# Adversarial Schrödinger bridges

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# Research

The high computational inference cost of diffusion-based Schrödinger Bridge approaches is investigated.

## Research objective —

suggest a method of solving Schrödinger Bridges without recovering entire Schrödinger Bridge process.

## Required to suggest

1. algorithm of solving Schrödinger bridges problem that doesn't require recovering entire Schrödinger Bridge process,
2. analysis of this algorithm.

## Solution

Use adversarial training instead of diffusion approaches.

# Problem statement

It is given

1. $\{x_i\}_{i=0}^N = X, \{y_j\}_{j=0}^N = Y$ – two unpaired datasets, where $x_i \sim \pi_0(x)$ and $y_j \sim \pi_1(y)$,
2. $\mathbb{W}^\gamma$ – path measure of reference Wiener process with drift $\sqrt{\gamma}$.

Dynamic Schrödinger Bridge Problem finds the closest path measure $\mathbb{Q}$ with prescribed marginals of $\pi_0(x)$, $\pi_1(y)$ at times 0 and 1, to reference path measure $\mathbb{W}^\gamma$ in sense of KL-divergence, i.e.:

$$\hat{\mathbb{Q}} = \arg \min_{\mathbb{Q} \in \mathcal{D}(\pi_0, \pi_1)} D_{KL}(\mathbb{Q} || \mathbb{W}^\gamma), \tag{1}$$

where $\mathcal{D}(\pi_0, \pi_1)$ is space of all path measures with prescribed marginals $\pi_0(x)$ and $\pi_1(y)$.

However, the solving of the problem requires to learn process that connects both marginals. It is computationally expensive, so, we propose a new approach that addresses this problem.

# Suggested Method

Firstly, instead of solving dynamic version [1] of Schrödinger Bridge Problem we consider a Static formulation. With the same sets $X$ and $Y$ it is given joint distribution $p^{\mathbb{W}^\gamma}(x, y)$ of the reference Wiener process. Static Schrödinger Bridge Problem:

$$\begin{cases} q^*(x, y) = \arg\min_{q(x,y)} D_{KL}(q(x, y)||p^{\mathbb{W}^\gamma}(x, y)), \\ \pi_0(x) = \int q(x, y)dy, \\ \pi_1(y) = \int q(x, y)dx \end{cases} \tag{2}$$

To solve (2) is used Iterational Proportional Fitting (IPF), which alternates between solving two half bridges (constrained only on one distribution).

# Suggested Method

To leverage adversarial approach we represent KL divergence in it's variational (dual) form (inspired by f-GAN [2]):

## Backward

$$\min_{p(x,y)\in\mathcal{D}(\cdot,\pi_1)} D_{KL}(p(x,y)||q^{i-1}(x,y)) = \min_{p(x,y)\in\mathcal{D}(\cdot,\pi_1)} \max_{D} \mathbb{E}_{(x,y)\sim p}\left[D(x,y)\right] - \\ -\mathbb{E}_{(x,y)\sim q^{i-1}}\left[e^{D(x,y)-1}\right]$$

## Forward

$$\min_{q(x,y)\in\mathcal{D}(\pi_0,\cdot)} D_{KL}(q(x,y)||p^{i}(x,y)) = \min_{q(x,y)\in\mathcal{D}(\pi_0,\cdot)} \max_{D} \mathbb{E}_{(x,y)\sim q}\left[D(x,y)\right] - \\ -\mathbb{E}_{(x,y)\sim p^{i}}\left[e^{D(x,y)-1}\right]$$

# Suggested Method

In sense of converting one sample to another we want to learn conditional distribution:

## Backward

$$\min_{p(x,y)\in\mathcal{D}(\cdot,\pi_1)} \max_D \mathbb{E}_{(x,y)\sim p}\left[D(x,y)\right] - \mathbb{E}_{(x,y)\sim q^{i-1}}\left[e^{D(x,y)-1}\right] =$$

$$= \min_{p(x|y)} \max_D \mathbb{E}_{y\sim\pi_1}\mathbb{E}_{x\sim p(x|y)}\left[D(x,y)\right] - \mathbb{E}_{x\sim\pi_0}\mathbb{E}_{y\sim q^{i-1}(y|x)}\left[e^{D(x,y)-1}\right]$$

## Forward

$$\min_{q(x,y)\in\mathcal{D}(\pi_0,\cdot)} \max_D \mathbb{E}_{(x,y)\sim q}\left[D(x,y)\right] - \mathbb{E}_{(x,y)\sim p^{i}}\left[e^{D(x,y)-1}\right] =$$

$$= \min_{q(y|x)} \max_D \mathbb{E}_{x\sim\pi_0}\mathbb{E}_{y\sim q(y|x)}\left[D(x,y)\right] - \mathbb{E}_{y\sim\pi_1}\mathbb{E}_{x\sim p^{i}(x|y)}\left[e^{D(x,y)-1}\right]$$

# Suggested Method

Using approach from Conditional GAN [3]:

## Backward

$$\min_{p(x|y)} \max_{D} \mathbb{E}_{y\sim\pi_1}\mathbb{E}_{x\sim p(x|y)}\left[D(x,y)\right] - \mathbb{E}_{x\sim\pi_0}\mathbb{E}_{y\sim q^{i-1}(y|x)}\left[e^{D(x,y)-1}\right] =$$

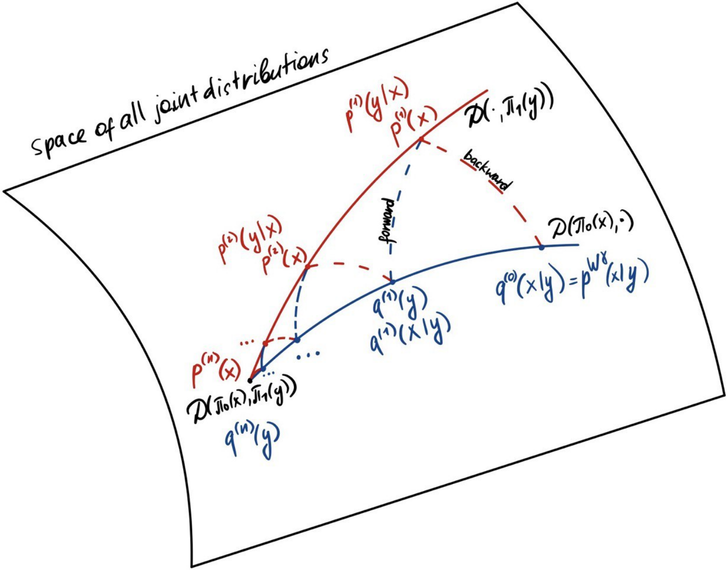
$$= \min_{G} \max_{D} \mathbb{E}_{y\sim\pi_1}\mathbb{E}_{z\sim p(z)}\left[D(G(z|y),y)\right] - \mathbb{E}_{x\sim\pi_0}\mathbb{E}_{z\sim p(z)}\left[e^{D(x,F^{i-1}(z|x))-1}\right]$$

## Forward

$$\min_{q(y|x)} \max_{D} \mathbb{E}_{x\sim\pi_0}\mathbb{E}_{y\sim q(y|x)}\left[D(x,y)\right] - \mathbb{E}_{y\sim\pi_1}\mathbb{E}_{x\sim p^{i}(x|y)}\left[e^{D(x,y)-1}\right] =$$

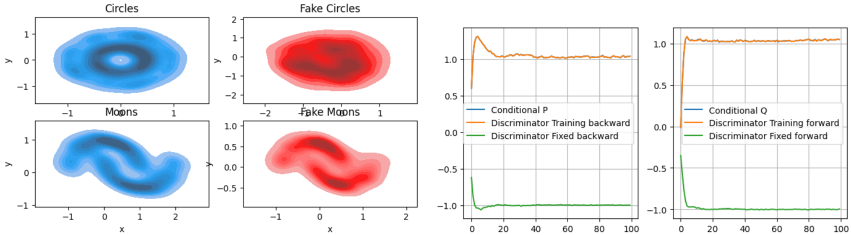
$$= \min_{F} \max_{D} \mathbb{E}_{x\sim\pi_0}\mathbb{E}_{z\sim p(z)}\left[D(x,F(z|x))\right] - \mathbb{E}_{y\sim\pi_1}\mathbb{E}_{z\sim p(z)}\left[e^{D(G^{i}(z|y),y)-1}\right]$$

# Suggested Method



# Analysis of suggested method

The graphs show the adversarial loss and final generative capabilities of adversarial bridges.




It was conducted several experiments on data from multi-modal normal distributions, and no mode collapse problem was encountered.

# Conclusion

1. Proposed method of solving Schrodinger Bridge Problem with adversarial training.
2. Conducted several experiments that shows great generative capability of proposed method,

# Related papers

1. Machine-learning approaches for the empirical Schrodinger bridge problem[1]
2. f-GAN: Training Generative Neural Samplers using Variational Divergence Minimization[2]
3. Conditional Generative Adversarial Nets[3]

[1] Machine-learning approaches for the empirical Schrodinger bridge problem, F. Vargas, 2021
[2] f-GAN: Training Generative Neural Samplers using Variational Divergence Minimization, S. Nowozin, 2016
[3] Conditional Generative Adversarial Nets, Mirza M. and Osindero S., 2014

# Future work plan

1. Analyse numerical error and stability of proposed algorithm;
2. Consider adversarial training without using IPF;
3. Investigate generalization of $D_{KL}$ to other distances or metrics.