

Issue II: Inductive Types

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Abstract

Impredicative Encoding of Inductive Types in HoTT.

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1 Inductive Types

1.1 Empty

1.2 Unit

1.3 Bool

1.4 Maybe

1.5 Either

1.6 Nat

1.7 List

2 Inductive Encodings

2.1 Church Encoding

You know Church encoding which also has its dependent analogue in CoC, however in Coq it is impossible to derive Inductive Principle as type system lacks fixpoint and functional extensionality. The example of working compiler of PTS languages are Om and Morte. Assume we have Church encoded NAT:

$$\text{nat} = (X:U) \rightarrow (X \rightarrow X) \rightarrow X \rightarrow X$$

where first parameter $(X \rightarrow X)$ is a *succ*, the second parameter X is *zero*, and the result of encoding is landed in X . Even if we encode the parameter

$$\text{list } (A: U) = (X:U) \rightarrow X \rightarrow (A \rightarrow X) \rightarrow X$$

and parameter A let's say live in 42 universe and X live in 2 universe, then by the signature of encoding the term will be landed in X , thus 2 universe. In other words such dependency is called impredicative displaying that landed term is not a predicate over parameters. This means that Church encoding is incompatible with predicative type checkers with predicative of predicative-cumulative hierarchies.

2.2 Impredicative Encoding

In HoTT n -types is encoded as n -groupoids, thus we need to add a predicate in which n -type we would like to land the encoding:

$$\text{NAT } (A: U) = (X:U) \rightarrow \text{isSet } X \rightarrow X \rightarrow (A \rightarrow X) \rightarrow X$$

Here we added *isSet* predicate. With this motto we can implement propositional truncation by landing term in *isProp* or even HIT by landing in *isGroupoid*:

```

TRUN (A:U) type = (X: U) → isProp X → (A → X) → X
S1 = (X:U) → isGroupoid X → ((x:X) → Path X x x) → X
MONOPL (A:U) = (X:U) → isSet X → (A → X) → X
NAT = (X:U) → isSet X → X → (A → X) → X

```

The main publication on this topic could be found at [1] and [2].

2.3 The Unit Example

Here we have the implementation of Unit impredicative encoding in HoTT.

```

upPath      (X Y:U)(f:X→Y)(a:X→X): X → Y = o X X Y f a
downPath    (X Y:U)(f:X→Y)(b:Y→Y): X → Y = o X Y Y b f
naturality  (X Y:U)(f:X→Y)(a:X→X)(b:Y→Y): U
  = Path (X→Y)(upPath X Y f a)(downPath X Y f b)

unitEnc': U = (X: U) → isSet X → X → X
isUnitEnc (one: unitEnc'): U
  = (X Y:U)(x:isSet X)(y:isSet Y)(f:X→Y) →
    naturality X Y f (one X x)(one Y y)

unitEnc: U = (x: unitEnc') * isUnitEnc x
unitEncStar: unitEnc = (\(X:U)(_:isSet X) →
  idfun X, \ (X Y: U) ( _:isSet X) ( _:isSet Y) → refl (X→Y))
unitEncRec (C: U) (s: isSet C) (c: C): unitEnc → C
  = \ (z: unitEnc) → z.1 C s c
unitEncBeta (C: U) (s: isSet C) (c: C)
  : Path C (unitEncRec C s c unitEncStar) c = refl C c
unitEncEta (z: unitEnc): Path unitEnc unitEncStar z = undefined
unitEncInd (P: unitEnc → U) (a: unitEnc): P unitEncStar → P a
  = subst unitEnc P unitEncStar a (unitEncEta a)
unitEncCondition (n: unitEnc'): isProp (isUnitEnc n)
  = \ (f g: isUnitEnc n) →
    <h> \ (x y: U) → \ (X: isSet x) → \ (Y: isSet y)
    → \ (F: x → y) → <i> \ (R: x) → Y (F (n x X R)) (n y Y (F R))
    (<j> f x y X Y F @ j R) (<j> g x y X Y F @ j R) @ h @ i

```

References

- [1] Steve Awodey. Impredicative encodings in hott. 2017.
- [2] Sam Speight. Impredicative encoding of inductive types in hott. 2017.