

# Monads and Descent

Jean Bénabou and Jacques Roubaud

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## Анотація

Using category theory, we interpret descent data to determine, in very general settings, whether a morphism is a descent morphism or an effective descent morphism.

## 1 Chevalley Bifibrations and Descent

Let  $P : \mathbf{M} \rightarrow \mathbf{A}$  denote a bifibrant functor [1]. For an object  $A \in \mathbf{A}$ , let  $\mathbf{M}(A)$  denote the fibre over  $A$ . We assume that  $\mathbf{A}$  has fibred products.

### 1.1 Monad Associated with an Arrow

Let  $\alpha : A_1 \rightarrow A_0$  be an arrow in  $\mathbf{A}$ . Denote by

$$\alpha^* : \mathbf{M}(A_0) \rightarrow \mathbf{M}(A_1) \quad [\text{resp. } \alpha_* : \mathbf{M}(A_1) \rightarrow \mathbf{M}(A_0)]$$

the inverse image functor (resp. direct image functor), and

$$\eta^\alpha : \text{Id}_{\mathbf{M}(A_1)} \rightarrow \alpha^* \alpha_*; \quad \varepsilon^\alpha : \alpha_* \alpha^* \rightarrow \text{Id}_{\mathbf{M}(A_0)}$$

the canonical natural transformations making  $\alpha_*$  a left adjoint to  $\alpha^*$ . This adjunction defines [2] on  $\mathbf{M}(A_1)$  the monad  $\mathbf{T}^\alpha = (\mathbf{T}^\alpha, \mu^\alpha, \eta^\alpha)$ , where

$$\mathbf{T}^\alpha = \alpha^* \alpha_* : \mathbf{M}(A_1) \rightarrow \mathbf{M}(A_1), \quad \mu^\alpha = \alpha^* \varepsilon^\alpha \alpha_* : \mathbf{T}^\alpha \circ \mathbf{T}^\alpha \rightarrow \mathbf{T}^\alpha.$$

Let  $\mathbf{M}^\alpha$  denote the category  $\mathbf{M}(A_1)^{(\mathbf{T}^\alpha)}$  of algebras over the monad  $\mathbf{T}^\alpha$ , and let

$$U^{\mathbf{T}^\alpha} : \mathbf{M}^\alpha \rightarrow \mathbf{M}(A_1), \quad \Phi^\alpha : \mathbf{M}(A_0) \rightarrow \mathbf{M}^\alpha$$

be the canonical functors.

## 1.2 Chevalley Property

**Definition 1.** The functor  $P$  is a *Chevalley functor* if it satisfies the following property (C):

(C) For every commutative diagram in  $\mathbf{M}$

$$\begin{array}{ccc} M_1 & \xrightarrow{k_1} & M_2 \\ \gamma \downarrow & & \downarrow \gamma' \\ M_3 & \xrightarrow{k_0} & M_4 \end{array}$$

whose image under  $P$  is a cartesian square in  $\mathbf{A}$ , if  $\gamma$  and  $\gamma'$  are cartesian and  $k_0$  is cocartesian, then  $k_1$  is cocartesian.

## 1.3 Characterization of Descent Data

Assume henceforth that  $P : \mathbf{M} \rightarrow \mathbf{A}$  is a Chevalley functor. Let  $\mathbf{a} : A_1 \rightarrow A_0$  be an arrow in  $\mathbf{A}$ . Let  $A_2$  be the fibred product  $A_1 \times_{A_0} A_1$ , with canonical projections  $\mathbf{a}_1, \mathbf{a}_2 : A_2 \rightarrow A_1$ . The property (C) defines, for every object  $M_1 \in \mathbf{M}(A_1)$ , a canonical bijection, natural in  $M_1$ ,

$$\mathrm{Hom}_{\mathbf{M}(A_2)}(\mathbf{a}_1^*(M_1), \mathbf{a}_2^*(M_1)) \rightarrow \mathrm{Hom}_{\mathbf{M}(A_1)}(\mathbf{T}^{\mathbf{a}}(M_1), M_1),$$

denoted  $\varphi \mapsto K^{\mathbf{a}}(\varphi)$ .

**Lemma 1.** An arrow  $\varphi : \mathbf{a}_1^*(M_1) \rightarrow \mathbf{a}_2^*(M_1)$  such that  $P(\varphi) = \mathrm{id}_{A_2}$  is a descent datum if and only if  $K^{\mathbf{a}}(\varphi)$  is an algebra over the monad  $\mathbf{T}^{\mathbf{a}}$ .

Let  $D(\mathbf{a})$  denote the category of descent data relative to  $\mathbf{a}$ , and let

$$\Psi^{\mathbf{a}} : \mathbf{M}(A_0) \rightarrow D(\mathbf{a}), \quad U^{\mathbf{a}} : D(\mathbf{a}) \rightarrow \mathbf{M}(A_1)$$

be the canonical functors.

**Theorem 1.** The correspondence  $\varphi \mapsto K^{\mathbf{a}}(\varphi)$  induces an equivalence of categories  $K^{\mathbf{a}} : D(\mathbf{a}) \rightarrow \mathbf{M}^{\mathbf{a}}$ , making the following diagram commute:

$$\begin{array}{ccccc} \mathbf{M}(A_0) & \xrightarrow{\Psi^{\mathbf{a}}} & D(\mathbf{a}) & \xrightarrow{K^{\mathbf{a}}} & \mathbf{M}^{\mathbf{a}} \\ & & \downarrow U^{\mathbf{a}} & \nearrow U^{\mathbf{T}^{\mathbf{a}}} & \\ & & \mathbf{M}(A_1) & & \end{array}$$

**Proposition 1.** The correspondence  $\varphi \mapsto K^a(\varphi)$  is universal. Precisely, for an arrow  $b_0 : A'_0 \rightarrow A_0$  in  $\mathbf{A}$ , consider the change-of-base diagram in  $\mathbf{A}$ :

$$\begin{array}{ccccc}
 A'_2 & \xrightarrow{a'_1} & A'_1 & \xrightarrow{b_2} & A_2 & \xrightarrow{a_1} & A_1 \\
 & \searrow a' & & & \searrow a & & \\
 A'_0 & \xrightarrow{b_0} & A_0 & & & & 
 \end{array}$$

For  $M_1 \in \mathbf{M}(A_1)$  and  $\varphi : a_1^*(M_1) \rightarrow a_2^*(M_1)$  in  $\mathbf{M}(A_2)$ ,

$$K^{a'}(b_2^*(\varphi)) = b_1^*(K^a(\varphi)).$$

In particular, taking  $A'_0 = A_1$  and  $b_0 = a$ , if  $\varphi$  is a descent datum, then  $b_2^*(\varphi)$  is an effective descent datum. The converse holds, yielding:

**Corollary 1.** An arrow  $\varphi : a_1^*(M_1) \rightarrow a_2^*(M_1) \in \mathbf{M}(A_2)$  is a descent datum if and only if its inverse image  $b_2^*(\varphi)$  under the canonical change of base  $b_0 = a : A'_0 = A_1 \rightarrow A_0$  is an effective descent datum.

This eliminates the need for the “cocycle condition” in subsequent arguments.

## 2 First Applications

Using Theorem 1, Beck’s criterion [2] provides necessary and sufficient conditions for  $\Psi^a$  to be faithful, fully faithful, or an equivalence of categories, in terms of commutation and reflection of certain cokernels by  $a^*$ .

**Proposition 2.** If cokernels of pairs of arrows exist in  $\mathbf{M}(A_0)$ , then  $\Psi^a$  has a left adjoint.

**Proposition 3.** The functor  $\Psi^a$  is faithful if and only if  $a^*$  is faithful.

**Proposition 4.** If  $a^*$  reflects cokernels, then  $\Psi^a$  is fully faithful. In particular, if all fibres of  $\mathbf{M}$  are abelian, then

$$\Psi^a \text{ faithful} \iff \Psi^a \text{ fully faithful} \iff a^* \text{ faithful}.$$

**Definition 2.** An arrow  $a : A_1 \rightarrow A_0$  is *faithfully flat* if  $a^*$  commutes with cokernels and reflects isomorphisms.

**Proposition 5.** If  $a : A_1 \rightarrow A_0$  is faithfully flat and cokernels exist in  $\mathbf{M}(A_0)$ , then  $\Psi^a$  is an equivalence of categories.

### 3 First Examples of Chevalley Functors

1. If  $\mathbf{A}$  is the dual of the category of commutative rings and  $\mathbf{M}$  is the dual of the category of modules over varying commutative rings, the obvious functor  $P : \mathbf{M} \rightarrow \mathbf{A}$  is Chevalley.
2. If  $\mathbf{A}$  is a category with fibred products and  $\mathbf{M} = \mathbf{Fl}(\mathbf{A})$  is the category of arrows in  $\mathbf{A}$ , the “target” functor  $P : \mathbf{M} \rightarrow \mathbf{A}$  is Chevalley.
3. If  $P : \mathbf{M} \rightarrow \mathbf{A}$  and  $Q : \mathbf{N} \rightarrow \mathbf{M}$  are Chevalley, their composite  $P \circ Q$  is Chevalley.
4. If  $P : \mathbf{M} \rightarrow \mathbf{A}$  is Chevalley and  $\mathbf{I}$  is any category, the functor  $P^{\mathbf{I}} : \mathbf{M}^{\mathbf{I}} \rightarrow \mathbf{A}^{\mathbf{I}}$  is Chevalley.
5. In a cartesian diagram of categories

$$\begin{array}{ccc} \mathbf{X}' & \xrightarrow{f^*} & \mathbf{M} \\ g \downarrow & & \downarrow P \\ \mathbf{X} & \xrightarrow{f} & \mathbf{A} \end{array}$$

if  $\mathbf{X}$  has fibred products,  $f$  preserves fibred products, and  $P$  is Chevalley, then  $f^*(P)$  is Chevalley.

In a future publication, we will provide further examples of Chevalley categories and more precise criteria for determining whether  $\Psi^a$  is faithful, fully faithful, or an equivalence when the fibres of  $\mathbf{M}$  are algebraic categories (e.g., categories of modules).

### Література

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- [3] C. Chevalley, Séminaire sur la descente, 1964–1965 (unpublished).