Issue XXXV: Cohomology and Spectra

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23 травня 2025 р.

Анотація

This article presents formal definitions and theorems for ordinary and generalized cohomology theories, unstable and stable spectra, and spectral sequences in Abelian categories, including the Serre, Atiyah-Hirzebruch, Leray, Eilenberg-Moore, Hochschild-Serre, Filtered Complex, Chromatic, Adams, and Bockstein spectral sequences. We define slopes, sheets, coordinates, quadrants, complex filtrations, and double complexes. Additionally, we explore the categorical foundations of cohomology theories and spectra, including their relationships to algebra, homological algebra, and stable homotopy theory, through isomorphisms, analogies, and instances.

Зміст

1 Stable Homotopy Type Theory

1.1 Ordinary Cohomology Theories

Definition 1. An *ordinary cohomology theory* on the category of topological spaces and pairs is a contravariant functor $H^*(-;G) : \mathrm{Top}^{\mathrm{op}} \to \mathrm{GrAb}$, assigning to each pair (X, A) a sequence of abelian groups $\{H^n(X, A; G)\}_{n\in\mathbb{Z}}$, with coefficient group G, satisfying:

1. *Homotopy*: If $f \simeq g : (X, A) \to (Y, B)$, then $f^* = g^* : H^n(Y, B; G) \to H^n(X, A; G)$.
2. *Exactness*: For (X, A), there is a long exact sequence: $\cdots \to H^n(X, A; G) \to H^n(X; G) \to H^n(A; G) \xrightarrow{\delta} H^{n+1}(X, A; G) \to \cdots$
3. *Excision*: For $U \subset A$ with $\overline{U} \subset \mathrm{int}(A)$, the inclusion $(X\backslash U, A\backslash U) \hookrightarrow (X, A)$ induces isomorphisms $H^n(X, A; G) \cong H^n(X \setminus U, A \setminus U; G)$.
4. *Additivity*: For $X = \bigsqcup X_i$, $H^n(X; G) \cong \bigoplus H^n(X_i; G)$.
5. *Dimension*: For a point pt, $H^n(\mathrm{pt}; G) = \begin{cases} G & n = 0 \\ 0 & n \neq 0 \end{cases}$.

1.2 Generalized Cohomology Theories

Definition 2. A *generalized cohomology theory* is a contravariant functor $h^* : \mathrm{Top}^{\mathrm{op}} \to \mathrm{GrAb}$, assigning to each pair (X, A) a sequence $\{h^n(X, A)\}_{n\in\mathbb{Z}}$, satisfying:

1. *Homotopy*, *Exactness*, *Excision*, and *Additivity* as in **Definition 1**.
2. *Suspension*: There is a natural isomorphism $h^n(X, A) \cong h^{n+1}(\Sigma X, \Sigma A)$, where Σ is the reduced suspension.

The groups $h^n(\mathrm{pt})$ form a graded ring, the coefficients of h^*.

Theorem 1. Every generalized cohomology theory h^* is representable by a spectrum $E = \{E_n, \sigma_n : \Sigma E_n \to E_{n+1}\}$, with $h^n(X) \cong [X, E_n]_*$, where $[-, -]_*$ denotes pointed homotopy classes.

1.3 Unstable and Stable Spectra

Definition 3. A *spectrum* is a sequence of pointed spaces $\{E_n\}_{n\in I}$, where $I \subseteq \mathbb{Z}$, with structure maps $\sigma_n : \Sigma E_n \to E_{n+1}$. It is:

- *Unstable* if $I \subseteq \mathbb{Z}_{\geq 0}$.
- *Stable* if $I = \mathbb{Z}$ and each σ_n is a homotopy equivalence.

Theorem 2. For an unstable spectrum E, the functor $X \mapsto [X, E_n]_*$ defines a cohomology theory on spaces of dimension $\leq n$. For a stable spectrum E, the functor $h^n(X) = [X, E_n]_*$ defines a generalized cohomology theory.

1.4 Categorical Interpretation

This section explores the categorical foundations of ordinary and generalized cohomology theories, their associated spectra, and the relationships between algebraic and topological categories through isomorphisms, analogies, and instances. We formalize these structures and highlight their categorical nuances isomorphisms and non-isomorphic relationships, drawing on frameworks like algebra, homological algebra, and stable homotopy theory.

Definition 4. The *category of spectra*, denoted Spectra, is the category whose objects are stable spectra $E = \{E_n, \sigma_n : \Sigma E_n \to E_{n+1}\}$, where E_n are pointed spaces and σ_n are homotopy equivalences. Morphisms are collections of maps $f_n : E_n \to F_n$ compatible with structure maps. The *stable homotopy category* is the localization of Spectra at weak equivalences (maps inducing isomorphisms on homotopy groups).

Definition 5. An *ordinary cohomology theory* is a functor $H^*(-; G) : \mathrm{Top}^{\mathrm{op}} \to \mathrm{GrAb}$ satisfying the Eilenberg-Steenrod axioms (Definition 1). Categorically, it is represented by the Eilenberg-MacLane spectrum H, where $A \in \mathrm{Ab}$, with $H^n(X; A) \cong [X, H_n]_*$.

Definition 6. A *generalized cohomology theory* is a functor $h^* : \mathrm{Top}^{\mathrm{op}} \to \mathrm{GrAb}$ satisfying the axioms of Definition 2. It is representable in Spectra, with $h^n(X) \cong [X, E_n]_*$ for a spectrum E.

Theorem 3 (Brown Representability)**.** Every generalized cohomology theory h^* on Top is representable by a spectrum $E \in$ Spectra, i.e., there exists E such that $h^n(X) \cong [X, E_n]_*$ for all $X \in$ Top.

Theorem 4. The stable homotopy category Spectra is a triangulated category, with distinguished triangles corresponding to cofiber sequences. It is equivalent to the category of spectra localized at weak equivalences.

Theorem 5. The functor $A \mapsto H$ from Ab to Spectra, mapping an abelian group to its Eilenberg-MacLane spectrum, is faithful but not full. The induced functor on ordinary cohomology theories to generalized cohomology theories is an embedding of categories.

1.4.1 Algebraic and Spectral Correspondences

Mathematics is unified through *isomorphisms* (categorical equivalences), *analogies* (functorial similarities), and *instances* (specific subcategories or objects). We present a correspondence table linking Algebra (Ab), Homological Algebra (Ch($\mathbb{Z}$)), Ordinary Cohomology, K-Theory, Superalgebra, and Stable Spectra (Spectra).

Definition 7 (Isomorphism)**.** An **isomorphism** in a category $\mathcal{C}$ is a morphism $f : A \to B$ with an inverse $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. For categories, an isomorphism is an equivalence, i.e., a functor $F : \mathcal{C} \to \mathcal{D}$ with a quasi-inverse $G : \mathcal{D} \to \mathcal{C}$.

Definition 8 (Analogy)**.** A non-isomorphic **analogy** is a structural similarity between objects or categories, captured by functors that preserve some properties but not all, ensuring no categorical equivalence.

Definition 9 (Instance)**.** An **instance** is a specific object or subcategory within a broader category, embedded via a faithful functor. A column in the table is an instance of another if its structures are special cases of the latter's, maintaining non-isomorphic distinctions from other categories.

Табл. 1: Algebraic and Spectral Correspondences

Category	Object	Ring	Initial Unit	Operations
Algebra	Abelian group	Ring	$\mathbb{Z}$	$\oplus, \otimes$
Homological Algebra	Chain complex	dg-ring	$\mathbb{Z}[0]$	$\oplus, \otimes$
Superalgebra	$\mathbb{Z}/2\mathbb{Z}$-graded Ab	$\mathbb{Z}/2\mathbb{Z}$-graded Ring	$\mathbb{Z}$	$\oplus, \otimes$
Ordinary Cohomology	Cohomology $H^*(-; A)$	Graded ring	$H^*(-; \mathbb{Z})$	$\oplus, \otimes$
Complex K-Theory	Graded abelian group	Graded ring	KU	$\vee, \wedge$
Real K-Theory	Graded abelian group	Graded ring	KO	$\vee, \wedge$
Stable Spectra	Stable spectrum	Ring spectrum	$\mathbb{S}$	$\vee, \wedge$

- *Isomorphisms*: Rare, e.g., Ab $\cong$ $\mathrm{Mod}_{\mathbb{Z}}$. Most relationships are non-isomorphic.
- *Analogies*: The tensor product $\otimes$ in Ab and smash product $\wedge$ in Spectra are analogous, but Ab $\ncong$ Spectra due to Spectra's triangulated structure.
- *Instances*: KU, KO, and H are instances of Spectra. Superalgebra is an instance of Ab via the forgetful functor.

Example 1. The functor $A \mapsto A[0]$ embeds Ab into $\mathrm{Ch}(\mathbb{Z})$, but $\mathrm{Ch}(\mathbb{Z}) \ncong$ Ab due to differentials. Similarly, $H : \mathrm{Ab} \to \mathrm{Spectra}$ embeds abelian groups as Eilenberg-MacLane spectra, but Spectra's stable phenomena (e.g., suspension equivalences) distinguish it.

Remark 1. Non-isomorphic analogies require careful handling. Conflating $\wedge$ in Spectra with $\otimes$ in Ab can lead to errors in spectral sequence computations, as $\wedge$ introduces higher Tor terms.

1.5 Spectral Sequences

Definition 10. A *spectral sequence* in an Abelian category $\mathcal{A}$ is a collection of objects $\{E_r^{p,q}\}_{r\geq 1, p,q\in\mathbb{Z}}$, $E_r^{p,q} \in \mathcal{A}$, with differentials:

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+a_r, q+b_r},$$

such that:

1. $d_r \circ d_r = 0$.
2. $E_{r+1}^{p,q} = H^{p,q}(E_r, d_r) = \ker(d_r^{p,q})/\mathrm{im}(d_r^{p-a_r, q-b_r})$.
3. There exists a graded object $H^n \in \mathcal{A}$ with filtration $F_p H^{p+q} \subseteq H^{p+q}$, such that:
$$E_\infty^{p,q} \cong F_p H^{p+q}/F_{p-1} H^{p+q}.$$

The sequence is *first-quadrant* if $E_r^{p,q} = 0$ for $p < 0$ or $q < 0$.

Definition 11. The r-*th sheet* of a spectral sequence is the collection $\{E_r^{p,q}\}_{p,q}$. The indices (p, q) are *coordinates*, with p the filtration degree and q the complementary degree, satisfying total degree $n = p + q$. The *slope* of $d_r : E_r^{p,q} \to E_r^{p+r, q-r+1}$ is $\frac{-r+1}{r}$.

Definition 12. A *filtered complex* in $\mathcal{A} = \mathrm{Ab}$ is a chain complex (C_*, ∂) with a filtration $\cdots \subseteq F_{p-1}C_n \subseteq F_p C_n \subseteq F_{p+1} C_n \subseteq \cdots$, compatible with ∂. A *double complex* is a bigraded object $C_{p,q}$ with differentials $d^h : C_{p,q} \to C_{p-1,q}$, $d^v : C_{p,q} \to C_{p,q-1}$, satisfying $d^h d^h = d^v d^v = d^h d^v + d^v d^h = 0$. The total complex is $\mathrm{Tot}(C)_n = \bigoplus_{p+q=n} C_{p,q}$.

Theorem 6. A filtered complex (C_*, F_p) induces a spectral sequence with:

$$E_0^{p,q} = F_p C_{p+q}/F_{p-1} C_{p+q}, \quad E_1^{p,q} = H_{p+q}(F_p C/F_{p-1} C) \implies H_{p+q}(C).$$

A double complex $C_{p,q}$ with filtration by p-index induces:

$$E_1^{p,q} = H_q^v(C_{p,*}), \quad d_1 = H(d^h) \implies H_{p+q}(\mathrm{Tot}(C)).$$

1.5.1 Serre Spectral Sequence

Theorem 7. For a fibration $F \to E \to B$ with B path-connected, there exists a first-quadrant spectral sequence:

$$E_2^{p,q} = H^p(B; H^q(F; \mathbb{Z})) \implies H^{p+q}(E; \mathbb{Z}),$$

with $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$.

1.5.2 Atiyah-Hirzebruch Spectral Sequence

Theorem 8. For a generalized cohomology theory h^* and a CW-complex X, there exists a spectral sequence:

$$E_2^{p,q} = H^p(X; h^q(\mathrm{pt})) \implies h^{p+q}(X),$$

with $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$.

1.5.3 Leray Spectral Sequence

Theorem 9. For a continuous map $f : X \to Y$ and a sheaf $\mathcal{F}$ on X, there exists a spectral sequence:

$$E_2^{p,q} = H^p(Y; R^q f_* \mathcal{F}) \implies H^{p+q}(X; \mathcal{F}),$$

with $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$.

1.5.4 Eilenberg-Moore Spectral Sequence

Theorem 10. For a pullback diagram with fibration $F \to E \to B$, there exists a spectral sequence:

$$E_2^{p,q} = \mathrm{Tor}^{p,q}_{H_*(B)}(H_*(F), R) \implies H_{p+q}(F; R),$$

with $d_r : E_r^{p,q} \to E_r^{p-r,q+r-1}$.

1.5.5 Hochschild-Serre Spectral Sequence

Theorem 11. For a group extension $1 \to N \to G \to Q \to 1$, there exists a spectral sequence:

$$E_2^{p,q} = H^p(Q; H^q(N; R)) \implies H^{p+q}(G; R),$$

with $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$.

1.5.6 Spectral Sequence of a Filtered Complex

Theorem 12. For a filtered complex (C_*, F_p), there exists a spectral sequence:

$$E_1^{p,q} = H_{p+q}(F_p C / F_{p-1} C) \implies H_{p+q}(C),$$

with $d_r : E_r^{p,q} \to E_r^{p-r,q+r-1}$.

1.5.7 Chromatic Spectral Sequence

Theorem 13. For a spectrum X, there exists a spectral sequence:

$$E_1^{n,k} = \pi_{n-k}(L_{K(k)}X) \implies \pi_{n-k}(X),$$

where $L_{K(k)}X$ is the localization at the k-th Morava K-theory, with $d_r : E_r^{n,k} \to E_r^{n+1,k-r}$.

1.5.8 Adams Spectral Sequence

Theorem 14. For a spectrum X and prime p, there exists a spectral sequence:

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\mathrm{Hom}_*(X, \mathbb{Z}/p), \mathbb{Z}/p) \implies \pi_{t-s}(X_{(p)}),$$

where A is the Steenrod algebra, with $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$.

1.5.9 Bockstein Spectral Sequence

Theorem 15. For a short exact sequence $0 \to R \to R' \to R'' \to 0$ of coefficient rings, there exists a spectral sequence:

$$E_1^{p,q} = H^{p+q}(X; R'') \implies H^{p+q}(X; R),$$

with $d_r : E_r^{p,q} \to E_r^{p+1,q-r}$.

Література

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