Issue IV: Higher Inductive Types

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Abstract

CW-complexes are fundamental objects in homotopy type theory and even included inside cubical type checker in a form of higher inductive types (HIT). Just like regular (co)-inductive types could be described as recursive terminating (well-founded) or non-terminating trees, higher inductive types could be described as CW-complexes. Defining HIT means to define some CW-complex directly using cubical homogeneous composition structure as an element of initial algebra inside cubical model.

Keywords: Collular Piecewise Topology, Cubical Type Theory, Higher Inductive Types

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1 Higher Inductive Types

CW-complexes are fundamental objects in homotopy type theory and even included inside cubical type checker in a form of higher (co)-inductive types (HITs). Just like regular (co)-inductive types could be described as recursive terminating (well-founded) or non-terminating trees, higher inductive types could be described as CW-complexes. Defining HIT means to define some CW-complex directly using cubical homogeneous composition structure as an element of initial algebra inside cubical model.

Definition 1. (Pushout). One of the notable examples is pushout as it's used to define the cell attachment formally, as others cofibrant objects.

```
data pushout (A B C: U) (f: C -> A) (g: C -> B)
   = po1 (_: A)
   | po2 (_: B)
   | po3 (c: C) <i> [ (i = 0) -> po1 (f c) ,
                      (i = 1) -> po2 (g c) ]
```

Definition 2. (Shperes and Disks). Here are some example of using dimensions to construct spherical shapes.

```
data S1
   = base
   | loop <i> [ (i = 0) -> base,
                (i = 1) -> base ]

data S2
   = point
   | surf <i j> [ (i = 0) -> point, (i = 1) -> point,
                  (j = 0) -> point, (j = 1) -> point ]
                  (j = 0) -> point, (j = 1) -> point ]
```

2 CW-Complexes

The definition of homotopy groups, a special role is played by the inclusions $S^{n-1} \hookrightarrow D^n$. We study spaces obtained iterated attachments of D^n along S^{n-1}.

Definition 3. (Attachment). Attaching n-cell to a space X along a map $f : S^{n-1} \to X$ means taking a pushout figure.

$$\begin{tikzcd} S^{n-1} \arrow[r, "k"] \arrow[d] & X \arrow[d] \\ D^n \arrow[r, "g"] & \cup_f D^n \end{tikzcd}$$

where the notation $X \cup_f D^n$ means result depends on homotopy class of f.

Definition 4. (CW-Complex). Inductively. The only CW-complex of dimention -1 is $\varnothing$. A CW-complex of dimension $\leqslant n$ on X is a space X obtained by attaching a collection of n-cells to a CW-complex of dimension $n-1$.

A CW-complex is a space X which is the $colimit(X_i)$ of a sequence $X_{-1} = \varnothing \hookrightarrow X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow ...X$ of CW-complexes X_i of dimension $\leqslant n$, with X_{i+1} obtained from X_i by i-cell attachments. Thus if X is a CW-complex, it comes with a filtration

$$\varnothing \hookrightarrow X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow ...X$$

where X_i is a CW-complex of dimension $\leqslant i$ called the i-skeleton, and hence the filtration is called the skeletal filtration.