Monads and Descent

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Анотація

Using category theory, we interpret descent data to determine, in very general settings, whether a morphism is a descent morphism or an effective descent morphism.

1 Chevalley Bifibrations and Descent

Let $P: \mathbf{M} \to \mathbf{A}$ denote a bifibrant functor [1]. For an object $A \in \mathbf{A}$, let $\mathbf{M}(A)$ denote the fibre over A. We assume that \mathbf{A} has fibred products.

1.1 Monad Associated with an Arrow

Let $a: A_1 \to A_0$ be an arrow in **A**. Denote by

$$\alpha^* : \mathbf{M}(A_0) \to \mathbf{M}(A_1)$$
 [resp. $\alpha_* : \mathbf{M}(A_1) \to \mathbf{M}(A_0)$]

the inverse image functor (resp. direct image functor), and

$$\eta^{\alpha}: \mathrm{Id}_{\mathbf{M}(A_1)} \to \alpha^* \alpha_*; \quad \varepsilon^{\alpha}: \alpha_* \alpha^* \to \mathrm{Id}_{\mathbf{M}(A_0)}$$

the canonical natural transformations making a_* a left adjoint to a^* . This adjunction defines [2] on $M(A_1)$ the monad $T^{\alpha} = (T^{\alpha}, \mu^{\alpha}, \eta^{\alpha})$, where

$$\mathsf{T}^{\mathfrak{a}} = \mathfrak{a}^* \mathfrak{a}_* : \mathbf{M}(\mathsf{A}_1) \to \mathbf{M}(\mathsf{A}_1), \quad \mu^{\mathfrak{a}} = \mathfrak{a}^* \varepsilon^{\mathfrak{a}} \mathfrak{a}_* : \mathsf{T}^{\mathfrak{a}} \circ \mathsf{T}^{\mathfrak{a}} \to \mathsf{T}^{\mathfrak{a}}.$$

Let \mathbf{M}^α denote the category $\mathbf{M}(A_1)^{(T^\alpha)}$ of algebras over the monad $T^\alpha,$ and let

$$U^{\mathsf{T}^{\alpha}}: \mathsf{M}^{\alpha} \to \mathsf{M}(\mathsf{A}_1), \quad \Phi^{\alpha}: \mathsf{M}(\mathsf{A}_0) \to \mathsf{M}^{\alpha}$$

be the canonical functors.

1.2 Chevalley Property

Definition 1. The functor P is a *Chevalley functor* if it satisfies the following property (C):

(C) For every commutative diagram in M

whose image under P is a cartesian square in A, if γ and γ' are cartesian and k_0 is cocartesian, then k_1 is cocartesian.

1.3 Characterization of Descent Data

Assume henceforth that $P: \mathbf{M} \to \mathbf{A}$ is a Chevalley functor. Let $\mathfrak{a}: A_1 \to A_0$ be an arrow in \mathbf{A} . Let A_2 be the fibred product $A_1 \times_{A_0} A_1$, with canonical projections $\mathfrak{a}_1, \mathfrak{a}_2: A_2 \to A_1$. The property (C) defines, for every object $M_1 \in \mathbf{M}(A_1)$, a canonical bijection, natural in M_1 ,

$$\operatorname{Hom}_{\mathbf{M}(A_2)}(\mathfrak{a}_1^*(M_1),\mathfrak{a}_2^*(M_1)) \to \operatorname{Hom}_{\mathbf{M}(A_1)}(\mathsf{T}^{\mathfrak{a}}(M_1),M_1),$$

denoted $\varphi \mapsto K^{\mathfrak{a}}(\varphi)$.

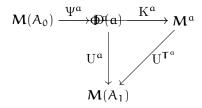
Lemma 1. An arrow $\varphi: \mathfrak{a}_1^*(M_1) \to \mathfrak{a}_2^*(M_1)$ such that $P(\varphi) = \mathrm{id}_{A_2}$ is a descent datum if and only if $K^{\mathfrak{a}}(\varphi)$ is an algebra over the monad $T^{\mathfrak{a}}$.

Let D(a) denote the category of descent data relative to a, and let

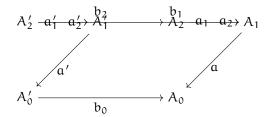
$$\Psi^{\alpha}: \mathbf{M}(A_0) \to \mathrm{D}(\alpha), \quad \mathrm{U}^{\alpha}: \mathrm{D}(\alpha) \to \mathbf{M}(A_1)$$

be the canonical functors.

Theorem 1. The correspondence $\phi \mapsto K^{\alpha}(\phi)$ induces an equivalence of categories $K^{\alpha}: D(\alpha) \to M^{\alpha}$, making the following diagram commute:



Proposition 1. The correspondence $\phi \mapsto K^{\alpha}(\phi)$ is universal. Precisely, for an arrow $b_0: A'_0 \to A_0$ in A, consider the change-of-base diagram in A:



For $M_1 \in \mathbf{M}(A_1)$ and $\varphi : \mathfrak{a}_1^*(M_1) \to \mathfrak{a}_2^*(M_1)$ in $\mathbf{M}(A_2)$,

$$K^{\alpha'}(b_2^*(\varphi)) = b_1^*(K^{\alpha}(\varphi)).$$

In particular, taking $A_0' = A_1$ and $b_0 = a$, if ϕ is a descent datum, then $b_2^*(\phi)$ is an effective descent datum. The converse holds, yielding:

Corollary 1. An arrow $\phi: \mathfrak{a}_1^*(M_1) \to \mathfrak{a}_2^*(M_1) \in M(A_2)$ is a descent datum if and only if its inverse image $\mathfrak{b}_2^*(\phi)$ under the canonical change of base $\mathfrak{b}_0 = \mathfrak{a}: A_0' = A_1 \to A_0$ is an effective descent datum.

This eliminates the need for the "cocycle condition" in subsequent arguments.

2 First Applications

Using Theorem 1, Beck's criterion [2] provides necessary and sufficient conditions for Ψ^a to be faithful, fully faithful, or an equivalence of categories, in terms of commutation and reflection of certain cokernels by \mathfrak{a}^* .

Proposition 2. If cokernels of pairs of arrows exist in $M(A_0)$, then Ψ^{α} has a left adjoint.

Proposition 3. The functor Ψ^{α} is faithful if and only if α^* is faithful.

Proposition 4. If α^* reflects cokernels, then Ψ^{α} is fully faithful. In particular, if all fibres of M are abelian, then

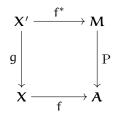
 Ψ^{α} faithful $\iff \Psi^{\alpha}$ fully faithful $\iff \alpha^*$ faithful.

Definition 2. An arrow $a:A_1\to A_0$ is faithfully flat if a^* commutes with cokernels and reflects isomorphisms.

Proposition 5. If $a: A_1 \to A_0$ is faithfully flat and cokernels exist in $\mathbf{M}(A_0)$, then Ψ^a is an equivalence of categories.

3 First Examples of Chevalley Functors

- 1. If A is the dual of the category of commutative rings and M is the dual of the category of modules over varying commutative rings, the obvious functor $P: M \to A$ is Chevalley.
- 2. If **A** is a category with fibred products and $\mathbf{M} = \mathsf{Fl}(\mathbf{A})$ is the category of arrows in **A**, the "target" functor $P: \mathbf{M} \to \mathbf{A}$ is Chevalley.
- 3. If $P: M \to A$ and $Q: N \to M$ are Chevalley, their composite $P \circ Q$ is Chevalley.
- 4. If $P:M\to A$ is Chevalley and I is any category, the functor $P^I:M^I\to A^I$ is Chevalley.
- 5. In a cartesian diagram of categories



if X has fibred products, f preserves fibred products, and P is Chevalley, then $f^*(P)$ is Chevalley.

In a future publication, we will provide further examples of Chevalley categories and more precise criteria for determining whether Ψ^{α} is faithful, fully faithful, or an equivalence when the fibres of M are algebraic categories (e.g., categories of modules).

Література

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- [3] C. Chevalley, Séminaire sur la descente, 1964–1965 (unpublished).