Issue XLI: Local Homotopy Type Theory

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15 травня 2025 р.

Анотація

Keywords: Motivic Stable Homotopy Theory

Зміст

1 Local Homotopy Type Theory

Motivic homotopy theory, introduced by Morel and Voevodsky, extends classical homotopy theory to the setting of algebraic geometry, treating schemes as analogous to topological spaces. A central object in this framework is the category of *T-spectra*, which generalizes the notion of spectra in stable homotopy theory to the motivic context, where the circle S^1 is replaced by the Tate object $\mathbb{T} = \mathbb{A}^1/(\mathbb{A}^1 \setminus \{0\})$. John F. Jardine's work on motivic symmetric spectra provides a categorical model for the motivic stable category, equipped with a symmetric monoidal smash product, enabling rich interactions between algebraic and topological structures [1].

This article formalizes the category of T-spectra, emphasizing Jardine's contributions. We define T-spectra and symmetric T-spectra, describe their model category structure, and present key theorems on stable equivalences and monoidal properties. Applications to algebraic geometry, such as the study of motivic cohomology and algebraic K-theory, are discussed.

We assume familiarity with basic category theory and algebraic geometry. Below, we outline essential concepts.

Definition 1. Let S be a Noetherian scheme of finite Krull dimension. The category Sm_S consists of smooth schemes of finite type over S, with morphisms being scheme morphisms over S.

Definition 2. A *simplicial presheaf* on Sm_S is a contravariant functor from Sm_S to the category of simplicial sets. The category of simplicial presheaves, denoted $\mathrm{Sp}(\mathrm{Sm}_S)$, is equipped with a proper closed simplicial model structure, as constructed by Morel and Voevodsky [3].

Remark 1. The *Nisnevich topology* on Sm_S, denoted Nis, is a Grothendieck topology coarser than the Zariski topology but finer than the étale topology. It is crucial for defining the motivic model category.

1.1 Definition of T-Spectra

In motivic homotopy theory, the Tate object $\mathbb{T}$ plays the role of the suspension functor. We define T-spectra as follows.

Definition 3. A *T-spectrum* over a scheme S is a sequence of pointed simplicial presheaves $E = \{E_n\}_{n\geq 0}$ on Sm_S, equipped with structure maps $\sigma_n : \mathbb{T} \wedge E_n \to E_{n+1}$, where $\wedge$ denotes the smash product of pointed presheaves. The category of T-spectra, denoted $\mathrm{Sp}_S^{\mathbb{T}}$, has morphisms given by sequences of maps $f_n : E_n \to F_n$ compatible with the structure maps.

Example 1. The *motivic sphere spectrum* $\mathbb{S}$ is a T-spectrum with $\mathbb{S}_n = \mathbb{T}^{\wedge n}$, where $\mathbb{T}^{\wedge n}$ is the n-fold smash product of $\mathbb{T}$, and structure maps given by the identity.

1.2 Symmetric T-Spectra

Jardine's work focuses on symmetric T-spectra, which incorporate symmetric group actions to define a robust smash product.

Definition 4. A *symmetric T-spectrum* over S is a T-spectrum $E = \{E_n\}_{n\geq 0}$ where each E_n is equipped with an action of the symmetric group Σ_n, and the structure maps $\sigma_n : \mathbb{T} \wedge E_n \to E_{n+1}$ are Σ_n-equivariant with respect to the trivial action on $\mathbb{T}$. The category of symmetric T-spectra is denoted $\mathrm{Sp}_S^{\mathbb{T},\Sigma}$.

Remark 2. The symmetric structure allows for a well-defined internal smash product, making $\mathrm{Sp}_S^{\mathbb{T},\Sigma}$ a symmetric monoidal category [1].

1.3 Model Category Structure

Jardine establishes a model category structure on $\mathrm{Sp}_S^{\mathbb{T}}$ and $\mathrm{Sp}_S^{\mathbb{T},\Sigma}$.

Theorem 1. The category $\mathrm{Sp}_S^{\mathbb{T}}$ admits a proper closed simplicial model structure where:

- *Weak equivalences* are maps $f : E \to F$ inducing isomorphisms on stable homotopy groups in the Nisnevich topology.

- *Cofibrations* are monomorphisms.
- *Fibrations* are defined via the right lifting property with respect to trivial cofibrations.

A Bousfield localization of this model structure with respect to stable weak equivalences yields the *motivic stable category* [1].

Theorem 2. The category $\mathrm{Sp}_S^{\mathbb{T},\Sigma}$ is a cofibrantly generated, symmetric monoidal model category satisfying the monoid axiom. The smash product $\wedge$ is an internal symmetric monoidal structure, with unit the sphere spectrum $\mathbb{S}$.

1.4 Key Theorems

Jardine's results provide a categorical foundation for motivic stable homotopy theory.

Theorem 3. The motivic stable category, obtained as the homotopy category of $\mathrm{Sp}_S^{\mathbb{T},\Sigma}$, is equivalent to the localization of $\mathrm{Sp}_S^{\mathbb{T}}$ at stable weak equivalences. Stable equivalences in this category are stable homotopy isomorphisms in the Nisnevich topology [1].

Theorem 4. The symmetric smash product $\wedge$ on $\mathrm{Sp}_S^{\mathbb{T},\Sigma}$ is associative, commutative, and unital up to homotopy, making the motivic stable category a symmetric monoidal category with unit $\mathbb{S}$.

Example 2. The Eilenberg-MacLane spectrum H for an abelian group A is a symmetric T-spectrum, with $H_n = K(A, n)$, the simplicial presheaf representing motivic cohomology. Its homotopy groups recover motivic cohomology groups.

1.5 Conclusion

T-spectra provide a framework for studying phenomena in algebraic geometry and stable homotopy theory: - *Motivic Cohomology*: The spectrum H represents motivic cohomology, connecting algebraic cycles to homotopy theory. - *Algebraic K-Theory*: Voevodsky's motivic spectrum for K-theory, refined by Jardine's symmetric structures, links K-theory to stable homotopy [1]. - *Algebraic Geometry*: The motivic stable category facilitates the study of Gysin triangles and oriented spectra, generalizing classical results in algebraic topology [1].

The category of T-spectra, as developed by Jardine, provides a powerful framework for motivic homotopy theory. By equipping T-spectra with a symmetric monoidal smash product and a robust model category structure, Jardine's work bridges algebraic geometry and stable homotopy theory. Future directions include exploring representability theorems for presheaves of spectra and applications to topological modular forms [2].

Література

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