Tower of Definitions in Cohesive Topos Theory

1. Category

A **category** $\mathcal{C}$ consists of:

- A class of **objects**, $\mathrm{Ob}(\mathcal{C})$,
- A class of **morphisms**, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, for each pair $X, Y \in \mathrm{Ob}(\mathcal{C})$,
- Composition maps $\circ : \mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$,
- Identity morphisms $\mathrm{id}_X \in \mathrm{Hom}(X, X)$ for each X,

satisfying associativity and identity laws.

2. Functor

A **functor** $F : \mathcal{C} \to \mathcal{D}$ assigns to each:

- Object $X \in \mathcal{C}$ an object $F(X) \in \mathcal{D}$,
- Morphism $f : X \to Y$ a morphism $F(f) : F(X) \to F(Y)$,

such that $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$.

3. Natural Transformation

A **natural transformation** $\eta : F \Rightarrow G$ between functors $F, G : \mathcal{C} \to \mathcal{D}$ consists of morphisms $\eta_X : F(X) \to G(X)$ such that for every $f : X \to Y$ in $\mathcal{C}$,

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y) \end{tikzcd}$$

commutes.

4. Adjunction

An **adjunction** between categories $\mathcal{C}$ and $\mathcal{D}$ consists of functors

$$F : \mathcal{C} \leftrightarrows \mathcal{D} : G$$

and natural transformations (unit η and counit ε)

$$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow G \circ F, \quad \varepsilon : F \circ G \Rightarrow \mathrm{Id}_{\mathcal{D}}$$

satisfying the triangle identities.

5. Topos

A **topos** $\mathcal{E}$ is a category that:

- Has all finite limits and colimits,
- Is Cartesian closed: has exponential objects $[X, Y]$,
- Has a subobject classifier Ω.

6. Geometric Morphism

A **geometric morphism** $f : \mathcal{E} \to \mathcal{F}$ between topoi consists of an adjoint pair

$$f^* : \mathcal{F} \leftrightarrows \mathcal{E} : f_*$$

with $f^* \dashv f_*$, where f^* preserves finite limits (i.e., is left exact).

7. Cohesive Topos

A **cohesive topos** is a topos $\mathcal{E}$ equipped with a quadruple of adjoint functors:

$$\Pi \dashv \Delta \dashv \Gamma \dashv \nabla : \mathcal{E} \leftrightarrows \mathbf{Set}$$

such that:

- Γ is the global sections functor,
- Δ is the constant sheaf functor,
- ∇ sends a set to a codiscrete object,
- Π is the shape or fundamental groupoid functor,
- Δ and ∇ are fully faithful,
- Δ preserves finite limits,
- Π preserves finite products (in some variants).

8. Cohesive Adjunction Tower Diagram

$$\mathcal{E}\ \begin{array}{c} \xrightarrow{\Pi\ \Gamma} \\ \xleftarrow{\qquad} \\ \dashv \\ \xleftarrow{\qquad} \\ \xrightarrow[\Delta\ \nabla]{} \end{array}\ \mathbf{Set}$$