Adversarial training of Schrödinger bridges

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Research

The problem of domain adaptation using Schrödinger bridge problem is investigated.

Research objective —

suggest a method of solving Schrödinger bridges using adversarial training.

Required to suggest

1. theory to connect adversarial training with Schrödinger bridges,
2. algorithm of solving Schrödinger bridges using adversarial training.

Problem statement

It is given

1. $\{x_i\}_{i=0}^N, \{y_j\}_{j=0}^N \in \mathbb{R}^d$ – two unpaired datasets, where $x_i \sim \pi_0(x)$ and $y_j \sim \pi_1(y)$,
2. $p^{\mathbb{W}^\gamma} = \mathcal{N}(0, \gamma\mathbb{I})$ – conditional distribution, given by Wiener process with drift γ.

We want to find joint distribution $p(x, y)$ that are constrained on $\pi_0(x)$ and $\pi_1(y)$. Static Schrödinger Bridge Problem find that distribution:

$$\begin{cases} p^*(x, y) = \arg\min_{p(x,y)} D_{KL}(p(x, y)||p^{\mathbb{W}^\gamma}(x, y)), \\ \pi_0(x) = \int p(x, y)dy, \\ \pi_1(y) = \int p(x, y)dx \end{cases} \tag{1}$$

Suggested Method

To solve (1) is used Iterational Proportional fitting, which is alternates solving between two half bridges (constrained only on one distribution). However it can be mentioned that such algorithm could be rewritten as:

Backward

$$\arg\min_{\phi,\psi} D_{KL}(p_\phi(x)||\pi_0(x)) + \mathbb{E}_{x\sim p_\phi(x)}\left[D_{KL}(p_\psi(y|x)||q^*_\omega(x|y))\right] - \mathbb{E}_{x\sim p_\phi(x)}[\ln \pi_0(x)]$$

Forward

$$\arg\min_{\theta,\omega} D_{KL}(q_\theta(x)||\pi_1(y)) + \mathbb{E}_{y\sim q_\theta(y)}\left[D_{KL}(q_\omega(x|y)||p^*_\psi(y|x))\right] - \mathbb{E}_{y\sim q_\theta(y)}[\ln \pi_1(y)]$$

Suggested Method

So, two marginal distributions could be learned with adversarial training and conditional distributions parameterised by normal distribution:

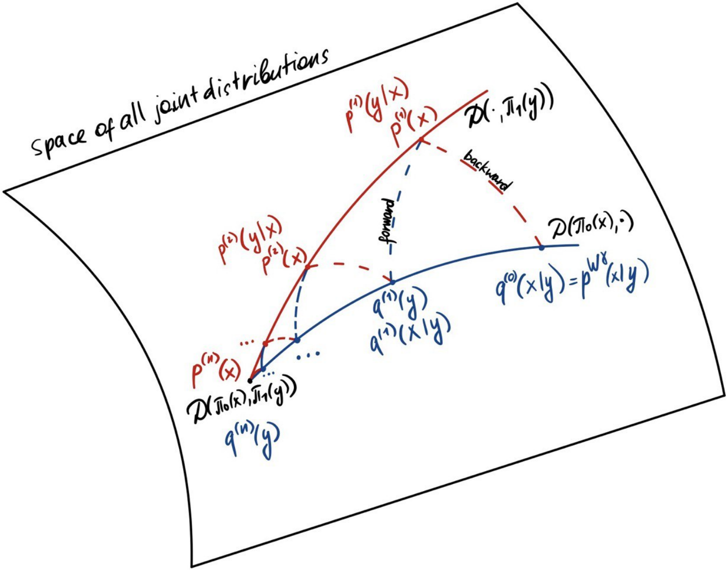
Backward

$$\arg\min_{\phi,\psi}\max_{D_0} \mathbb{E}_{x\sim\pi_0(x)}[D_0(x)] - \mathbb{E}_{x\sim p_\phi(x)}\left[e^{D_0(x)-1}\right] +$$
$$+\mathbb{E}_{x\sim p_\phi(x)}\left[D_{KL}(\mathcal{N}(y;\mu_\psi(x),\Sigma_\psi(x))||\mathcal{N}(x;\mu^*_\omega(y),\Sigma^*_\omega(y)))\right]-$$
$$-\mathbb{E}_{x\sim p_\phi(x)}[\ln\pi_0(x)]$$

Forward

$$\arg\min_{\theta,\omega}\max_{D_1} \mathbb{E}_{y\sim\pi_1(y)}[D_1(y)] - \mathbb{E}_{y\sim q_\theta(y)}\left[e^{D_1(y)-1}\right] +$$
$$+\mathbb{E}_{y\sim q_\theta(y)}\left[D_{KL}(\mathcal{N}(x;\mu_\omega(y),\Sigma_\omega(y))||\mathcal{N}(y;\mu^*_\psi(x),\Sigma^*_\psi(x)))\right]-$$
$$-\mathbb{E}_{y\sim q_\theta(y)}[\ln\pi_1(y)]$$

Suggested Method



Related papers

1. Machine-learning approaches for the empirical Schrodinger bridge problem[1]
2. f-GAN: Training Generative Neural Samplers using Variational Divergence Minimization[2]

[1] Machine-learning approaches for the empirical Schrodinger bridge problem, F. Vargas, 2021

[2] f-GAN: Training Generative Neural Samplers using Variational Divergence Minimization, S. Nowozin, 2016

Additional information

Schrödinger Bridge Problem arises from Sanov's theorem, that allows us to measure probability of prior empirical stochastic process $\hat{\mathbb{W}}$ to be bounded between two given distributions $\pi_0(x)$ and $\pi_1(y)$:

$$P\left(\hat{\mathbb{W}} \in \mathcal{D}(\pi_0, \pi_1)\right) \xrightarrow{N \to \infty} \exp\left(-N \inf_{\mathbb{P} \in \mathcal{D}(\pi_0, \pi_1)} D_{KL}(\mathbb{P} || \mathbb{W})\right)$$

So the question arises: Is it possible to change somehow KL divergence to other distances or metrics?

Future work plan

1. Make experiments with proposed adversarial training;
2. Investigate generalization of D_{KL} to other distances or metrics.