

# Issue XLI: Local Homotopy Type Theory

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## Анотація

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## 1 Local Homotopy Type Theory

Motivic homotopy theory, introduced by Morel and Voevodsky, extends classical homotopy theory to the setting of algebraic geometry, treating schemes as analogous to topological spaces. A central object in this framework is the category of *T-spectra*, which generalizes the notion of spectra in stable homotopy theory to the motivic context, where the circle  $S^1$  is replaced by the Tate object  $\mathbb{T} = \mathbb{A}^1/(\mathbb{A}^1 \setminus \{0\})$ . John F. Jardine's work on motivic symmetric spectra provides a categorical model for the motivic stable category, equipped with a symmetric monoidal smash product, enabling rich interactions between algebraic and topological structures [1].

This article formalizes the category of T-spectra, emphasizing Jardine's contributions. We define T-spectra and symmetric T-spectra, describe their model category structure, and present key theorems on stable equivalences and monoidal properties. Applications to algebraic geometry, such as the study of motivic cohomology and algebraic K-theory, are discussed.

We assume familiarity with basic category theory and algebraic geometry. Below, we outline essential concepts.

**Definition 1.** Let  $S$  be a Noetherian scheme of finite Krull dimension. The category  $\mathrm{Sm}_S$  consists of smooth schemes of finite type over  $S$ , with morphisms being scheme morphisms over  $S$ .

**Definition 2.** A *simplicial presheaf* on  $\mathrm{Sm}_S$  is a contravariant functor from  $\mathrm{Sm}_S$  to the category of simplicial sets. The category of simplicial presheaves, denoted  $\mathrm{Sp}(\mathrm{Sm}_S)$ , is equipped with a proper closed simplicial model structure, as constructed by Morel and Voevodsky [3].

**Remark 1.** The *Nisnevich topology* on  $\mathrm{Sm}_S$ , denoted  $\mathrm{Nis}$ , is a Grothendieck topology coarser than the Zariski topology but finer than the étale topology. It is crucial for defining the motivic model category.

### 1.1 Definition of T-Spectra

In motivic homotopy theory, the Tate object  $\mathbb{T}$  plays the role of the suspension functor. We define T-spectra as follows.

**Definition 3.** A *T-spectrum* over a scheme  $S$  is a sequence of pointed simplicial presheaves  $E = \{E_n\}_{n \geq 0}$  on  $\mathrm{Sm}_S$ , equipped with structure maps  $\sigma_n : \mathbb{T} \wedge E_n \rightarrow E_{n+1}$ , where  $\wedge$  denotes the smash product of pointed presheaves. The category of T-spectra, denoted  $\mathrm{Sp}_S^{\mathbb{T}}$ , has morphisms given by sequences of maps  $f_n : E_n \rightarrow F_n$  compatible with the structure maps.

**Example 1.** The *motivic sphere spectrum*  $\mathbb{S}$  is a T-spectrum with  $\mathbb{S}_n = \mathbb{T}^{\wedge n}$ , where  $\mathbb{T}^{\wedge n}$  is the  $n$ -fold smash product of  $\mathbb{T}$ , and structure maps given by the identity.

### 1.2 Symmetric T-Spectra

Jardine's work focuses on symmetric T-spectra, which incorporate symmetric group actions to define a robust smash product.

**Definition 4.** A *symmetric T-spectrum* over  $S$  is a T-spectrum  $E = \{E_n\}_{n \geq 0}$  where each  $E_n$  is equipped with an action of the symmetric group  $\Sigma_n$ , and the structure maps  $\sigma_n : \mathbb{T} \wedge E_n \rightarrow E_{n+1}$  are  $\Sigma_n$ -equivariant with respect to the trivial action on  $\mathbb{T}$ . The category of symmetric T-spectra is denoted  $\mathrm{Sp}_S^{\mathbb{T}, \Sigma}$ .

**Remark 2.** The symmetric structure allows for a well-defined internal smash product, making  $\mathrm{Sp}_S^{\mathbb{T}, \Sigma}$  a symmetric monoidal category [1].

### 1.3 Model Category Structure

Jardine establishes a model category structure on  $\mathrm{Sp}_S^{\mathbb{T}}$  and  $\mathrm{Sp}_S^{\mathbb{T}, \Sigma}$ .

**Theorem 1.** The category  $\mathrm{Sp}_S^{\mathbb{T}}$  admits a proper closed simplicial model structure where:

- *Weak equivalences* are maps  $f : E \rightarrow F$  inducing isomorphisms on stable homotopy groups in the Nisnevich topology.

- *Cofibrations* are monomorphisms.
- *Fibrations* are defined via the right lifting property with respect to trivial cofibrations.

A Bousfield localization of this model structure with respect to stable weak equivalences yields the *motivic stable category* [1].

**Theorem 2.** The category  $\mathrm{Sp}_S^{\mathbb{T}, \Sigma}$  is a cofibrantly generated, symmetric monoidal model category satisfying the monoid axiom. The smash product  $\wedge$  is an internal symmetric monoidal structure, with unit the sphere spectrum  $\mathbb{S}$ .

## 1.4 Key Theorems

Jardine’s results provide a categorical foundation for motivic stable homotopy theory.

**Theorem 3.** The motivic stable category, obtained as the homotopy category of  $\mathrm{Sp}_S^{\mathbb{T}, \Sigma}$ , is equivalent to the localization of  $\mathrm{Sp}_S^{\mathbb{T}}$  at stable weak equivalences. Stable equivalences in this category are stable homotopy isomorphisms in the Nisnevich topology [1].

**Theorem 4.** The symmetric smash product  $\wedge$  on  $\mathrm{Sp}_S^{\mathbb{T}, \Sigma}$  is associative, commutative, and unital up to homotopy, making the motivic stable category a symmetric monoidal category with unit  $\mathbb{S}$ .

**Example 2.** The Eilenberg-MacLane spectrum  $H$  for an abelian group  $A$  is a symmetric  $T$ -spectrum, with  $H_n = K(A, n)$ , the simplicial presheaf representing motivic cohomology. Its homotopy groups recover motivic cohomology groups.

## 1.5 Conclusion

$T$ -spectra provide a framework for studying phenomena in algebraic geometry and stable homotopy theory: - *Motivic Cohomology*: The spectrum  $H$  represents motivic cohomology, connecting algebraic cycles to homotopy theory. - *Algebraic K-Theory*: Voevodsky’s motivic spectrum for  $K$ -theory, refined by Jardine’s symmetric structures, links  $K$ -theory to stable homotopy [1]. - *Algebraic Geometry*: The motivic stable category facilitates the study of Gysin triangles and oriented spectra, generalizing classical results in algebraic topology [1].

The category of  $T$ -spectra, as developed by Jardine, provides a powerful framework for motivic homotopy theory. By equipping  $T$ -spectra with a symmetric monoidal smash product and a robust model category structure, Jardine’s work bridges algebraic geometry and stable homotopy theory. Future directions include exploring representability theorems for presheaves of spectra and applications to topological modular forms [2].

## Література

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