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Introduction to Formalization of Mathematics

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Зміст

Issue XXV: Category Theory

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Анотація

Formal definition of Category.

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1 Category Theory

Category Theory provides a rigorous framework for abstracting and unifying mathematical structures. Developed in the 1940s by Samuel Eilenberg and Saunders Mac Lane to address coherence problems in algebraic topology, it generalizes relationships between mathematical objects across diverse fields like algebra, geometry, and computer science. Category Theory captures objects and their morphisms—functions preserving structure—as a universal systems theory, akin to a universal algebra of functions, emphasizing composition and transformation. Interpreted as a foundational language, a tool for structural analysis, or a bridge to computer-aided formalization, it solves problems of abstraction and generalization. Categories serve as a stepping stone to topos theory, which enriches logical and geometric insights, and higher cohesive topos theory, extending to infinity-categories for advanced applications.

1.1 Category

First of all very simple category theory up to pullbacks is provided. We give here all definitions only to keep the context valid.

A **category** $\mathcal{C}$ consists of:

- A class of **objects**, $\text{Ob}(\mathcal{C})$,
- A class of **morphisms**, $\text{Hom}_{\mathcal{C}}(X, Y)$, for each pair $X, Y \in \text{Ob}(\mathcal{C})$,
- Composition maps $\circ : \text{Hom}(Y, Z) \times \text{Hom}(X, Y) \to \text{Hom}(X, Z)$,
- Identity morphisms $\text{id}_X \in \text{Hom}(X, X)$ for each X,

satisfying associativity and identity laws.

Definition 1. (Category Signature). The signature of category is a $\sum_{A:U} A \to A \to U$ where U could be any universe. The pr_1 projection is called Ob and pr_2 projection is called $\text{Hom}(a, b)$, where $a, b : \text{Ob}$.

```
cat: U = (A: U) * (A -> A -> U)
```

Definition 2. (Precategory). More formal, precategory C consists of the following. (i) A type Ob_C, whose elements are called objects; (ii) for each $a, b : \text{Ob}_C$, a set $\text{Hom}_C(a, b)$, whose elements are called arrows or morphisms. (iii) For each $a : \text{Ob}_C$, a morphism $1_a : \text{Hom}_C(a, a)$, called the identity morphism. (iv) For each $a, b, c : \text{Ob}_C$, a function $\text{Hom}_C(b, c) \to \text{Hom}_C(a, b) \to \text{Hom}_C(a, c)$ called composition, and denoted $g \circ f$. (v) For each $a, b : \text{Ob}_C$ and $f : \text{Hom}_C(a, b)$, $f = 1_b \circ f$ and $f = f \circ 1_a$. (vi) For each $a, b, c, d : A$ and $f : \text{Hom}_C(a, b)$, $g : \text{Hom}_C(b, c)$, $h : \text{Hom}_C(c, d)$, $h \circ (g \circ f) = (h \circ g) \circ f$.

```
def cat : U₁
 := Σ (ob: U) (hom: ob -> ob -> U), unit

def isPrecategory (C: cat) : U := Σ
    (id:       Π (x: C.ob), C.hom x x)
    (∘:        Π (x y z: C.ob),
                 C.hom x y -> C.hom y z -> C.hom x z)
    (homSet:   Π (x y: C.ob), isSet (C.hom x y))
    (∘-left:   Π (x y: C.ob) (f: C.hom x y),
               = (C.hom x y) (∘ x x y (id x) f) f)
    (∘-right:  Π (x y: C.ob) (f: C.hom x y),
               = (C.hom x y) (∘ x y y f (id y)) f)
    (∘-assoc:  Π (x y z w: C.ob) (f: C.hom x y)
                 (g: C.hom y z) (h: C.hom z w),
               = (C.hom x w) (∘ x z w (∘ x y z f g) h)
                             (∘ x y w f (∘ y z w g h))), 1
def precategory: U₁ := Σ (C: cat) (P: isPrecategory C), unit
```

Univalent Categories:

```
def isoCat (P: precategory) (A B: P.C.ob) : U := Σ
     (f: P.C.hom A B)
     (g: P.C.hom B A)
     (retract: Path (P.C.hom A A) (P.P.∘ A B A f g) (P.P.id A))
     (section: Path (P.C.hom B B) (P.P.∘ B A B g f) (P.P.id B)), 1

def isCategory (P: precategory): U
 := Σ (A: P.C.ob), isContr (Π (B: P.C.ob), isoCat P A B)

def category: U₁
 := Σ (P: precategory), isCategory P
```

1.2 Pullback

Definition 3. (Categorical Pullback). The pullback of the cospan $A \xrightarrow{f} C \xleftarrow{g} B$ is a object $A \times_C B$ with morphisms $pb_1 : \times_C \to A$, $pb_2 : \times_C \to B$, such that diagram commutes:

$$\begin{tikzcd}
A \times_C B \arrow[r, "pb_2"] \arrow[d, "f"'] \arrow[dr] & B \arrow[d, "pb_1"] \\
A \arrow[r, "g"] & C
\end{tikzcd}$$

Pullback $(\times_C, pb_1, pb_2)$ must be universal, means for any (D, q_1, q_2) for which diagram also commutes there must exists a unique $u : D \to \times_C$, such that $pb_1 \circ u = q_1$ and $pb_2 \circ q_2$.

```
def homTo  (P: precategory) (X: P.C.ob): U
 := Σ (Y: P.C.ob), P.C.hom Y X

def cospan (P: precategory): U
 := Σ (X: P.C.ob) (_: homTo P X), homTo P X

def hasCospanCone (P: precategory) (D: cospan P) (w: P.C.ob) : U
 := Σ (f: P.C.hom w D.2.1.1) (g: P.C.hom w D.2.2.1),
    = (P.C.hom w D.1) (P.P.∘ w D.2.1.1 D.1 f D.2.1.2)
                      (P.P.∘ w D.2.2.1 D.1 g D.2.2.2)
def cospanCone (P: precategory) (D: cospan P): U
 := Σ (w: P.C.ob), hasCospanCone P D w

def isCospanConeHom (P: precategory) (D: cospan P)
    (E1 E2: cospanCone P D) (h: P.C.hom E1.1 E2.1) : U
 := Σ (_ : = (P.C.hom E1.1 D.2.1.1)
             (P.P.∘ E1.1 E2.1 D.2.1.1 h E2.2.1) E1.2.1),
           = (P.C.hom E1.1 D.2.2.1)
             (P.P.∘ E1.1 E2.1 D.2.2.1 h E2.2.2.1) E1.2.2.1

def cospanConeHom (P: precategory) (D: cospan P) (E1 E2: cospanCone P D) : U
 := Σ (h: P.C.hom E1.1 E2.1), isCospanConeHom P D E1 E2 h

def isPullback (P: precategory) (D: cospan P) (E: cospanCone P D) : U
 := Σ (h: cospanCone P D), isContr (cospanConeHom P D h E)

def hasPullback (P: precategory) (D: cospan P) : U
 := Σ (E: cospanCone P D), isPullback P D E
```

1.3 Functor

A **functor** $F : \mathcal{C} \to \mathcal{D}$ assigns to each:

- Object $X \in \mathcal{C}$ an object $F(X) \in \mathcal{D}$,
- Morphism $f : X \to Y$ a morphism $F(f) : F(X) \to F(Y)$,

such that $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$.

Definition 4. (Category Functor). Let A and B be precategories. A functor $F : A \to B$ consists of: (i) A function $F_{Ob} : Ob_h A \to Ob_B$; (ii) for each $a, b : Ob_A$, a function $F_{Hom} : Hom_A(a, b) \to Hom_B(F_{Ob}(a), F_{Ob}(b))$; (iii) for each $a : Ob_A$, $F_{Ob}(1_a) = 1_{F_{Ob}}(a)$; (iv) for $a, b, c : Ob_A$ and $f : Hom_A(a, b)$ and $g : Hom_A(b, c)$, $F(g \circ f) = F_{Hom}(g) \circ F_{Hom}(f)$.

```
catfunctor (A B: precategory): U
  = (ob: carrier A -> carrier B)
  * (mor: (x y:carrier A)->hom A x y->hom B(ob x)(ob y))
  * (id: (x: carrier A) -> Path (hom B (ob x) (ob x))
    (mor x x (path A x)) (path B (ob x)))
  * ((x y z: carrier A) -> (f: hom A x y) -> (g: hom A y z) ->
    Path (hom B (ob x) (ob z)) (mor x z (compose A x y z f g))
         (compose B (ob x) (ob y) (ob z) (mor x y f) (mor y z g)))
```

1.4 Terminals

Definition 5. (Terminal Object). Is such object Ob_{C}, that

$$\prod_{x,y:\mathrm{Ob}_{\mathrm{C}}} \mathrm{isContr}(\mathrm{Hom}_{\mathrm{C}}(y,x)).$$

```
isTerminal (C: precategory) (y: carrier C): U
  = (x: carrier C) -> isContr (hom C x y)
terminal (C: precategory): U
  = (y: carrier C) * isTerminal C y
```

1.5 Natural Transformation

A **natural transformation** $\eta : \mathsf{F} \Rightarrow \mathsf{G}$ between functors $\mathsf{F}, \mathsf{G} : \mathcal{C} \to \mathcal{D}$ consists of morphisms $\eta_{\mathsf{X}} : \mathsf{F}(\mathsf{X}) \to \mathsf{G}(\mathsf{X})$ such that for every $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ in $\mathcal{C}$,

$$\begin{tikzcd}
\mathsf{F}(\mathsf{X}) \arrow[r, "\eta_{\mathsf{X}}"] \arrow[d, "\mathsf{F}(\mathsf{f})"'] & \mathsf{G}(\mathsf{X}) \arrow[d, "\mathsf{G}(\mathsf{f})"] \\
\mathsf{F}(\mathsf{Y}) \arrow[r, "\eta_{\mathsf{Y}}"'] & \mathsf{G}(\mathsf{Y})
\end{tikzcd}$$

commutes.

```
isNaturalTrans (C D: precategory)
               (F G: catfunctor C D)
               (eta: (x: carrier C) -> hom D (F.1 x) (G.1 x)): U
  = (x y: carrier C) (h: hom C x y) ->
    Path (hom D (F.1 x) (G.1 y))
         (compose D (F.1 x) (F.1 y) (G.1 y) (F.2.1 x y h) (eta y))
         (compose D (F.1 x) (G.1 x) (G.1 y) (eta x) (G.2.1 x y h))

ntrans (C D: precategory) (F G: catfunctor C D): U
    = (eta: (x: carrier C) -> hom D (F.1 x) (G.1 x))
    * (isNaturalTrans C D F G eta)
```

1.6 Adjunction

An **adjunction** between categories $\mathcal{C}$ and $\mathcal{D}$ consists of functors

$$\mathsf{F} : \mathcal{C} \leftrightarrows \mathcal{D} : \mathsf{G}$$

and natural transformations (unit η and counit ε)

$$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \mathsf{G} \circ \mathsf{F}, \quad \varepsilon : \mathsf{F} \circ \mathsf{G} \Rightarrow \mathrm{Id}_{\mathcal{D}}$$

satisfying the triangle identities.

```
ntransL (C D: precategory) (F G: catfunctor C D)
        (f: ntrans C D F G) (B: precategory) (H: catfunctor B C)
      : ntrans B D (compFunctor B C D H F) (compFunctor B C D H G)
      = (eta, p) where
        F': catfunctor B D = compFunctor B C D H F
        G': catfunctor B D = compFunctor B C D H G
        eta (x: carrier B): hom D (F'.1 x) (G'.1 x) = f.1 (H.1 x)
        p (x y: carrier B) (h: hom B x y): Path (hom D (F'.1 x) (G'.1 y))
            (compose D (F'.1 x) (F'.1 y) (G'.1 y) (F'.2.1 x y h) (eta y))
            (compose D (F'.1 x) (G'.1 x) (G'.1 y) (eta x) (G'.2.1 x y h))
          = f.2 (H.1 x) (H.1 y) (H.2.1 x y h)

ntransR (C D: precategory) (F G: catfunctor C D)
    (f: ntrans C D F G) (E: precategory) (H: catfunctor D E)
  : ntrans C E (compFunctor C D E F H) (compFunctor C D E G H)
  = (eta, p) where
    F': catfunctor C E = compFunctor C D E F H
    G': catfunctor C E = compFunctor C D E G H
    eta (x: carrier C): hom E (F'.1 x) (G'.1 x)
      = H.2.1 (F.1 x) (G.1 x) (f.1 x)
    p (x y: carrier C) (h: hom C x y): Path (hom E (F'.1 x) (G'.1 y))
        (compose E (F'.1 x) (F'.1 y) (G'.1 y) (F'.2.1 x y h) (eta y))
        (compose E (F'.1 x) (G'.1 x) (G'.1 y) (eta x) (G'.2.1 x y h))
      = <i> comp (<_> hom E (F'.1 x) (G'.1 y))
                 (H.2.1 (F.1 x) (G.1 y) (f.2 x y h @ i))
        [ (i = 0) -> H.2.2.2 (F.1 x) (F.1 y) (G.1 y) (F.2.1 x y h) (f.1 y),
          (i = 1) -> H.2.2.2 (F.1 x) (G.1 x) (G.1 y) (f.1 x) (G.2.1 x y h) ]
```

```
areAdjoint (C D: precategory)
           (F: catfunctor D C)
           (G: catfunctor C D)
           (unit: ntrans D D (idFunctor D) (compFunctor D C D F G))
           (counit: ntrans C C (compFunctor C D C G F) (idFunctor C)): U
  = prod  ((x: carrier C) -> = (hom D (G.1 x) (G.1 x))
                               (path D (G.1 x)) (h0 x))
          ((x: carrier D) -> = (hom C (F.1 x) (F.1 x))
                               (path C (F.1 x)) (h1 x)) where
    h0 (x:  carrier C) : hom D (G.1 x) (G.1 x)
                = compose D (G.1 x) (G.1 (F.1 (G.1 x))) (G.1 x)
          ((ntransL D D (idFunctor D)
                        (compFunctor D C D F G) unit C G).1 x)
          ((ntransR C C (compFunctor C D C G F)
                        (idFunctor C) counit D G).1 x)
    h1 (x: carrier D) : hom C (F.1 x) (F.1 x)
               = compose C (F.1 x) (F.1 (G.1 (F.1 x))) (F.1 x)
          ((ntransR D D (idFunctor D)
                        (compFunctor D C D F G) unit C F).1 x)
          ((ntransL C C (compFunctor C D C G F)
                        (idFunctor C) counit D F).1 x)

adjoint (C D: precategory) (F: catfunctor D C) (G: catfunctor C D): U
  = (unit: ntrans D D (idFunctor D) (compFunctor D C D F G))
  * (counit: ntrans C C (compFunctor C D C G F) (idFunctor C))
  * areAdjoint C D F G unit counit
```

Issue XXXI: Categories with Families

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Анотація

Категорійна семантика залежної теорії типів.

Ключові слова: теорія категорій, категорії з сім'ями, залежна теорія типів

2 Categories with Families

Тут подано короткий неформальний опис категорійної семантики залежної теорії типів, запропонований Пітером Диб'єром. Категоріальна абстрактна машина Диб'єра на Haskell описана тут[1].

2.1 Основні визначення

Definition 6 (Fam)**.** Категорія Fam — це категорія сімей множин, де об'єкти є залежними функціональними просторами $(x : A) \to B(x)$, а морфізми з доменом $\Pi(A, B)$ і кодоменом $\Pi(A', B')$ — це пари функцій $\langle f : A \to A', g(x : A) : B(x) \to B'(f(x))\rangle$.

Definition 7 (П-похідність)**.** Для контексту Γ і типу A позначимо $\Gamma \vdash A = (\gamma : \Gamma) \to A(\gamma)$.

Definition 8 (Σ-охоплення)**.** Для контексту Γ і типу A маємо $\Gamma; A = (\gamma : \Gamma) * A(\gamma)$. Охоплення не є асоціативним:

$$\Gamma; A; B \neq \Gamma; B; A$$

Definition 9 (Контекст)**.** Категорія контекстів C — це категорія, де об'єкти є контекстами, а морфізми — підстановками. Термінальний об'єкт $\Gamma = 0$ у C називається порожнім контекстом. Операція охоплення контексту $\Gamma; A = (x : \Gamma) * A(x)$ має елімінатори: $p : \Gamma; A \vdash \Gamma$, $q : \Gamma; A \vdash A(p)$, що задовольняють універсальну властивість: для будь-якого $\Delta : ob(C)$, морфізму $\gamma : \Delta \to \Gamma$ і терму $a : \Delta \to A$ існує єдиний морфізм $\theta = \langle \gamma, a\rangle : \Delta \to \Gamma; A$, такий що $p \circ \theta = \gamma$ і $q(\theta) = a$. Твердження: підстановка є асоціативною:

$$\gamma(\gamma(\Gamma, x, a), y, b) = \gamma(\gamma(\Gamma, y, b), x, a)$$

Definition 10 (CwF-об'єкт)**.** CwF-об'єкт — це пара $\Sigma(C, C \to \mathsf{Fam})$, де C — категорія контекстів з об'єктами-контекстами та морфізмами-підстановками, а $T : C \to \mathsf{Fam}$ — функтор, який відображає контекст Γ у C на сім'ю множин термів $\Gamma \vdash A$, а підстановку $\gamma : \Delta \to \Gamma$ — на пару функцій, що виконують підстановку γ у термах і типах відповідно.

Definition 11 (CwF-морфізм)**.** Нехай $(C, T) : ob(C)$, де $T : C \to \mathsf{Fam}$. CwF-морфізм $m : (C, T) \to (C', T')$ — це пара $\langle F : C \to C', \sigma : T \to T'(F)\rangle$, де F — функтор, а σ — натуральна трансформація.

Definition 12 (Категорія типів)**.** Для CwF з об'єктами (C, T) і морфізмами $(C, T) \to (C', T')$, для заданого контексту $\Gamma \in Ob(C)$ можна побудувати категорію $\mathsf{Type}(\Gamma)$ — категорію типів у контексті Γ, де об'єкти — множина типів у контексті, а морфізми — функції $f : \Gamma; A \to B(p)$.

[1] https://www.cse.chalmers.se/~peterd/papers/Ise2008.pdf

2.2 Семантика залежної теорії типів

Definition 13 (Терми та типи)**.** У CwF для контексту Γ терми $\Gamma \vdash a : A$ є елементами множини $A(\gamma)$, де $\gamma : \Gamma$. Типи $\Gamma \vdash A$ є об'єктами в $\mathsf{Type}(\Gamma)$, а підстановка $\gamma : \Delta \to \Gamma$ діє на типи та терми через функтор T.

Theorem 1 (Композиція підстановок)**.** Підстановки в категорії контекстів C є асоціативними та мають одиницю (ідентичну підстановку). Формально, для $\gamma : \Delta \to \Gamma$, $\delta : \Theta \to \Delta$ і $\epsilon : \Gamma \to \Lambda$ виконується:

$$(\gamma \circ \delta) \circ \epsilon = \gamma \circ (\delta \circ \epsilon), \quad \mathsf{id}_\Gamma \circ \gamma = \gamma, \quad \gamma \circ \mathsf{id}_\Delta = \gamma.$$

Доведення. Асоціативність випливає з універсальної властивості охоплення контексту (Визначення 1.4). Для будь-яких γ, δ, ϵ композиція морфізмів у C відповідає послідовному застосуванню підстановок, що зберігає структуру контекстів. Ідентична підстановка id_Γ діє як нейтральний елемент, оскільки $\mathsf{p} \circ \mathsf{id}_\Gamma = \mathsf{id}_\Gamma$ і $\mathsf{q}(\mathsf{id}_\Gamma) = \mathsf{q}$. □

Definition 14 (Залежні типи)**.** Залежний тип у контексті Γ — це відображення $\Gamma \to \mathsf{Fam}$, де для кожного $\gamma : \Gamma$ задається множина $A(\gamma)$. У категорії $\mathsf{Type}(\Gamma)$ залежні типи є об'єктами, а морфізми між A і B — це функції $f : \Gamma; A \to B(\mathsf{p})$, що зберігають структуру підстановок.

Theorem 2 (Універсальна властивість залежних типів)**.** Для будь-якого контексту Γ, типу A і терму $a : \Gamma \vdash A$ існує унікальний морфізм $\theta : \Gamma \to \Gamma; A$, який задовольняє $\mathsf{p} \circ \theta = \mathsf{id}_\Gamma$ і $\mathsf{q}(\theta) = a$. Це забезпечує коректність залежної типізації в CwF.

Доведення. За Визначенням 1.4, універсальна властивість охоплення контексту гарантує існування $\theta = \langle \mathsf{id}_\Gamma, a \rangle$. Унікальність випливає з того, що будь-який інший морфізм θ' з тими ж властивостями ($\mathsf{p} \circ \theta' = \mathsf{id}_\Gamma$, $\mathsf{q}(\theta') = a$) збігається з θ через єдиність композиції в C. □

2.3 Формалізація в Anders

Для формалізації CwF у Agda чи Lean необхідно визначити категорію C як запис із полями для об'єктів, морфізмів, композиції та ідентичності, а також функтор $\mathsf{T} : \mathsf{C} \to \mathfrak{Fam}$. Нижче наведено псевдокод для Anders[2]:

```
def algebra : U₁ := Σ
    -- a semicategory of contexts and substitutions:
    (Con: U)
    (Sub: Con → Con → U)
    (◊: Π (Γ Θ Δ : Con), Sub Θ Δ → Sub Γ Θ → Sub Γ Δ)
    (◊-assoc: Π (Γ Θ Δ Φ : Con) (σ: Sub Γ Θ) (δ: Sub Θ Δ)
         (ν: Sub Δ Φ), PathP (<_>Sub Γ Φ) (◊ Γ Δ Φ ν (◊ Γ Θ Δ δ σ))
                                          (◊ Γ Θ Φ (◊ Θ Δ Φ ν δ) σ))
    -- identity morphisms as identity substitutions:
    (id: Π (Γ : Con), Sub Γ Γ)
    (id-left: Π (Θ Δ : Con) (δ : Sub Θ Δ),
              Path (Sub Θ Δ) δ (◊ Θ Δ Δ (id Δ) δ))
    (id-right: Π (Θ Δ : Con) (δ : Sub Θ Δ),
               Path (Sub Θ Δ) δ (◊ Θ Θ Δ δ (id Θ)))
    -- a terminal oject as empty context:
    (•: Con)
    (ε: Π (Γ : Con), Sub Γ •)
    (•-η: Π (Γ: Con) (δ: Sub Γ •), Path (Sub Γ •) (ε Γ) δ)
    (Ty: Con → U)
    (_|_|ᵀ: Π (Γ Δ : Con), Ty Δ → Sub Γ Δ → Ty Γ)
    (|id|ᵀ: Π (Δ : Con) (A : Ty Δ), Path (Ty Δ) (_|_|ᵀ Δ Δ A (id Δ)) A)
    (|◊|ᵀ: Π (Γ Δ Φ: Con) (A : Ty Φ) (σ : Sub Γ Δ) (δ : Sub Δ Φ),
         PathP (<_>Ty Γ) (_|_|ᵀ Γ Φ A (◊ Γ Δ Φ δ σ))
                         (_|_|ᵀ Γ Δ (_|_|ᵀ Δ Φ A δ) σ))
    -- a (covariant) presheaf on the category of elements as terms:
    (Tm: Π (Γ : Con), Ty Γ → U)
    (_|_|ᵗ: Π (Γ Δ : Con) (A : Ty Δ) (B : Tm Δ A)
              (σ: Sub Γ Δ), Tm Γ (_|_|ᵀ Γ Δ A σ))
    (|id|ᵗ: Π (Δ : Con) (A : Ty Δ) (t: Tm Δ A),
            PathP (<i> Tm Δ (|id|ᵀ Δ A @ i))
                  (_|_|ᵗ Δ Δ A t (id Δ)) t)
    (|◊|ᵗ: Π (Γ Δ Φ: Con) (A : Ty Φ) (t: Tm Φ A)
             (σ : Sub Γ Δ) (δ : Sub Δ Φ),
             PathP (<i> Tm Γ (|◊|ᵀ Γ Δ Φ A σ δ @ i))
                   (_|_|ᵗ Γ Φ A t (◊ Γ Δ Φ δ σ))
          (_|_|ᵗ Γ Δ (_|_|ᵀ Δ Φ A δ) (_|_|ᵗ Δ Φ A t δ) σ))
```

Ця структура дозволяє реалізувати Визначення 1.1–1.11, а Теореми 1.10 і 1.12 доводяться через перевірку асоціативності та універсальних властивостей.

[2] `https://anders.groupoid.space/lib/mathematics/categories/meta/kraus.anders`

2.4 Висновки

Категорії з сім'ями (CwF) є потужним інструментом для моделювання залежної теорії типів. Вони забезпечують чітку семантику для контекстів, підстановок і залежних типів, що полегшує аналіз і формалізацію.

Issue XXVIX: Formal Topos on Category of Sets

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Анотація

The purpose of this work is to clarify all topos definitions using type theory. Not much efforts was done to give all the examples, but one example, a topos on category of sets, is constructively presented at the finale.

As this cricial example definition is used in presheaf definition, the construction of category of sets is a mandatory excercise for any topos library. We propose here cubicaltt[1] version of elementary topos on category of sets for demonstration of categorical semantics (from logic perspective) of the fundamental notion of set theory in mathematics.

Other disputed foundations for set theory could be taken as: ZFC, NBG, ETCS. We will disctinct syntetically: i) category theory; ii) set theory in univalent foundations; iii) topos theory, grothendieck topos, elementary topos. For formulation of definitions and theorems only Martin-Löf Type Theory is requested. The proofs involve cubical type checker primitives.

Keywords: Homotopy Type Theory, Topos Theory

[1] Cubical Type Theory, http://github.com/mortberg/cubicaltt

3 Topos Theory

One can admit two topos theory lineages. One lineage takes its roots from published by Jean Leray in 1945 initial work on sheaves and spectral sequences. Later this lineage was developed by Henri Paul Cartan, André Weil. The peak of lineage was settled with works by Jean-Pierre Serre, Alexander Grothendieck, and Roger Godement.

Second remarkable lineage take its root from William Lawvere and Myles Tierney. The main contribution is the reformulation of Grothendieck topology by using subobject classifier.

3.1 Set Theory

Here is given the ∞-groupoid model of sets.

Definition 15. (Mere proposition, PROP). A type P is a mere proposition if for all $x, y : P$ we have $x = y$:

$$\text{isProp}(P) = \prod_{x,y:P} (x = y).$$

Definition 16. (0-type). A type A is a 0-type is for all $x, y : A$ and $p, q : x =_A y$ we have $p = q$.

Definition 17. (1-type). A type A is a 1-type if for all $x, y : A$ and $p, q : x =_A y$ and $r, s : p =_{=_A} q$, we have $r = s$.

Definition 18. (A set of elements, SET). A type A is a SET if for all $x, y : A$ and $p, q : x = y$, we have $p = q$:

$$\text{isSet}(A) = \prod_{x,y:A} \prod_{p,q:x=y} (p = q).$$

Definition 19. `data N = Z | S (n: N)`

```
n_grpd (A: U) (n: N): U = (a b: A) -> rec A a b n where
  rec (A: U) (a b: A) : (k: N) -> U
    = split { Z -> Path A a b ; S n -> n_grpd (Path A a b) n }

isContr (A: U): U = (x: A) * ((y: A) -> Path A x y)
isProp  (A: U): U = n_grpd A Z
isSet   (A: U): U = n_grpd A (S Z)
PROP  : U = (X:U) * isProp X
SET   : U = (X:U) * isSet X
```

Definition 20. (Π-Contractability). If fiber is set thene path space between any sections is contractible.

```
setPi (A: U) (B: A -> U) (h: (x: A) -> isSet (B x)) (f g: Pi A B)
      (p q: Path (Pi A B) f g)
    : Path (Path (Pi A B) f g) p q
```

Definition 21. (Σ-Contractability). If fiber is set then Σ is set.

```
setSig (A:U) (B: A -> U) (base: isSet A)
       (fiber: (x:A) -> isSet (B x)) : isSet (Sigma A B)
```

Definition 22. (Unit type, 𝟙). The unit 𝟙 is a type with one element.

```
data unit = tt
unitRec (C: U) (x: C): unit -> C = split tt -> x
unitInd (C: unit -> U) (x: C tt): (z:unit) -> C z
  = split tt -> x
```

Theorem 3. (Category of Sets, ***Set***). Sets forms a Category. All compositional theorems proved by using reflection rule of internal language. The proof that Hom forms a set is taken through Π-contractability.

```
Set: precategory = ((Ob,Hom),id,c,HomSet,L,R,Q) where
  Ob: U = SET
  Hom (A B: Ob): U = A.1 -> B.1
  id (A: Ob): Hom A A = idfun A.1
  c (A B C: Ob) (f: Hom A B) (g: Hom B C): Hom A C
    = o A.1 B.1 C.1 g f
  HomSet (A B: Ob): isSet (Hom A B) = setFun A.1 B.1 B.2
  L (A B:Ob) (f:Hom A B): Path (Hom A B)(c A A B (id A)f)f
    = refl (Hom A B) f
  R (A B:Ob) (f:Hom A B): Path (Hom A B)(c A B B f(id B))f
    = refl (Hom A B) f
  Q (A B C D: Ob) (f:Hom A B) (g:Hom B C) (h:Hom C D)
    : Path (Hom A D) (c A C D (c A B C f g) h)
                     (c A B D f (c B C D g h))
    = refl (Hom A D) (c A B D f (c B C D g h))
```

3.2 Topological Structure

Topos theory extends category theory with notion of topological structure but reformulated in a categorical way as a category of sheaves on a site or as one that has cartesian closure and subobject classifier. We give here two definitions.

Definition 23. (Topology). The topological structure on A (or topology) is a subset S ∈ A with following properties: i) any finite union of subsets of S is belong to S; ii) any finite intersection of subsets of S is belong to S. Subets of S are called open sets of family S.

```
Structure topology (A : Type) := {
          open :> (A -> Prop) -> Prop;
          empty_open: open (empty _);
          full_open:  open (full _);
          inter_open: forall u,
                      open u -> forall v, open v
                             -> open (inter A u v) ;
          union_open: forall s, (subset _ s open)
                             -> open (union A s) }.
```

For fully functional general topology theorems and Zorn lemma you can refer to the Coq library [2]topology by Daniel Schepler.

[2] https://github.com/verimath/topology

3.3 Grothendieck Topos

Grothendieck Topology is a calculus of coverings which generalizes the algebra of open covers of a topological space, and can exist on much more general categories. There are three variants of Grothendieck topology definition: i) sieves; ii) coverage; iii) covering families. A category have one of these three is called a Grothendieck site.

Examples: Zariski, flat, étale, Nisnevich topologies.

A sheaf is a presheaf (functor from opposite category to category of sets) which satisties patching conditions arising from Grothendieck topology, and applying the associated sheaf functor to preashef forces compliance with these conditions.

The notion of Grothendieck topos is a geometric flavour of topos theory, where topos is defined as category of sheaves on a Grothendieck site with geometric moriphisms as adjoint pairs of functors between topoi, that satisfy exactness properties. [**?**]

As this flavour of topos theory uses category of sets as a prerequisite, the formal construction of set topos is cricual in doing sheaf topos theory.

Definition 24. (Sieves). Sieves are a family of subfunctors

$$\mathsf{R} \subset \mathsf{Hom}_{\mathsf{C}}(\_, \mathsf{U}), \mathsf{U} \in \mathrm{C},$$

such that following axioms hold: i) (base change) If $\mathsf{R} \subset \mathsf{Hom}_{\mathsf{C}}(\_, \mathsf{U})$ is covering and $\phi : \mathsf{V} \to \mathsf{U}$ is a morphism of C, then the subfuntor

$$\phi^{-1}(\mathsf{R}) = \{\gamma : W \to \mathsf{V} \| \phi \cdot \gamma \in \mathsf{R}\}$$

is covering for V; ii) (local character) Suppose that $\mathsf{R}, \mathsf{R}' \subset \mathsf{Hom}_{\mathsf{C}}(\_, \mathsf{U})$ are subfunctors and R is covering. If $\phi^{-1}(\mathsf{R}')$ is covering for all $\phi : \mathsf{V} \to \mathsf{U}$ in R, then R' is covering; iii) $\mathsf{Hom}_{\mathsf{C}}(\_, \mathsf{U})$ is covering for all $\mathsf{U} \in \mathrm{C}$.

Definition 25. (Coverage). A coverage is a function assigning to each Ob_C the family of morphisms $\{f_i : U_i \to U\}_{i \in I}$ called covering families, such that for any $g : V \to U$ exist a covering family $\{h : V_j \to V\}_{j \in J}$ such that each composite $h_j \circ g$ factors some f_i:

$$\begin{tikzcd}
V_j \arrow[r, "k"] \arrow[d, "h"] & U_i \arrow[d, "f_i"] \\
V \arrow[r, "g"] & U
\end{tikzcd}$$

```
Co (C: precategory) (cod: carrier C) : U
  = (dom: carrier C)
  * (hom C dom cod)

Delta (C: precategory) (d: carrier C) : U
  = (index: U)
  * (index -> Co C d)

Coverage (C: precategory): U
  = (cod: carrier C)
  * (fam: Delta C cod)
  * (coverings: carrier C -> Delta C cod -> U)
  * (coverings cod fam)
```

Definition 26. (Grothendieck Topology). Suppose category C has all pullbacks. Since C is small, a pretopology on C consists of families of sets of morphisms

$$\{\phi_\alpha : U_\alpha \to U\}, U \in C,$$

called covering families, such that following axioms hold: i) suppose that $\phi_\alpha : U_\alpha \to U$ is a covering family and that $\psi : V \to U$ is a morphism of C. Then the collection $V \times_U U_\alpha \to V$ is a cvering family for V. ii) If $\{\phi_\alpha : U_\alpha \to U\}$ is covering, and $\{\gamma_{\alpha,\beta} : W_{\alpha,\beta} \to U_\alpha\}$ is covering for all α, then the family of composites

$$W_{\alpha,\beta} \xrightarrow{\gamma_{\alpha,\beta}} U_\alpha \xrightarrow{\phi_\alpha} U$$

is covering; iii) The family $\{1 : U \to U\}$ is covering for all $U \in C$.

Definition 27. (Site). Site is a category having either a coverage, grothendieck topology, or sieves.

```
site (C: precategory): U
  = (C: precategory) * Coverage C
```

Definition 28. (Presheaf). Presheaf of a category C is a functor from opposite category to category of sets: $C^{op} \to \mathrm{Set}$.

```
presheaf (C: precategory): U
  = catfunctor (opCat C) Set
```

Definition 29. (Presheaf Category, **PSh**). Presheaf category **PSh** for a site C is category were objects are presheaves and morphisms are natural transformations of presheaf functors.

Definition 30. (Sheaf). Sheaf is a presheaf on a site. In other words a presheaf $F : C^{op} \to \boldsymbol{Set}$ such that the cannonical map of inverse limit

$$F(U) \to \overset{\leftarrow}{\lim_{V \to U \in R}} F(V)$$

is an isomorphism for each covering sieve $R \subset \mathsf{Hom}_C(\_, U)$. Equivalently, all induced functions

$$\mathsf{Hom}_C(\mathsf{Hom}_C(\_, U), F) \to \mathsf{Hom}_C(R, F)$$

should be bejections.

```
sheaf (C: precategory): U
  = (S: site C)
  * presheaf S.1
```

Definition 31. (Sheaf Category, **Sh**). Sheaf category **Sh** is a category where objects are sheaves and morphisms are natural transformation of sheves. Sheaf category is a full subcategory of category of presheaves **PSh**.

Definition 32. (Grothendieck Topos). Topos is the category of sheaves $\mathbf{Sh}(C, J)$ on a site C with topology J.

Theorem 4. (Giraud). A category C is a Grothiendieck topos iff it has following properties: i) has all finite limits; ii) has small disjoint coproducts stable under pullbacks; iii) any epimorphism is coequalizer; iv) any equivalence relation $R \to E$ is a kernel pair and has a quotient; v) any coequalizer $R \to E \to Q$ is stably exact; vi) there is a set of objects that generates C.

Definition 33. (Geometric Morphism). Suppose that C and D are Grothendieck sites. A geometric morphism

$$f : \mathbf{Sh}(C) \to \mathbf{Sh}(D)$$

consist of functors $f_* : \mathbf{Sh}(C) \to \mathbf{Sh}(D)$ and $f^* : \mathbf{Sh}(D) \to \mathbf{Sh}(C)$ such that f^* is left adjoint to f_* and f^* preserves finite limits. The left adjoint f^* is called the inverse image functor, while f_* is called the direct image. The inverse image functor f^* is left and right exact in the sense that it preserves all finite colimits and limits, respectively.

Definition 34. (Cohesive Topos). A topos E is a cohesive topos over a base topos S, if there is a geometric morphism $(p^*, p_*) : E \to S$, such that: i) exists adjunction $p^! \vdash p_*$ and $p^! \dashv p_*$; ii) p^* and $p^!$ are full faithful; iii) $p_!$ preserves finite products.

This quadruple defines adjoint triple:

$$\int \dashv \flat \dashv \sharp$$

3.4 Elementary Topos

Giraud theorem was a synonymical topos definition involved only topos properties but not a site properties. That was step forward on predicative definition. The other step was made by Lawvere and Tierney, by removing explicit dependance on categorical model of set theory (as category of set is used in definition of presheaf). This information was hidden into subobject classifier which was well defined through categorical pullback and property of being cartesian closed (having lambda calculus as internal language).

Elementary topos doesn't involve 2-categorical modeling, so we can construct set topos without using functors and natural transformations (what we need in geometrical topos theory flavour). This flavour of topos theory more suited for logic needs rather that geometry, as its set properties are hidden under the predicative predicative pullback definition of subobject classifier rather that functorial notation of presheaf functor. So we can simplify proofs at the homotopy levels, not to lift everything to 2-categorical model.

Definition 35. (Monomorphism). An morphism $f : Y \to Z$ is a monic or mono if for any object X and every pair of parralel morphisms $g_1, g_2 : X \to Y$ the

$$f \circ g_1 = f \circ g_2 \to g_1 = g_2.$$

More abstractly, f is mono if for any X the Hom$(X, _)$ takes it to an injective function between hom sets Hom$(X, Y) \to$ Hom(X, Z).

```
mono (P: precategory) (Y Z: carrier P) (f: hom P Y Z): U
   = (X: carrier P) (g1 g2: hom P X Y)
  -> Path (hom P X Z) (compose P X Y Z g1 f)
                      (compose P X Y Z g2 f)
  -> Path (hom P X Y) g1 g2
```

Definition 36. (Subobject Classifier[?]). In category C with finite limits, a subobject classifier is a monomorphism $\mathsf{true} : 1 \to \Omega$ out of terminal object 1, such that for any mono $U \to X$ there is a unique morphism $\chi_U : X \to \Omega$ and pullback diagram:

$$\begin{tikzcd}
U \arrow[r, "k"] \arrow[d] & 1 \arrow[d, "\mathsf{true}"] \\
X\Omega \arrow[r, "\chi_U"] & \Omega
\end{tikzcd}$$

```
subobjectClassifier (C: precategory): U
  = (omega: carrier C)
  * (end: terminal C)
  * (trueHom: hom C end.1 omega)
  * (chi: (V X: carrier C) (j: hom C V X) -> hom C X omega)
  * (square: (V X: carrier C) (j: hom C V X) -> mono C V X j
    -> hasPullback C (omega,(end.1,trueHom),(X,chi V X j)))
  * ((V X: carrier C) (j: hom C V X) (k: hom C X omega)
    -> mono C V X j
    -> hasPullback C (omega,(end.1,trueHom),(X,k))
    -> Path (hom C X omega) (chi V X j) k)
```

Theorem 5. (Category of Sets has Subobject Classifier).

Definition 37. (Cartesian Closed Categories). The category C is called cartesian closed if exists all: i) terminals; ii) products; iii) exponentials. Note that this definition lacks beta and eta rules which could be found in embedding **MLTT**.

```
isCCC (C: precategory): U
  = (Exp:    (A B: carrier C) -> carrier C)
  * (Prod:   (A B: carrier C) -> carrier C)
  * (Apply:  (A B: carrier C) -> hom C (Prod (Exp A B) A) B)
  * (P1:     (A B: carrier C) -> hom C (Prod A B) A)
  * (P2:     (A B: carrier C) -> hom C (Prod A B) B)
  * (Term:   terminal C)
  * unit
```

Theorem 6. (Category of Sets is cartesian closed). As you can see from exp and pro we internalize Π and Σ types as SET instances, the isSet predicates are provided with contractability. Exitense of terminals is proved by propPi. The same technique you can find in **MLTT** embedding.

```
cartesianClosure : isCCC Set
  = (expo,prod,appli,proj1,proj2,term,tt) where
    exp (A B: SET): SET = (A.1    -> B.1, setFun A.1 B.1 B.2)
    pro (A B: SET): SET = (prod A.1 B.1, setSig A.1 (\(_ : A.1)
                          -> B.1) A.2 (\(_ : A.1) -> B.2))
    expo:  (A B: SET) -> SET = \(A B: SET) -> exp A B
    prod:  (A B: SET) -> SET = \(A B: SET) -> pro A B
    appli: (A B: SET) -> hom Set (pro (exp A B) A) B
        = \(A B: SET) -> \(x:(pro(exp A B)A).1)-> x.1 x.2
    proj1: (A B: SET) -> hom Set (pro A B) A
        = \(A B: SET) (x: (pro A B).1) -> x.1
    proj2: (A B: SET) -> hom Set (pro A B) B
        = \(A B: SET) (x: (pro A B).1) -> x.2
    unitContr (x: SET) (f: x.1 -> unit) : isContr (x.1 -> unit)
      = (f, \(z: x.1 -> unit) -> propPi x.1 (\(_:x.1)->unit)
            (\(x:x.1) -> propUnit) f z)
    term: terminal Set = ((unit,setUnit),
            \(x: SET) -> unitContr x (\(z: x.1) -> tt))
```

Note that rules of cartesian closure forms a type theoretical langage called lambda calculus.

Definition 38. (Elementary Topos). Topos is a precategory which is cartesian closed and has subobject classifier.

```
Topos (cat: precategory) : U
  = (cartesianClosure: isCCC cat)
  * subobjectClassifier cat
```

Theorem 7. (Topos Definitions). Any Grothendieck topos is an elementary topos too. The proof is sligthly based on results of Giraud theorem.

Theorem 8. (Category of Sets forms a Topos). There is a cartesian closure and subobject classifier for a categoty of sets.

```
internal : Topos Set
         = (cartesianClosure ,hasSubobject)
```

Theorem 9. (Freyd). Main theorem of topos theory[**?**]. For any topos C and any $\mathsf{b} : \mathrm{Ob}_{\mathsf{C}}$ relative category $\mathsf{C} \downarrow \mathsf{b}$ is also a topos. And for any arrow $\mathsf{f} : \mathsf{a} \to \mathsf{b}$ inverse image functor $\mathsf{f}^* : \mathsf{C} \downarrow \mathsf{b} \to \mathsf{c} \downarrow \mathsf{a}$ has left adjoint $\sum_{\mathsf{f}}$ and right adjoin $\prod_{\mathsf{f}}$.

Conclusion

We gave here constructive definition of topology as finite unions and intersections of open subsets. Then make this definition categorically compatible by introducing Grothendieck topology in three different forms: sieves, coverage, and covering families. Then we defined an elementary topos and introduce category of sets, and proved that **Set** is cartesian closed, has object classifier and thus a topos.

This intro could be considered as a formal introduction to topos theory (at least of the level of first chapter) and you may evolve this library to your needs or ask to help porting or developing your application of topos theory to a particular formal construction.

Issue XXVII: Cohesive Topos

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Анотація

Formal definition of Cohesive Topos.

Keywords: Topos Theory

4 Cohesive Topos Theory

4.1 Category

A **category** $\mathcal{C}$ consists of:

- A class of **objects**, $\mathrm{Ob}(\mathcal{C})$,
- A class of **morphisms**, $\mathrm{Hom}_{\mathcal{C}}(\mathsf{X}, \mathsf{Y})$, for each pair $\mathsf{X}, \mathsf{Y} \in \mathrm{Ob}(\mathcal{C})$,
- Composition maps $\circ : \mathrm{Hom}(\mathsf{Y}, \mathsf{Z}) \times \mathrm{Hom}(\mathsf{X}, \mathsf{Y}) \to \mathrm{Hom}(\mathsf{X}, \mathsf{Z})$,
- Identity morphisms $\mathrm{id}_{\mathsf{X}} \in \mathrm{Hom}(\mathsf{X}, \mathsf{X})$ for each X,

satisfying associativity and identity laws.

4.2 Functor

A **functor** $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ assigns to each:

- Object $\mathsf{X} \in \mathcal{C}$ an object $\mathsf{F}(\mathsf{X}) \in \mathcal{D}$,
- Morphism $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ a morphism $\mathsf{F}(\mathsf{f}) : \mathsf{F}(\mathsf{X}) \to \mathsf{F}(\mathsf{Y})$,

such that $\mathsf{F}(\mathrm{id}_{\mathsf{X}}) = \mathrm{id}_{\mathsf{F}(\mathsf{X})}$ and $\mathsf{F}(\mathsf{g} \circ \mathsf{f}) = \mathsf{F}(\mathsf{g}) \circ \mathsf{F}(\mathsf{f})$.

4.3 Natural Transformation

A **natural transformation** $\eta : F \Rightarrow G$ between functors $F, G : \mathcal{C} \to \mathcal{D}$ consists of morphisms $\eta_X : F(X) \to G(X)$ such that for every $f : X \to Y$ in $\mathcal{C}$,

$$\begin{tikzcd}
F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\eta_Y"'] & G(Y)
\end{tikzcd}$$

commutes.

4.4 Adjunction

An **adjunction** between categories $\mathcal{C}$ and $\mathcal{D}$ consists of functors

$$F : \mathcal{C} \leftrightarrows \mathcal{D} : G$$

and natural transformations (unit η and counit ε)

$$\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow G \circ F, \quad \varepsilon : F \circ G \Rightarrow \mathrm{Id}_{\mathcal{D}}$$

satisfying the triangle identities.

4.5 Topos

A **topos** $\mathcal{E}$ is a category that:

- Has all finite limits and colimits,
- Is Cartesian closed: has exponential objects $[X, Y]$,
- Has a subobject classifier Ω.

4.6 Geometric Morphism

A **geometric morphism** $f : \mathcal{E} \to \mathcal{F}$ between topoi consists of an adjoint pair

$$f^* : \mathcal{F} \leftrightarrows \mathcal{E} : f_*$$

with $f^* \dashv f_*$, where f^* preserves finite limits (i.e., is left exact).

4.7 Cohesive Topos

A **cohesive topos** is a topos $\mathcal{E}$ equipped with a quadruple of adjoint functors:

$$\Pi \dashv \Delta \dashv \Gamma \dashv \nabla : \mathcal{E} \leftrightarrows \boldsymbol{Set}$$

such that:

- Γ is the global sections functor,
- Δ is the constant sheaf functor,
- ∇ sends a set to a codiscrete object,
- Π is the shape or fundamental groupoid functor,
- Δ and ∇ are fully faithful,
- Δ preserves finite limits,
- Π preserves finite products (in some variants).

4.8 Cohesive Adjunction Diagram and Modalities

$$\begin{tikzcd}
\mathcal{E} \arrow[r, shift left=3, "\Pi"] \arrow[r, shift right=3, "\Delta"', "\nabla"'] \arrow[r, phantom, "\dashv"] & \boldsymbol{Set} \arrow[l, shift right=1] \arrow[l, shift left=1]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{E} \arrow[r, bend left, "\int"] \arrow[r, bend right, "\sharp"'] \arrow[r, phantom, "\overset{\|}{\flat}\Downarrow"] & \mathcal{E}
\end{tikzcd}$$

4.9 Cohesive Modalities

The above adjoint quadruple canonically induces a triple of endofunctors on $\mathcal{E}$:

$$(\int \dashv \flat \dashv \sharp) : \mathcal{E} \to \mathcal{E}$$

defined as follows:

$$\int := \Delta \circ \Pi$$
$$\flat := \Delta \circ \Gamma$$
$$\sharp := \nabla \circ \Gamma$$

This yields an **adjoint triple** of endofunctors on $\mathcal{E}$:

$$\int \dashv \flat \dashv \sharp$$

These are:

- $\int$ — the **shape modality**: captures the fundamental shape or homotopy type,

- $\flat$ — the **flat modality**: forgets cohesive structure while remembering discrete shape,

- $\sharp$ — the **sharp modality**: codiscretizes the structure, reflecting the full cohesion.

Each of these is an **idempotent** (co)monad, hence a *modality* in the internal language (type theory) of $\mathcal{E}$.

4.10 Differential Cohesion

A **differential cohesive topos** is a cohesive topos $\mathcal{E}$ equipped with an additional adjoint triple of endofunctors:

$$(\mathfrak{R} \dashv \mathfrak{I} \dashv \&) : \mathcal{E} \to \mathcal{E}$$

These are:

- $\mathfrak{R}$: the **reduction modality** — forgets nilpotents,
- $\mathfrak{I}$: the **infinitesimal shape modality** — retains infinitesimal data,
- $\&$: the **infinitesimal flat modality** — reflects formally smooth structure.

Important object classes:

- An object X is **reduced** if $\mathfrak{R}(\mathsf{X}) \cong \mathsf{X}$.
- It is **coreduced** if $\&(\mathsf{X}) \cong \mathsf{X}$.
- It is **formally smooth** if the unit map $\mathsf{X} \to \&\mathsf{X}$ is an effective epimorphism.

Formally étale maps are those morphisms $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ such that the square

$$\begin{tikzcd}
\mathsf{X} \arrow[r] \arrow[d, "\mathsf{f}"'] & \mathfrak{I}\mathsf{X} \arrow[d, "\mathfrak{I}(\mathsf{f})"] \\
\mathsf{Y} \arrow[r] & \mathfrak{I}\mathsf{Y}
\end{tikzcd}$$

is a pullback.

4.11 Graded Differential Cohesion

In **graded differential cohesion**, such as used in synthetic supergeometry, one introduces an adjoint triple:

10) $\rightrightarrows \dashv \rightsquigarrow \dashv \mathsf{Rh}$

$$(\rightrightarrows \dashv \rightsquigarrow \dashv \mathsf{Rh}) : \mathcal{E} \to \mathcal{E}$$

These are:

- $\rightrightarrows$: the **fermionic modality** — captures anti-commuting directions,
- $\rightsquigarrow$: the **bosonic modality** — filters out fermionic directions,
- Rh: the **rheonomic modality** — encodes constraint structures.

These modal operators form part of the internal logic of supergeometric or supersymmetric type theories.

Issue XXVIII: Modalities and Identity Systems

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Анотація

This article explores the interplay between modalities, identity systems, and homologies in the framework of Homotopy Type Theory (HoTT). We formalize modalities and identity systems as structures within (∞,1)-categories and investigate the homological properties arising when their functor compositions are treated as groups. Special attention is given to topological structures, such as the Möbius strip, that emerge from non-trivial compositions, and their role in generating non-trivial fundamental groups. A classification of generators is provided, highlighting their categorical and homotopical properties.

5 Modalities and Identity Systems

Homotopy Type Theory (HoTT) provides a powerful framework for studying categorical structures through the lens of types, paths, and higher homotopies. In this context, *modalities* and *identity systems* serve as fundamental constructs that encode localization and identification properties, respectively. When compositions of their associated functors are interpreted as groups, they give rise to homological structures, such as fundamental groups, that can model complex topological spaces like the Möbius strip. This article formalizes these concepts and explores their implications in (∞,1)-toposes, with a focus on the emergence of CW-complexes and homologies.

5.1 Modality

Definition 39 (Modality)**.** A modality in HoTT is a structure comprising:

```
def Modality :=
  Σ (modality: U → U)
    (isModal : U → U)
    (eta:      Π (A : U), A → modality A)
    (elim:     Π (A : U) (B : modality A → U)
                 (B–Modal : Π (x : modality A), isModal (B x))
                 (f: Π (x : A), (B (eta A x))),
                 (Π (x : modality A), B x))
    (elim–β :  Π (A : U) (B : modality A → U)
                 (B-Modal : Π (x : modality A), isModal (B x))
                 (f : Π (x : A), (B (eta A x))) (a : A),
                 PathP (<_>B (eta A a)) (elim A B B-Modal f (eta A a)) (f a))
  (modalityIsModal : Π (A : U), isModal (modality A))
  (propIsModal : Π (A : U), Π (a b : isModal A),
                     PathP (<_>isModal A) a b)
  (=-Modal : Π (A : U) (x y : modality A),
                 isModal (PathP (<_>modality A) x y)), 1
```

where 𝒰 is a universe of types, η is a natural inclusion, and elim provides a universal property for modal types (see [1] for details).

Modalities act as localization functors, projecting types onto subcategories of modal types. For instance, the *discrete modality* (♭) trivializes higher homotopies, while the *codiscrete modality* (♯) makes types contractible.

5.2 Identity Systems

Definition 40 (Identity System)**.** For a type $A : \mathcal{U}$, an identity system is defined as:

```
def IdentitySystem (A : U) : U :=
  Σ (=-form : A → A → U)
    (=-ctor : Π (a : A), =-form a a)
    (=-elim : Π (a : A) (C: Π (x y : A)
                (p : =-form x y), U)
                (d : C a a (=-ctor a)) (y : A)
                (p : =-form a y), C a y p)
    (=-comp : Π (a : A) (C: Π (x y : A)
                (p : =-form x y), U)
                (d : C a a (=-ctor a)),
                Path (C a a (=-ctor a)) d
                     (=-elim a C d a (=-ctor a))), 1
```

where = -form generalizes the identity type, and = -ctor ensures reflexivity.

Identity systems generalize paths in HoTT, allowing the construction of types with non-trivial fundamental groups, such as the Möbius strip, where identifications generate $\mathbb{Z}$.

5.3 Classification of Generators

The following table classifies key generators, including modalities and identity systems, based on their categorical and homotopical properties.

5.4 Homologies from Functor Compositions

When functor compositions of modalities and identity systems are treated as groups, they generate homological structures, such as fundamental groups or homology groups. For example, consider the composition $\flat \circ \sharp \circ \flat$. In a topological context, this may correspond to a localization that preserves certain homotopical features, potentially yielding a CW-complex like the Möbius strip.

Табл. 1: Classification of Generators in Homotopy Type Theory

Generator	**Notation**	**Type**	**Adjunction**
Discrete	$\flat$	Modality	$\flat \dashv \sharp$
Codiscrete	$\sharp$	Comodality	$\flat \dashv \sharp$
Bosonic	$\bigcirc$	Modality	$\bigcirc \dashv \bigcirc^{+}$
Fermionic/Infinitesimal	$\Im$	Modality	$\Im \dashv \Im^{+}$
Rheonomic	Rh	Modality	—
Reduced	$\Re$	Modality	—
Polynomial	W	Inductive	—
Polynomial	M	Coinductive	—
Higher Inductive	HIT	Inductive	HIT $\dashv$ Path
Higher Coinductive	CoHIT	Coinductive	Path $\dashv$ CoHIT
Path Spaces	Path	Identification	HIT $\dashv$ $\Im$
Identity	$=, \simeq, \cong$	Identification	—
Isomporphism	$=, \simeq, \cong$	Identification	—
Equality	$=, \simeq, \cong$	Identification	—

Theorem 10. Let $\mathcal{C}$ be an $(\infty,1)$-topos, and let $\mathsf{F} = \flat \circ \sharp \circ \flat$ be a functor composition treated as a group action. The resulting structure induces a fundamental group isomorphic to $\mathbb{Z}$ for types modeling the Möbius strip.

Sketch. The Möbius strip can be constructed as a higher inductive type (HIT) with an identity system generating $\mathbb{Z}$. The functor $\flat$ discretizes the type, $\sharp$ contracts it, and the second $\flat$ reintroduces discrete structure, preserving the nontrivial loop in the identification system. The resulting type has a fundamental group $\pi_1 \cong \mathbb{Z}$. □

5.5 Topological Interpretation

The Möbius strip, as a CW-complex, arises naturally in this framework. Its nontrivial fundamental group is generated by an identity system, while modalities like $\Im$ or $\bigcirc$ introduce twisting or orientation properties. This connects to topological quantum field theories (TQFTs), where surfaces like the Möbius strip encode non-trivial symmetries.

5.6 Conclusion

Modalities and identity systems in HoTT provide a rich framework for modeling categorical and topological phenomena. By treating functor compositions as groups, we uncover homological structures that bridge type theory and topology. Future work may explore applications in TQFT and synthetic differential geometry.

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Issue XXVI: Structure Preserving Theorems

Namdak Tonpa

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Анотація

This article unifies algebra and geometry by characterizing algebra as the domain of homomorphisms preserving structure and geometry as the domain of inverse images of homomorphisms preserving structure. We introduce two new theorems: the Homomorphism Preservation Theorem (HPT) for Algebraic Categories and the Inverse Image Preservation Theorem (IIPT) for Geometric Categories. These build on foundational results like the First Isomorphism Theorem, Continuity Theorem, Pullback Theorem, Stone Duality, Gelfand Duality, and Adjoint Functor Theorem. Aimed at advanced graduate students, this exposition uses category theory to illuminate the algebraic-geometric duality.

6 Algebra and Geometry

Algebra and geometry, foundational to pure mathematics, differ in focus: algebra on abstract structures and their transformations, geometry on spatial properties and invariants. We propose a unifying perspective: algebra is defined by homomorphisms preserving structure, and geometry by the inverse images of homomorphisms preserving structure. This article formalizes this view through two explicit theorems—the Homomorphism Preservation Theorem (HPT) for Algebraic Categories and the Inverse Image Preservation Theorem (IIPT) for Geometric Categories—building on established results. Assuming familiarity with category theory, algebraic topology, and commutative algebra, we provide a framework for graduate students to explore these fields' interplay.

6.1 Homomorphisms in Algebra

Definition 41. Let $\mathcal{C}$ be a category, and let A, B be objects in $\mathcal{C}$. A *homomorphism* $\phi : A \to B$ is a morphism in $\mathcal{C}$ that preserves the structure defined by the category's operations and relations.

In algebraic categories (e.g., $\mathbf{Grp}$, $\mathbf{Ring}$, $\mathbf{Mod}_R$), homomorphisms preserve operations like group multiplication or module scalar multiplication.

Example 1. In $\mathbf{Grp}$, a group homomorphism $\phi : G \to H$ satisfies $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$ for all $g_1, g_2 \in G$, preserving the group operation.

Theorem 11 (First Isomorphism Theorem)**.** Let $\phi : G \to H$ be a group homomorphism with kernel $K = \ker(\phi)$. Then $G/K \cong \mathrm{im}(\phi)$.

Theorem 12 (Universal Property of Free Objects)**.** In an algebraic category (e.g., $\mathbf{Grp}$, $\mathbf{Ring}$), for a free object $F(X)$ on a set X, any map $f : X \to A$ (where A is an object) extends uniquely to a homomorphism $\phi : F(X) \to A$.

We now introduce a theorem encapsulating the algebraic perspective.

Theorem 13 (Homomorphism Preservation Theorem for Algebraic Categories)**.** Let $\mathcal{C}$ be an algebraic category (e.g., $\mathbf{Grp}$, $\mathbf{Ring}$, $\mathbf{Mod}_R$) with a forgetful functor $U : \mathcal{C} \to \mathbf{Set}$. For any surjective homomorphism $\phi : A \to B$ in $\mathcal{C}$ with kernel K (a normal subobject), there exists an isomorphism $\psi : A/K \to B$ such that $\psi \circ \pi = \phi$, where $\pi : A \to A/K$ is the canonical projection. Moreover, any object A can be generated by a free object $F(X)$ via a surjective homomorphism whose structure is preserved by ϕ.

Доведення. The first part follows from the First Isomorphism Theorem [1]: for a surjective homomorphism $\phi : A \to B$ with kernel K, the quotient $A/K \cong B$ via the isomorphism $\psi : aK \mapsto \phi(a)$. The second part follows from the Universal Property of Free Objects [2]: for any object A, there exists a set X and a free object $F(X)$ with a surjective homomorphism $\eta : F(X) \to A$, and any homomorphism $\phi : A \to B$ extends the structure-preserving maps from $F(X)$. □

Remark 1. The HPT formalizes that homomorphisms in algebraic categories preserve structure forward, inducing isomorphisms on quotients and respecting generators, unifying the First Isomorphism Theorem and Universal Property. The name avoids confusion with the Structure-Identity Principle in category theory [2].

6.2 Homomorphisms in Geometry

Geometry emphasizes spaces where structure is preserved under inverse images of homomorphisms, as in $\mathbf{Top}$ or $\mathbf{Sch}$.

Definition 42. Let $\phi : X \to Y$ be a morphism in a category $\mathcal{C}$. The *inverse image* of a subobject $S \subseteq Y$ (if it exists) is the subobject $\phi^{-1}(S) \subseteq X$ defined via the pullback of $S \hookrightarrow Y$ along ϕ.

Example 2. In $\mathbf{Top}$, a continuous map $\phi : X \to Y$ ensures that $\phi^{-1}(V) \subseteq X$ is open for every open set $V \subseteq Y$.

Theorem 14 (Continuity in Topology)**.** A function $\phi : X \to Y$ between topological spaces is continuous if and only if for every open set $V \subseteq Y$, the inverse image $\phi^{-1}(V)$ is open in X.

Theorem 15 (Pullback Theorem in Sheaf Theory)**.** For a morphism $\phi : X \to Y$ in a category with sheaves (e.g., $\mathbf{Top}$, $\mathbf{Sch}$), the inverse image functor ϕ^{-1} : $\mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is exact, preserving the structure of sheaves.

We now define a theorem for geometric categories.

Theorem 16 (Inverse Image Preservation Theorem for Geometric Categories)**.** Let $\mathcal{C}$ be a geometric category (e.g., $\mathbf{Top}$, $\mathbf{Sch}$) with pullbacks. For any morphism $\phi : X \to Y$ in $\mathcal{C}$, the inverse image functor $\phi^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ preserves the lattice structure of subobjects. If $\mathcal{C}$ admits sheaves, $\phi^{-1} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is exact and preserves sheaf isomorphisms, ensuring that the geometric structure of Y is reflected in X.

Доведення. In $\mathbf{Top}$, the Continuity Theorem [4] ensures that $\phi : X \to Y$ is continuous if and only if $\phi^{-1}(V)$ is open for every open set $V \subseteq Y$, so ϕ^{-1} preserves the lattice of open sets. In categories with sheaves (e.g., $\mathbf{Top}$, $\mathbf{Sch}$), the Pullback Theorem [5] guarantees that $\phi^{-1} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is exact, preserving sheaf structures. For schemes, ϕ^{-1} maps prime ideals to prime ideals [3], preserving geometric properties. Since ϕ^{-1} is functorial and preserves monomorphisms, it maintains isomorphisms of subobjects or sheaves. □

Remark 2. The IIPT captures the geometric essence of inverse images preserving structure, unifying the Continuity Theorem and Pullback Theorem. The name distinguishes it from the Structure-Identity Principle [2].

Example 3. For a morphism of schemes $\phi : X \to Y$, the inverse image of a prime ideal under the induced map on stalks is prime, preserving geometric structure [3].

6.3 Categorical Unification

Category theory bridges algebra and geometry through dualities, where the HPT and IIPT interplay.

Theorem 17 (Stone Duality)**.** The category of Boolean algebras, $\mathbf{BoolAlg}$, is dually equivalent to the category of Stone spaces, $\mathbf{Stone}$, via the spectrum functor.

Theorem 18 (Gelfand Duality)**.** The category of commutative C*-algebras is dually equivalent to the category of compact Hausdorff spaces via the spectrum functor.

Theorem 19 (Adjoint Functor Theorem)**.** In a complete category, a functor has a left adjoint if it preserves limits, and a right adjoint if it preserves colimits.

Remark 3. Stone and Gelfand Dualities [6, 7] connect algebraic homomorphisms (HPT) to geometric inverse images (IIPT). The Adjoint Functor Theorem [2] underpins dualities like Spec, where algebraic and geometric structures are preserved [3].

Example 4. The Spec functor maps a ring homomorphism $\phi : R \to S$ to a morphism $\mathbf{Spec}S \to \mathbf{Spec}R$, with inverse images of prime ideals preserving geometric structure.

6.4 Applications and Implications

The HPT and IIPT, supported by prior results, impact advanced research:

- **Algebraic Topology**: The HPT governs homology maps, while the IIPT defines covering spaces.
- **Algebraic Geometry**: The IIPT underpins étale cohomology via inverse images, while the HPT applies to ring homomorphisms.
- **Category Theory**: Stone, Gelfand, and Adjoint Functor Theorems reveal algebra-geometry correspondences.

Corollary 1. In any category with pullbacks, $\phi^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ preserves subobject lattices, as per the IIPT.

6.5 Conclusion

The Homomorphism Preservation Theorem and Inverse Image Preservation Theorem formalize that algebra preserves structure via homomorphisms and geometry via inverse images. Building on the First Isomorphism Theorem, Continuity Theorem, Pullback Theorem, and dualities, these theorems unify pure mathematics. Graduate students are encouraged to apply this framework to algebraic topology, algebraic geometry, and category theory, deepening their research.

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Grothendieck Schemes

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Анотація

We present Grothendieck's functorial definition of schemes as sheaves on the category of affine schemes, structured according to the functor of points perspective. We also outline a path toward formalizing these objects within Homotopy Type Theory (HoTT).

7 Grothendieck Schemes

We view schemes as **sheaves on the category of affine schemes**, satisfying a gluing condition analogous to the usual descent condition in topology.

7.1 Affine Schemes

Let:

$$\mathbf{Aff} := (\mathbf{CRing})^{\mathrm{op}}$$

denote the category of affine schemes, i.e., the opposite of the category of commutative rings.

An affine scheme is of the form $\mathrm{Spec}(A)$, for a commutative ring A.

7.2 Zariski Covers

A **presheaf of sets** on $\mathbf{Aff}$ is a functor:

$$F : \mathbf{Aff}^{\mathrm{op}} \to \mathbf{Set}.$$

This is the *functor of points* perspective: each affine scheme $\mathrm{Spec}(A)$ represents the "test ring"A, and $F(\mathrm{Spec}(A))$ can be thought of as the A-points of F.

A **Zariski sheaf** is a presheaf that satisfies descent for Zariski covers: if $\{\mathrm{Spec}(A_{f_i}) \to \mathrm{Spec}(A)\}$ is a Zariski open affine cover, then the diagram

$$F(\mathrm{Spec}(A)) \to \mathrm{Eq}\left(\prod_i F(\mathrm{Spec}(A_{f_i})) \rightrightarrows \prod_{i,j} F(\mathrm{Spec}(A_{f_i f_j}))\right)$$

is an equalizer diagram.

7.3 Grothendieck Scheme

A **scheme** is a Zariski sheaf

$$\mathsf{F} : \mathbf{Aff}^{\mathrm{op}} \to \boldsymbol{Set}$$

such that:

- There exists a Zariski cover $\{U_i \to F\}$ where each U_i is **representable**, i.e., $U_i \cong \mathrm{Spec}(A_i)$ for some ring A_i.
- Each morphism $U_i \to F$ is an **open immersion** (in the sheaf-theoretic sense).

This means F is **locally isomorphic to affine schemes** and satisfies Zariski descent.

Equivalently: Schemes are Zariski sheaves on $\mathbf{Aff}$ that are **locally representable by affine schemes**.

7.4 Formalization in HoTT

Categories and Presheaves in HoTT

In HoTT, a category can be defined as a type of objects together with types of morphisms and operations satisfying associativity and identity laws up to higher homotopies. A presheaf is then a functor:

$$F : \mathcal{C}^{\mathrm{op}} \to \mathcal{U}_0$$

where $\mathcal{U}_0$ is the universe of 0-types (sets). For $\mathcal{C} = \mathbf{Aff}$, this gives us the functor-of-points view.

Sheaf Conditions in HoTT

A sheaf in HoTT is a presheaf that satisfies a descent condition with respect to a Grothendieck topology, formalized via homotopy limits or truncations, depending on the level of the types involved.

Defining Schemes in HoTT

Within HoTT, a scheme is a sheaf $F : \mathbf{Aff}^{\mathrm{op}} \to \mathcal{U}_0$ satisfying:

- A Zariski descent condition.
- Local representability: there exists a family of open immersions $\{\mathrm{Spec}(A_i) \to F\}$ covering F.

This mirrors the classical definition but is grounded in type-theoretic and higher-categorical constructions.

7.5 Conclusion

Grothendieck's functorial approach to schemes provides a clean and general definition that is well-suited for formalization in Homotopy Type Theory. This opens the way for a synthetic and structured foundation for algebraic geometry in type-theoretic settings.