Math Review Notes—Real Analysis

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1 Real Analysis

These are my notes from Math 4650: Analysis I at Cal State LA as well as Prof. Steven Heilman's notes from Math 541A at USC.

1.1 Midterm 1

1.1.1 Homework 1

Definition: Let $S \subseteq \mathbb{R}$. We say that S is **bounded from above** if $\exists\, b \in \mathbb{R}$ where

$$s \leq b \;\forall\; s \in S$$

If this is the case, we call b an **upper bound** of S.

If $b \leq c$ for all upper bounds c of S, we call b the **supremum** of S: $b = \sup(S)$.

We say that S is **bounded from below** if $\exists\, a \in \mathbb{R}$ where

$$s \geq a \;\forall\; s \in S$$

If this is the case, we call a a **lower bound** of S.

If $a \geq d$ for all lower bounds d of S, we call a the **infimum** of S: $a = \inf(S)$.

Useful Sup/Inf Fact: Let $S \in \mathbb{R}$, $S \neq \emptyset$.

(1) Suppose S is bounded from above by an element b. Then $b = \sup(S) \iff \forall\, \epsilon > 0 \;\exists\, x \in S$ with

$$b - \epsilon < x \leq b$$

(2) Suppose S is bounded from below by an element a. Then $a = \inf(S) \iff \forall\, \epsilon > 0 \;\exists\, x \in S$ with

$$a \leq x < a + \epsilon$$

Completeness Axiom: Let S be a nonempty subset of $\mathbb{R}$. If S is bounded from above, then $\sup(S)$ exists. If S is bounded from below, then $\inf(S)$ exists.

Facts about absolute value:

(1) $|x - y| < \epsilon \iff y - \epsilon < x < y + \epsilon$ (proof: in notes 08/23)

(2) $|ab| = |a||b|$ (proof: 7(c) in Homework 1)

(3) Let $\epsilon > 0$. Then $|a| < \epsilon \iff -\epsilon < a < \epsilon$. (Proof: follows from (1) if $x = a$, $y = 0$.)

(4) $-|a| \leq a \leq |a|$ (proof: Follows from (1) if $x = a$, $y = 0$, $\epsilon = |a|$.)

(5) **Triangle Inequality:** $|a + b| \leq |a| + |b|$ (Proof in notes 08/23)

(6) $|\,|a| - |b|\,| \leq |a - b|$ (Proof: 7(d) in Homework 1)

(7) **Triangle Inequality:** $|a-b| \leq |a| + |b|$ (Proof: follows from (5), let $b = -b$.)

(8) If $a < x < b$ and $a < y < b$ then $|x - y| < b - a$. (Proof: 7(a) in Homework 1)

(9) $|a - b| = |b - a|$ (Proof: 7(b) in Homework 1.)

1.1.2 Homework 2

Definition: A sequence (a_n) of real numbers is said to **converge** to a **limit** $L \in \mathbb{R}$ if $\forall\ \epsilon > 0\ \exists\ N > 0$ where

$$n \geq N \implies |a_n - L| < \epsilon$$

We say that (a_n) **diverges** if it does not converge.

Definition: A sequence (a_n) of real numbers is **bounded** if $\exists\ M > 0$ where $\forall\ n \in \mathbb{N}$

$$|a_n| \leq M$$

Theorem. If (a_n) converges then (a_n) is bounded.

Definition: Let (a_n) be a sequence of real numbers. We say that (a_n) is a **Cauchy sequence** if $\forall\ \epsilon > 0\ \exists\ N$ where

$$n, m \geq N \implies |a_n - a_m| < \epsilon$$

Theorem. (a_n) is Cauchy if and only if (a_n) converges.

Corollary. If (a_n) is Cauchy then (a_n) is bounded.

1.2 Midterm 2

1.2.1 Homework 3

Limits of functions at infinity. Let f be a real-valued function defined on some set D where D contains an interval of the form (a, ∞). Let $L \in \mathbb{R}$. We say

$$\lim_{x \to \infty} f(x) = L$$

if $\forall\ \epsilon > 0\ \exists\ N \in \mathbb{R}$ where

$$x \geq N \implies |f(x) - L| < \epsilon$$

Definition: Let $D \subseteq \mathbb{R}$. Let $a \in \mathbb{R}$. We say that a is a **limit point** (or "cluster point," or "accumulation point") of D if $\forall\ \delta > 0\ \exists\ x \in D$ where

$$x \neq a \text{ and } |x - a| < \delta$$

(Note that a may or may not be contained in D.)

Limit of a function at a: Let $D \subseteq \mathbb{R}$ and $f : d \to \mathbb{R}$. Let a be a limit point of D. Let $x \in D$. We say that f has a *limit as* x *tends to* a if $\exists\ L \in \mathbb{R}$ where $\forall\ \epsilon > 0\ \exists\ \delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$

and we write

$$\lim_{x \to a} f(x) = L$$

Properties of Limits: Let $D \in \mathbb{R}$ and let a be a limit point of D. Suppose $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$. Let $\alpha \in \mathbb{R}$.

(1) If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ then

(a)
$$\lim_{x \to a} \alpha = \alpha$$

(b)
$$\lim_{x \to a} [f(x) + g(x)] = L + M$$

(c)
$$\lim_{x \to a} [f(x) - g(x)] = L - M$$

(d)
$$\lim_{x \to a} [f(x) \cdot g(x)] = L \cdot M$$

(e)
$$\lim_{x \to a} [\alpha \cdot f(x)] = \alpha \cdot L$$

(2) If $h : D \to \mathbb{R}$ and $h(x) \neq 0 \; \forall \; x \in D$ and $\lim_{x\to a} h(x) = H \neq 0$, then

$$\lim_{x\to a} \frac{1}{h(x)} = \frac{1}{H}$$

Note that properties (2) and (1)(d) combined imply

$$\lim_{x\to a} \frac{f(x)}{h(x)} = \frac{L}{H}$$

1.2.2 Homework 4

Continuity: Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ and $a \in D$. Then f is **continuous** at a if $\lim_{x\to a} f(x)$ exists and

$$\lim_{x\to a} f(x) = f(a)$$

(Note: if f is continuous at a, then we can say $\forall \; \epsilon > 0 \; \exists \; \delta > 0$ such that

$$|x - a| < \delta \implies |f(x) - L| < \epsilon$$

that is, we don't need to say $0 < |x - a| < \delta$).

If $B \subseteq D$, then f is **continuous on B** if f is continuous at every $b \in B$.

Intermediate Value Theorem: Let f be continuous on $[a, b]$ and suppose $f(a) < f(b)$. $\forall \; d$ such that

$$f(a) < d < f(b)$$

$\exists \; c \in \mathbb{R}$ where

$$a < c < b, \;\; f(c) = d$$

1.3 Final

1.3.1 Homework 5

Definition: Let $S \subseteq \mathbb{R}$. We say $x \in \mathbb{R}$ is an **interior point** of S if there exists an open interval (a, b) where

$$x \in (a, b) \text{ and } (a, b) \subseteq S$$

Open sets: Let $S \subseteq \mathbb{R}$. We say S is **open** if every $x \in S$ is an interior point of S.

Closed sets: Let $S \subseteq \mathbb{R}$. We say S is **closed** if $\mathbb{R} \setminus S$ is open.

Theorem. A set is closed if and only if it contains all of its limit points.

Facts about open and closed sets: Suppose $a, b \in \mathbb{R}$. Then

- (a, ∞) is open (Proof: Homework 5 problem 5b).
- $(-\infty, b)$ is open (Proof: Homework 5 problem 5a).
- (a, b) is open (Proof: class notes).
- If $a < b$, then $[a, b]$ is closed (Proof: Homework 5 problem 5c).
- If A and B are open, then $A \cup B$ and $A \cap B$ are open (Proof: Homework 5 problem 3).
- If A and B are closed, then $A \cup B$ and $A \cap B$ are closed (Proof: Homework 5 problem 4).
- $\mathbb{R}$ is open (Proof: Homework 5 problem 1) and closed (Proof: $\mathbb{R} \setminus \mathbb{R} = \emptyset$ is open).
- $\emptyset$ is open (Proof: Homework 5 problem 2) and closed (Proof: $\mathbb{R} \setminus \emptyset = \mathbb{R}$ is open).

Definition: Let $S \subseteq \mathbb{R}$. An **open cover** of S is a collection $X = \{\mathcal{O}_\alpha \mid \alpha \in I\}$ where each set $\mathcal{O}_\alpha$ is an open subset of $\mathbb{R}$ such that

$$S \subseteq \bigcup_{\alpha \in I} \mathcal{O}_\alpha$$

(Here I is some set that indexes the $\mathcal{O}_\alpha$).

If $X' \subseteq X$ such that

$$S \subseteq \bigcup_{\mathcal{O}_\alpha \in X'} \mathcal{O}_\alpha$$

then X' is called a **subcover** of S contained in X. In addition, if X' is finite then we call X' a **finite subcover** of S contained in X.

Compactness: Let $S \subseteq \mathbb{R}$. We say that S is **compact** if every open cover of S contains a finite subcover.

Definition: Let $S \subseteq \mathbb{R}$. We say that S is **bounded** if $\exists\ M > 0$ where $S \subseteq [-M, M]$.

Note: S is bounded if and only if $|s| \leq M\ \forall\ s \in S$.

Heine-Borel Theorem. Let $S \subseteq \mathbb{R}$. S is compact if and only if S is closed and bounded.

Theorem. Let $f : D \to \mathbb{R}$ be continuous on D. If $X \subseteq D$ and X is compact (closed and bounded), then

$$f(\bar{x}) = \{f(x) \mid x \in X\}$$

is compact (closed and bounded).

Corollary: Suppose $f : D \to \mathbb{R}$ where D is closed and bounded. Then there exists $a, b \in D$ where $f(a)$ is the min of f on D and $f(b)$ is the max of f on D.

1.3.2 Homework 6

Uniform Continuity: Let $D \subseteq \mathbb{R}$ and let $f : D \to \mathbb{R}$. We say that f is **uniformly continuous** on D if $\forall\ \epsilon > 0\ \exists\ \delta > 0$ where

$$x, y \in D \text{ and } 0 < |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Uniform continuity implies continuity. Suppose $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$. If f is uniformly continuous on D, then f is continuous at every $a \in D$.

1.4 More Theorems

Theorem 1. Fubini's Theorem. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that $\int\int_{\mathbb{R}^2} |h(x, y)| dxdy < \infty$. Then

$$\int\int_{\mathbb{R}^2} h(x, y)dxdy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(x, y)dx \right) dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(x, y)dy \right) dx$$

1.5 Problems from Practice Math GRE Subject Tests