

# Math Review Notes—Stochastic Processes

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# 1 Stochastic Processes

These notes are based on my notes from ISE 620 at USC taught by Sheldon Ross (along with the textbooks *Stochastic Processes* [Ross, 2008] and *Introduction to Probability Models* [Ross, 2014] by Sheldon Ross) *Time Series and Panel Data Econometrics* (1st edition) by M. Hashem Pesaran [Pesaran, 2015] and coursework for Economics 613: Economic and Financial Time Series I at USC, as well as notes from *Probability and Random Processes* by Grimmett and Stirzaker [Grimmett and Stirzaker, 2001].

## 1.1 Preliminaries

**Definition 1.1.** A **stochastic process** is a collection of random variables  $X(t), t \geq 0$  (in the continuous case) or  $X_1, X_2, \dots$  in the discrete case such that ...

**Definition 1.2.** A stochastic process  $\{N(t), t \geq 0\}$  is said to be a **counting process** if  $N(t)$  represents the total number of events that have occurred up to time  $t$ . Hence, a counting process  $N(t)$  must satisfy

- (i)  $N(t) \geq 0$
- (ii)  $N(t)$  is integer-valued.
- (iii) If  $s < t$  then  $N(s) \leq N(t)$ .
- (iv) For  $s < t$ ,  $N(s) - N(t)$  equals the number of events that have occurred in the interval  $(s, t]$ .

**Definition 1.3.** We say a counting process  $\{N(t), t \geq 0\}$  has **independent increments** if the numbers of events that occur in disjoint time intervals are independent; that is, for all  $t_0 < t_1 < \dots < t_n$ ,  $N(t_1) - N(t_0), \dots, N(t_n) - N(t_{n-1})$  are independent.

**Definition 1.4.** A counting process  $\{N(t), t \geq 0\}$  has **stationary increments** if the distribution of the number of events that occur in any interval of time depends only on the length of the time interval. That is, if  $N(t + s) - N(s)$  has a distribution that does not depend on  $s$  (is the same for all  $s$ ; only depends on  $t$ ).

## 1.2 Poisson Processes

**Definition 1.5 (Poisson Process, Grimmett and Stirzaker definition).** A **Poisson process with intensity  $\lambda$**  is a process  $N = \{N(t) : t \geq 0\}$  taking values in  $S = \{0, 1, 2, \dots\}$  such that

- (a)  $N(0) = 0$ ; if  $s < t$  then  $N(s) \leq N(t)$ .
- (b) 
$$\Pr(N(t+h) = n+m \mid N(t) = n) = \begin{cases} \lambda h + o(h) & \text{if } m = 1, \\ o(h) & \text{if } m > 1; \\ 1 - \lambda h + o(h) & \text{if } m = 0 \end{cases}$$
- (c) If  $s < t$ , the number  $N(t) - N(s)$  of emissions in the interval  $(s, t]$  is independent of the times of emissions during  $[0, s]$ .

**Remark.**  $\lambda$  can be interpreted as the average or long-run frequency of the Poisson process.

**Definition 1.6 (Poisson Process, Ross definition, 2.1.2 in Stochastic Processes).** The counting process  $\{N(t) : t \geq 0\}$  is said to be a **Poisson process with rate  $\lambda$**  if

- (a)  $N(0) = 0$
- (b)  $\{N(t) : t \geq 0\}$  has independent increments
- (c)  $\Pr(N(t+h) - N(t) = 1) = \lambda h + o(h)$
- (d)  $\Pr(N(t+h) - N(t) \geq 2) = o(h)$

**Remark.** Note that a Poisson process has stationary increments.

**Lemma 1 (ISE 620 Ross Lemma 1).** Let  $N(t)$  be a Poisson process. For a fixed time  $s$ , let  $N_s(t) = N(s+t) - N(s)$  be the difference between two Poisson processes. Then  $\{N_s(t), t \geq 0\}$  is a Poisson process.

*Proof.* We verify the conditions in Definition 1.6.

- (a)  $N_s(0) = N(s) - N(s) = 0$ , QED.
- (b)  $\{N_s(t) : t \geq 0\}$  has independent increments, yes.
- (c)  $\Pr(N_s(t+h) - N_s(t) = 1) = \Pr(N(s+t+h) - N(s+t) = 1) = \lambda h + o(h)$ , QED.
- (d)  $\Pr(N_s(t+h) - N_s(t) \geq 2) = (\Pr(N(s+t+h) - N(s+t) \geq 2) = o(h)$ , QED.

□

**Lemma 2 (ISE 620 Ross Lemma 2).** Let  $N(t)$  be a Poisson process. Then  $P_0(t) = \Pr(N(t) = 0) = e^{-\lambda t}$ .

*Proof.*

$$\begin{aligned} P_0(t+h) &= \Pr(N(t+h) = 0) = \Pr(N(t+h) = 0, N(t) = 0) \\ &= \Pr(N(t) = 0) \Pr(N(t+h) - N(t) = 0) \text{ (by independent increments)} \end{aligned}$$

Using conditions (iii) and (iv) of Definition 1.6,

$$P_0(t+h) = P_0(t)(1 - \lambda h - o(h)) = P_0(t)(1 - \lambda h) + o(h)$$

$$\iff \frac{P_0(t+h) - P_0(t)}{h} = \frac{-\lambda h P_0(t)}{h} + \frac{o(h)}{h}$$

Taking the limit as  $h \rightarrow 0$ , we have

$$P'_0(t) = -\lambda P_0(t) \iff \frac{P'_0(t)}{P_0(t)} = -\lambda \iff \log(P_0(t)) = -\lambda t + C$$

$$\Longleftrightarrow P_0(t) = \lambda e^{-\lambda t} \Longleftrightarrow P_0(0) = 1$$

□

**Theorem 3.** [Grimmett and Stirzaker theorem 6.8.2] Let  $N(t)$  be a Poisson process with intensity  $\lambda$ . Then  $N(t)$  has the Poisson distribution with parameter  $\lambda t$ ; that is,

$$\Pr(N(t) = j) = \frac{(\lambda t)^j \exp(-\lambda t)}{j!}, \quad j = 0, 1, 2, \dots$$

*Proof.* See Grimmett and Stirzaker section 6.8.2, page 247. Ross proof:

$$\Pr(N(t) = n) = \frac{1}{\Pr(X_{n+1} = t - s \mid X_n = s)} \int_0^t \Pr(N(t) = n \mid S_n = s) \lambda e^{-\lambda s} \frac{(\lambda s)^{n-1}}{(n-1)!} ds = 1$$

$$= \Pr(X_{n+1} > t - s) = e^{-\lambda(t-s)}$$

$$\Pr(N(t) = n) = \int_0^t e^{-\lambda(t-s)} \lambda e^{-\lambda s} \frac{(\lambda s)^{n-1}}{(n-1)!} ds$$

$$= \frac{e^{-\lambda t} \lambda^n}{(n-1)!} \int_0^t s^{n-1} ds = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

□

**Corollary 4.**  $N(t + s) - N(s) \sim \text{Poisson}(\lambda t)$

*Proof.* By Lemma 1, ?? is a Poisson process. ...

□

**Definition 1.7 (Poisson Process, Ross definition, 2.1.1 in Stochastic Processes).** The counting process  $\{N(t) : t \geq 0\}$  is said to be a **Poisson process with rate  $\lambda$**  if

- (a)  $N(0) = 0$
- (b)  $\{N(t) : t \geq 0\}$  has independent increments
- (c) The number of events in any interval of length  $t$  is Poisson distributed with mean  $\lambda t$ . That is, for all  $s, t \geq 0$ ,

$$\Pr(N(t + s) - N(s) = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \quad n = 0, 1, \dots$$

**Remark.** Note that it follows from condition (iii) in Definition 1.7 that a Poisson process has stationary increments and also that  $\mathbb{E}(N(t)) = \lambda t$ .

**Definition 1.8 (Ross Stochastic Processes definition, Section 2.2).** Let  $X_n$  denote the time between the  $n - 1$ st and  $n$ th event in a Poisson process. The sequence  $\{X_n, n \geq 1\}$  is called the **sequence of interarrival times**.

**Definition 1.9 (Grimmett and Stirzaker definition).** Let  $N(t)$  be a Poisson process with intensity  $\lambda$ . Let  $T_0, T_1, \dots$  be given by

$$T_0 = 0, T_n = \inf\{t : N(t) = n\} \quad (1)$$

so that  $T_n$  is the time of the  $n$ th arrival. The **interarrival times** are the random variables  $X_1, X_2, \dots$  given by

$$X_n = T_n - T_{n-1}. \quad (2)$$

**Remark.** From knowledge of  $N$ , we can find the values of  $X_1, X_2, \dots$  by (1) and (2). Conversely, we can construct  $N$  from knowledge of the  $X_i$  by

$$T_n = \sum_{i=1}^n X_i, \quad N(t) = \max\{n : T_n \leq t\} \quad (3)$$

**Theorem 5. (Grimmett and Stirzaker theorem 6.8.10.)** Let  $N(t)$  be a Poisson process with intensity  $\lambda$ . Let  $T_0, T_1, \dots$  be given by (1) and let  $X_n$  be given by (2). Then the random variables  $\{X_n\}$  are independent, each having an exponential distribution with parameter  $\lambda$ .

*Proof.* See Grimmett and Stirzaker section 6.8.2, page 249. □

**Corollary 6.** Let  $N(t)$  be a Poisson process with intensity  $\lambda$ . Let  $T_0, T_1, \dots$  be given by (1). Then  $T_n \sim \text{Gamma}(n, \lambda^{-1})$ .

*Proof.* By (3),  $T_n = \sum_{i=1}^n X_i$ .  $X_i \sim \text{Exponential}(\lambda)$  by Theorem 5, which means  $X_i \sim \text{Gamma}(1, \lambda^{-1})$ . Then by Proposition ??,  $T_n \sim \text{Gamma}(n, \lambda^{-1})$ . □

**Lemma 7 (ISE 620 Ross in-class Lemma 3, Proposition 2.2.1 in Stochastic Processes).** Let  $N(t)$  be a Poisson process. Let  $X_1, X_2, \dots$  be the interarrival times. Then  $X_1, X_2, \dots$  are independently and identically distributed as  $\text{Exponential}(\lambda)$ . Since the sum of exponential distributions is Gamma, we have  $S_n = \sum_{i=1}^n X_i \sim \text{Gamma}(n, 1/\lambda)$ .

*Proof.* See proof on page 64 (section 2.2) of *Stochastic Processes*. Follows from Lemmas 1 and 2. □

**Remark.** If the arrival times are IID exponential, then the process is Poisson. (Will come later in notes.)

**Remark.**

$$X_1 < t \iff N(t) = t$$

$$\Pr(X_2 > t \mid X_1 = s) = \Pr(N(s+t) - N(s) = 0 \mid X_1 = s) = \Pr(N(s+t) - N(s) = 0) = \Pr(N_s(t) = 0) = e^{-\lambda t}$$

**Theorem 8.** Suppose events occur in a Poisson process with parameter  $\lambda$ ,  $PP(\lambda)$ . Let  $N(t)$  be the number of events that occur by time  $t$ . Suppose each event is independent of all that has previously occurred; that is, each event is type  $i = 1, \dots, r$  with probability  $p_i$ , and  $\sum_{i=1}^r p_i = 1$ . Let  $N_i(t)$  be the number of type  $i$  events that occur by time  $t$ .  $\{N_i(t)\} \sim PP(\lambda p_i)$ . Moreover, these processes are independent for different  $i$ .

*Proof.* Verify axioms of Poisson process:

1.  $N_i(0) = 0$
2. independent increments? Yes, the number of type  $i$  events in each interval is totally independent of what happened in previous intervals.
3.  $\Pr(N_i(t+h) - N_i(t) = 1) = \Pr(N(t+h) - N(t) = 1, \text{ event is type } i + \dots$   
probability of two or more events (and then one of them is type  $i$ ) is  $o(h)$ .

$$= \lambda h p_i + o(h)$$

4.  $\Pr(N_i(t+h) - N_i(t) \geq 2) \leq \Pr(N(t_h) - N(t) \geq 2) = o(h)$

Therefore  $N_i(t)$  is a Poisson process with rate  $\lambda p_i$ . We could have also proven this by looking at the inter-arrival times and showing that they are i.i.d. exponential.

□

*Proof: Alternative, stronger statement.*

$$\Pr(N_1(t) = n_1, N_2(t) = n_2, \dots, N_r(t) = n_r)$$

we want to show that these random variables are independent. Let  $n = \sum_{i=1}^r n_i$ . Then

$$= \Pr(N_1(t) = n_1, \dots, N_r(t) = n_r \mid N(t) = n) \Pr(N(t) = n)$$

But this is a Multinomial distribution. So we have

$$= \frac{n!}{n_1! \cdots n_r!} \prod_{i=1}^r p_i^{n_i} \cdot e^{-\lambda t} \frac{(\lambda t)^{\sum_i n_i}}{n!}$$

$$\prod_{i=1}^r e^{-\lambda t p_i} \frac{(\lambda t p_i)^{n_i}}{n_i!}$$

And these are just Poisson probabilities.

Then using Proposition 9, we have independence (???)

□



**Proposition 9** (offhand claim Ross made). If  $\Pr(X = n, Y = m) = g(n)h(m)$  (that is, if probability can be broken into products of functions) then variables are independent.

*Proof.*

$$\Pr(X = n) = \sum_m g(n)h(m) = g(n)C_h$$

$$\Pr(Y = m) = h(m) \sum_n g(n) = h(m)C_g$$

$$1 = \sum_n \sum_m g(n)h(m) = C_g C_h$$

So this is true as long as  $C_g C_h = 1$  which it does. □

**Example 1.1.** Suppose in particular  $\lambda = 10$  so we expect 10 people to arrive every hour, either men or women. What is the expected number of women to arrive given that 7 men arrived? (5—women and men's arrivals are independent.)

**Example 1.2 (Coupon collecting problem, see also probability notes.).**  $r$  types of coupons collected with probability  $p_1, \dots, p_r$ . Let  $N$  be the number of coupons you collect until you have a complete set. What is  $\mathbb{E}(N)$ ?

**Solution.** Let  $m(\mathcal{S})$  be the mean number of draws required to obtain at least one coupon of type  $i$  for each  $i \in \mathcal{S}$ . Note that  $m(\emptyset) = 0$ , and

$$\begin{aligned} m(\mathcal{S}) &= 1 + \sum_{i \in \mathcal{S}} p_i m(\mathcal{S} \setminus i) + \sum_{i \notin \mathcal{S}} p_i m(\mathcal{S}) \\ \implies m(\mathcal{S}) &= \frac{1 + \sum_{i \in \mathcal{S}} p_i m(\mathcal{S} \setminus i)}{\sum_{i \notin \mathcal{S}} p_i} \end{aligned}$$

But unless  $r$  is small this is going to be hard to compute, so this doesn't really work.

**Solution.** New idea: Let  $N_i$  be the number of draws required to obtain types  $1, \dots, i$ . We want  $N_r$ . Note that  $\mathbb{E}(N_1) = 1/p_1$ . We have  $N_{i+1} = N_i + A$  where  $A$  is the additional time required.

$$\mathbb{E}(N_{i+1}) = \mathbb{E}(N_i) + \mathbb{E}(A)$$

if there is a type  $i + 1$  in the original group, then  $\mathbb{E}(A) = 0$ . If not, note that

$$\mathbb{E}(A) = \begin{cases} 1/(p_i + 1) & \Pr(\{\text{a type } i + 1 \text{ coupon has not already been collected}\}) \\ 0 & \Pr(\{\text{a type } i + 1 \text{ coupon has already been collected}\}) \end{cases}$$

$$\implies \mathbb{E}(A) = \frac{1}{p_i + 1} \cdot \Pr(\{\text{a type } i + 1 \text{ coupon has not already been collected}\})$$

$$\implies \mathbb{E}(A) = \frac{1}{p_i + 1} \cdot \Pr(\{\text{type } i + 1 \text{ is the last of types } 1, \dots, i + 1 \text{ to be collected}\})$$

$$= \frac{1}{p_i + 1} \cdot \frac{\sum_{j=1}^{i+1} p_j}{p_{i+1}}$$

this didn't work either.

**Solution.**

Let  $N_i$  be the number to get type  $i$ . So  $N_i \sim \text{Geometric}(p_i)$ . Then  $N = \max\{N_i\}$ . So

$$\Pr(N \leq k) = \Pr(N_1 \leq k, \dots, N_r \leq k)$$

but you can't multiply out probabilities because  $N_i$  is not independent.

### 1.2.1 Poissonization Trick

Suppose we collect coupons at times distributed according to a Poisson process with rate  $\lambda = 1$ . Let event type  $i$  be the event of collecting a type  $i$  coupon. Let  $T_i$  be the time until collecting a type  $i$  coupon. Note that by Theorem 8 and Lemma 7,  $T_i \sim \text{Exponential}(\lambda \cdot p_i) = \text{Exponential}(p_i)$ . Then the time to collect at least one of every type is  $T = \max_i\{T_i\}$ . And by Theorem 8, the  $T_i$  are all independent. Return to the coupon collecting problem:

**Solution.** Recall the layer cake formulation for expected value:

$$\mathbb{E}(T) = \int_0^\infty \Pr(T > t) dt$$

We have (using independence of the  $T_i$ )

$$\Pr(T > t) = 1 - \Pr(T \leq t) = 1 - \Pr(T_1 \leq t, \dots, T_r \leq t) = 1 - \prod_{i=1}^r \Pr(T_i \leq t) = 1 - \prod_{i=1}^r (1 - e^{-p_i t})$$

$$\implies \boxed{\mathbb{E}(T) = \int_0^\infty \left(1 - \prod_{i=1}^r (1 - e^{-p_i t})\right) dt}$$

$$= \sum_i \frac{1}{p_i} - \sum_{i < j} \frac{1}{p_i + p_j} + \sum_{i, j, k} \frac{1}{p_i + p_j + p_k} - \dots + \frac{(-1)^{r+1}}{p_1 + \dots + p_r}$$

But we want  $\mathbb{E}(N)$ . Recall that  $T$  is the time of the  $N$ th event. So

$$T = \sum_{i=1}^N X_i$$

where  $X_1, X_2, \dots$  are the interarrival times of a Poisson process with rate  $\lambda = 1$ . But  $N$  is independent of  $X_1, X_2, \dots$  (the type of coupons you get has nothing to do with the time between coupons). So  $T$  is a compound random variable. Therefore

$$\mathbb{E}(T) = \mathbb{E}(N)\mathbb{E}(X) = \mathbb{E}(N) \cdot \frac{1}{\lambda} = \mathbb{E}(N)$$

Therefore we have  $\mathbb{E}(N) = \int_0^\infty \left(1 - \prod_{i=1}^r (1 - e^{-p_i t})\right) dt$ .

**Example 1.3.** A type is a *singleton* if after you get a complete set, you have only one object of that type. Suppose we collect coupons until we have at least one of every type, and that all coupons are equally likely to be collected on each draw. What is the expected number of singletons when you stop?

**Solution.** Let  $X$  be the number of singletons when you stop. Let  $I_j$  be an indicator variable for type  $j$  being a singleton. Then

$$\mathbb{E}(X) = \mathbb{E}\left[\sum_{j=1}^r I_j\right] = \sum_{j=1}^r \mathbb{E}(I_j) = \sum_{j=1}^r \Pr(\{j \text{ is a singleton}\})$$

$$\vdots$$

$$= \frac{1}{r} \sum_{i=1}^r \frac{1}{r-i+1}$$

$$\vdots$$

Let  $T_i$  be the time you get your first type  $i$  coupon. The probability we want is that at the time when you have at least one of every type except  $j$ , you have either 0 or 1 coupons of type  $j$ . That is, if  $S_2^{(j)}$  is the time the second card of type  $j$  is selected, we must have  $S_2^{(j)} > \max_{i \neq j} \{T_i\}$ . So we seek  $\Pr(S_2^{(j)} > \max_{i \neq j} \{T_i\})$ . Note that  $S_2^{(j)} \sim \text{Gamma}(2, p_j)$  by Corollary 6, so

$$f_{S_2^{(j)}}(s) = \frac{1}{p_j^{-2}} s^{2-1} e^{-p_j s} = p_j e^{-p_j s} p_j \cdot s.$$

So we have

$$\Pr(I_j = 1) = \Pr(S_2^{(j)} > \max_{i \neq j} \{T_i\}) = \int_0^\infty \Pr(\max_{i \neq j} \{T_i\} < S_2^{(j)} \mid S_2^{(j)} = s) \cdot f_{S_2^{(j)}}(s) ds$$

$$= \int_0^\infty \prod_{i \neq j} (1 - e^{-p_i s}) \cdot p_j e^{-p_j s} p_j \cdot s \cdot ds$$

which yields

$$\mathbb{E}(X) = \sum_{j=1}^r p_j^2 \int_0^\infty e^{-p_j s} \cdot s \cdot \prod_{i \neq j} (1 - e^{-p_i s}) \cdot ds$$

**Remark.** Per Example 1.4, suppose people arrive at a bus stop according to a Poisson process with rate  $\lambda$ . A bus arrives at (fixed) time  $T$ . Let  $W$  be the sum of the waiting times for everyone at the bus stop. Let  $S_j$  be the arrival time of the  $j$ th person. (Note that if  $X_i$  are the interarrival times then  $S_j = \sum_{i=1}^j X_i$ .) The number of people who get on the bus is  $N(t)$ , the Poisson counting process. So

$$W = \sum_{i=1}^{N(T)} (T - S_i) = N(T)T - \sum_{i=1}^{N(T)} S_i$$

**Proposition 10.** In a Poisson process, if we know that one event has occurred by time  $t$ , then the distribution of times of the event is uniform between 0 and  $t$ . That is,

$$S_1 < x \mid N(t) = 1 \sim U(0, t)$$

*Proof.*

$$\begin{aligned} \Pr(S_1 < x \mid N(t) = 1) &= \frac{\Pr(S_1 < x \cap N(t) = 1)}{\Pr(N(t) = 1)} = \frac{\Pr(N(x) = 1 \cap N(t) - N(x) = 0)}{\exp(-\lambda t)\lambda t} \\ &= \frac{\Pr(N(x) = 1) \Pr(N(t) - N(x) = 0)}{\exp(-\lambda t)\lambda t} = \frac{\lambda x e^{-\lambda x} \cdot e^{-\lambda(t-x)}}{\exp(-\lambda t)\lambda t} = \frac{x}{t} \end{aligned}$$

which is the cdf for a random variable distributed as  $U(0, t)$ .

□

**Remark.** If  $X_i \sim U(0, t)$ , then  $f(x) = 1/t$ ,  $0 < x < t$ , so per Proposition ??,

$$f_{X_{(1)}, \dots, X_{(n)}}(x_1, \dots, x_n) = \frac{n!}{t^n}, \quad 0 < x < t$$

**Theorem 11.** In a Poisson process, if we know that  $n$  events have occurred by time  $t$  ( $N(t) = n$ ), then the set of event times  $\{S_1, \dots, S_n\}$  are distributed as a set of  $n$  i.i.d. uniform random variables between 0 and  $t$ . That is, the unordered times are distributed uniformly on the interval.

More precisely: Given  $N(t) = n$ ,  $S_1, \dots, S_n$  are distributed as order statistics of  $n$  i.i.d.  $U(0, t)$  random variables.

*Proof.* We will examine the density function

$$f_{S_1, \dots, S_n, |N(t)=n}(t_1, \dots, t_n), \quad 0 < t_1 < \dots < t_n < t$$

Per the above remark, we want to show that this is  $n!/t^n$ . Note that by Bayes' Theorem

$$f_{S_1, \dots, S_n, |N(t)=n}(t_1, \dots, t_n) = \frac{f_{S_1, \dots, S_n}(t_1, \dots, t_n) \cdot \Pr(N(t) = n \mid S_1 = t_1, \dots, S_n = t_n)}{\Pr(N(t) = n)}. \quad (4)$$

Let  $X_1, X_2, \dots$  be the interarrival times. So the condition above is equivalent to  $X_1 = t_1, X_2 = t_2 - t_1, \dots, X_n = t_n - t_{n-1}$ . Recall that all the  $\{X_i\}$  are independent. So we have

$$f_{S_1, \dots, S_n}(t_1, \dots, t_n) = f_{X_1, \dots, X_n}(t_1, t_2 - t_1, \dots, t_n - t_{n-1})$$

Also, we can interpret  $\Pr(N(t) = n \mid S_1 = t_1, \dots, S_n = t_n)$  as the probability that there are 0 arrivals between times  $t_n$  and  $t$ ; that is,

$$\Pr(N(t) = n \mid S_1 = t_1, \dots, S_n = t_n) = e^{-\lambda(t-t_n)}$$

So we can write (4) as

$$\begin{aligned} f_{S_1, \dots, S_n, |N(t)=n}(t_1, \dots, t_n) &= \frac{f_{X_1, \dots, X_n}(t_2 - t_1, \dots, t_n - t_{n-1}) \cdot e^{-\lambda(t-t_n)}}{e^{-\lambda t}(\lambda t)^n/n!} \\ &= \frac{\lambda e^{-\lambda t_1} \lambda e^{-\lambda(t_2-t_1)} \dots \lambda e^{-\lambda(t_n-t_{n-1})} \cdot e^{-\lambda(t-t_n)}}{e^{-\lambda t}(\lambda t)^n/n!} \\ &= \frac{e^{-\lambda t} \lambda^n}{e^{-\lambda t}(\lambda t)^n/n!} = \frac{n!}{t^n} \end{aligned}$$

□

**Example 1.4.** Suppose people arrive at a bus stop according to a Poisson process with rate  $\lambda$ . A bus arrives at (fixed) time  $T$ . What is the expected value of  $W$ , the sum of the waiting times for everyone at the bus stop?

**Solution.** Let  $S_j$  be the arrival time of the  $j$ th person. (Note that if  $X_i$  are the interarrival times then  $S_j = \sum_{i=1}^j X_i$ .) The number of people who get on the bus is  $N(t)$ , the Poisson counting process. So

$$\begin{aligned} W &= \sum_{j=1}^{N(T)} (T - S_j) = N(T)T - \sum_{j=1}^{N(T)} S_j \\ \mathbb{E}(W) &= \mathbb{E}(N(T)T) - \mathbb{E}\left(\sum_{j=1}^{N(T)} S_j\right) = \lambda T^2 - \mathbb{E}\left(\sum_{j=1}^{N(T)} S_j\right) \end{aligned}$$

Note that by Theorem 11, if  $\{U_i\}$  are i.i.d. uniform random variables on  $[0, t]$ ,

$$\mathbb{E}\left(\sum_{j=1}^{N(T)} S_j \mid N(T) = n\right) = \mathbb{E}\left(\sum_{j=1}^n S_j \mid N(T) = n\right) = \mathbb{E}\left(\sum_{j=1}^n U_i\right);$$

that is, if we know that  $n$  arrivals have occurred by time  $T$ , then the (unordered) arrival times are distributed uniformly on  $[0, T]$ . But

$$\mathbb{E}\left(\sum_{j=1}^n U_i\right) = \sum_{j=1}^n \mathbb{E}U_i = \frac{nT}{2}$$

Also note that because the  $U_i$  is independent from  $N(T)$  (the arrival time of the  $j$ th person has nothing to do with how many people arrive by time  $T$ ),  $\sum_{j=1}^{N(T)} S_j$  is a compound random variable, which means

$$\mathbb{E}\left(\sum_{j=1}^{N(T)} S_j\right) = \mathbb{E}\left[\mathbb{E}\left(\sum_{j=1}^{N(T)} S_j \mid N(T) = n\right)\right] = \mathbb{E}\left(\frac{N(T)T}{2}\right) = \boxed{\frac{\lambda T^2}{2}}$$

$\vdots$

Note that the sum of the ordered values is equal to the sum of the unordered values, so we have

$$\mathbb{E}\left(\sum_{i=1}^n X_{(i)}\right) = \mathbb{E}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \mathbb{E}(X_i) = n\mu$$

**Example 1.5.** Another thing: Let  $I_j$  be an indicator variable for the event  $X_{(j)} < t$ . We are interested in the distribution of  $\sum_{j=1}^n I_j$ .

**Solution.** Note that the  $I_j$  is not independent because if  $I_1 = 0$  (so the smallest one is greater than  $t$ ) then it must be the case that  $I_2 = 0$  too. Also note that the sum of the ordered values less than  $t$  is the same as the sum of the unordered values less than  $t$ . So this is distributed as binomial with parameters  $n$  and  $p = F(t)$ .

**Example 1.6.** Suppose that shocks occur according to a Poisson process with rate  $\lambda$ . Let  $D_i$  be the damage caused by shock  $i$ , where  $\{D_i\}$  are i.i.d. and independent of  $\{N(t)\}$ . The damages dissipate at an exponential rate  $\alpha$ ; that is, damage of value  $d$  has value  $de^{-\alpha s}$  after a time  $s$ . Damages are cumulative. What is the total damage by time  $t$ ?

**Solution.** We have

$$D(t) = \sum_{i=1}^{N(t)} D_i e^{-\alpha(t-S_i)}$$

so

$$\begin{aligned}
\mathbb{E}(D(t) \mid N(t) = n) &= \mathbb{E}\left(\sum_{i=1}^{N(t)} D_i e^{-a(t-S_i)} \mid N(t) = n\right) = \sum_{i=1}^{N(t)} \mathbb{E}(D_i e^{-a(t-S_i)} \mid N(t) = n) \\
&= \sum_{i=1}^{N(t)} \mathbb{E}(D_i) \mathbb{E}(e^{-a(t-S_i)} \mid N(t) = n) = \mathbb{E}(D_i) e^{-at} \sum_{i=1}^n \mathbb{E}(e^{aS_i} \mid N(t) = n)
\end{aligned}$$

Let  $\{U_i\}$  be i.i.d. random variables on  $[0, t]$ . Then by Theorem 11, we can write this as

$$\begin{aligned}
&= \mathbb{E}(D_i) e^{-at} \sum_{i=1}^n \mathbb{E}(e^{aU_i} \mid N(t) = n) = \mathbb{E}(D_i) e^{-at} \cdot n \int_0^t \frac{e^{ax}}{t} dx = \mathbb{E}(D_i) e^{-at} \cdot \frac{n}{at} (e^{at} - 1) \\
&= \mathbb{E}(D_i) \cdot \frac{n}{at} (1 - e^{-at})
\end{aligned}$$

So finally we have

$$\mathbb{E}(D(t)) = \mathbb{E}[\mathbb{E}(D(t) \mid N(t) = n)] = \mathbb{E}\left[\mathbb{E}(D_i) \cdot \frac{N(t)}{at} (1 - e^{-at})\right] = \mathbb{E}(D_i) \cdot \frac{\mathbb{E}(N(t))}{at} (1 - e^{-at}) = \boxed{\mathbb{E}(D_i) \cdot \frac{\lambda}{a} (1 - e^{-at})}$$

### 1.2.2 Time Sampling Poisson Processes

**Proposition 12 (Time Sampling (Proposition 5.3 in Introduction to Probability Models)).**

Suppose we have events that happen as a Poisson process with rate  $\lambda$ . Each event is of type  $1, \dots, r$  independently of what has come before. An event at time  $s$  is type  $i$  with probability  $P_i(s)$ . Let  $N_i(t)$  be the number of type  $i$  events by time  $t$ . Then  $N_1(t), \dots, N_r(t)$  are independent Poisson random variables with mean  $\mathbb{E}[N_i(t)] = \lambda \int_0^t P_i(s) ds$ .

*Proof.* I'm not sure this proof makes sense? Note that

$$\Pr(N_1(t) = n_1, \dots, N_r(t) = n_r) = \Pr(N_1(t) = n_1, \dots, N_r(t) = n_r \mid N(t) = \sum_{i=1}^r n_i) \cdot \Pr\left(N(t) = \sum_{i=1}^r n_i\right).$$

Given  $N(t) = n$ , the  $n$  event times are i.i.d.  $U(0, t)$  by Theorem 11. Let  $P_i$  be the probability that a particular event is of type  $i$  given a time  $t$  and the fact that  $N(t) = n$ . Then

$$P_i = \int_0^t P_i(s) \cdot f_U(s) ds = \int_0^t \frac{P_i(s)}{t} ds$$

Since  $\Pr\left(N(t) = \sum_{i=1}^r n_i\right) = e^{-\lambda t}$ , we have

$$\Pr(N_1(t) = n_1, \dots, N_r(t) = n_r) = \frac{n!}{n_1! \dots n_r!} P_1^{n_1} \dots P_r^{n_r} e^{-\lambda t}$$

□

**Remark.** Notation for Queueing: in many cases we suppose times between successive arrivals are independent with a common distribution  $F$ . Then it is a renewal process. (If they are exponential, we get a Poisson process.) We have the following notation for a renewal process if  $k$  number of servers and  $G$  is the distribution of the service time:

$$F/G/k$$

If the distribution is exponential, we use  $M$  (for “memoryless” or “Markovian”). ( $E$  stands for Erlang, sum of i.i.d. exponentials).

**Example 1.7.** We have an  $M/G/\infty$  queueing system. Arrivals are a Poisson process with rate  $\lambda$ . Fix  $t$  and let  $X(t)$  be the number of people in the system at time  $t$ . Let  $Y(t)$  be the number of people who have completed service at time  $t$ . Let the events be arrivals of customers, and call it a Type 1 event if the customer is still in the system at time  $t$ , and Type 2 if the customer completes service by time  $t$  (and Type 3 otherwise).

Suppose a customer arrives at time  $s < t$ . Then the customer is Type 1 if their service time exceeds  $t - s$ , which happens with probability  $P_1(s) = \bar{G}(t - s)$ . Similarly  $P_2(s) = G(t - s)$ . Let  $N_1(t)$  be the number of Type I events that happen by time  $t$  and similarly for  $N_2(t)$ . Then by Proposition 12 we have

$$X(t) = N_1(t) \sim \text{Poisson} \left( \lambda \int_0^t \bar{G}(t - s) ds \right) = \text{Poisson} \left( \lambda \int_0^t \bar{G}(y) dy \right)$$

and

$$Y(t) = N_2(t) \sim \text{Poisson} \left( \lambda \int_0^t G(t - s) ds \right) = \text{Poisson} \left( \lambda \int_0^t G(y) dy \right)$$

which implies the independence of  $X(t)$  and  $Y(t)$ .

### 1.2.3 Nonstationary Poisson Processes

**Definition 1.10 (Nonstationary Poisson process.).** The counting process  $\{N(t), t \geq 0\}$  is said to be a **nonstationary Poisson process** (or **nonhomogeneous Poisson process**) with intensity function  $\lambda(t), t \geq 0$  if

- (i)  $N(0) = 0$ .
- (ii)  $\{N(t)\}$  has independent increments.



$$(iii) \Pr(N(t+h) - N(t) = 1) = \lambda(t)h + o(h).$$

$$(iv) \Pr(N(t+h) - N(t) \geq 2) = o(h).$$

**Remark.** Note the similarities to Definition 1.6.

**Lemma 13.** Let  $\{N(t), t \geq 0\}$  be a nonstationary Poisson process. Let  $N_s(t) = N(s+t) - N(s)$ . Then  $\{N_s(t), t \geq 0\}$  is a nonstationary Poisson Process with intensity  $\lambda_s(t) = \lambda(s+t)$ .

**Remark.** Note the similarity to Lemma 1.

*Proof.* Note the parts of Definition 1.10 above:

$$(i) N_s(0) = N(s) - N(s) = 0.$$

$$(ii) \{N_s(t)\} = \{N(s+t) - N(s)\} \text{ has independent increments since } N(t) \text{ has independent increments.}$$

$$(iii) \text{ **Not sure about this part?** } \Pr(N_s(t+h) - N_s(t) = 1) = \Pr(N(s+t+h) - N(s) - [N(s+t) - N(s)] = 1) = \Pr(N(s+t+h) - N(s+t) = 1) = \lambda(t)h + o(h).$$

$$(iv) \Pr(N_s(t+h) - N_s(t) \geq 2) = o(h) \text{ by similar argument to (iii).}$$

□

**Definition 1.11.** Let  $m(t) = \int_0^t \lambda(s)ds$ . Note that  $m'(t) = \lambda(t)$ . We call  $m(t)$  the **mean value function**.

**Remark.** Note that the mean value function for  $N_s(t)$  is

$$m_s(t) = \int_0^t \lambda_s(y)dy = \int_0^t \lambda(s+y)dy = \int_s^{s+t} \lambda(x)dx = m(t+s) - m(s).$$

**Lemma 14.** Let  $N(t)$  be a nonstationary Poisson process. Then  $\Pr(N(t) = 0) = e^{-m(t)}$  for any  $t \geq 0$ .

**Remark.** Note the similarity to Lemma 2.

*Proof.* Let  $P(t) = \Pr(N(t) = 0)$ . Then by independent increments

$$\Pr(N(t+h) = 0) = \Pr(N(t) = 0 \cap N(t+h) - N(t) = 0) = \Pr(N(t) = 0) \cdot \Pr(N(t+h) - N(t) = 0) \quad (5)$$

Let  $P_0(t+h) = \Pr(N(t+h) = 0)$ . Then using Definition 1.10 we have

$$\Pr(N(t+h) - N(t) = 0) = 1 - \Pr(N(t+h) - N(t) = 1) - \Pr(N(t+h) - N(t) \geq 2) = 1 - \lambda(t+h) + o(h)$$

so we can write (5) as

$$P_0(t+h) = P_0(t)[1 - \lambda(t+h) + o(h)]$$

$$\Longleftrightarrow \frac{P_0(t+h) - P_0(t)}{h} = -\lambda(t+h) \frac{P_0(t)}{h} + \frac{o(h)}{h}$$

Taking the limit as  $h \rightarrow 0$  yields

$$\begin{aligned} P'_0(t) = -\lambda(t)P_0(t) &\Longleftrightarrow \int_0^s \frac{P'_0(t)}{P_0(t)} ds = \int_0^s -\lambda(t)dt \Longleftrightarrow \log P_0(t) = \int_0^s -\lambda(t)dt \Big|_0^s \\ &\Longleftrightarrow P_0(s) = e^{-m(s)} \end{aligned}$$

□

**Remark (Interarrival times).** Let  $T_1$  be the time of the first event. Note that  $T_1 > t \Longleftrightarrow N(t) = 0$ . So

$$\bar{F}_{T_1}(t) = \Pr(T_1 > t) = \Pr(N(t) = 0) = e^{-m(t)}$$

which means

$$f_{T_1}(t) = \frac{d}{dt}(-\bar{F}_{T_1}(t)) = m'(t)e^{-m(t)} = \lambda(t)e^{-m(t)}.$$

Note that because  $\lambda(t)$  is not constant, the interarrival times are not i.i.d.

**Theorem 15.** Let  $N(t)$  be the counting process for a nonstationary Poisson process. Then  $N(t) \sim \text{Poisson}(m(t))$ .

*Proof.* We must show that

$$\Pr(N(t) = n) = e^{-m(t)} \frac{(m(t))^n}{n!}, \quad n = 0, 1, \dots$$

We have already shown that  $\Pr(N(t) = 0) = e^{-m(t)} \frac{(m(t))^0}{0!} = e^{-m(t)}$  this is true when  $n = 0$  in Lemma 14. We will show that this expression holds for  $n = 1, 2, \dots$  by induction. Assume for a fixed  $n$

$$\Pr(N(t) = n) = e^{-m(t)} \frac{(m(t))^n}{n!}.$$

We seek

$$\Pr(N(t) = n+1) = \int_0^t \Pr(N(t) = n+1 \mid T_1 = s) f_{T_1}(s) ds \quad (6)$$

Note that (using the property of independent increments)

$$\Pr(N(t) = n+1 \mid T_1 = s) = \Pr(N(t) - N(s) = n \mid T_1 = s) = \Pr(N(t) - N(s) = n)$$

$$= \Pr(N_s(t-s) = n)$$

By Lemma 13 above,  $N_s(\cdot)$  is a Poisson process. So using that and the induction hypothesis, we have

$$\Pr(N(t) = n+1 \mid T_1 = s) = e^{-m_s(t-s)} \frac{(m_s(t-s))^n}{n!} = e^{-[m(t)-m(s)]} \frac{[m(t)-m(s)]^n}{n!}.$$

Substituting this expression back in to (6) and using  $f_{T_1}(t) = \lambda(t)e^{-m(t)}$ , we have

$$\begin{aligned} \Pr(N(t) = n+1) &= \int_0^t e^{-[m(t)-m(s)]} \frac{[m(t)-m(s)]^n}{n!} \cdot \lambda(s)e^{-m(s)} ds \\ &= \frac{e^{-m(t)}}{n!} \int_0^t [m(t)-m(s)]^n \cdot \lambda(s) ds \end{aligned}$$

Substituting  $y = m(t) - m(s) \implies dy = -\lambda(s)ds$ , we have

$$= \frac{e^{-m(t)}}{n!} \int_{m(t)}^0 -y^n dy = \frac{e^{-m(t)}}{n!} \int_0^{m(t)} y^n dy = e^{-m(t)} \frac{(m(t))^{n+1}}{(n+1)!}.$$

Therefore the result follows by induction. □

**Corollary 16.**  $N(t+s) - N(s) \sim \text{Poisson}(m(t+s) - m(s)) = \text{Poisson}(\int_s^{s+t} \lambda(y)dy)$ .

*Proof.* Follows almost immediately from Theorem 15, since if  $N(t) \sim \text{Poisson}(m(t))$ , □

**Proposition 17.** Suppose events occur according to a Poisson process with rate  $\lambda$ .  $\{N(t)\}$  is the counting process. An event at time  $s$  is type 1 with probability  $p(s)$ . Let  $N_1(t)$  be the number of type 1 events by time  $t$ . Then  $\{N_1(t)\}$  is a nonhomogeneous Poisson Process with intensity  $\lambda(t) = \lambda p(t)$ .

**Remark.** Note the similarity to Proposition 12.

*Proof.* Note the parts of Definition 1.10 above:

- (i)  $N_1(0) = 0$ : yes.
- (ii)  $\{N_1(t)\}$  has independent increments: yes, follows pretty much immediately.
- (iii) Since the probability of exactly one event in the interval that happens to be type 1 is  $\lambda h P_1(t)$ , we have  $\Pr(N_1(t+h) - N_1(t) = 1) = \lambda h P_1(t) = \lambda(t)h + o(h)$ .
- (iv)  $\Pr(N_1(t+h) - N_1(t) \geq 2) = o(h)$  by similar argument to (iii).

□

So now we have that  $N_1(t) \sim \text{Poisson}(\int_0^t \lambda p_1(s) ds)$ . So if every time an event happens it is of type  $i$  with a certain probability that varies over time, the counting process is for each is a nonstationary Poisson process, and all the processes are independent. (Another way to understand the time sampling result from before.)

What if the arrival process is nonstationary and the probability of a type  $i$  event is also time varying?  $\lambda(t), p(t)$ . It's a nonstationary PP with intensity  $\lambda(t)p(t)$ . (Probability in an interval is  $\lambda(t)hp(t) + o(h)$ ).

One more result about  $M/G/\infty$  processes:

**Theorem 18 (Example 5.25 in *Introduction to Probability Models*).** The departure process from an  $M/G/\infty$  queue is a nonstationary Poisson process with intensity function  $\lambda(t) = \lambda G(t)$ .

*Proof.* Let  $D(t)$  be the number of departures by time  $t$ . Examine the axioms of Definition 1.10 above:

- (i)  $D(0) = 0$ : yes.
- (ii)  $\{D(t)\}$  has independent increments: think of each arrival as an event. Types: type 1 if they depart in interval 1, 2 if they depart in 2, 3 if they depart in 3, 4 if they depart elsewhere, where 1, 2, and 3 are sequential nonoverlapping intervals. Note that for a given arrival time, the type of the event is dependent only on the service time. Since the arrival times are independent by assumption and the service times are also independent, the increments are independent.
- (iii)  $\Pr(D(t+h) - D(t) = 1)$ : Call an event type 1 if they depart in the interval  $(t, t+h)$ . Then  $P_1(s) = G(t+h+s) - G(t+s) = g(t-s) \cdot h + o(h)$ . So

$$D(t+h) - D(t) \sim \text{Poisson}(\lambda h \int_0^t g(t-s) ds + o(h)) = \text{Poisson}(\lambda h \int_0^t g(y) dy + o(h)) \text{Poisson}(\lambda h G(t) + o(h))$$

which means  $\Pr(D(t+h) - D(t) = 1) = \lambda G(t) h e^{-\lambda G(t)h} =$  (by Taylor expansion of exp)  $\lambda G(t)h + o(h)$  which is what we wanted to show.

- (iv)  $\Pr(D(t+h) - D(t) \geq 2) = 1 - (\Pr(D(t+h) - D(t) = 0) - (\Pr(D(t+h) - D(t) = 1) = 1 - e^{-\lambda G(t)h} - \lambda G(t)h + o(h) =$  (by Taylor expansion of exp)  $1 + \lambda G(t)h - \lambda G(t)h + o(h) = o(h)$ .

□

**Remark.** Notice in the limit as  $t \rightarrow \infty$  it converges to a Poisson process with rate  $\lambda$ .

**Example 1.8.** Poisson process, event  $i$  is associated with reward  $V_i$ .  $\{X(t), t \geq 0\}$  is amount of money you get at time  $t$ .

$$X(t) = \sum_{i=1}^r V_i N_i(t) \tag{7}$$

or

$$\mathbb{E}(X(t)) = \sum V_i \mathbb{E}(N_i(t)) = \sum V_i \lambda \alpha_i t = \lambda t \sum \alpha_i v_i = \lambda t \mathbb{E}(X)$$

nice thing about this representation is  $V_i N_i(t)$  are independent, so variance is sum of variances. Using  $\text{Var}(N_i(t)) = \lambda \alpha_i t$ ,

$$\text{Var}(X(t)) = \sum_{i=1}^r V_i^2 \lambda \alpha_i t = \lambda t \sum V_i^2 \alpha_i = \lambda t \mathbb{E}(X^2)$$

so we verified the formulas we derived a different way.

Note that  $N_i(t)$  is approximately Gaussian for large  $t$  (because Poisson goes to normal for large  $t$ ). Therefore when  $t$  is large  $X(t)$  is also approximately Gaussian.

#### 1.2.4 Queueing Systems

Say we have a  $M/G/1$  queueing process. That is, arrivals are  $PP(\lambda)$ , the service time has distribution  $G$ , and there is one server. Note that if no one is in line, the waiting time (length of an “idle period”) until the next arrival is exponential (since it’s a Poisson process). Consider the “busy periods,” that is, the time from when a customer arrives until the time the next idle period starts.

Let  $B$  be the length of a busy period. Let  $S$  be the service time of the initial customer. Note that  $B$  is independent from what happened before. Let  $A$  be the additional time after the first customer is served until the next idle period so that  $B = S + A$ . Let  $N(S)$  be the number of arrivals during  $S$ . Note that  $A$  depends on  $N(S)$ .

Suppose  $N(S) = n$ . If  $n = 0$ , then  $A = 0$ . If  $n = 1$ ,  $A$  has the same distribution as  $B_1$ . If  $n = 2$ ,  $A$  has the same distribution as the sum of two  $B$ s. And so on. This tells us we can say

$$B = S + \sum_{i=1}^{N(S)} B_i$$

If we condition on  $S = s$ ,

$$\{B \mid S = s\} = s + \sum_{i=1}^{N(s)} B_i$$

Note that  $\sum_{i=1}^{N(s)} B_i$  is a compound Poisson random variable. So

$$\mathbb{E}(B \mid S = s) = s + \mathbb{E}(N(s))\mathbb{E}(B_i) = s + \mathbb{E}(B)\lambda s$$

$$\text{Var}(B \mid S = s) = \lambda s \mathbb{E}(B^2)$$

or

$$\mathbb{E}(B \mid S) = S + \mathbb{E}(N(S))\mathbb{E}(B_i) = S + \mathbb{E}(B)\lambda S$$

$$\text{Var}(B \mid S) = \lambda S \mathbb{E}(B^2)$$

So

$$\mathbb{E}(B) = \mathbb{E}[\mathbb{E}(B \mid S)] = \mathbb{E}(S) + \lambda \mathbb{E}(S)\mathbb{E}(B) \implies \boxed{\mathbb{E}(B) = \frac{\mathbb{E}(S)}{1 - \lambda \mathbb{E}(S)}}$$

Note the similarity to the sum of an infinite geometric series. Similarly, if  $\lambda \mathbb{E}(S) \geq 1 \iff \mathbb{E}(S) \geq 1/\lambda$  then the expected waiting time is infinite because the arrival times under the Poisson process ( $1/\lambda$  in expectation) are faster in expectation than the service times. Also,

$$\text{Var}(B) = \lambda \mathbb{E}(S) \mathbb{E}(B^2) + (1 + \lambda \mathbb{E}(B))^2 \text{Var}(S)$$

Note that  $\text{Var}(B) = \mathbb{E}(B)^2$ , etc., then you work out the answer (**see textbook for complete answer.**)

**End of Poisson processes**

### 1.3 Renewal Processes

**Definition 1.12 (Stieltjes Integral).**

$$\int_a^b h(x) dF(x) = \lim_{n \rightarrow \infty} \sum_{i=1}^n h(x_i) (F(x_i) - F(x_{i-1}))$$

In particular, we often suppose that  $F(x)$  is the distribution function for a random variable  $X$ . Then we have (if  $X$  is continuous)

$$\int_a^b h(x) dF(x) = \lim_{n \rightarrow \infty} \sum_{i=1}^n h(x_i) (\Pr(x_{i-1} < X \leq x_i)) = \int_a^b h(x) f_X(x) dx = \mathbb{E}(h(x))$$

or if  $X$  is discrete

$$\int_a^b h(x) dF(x) = \lim_{n \rightarrow \infty} \sum_{i=1}^n h(x_i) (\Pr(x_{i-1} < X \leq x_i)) = \mathbb{E}(h(x)).$$

Generalization of Poisson processes. A counting process where the interarrival times are i.i.d. Let  $X_1, X_2, \dots$  be i.i.d. non-negative random variables with distribution function  $F$ . We require  $F(0) = \Pr(X \leq 0) = \Pr(X = 0) < 1$ . Let

$$\mathbb{E}(X_1) = \mu = \int_0^\infty x dF(x) = \int_0^\infty \bar{F}(t) dt$$

where  $0 < \mu \leq \infty$ . Let  $N(t)$  be the counting process (this is the largest value of  $n$  for which the  $n$ th event has occurred at time  $t$ , so  $N(t) = \max\{n : S_n \leq t\}$ ). Let  $S_n$  be the arrival time for the  $n$ th event. Define  $S_0 = 0, S_n = \sum_{i=1}^n X_i$ .

**Definition 1.13 (Renewal process; Definition 3.1.1 in *Stochastic Processes*).** Let  $\{X_n, n = 1, 2, \dots\}$  be a sequence of nonnegative independent random variables with a common distribution  $F$ . To avoid trivialities, suppose that  $F(0) = \Pr(\{X_n = 0\}) < 1$ . We shall interpret  $X_n$  as the time between the  $(n-1)$ st and  $n$ th event. Let

$$S_0 = 0, \quad S_n = \sum_{i=1}^n X_i, n \geq 1$$

so that  $S_n$  is the time of the  $n$ th event. As the number of events by time  $t$  will equal the largest value of  $n$  for which the  $n$ th event occurs before or at time  $t$ , we have that  $N(t)$ , the number of events by time  $t$ , is given by

$$N(t) = \sup\{n : S_n \leq t\}.$$

The counting process  $\{N(t), t \geq 0\}$  is called a **renewal process**. We can write this as  $\{N(t), t \geq 0\}$  as a renewal process with intensity distribution  $F$ .

**Remark.** Let  $\mu = \mathbb{E}(X_n) = \int_0^\infty x dF(x)$  denote the mean time between successive events and note that from the assumptions  $X_n \geq 0$  and  $F(0) < 1$  it follows that  $0 < \mu \leq \infty$ .

Every time an event occurs is a “renewal:” from that time on, all the arrivals are i.i.d. (Between events it is unclear what is going on until the next event happens; depends on the nature of  $F$ . If we have a Poisson process, then it is memoryless, but otherwise it is not.)

**Example 1.9 (St. Petersburg Paradox).** Idea: you play a game, you get the amount of money  $X$ . In order to play the game you have to pay, so what’s a fair amount to pay? Belief:  $\mathbb{E}(X)$  (that’s a rational price).

Game: fair coin, flip until heads occurs. If it occurs on trial  $n$ , you win  $2^n$  dollars. Note that

$$\mathbb{E}(X) = \sum_{i=1}^{\infty} 2^i \cdot \left(\frac{1}{2}\right)^i = \infty.$$

**Proposition 19.** With probability 1,  $N(t) < \infty$  for all  $t$ .

*Proof.*

$$\frac{S_n}{n} = \frac{X_1 + \dots + X_n}{n} \xrightarrow{a.s.} \mu > 0 \text{ as } n \rightarrow \infty$$

Because the denominator goes to  $\infty$  and the ratio doesn't go to 0, we must have  $S_n \xrightarrow{a.s.} \infty$ . Therefore  $\Pr(N(t) < \infty) = 1$  for all  $t$ .

□

**Proposition 20.** With probability 1,  $N(\infty) = \lim_{t \rightarrow \infty} N(t) = \infty$ . (There's never a last renewal—there will always be another.)

*Proof.* Recall **Boole's Inequality** (which can be proven in a similar manner to Theorem 66):

$$\Pr\left(\bigcup_{n=1}^{\infty} \{A_n\}\right) \leq \sum_{n=1}^{\infty} \Pr(A_n)$$

(with equality if the events are disjoint). Then

$$\Pr(N(\infty) < \infty) = \Pr(X_n = \infty \text{ (for some } n)) = \Pr\left(\bigcup_{n=1}^{\infty} \{X_n = \infty\}\right) \leq \sum_{n=1}^{\infty} \Pr(X_n = \infty) = 0$$

□

**Definition 1.14 (Notation for  $n$ -fold self-convolution).** Recall the notation for convolution: if  $X \sim F \perp\!\!\!\perp Y \sim G$ , we have  $X + Y \sim F * G$ . Then

$$S_n \sim F * F * \dots * F(t).$$

Let

$$F_n(t) := F * F * \dots * F(t)$$

the  $n$ -fold convolution of  $F$  with itself. (In general, this is very hard to get a closed-form notation for except in some simple cases. More of a theoretical construct than something practical to compute.)

**Proposition 21.**  $\Pr(N(t) = n) = F_n(t) - F_{n+1}(t)$ .

*Proof.*

$$N(t) \geq n \iff S_n \leq t.$$

$$\implies \Pr(N(t) \geq n) = \Pr(S_n \leq t)$$

Then we have

$$\Pr(N(t) = n) = \Pr(N(t) \geq n) - \Pr(N(t) \geq n+1) = F_n(t) - F_{n+1}(t).$$

(Again, this is more of a theoretical construct than something practical to compute.)

□



**Definition 1.15 (Renewal function).** Let  $m(t) = \mathbb{E}(N(t))$ . We call  $m(t)$  the **renewal function**.

**Remark.**

$$m(t) = \sum_{n=1}^{\infty} \Pr(N(t) \geq n) = \sum_{n=1}^{\infty} F_n(t).$$

**Lemma 22.**  $m(t) < \infty$  for all  $t$ .

*Proof.* Skipped in class, not very enlightening.

□

**Proposition 23 (Renewal Equation).**

$$m(t) = F(t) + \int_0^t m(t-s)dF(s).$$

*Proof.*

$$m(t) = \mathbb{E}(N(t)) = \int_0^{\infty} \mathbb{E}(N(t) \mid X_1 = s)dF(s)$$

Note that

$$\mathbb{E}(N(t) \mid X_1 = s) = \begin{cases} 1 + m(t-s) & s \leq t \\ 0 & s > t \end{cases}$$

so we have

$$m(t) = \int_0^t (1 + m(t-s))dF(s) = F(t) + \int_0^t m(t-s)dF(s)$$

□

**Remark.** Remember we did a problem where  $X_i \sim \text{Unif}(0, 1)$  and asked how many you have to sum until the number of greater than 1. We ended up with  $1/e$ . This is the smallest value of  $n$  for which the event occurred after time  $n$  (or something?)

$$N = \min\{X_1 + \dots + X_n > 1\} = N(1) + 1$$

(the first event that occurred after time 1. We showed that this was equal to  $e$  basically by solving the renewal equation. Only feasible because of uniform distribution between 0 and 1.

**Remark.** Note that

$$S_{N(t)} := (\text{time of the most recent event before or at time } t),$$

$$S_{N(t+1)} := (\text{time of the first event after time } t).$$

At what rate do renewals occur?

**Theorem 24 (Strong Law for Renewal Processes).**

$$\frac{N(t)}{t} \xrightarrow{a.s.} \frac{1}{\mu}$$

(Intuitively: since  $X_i$  are the time between renewals, the average time between renewals is  $\mathbb{E}(X_i) = \mu$ . So the average rate of renewals is equal to one over the average time between events.)

*Proof.* Note that

$$S_{N(t)} \leq t < S_{N(t)+1}$$

$$\iff \frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}$$

As  $t \rightarrow \infty$ , the quantity on the left converges to the mean by the Strong Law of Large Numbers (see the proof of Proposition 19 above). For the quantity on the right, we have

$$\frac{S_{N(t)+1}}{N(t)} = \frac{S_{N(t)+1}}{N(t)+1} \cdot \frac{N(t)+1}{N(t)} \xrightarrow{a.s.} \mu \cdot 1$$

which means

$$\frac{t}{N(t)} \xrightarrow{a.s.} \mu \iff \frac{N(t)}{t} \xrightarrow{a.s.} \frac{1}{\mu}.$$

□

**Remark.** In many applications:  $X_1, X_2, \dots$  are independent non-negative random variables.  $X_1 \sim G$ ,  $X_i \sim F, i \geq 2$ . (That is, once the first event occurs we have a renewal process, but before then we don't.)

**Definition 1.16 (Delayed renewal process).** Define

$$N_d(t) := \max\{n : X_1 + \dots + X_n \leq t\}.$$

We call  $\{N_d(t), t \geq 0\}$  a **delayed renewal process**.

**Remark.** Almost all limiting results for renewal processes apply for delayed renewal processes as well. (In the long run, that first waiting period doesn't really make a difference.) For example, consider the Strong Law for Renewal Processes (Theorem 24) (quick note: let  $N(s) = -1, s < 0$ ):

$$N_d(t) = 1 + N(t - X_1) \iff \frac{N_d(t)}{t} = \frac{1}{t} + \frac{N(t - X_1)}{t - X_1} \frac{t - X_1}{t}$$

By the Strong Law for Renewal processes,

$$\frac{N(t - X_1)}{t - X_1} \xrightarrow{a.s.} \frac{1}{\mu_F}$$

where  $\mu_F = \int_0^\infty \bar{F}(t)dt$ . Since  $\lim_{t \rightarrow \infty} \frac{t - X_1}{t} = 1$ , we have the same result as for the Strong Law for Renewal Processes:

$$\frac{N_d(t)}{t} \xrightarrow{a.s.} \frac{1}{\mu_F}$$

**Example 1.10.** Arrivals to a single server queue according to a Poisson process with rate  $\lambda$ . However, they will only enter if the server is free when they arrive. Service time has distribution  $G$ . (This is often called a **loss model** or **Erlang loss model**, more generally with  $k$  servers.)

- (a) At what rate do customers enter the system?
- (b) What proportion of arrivals enter the system?

**Solution.**

- (a) Note that the events of customers entering are a renewal process because everything probabilistically starts all over again when a customer arrived. Specifically it's a delayed renewal process because the time of the first arrival is exponential with rate  $\lambda$ , but the rest of them are more complicated because there must be a service and an arrival (the sum of two random variables). So the rate of arrivals is 1 over the expected time between events, or

$$\frac{1}{\mathbb{E}(S + I)}$$

where  $S$  is the service time and  $I$  is the interarrival time for the next customer. Let  $\mathbb{E}(X) = \mu_G = \int_0^\infty \bar{G}(t)dt$ . Then

$$\mathbb{E}(S + I) = \mu_G + \frac{1}{\lambda}$$

so the answer is

$$\frac{1}{\mu_G + \frac{1}{\lambda}} = \boxed{\frac{\lambda}{1 + \lambda\mu_G}}.$$

- (b) Rate of service arrivals divided by overall rates of arrivals:

$$\frac{\lambda}{1 + \lambda\mu_G} \bigg/ \lambda = \boxed{\frac{1}{1 + \lambda\mu_G}}.$$

**Example 1.11.** Suppose we have a bin with an infinite number of coins. Each bin has value  $p$  of landing on heads, and suppose the value of  $p$  for a randomly chosen coin is distributed as Uniform(0, 1). We draw coins and flip them. Every turn we can either draw a new coin or flip one of the coins we already have. If we want to maximize the proportion of coins that flip heads, what is the optimal strategy?

**Solution.** Consider a strategy of giving up on a coin if it comes up tails once. Every time you flip tails, the process renews. Let  $N(n)$  be the number of tails in the first  $n$  flips. Then  $N(n)$  is a renewal process. So we know that

$$\frac{N(n)}{n} \xrightarrow{a.s.} \frac{1}{\mathbb{E}(T)}$$

where  $T$  is the time between tails. Let  $P$  be a random variable for the probability of a coin flipping heads and note that

$$\mathbb{E}(T) = \int_0^1 \mathbb{E}(T \mid P = p) dp = \int_0^1 \frac{1}{1-p} dp = -\log(1-p) \Big|_0^1 = \infty$$

so the long-run proportion of coins that land heads is 1.

**Proposition 25 (Homework problem; *Introduction to Probability Models* Ch. 7 Problem 30).**

For a renewal process, let  $A(t) = t - S_{N(t)}$  be the age at time  $t$ . If  $\mu < \infty$ , then

$$\frac{A(t)}{t} \xrightarrow{w.p.1} 0.$$

*Proof.* We hope to show that

$$\frac{t - S_{N(t)}}{t} \xrightarrow{w.p.1} 0.$$

Note that

$$\frac{t - S_{N(t)}}{t} = \frac{t - S_{N(t)}}{N(t)} \cdot \frac{N(t)}{t} = \left( \frac{t}{N(t)} - \frac{S_{N(t)}}{N(t)} \right) \frac{N(t)}{t}$$

By the Strong Law for Renewal Processes (Theorem 24),  $\frac{N(t)}{t} \xrightarrow{w.p.1} \mu^{-1}$ , where  $\mu = \mathbb{E}(X)$  is the expected interarrival time. Since the expected interarrival time is finite,  $N(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . Therefore

$$\frac{S_{N(t)}}{N(t)} = \frac{X_1 + \dots + X_{N(t)}}{N(t)} \xrightarrow{w.p.1} \mu.$$

Lastly, since  $N(t) > 0$  with probability 1 as  $t \rightarrow \infty$  and  $t > 0$ , by the Continuous Mapping Theorem (Theorem asym.contmappthm),

$$\frac{t}{N(t)} = \left( \frac{N(t)}{t} \right)^{-1} \xrightarrow{w.p.1} (\mu^{-1})^{-1} = \mu.$$

Therefore by another application of the Continuous Mapping Theorem (Theorem asym.contmappthm),

$$\frac{t - S_{N(t)}}{t} = \left( \frac{t}{N(t)} - \frac{S_{N(t)}}{N(t)} \right) \frac{N(t)}{t} \xrightarrow{w.p.1} (\mu - \mu) \cdot \mu^{-1} = 0.$$

□

### 1.3.1 Stopping Times

**Definition 1.17 (Stopping times).** Let  $X_1, X_2, \dots$  be independent random variables. We say the non-negative integer valued random variable  $N$  is a **stopping time** for  $X_1, X_2, \dots$  if the event  $\{N = n\}$  is independent of  $X_{n+1}, X_{n+2}, \dots$  (This definition is more general than saying it must depend on previous times because it could be random and not depend on anything (except the current one) or you could have a finite memory, etc.)

**Theorem 26 (Wald's Equation).** Suppose  $X_1, X_2, \dots$  are i.i.d. with  $\mathbb{E}(X) < \infty$ . Let  $N$  be a stopping time for  $X_1, X_2, \dots$  such that  $\mathbb{E}(N) < \infty$ . Then

$$\mathbb{E}\left(\sum_{i=1}^N X_i\right) = \mathbb{E}(N)\mathbb{E}(X).$$

**Remark.** Note the similarity to compound random variables. But the proof differs because  $N$  is not independent of the  $\{X_i\}$ .

*Proof.* Let  $I_i$  be an indicator variable for  $I \leq N$ . Then

$$\sum_{i=1}^N X_i = \sum_{i=1}^{\infty} X_i I_i \implies \mathbb{E}\left(\sum_{i=1}^N X_i\right) = \mathbb{E}\left(\sum_{i=1}^{\infty} X_i I_i\right) = \sum_{i=1}^{\infty} \mathbb{E}(X_i I_i)$$

Note that  $I_i$  depends on  $X_1, \dots, X_{i-1}$  but not on  $X_i$  (it only depends on whether you play the  $i$ th game, not the outcome of that game. You only decide whether to play the  $i$ th game based on the outcomes of the previous games.). Therefore  $X_i \perp I_i$ . So we have

$$= \sum_{i=1}^{\infty} \mathbb{E}(X_i)\mathbb{E}(I_i) = \mathbb{E}(X) \sum_{i=1}^{\infty} \mathbb{E}(I_i) = \mathbb{E}(X) \sum_{i=1}^{\infty} \Pr(N \geq i) = \mathbb{E}(X)\mathbb{E}(N)$$

or we can write

$$\mathbb{E}(X) \sum_{i=1}^{\infty} \mathbb{E}(I_i) = \mathbb{E}(X) \mathbb{E}\left(\sum_{i=1}^{\infty} I_i\right) = \mathbb{E}(X)\mathbb{E}(N).$$

□

**Remark.** How do we justify

$$\mathbb{E}\left(\sum_{i=1}^{\infty} X_i I_i\right) = \sum_{i=1}^{\infty} \mathbb{E}(X_i I_i)?$$

Do the same thing again but replace all the  $X_i$  with  $|X_i|$ . Then the proof works, and Wald's Equation follows by Lebesgue's Dominated Convergence Theorem.

**Example 1.12.** Let

$$X_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{otherwise} \end{cases}$$

One stopping time could be  $N_1 = \min\{n : X_1 + \dots + X_n = k\}$ . Then by Wald's Equation (Theorem 26),

$$\mathbb{E}\left(\sum_{i=1}^N X_i\right) = \mathbb{E}(N)\mathbb{E}(X) \iff k = \mathbb{E}(N)(p) \implies \mathbb{E}(N) = \frac{k}{p}.$$

Another could be  $N_2 = \min\{n : X_{n-1} = X_n = 1\}$ .

**Example 1.13.** Let

$$X_i = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1 - p \end{cases}$$

with  $p > 1/2$ . Note that  $\mathbb{E}(X_i) = 2p - 1 > 0$ . One stopping time could be  $N = \min\{n : X_1 + \dots + X_n = 10\}$ . (Note that by the Strong Law of Large Numbers, with probability 1 this will eventually happen.) Then by Wald's Equation (Theorem 26),

$$\mathbb{E}\left(\sum_{i=1}^N X_i\right) = \mathbb{E}(N)\mathbb{E}(X) \iff 10 = \mathbb{E}(N)(2p - 1) \implies \mathbb{E}(N) = \frac{10}{2p - 1}.$$

What's the mean amount of time until you're up by one dollar?

$$m = 1 + (1 - p)\mathbb{E}(\text{up } 2) = 1 + (1 - p)2m$$

Possible stopping rule: stop when you're winning money.

$$1 = \sum_{i=1}^N X_i = \mathbb{E}(N)\mathbb{E}(X) = 0$$

but it turns out Wald's equation doesn't apply because  $\mathbb{E}(N) = \infty$  (the mean number of plays to get ahead is infinite).

**Example 1.14.** Let

$$X_i = \begin{cases} 1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$$

with  $p > 1/2$ . Note that  $\mathbb{E}(X_i) = 2p - 1 > 0$ . One stopping time could be  $N = \min\{n : X_1 + \dots + X_n = 1\}$ . It turns out from Markov chain theory that this will eventually happen with probability 1. We can't apply Wald's Equation (Theorem 26) because  $\mathbb{E}(N)$  is not finite. Note that if we try, a contradiction results:

$$1 = \mathbb{E}(N) \cdot 0 = 0$$

**Example 1.15.** Let  $U_i \sim \text{Uniform}(0, 1)$ ,  $U_i$  i.i.d. One stopping time is  $N = \min\{n : U_n > 0.8\}$ . Then by Wald's Equation (Theorem 26),

$$\mathbb{E}\left(\sum_{i=1}^N X_i\right) = \mathbb{E}(N)\mathbb{E}(X) = 10 \cdot 0 = 0$$

so the expected winnings when you stop is 0 (regardless of your stopping time, since it's a fair game).

Back to renewal theory: Let  $X_1, X_2, \dots$  be a sequence of random variables with interarrival times distributed as  $F$ . Note that  $N(t) = 5 \iff X_1 + \dots + X_5 \leq t, X_1 + \dots + X_6 > t$  so this is not a stopping time. But we could stop at the first event that occurs after time  $t$ , so  $N(t) + 1$  is a stopping time. Note that

$$N(t) + 1 = n \iff N(t) = n - 1, X_1 + \dots + X_{n-1} \leq t, X_1 + \dots + X_n > t.$$

(You can't say "I'll stop at the last event before  $t$ " because at the time of that event you wouldn't have been able to know it was the last event before  $t$ .)

**Corollary 27 (Corollary to Wald's Equation).**

$$\mathbb{E}\left(\sum_{i=1}^{N(t)+1} X_i\right) = \mu(m(t) + 1)$$

**Theorem 28 (Elementary Renewal Theorem).**

$$\frac{m(t)}{t} \xrightarrow{a.s.} \frac{1}{\mu}.$$

*Proof.* Note that

$$S_{N(t)+1} > t$$

We have

$$\mathbb{E}(S_{N(t)+1}) > t$$

Using Corollary 27,

$$\mu(m(t) + 1) > t \iff m(t) + 1 > \frac{t}{\mu} \iff \frac{m(t)}{t} > \frac{1}{\mu} - \frac{1}{t}$$

Then as  $t \rightarrow \infty$

$$\liminf \frac{m(t)}{t} \geq \frac{1}{\mu}.$$

So if we can show  $\liminf \frac{m(t)}{t} \leq \frac{1}{\mu}$ , we're done. Assume  $\Pr(X_i \leq M) = 1$  for some  $M \in \mathbb{R}$ . Then

$$S_{N(t)+1} < t + M \implies \mu(m(t) + 1) < t + M \implies m(t) < \frac{t}{\mu} + \frac{M}{\mu} - 1$$

$$\implies \frac{m(t)}{t} < \frac{1}{\mu} + \frac{M}{t\mu} - \frac{1}{t}$$

$$\implies \limsup \frac{m(t)}{t} \leq \frac{1}{\mu}.$$

Now consider the general case. Consider  $X_1, X_2, \dots$ . Let

$$X_i^* = \begin{cases} X_i & \text{if } X_i \leq M \\ M & \text{if } X_i > M \end{cases}$$

and let  $N^*(t) = \max\{n : X_1^* + \dots + X_n^* \leq t\}$ . Then since  $N^*(t)$  has smaller interarrival times than  $N(t)$ ,

$$N(t) \leq N^*(t) \implies \frac{\mathbb{E}(N(t))}{t} \leq \frac{\mathbb{E}(N^*(t))}{t}$$

$$\implies \lim \frac{m(t)}{t} \leq \lim \frac{\mathbb{E}(N^*(t))}{t} = \frac{1}{\mathbb{E}(X_i^*)}$$

Note that (since  $X_i^* = X_i$  if  $X_i \leq M$ )

$$\mathbb{E}(X_i^*) = \int_0^M x dF(x) + M\bar{F}(M) \rightarrow \mu + 0 = \mu \text{ as } M \rightarrow \infty$$

so

$$\lim \frac{m(t)}{t} \leq \frac{1}{\mathbb{E}(\min\{X, M\})}$$

which is true for every  $M$ . Letting  $M \rightarrow \infty$ , we have

$$\mathbb{E}(\min\{X, M\}) \rightarrow \mathbb{E}(X) \text{ as } M \rightarrow \infty$$

by Lebesgue's Monotone convergence theorem. ("if you have a sequence of variables  $X_n \leq X_{n+1} \leq \dots$  then  $\mathbb{E}(\lim_{n \rightarrow \infty} X_n) = \lim_{n \rightarrow \infty} \mathbb{E}(X_n)$ ").



□

**Remark.** This technique is called “coupling:” relating something you don’t know to something you do.

**Remark.** Why doesn’t the Elementary Renewal Theorem follow from the Strong Law for Renewal Processes (Theorem 24)? Consider this counterexample: Let  $U \sim \text{Uniform}(0, 1)$  and let  $X_n$  be a random variable such that

$$X_n = \begin{cases} n & U < 1/n \\ 0 & U > 1/n \end{cases}$$

Of course with probability 1,  $U > 0$ ; that is,  $U = \epsilon > 0$ . Note that  $X_n = 0$  for all  $n$  sufficiently large ( $1/n < \epsilon$ ). So we see that  $X_n \xrightarrow{a.s.} 0$ . But for all  $n$  sufficiently large

$$\mathbb{E}(X_n) = n \cdot \frac{1}{n} = 1.$$

So we have

$$\lim_{n \rightarrow \infty} X_n = 0 = \mathbb{E}(\lim_{n \rightarrow \infty} X_n) \neq \lim_{n \rightarrow \infty} \mathbb{E}(X_n) = 1$$

In summary, just because  $\frac{N(t)}{t} \xrightarrow{a.s.} \frac{1}{\mu}$  doesn’t mean  $\frac{m(t)}{t} \xrightarrow{a.s.} \frac{1}{\mu}$ .

**Proposition 29.** Consider a delayed renewal process  $N_d(t)$  and let  $m_d(t) = \mathbb{E}(N_d(t))$ . Recall that the first arrival has distribution  $G$  and the rest have distribution  $F$ . Then  $m_d(t) \xrightarrow{a.s.} 1/\mu$ .

*Proof.*

$$m_d(t) = \int_0^t \mathbb{E}(N_d(t) \mid X_1 = s) dG(s)$$

Note that

$$\mathbb{E}(N_d(t) \mid X_1 = s) = 1 + m(t - s)$$

so we have

$$m_d(t) = \int_0^t (1 + m(t - s)) dG(s) = G(t) + \int_0^t m(t - s) dG(s)$$

$$\iff \frac{m_d(t)}{t} = \frac{G(t)}{t} + \frac{1}{t} \int_0^t m(t - s) dG(s) \xrightarrow{a.s.} \frac{1}{\mu}$$

by using  $m(y)/y \xrightarrow{a.s.} 1/\mu$ . (Details can be fleshed out.)

□

**Proposition 30.** Suppose  $X_1, X_2, \dots$  are integer valued, and there is only one renewal at a time. Suppose  $\Pr(X_i = 0) = 0$ . Then

*Proof.* Note that

$$\frac{\mathbb{E}(N_d(t))}{n} \rightarrow \frac{1}{\mu}$$

by the Strong Law for Renewal Processes (Theorem 24). Note that

$$N_d(n) = \sum_{j=1}^n I_j$$

where  $I_j$  is an indicator variable for whether there is a renewal at time  $j$ . Then

$$\mathbb{E}(N_d(n)) = \sum_{j=1}^n \mathbb{E}(I_j) = \sum_{j=1}^n \Pr(I_j = 1)$$

Then by the Elementary Renewal Theorem (Theorem 28),

$$\frac{1}{n} \sum_{j=1}^n \Pr(I_j = 1) \xrightarrow{a.s.} \frac{1}{\mathbb{E}(X_i)}$$

□

Recall we say that  $a_n \rightarrow a$  pointwise if  $\lim_{n \rightarrow \infty} a_n = a$ .

**Definition 1.18 (Caesaro Convergence).** We say  $a_n$  Caesaro converges to  $a$  if

$$\frac{a_1 + \dots + a_n}{n} \rightarrow a \text{ as } n \rightarrow \infty.$$

**Remark.** Pointwise convergence implies Caesaro convergence, but not the other way around.

When does  $\Pr(\{\text{renewal at } n\}) \rightarrow 1/\mu$ ? Let  $\Pr(\{\text{renewal at } j\}) = P_j$  and suppose  $P_n \rightarrow P^*$ . This implies Caesaro convergence

$$\frac{P_1 + \dots + P_n}{n} \rightarrow p^*$$

and  $p^* = 1/\mu$  where  $\mu$  is the expected time between renewals.

**These examples are very confusing—look for explanations in textbook? Seems to be same as Section 7.9.2 in *Introduction to Probability Models*.**

**Example 1.16.** Let  $Y_1, Y_2, \dots$  i.i.d. Suppose  $p_j = \Pr(Y_i = j)$ . We will keep doing draws until we observe the pattern 1, 2, 1, 3, 4, 1, 2, 1. Let  $N$  be the number of trials until this occurs. What is  $\mathbb{E}(N)$ ?

**Solution.** Suppose we consider the process as continuing to happen forever even after a pattern appears, and treat a pattern arriving as an event. let  $N(n)$  be the number of events by  $n$ . Is  $\{N(n)\}$  a renewal process? That is, once an event occurs does everything start all over? Yes, so it's either an ordinary or delayed renewal process. Note that it's delayed because the last three digits of the pattern match the first three, so once a pattern has just happened, another one could happen in 5 draws. Note that

$$\Pr(\text{pattern appears at time } n) = P_1 P_2 \dots = P_1^4 P_2^2 P_3 P_4$$

so the expected time between renewals is  $1/(P_1^4 P_2^2 P_3 P_4)$ , except that we need to take into account the fact that we already have the first three digits of the pattern if we just got one. Note that

$$\Pr(\text{pattern appears at time } n \mid \text{just had pattern}) = P_1^2 P_2 P_3 P_4$$

We can look at this a different way. The time until we get a pattern  $T_p$  is the time until we get 1,2,1  $T_{121}$  plus the time until we get the rest of the pattern  $T_{rest}$ .

$$T_p = T_{121} + T_{rest} \implies \mathbb{E}(T_p) = \mathbb{E}(T_{121}) + 1/(P_1^4 P_2^2 P_3 P_4)$$

Then  $\Pr(\text{renewal at } n) = P_1^2 P_2$  so the expected time between renewals is  $1/(P_1^2 P_2)$ . Then the time between renewals is the additional time between renewals given that you currently have 1, so

$$\mathbb{E}(\text{time between renewals}) = \mathbb{E}(\text{additional time when you have a 1}) = 1/(P_1^2 P_2)$$

Do it again: time to get to 1,2,1 is time to get to 1 plus additional:

$$T_{121} = T_1 + T_{21} \implies \mathbb{E}(T_{121}) = \mathbb{E}(T_1) + 1/(P_1^2 P_2) = 1/P_1 + 1/(P_1^2 P_2)$$

putting this all together, we have

$$\mathbb{E}(T_p) = 1/P_1 + 1/(P_1^2 P_2) + 1/(P_1^4 P_2^2 P_3 P_4).$$

**Example 1.17.** Flips coins, get heads with probability  $P$ . Then  $T_k$  is the time to land  $k$  heads in a row. What is the expected time until  $k$  heads in a row?

**Solution.** Note that the probability of a renewal at  $n$  is  $p^k$ . So the expected time between renewals is equal to  $1/p^k$  not taking into account that the next time we could use the heads we already have. Note that

$$T_k = T_{k-1} + T_{rest}$$

so

$$\mathbb{E}(T_k) = \mathbb{E}(T_{k=1} + 1/p^k = 1/p^k + \mathbb{E}(T_{k-2}) + 1/(p^{k-1}) + \dots$$

## 1.4 ISE 620 Midterm Solutions

**Exercise 1.**

$$\text{Var}\left(\sum_{i=1}^n X_i\right)$$

**Solution.**

$$\text{Var}\left(\sum_{i=1}^n X_i\right) = \text{Var}\left(\sum_{i=1}^n (n+1-i)X_i\right) = \sum_{i=1}^n \frac{(n+1-i)^2}{\lambda^2} = \sum_{j=1}^n j^2/\lambda^2$$

or

$$\text{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \text{Var}(S_i) + 2 \sum_{1 \leq i < j \leq n} \text{Cov}(S_i, S_j)$$

Note that

$$\text{Cov}(S_i, S_j) = \text{Cov}\left(S_i, S_i + \sum_{k=i+1}^j X_k\right) = \text{Var}(S_i) + 0 = i/\lambda^2$$

**Exercise 2.**

$$\text{Var}\left(\sum_{i=1}^{N(t)} S_i \mid N(t)\right)$$

**Solution.**

$$\text{Var}\left(\sum_{i=1}^{N(t)} S_i \mid N(t) = n\right) = \text{Var}\left(\sum_{i=1}^n U_i\right)$$

where  $U_1, \dots, U_n \sim \text{i.i.d. } U(0, t)$  and they are ordered. Since the sum of the ordered values is the same as the sum of the unordered values and they are i.i.d., we have

$$\text{Var}\left(\sum_{i=1}^n U_i\right) = \sum_{i=1}^n \text{Var}(U_i) = \frac{nt^2}{12}.$$

**Exercise 3.** A wins point with probability  $p_1$  if  $A$  is server,  $A$  wins with probability  $p_2$  if  $B$  serves. (a) serve next if you just won. (b) alternate serve.

**Solution.**

- (a) A serving is a renewal. Suppose  $A$  just won a point. Let  $X$  be the number of games until  $A$  wins again. Frequency with which  $A$  wins converges to  $1/\mathbb{E}(X)$  by the Strong Law for Renewal Processes (Theorem 24). Condition on who wins next game. Let  $Y$  be an indicator variable for  $A$  winning the next game. Then

$$\mathbb{E}(X) = \mathbb{E}(X | Y = 1)p_1 + \mathbb{E}(X | Y = 0)(1 - p_1)$$

Note that if  $Y = 1$ ,  $X = 1$ . For  $\mathbb{E}(X | Y = 0)$ ,  $B$  keeps serving until  $A$  wins, so the number of games until  $A$  wins is a geometric random variable with probability  $p_2$ , so the mean is  $1/p_2$ . So we have

$$\mathbb{E}(X) = 1 \cdot p_1 + (1 + 1/p_2)(1 - p_1) = 1 + \frac{1 - p_1}{p_2} = \frac{p_2 + 1 - p_1}{p_2}$$

and the rate is

$$\frac{p_2}{p_2 + 1 - p_1}$$

- (b) Not a renewal—everything doesn't start all over again when  $A$  wins a point, because just because  $A$  just won doesn't tell us anything about who wins. A renewal would happen every time  $A$  wins on her serve. Let  $X$  be the number of games until the next renewal. Then the answer we seek is  $1/\mathbb{E}(X)$ . Let  $Y$  be an indicator variable for  $A$  winning on her next serve. Note that

$$\mathbb{E}(X) = \mathbb{E}(X | Y = 1)p_1 + \mathbb{E}(X | Y = 0)(1 - p_1)$$

Note that  $\mathbb{E}(X | Y = 1) = 2$ ,  $\mathbb{E}(X | Y = 0) = 2 + \mathbb{E}(X)$ . So we have

$$\mathbb{E}(X) = 2p_1 + (2 + \mathbb{E}(X))(1 - p_1) \iff p_1\mathbb{E}(X) = 2p_1 + 2 - 2p_1 \iff \mathbb{E}(X) = 2/p_1$$

Do the same thing again for when  $A$  wins on  $B$ 's serves. Then add them together.

A renewal would happen every time  $A$  wins on  $B$ 's serve. Let  $Z$  be the number of games until the next renewal. Then the answer we seek is  $1/\mathbb{E}(Z)$ . Let  $Y$  be an indicator variable for  $A$  winning on  $B$ 's next serve. Note that

$$\mathbb{E}(Z) = \mathbb{E}(Z | Y = 1)p_2 + \mathbb{E}(Z | Y = 0)(1 - p_2)$$

Note that  $\mathbb{E}(Z | Y = 1) = 2$ ,  $\mathbb{E}(Z | Y = 0) = 2 + \mathbb{E}(Z)$ . So we have

$$\mathbb{E}(Z) = 2p_2 + (2 + \mathbb{E}(Z))(1 - p_2) \iff p_1\mathbb{E}(Z) = 2p_2 + 2 - 2p_2 \iff \mathbb{E}(Z) = 2/p_2$$

So the long-run proportion of games won by  $A$  is

$$\frac{1}{\mathbb{E}(X)} + \frac{1}{\mathbb{E}(Z)} = \frac{p_1 + p_2}{2}.$$

**Exercise 4.**  $M/G/\infty$ ,  $\Pr(X(t) = j | N(t) = n)$ .

**Solution.**

The arrival times are i.i.d. uniform on  $(0, t)$ . So each one has the same probability of being in the system at time  $t$ , and the number who are in the system is a binomial random variable.

$$\Pr(X(t) = j \mid N(t) = n) = \binom{n}{j} p^j (1-p)^{n-j}$$

where  $p$  is the probability of being in the system at time  $t$ :

$$p = \int_0^t \Pr(\text{in system} \mid \text{arrived at } s) \cdot (1/t) ds = \frac{1}{t} \int_0^t \bar{G}(t-s) ds.$$

**Exercise 5.** number of claims  $N$  poisson with mean lambda, claim amounts 'are uniformly distributed on  $(0, 1)$ , we want  $\mathbb{E}(\max\{X_1, \dots, X_N\})$ .

**Solution.** Note that

$$\Pr(\max\{X_1, \dots, X_n\} \leq t) = t^n \implies \Pr(\max\{X_1, \dots, X_n\} > t) = 1 - t^n$$

$$\implies \mathbb{E}(\max\{X_1, \dots, X_n\}) = \int_0^1 (1 - t^n) dt = \frac{n}{n+1} = 1 - \frac{1}{n+1}$$

so this is the expected value if we condition on  $n$ . Therefore

$$\mathbb{E}(\max\{X_1, \dots, X_N\}) = \mathbb{E}(\max\{X_1, \dots, X_N\} \mid N = n) \Pr(N = n)$$

$$= \sum_{n=0}^{\infty} \left(1 - \frac{1}{n+1}\right) \cdot e^{-\lambda} \frac{\lambda^n}{n!} = \dots$$

**Exercise 6.**

$$\mathbb{E}(X(t) \mid N(s) = n)$$

where

$$X(t) = \sum_{i=1}^{N(t)} X_i$$

**Solution.**

We have

$$X(t) \mid N(s) = n = \sum_{i=1}^{N(s)} X_i + \sum_{i=N(s)+1}^{N(t)} X_i = \sum_{i=1}^n X_i + \sum_{i=N(s)+1}^{N(t)} X_i$$

$$\implies \mathbb{E}(X(t) \mid N(s)) = \mathbb{E}\left(\sum_{i=1}^n X_i\right) + \mathbb{E}\left(\sum_{i=N(s)+1}^{N(t)} X_i\right) = n\mathbb{E}(X) + \mathbb{E}(X)\lambda(t-s)$$

where  $\lambda(t-s)$  is the expected number of events between  $s$  and  $t$ .

### 1.5 Renewal Reward Processes (Section 4.7 in *Introduction to Probability Models*, 3.6 *Stochastic Processes*)

**Definition 1.19 (Renewal reward process).** Suppose we have a renewal process with interarrival times  $X_1, X_2, \dots$ . Suppose we receive a reward  $R_i$  when renewal  $i$  occurs (after time  $X_i$  from the previous renewal). The reward is allowed to depend on what happens during the previous period  $X_i$ , but every time renewal happens, process starts again. So we assume the vectors  $(X_i, R_i), i \geq 1$  are i.i.d. Let  $R(t)$  be the total reward by time  $t$ ; that is,

$$R(t) = \sum_{i=1}^{N(t)} R_i.$$

Then  $\{R(t), t \geq 0\}$  is a **renewal reward process**.

**Proposition 31 (Proposition 7.3 in *Introduction to Probability Models*; sometimes called Renewal Reward Theorem).**

(a)

$$\frac{R(t)}{t} \xrightarrow{a.s.} \frac{\mathbb{E}(R)}{\mathbb{E}(X)}$$

(generalization of strong law)

(b)

$$\frac{\mathbb{E}(R(t))}{t} \xrightarrow{a.s.} \frac{\mathbb{E}(R)}{\mathbb{E}(X)}$$

*Proof.* (a)

$$\frac{R(t)}{t} = \frac{1}{t} \sum_{i=1}^{N(t)} R_i = \frac{1}{N(t)} \sum_{i=1}^{N(t)} R_i \cdot \frac{N(t)}{t}$$

Since  $N(t) \rightarrow \infty$  as  $t \rightarrow \infty$ , by Strong Law of Large Numbers (Theorem ??)

$$\frac{1}{N(t)} \sum_{i=1}^{N(t)} R_i \xrightarrow{a.s.} \mathbb{E}(R)$$

By Strong Law for Renewal Processes (Theorem 24),

$$\frac{N(t)}{t} \xrightarrow{a.s.} 1/\mathbb{E}(X).$$

Then by the Continuous Mapping Theorem (Theorem ??), the result follows.

(b)

□

**Example 1.18 (Same as exam question).** A wins point with probability  $p_1$  if  $A$  is server,  $A$  wins with probability  $p_2$  if  $B$  serves. alternate serve.

**Solution.** Every time  $A$  serves is a renewal. 1 reward each time  $A$  wins a point. Expected earning per cycle is  $p_1 + p_2$ . Cycles are length 2. Works out to  $(p_1 + p_2)/2$ .

**Proposition 32.** We've assumed that the renewal is earned at the end of the cycle. But the result holds even if the reward is earned during the interval gradually, not all at once at the end of the renewal interval (assuming all rewards are nonnegative). So

$$\frac{R(t)}{t} \xrightarrow{a.s.} \frac{\mathbb{E}(R)}{\mathbb{E}(X)}.$$

Also,

$$\mathbb{E} \left[ \frac{R(t)}{t} \right] \xrightarrow{a.s.} \frac{\mathbb{E}(R)}{\mathbb{E}(X)}.$$

*Proof.*  $R(t)$  is the total reward you get by time  $t$ . So  $\sum_{i=1}^{N(t)} R_i \leq R(t)$  (the inequality holds if you've earned a bit since the last renewal). We have

$$\sum_{i=1}^{N(t)} R_i \leq R(t) \leq \sum_{i=1}^{N(t)+1} R_i$$

Dividing through by  $t$ :

$$\frac{1}{t} \sum_{i=1}^{N(t)} R_i \leq \frac{R(t)}{t} \leq \frac{1}{N(t)+1} \sum_{i=1}^{N(t)+1} R_i \cdot \frac{N(t)+1}{t}$$

But by Proposition 31

$$\frac{1}{t} \sum_{i=1}^{N(t)} R_i \xrightarrow{a.s.} \mathbb{E}(R)/\mathbb{E}(X),$$

$$\frac{1}{N(t)+1} \sum_{i=1}^{N(t)+1} R_i \xrightarrow{a.s.} \mathbb{E}(R)$$

$$\frac{N(t)+1}{t} \xrightarrow{a.s.} 1/\mathbb{E}(X)$$

So by the Continuous Mapping Theorem,

$$\frac{R(t)}{t} \xrightarrow{a.s.} \frac{\mathbb{E}(R)}{\mathbb{E}(X)}.$$

□



**Example 1.19 (Similar to Example 7.15 in *Introduction to Probability Models*).** People arrive to train station in a Poisson process with rate  $\lambda$ .  $K$  is the cost to dispatch the train. Suppose we incur a cost  $c$  per unit of time waiting per customer. What dispatching policy minimizes the long-run average cost per unit time?

**Solution.**

The structure of the optimal policy is to wait until a certain number of people arrive and then dispatch a train ( $n$  policy). So we want to determine this number. (Another policy is a  $t$  policy: dispatch a train every  $t$  units of time. But it turns out the best  $n$  policy is better than the best  $t$  policy. We may be interested in the difference in minimal costs because  $t$  policies are easier to employ and more convenient for customers.)

- (a)  **$n$ -policy:** Note that every time we dispatch a train, there is a renewal. At every renewal time there is a cost incurred. The long run average cost per unit time is the expected cost during a cycle divided by the expected time of a cycle. Note that the expected time between cycles is the expected amount of time for  $n$  people to arrive, which is simply  $n/\lambda$ . The cost of a cycle is

$$cX_2 + 2cX_3 + 3cX_4 + \dots + (n-1)cX_n + K$$

so the expected cost is

$$\mathbb{E}(X)(c + 2c + 3c + \dots + (n-1)c) + K = \frac{c(n-1)n}{\lambda} + K$$

Therefore taking the ratio, the average cost per cycle is

$$\left( \frac{1}{\lambda} \frac{c(n-1)n}{2} + K \right) / \left( \frac{n}{\lambda} \right) = \frac{c(n-1)}{2} + \frac{\lambda K}{n}.$$

Now we want to pick  $n$  to minimize this quantity. Treat  $n$  as continuous and take the derivative.

$$\frac{c}{2} - \frac{\lambda K}{n^2} = 0 \iff n = \sqrt{\frac{2\lambda K}{c}}$$

which makes the minimum average cost

$$\frac{1}{2} \left( c \sqrt{\frac{2\lambda K}{c}} - c \right) + K \sqrt{\frac{c}{2\lambda K}} = \sqrt{2\lambda c K} - \frac{c}{2}.$$

- (b)  **$t$ -policy:** Dispatch a train every  $t$  time units. Note that we have a renewal every time you dispatch a train. The average cost per unit time is the expected cost per cycle  $\mathbb{E}(C)$  divided by  $t$ . Note that

$$\begin{aligned} C &= c \sum_{i=1}^{N(t)} (t - S_i) = c \left( tN(t) - \sum_{i=1}^{N(t)} S_i \right) + K \\ \implies \mathbb{E}(C) &= c \mathbb{E} \left( tN(t) - \sum_{i=1}^{N(t)} S_i \right) + K = c \left[ \lambda t^2 - \mathbb{E} \left( \sum_{i=1}^{N(t)} S_i \right) \right] + K \end{aligned}$$

Note that

$$\mathbb{E}\left(\sum_{i=1}^{N(t)} S_i \mid N(t) = n\right) = \mathbb{E}\left(\sum_{i=1}^n S_i\right) = \mathbb{E}\left(\sum_{i=1}^n U_i\right) = \frac{nt}{2} \implies \mathbb{E}\left(\sum_{i=1}^{N(t)} S_i \mid N(t)\right) = \frac{tN(t)}{2}$$

where  $U_i \sim U(0, t)$ .

$$\begin{aligned} \implies \mathbb{E}(C) &= c\lambda t^2 - c\mathbb{E}\left[\mathbb{E}\left(\sum_{i=1}^{N(t)} S_i \mid N(t)\right)\right] + K = c\lambda t^2 - c\mathbb{E}\left[\frac{tN(t)}{2}\right] + K = c\lambda t^2 - c\frac{\lambda t^2}{2} + K \\ &\implies \mathbb{E}(C) = \frac{c\lambda t^2}{2} + K \end{aligned}$$

Therefore the average cost per cycle is

$$\frac{1}{t} \cdot \left(\frac{c\lambda t^2}{2} + K\right) = \frac{\lambda c t}{2} + \frac{K}{t}$$

We seek the best  $t$  to minimize this cost. Take the derivative with respect to  $t$ .

$$\frac{\lambda c}{2} - \frac{K}{t^2} = 0 \implies t = \sqrt{\frac{2K}{\lambda c}}$$

So the minimum average cost is

$$\sqrt{\frac{2K}{\lambda c}} \frac{\lambda c}{2} + \frac{K}{\sqrt{\frac{2K}{\lambda c}}} = K \sqrt{\frac{\lambda c}{2K}}.$$

**Example 1.20.** It costs  $C$  to buy a new car. The car lasts for a time  $L$  having distribution  $F$  before it fails. If the car fails, you incur cost  $K$ . Suppose you have a policy of buying a new car when the old car either fails or reaches a certain age  $t$ .

**Solution.** Note that a renewal occurs when you get a new car or when it fails. The cost per cycle is  $C$  if  $L > t$ ,  $C + K$  if  $L \leq t$ . The time for a cycle is  $\min\{L, t\}$ . Therefore the average cost per unit time is the expected cost of a cycle  $C + K \Pr(L < t) = C + KF(t)$ . The average lifetime is

$$\int_0^t x dF(x) + \int_t^\infty t dF(x) = \int_0^t x dF(x) + t\bar{F}(t).$$

Therefore the average cost per cycle is

$$\left(c + KF(t)\right) / \left(\int_0^t x dF(x) + t\bar{F}(t)\right).$$

**Definition 1.20 (Age of a renewal process).** We call  $A(t) = t - S_{N(t)}$  the **age of the renewal process at time  $t$** .

**Definition 1.21 (Excess (or residual) lifetime at time  $t$ ).** We call  $Y(t) = S_{N(t)+1} - t$  be the **excess (or residual) lifetime of a renewal process at time  $t$** .

**Proposition 33.** Let  $X$  be the interarrival time for a renewal reward process. Then

(a) with probability 1,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t A(s) ds = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)}.$$

(b) With probability 1,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t Y(s) ds = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)}.$$

(c) With probability 1,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{E}[A(s)] ds = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)}.$$

(d) With probability 1,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{E}[Y(s)] ds = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)}.$$

*Proof.* (a) Imagine we earn a reward at a rate equal to the age of the renewal process. Then the amount earned by time  $t$  is

$$\int_0^t A(s) ds.$$

When there is a new renewal, the process starts over again, so this is a renewal reward process. The reward earned during each cycle is

$$\int_0^X t dt = \frac{X^2}{2}.$$

Taking expectations and taking the ratio of these in the limit, we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t A(s) ds = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)}.$$

For more detail, see Example 7.18 in *Introduction to Probability Models*.

(b) Imagine we earn a reward at a rate equal to the residual lifetime of the renewal process. Then the amount earned by time  $t$  is

$$\int_0^t Y(s) ds.$$

When there is a new renewal, the process starts over again, so this is a renewal reward process. The reward earned during each cycle is

$$\int_0^X (X - t) dt = X^2 - \frac{X^2}{2} = \frac{X^2}{2}.$$

Taking expectations and taking the ratio of these in the limit, we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t A(s) ds = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)}.$$

For more detail, see Example 7.19 in *Introduction to Probability Models*.

(c) Similar to (a).

(d) Similar to (c).

□

**Remark.** This makes sense because if the process is infinitely long, if you look at it backwards in time the distribution of renewals is the same as if you look forward in time. But when you switch direction, the residual lifetimes and ages reverse meaning. Therefore since the distribution doesn't change depending on the direction you look at it from, the long-term expected residual lifetime and age ought to be equal.

**Proposition 34 (Homework 7, *Introduction to Probability Models* Ch. 7 Problem 30).**

$$\frac{A(t)}{t} \xrightarrow{a.s.} 0.$$

*Proof.*  $A(t)$  is the time since the last renewal; that is,  $A(t) = t - S_{N(t)}$ . So we hope to show that with probability one,

$$\frac{t - S_{N(t)}}{t} \rightarrow 0 \text{ as } t \rightarrow \infty$$

(or, equivalently)

$$\frac{t - S_{N(t)}}{t} \xrightarrow{w.p.1} 0.$$

Note that

$$\frac{t - S_{N(t)}}{t} = \frac{t - S_{N(t)}}{N(t)} \cdot \frac{N(t)}{t} = \left( \frac{t}{N(t)} - \frac{S_{N(t)}}{N(t)} \right) \frac{N(t)}{t}$$

By the Strong Law for Renewal Processes (Theorem 24),  $\frac{N(t)}{t} \xrightarrow{w.p.1} \mu^{-1}$ , where  $\mu = \mathbb{E}(X)$  is the expected interarrival time. Since the expected interarrival time is finite,  $N(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . Therefore

$$\frac{S_{N(t)}}{N(t)} = \frac{X_1 + \dots + X_{N(t)}}{N(t)} \xrightarrow{w.p.1} \mu.$$

Lastly, since  $N(t) > 0$  with probability 1 as  $t \rightarrow \infty$  and  $t > 0$ , by the Continuous Mapping Theorem

$$\frac{t}{N(t)} = \left( \frac{N(t)}{t} \right)^{-1} \xrightarrow{w.p.1} (\mu^{-1})^{-1} = \mu.$$

Therefore by another application of the Continuous Mapping Theorem,

$$\frac{t - S_{N(t)}}{t} = \left( \frac{t}{N(t)} - \frac{S_{N(t)}}{N(t)} \right) \frac{N(t)}{t} \xrightarrow{w.p.1} (\mu - \mu) \cdot \mu^{-1} = 0.$$

□

**Example 1.21 (Similar to Example 7.20 in *Introduction to Probability Models*).** People arrive to a bus stop in a Poisson process with rate  $\lambda$ . Buses arrive in an independent renewal process after time  $T$  with distribution  $F$  where  $\mathbb{E}(T) = \mu$ .

- (a) What is the long-run average number of people picked up by a bus when it arrives?
- (b) Suppose busses arrive according to a Poisson process with rate  $\lambda$ ; that is,  $F(t) = 1 - e^{-\lambda t}$ . What is the long-run average number of people waiting at any given time (averaged over all time)?
- (c) What is the average amount of time a person waits for a bus (averaged over all people)? (Let  $W_i$  be the waiting time of person  $i$ . We seek  $\lim_{n \rightarrow \infty} \sum_{i=1}^n W_i / n$ .)

**Solution.**

- (a) Every time a bus arrives, we have a renewal. The number of people picked up by a bus is  $N(T)$ . Note that  $\{N(t)\} \sim PP(\lambda)$ . Note that

$$\mathbb{E}(N(T) \mid T = t) = \mathbb{E}(N(t) \mid T = t) = (\text{by independence}) \mathbb{E}(N(t)) = \lambda t \implies \mathbb{E}(N(T) \mid T) = \lambda T$$

$$\implies \mathbb{E}(N(T)) = \mathbb{E}(\mathbb{E}(N(T) \mid T)) = \lambda \mathbb{E}(T) = \lambda \mu.$$

- (b) Let  $N_s(s)$  be the number of people waiting at time  $s$ . We seek

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t N_w(s) ds.$$

Imagine that at time  $s$  we earn at a rate  $N_w(s)$ . Then we want the average reward per unit time. Then this is a renewal reward process, where renewals happen when a bus arrive. The expected reward earned per cycle is

$$\frac{1}{\mathbb{E}(T)} \cdot \mathbb{E} \left( \int_0^T N(s) ds \right) \tag{8}$$

Note that

$$\begin{aligned} \mathbb{E} \left( \int_0^T N(s) ds \mid T = t \right) &= \mathbb{E} \left( \int_0^t N(s) ds \right) = (\text{by Fubini's Theorem}) \int_0^t \mathbb{E}(N(s)) ds = \int_0^t \lambda s ds = \frac{\lambda t^2}{2} \\ \implies \mathbb{E} \left( \int_0^T N(s) ds \mid T \right) &= \frac{\lambda T^2}{2} \end{aligned}$$

Therefore we have that the expected reward earned per cycle is (plugging into (8))

$$\frac{\lambda \mathbb{E}(T^2)}{2\mathbb{E}(T)}.$$

Because arrivals occur in a Poisson process ( $F(t) = 1 - e^{-\alpha t}$ ), we have  $\mathbb{E}(T) = \alpha^{-1}$ ,  $\mathbb{E}(T^2) = 2\alpha^{-2}$ . Therefore the expected reward earned per cycle is

$$\frac{\lambda 2\alpha^{-2}}{2\alpha^{-1}} = \frac{\lambda}{\alpha}.$$

(c)  $\frac{\lambda}{\alpha}$ .

**Remark.** Note that the average number of people picked up by a bus now matches the average number of people waiting at any given time. This is counterintuitive because you would think at the time the bus comes, the most people are there, so the average number of people picked up should be larger than the average number of people waiting in general.

This illustrates the **PASTA principle** and the **inspection paradox**. The paradox is resolved by the fact that the average in (b) is averaged over all time, while the average in (c) is averaged over all people.

**Theorem 35.** Queue, customers arrive in a renewal process with distribution  $F$ , so

$$\lambda^{-1} = \int_0^\infty \bar{F}(t) dt.$$

Each customer eventually leaves. Let  $L$  be the average number of customers in the system, averaged over all time. Let  $X(s)$  be the number of customers in the system at time  $s$ . Note that

$$L = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t X(s) ds.$$

Let  $W$  be the average time a customer spends in the system. Let  $W_i$  be the amount of time customer  $i$  spends in the system. Note that

$$W = \lim_{n \rightarrow \infty} \frac{W_1 + \dots + W_n}{n}.$$

Then  $L = \lambda W$ .

*Proof.*

$$L = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t X(s) ds$$

Suppose we are paid at any time at a rate equal to the number of customers in the system. Then we can interpret  $L$  as the average reward per unit time. We can consider a renewal as occurring upon the arrival of the first customer when the system was previously empty. Then we have a renewal reward process. Let  $T$  be the time between renewals. Assume the first arrival occurs at time  $t = 0$  (so this is not a delayed renewal reward process). Note that

$$L = \frac{1}{\mathbb{E}(T)} \mathbb{E} \left( \int_0^T X(s) ds \right) \iff \mathbb{E} \left( \int_0^T X(s) ds \right) = L \mathbb{E}(T) \quad (9)$$

Note that

$$W = \lim_{n \rightarrow \infty} \frac{W_1 + \dots + W_n}{n}$$

Let  $N$  be the number of customers served in a busy period. Then when customer  $N+1$  arrives, the system is empty and there is a renewal, so  $T$  is the time of the  $N+1$ st arrival. So

$$T = \sum_{i=1}^N X_i$$

where  $X_i$  are the interarrival times ( $X_i$  is the time between the  $i$ th and  $i+1$ st arrival). Note that  $N$  depends on the interarrival times; if the interarrival times are small,  $N$  will be larger. So this is not a compound random variable, but we can use Wald's Equation (Theorem 26) if  $N$  is a stopping time for  $X_1, \dots, X_n$ . Note that  $N = 1$  if and only if  $X_1$  is greater than the service time of the initial customer. And in general,  $N = k$  depends only on the arrival times up to  $X_k$ . So  $N$  is a stopping time and we can use Wald's Equation. Therefore

$$\mathbb{E}(T) = \frac{\mathbb{E}(N)}{\lambda}. \quad (10)$$

Suppose each customer pays us 1 dollar per unit time. Then at time  $s$  we are earning at rate  $X(s)$ , so

$$\int_0^T X(s) ds$$

is the total amount you earn by time  $T$ . But we can also calculate this amount by adding up the amount of time each customer spends in the system:

$$\int_0^T X(s) ds = \sum_{i=1}^N W_i.$$

So we have (using (9) and (10))

$$\begin{aligned} \mathbb{E} \left( \int_0^T X(s) ds \right) &= \mathbb{E} \sum_{i=1}^N W_i \iff L \mathbb{E}(T) = \mathbb{E} \sum_{i=1}^N W_i \iff L = \frac{\mathbb{E}(\sum_{i=1}^N W_i)}{\mathbb{E}(N)/\lambda} \\ &= (\text{by Wald's Equation}) \frac{\lambda \mathbb{E}(N) \mathbb{E}(W_i)}{\mathbb{E}(N)} = \lambda W. \end{aligned}$$

□

**Exercise 7.** Queue, customers arrive in a renewal process with distribution  $F$ , so

$$\lambda^{-1} = \int_0^\infty \bar{F}(t) dt.$$

Each customer eventually leaves. Let  $L$  be the average number of customers in the system, averaged over all time. Let  $X(s)$  be the number of customers in the system at time  $s$ . Note that

$$L = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t X(s) ds.$$

Let  $W$  be the average time a customer spends in the system. Let  $W_i$  be the amount of time customer  $i$  spends in the system. Note that

$$W = \lim_{n \rightarrow \infty} \frac{W_1 + \dots + W_n}{n}.$$

**Solution.**

### 1.5.1 Alternating Renewal Processes (Section 7.5.1 in *Introduction to Probability Models*)

**Definition 1.22 (Alternating renewal process).** Suppose we have a process that is on for time  $Y_1$ , off for time  $Z_1$ , on for time  $Y_2$ , off for time  $Z_2$ , etc if  $(X_i, Y_i)$ ,  $i \geq 1$  are i.i.d. Call  $\{X(t), t \geq 0\}$  an **alternating renewal process**, where  $X(t)$  is an indicator variable for the process being on at time  $t$ .

**Proposition 36 (Proposition 7.4 in *Introduction to Probability Models*; similar to Theorem 37 (Theorem 3.4.4 in *Stochastic Processes*)).**

(a) The long-run proportion of time that the system is on is

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t I\{X(s) = 1\} ds = \frac{\mathbb{E}(Y)}{\mathbb{E}(Y) + \mathbb{E}(Z)}.$$

(b)

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \Pr(X(s) = 1) ds = \frac{\mathbb{E}(Y)}{\mathbb{E}(Y) + \mathbb{E}(Z)}.$$

*Proof.* (a) Imagine we earn 1 dollar per unit time when the system is on and nothing when the system is off. Note that we have a renewal when the system turns back on after having been turned off. So this is a renewal reward process.

(b) Something about

$$\frac{R(t)}{t} \rightarrow \frac{\mathbb{E}(R)}{\mathbb{E}(T)}$$



and

$$\frac{\mathbb{E}R(t)}{t} \rightarrow \frac{\mathbb{E}(R)}{\mathbb{E}(T)}$$

□

**Proposition 37** (Theorem 3.4.4 in *Stochastic Processes*; Similar to Proposition 36 (Proposition 7.4 in *Introduction to Probability Models*)). If the cycle distribution is not lattice (see Definition ??), then

$$\Pr(\text{on at } t) \rightarrow \frac{\mathbb{E}(Y)}{\mathbb{E}(Y) + \mathbb{E}(Z)} \text{ as } t \rightarrow \infty$$

where  $\mathbb{E}(Y)$  is the expected length of time intervals when the system is on and  $\mathbb{E}(Z)$  is the expected length of time intervals when the system is off.

**Example 1.22.** Suppose you have insurance and you pay at a rate  $r_1$  until you have an accident. Then the rate is  $r_2$  until  $s$  units pass without an accident, in which case you go back to paying rate  $r_1$ . Suppose accidents occur according to a Poisson process with rate  $\lambda$ . Then the probability you have to pay at rate  $r_1$  is

$$\frac{\mathbb{E}(\text{on})}{\mathbb{E}(\text{on}) + \mathbb{E}(\text{off})}.$$

where we define the process to be “on” when your pay rate is  $r_1$  and “off” when your pay rate is  $r_2$ . Note that

$$\mathbb{E}(\text{on}) = 1/\lambda$$

$$\mathbb{E}(\text{off}) = \int_0^\infty \mathbb{E}(\text{off} \mid T = x) \lambda e^{-\lambda x} dx$$

We have

$$\mathbb{E}(\text{off} \mid T = x) = \begin{cases} x + \mathbb{E}(\text{off}) & x < s \\ s & x > s \end{cases}$$

Substituting this in yields

$$\mathbb{E}(\text{off}) = \int_0^s (x + \mathbb{E}(\text{off})) \lambda e^{-\lambda x} dx + s \int_s^\infty \lambda e^{-\lambda x} dx = \int_0^s x \lambda e^{-\lambda x} dx + \mathbb{E}(\text{off})(1 - e^{-\lambda s}) + s e^{-\lambda s}.$$

Recall that the age of the renewal process at time  $t$  is  $A(t) - t - S_{N(t)}$  and the excess time at  $t$  is  $Y(t) = S_{N(t)+1} - t$ . We have already shown (in Proposition 33) that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t A(s) ds = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t Y(s) ds = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)}.$$

**Example 1.23 (Example 7.28 in *Introduction to Probability Models*).**  $M/G/\infty$  queue, (arrivals are a Poisson Process). Consider the system to be “on” if the system is empty and “off” when the system is not empty (busy period). Note that  $\mathbb{E}(\text{on}) = \lambda^{-1}$ . Let  $\mathbb{E}(B)$  be the expected length of the “off” time (busy period). Find  $\mathbb{E}(B)$ .

**Solution.** A busy time starts when a first customer arrives. The long-run proportion of time the system is on is

$$\frac{\mathbb{E}(\text{on})}{\mathbb{E}(\text{on}) + \mathbb{E}(\text{off})}.$$

By Proposition 37, since this is nonlattice, this equals the limiting probability that the system is on at time  $t$ . That is,

$$\lim_{t \rightarrow \infty} \Pr(\text{on at } t) = \frac{\mathbb{E}(\text{on})}{\mathbb{E}(\text{on}) + \mathbb{E}(\text{off})}.$$

Let  $X(t)$  be the number remaining in the system at time  $t$ . By Proposition 12,

$$X(T) \sim \text{Poisson} \left( \lambda \int_0^t \overline{G}(s) ds \right).$$

So

$$\lim_{t \rightarrow \infty} \Pr(\text{on at } t) = \Pr(X(t) = 0) = \exp \left( - \lambda \int_0^t \overline{G}(s) ds \right)$$

Plugging in we have **might have made a mistake here:**  $S \sim G$ , then  $\lim_{t \rightarrow \infty} \Pr(\text{on at } t) = \mathbb{E}(S)$

$$\exp \left( - \lambda \int_0^t \overline{G}(s) ds \right) = \frac{\lambda^{-1}}{\lambda^{-1} + \mathbb{E}(\text{off})}.$$

**think this bottom one below might be the right one**

$$\exp \left( - \mathbb{E}(S) \right) = \frac{\lambda^{-1}}{\lambda^{-1} + \mathbb{E}(\text{off})}.$$

**Definition 1.23 (Equilibrium distribution of a renewal process).** We call

$$F_e(x) = \frac{1}{\mu} \int_0^x \overline{F}(t) dt$$

the **equilibrium distribution** of a renewal process with interarrival distribution  $F$ , where  $\mu$  is the expected length of a cycle in the renewal process.

**Proposition 38** (Age of equilibrium distribution; restatement of Theorem 37 (Theorem 3.4.4 in *Stochastic Processes*); similar to Examples 7.26 and 7.27 in *Introduction to Probability Models*; Similar to Proposition 36 (Proposition 7.4 in *Introduction to Probability Models*)).  
 With probability 1, the long-run proportion of time that the age is less than  $x$  is

$$\lim_{s \rightarrow \infty} \frac{1}{s} \int_0^s I\{A(t) < x\} dt = F_e(x)$$

and the long-run proportion of the time that the excess is less than  $x$

$$\lim_{s \rightarrow \infty} \frac{1}{s} \int_0^s I\{Y(t) < x\} dt = F_e(x)$$

both equal  $F_e(x) = \frac{1}{\mu} \int_0^x \bar{F}(t) dt$ . These are also equal to

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \Pr(A(s) < x) ds = F_e(x)$$

and

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \Pr(Y(s) < x) ds = F_e(x)$$

If  $F$  is not lattice, then

$$\lim \Pr(A(t) < a) = \lim \Pr(Y(t) < a) = F_e(t).$$

*Proof.* Say the system is “on” at time  $t$  if  $A(t) < x$ , “off” otherwise. A cycle is a renewal. So the system is “on” for the first  $x$  units of a renewal cycle, and then it is “off.” Once a renewal occurs, the age goes back to 0, so this is an alternating renewal process. The “on” time in a cycle is  $\min\{x, X\}$  because it’s on for the first  $x$  units of time the item is in use of a cycle, unless the cycle is less than  $x$ , in which case it’s on for the whole length of the cycle. Therefore we have that the expected “on” time is

$$\frac{\mathbb{E}(\text{on time in cycle})}{\mu} = \frac{\mathbb{E}(\min\{x, X\})}{\mu} = F_e(x)$$

where  $\mu$  is the average length of a cycle. Note that

$$\mathbb{E}(\min\{x, X\}) = \int_0^\infty \Pr(\min\{x, X\} > t) dt = \int_0^x \bar{F}(t) dt$$

so we have that the expected “on” time is

$$\frac{1}{\mu} \int_0^x \bar{F}(t) dt = F_e(x)$$

Now we will show that the expected excess is the same as the expected age. Say that the system is “on” at  $t$  if  $Y(t) > x$  and “off” otherwise. A cycle happens every time a renewal occurs. So if a renewal cycle is longer than  $x$ , the system will be initially on and then become off for the last  $x$  time units of a renewal. So the off time is  $\min\{x, X\}$ . Again, by a similar argument as above we get the same result. (Intuitively this makes sense because the distribution of interarrival times is the same if you look backward in time as it is if you look forward in time.)

□

Recall Proposition 33: The average age and average excess both equal  $\mathbb{E}(X^2)/(2\mathbb{E}(X))$ . Note that

$$A(t) + Y(t) = S_{N(t)+1} - S_{N(t)} = X_{N(t)+1}.$$

**Proposition 39** ([Proposition 3.4.6 in *Stochastic Processes*). With probability 1, the average value of  $X_{N(t)+1}$  is

$$\frac{\mathbb{E}(X^2)}{\mathbb{E}(x)} > \mathbb{E}(X)$$

*Proof.* Suppose  $X_e \sim F_e$  where

$$F_e(x) = \frac{1}{\mu} \int_0^x \bar{F}(y) dy$$

Note that

$$\frac{dF_e(x)}{dx} = \frac{\bar{F}(x)}{\mu}$$

We have

$$\begin{aligned} \mathbb{E}(X_e) &= \int_0^\infty x dF_e(x) = \int_0^\infty x \frac{\bar{F}(x)}{\mu} dx = \int_0^\infty \frac{x}{\mu} \int_x^\infty dF(y) dx = (\text{by Fubini's Theorem}) \int_0^\infty \int_0^y \frac{x}{\mu} dx dF(y) \\ &= \int_0^\infty \frac{y^2}{2\mu} dF(y) = \frac{\mathbb{E}(X^2)}{2\mu} \end{aligned}$$

□

**Proposition 40** (Problem on homework 7; inspection paradox).

$$\Pr(X_{N(t)+1} > x) > \bar{F}(x).$$

### 1.5.2 Equilibrium renewal processes

**Definition 1.24 (Equilibrium renewal process).** Suppose we have a delayed renewal process where  $X_1 \sim F_e$ ,  $X_i \sim F \forall i > 1$ . Then  $\{N_e(t), t \geq 0\}$  is called an **equilibrium renewal process**. (the time until the first event has the equilibrium distribution.) We let  $m_e(t) = \mathbb{E}(N_e(t))$ .

**Example 1.24.** Suppose you arrive at some time  $t$  during a renewal process with finite expected renewal time  $\mu$ . Then you will have to wait  $Y(t)$  time until the next renewal. Let  $F_t$  be the distribution function of  $Y(t)$ . After that first arrival, all of the other arrival times will have distribution  $F$ . If the time  $t$  at which you start observing is very large, you observe an equilibrium renewal process (because  $F_t \rightarrow F_e$  as  $t \rightarrow \infty$ ).

**Proposition 41 (Theorem 3.5.2 from *Stochastic Processes*).** Let  $\{N_e(t), t \geq 0\}$  be an equilibrium renewal process. Let the excess at time  $t$  of  $\{N_e(t)$  be  $Y_e(t)$ .

- (1)  $Y_e(t) \sim F_e$  for all  $t$ .
- (2)  $N_e(t+s) - N_e(t)$  has the same distribution for all  $t$  (this counting process has stationary increments—see Definition 1.4).
- (3)  $m_e(t) = t/\mu$ .

*Proof (More of an intuitive argument than a complete proof).* (1) Consider an ordinary renewal process with interarrival times distributed as  $F$ . Suppose we start observing at a very large time  $t'$ , so we are observing an equilibrium renewal process. That is, the time until the first event is distributed as  $F_e$ . We'd like to know the time until the next event after you've waited a time  $t$ . We have

$$Y_e(t) = Y(t+t') \sim F_e$$

since  $t'$  is very very large.

- (2) Similar argument—after arriving after a very large amount of time, the distribution is in equilibrium, so it makes no difference when you start watching, just matters how large  $s$  is.
- (3)

$$N_e(t+s) = N_e(t+s) - N_e(t) + N_e(t)$$

$$\iff \mathbb{E}(N_e(t+s)) = \mathbb{E}[N_e(t+s) - N_e(t)] + \mathbb{E}[N_e(t)] \iff m_e(t+s) = m_e(s) + m_e(t)$$

where  $\mathbb{E}(N_e(t+s)) = m_e(t+s)$  by definition,  $\mathbb{E}[N_e(t+s) - N_e(t)] = m_e(s)$  by the result from part (2), and  $\mathbb{E}[N_e(t)] = m_e(t)$  by definition. It is the case that  $f(x+y) = f(x) + f(y) \implies f(x) = cx$  for some constant  $c$  if  $f$  is measurable. Because of that, we have that

$$m_e(t) = ct$$

where  $c$  is some constant. By the Elementary Renewal Theorem (Theorem 28),

$$\frac{m_e(t)}{t} \rightarrow \frac{1}{\mu}$$

which implies that  $c$  is  $\mu^{-1}$ .

□

Proving that the previous statement is true for any measurable function requires measure theory, but it is easy to prove it is true if  $f$  is differentiable.

**Lemma 42.** If  $f$  is differentiable,  $f(x+h) = f(x) + f'(x)h + o(h)$ .

*Proof.* Note that

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

Note that

$$\lim_{h \rightarrow 0} \frac{f(h)}{h} = f'(0)$$

Then  $f'(x) = c$ ,  $f(x) = cx + k$ , so  $f(x) = cx$ .

□

**Theorem 43 (Blackwell's Theorem (Theorem 3.4.1 in *Stochastic Processes*, p.110)).** (i) If  $F$  is not lattice, then

$$m(t+a) - m(t) \rightarrow \frac{a}{\mu} \text{ as } t \rightarrow \infty$$

(ii) If  $F$  is lattice with period  $d$ , then

$$\mathbb{E}(\text{number of renewals at } nd) \rightarrow \frac{d}{\mu} \text{ as } n \rightarrow \infty.$$

**Remark.** If  $\Pr(X_i = 0) = 0$  then then this means that  $\Pr(\text{renewal at } nd) \rightarrow d/\mu$  as  $n \rightarrow \infty$ .

*Proof (intuitive; rigorous proof is quite technical).* (i) If you arrive at time  $t$ , then  $\{N(s), s \geq 0\}$  is a delayed renewal process. That is,  $X_1$  has the distribution of  $Y(t) \sim F_t$ , and the remaining  $X_i \sim F$ . So

$$m(t+a) - m(t) = \mathbb{E}[N_t(a)].$$

We already know that as  $t \rightarrow \infty$ ,  $F_t \rightarrow F_e$ . So as  $t \rightarrow \infty$ ,  $m(t+a) - m(t) = \mathbb{E}[N_t(a)] \rightarrow \mathbb{E}[N_e(a)]$ . But by Proposition 41, **the result follows (??)**.

(ii) Suppose we have a renewal process where

$$\sum_{i=0}^{\infty} \Pr(X = i) = 1.$$

Then  $m(n)$  is the expected number of renewals by time  $n$ . That is,

$$m(n) = \mathbb{E} \sum_{i=0}^n (\text{number of renewals at } i) = \sum_{i=0}^n \mathbb{E}(\text{number of renewals at } i)$$

Then by the Elementary Renewal Theorem (Theorem 28),

$$\frac{m(n)}{n} = \frac{1}{n} \sum_{i=0}^n \mathbb{E}(\text{number of renewals at } i) \rightarrow \frac{1}{\mu}$$

In general, this does not imply that

$$\mathbb{E}(\text{number of renewals at } nd) \rightarrow \frac{d}{\mu} \text{ as } n \rightarrow \infty.$$

For example, consider the discrete random variable

$$X_i = \begin{cases} 2 & \text{with probability } 1/2 \\ 4 & \text{with probability } 1/2 \end{cases}$$

Then

$$\mathbb{E}(\text{number of renewals at } n) = m(n) = \frac{1}{n} \sum_{i=1}^n \mathbb{E}(\text{number of renewals at } i) \rightarrow \frac{1}{\mu}$$

But of course all of these sums are 0 when  $i$  is odd. So we could also write this as

$$\begin{aligned} m(n) &= \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{1}{n} \mathbb{E}(\text{number of renewals at } 2i) \rightarrow \frac{1}{\mu} \\ \iff m(n) &= \frac{1}{2} \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{1}{n/2} \mathbb{E}(\text{number of renewals at } 2i) \rightarrow \frac{1}{\mu} \\ \iff m(n) &= \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{1}{n/2} \mathbb{E}(\text{number of renewals at } 2i) \rightarrow \frac{2}{\mu} \\ &\vdots \end{aligned}$$

You get pointwise convergence in the lattice case with period  $d$  when  $d$  is the period.

□

Recall Proposition 33: The average age and average excess both equal  $\mathbb{E}(X^2)/(2\mathbb{E}(X_e))$ , where  $X_e \sim F_e$ .

**Theorem 44 (Proposition 3.4.6 in *Stochastic Processes*).** If  $F$  is not lattice,

$$\lim_{t \rightarrow \infty} \mathbb{E}(A(t)) = \lim_{t \rightarrow \infty} \mathbb{E}(Y(t)) = \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)}.$$

*Proof.* Won't prove; requires Key Renewal Theorem, which is beyond scope of this course. Some notes: we know that  $F_{Y(t)} \rightarrow F_e$ , or  $Y(t) \rightarrow Y(\infty)$  where  $Y(\infty) \sim F_e$ . Then

$$\lim \mathbb{E}(Y(t)) = (? \text{ can't justify now}) \mathbb{E}(\lim(Y(t))) = \mathbb{E}(X_\infty)$$

□

**Corollary 45** (Corollary 3.4.7 in *Stochastic Processes*, p. 121). If  $\mathbb{E}(X^2) < \infty$  and  $F$  is nonlattice, then

$$m(t) - \frac{t}{\mu} \rightarrow \frac{\mathbb{E}(X^2)}{2\mu^2} - 1 \text{ as } t \rightarrow \infty$$

*Proof.* Recall that by Wald's Equation (Theorem 26),

$$\mathbb{E}\left(\sum_{i=1}^{N(t)+1} X_i\right) = \mu(m(t) + 1)$$

So

$$\mu(m(t) + 1) = \mathbb{E}(t + Y(t)) = t + \mathbb{E}(Y(t)) \iff m(t) + 1 = \frac{t}{\mu} + \frac{\mathbb{E}(Y(t))}{\mu} \iff m(t) - \frac{t}{\mu} = \frac{\mathbb{E}(Y(t))}{\mu} - 1$$

$$\implies m(t) - \frac{t}{\mu} \rightarrow \frac{\mathbb{E}(X^2)}{2\mu^2} - 1 \text{ as } t \rightarrow \infty$$

$$\implies m(t) \approx \frac{t}{\mu} + \frac{\mathbb{E}(X^2)}{2\mu^2} - 1$$

□

**Theorem 46** (Central Limit Theorem for Renewal Processes; Theorem 7.3 in *Introduction to Probability Models*). Let  $N(t)$  be a renewal process. Then

$$N(t) \xrightarrow{d} \mathcal{N}\left(\frac{t}{\mu}, \frac{t\sigma^2}{\mu^3}\right)$$

where  $\mu = \mathbb{E}(X_i)$  and  $\sigma^2 = \text{Var}(X_i)$ .

**Remark.** If the renewal process is a Poisson process with rate  $\lambda$ , then  $F(x) = 1 - e^{-\lambda x}$ . Then  $\mu = \mathbb{E}(X) = \lambda^{-1}$ ,  $\sigma^2 = \text{Var}(X) = \lambda^{-2}$ ,  $t/\mu = \lambda t$ ,  $t\sigma^2/\mu^2 = t\lambda^3/\lambda^2 = \lambda t$ , which implies the limiting distribution is  $\mathcal{N}(\lambda t, \lambda t)$ , as expected.

*Proof.* Note that  $N(t) < n$  if and only if the  $n$ th event occurred after  $t$ , or  $X_1 + \dots + X_n > t$ . So

$$\Pr(N(t) < n) = \Pr(X_1 + \dots + X_n > t)$$

Then by the Central Limit Theorem,

$$X_1 + \dots + X_n \sim \mathcal{N}(n\mu, n\sigma^2) \text{ (approximately)}$$



$$\vdots$$

□

### 1.5.3 Regenerative Processes

**Theorem 47 (Theorem 3.7.1 in *Stochastic Processes*).** Let  $\{X(t), t \geq 0\}$  be a stochastic process with state space  $\{0, 1, 2, \dots\}$  having the property that there exist time points at which the state restarts itself (with probability 1). Let  $S_1, S_2, \dots$  constitute the event times of a renewal process. We call  $X(t)$  a **regenerative process**. So  $\{S_1, S_2, \dots\}$  constitute the event times of a renewal process. We say a cycle is completed every time a renewal occurs. Let  $N(t) = \max\{n : S_n \leq t\}$  denote the number of cycles by time  $t$ .

If  $F$ , the distribution of a cycle, has a density over some interval, and if  $\mathbb{E}(S_1) < \infty$ , then

$$P_j = \lim_{t \rightarrow \infty} \Pr(X(t) = j) = \frac{\mathbb{E}(\text{amount of time in state } j \text{ in a cycle})}{\mathbb{E}(\text{time of a cycle})}.$$

### 1.5.4 Example of Renewal Reward Processes to Patterns

**Example 1.25 (Seems to be similar to Section 7.9.2 in *Introduction to Probability Models*).** Suppose we have i.i.d. data. Number  $i$  appears with probability  $p_i$ . We want to see the mean time until we see the pattern 1213121. An event is when the pattern appears. The next event is the next time the pattern appears, but we can't use data from the last event. Earn \$1 whenever the last 7 values are 1313121 (so you *are* allowed to use previous values for the reward process). That is, if the cycle is

$$\dots 1213121 \mid 312\mathbf{1} \dots 1213121 \mid$$

the events happen at the  $\mid$  marks, but in between there were two rewards (one at the second event, one at the bold 1). So this is a delayed renewal reward process  $(X_i, R_i), i \geq 2$ . Note that the limiting results will be the same for the delayed renewal process as for the normal one. So the expected average reward per unit time is the expected reward during the cycle divided by the expected time of a cycle;

$$\mathbb{E}(\text{average reward per unit time}) = \frac{\mathbb{E}(\text{reward during the cycle})}{\mathbb{E}(T)}$$

We want  $\mathbb{E}(T)$ , the expected length of a cycle. We know  $\mathbb{E}(\text{average reward per unit time})$  is

$$\mathbb{E}(\text{average reward per unit time}) = p_1^4 p_2^2 p_3$$

Now we want  $\mathbb{E}(\text{reward during the cycle})$ . Suppose a cycle just ended (the pattern just appeared). Then we could earn a reward after 4 values (if the last 4 terms of the pattern appear again) after 6 values (if the

last 6 terms of the pattern appear again). And we will earn a reward when the cycle ends (after the full pattern appears again). So if  $Y_i$  is the reward earned during after  $i$  new values,

$$\text{reward during the cycle} = Y_4 + Y_6 + 1 \implies \mathbb{E}(\text{reward during the cycle}) = \mathbb{E}(Y_4) + \mathbb{E}(Y_6) + 1 = p_1^2 p_2 p_3 + p_1^3 p_2^2 p_3 + 1$$

$$\implies \mathbb{E}(T) = \frac{p_1^2 p_2 p_3 + p_1^3 p_2^2 p_3 + 1}{p_1^4 p_2^2 p_3} = \frac{1}{p_1^2 p_2} + \frac{1}{p_1} + \frac{1}{p_1^4 p_2^2 p_3}$$

Note that the mean time between renewals is  $\frac{1}{p_1^4 p_2^2 p_3}$ , the mean time to get 121 is  $\frac{1}{p_1^2 p_2}$ , and the mean time to get 1 is  $\frac{1}{p_1}$ .

**Example 1.26 (Similar to example 7.38 in *Introduction to Probability Models*).** Suppose we are flipping coins, probability of flipping heads is  $p$ . The pattern is  $n$  heads in a row. Cycle is a pattern without using data from last cycle. Get a reward of 1 each time the last  $n$  values were all heads. Then

$$p^n = \mathbb{E}(\text{average reward per unit time}) = \frac{\mathbb{E}(\text{reward during the cycle})}{\mathbb{E}(T)}$$

Note that

$$\text{reward during the cycle} = Y_1 + \dots + Y_{n-1} + 1 \implies \mathbb{E}(\text{reward during the cycle}) = \sum_{i=1}^{n-1} \mathbb{E}(Y_i) + 1$$

$$= \sum_{i=1}^{n-1} p^i + 1$$

$$\implies p^n = \frac{1}{\mathbb{E}(T)} \left( \sum_{i=1}^{n-1} p^i + 1 \right) \implies \mathbb{E}(T) = p^{-n} \left( \sum_{i=1}^{n-1} p^i + 1 \right) = \left( \frac{1}{p} \right)^n + \left( \frac{1}{p} \right)^{n-1} + \dots + \frac{1}{p}$$

## 1.6 Markov Chains (Chapter 4 of *Stochastic Processes*; Chapter 4 of *Introduction to Probability Models*)

Suppose we have  $X_0, X_1, X_2, \dots$ , where  $X_n$  is the state of the system at time  $n$ . The set of possible values of states are the nonnegative integers (the number of possible states will be either finite or countable). In a Markov chain, the probability of reaching state  $j$  given that you are currently in state  $i$  is  $P_{ij}$ ; that is,

$$\Pr(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = P_{ij}$$

That is, the next state depends only on the current state, not any of the previous states (all the information is contained in the current state). In other words, every starts over again when you reach state  $i$ ; so there is a renewal every time you reach state  $i$ . We say a system has the **Markovian property** if given the present state  $X_n$ , the next state  $X_{n+1}$  is independent of the past  $X_{n-1}, X_{n-2}, \dots$

**Definition 1.25 (Markov chain).** Let  $S$  be the set of possible values of a system.  $S$  must be either finite or countably infinite (in general we will take  $S = \mathbb{N}$ ). Then  $\{X_0, X_1, \dots\}$  is said to be a **Markov chain** with transition probabilities  $\Pr(X_{n+1} = j \mid X_n = i)P_{i,j}$ ,  $i, j \in S$  if

$$\Pr(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = \Pr(X_{n+1} = j \mid X_n = i) = P_{i,j}.$$

We call the matrix

$$P = [P_{i,j}]$$

the **transition probability matrix**. Note that  $\sum_j P_{i,j} = 1$ , so the rows of the transition probability matrix all add to 1.

**Remark.** Given the initial state  $\Pr(X_0 = i_0)$  and the probability transition matrix, all probabilities about  $X_0, X_1, \dots$  can (in theory) be determined. That is,

Note that

$$\Pr(X_n = i_n) = \sum_{i_0, i_1, \dots, i_{n-1}} \Pr(X_0 = i_0)P_{i_0, i_1}P_{i_1, i_2} \dots P_{i_{n-1}, i_n}.$$

**Example 1.27 (Examples 4.1 and 4.4 in *Introduction to Probability Models*).** Suppose we have a state based on weather today and yesterday. Then probability of rain tomorrow is the following:

$$\begin{cases} dd \rightarrow r & \text{with probability 0.3} \\ rd \rightarrow r & \text{with probability 0.4} \\ dr \rightarrow r & \text{with probability 0.6} \\ rr \rightarrow r & \text{with probability 0.7} \end{cases}$$

The the transition matrix looks as follows:

$$P = \begin{bmatrix} & dd & rd & dr & rr \\ dd & 0.7 & 0 & 0.3 & 0 \\ rd & 0.6 & 0 & 0.4 & 0 \\ dr & 0 & 0.4 & 0 & 0.6 \\ rr & 0 & 0.3 & 0 & 0.7 \end{bmatrix}$$

**Example 1.28 (Random walk; examples 4.5 and 4.6 in *Introduction to Probability Models*).**

$$P_{i,i+1} = p, P_{i,i-1} = 1 - p, \quad S = \mathbb{Z}$$

One example is Gambler's Ruin:  $X_0 = i$ , stop when fortune is either 0 or  $N$ . (See Example 1.41 and several solutions in Grimmett and Stirzaker.)

**Example 1.29 (Similar to example 4.10 in *Introduction to Probability Models*, similar to homework problem from Math 505A).** Urn with 2 balls, red and blue. Each period, choose ball from urn at random. If red then replace with blue. If blue then replace with red with probability 0.7, with blue with probability 0.3. What is the long-run proportion of time the chosen ball is red?

**Solution.** Let  $X_n$  be an indicator variable for the  $n$ th ball chosen to be red. Note that if you just chose a red ball, it is not clear what the probability of choosing a red ball next is, because it depends on what color the other ball in the urn is. So consider the state of the Markov chain to be the number of red balls in the urn, and let  $X_n$  be the number of red balls in the urn before the  $n$ th withdrawal. Then we have the following transition matrix:

$$P = \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.5 & 0.5 \cdot 0.3 & 0.5 \cdot 0.7 \\ 0 & 1 & 0 \end{bmatrix}$$

Let  $\pi_j$  be the long-run proportion of the time the system is in state  $j$ . Note that every time we reach state  $j$  everything starts all over again, so that's a renewal (ordinary renewal process) if you are certain to eventually return to state  $j$  (we will assume this is true for now). We have

$$\pi_0 \cdot 0 + \pi_1 \cdot \frac{1}{2} + \pi_2 = \pi_2 + \frac{1}{2}\pi_1$$

is the long-run probability of choosing a red ball.

**Example 1.30 (Embedded Markov chain; example 4.1(a) in *Stochastic Processes*).**  $M/G/1$  queueing process. The number of people currently in the system is not Markovian, but it would be if the service time were exponential (since then it wouldn't matter how long the customer had been in service). But the number of people in the system immediately after the  $n$ th service completion is Markovian. (Called an **embedded Markov chain** because it is a Markov chain if you only look at it at certain times.)

Note that

$$\begin{aligned} P_{0j} &= a_j = \Pr(j \text{ arrivals during service time}) = \int_0^\infty \Pr(j \text{ arrivals during service time } t) dG(t) \\ &= \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^j}{j!} dG(t) \end{aligned}$$

More generally, note that for  $i > 0, j \geq i - 1$

$$\begin{aligned} P_{ij} &= a_{j-i+1} = \Pr(j-i+1 \text{ arrivals during service time}) = \int_0^\infty \Pr(j-i+1 \text{ arrivals during service time } t) dG(t) \\ &= \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^{j-i+1}}{(j-i+1)!} dG(t). \end{aligned}$$

**Example 1.31 (Example 4.1(b) in *Stochastic Processes*).** if  $X_n$  is the number of people in the system as seen by the  $n$ th arrival, then this is an (embedded) Markov chain.

⋮

We want  $P_{ij}, j > 0$ . If  $n$  is the number of people served, then

$$i + 1 - n = j \iff n = i + 1 - j$$

Then

$$P_{ij} = \int \Pr(i + 1 - j \text{ services in time } T \mid T = s) dF(s) = \int e^{-\mu s} \frac{(\mu s)^{i+1-j}}{(1 + 1 - j)!} dF(s)$$

$P_{i0}$  is different because if we end with 0 people, all of the previous people had to depart. But they could have departed at any time before the next person arrived. Imagine that when all the customers leave, the server keeps serving “imaginary” customers. Then  $P_{i0}$  is the probability of at least  $i + 1$  customers (real or imaginary) being served.

**Definition 1.26** ( $n$ -step transition probabilities).

$$P_{ij}^n = \Pr(X_n = j \mid X_0 = i)$$

**1.6.1 Chapman-Kolmogorov Equations**—section 4.2 of *Introduction to Probability Models*, section 4.2 of *Stochastic Processes* (p. 178 of pdf)

**Proposition 48** (Chapman-Kolmogorov Equations).

**Definition 1.27** (Matrix of  $n$ -step transition probabilities).

$$P^{(n)} = [P_{ij}^n]$$

**Example 1.32** (Similar to example 4.10 in *Introduction to Probability Models*). Urn initially has 1 red ball 1 blue. At each stage choose a ball at random. If red, replace with blue. If blue, replace with blue with probability 1/3 or red with probability 2/3. What is the probability that the 5th ball selected is red?

**Solution.** Note that per the example from last time, knowing the ball you just drew is not sufficient to have a Markov chain (need to know color of other ball in urn to get probabilities of next state). So let  $X_n$  be the number of red balls in the urn after you draw the  $n$ th ball, and note that  $X_0 = 1$ . Note that there are three states. The probability transition matrix is

$$P = \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 \end{bmatrix}$$

Note that the probability the 5th ball selected is red is equal to

$$\sum_{i=1}^2 \Pr(5th \text{ ball is red} \mid X_4 = i) P_{1i}^4 = 0 \cdot P_{10}^4 + \frac{1}{2} P_{11}^4 + 1 \cdot P_{12}^4$$

Now we calculate  $P^4$ .

$$P^2 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{9} & \frac{3}{9} & \frac{2}{9} \\ \frac{3}{12} & \frac{25}{36} & \frac{2}{36} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{bmatrix}$$

$$P^4 = P^2 P^2 = \begin{bmatrix} \frac{4}{9} & \frac{3}{9} & \frac{2}{9} \\ \frac{3}{12} & \frac{25}{36} & \frac{2}{36} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{4}{9} & \frac{3}{9} & \frac{2}{9} \\ \frac{3}{12} & \frac{25}{36} & \frac{2}{36} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{bmatrix} = \dots$$

**Example 1.33** (Example 4.12 from *Introduction to Probability Models*). Part (c) (not in book):  $N$  is the number of flips until either a run of 3 heads or 3 tails.  $\Pr(N = 8)$ ?

**Solution.** Say start at state 0, state  $i$  if you are on a run of  $i$  heads for  $i = 1, 2, 3$ , and state  $i$  if you are on a state of  $i - 3$  tails,  $i = 4, 5, 6$ . Then we have

$$P = \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} & P_{04} & P_{05} & P_{06} \\ P_{10} & P_{11} & P_{12} & P_{13} & P_{14} & P_{15} & P_{16} \\ P_{20} & P_{21} & P_{22} & P_{23} & P_{24} & P_{25} & P_{26} \\ P_{30} & P_{31} & P_{32} & P_{33} & P_{34} & P_{35} & P_{36} \\ P_{40} & P_{41} & P_{42} & P_{43} & P_{44} & P_{45} & P_{46} \\ P_{50} & P_{51} & P_{52} & P_{53} & P_{54} & P_{55} & P_{56} \\ P_{60} & P_{61} & P_{62} & P_{63} & P_{64} & P_{65} & P_{66} \end{bmatrix} = \begin{bmatrix} 0 & p & 0 & 0 & 1-p & 0 & 0 \\ 0 & 0 & p & 0 & 1-p & 0 & 0 \\ 0 & 0 & 0 & p & 1-p & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 1-p & 0 \\ 0 & p & 0 & 0 & 0 & 0 & 1-p \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

In general, if  $T_i$  is the number of transitions to enter state  $i$  even state 0 is 0, we can change state  $i$  into an absorbing state.

### 1.6.2 Classification of States (Section 4.3 of *Introduction to Probability Models*)

We say state  $j$  is *accessible* from state  $i$  if for some  $n \geq 0$   $P_{ij}^n > 0$ . Note that  $P_{ij}^n = \Pr(X_n = j \mid X_0 = i)$ . So  $j$  is accessible from  $i$  if and only if starting in  $i$  it is possible that the Markov chain is every in  $j$ . So if  $j$  is not accessible from  $i$

$$\Pr(\text{ever in } j \mid X_0 = i) = \Pr\left(\bigcup_n \{X_n = j\} \mid X_0 = i\right) \leq \sum_n \Pr(\{X_n = j\} \mid X_0 = i) = 0$$

Two states  $i$  and  $j$  that are accessible to each other are said to *communicate*, and we write  $i \leftrightarrow j$ . Two states that communicate are said to be in the same *class*.

A Markov chain is **irreducible** if there is only one class.

**Example 1.34.**

$$P = \begin{bmatrix} 0.2 & 0.8 & 0 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 & 0 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0.1 & 0.9 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

Then the classes are  $\{0, 1\}$ ,  $\{2\}$ , and  $\{3, 4\}$ . Note that for the classes  $\{0, 1\}$  and  $\{3, 4\}$  you will stay there forever—they are essentially Markov chains themselves. But if you start in state 2 you will eventually go to one of the other states and never go back. In contrast consider

$$P = \begin{bmatrix} 0.2 & 0.8 & 0 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0.9 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

where all three states have this property. We can merge classes  $\{2\}$  and  $\{3, 4\}$  like this:

$$P = \begin{bmatrix} 0.2 & 0.8 & 0 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 & 0 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0.1 & 0.9 \\ 0 & 0 & 0.1 & 0.5 & 0.4 \end{bmatrix}$$

Let  $N_k$  be the number of transitions until you first enter state  $k$ . (discussion on p. 195 - 196 of *Introduction to Probability Models*.)

**Example 1.35** (Example 4.12 in *Introduction to Probability Models*).

$$\Pr(N = 8) = P_{02}^7 p = P_{00}^5 p^3$$

**Definition 1.28.** For any states  $i$  and  $j$  in a Markov process, let  $f_{ij}^n$  be the probability that starting in  $i$  the first transition into  $j$  occurs at time  $n$ . Formally,

$$f_{ij}^0 = 0, \quad f_{ij}^n = \Pr(X_n = j, X_k \neq j, k = 1, \dots, n-1 \mid X_0 = i).$$

Let

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^n.$$

Then  $f_{ij}$  denotes the probability of ever making a transition into state  $j$  given that the process starts in  $i$ .

**Proposition 49** (Homework problem; Problem 4.4 in *Stochastic Processes*).

$$P_{ij}^n = \sum_{k=0}^n f_{ij}^k P_{jj}^{n-k}.$$

*Proof.* Recall that  $P_{ij}^n$  is the probability that a process in state  $i$  will be in state  $j$  after  $n$  additional transitions. Also,  $f_{ij}^n$  is the probability that starting in  $I$  the first transition into  $j$  occurs at time  $n$ . Note that

$$P_{ij}^n = \sum_{k=0}^n \Pr(\text{first reach } j \text{ at time } k) \cdot \Pr(\text{end at } j \text{ starting from } j \text{ in } n-k \text{ steps}) = \sum_{k=0}^n f_{ij}^k P_{jj}^{n-k}.$$

□

**Proposition 50 (Homework problem; Problem 4.5 in *Stochastic Processes*).** Let

$$P_{ij/k} = \Pr(X_n = j, X_\ell \neq k, \ell = 1, \dots, n-1 \mid X_0 = i).$$

(This is the probability that at time  $n$  we are in state  $j$  and we never reached state  $k$  between time 0 and  $n$ , given that we started in state  $i$ .) Then for  $i \neq j$ ,

$$P_{ij}^n = \sum_{k=0}^n P_{ii}^k P_{ij/i}^{n-k}.$$

*Proof.* There are two classes of ways we can reach  $j$  starting from  $i$ : we either go back to  $i$  at some point and then end up at  $j$ , or we go straight to  $j$  without ever going back to  $i$ . If we go back to  $i$  for the last time at time  $k \in \{1, \dots, n-1\}$ , we can end up at  $j$  at time  $n$  with probability  $P_{ij/i}^{n-k}$ . We end up back at state  $i$  for the last time at time  $k$  before time  $n$  (regardless of what happened before time  $k$ ) with probability  $P_{ii}^k$ . Therefore,

$$\text{for } i \neq j, P_{ij}^n = \sum_{k=0}^n P_{ii}^k P_{ij/i}^{n-k}.$$

□

**Definition 1.29 (Recurrent state).** We call state  $i$  **recurrent** if starting in  $i$  the Markov Chain returns to  $i$  with probability 1.

**Definition 1.30 (Transient state).** We call state  $i$  **transient** if starting in  $i$  the probability the Markov Chain returns to  $i$  is less than 1.

**Proposition 51 (Proposition 4.1 in *Introduction to Probability Models*, Proposition 4.2.3 in *Stochastic Processes*).** In a Markov chain, state  $i$  is recurrent if

$$\sum_{n=1}^{\infty} P_{ii}^n = \infty$$

and transient if

$$\sum_{n=1}^{\infty} P_{ii}^n < \infty.$$

*Proof.* State  $i$  is recurrent if with probability 1 a process starting at  $i$  will eventually return to  $i$ . However, by the Markovian property it follows that this process will return again to  $i$  with probability 1, and so on. There is no last term, so with probability 1, the number of visits to  $i$  will be infinite.



On the other hand, suppose  $i$  is transient. Then each time the process returns to  $i$  there is a positive probability  $1 - f_{ii}$  that it will never again return. That is, the probability the total time in  $i$  is  $n$  periods given you start in  $i$  is

$$f_{ii}^{n_i-1}(1 - f_{ii}).$$

Therefore the number of visits is geometric with finite mean  $(1 - f_{ii})^{-1}$ . Let  $I_n$  be an indicator variable for  $X_n = i$ . Then the total number of visits to state  $i$  is  $\sum_{n=0}^{\infty} I_n$  and the expected number of visits if we start at  $i$  is

$$\mathbb{E} \sum_{n=0}^{\infty} \{I_n = 1 \mid X_0 = i\} = \sum_{n=0}^{\infty} \mathbb{E}\{I_n = 1 \mid X_0 = i\} = \sum_{n=0}^{\infty} \Pr\{X_n = i \mid X_0 = i\}$$

□

**Corollary 52** (Corollary 4.2 in *Introduction to Probability Models*, Corollary 4.2.4 in *Stochastic Processes*). If state  $i$  is recurrent and state  $i$  communicates with state  $j$ , then state  $j$  is recurrent.

*Proof.* Let  $m$  and  $n$  be such that  $P_{ij}^n < 0$ ,  $P_{ji}^m > 0$  (these values exist since  $i$  and  $j$  communicate). Then for any  $s \geq 0$ ,

$$P_{jj}^{m+n+s} \geq P_{ji}^m P_{ii}^s P_{ij}^n.$$

Therefore

$$\sum_{s=0}^{\infty} P_{jj}^{m+n+s} \geq P_{ji}^m P_{ij}^n \sum_{s=0}^{\infty} P_{ii}^s = \infty \implies \sum_{n=0}^{\infty} P_{jj}^n = \infty$$

where  $\sum_{s=0}^{\infty} P_{ii}^s = \infty$  by definition of  $i$  being recurrent.

□

**Corollary 53** (Corollary 4.2.5 in *Stochastic Processes*). If state  $i$  is recurrent and state  $i$  communicates with state  $j$ , then

$$f_{ij} = \Pr(\text{ever enter } j \mid X_0 = i) = 1$$

*Proof.* Suppose  $X_0 = i$ , and let  $n$  be such that  $P_{ij}^n > 0$ . Say that we miss opportunity 1 if  $X_n \neq j$ . If we miss opportunity 1, then let  $T_1$  denote the next time we enter  $i$ . Note that  $T_1$  is finite with probability 1 by Corollary 52. Say we miss opportunity 2 if  $X_{T_1+n} \neq j$ . If opportunity 2 is missed, let  $T_2$  denote the next time we enter  $i$  and say we miss opportunity 3 if  $X_{T_2+n} \neq j$ , and so on. It is easy to see that the opportunity number of the first success is a geometric random variable with mean  $1/P_{ij}^n$ , and is thus finite with probability 1. The result follows since  $i$  being recurrent implies that the number of potential opportunities is infinite.

□

So we have the recurrence and transience are class properties (if one element in the class has them, they all do).

**Example 1.36** (Simple random walk; example 4.18 in *Introduction to Probability Models*).

$$\frac{(2n)!}{n!n!} \frac{(2n)^{2n+1/2} e^{-2n} \sqrt{2\pi}}{n^{2n+1} e^{-2n} 2\pi} = \frac{4^n}{\sqrt{n\pi}}$$

Homework problem: If  $R$  is a recurrent class,  $i \in R$ ,  $j \notin R$ , then  $P_{ij} = 0$ . Why? Suppose  $P_{ij} > 0$ . But since  $i$  doesn't communicate with  $j$ ,  $P_{ij}^n = 0$  for all  $n$ . Contradiction.

### 1.6.3 Long-Run Proportions and Limiting Probabilities (Limit Theorems) (4.4 in *Introduction to Probability Models*, 4.3 in *Stochastic Processes*)

Let  $\pi_j$  be the long-run proportion of the time you are in state  $j$ . That is,

$$\pi_j = \lim_{n \rightarrow \infty} \frac{N_n(j)}{n}$$

where  $N_n(j)$  is the time in  $j$  during the first  $n$  periods. Then this equals  $1/m_j$  where  $m_j$  is the expected time to re-enter  $j$  given that we started in  $j$  if state  $j$  is recurrent.

Bad approach to find:

$$m_j = \sum_k \mathbb{E}[\text{return to } j \mid X_1 = k] P_{jk} = P_{jj} + \sum_{k \neq j} (1 + m_{kj}) P_{jk} = 1 + \sum_{k \neq j} m_{kj} P_{jk}$$

$$m_{ij} = 1 + \sum_{k \neq j} P_{ik} m_{kj}$$

for  $|S|$  states.

**Proposition 54** (Proposition 4.4 in *Introduction to Probability Models*). If the Markov chain is irreducible and recurrent, then for any initial state,

$$\pi_j = \frac{1}{m_j}.$$

**Proposition 55** (Proposition 4.5 in *Introduction to Probability Models*).

**Theorem 56** (Theorem 4.1 in *Introduction to Probability Models*). In an irreducible Markov chain is positive recurrent, then the long-run proportions are the unique solution of the equations

$$\pi_j = \sum_i \pi_i P_{ij}, \quad j \geq 1$$

$$\sum_j \pi_j = 1.$$

Moreover, if there is no solution of the preceding linear equations, then the Markov chain is either transient or null recurrent and all  $\pi_j = 0$ .

*Proof.*  $\pi_i$  is the long-run proportion of time you are in state  $i$ ; think of it as the long-run proportion of transitions that come from  $i$ . These transitions go to  $j$  with probability  $P_{ij}$ . So  $\pi_i P_{ij}$  is the long-run proportion of transitions that go from  $i$  to  $j$ . If we sum over all  $i$  then we get the long-run proportion of transitions that go into  $j$ , which is the same as the proportion of time you're in  $j$ . So

$$\pi_j = \sum_i \pi_i P_{ij}$$

□

For an irreducible Markov chain, let  $N_n(j)$  be the number of transitions into  $j$  by time  $n$ . Every time we reach state  $j$  we have a renewal. By the Strong Law for Renewal Processes (Theorem 24),

$$\frac{N_n(t)}{n} \xrightarrow{a.s.} \frac{1}{m_{jj}}$$

where  $m_{jj}$  is the expected number of transitions returning to  $j$  given that  $X_0 = j$ . By the Elementary Renewal Theorem (Theorem 28),

$$\lim_{s \rightarrow \infty} \frac{\mathbb{E}(N_n(s))}{n} = \frac{1}{m_{jj}}.$$

Note that

$$N_n(j) = \sum_{k=1}^n I_k$$

where  $I_k$  is an indicator variable for  $X_k = j$ .

$$\implies \frac{1}{n} \sum_{k=1}^n \Pr(X_k = j) \xrightarrow{a.s.} \frac{1}{m_{jj}}.$$

Let  $\pi_j = \frac{1}{m_{jj}}$ .

**Definition 1.31.** If state  $j$  is recurrent, call it **positive recurrent** if  $m_{jj} < \infty$  and **null recurrent** if  $m_{jj} = \infty$ .

**Remark.** Being positive recurrent is equivalent to  $\pi_j > 0$ .

**Proposition 57 (Proposition 4.3.2 in *Stochastic Processes* (p.185 of pdf)).** Positive (null) recurrence is a class property.

*Proof.* Suppose state  $i$  is positive recurrent; that is  $\pi_i > 0$ . We will show that if it communicates with another state  $j$ , that state must be recurrent. Let  $n$  be such that  $P_{ij}^n > 0$ ; such an  $n$  exists since  $i$  and  $j$  communicate.. Then  $\pi_i P_{ij}^n$  is the long-run proportion of the time the chain is in state  $i$  and will be in state  $j$   $n$  time periods later (or the proportion of time that the Markov chain is in state  $j$  and was in state  $i$   $n$  time periods earlier). Since this is less than or equal to the proportion of time the Markov chain is in state  $j$ , we have

$$\pi_i P_{ij}^n \leq \pi_j$$

But since  $\pi_i > 0$ , we have

$$0 < \pi_i P_{ij}^n \leq \pi_j \implies \pi_j > 0.$$

□

**Remark.** This also shows that null recurrence is a class property (because if one element of the class is not positive recurrent, it must be that they are all not positive recurrent).

Note that if we have a Markov chain with  $m < \infty$  states. we have to have

$$\begin{aligned} \sum_{j=1}^m N_n(0) = n &\iff \frac{1}{n} \sum_{j=1}^m N_n(0) = 1 \\ \implies 1 &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^m N_n(0) = \sum_{j=1}^m \lim_{n \rightarrow \infty} \frac{N_n(0)}{n} = \sum_{j=1}^m \pi_j \end{aligned}$$

so all the  $\pi_j$  have to add up to 1.

**Definition 1.32.** The non-negative numbers  $x_j, j \in S$  are said to be **stationary probabilities** of a Markov chain if

$$x_j = \sum_i x_i P_{ij}, \quad i \in S$$

$$\sum_{j \in S} x_j = 1.$$

**Definition 1.33** (p. 185 of *Stochastic Processes pdf*). A probability distribution  $\{P_j, j \geq 0\}$  is said to be **stationary** for the Markov chain if

$$P_j = \sum_{i=0}^{\infty} P_i P_{ij}, \quad j \geq 0.$$

**Lemma 58.** Suppose  $\{x_i, i \in S\}$  is a stationary probability vector for a Markov chain. If  $\Pr(X_0 = i) = x_i, i \in S$   $\Pr(X_n = j) = x_j, j \in S$  for all  $n$ .

*Proof.* We will prove this by induction on  $n$  for  $X_n$ . The case  $n = 0$  is true by assumption. Assume  $\Pr(X_n = i) = x_i, i \in S$ . Then

$$\Pr(X_{n+1} = j) = \sum_{i \in S} \Pr(X_{n+1} = j \mid X_n = i) \Pr(X_n = i) = \sum_{i \in S} P_{ij} x_i = x_j$$

where the last step follows by Definition 1.32.  $\square$

By the Elementary Renewal Theorem (Theorem 28),

$$\sum_{k=1}^n \frac{1}{n} \Pr(X_k = j) \xrightarrow{a.s.} \frac{1}{m_{jj}}.$$

Suppose the initial state is chosen randomly according to the probability vector.

**Proposition 59 (Stationary probability vector is  $\pi_j$ ).** If  $x_j$  is a stationary probability vector then  $x_j = \pi_j, j \in S$ .

*Proof.* Let  $x_j$  be a stationary probability vector. Suppose  $\Pr(X_0 = i) = x_i, i \in S$ . Then  $\Pr(X_k = j) = x_j$  for all  $k$  by Lemma 58. By the Elementary Renewal Theorem (Theorem 28),

$$\frac{1}{n} \Pr(X_k = j) \xrightarrow{a.s.} \pi_j.$$

But also

$$\lim_{n \rightarrow \infty} \frac{1}{n} \Pr(X_k = j) = x_j$$

so  $\pi_j = x_j$ .  $\square$

**Definition 1.34 (Ergodic Markov chain state; p.174 of pdf of *Stochastic Processes*).** A positive recurrent, aperiodic state is called **ergodic**.

**Definition 1.35 (Ergodic Markov chain; p.188 of pdf of *Stochastic Processes*).** We say a Markov chain is **ergodic** if all states are positive recurrent; that is,

$$\pi_j = \lim_{n \rightarrow \infty} P_{ij}^n > 0.$$

In this case,  $\{\pi_j, j = 0, 1, 2, \dots\}$  is a stationary distribution and there exists no other stationary distribution.

**Definition 1.36.** A Markov chain that can only return to a state in a multiple of  $d > 1$  steps is said to be **periodic** and does not have limiting probabilities. That is,  $P_{ij}^n = 0$  except when  $n$  is a multiple of  $d$  and  $d$  is the largest such integer.

An irreducible chain that is not periodic is said to be **aperiodic**.

**Lemma 60.** Periodicity is a class property. That is, if  $i$  communicates with  $j$  then  $i$  and  $j$  have the same period.

*Proof.* Requires non-trivial results from algebra to prove. □

Also,  $P_{jj}^{nd} \rightarrow d/m_{jj}$ . For an aperiodic irreducible Markov chain,  $P_{ij}^n \rightarrow 1/m_{jj} = \pi_j$ . (Related to Blackwell's Theorem.)

**Example 1.37 (Example 4.18 in *Introduction to Probability Models*).** Stopping time:

$$N = \min\{n : X_1 + \dots + X_n = 1\} \implies 1 = X_1 + \dots + X_N$$

Apply Wald's Equation (Theorem 26):

$$1 = \mathbb{E}[X_1 + \dots + X_N] = \mathbb{E}(X)\mathbb{E}(N) = 0$$

So Wald's equation leads to a contradiction, meaning its assumptions weren't satisfied, which must be because  $\mathbb{E}(N) = \infty$ . See also Example ?? in the Probability notes.

**Example 1.38 (Example 4.24 in *Introduction to Probability Models*).** Not an alternating renewal process because in general it depends on which “up” state you enter into. Typically use renewal reward processes to find average length of “up” and “down” periods.

**Theorem 61 (Theorem 4.3.3 in *Stochastic Processes*, p. 186 of pdf).** Suppose we have an irreducible aperiodic Markov chain. If it is either transient or null recurrent, then there are no solutions of the equations

$$x_j = \sum_{i \in S} x_i P_{ij}, \quad j \in S,$$

$$\sum_j x_j = 1.$$

If it is positive recurrent, then  $\{\pi_j, j \in S\}$  uniquely satisfy the preceding equations. (There is no irreducible finite state Markov chain.)

*Proof.* Suppose a Markov chain is positive recurrent. Fix a state 0 and consider the Markov chain to start a new cycle each time it transitions into state 0. Think of this as a renewal reward process: suppose you earn 1 each time a transition into state  $j$  occurs (renewal each time you reach state 0 again). Then the average reward per unit time is  $\pi_j$ . But this also equals

$$\frac{\mathbb{E}(N_j)}{\mathbb{E}(N)}$$

where  $N_j$  is the number of transitions into state  $j$  during a cycle and  $N$  is the number of transitions in a cycle, so  $N = \sum_j N_j$ . Let  $N_{ij}$  be the number of transitions from state  $i$  to  $j$  in a cycle, so  $N_j = \sum_{i \in S} N_{ij}$ . Then we have

$$\mathbb{E}(N_j) = \sum_{i \in S} \mathbb{E}(N_{ij})$$

Let  $I_k$  is an indicator variable for transitioning from  $i$  to  $j$  with  $\mathbb{E}(I_k) = P_{ij}$ . Then  $N_{ij} = \sum_{k=1}^{N_i} I_k$ . Note that  $N_i$  is a stopping time for  $I_1, I_2, \dots$  because it depends on what happened before but not on the future. Therefore by Wald's Equation (Theorem 26),

$$\mathbb{E}(N_{ij}) = \mathbb{E} \sum_{k=1}^{N_i} I_k = \mathbb{E}(I_k) \mathbb{E}(N_i) = \mathbb{E}(N_i) P_{ij}.$$

Using that we have

$$\begin{aligned} \mathbb{E}(N_j) &= \sum_{i \in S} \mathbb{E}(N_{ij}) = \sum_{i \in S} \mathbb{E}(N_i) P_{ij} \\ \implies \frac{\mathbb{E}(N_j)}{\mathbb{E}(N)} &= \sum_{i \in S} \frac{\mathbb{E}(N_i)}{\mathbb{E}(N)} P_{ij} \iff \pi_j = \sum_i \pi_i P_{ij}. \end{aligned}$$

Also note that

$$\sum_j \pi_j = \sum_j \frac{\mathbb{E}(N_j)}{\mathbb{E}(N)} = \frac{\sum_j \mathbb{E}(N_j)}{\sum_j \mathbb{E}(N_j)} = 1.$$

□

**Example 1.39** (Similar to section 4.1.1 in *Introduction to Probability Models*). Suppose we have a 2 state Markov chain for the weather. 0 means dry, 1 means rain. dry to dry with probability 0.7, rain to rain with probability 0.6. So

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

Then  $\pi_j$  are the unique solutions to

$$\pi_j = \sum_i \pi_i P_{ij}, \quad \sum_j \pi_j = 1.$$

So we have

$$\pi_0 = \pi_0 P_{00} + \pi_1 P_{10}, \quad \pi_1 = \pi_0 P_{01} + \pi_1 P_{11}, \quad \pi_0 + \pi_1 = 1.$$

$$\implies \begin{bmatrix} (1-P_{00}) & -P_{10} \\ -P_{01} & (1-P_{11}) \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \pi_0 \\ \pi_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solving yields  $\pi_0 = 4/7$ ,  $\pi_1 = 3/7$ .

**Definition 1.37.** A Markov chain is said to be **doubly stochastic** if the sums of the columns of the transition probability matrix also add to 1 (the rows must always add to 1). That is,

$$\sum_i P_{ij} = 1, \quad \forall j \in S$$

**Proposition 62.** Suppose you have an irreducible Markov chain with  $m < \infty$  states that is doubly stochastic. Then

$$\pi_j = \frac{1}{m}, \quad j = 1, \dots, m$$

That is, all the long-run proportions are the same.

*Proof.* Recall that  $\pi_j$  is the unique solution to  $\pi_j = \sum_i \pi_i P_{ij}$  and that  $\sum_j \pi_j = 1$ . For  $\pi_j = 1/m$ , clearly the second condition is satisfied, so we need to verify the first state. That is we need to verify that

$$\frac{1}{m} = \frac{1}{m} \sum_i P_{ij} \iff \sum_i P_{ij} = 1$$

which is true if the Markov chain is doubly stochastic.

□

**Remark.** In practice if there is a Markov chain with more than three states and you are asked to find the long-run proportions, it is likely a doubly-stochastic Markov chain.

**Example 1.40 (Example 4.25 in *Introduction to Probability Models*).** Hotel, number of new guests every day is Poisson with mean  $\lambda$ . Each guest that stays the night will independently check out the next day with probability  $p = 1 - \alpha$ . (So the number of days each guest stays are independent geometric random variables.)

$$X_n = i \implies X_{n+1} = \text{Bin}(i, \alpha) * \text{Poisson}(\lambda)$$

where  $*$  is being used for a sum to stress the independence of the random variables.

$$P_{ij} = \Pr(\text{Bin}(i, \alpha) + \text{Poisson}(\lambda) = j)$$

$$\vdots$$



Part (c): Suppose the number of people initially there  $X_0$  is distributed  $\text{Poisson}(\beta)$ . Then the number of that cohort who remains the next day is  $\text{Poisson}(\alpha\beta)$ . So  $X_1 \sim \text{Poisson}(\alpha\beta + \lambda)$ . So to find the limiting distribution (where it's the same every day from now on), we need

$$\beta = \lambda + \alpha\beta \iff \beta = \frac{\lambda}{1-\alpha} \implies \pi_j = \exp\left(-\frac{\lambda}{1-\alpha}\right) \left(\frac{\lambda}{1-\alpha}\right)^j / j!$$

**Proposition 63 (Proposition 4.4.2 in *Stochastic Processes*; relevant to Gambler's Ruin).** If  $j$  is a recurrent state in a Markov chain, then the set of probabilities  $\{f_{ij}, i \in T\}$  satisfies

$$f_{ij} = \sum_{k \in T} P_{ik} f_{kj} + \sum_{k \in R} P_{ik}, \quad i \in T$$

where  $R$  denotes the set of states communicating with  $j$ .

**Example 1.41 (Gambler's Ruin; Section 4.5.1 of *Introduction to Probability Models*, Example 4.4(A) in *Stochastic Processes* (p. 197 of pdf)).** Also discussed drug testing example immediately afterward in section 4.5.1 of *Introduction to Probability Models*, and what happens as  $N \rightarrow \infty$  (p. 199 of pdf of *Stochastic Processes*).

#### 1.6.4 Branching Processes (Section 4.7 of *Introduction to Probability Models*, Section 4.5 of *Stochastic Processes*)

**Proposition 64 (Theorem 4.5.1 in *Stochastic Processes* (p. 203 of pdf)).** (i) If  $\mu \leq 1$ , then  $\pi_0 = 1$ . (ii) If  $\mu > 1$ ,  $\pi_0 < 1$ . (iii)  $\pi_0$  is the smallest positive number satisfying

$$\pi_0 = \sum_{j=0}^{\infty} \pi_0^j P_j.$$

*Proof.* (i) Using

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i,$$

we have

$$\mathbb{E}(X_n | X_{n-1}) = X_{n-1} \cdot \sum_{j=0}^{\infty} j P_j = X_{n-1} \mu$$

$$\implies \mathbb{E}(X_n) = \mu \mathbb{E}(X_{n-1}) = \mu^2 \mathbb{E}(X_{n-2}) = \dots = \mu^n \mathbb{E}(X_0) = \mu^n.$$

Suppose  $\mu < 1$ . By Markov's Inequality (Lemma ??),

$$\Pr(X_n \geq 1) \leq \mathbb{E}(X_n) = \mu^n \implies \lim_{n \rightarrow \infty} \Pr(X_n \geq 1) = 0.$$

When  $\mu = 1$ , can prove using a convexity argument and constraints  $P_0 > 0, P_0 + P_1 < 1$  (see *Stochastic Processes*).

- (ii) In *Stochastic Processes*.
- (iii) Suppose  $x > 0$  satisfies

$$x = \sum_{j=0}^{\infty} x^j P_j.$$

We will show by induction that  $x \geq \Pr(X_n = 0 \mid X_0 = 1)$  for all  $n$ . To start we need to show that  $x \geq \Pr(X_0 = 0 \mid X_0 = 1)$ . But

$$\Pr(X_0 = 0 \mid X_0 = 1) = P_0 \leq \sum_{j=0}^{\infty} x^j P_j = x$$

so that follows. Now we assume  $x \geq \Pr(X_n = 0 \mid X_0 = 1)$ . We would like to show that this implies  $x \geq \Pr(X_{n+1} = 0 \mid X_0 = 1)$ . Note that

$$\Pr(X_{n+1} = 0 \mid X_0 = 1) = \sum_{j=0}^{\infty} \Pr(X_{n+1} = 0 \mid X_1 = j, X_0 = 1) = \sum_{j=0}^{\infty} \Pr(X_{n+1} = 0 \mid X_1 = j) P_j$$

But  $\Pr(X_{n+1} = 0 \mid X_1 = j) = (\Pr(X_{n+1} = 0 \mid X_1 = 1))^j$  for the following reason: if there are  $j$  people to begin with, imagine they each had their own independent branching process. The probability that you reach 0 eventually is equal to the joint probability that all  $j$  processes reach 0 eventually. So we have

$$\sum_{j=0}^{\infty} \Pr(X_{n+1} = 0 \mid X_1 = j) P_j = \sum_{j=0}^{\infty} P_j (\Pr(X_{n+1} = 0 \mid X_1 = 1))^j \leq \sum_{j=0}^{\infty} P_j x^j = x$$

where the second to last step follows by the inductive hypothesis.

□

### 1.6.5 A Markov Chain Model of Algorithmic Efficiency (Section 4.6.1 of *Stochastic Processes*)

**Definition 1.38.** A set of events  $A_2, A_3, \dots$  is said to be an **increasing sequence** if  $A_n \subseteq A_{n+1}$  and a **decreasing sequence** if  $A_n \supseteq A_{n+1}$ . If  $A_n, n \geq 1$  is increasing, define

$$\lim_{n \rightarrow \infty} A_n = \bigcup_{n=1}^{\infty} A_n.$$

If  $A_n, n \geq 1$  is decreasing, define

$$\lim_{n \rightarrow \infty} A_n = \bigcap_{n=1}^{\infty} A_n.$$

The **limit infimum** is the set of all points contained in all but a finite number of  $A_1, A_2, \dots$ . The **limit supremum** is the set of all points contained in an infinite number of  $A_1, A_2, \dots$ . Note that for an increasing sequence the limit infimum is a subset of the limit supremum.

**Lemma 65.** For any events  $A_1, A_2, \dots$  there are events  $B_1, B_2, \dots$  that are mutually exclusive ( $B_i \cap B_j = \emptyset, i \neq j$ ) such that

$$B_i \cap B_j = \emptyset, \quad \bigcup_{i=1}^n B_i = \left(\bigcup_{i=1}^n A_i\right) \text{ for } n = 1, 2, \dots, \quad \bigcup_{i=1}^{\infty} B_i = \left(\bigcup_{i=1}^{\infty} A_i\right).$$

*Proof.* Let  $B_1 = A_1, B_2 = A_2 \setminus A_1, B_3 = A_3 \setminus \{A_1 \cup A_2\}, \dots, B_n = A_n \setminus \{A_1 \cup \dots \cup A_{n-1}\} = A_n \setminus \left\{\bigcup_{i=1}^{n-1} A_i\right\}$ .  $\square$

**Theorem 66.** Probability is a continuous set function. That is, if  $A_n$  is an increasing or decreasing sequence then

$$\lim_{n \rightarrow \infty} \Pr(A_n) = \Pr\left(\bigcup_{n=1}^{\infty} A_n\right) = \Pr\left(\lim_{n \rightarrow \infty} A_n\right).$$

*Proof.* Let  $A_1, A_2, \dots$  be events. Per Lemma 65, define  $B_1, B_2, \dots$  such that  $B_i \cap B_j = \emptyset$  and  $\bigcup_{i=1}^n B_i = \left(\bigcup_{i=1}^n A_i\right)$  for  $n = 1, 2, \dots, \bigcup_{i=1}^{\infty} B_i = \left(\bigcup_{i=1}^{\infty} A_i\right)$ . Suppose  $A_n$  is increasing, so that  $A_n \subseteq A_{n+1}$ . Then

$$\Pr\left(\lim_{n \rightarrow \infty} A_n\right) = \Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \Pr\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \Pr(B_i)$$

where the last equality follows since the  $B_i$  are disjoint. Continuing we have

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \Pr(B_i) = \lim_{n \rightarrow \infty} \Pr\left(\bigcup_{i=1}^n B_i\right) = \lim_{n \rightarrow \infty} \Pr\left(\bigcup_{i=1}^n A_i\right) = \lim_{n \rightarrow \infty} \Pr(A_n)$$

where the last step follows since  $A_n$  is an increasing sequence.  $\square$

**Remark.** Can prove Boole's Inequality by a similar logic:

$$\Pr\left(\bigcup_{i=1}^n A_i\right) = \Pr\left(\bigcup_{i=1}^n B_i\right) = \sum_{i=1}^n \Pr(B_i) \leq \sum_{i=1}^n \Pr(A_i)$$

where the last step follows since  $B_i \subseteq A_i$ .

**Remark.** Consider the Gambler's Ruin problem (Example 1.41). Let  $A_N$  be the event of reaching 0 before  $N$  given  $X_0 = i$ . Note that  $A_N \subseteq A_{N+1}$ . So we have

$$\lim_{N \rightarrow \infty} \Pr(A_N) = \Pr\left(\lim_{N \rightarrow \infty} A_N\right) = \Pr\left(\bigcup_{N=1}^{\infty} A_N\right)$$

where the last quantity is clearly the probability that the gambler eventually goes to 0, matching our interpretation in Example 1.41.

Similarly, in the proof of Proposition 64 for branching processes, we claimed

$$\Pr(\text{population dies out}) = \lim_{n \rightarrow \infty} \Pr(X_n = 0).$$

Note that  $\{X_n = 0\} \subset \{X_{n+1} = 0\}$  so

$$\lim_{n \rightarrow \infty} \{X_n = 0\} = \bigcup_{n=1}^{\infty} \{X_n = 0\}$$

again matching what we assumed.

### 1.6.6 Time Reversible Markov Chains (Section 4.8 of *Introduction to Probability Models*, Section 4.7 of *Stochastic Processes*)

**Proposition 67 (Unlabeled at beginning of each section).** Given a stationary Markov chain  $X_n, X_{n+1}, X_{n+2}, \dots$ , the reverse process  $X_n, X_{n-1}, X_{n-2}, \dots$  is a Markov chain with transition probabilities

$$Q_{ij} = P_{ij}^* = \frac{\pi_j P_{ji}}{\pi_i}.$$

*Proof.*

$$\begin{aligned} Q_{ij} = \Pr(X_m = j \mid X_{m+1} = i) &= \frac{\Pr(X_m = j \cap X_{m+1} = i)}{\Pr(X_{m+1} = i)} = \frac{\Pr(X_m = j) \Pr(X_{m+1} = i \mid X_m = j)}{\Pr(X_{m+1} = i)} \\ &= \frac{\pi_j P_{ji}}{\pi_i}. \end{aligned} \tag{11}$$

□

**Definition 1.39 (Time reversible Markov chain).** Suppose for a Markov chain, the reverse transition probabilities  $Q_{ij}$  equal the corresponding forward transition probabilities  $P_{ij}$  for all  $i, j$ . Then the Markov chain is said to be **time reversible**. The condition  $Q_{ij} = P_{ij}$  can also be written (per (11)) as

$$\pi_i P_{ij} = \pi_j P_{ji}, \quad \forall i, j.$$

**Theorem 68 (Theorem 4.7.2 in *Stochastic Processes* (p. 220 of pdf)).** A stationary Markov chain is time reversible if and only if starting in state  $i$ , any path back to  $i$  has the same probability as the reverse path, for all  $i$ . That is, if

$$P_{i,i_1} P_{i_1,i_2} \cdots P_{i_k,i} = P_{i,i_k} P_{i_k,i_{k-1}} \cdots P_{i_1,i}$$

for all states  $i, i_1, \dots, i_k$ .

**Theorem 69** (Theorem 4.2 in *Introduction to Probability Models*). An ergodic (all states positive recurrent; see Definition 1.35) Markov chain for which  $P_{ij} = 0$  whenever  $P_{ji} = 0$  is time reversible if and only if starting in state  $i$ , any path back to  $i$  has the same probability as the reverse path, for all  $i$ . That is, if

$$P_{i,i_1}P_{i_1,i_2}\cdots P_{i_k,i} = P_{i,i_k}P_{i_k,i_{k-1}}\cdots P_{i_1,i}$$

for all states  $i, i_1, \dots, i_k$ .

**Proposition 70** (Proposition 4.9 in *Introduction to Probability Models*, Theorem 4.7.3 in *Stochastic Processes* (p. 222 of pdf)). Consider an irreducible Markov chain  $\{X_n\}$  with transition probabilities  $P_{ij}$ . If  $\{X_j, j \in S\}$  is stationary, and there exist positive numbers  $x_i, i \geq 0$  and a transition probability matrix  $Q$  such that

$$x_i P_{ij} = x_j Q_{ji}, \quad i \neq j$$

$$\sum x_i = 1$$

then the Markov chain is reversible,  $\pi_j = x_j$  are the stationary probabilities for both the original and the reversed chain, and the  $Q_{ij}$  are the transition probabilities of the reversed chain.

*Proof.* Assume the above. we have

$$\sum_i x_i P_{ij} = \sum_i x_j P_{ji} = x_j$$

so these  $x_j$  satisfy the stationarity equations and the solution is unique, so  $x_j = \pi_j$ . □

**Example 1.42** (Similar to Example 4.35 in *Introduction to Probability Models*, Gambler's Ruin-type problem).

**Example 1.43** (Similar to Example 4.36 in *Introduction to Probability Models*, Proposition 4.7.1 in *Stochastic Processes* (p. 217 of pdf; weighted graph problem)).

**Example 1.44** (Example 4.37 in *Introduction to Probability Models*, Example 4.7(b) in *Stochastic Processes* (p. 221 of pdf; list, items requested, then move up one position in the list; average position)). Note the similarity to *Introduction to Probability Models* problem 4.26 (from Homework 10).

### 1.6.7 Semi-Markov Processes (Section 4.8 of *Stochastic Processes* (p. 224 of pdf), Section 7.6 of *Introduction to Probability Models*)

**Definition 1.40 (Semi-Markov Process).** Suppose the stochastic process  $\{X(t), t \geq 0\}$  takes on values  $\{0, 1, \dots, M\}$ . Then  $\{X(t), t \geq 0\}$  is called a **semi-Markov process**.

Why semi-Markov? What happens next doesn't just depend on your current state, it also depends on how long you're been there. But it's semi-Markov because just after a transition you know everything.

## 1.7 ISE 620

**Exercise 8. (“Best Prize” problem.)** There are  $n$  prizes that are presented one at a time in a random order. Each prize has a defined value, and there is a defined ordering of the value of the prizes. Each time you see a prize, you only know the value of that prize relative to the prizes already seen—the value of later prizes remains unknown. Each time a prize is presented, we either accept it or reject it. You only get one prize, so once you accept it, you’re done. Once you reject a prize, you can’t get it back. Your goal is to maximize the probability of accepting the best prize. What do you do (what is the optimal policy)?

**Solution.** The only strategy that makes sense is to accept a prize if it is a *candidate*—that is, the best you’ve seen so far. You should never accept a prize that isn’t a candidate unless you get to the last prize. Let  $P_n$  be the probability of getting the best prize. Then we expect  $P_n$  to approach 0 as  $n$  approaches infinity. Note that if it were good to accept the  $k$ th prize if it were a candidate, then it would definitely be better to accept the  $k + 1$ th candidate if it were a candidate. So a good policy ( $k$ -policy) is to let  $k$  prizes go by, then accept the first candidate to come afterward.

Let  $X$  be the position of the best prize. We will condition on the best prize being in position  $i$ . Then

$$P_k(\text{best}) = \sum_{i=1}^n P_k(\text{best} \mid X = i) \Pr(X = i) = \frac{1}{n} \sum_{i=1}^n P_k(\text{best} \mid X = i)$$

Note that because in order for you to win, none of the prizes between position  $k$  and  $i$  can be candidates (or else you would accept them and not get the best prize). So the best of the first  $i - 1$  prizes must be among the first  $k$  prizes in order for you to get the best prize.

$$P_k(\text{best} \mid X = i) = \begin{cases} 0 & i \leq k \\ \Pr(\{\text{best of } 1, \dots, i - 1 \text{ is among the first } k\}) & k > k \end{cases}$$

and

$$\Pr(\{\text{best of } 1, \dots, i - 1 \text{ is among the first } k\}) = \frac{k}{i - 1}$$

so

$$P_k(\text{best}) = \frac{1}{n} \sum_{i=k+1}^n \frac{k}{i - 1} = \frac{k}{n} \sum_{j=k}^{n-1} \frac{1}{j} \approx \frac{k}{n} \int_k^{n-1} \frac{dx}{x} = \frac{k}{n} \log(x) \Big|_k^{n-1} = \frac{k}{n} \log\left(\frac{n-1}{k}\right) \approx \boxed{\frac{k}{n} \log\left(\frac{n}{k}\right)}.$$

Now we want to choose  $k$  that maximizes this quantity. Generalize to letting  $x$  equal any real number in the expression

$$f(x) = \frac{x}{n} \log\left(\frac{n}{x}\right) \implies f'(x) = \frac{x}{n} \frac{x}{x^2} \left(-\frac{n}{x^2}\right) + \log\left(\frac{n}{x}\right) \frac{1}{n}$$

Setting equal to 0 we have

$$\frac{1}{n} = \frac{1}{n} \log\left(\frac{n}{x}\right) \implies \frac{n}{x} = e \implies \boxed{x = \frac{n}{e}}$$

so the optimal strategy is to let about  $1/e$  of the prizes go by, then choose the first candidate to come by afterward.

**Remark.** The probability of getting the best prize is then

$$f\left(\frac{n}{e}\right) = \frac{1}{e} \log(e) = \frac{1}{e}$$

regardless of  $n$ !

**Exercise 9. (Ballot problem.)** Suppose we have candidates  $A$  and  $B$  with votes counted in random order.  $A$  has  $n$  votes,  $B$  has  $m$  with  $n > m$ . What is the probability that  $A$  is always ahead in the count at every stage of the vote?

**Solution.** Let  $\Pr(\{A \text{ is always ahead}\}) = P(n, m) = P_{n,m}$ . Then

$$\begin{aligned} P_{n,m} &= \Pr(\{A \text{ is always ahead}\} \mid \{A \text{ receives the first vote}\}) \cdot \frac{n}{n+m} \\ &\quad + \Pr(\{A \text{ is always ahead}\} \mid \{B \text{ receives the first vote}\}) \cdot \frac{m}{n+m} \\ &= \Pr(\{A \text{ is always ahead}\} \mid \{A \text{ receives the first vote}\}) \cdot \frac{n}{n+m} + 0 \\ &= Q_{n-1,m} \cdot \frac{n}{n+m} \end{aligned}$$

where  $Q$  represents the probability that  $A$  is never behind (since going forward ties would be ok because  $A$  starts out ahead). We have

$$Q_{nm} = \Pr(\{A \text{ is never behind}\}) = \frac{n}{n_m} \Pr(\{A \text{ is never behind}\} \mid \{A \text{ gets first vote}\})$$

Note that  $\Pr(\{A \text{ is never behind}\} \mid \{A \text{ gets first vote}\}) = Q_{n-1,m-1}$ . But now things are more complicated because  $A$  could afford to be behind by 1 going forward. So this strategy is not working. Try instead to condition on who gets the last vote.

$$\begin{aligned} P_{n,m} &= \Pr(\{A \text{ is always ahead}\} \mid \{A \text{ receives the last vote}\}) \cdot \frac{n}{n+m} \\ &\quad + \Pr(\{A \text{ is always ahead}\} \mid \{B \text{ receives the last vote}\}) \cdot \frac{m}{n+m} \end{aligned}$$

$$= P_{n-1,m} \cdot \frac{n}{n+m} + P_{n,m-1} \cdot \frac{m}{n+m}$$

We can work this out recursively using the boundary conditions

$$P_{n,n} = 2, \quad P_{n,0} = 1, \quad n > 0$$

We will try to guess the answer.

$$P_{2,1} = \Pr(\{A \text{ gets the first two votes}\}) = \frac{1}{3}$$

$$P_{n,1} = \Pr(\{\text{first two votes are for } A\}) = \frac{n}{n+1} \cdot \frac{n-1}{n} = \frac{n-1}{n+1}$$

$$P_{3,2} = \frac{3}{5} \cdot \frac{2}{4} \cdot \Pr(\{A \text{ is not the last of the remaining votes}\}) = \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} = \frac{1}{5}$$

$$P_{4,2} = \frac{4}{6} \cdot \frac{3}{5} \cdot \Pr(\{B \text{ does not get next two votes}\}) = \frac{4}{6} \cdot \frac{3}{5} \cdot \left(1 - \frac{2}{4} \cdot \frac{1}{3}\right) = \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{5}{6} = \frac{1}{3}$$

$$\begin{aligned} P_{4,3} &= \frac{4}{7} \cdot \frac{3}{6} \cdot \left( \Pr(\{\text{next is } A\}) \cdot \Pr(\{\text{not 3 } B \text{ votes in a row}\}) + \Pr(\{\text{next is } B\}) \cdot \Pr(\{ \right. \\ &= \frac{4}{7} \cdot \frac{3}{6} \cdot \left( \frac{2}{5} \cdot \frac{3}{4} + \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} \right) \end{aligned}$$

$\vdots$

It seems that

$$\boxed{P_{n,m} = \frac{n-m}{n+m}}$$

We can argue this solution is unique or we can argue it is true by induction.

## 1.8 DSO Statistics Group Screening Exam Problems

**Exercise 10 (2016 DSO Statistics Group In-Class Screening Exam, Question 1).** Let  $\{V_n\}$  be i.i.d. non-negative random variables. Fixing  $r > 0$  and  $q \in (0, 1]$ , consider the sequence  $W_0 = 1$  and  $W_n = (qr + (1-q)V_n)W_{n-1}$ ,  $n \geq 1$ . A motivating example of  $W_n$  is recording the relative growth of a portfolio where a constant fraction  $q$  of one's wealth is re-invested each year in a risk-less asset that grows by  $r$  per year, with the remainder re-invested in a risky asset whose annual growth factors are random  $V_n$ .



- (a) Show that  $n^{-1} \log W_n \xrightarrow{a.s.} w(q)$ , where  $w(q) = \mathbb{E} \log(qr + (1-q)V_1)$ .
- (b) Show that  $q \rightarrow w(q)$  is concave on  $(0, 1]$ .
- (c) Using Jensen's Inequality, show that  $w(q) \leq w(q)$  in case  $\mathbb{E}V_1 \leq r$ . Further, show that if  $\mathbb{E}V_1^{-1} \leq r^{-1}$  then the almost sure convergence also applies for  $q = 0$  and  $w(q) \leq w(0)$ .

**Solution.**

- (a) Note that

$$W_1 = (qr + (1-q)V_1)W_0 = qr + (1-q)V_1$$

$$W_2 = (qr + (1-q)V_2)W_1 = (qr + (1-q)V_2)(qr + (1-q)V_1)$$

$$W_3 = (qr + (1-q)V_3)W_2 = (qr + (1-q)V_3)(qr + (1-q)V_2)(qr + (1-q)V_1) = \prod_{i=1}^3 (qr + (1-q)V_i)$$

$\vdots$

$$W_n = \prod_{i=1}^n (qr + (1-q)V_i)$$

$$\implies n^{-1} \log(W_n) = \frac{1}{n} \sum_{i=1}^n \log(qr + (1-q)V_i)$$

$\{qr + (1-q)V_i\}$  is a sequence of i.i.d. nonnegative random variables, so by the Strong Law of Large Numbers, the conclusion follows if  $\mathbb{E}[\log(qr + (1-q)V_1)] = \mathbb{E}[\log(qr + (1-q)V_1)] = \log(qr + (1-q)\mathbb{E}(V_1)) < \infty$ .

- (b) Since  $w(q)$  is twice differentiable, a sufficient condition for concavity is  $w''(q) \leq 0$  for all  $q \in (0, 1]$ .

$$w(q) = \mathbb{E} \log(qr + (1-q)V_1) = \mathbb{E} \log[q(r - V_1) + V_1]$$

$$w'(q) = \frac{\partial}{\partial q} \mathbb{E} \log[q(r - V_1) + V_1] = \mathbb{E} \left( \frac{\partial}{\partial q} \log[q(r - V_1) + V_1] \right) = \mathbb{E} \left( \frac{r - V_1}{q(r - V_1) + V_1} \right) \quad (12)$$

$$w''(q) = \mathbb{E} \left( \frac{\partial}{\partial q} \frac{r - V_1}{q(r - V_1) + V_1} \right) = \mathbb{E} \left( (r - V_1) \cdot \frac{\partial}{\partial q} [q(r - V_1) + V_1]^{-1} \right)$$

$$= \mathbb{E} \left( (r - V_1) \cdot (-1) [q(r - V_1) + V_1]^{-2} \cdot (r - V_1) \right) = -\mathbb{E} \left( \left[ \frac{(r - V_1)}{q(r - V_1) + V_1} \right]^2 \right) \quad (13)$$

This is  $-1$  times the expectation of a nonnegative random variable, so by Markov's Inequality we have

$$\mathbb{E} \left( \left[ \frac{(r - V_1)}{q(r - V_1) + V_1} \right]^2 \right) \geq 0 \iff w''(q) \leq 0 \quad \forall q \in (0, 1],$$

proving concavity.

(c) By Jensen's Inequality and the concavity of  $q \rightarrow \log[qr + (1 - q)V_1]$ , we have

$$\begin{aligned} w(q) &= \mathbb{E} \log[qr + (1 - q)V_1] \leq \log \mathbb{E}[qr + (1 - q)V_1] = \log(qr + (1 - q)\mathbb{E}[V_1]) \\ &\leq \log(qr + (1 - q)r) = \log(r) = \mathbb{E} \log(r) = w(1). \end{aligned}$$

where the third step used  $\mathbb{E}(V_1) \leq r$  and the second-to-last step used the fact that  $r$  is non-random. For  $q = 0$ , we have

$$n^{-1} \log(W_n) = \frac{1}{n} \sum_{i=1}^n V_i.$$

$\{V_i\}$  is a sequence of i.i.d. nonnegative random variables, so by the Strong Law of Large Numbers, almost sure convergence applies if  $\mathbb{E}[|V_1|] = \mathbb{E}[V_1] < \infty$ . **But this is the same condition as in the case  $q \in (0, 1]$ .**

To show  $w(q) \leq w(0)$ , we will show that  $w(q)$  is nonincreasing on  $[0, 1]$ . Recall from (12)

$$w'(q) = \mathbb{E} \left( \frac{r - V_1}{q(r - V_1) + V_1} \right)$$

We will show that (12) is upper-bounded by 0 on  $[0, 1]$ . Plugging in  $q = 0$  yields

$$w'(0) = \mathbb{E} \left( \frac{r - V_1}{V_1} \right) = \mathbb{E}(rV_1^{-1} - 1) = r\mathbb{E}(V_1^{-1}) - 1 \leq 1 - 1 = 0.$$

where we used the assumption  $\mathbb{E}(V_1^{-1}) \leq r$  on the second-to-last step. Recall that the second derivative (13) is continuous and nonpositive on  $(0, 1]$ . Therefore the first derivative (12) never exceeds  $q(0) = 0$  on  $[0, 1]$ , so  $w(q) \leq w(0)$ .

**Exercise 11 (2018 DSO Statistics Group In-Class Screening Exam, Question 1).** (a) For each  $x > 0$ , let  $M(x)$  be a real-valued random variable and set  $M(0) = 0$ . Assume that the random function  $M(x)$  is monotone non-decreasing on  $[0, \infty)$ . Define  $T(y) := \inf\{x \geq 0 : M(x) \geq y\}$ . Suppose that  $e^{-1}T(y)$  converges in distribution to an  $\text{Exponential}(\lambda)$  random variable when  $y \rightarrow \infty$ .

- (i) Find non-random  $a(x)$  and  $b(x) > 0$  such that  $(M(x) - a(x))/b(x)$  converges in distribution for  $x \rightarrow \infty$  to a non-degenerate random variable.
  - (ii) Provide the distribution function of the limit random variable. What is the name of this distribution?
- (b) Let  $X \sim \text{Bin}(n, p)$ . Find  $\mathbb{E}(1/(i + X))$ , for  $i = 1, 2$ . Hint: Recall that  $\int x^\alpha dx = x^{a+1}/(a+1) + C$  for  $a \neq -1$ .

## 1.9 Simple Random Walk

**Definition 1.41.** Let  $\{X_i\}$  be i.i.d. We have

$$X_i = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1 - p \end{cases}$$

Let  $S_k = \sum_{i=1}^k X_i$ . Then  $\{S_k\}$  is a **simple random walk**. (It is simple because the outcomes are either 1 or -1.)

## 1.10 Martingales

**Definition.** Let  $\{y_t\}_{t=0}^\infty$  be a sequence of random variables, and let  $\Omega_t$  denote the information set available at date  $t$ , which at least contains  $\{y_t, y_{t-1}, y_{t-2}, \dots\}$ . If  $\mathbb{E}(y_t \mid \Omega_{t-1}) = y_{t-1}$  holds then  $\{y_t\}$  is a martingale process with respect to  $\Omega_t$ .

**Definition.** Let  $\{y_t\}_{t=1}^\infty$  be a sequence of random variables, and let  $\Omega_t$  denote the information set available at date  $t$ , which at least contains  $\{y_t, y_{t-1}, y_{t-2}, \dots\}$ . If  $\mathbb{E}(y_t \mid \Omega_{t-1}) = 0$ , then  $\{y_t\}$  is a martingale difference process with respect to  $\Omega_t$ .

## 1.11 Brownian Motion

**Appendix B.13, Brownian motion.** A standard Brownian motion  $b(\cdot)$  is a continuous-time stochastic process associating each date  $a \in [0, 1]$  with the scalar  $b(a)$  such that

- (i)  $b(0) = 0$
- (ii) For any dates  $0 \leq a_1 \leq a_2 \leq \dots \leq a_k \leq 1$  the changes  $[b(a_2) - b(a_1)], [b(a_3) - b(a_2)], \dots, [b(a_k) - b(a_{k-1})]$  are independent multivariate Gaussian with  $b(a) - b(s) \sim \mathcal{N}(0, a - s)$ .
- (iii) For any given realization,  $b(a)$  is continuous in  $a$  with probability 1.

Other continuous time processes can be generated from the standard Brownian motion. For example, a Brownian motion with variance  $\sigma^2$  can be obtained as

$$w(a) = \sigma b(a)$$

where  $b(a)$  is a standard Brownian motion.

The continuous time process

$$\mathbf{w}(a) = \boldsymbol{\Sigma}^{1/2} \mathbf{b}(a)$$

is a Brownian motion with covariance matrix  $\Sigma$ .

**Definition 26 (Wiener process).** Let  $\Delta w(t)$  be the change in  $w(t)$  during the time interval  $dt$ . Then  $w(t)$  is said to follow a Wiener process if

$$\Delta w(t) = \epsilon_t \sqrt{dt}, \quad \epsilon_t \sim IID(0, 1)$$

and  $w(t)$  denotes the value of the  $w(\cdot)$  at date  $t$ . Clearly,

$$\mathbb{E}[\Delta w(t)] = 0, \text{ and } \text{Var}[\Delta w(t)] = dt$$

**Theorem 71. Donsker's Theorem, Theorem 43, p.335, Section 15.6.3.** Let  $a \in [0, 1)$ ,  $t \in [0, T]$ , and suppose  $(J-1)/T \leq a < J/T$ ,  $J = 1, 2, \dots, T$ . Define

$$R_T(a) = \frac{1}{\sqrt{T}} s_{[Ta]}$$

where

$$s_{[Ta]} = \epsilon_1 + \epsilon_2 + \dots + \epsilon_{[Ta]}$$

$[Ta]$  denotes the largest integer part of  $Ta$  and  $s_{[Ta]} = 0$  if  $[Ta] = 0$ . Then  $R_T(a)$  weakly converges to  $w(a)$ , i.e.,

$$R_T(a) \rightarrow w(a)$$

where  $w(a)$  is a Wiener process. Note that when  $a = 1$ ,  $R_T(1) = 1/\sqrt{T} \cdot S_{[T]} = 1/\sqrt{T} \cdot (\epsilon_1 + \epsilon_2 + \dots + \epsilon_T)$ . Since  $\epsilon_t$ 's are IID, by the central limit theorem,  $R_T(1) \rightarrow \mathcal{N}(0, 1)$ .

Similar (Theorem 2.1 in [Phillips and Durlauf \[1986\]](#)): Let  $\{u_t\}$  be a sequence satisfying  $\mathbb{E}(u_t) = 0$ ,  $\gamma(0) = \mathbb{E}(T^{-1}S_t^2) \rightarrow \sigma^2 < \infty$  as  $T \rightarrow \infty$ ,  $\{u_t\}$  is square summable,  $\sup_t \{\mathbb{E}(|u_t|^\beta)\} < \infty$  for some  $2 \leq \beta < \infty$  and all  $t$ ,  $\gamma(h) = \mathbb{E}(T^{-1}(y_t - y_{t-h})^2) \rightarrow K_h < \infty$  as  $\min\{h, T\} \rightarrow \infty$ . Then  $X_T(t) \Rightarrow W(t)$  as  $T \rightarrow \infty$ , where  $W(t)$  is a Wiener process.

**Theorem 72. Continuous Mapping Theorem (Theorem 44 of Pesaran in 15.6.3).** Let  $a \in [0, 1)$ ,  $i \in [0, n]$ , and suppose  $(J-1)/n \leq a < J/n$ ,  $J = 1, 2, \dots, n$ . Define  $R_n(a) = n^{-1/2} S_{[na]}$ . If  $f(\cdot)$  is continuous over  $[0, 1)$ , then

$$f[R_n(a)] \xrightarrow{d} f[w(a)]$$

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