : T_n \leq t\} \tag{3}$$

Theorem 2. (Grimmett and Stirzaker theorem 6.8.10.) Let $N(t)$ be a Poisson process with intensity λ. Let $T_0, T_1, \ldots$ be given by (1) and let X_n be given by (2). Then then random variables $\{X_n\}$ are independent, each having an exponential distribution with parameter λ.

Proof. See Grimmett and Stirzaker section 6.8.2, page 249. □

Corollary 3. Let $N(t)$ be a Poisson process with intensity λ. Let $T_0, T_1, \ldots$ be given by (1). Then $T_n \sim \text{Gamma}(n, \lambda^{-1})$.

Proof. By (3), $T_n = \sum_{i=1}^{n} X_i$. $X_i \sim \text{Exponential}(\lambda)$ by Theorem 2, which means $X_i \sim \text{Gamma}(1, \lambda^{-1})$. Then by Proposition **??**, $T_n \sim \text{Gamma}(n, \lambda^{-1})$. □

1.2 Martingales

Definition. Let $\{y_t\}_{t=0}^{\infty}$ be a sequence of random variables, and let Ω_t denote the information set available at date t, which at least contains $\{y_t, y_{t-1}, y_{t-2}, \ldots\}$. If $\mathbb{E}(y_t \mid \Omega_{t-1}) = y_{t-1}$ holds then $\{y_t\}$ is a martingale process with respect to Ω_t.

Definition. Let $\{y_t\}_{t=1}^{\infty}$ be a sequence of random variables, and let Ω_t denote the information set available at date t, which at least contains $\{y_t, y_{t-1}, y_{t-2}, \ldots\}$. If $\mathbb{E}(y_t \mid \Omega_{t-1}) = 0$, then $\{y_t\}$ is a martingale difference process with respect to Ω_t.

1.3 Brownian Motion

Appendix B.13, Brownian motion. A standard Brownian motion $b(\cdot)$ is a continuous-time stochastic process associating each date $a \in [0, 1]$ with the scalar $b(a)$ such that

(i) b(0) = 0

(ii) For any dates $0 \leq a_1 \leq a_2 \leq \ldots \leq a_k \leq 1$ the changes $[b(a_2) - b(a_1)]$, $[b(a_3) - b(a_2)], \ldots, [b(a_k) - b(a_k - 1)]$ are independent multivariate Gaussian with $b(a) - b(s) \sim \mathcal{N}(0, a - s)$.

(iii) For any given realization, $b(a)$ is continuous in a with probability 1.

Other continuous time processes can be generated from the standard Brownian motion. For example, a Brownian motion with variance σ^2 can be obtained as

$$w(a) = \sigma b(a)$$

where $b(a)$ is a standard Brownian motion.

The continuous time process

$$\boldsymbol{w}(a) = \boldsymbol{\Sigma}^{1/2} \boldsymbol{b}(a)$$

is a Brownian motion with covariance matrix $\mathbf{\Sigma}$.

Definition 26 (Wiener process). Let $\Delta w(t)$ be the change in $w(t)$ during the time interval dt. Then $w(t)$ is said to follow a Wiener process if

$$\Delta w(t) = \epsilon_t \sqrt{dt}, \quad \epsilon_t \sim IID(0,1)$$

and $w(t)$ denotes the value of the $w(\cdot)$ at date t. Clearly,

$$\mathbb{E}[\Delta w(t)] = 0, \text{ and } \mathrm{Var}[\Delta w(t)] = dt$$

Theorem 4. Donsker's Theorem, Theorem 43, p.335, Section 15.6.3. Let $a \in [0,1)$, $t \in [0,T]$, and suppose $(J-1)/T \leq a < J/T, J = 1,2,\dots,T$. Define

$$R_T(a) = \frac{1}{\sqrt{T}} s_{\left[Ta\right]}$$

where

$$s_{\left[Ta\right]} = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_{\left[Ta\right]}$$

$\left[Ta\right]$ denotes the largest integer part of Ta and $s_{\left[Ta\right]} = 0$ if $\left[Ta\right] = 0$. Then $R_T(a)$ weakly converges to $w(a)$, i.e.,

$$R_T(a) \to w(a)$$

where $w(a)$ is a Wiener process. Note that when $a = 1$, $R_T(1) = 1/\sqrt{T} \cdot S_{\left[T\right]} = 1/\sqrt{T} \cdot (\epsilon_1 + \epsilon_2 + \ldots + \epsilon_T$. Since ϵ_t's are IID, by the central limit theorem, $R_T(1) \to \mathcal{N}(0,1)$.

Similar (Theorem 2.1 in Phillips and Durlaf (1986)): Let $\{u_t\}$ be a sequence satisfying $\mathbb{E}(u_t) = 0$, $\gamma(0) = \mathbb{E}(T^{-1}S_t^2) \to \sigma^2 < \infty$ as $T \to \infty$, $\{u_t\}$ is square summable, $\sup_t\{\mathbb{E}(|u_t|^\beta)\} < \infty$ for some $2 \leq \beta < \infty$ and all t, $\gamma(h) = \mathbb{E}(T^{-1}(y_t - y_{t-h})^2) \to K_h < \infty$ as $\min\{h,T\} \to \infty$. Then $X_T(t) \implies W(t)$ as $T \to \infty$, where $W(t)$ is a Wiener process.

Theorem 5. Continuous Mapping Theorem (Theorem 44 of Pesaran in 15.6.3). Let $a \in [0,1)$, $i \in [0,n]$, and suppose $(J-1)/n \leq a < J/n, J = 1,2,\dots,n$. Define $R_n(a) = n^{-1/2} S_{\left[n \cdot a\right]}$. If $f(\cdot)$ is continuous over $[0,1)$, then

$$f[R_n(a)] \xrightarrow{d} f[w(a)]$$