## Monads and Descent

### Jean Bénabou and Jacques Roubaud Communicated by Henri Cartan

Received 5 January 1970

#### Анотація

Using category theory, we interpret descent data to determine, in very general settings, whether a morphism is a descent morphism or an effective descent morphism.

## 1 Chevalley Bifibrations and Descent

Let  $P: \mathbf{M} \to \mathbf{A}$  denote a bifibrant functor [?]. For an object  $A \in \mathbf{A}$ , let  $\mathbf{M}(A)$  denote the fibre over A. We assume that  $\mathbf{A}$  has fibred products.

#### 1.1 Monad Associated with an Arrow

Let  $a: A_1 \to A_0$  be an arrow in **A**. Denote by

$$a^* : \mathbf{M}(A_0) \to \mathbf{M}(A_1) \quad [\text{resp. } a_* : \mathbf{M}(A_1) \to \mathbf{M}(A_0)]$$

the inverse image functor (resp. direct image functor), and

$$\eta^{\alpha}: \mathrm{Id}_{\mathbf{M}(A_1)} \to \alpha^* \alpha_*; \quad \varepsilon^{\alpha}: \alpha_* \alpha^* \to \mathrm{Id}_{\mathbf{M}(A_0)}$$

the canonical natural transformations making  $a_*$  a left adjoint to  $a^*$ . This adjunction defines [?] on  $M(A_1)$  the monad  $T^{\alpha} = (T^{\alpha}, \mu^{\alpha}, \eta^{\alpha})$ , where

$$\mathsf{T}^{\mathfrak{a}} = \mathfrak{a}^* \mathfrak{a}_* : \mathbf{M}(\mathsf{A}_1) \to \mathbf{M}(\mathsf{A}_1), \quad \mu^{\mathfrak{a}} = \mathfrak{a}^* \varepsilon^{\mathfrak{a}} \mathfrak{a}_* : \mathsf{T}^{\mathfrak{a}} \circ \mathsf{T}^{\mathfrak{a}} \to \mathsf{T}^{\mathfrak{a}}.$$

Let  $M^\alpha$  denote the category  $M(A_1)^{(T^\alpha)}$  of algebras over the monad  $T^\alpha,$  and let

$$U^{\mathsf{T}^{\alpha}}: \mathsf{M}^{\alpha} \to \mathsf{M}(\mathsf{A}_1), \quad \Phi^{\alpha}: \mathsf{M}(\mathsf{A}_0) \to \mathsf{M}^{\alpha}$$

be the canonical functors.

### 1.2 Chevalley Property

**Definition 1.** The functor P is a *Chevalley functor* if it satisfies the following property (C):

(C) For every commutative diagram in M

whose image under P is a cartesian square in A, if  $\gamma$  and  $\gamma'$  are cartesian and  $k_0$  is cocartesian, then  $k_1$  is cocartesian.

#### 1.3 Characterization of Descent Data

Assume henceforth that  $P: \mathbf{M} \to \mathbf{A}$  is a Chevalley functor. Let  $\mathfrak{a}: A_1 \to A_0$  be an arrow in  $\mathbf{A}$ . Let  $A_2$  be the fibred product  $A_1 \times_{A_0} A_1$ , with canonical projections  $\mathfrak{a}_1, \mathfrak{a}_2: A_2 \to A_1$ . The property (C) defines, for every object  $M_1 \in \mathbf{M}(A_1)$ , a canonical bijection, natural in  $M_1$ ,

$$\operatorname{Hom}_{\mathbf{M}(A_2)}(\mathfrak{a}_1^*(M_1),\mathfrak{a}_2^*(M_1)) \to \operatorname{Hom}_{\mathbf{M}(A_1)}(\mathsf{T}^{\mathfrak{a}}(M_1),M_1),$$

denoted  $\varphi \mapsto K^{\mathfrak{a}}(\varphi)$ .

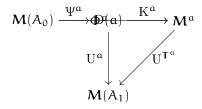
**Lemma 1.** An arrow  $\varphi: \mathfrak{a}_1^*(M_1) \to \mathfrak{a}_2^*(M_1)$  such that  $P(\varphi) = \mathrm{id}_{A_2}$  is a descent datum if and only if  $K^{\mathfrak{a}}(\varphi)$  is an algebra over the monad  $T^{\mathfrak{a}}$ .

Let D(a) denote the category of descent data relative to a, and let

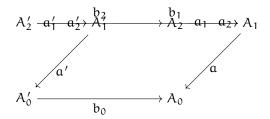
$$\Psi^{\alpha}: \mathbf{M}(A_0) \to \mathrm{D}(\alpha), \quad \mathrm{U}^{\alpha}: \mathrm{D}(\alpha) \to \mathbf{M}(A_1)$$

be the canonical functors.

**Theorem 1.** The correspondence  $\phi \mapsto K^{\alpha}(\phi)$  induces an equivalence of categories  $K^{\alpha}: D(\alpha) \to M^{\alpha}$ , making the following diagram commute:



**Proposition 1.** The correspondence  $\phi \mapsto K^{\alpha}(\phi)$  is universal. Precisely, for an arrow  $b_0: A'_0 \to A_0$  in A, consider the change-of-base diagram in A:



For  $M_1 \in \mathbf{M}(A_1)$  and  $\varphi : \mathfrak{a}_1^*(M_1) \to \mathfrak{a}_2^*(M_1)$  in  $\mathbf{M}(A_2)$ ,

$$K^{\alpha'}(b_2^*(\varphi)) = b_1^*(K^{\alpha}(\varphi)).$$

In particular, taking  $A_0'=A_1$  and  $b_0=\mathfrak{a}$ , if  $\phi$  is a descent datum, then  $b_2^*(\phi)$  is an effective descent datum. The converse holds, yielding:

Corollary 1. An arrow  $\phi: \mathfrak{a}_1^*(M_1) \to \mathfrak{a}_2^*(M_1) \in M(A_2)$  is a descent datum if and only if its inverse image  $\mathfrak{b}_2^*(\phi)$  under the canonical change of base  $\mathfrak{b}_0 = \mathfrak{a}: A_0' = A_1 \to A_0$  is an effective descent datum.

This eliminates the need for the "cocycle condition" in subsequent arguments.

# 2 First Applications

Using Theorem ??, Beck's criterion [?] provides necessary and sufficient conditions for  $\Psi^{\alpha}$  to be faithful, fully faithful, or an equivalence of categories, in terms of commutation and reflection of certain cokernels by  $\alpha^*$ .

**Proposition 2.** If cokernels of pairs of arrows exist in  $M(A_0)$ , then  $\Psi^{\alpha}$  has a left adjoint.

**Proposition 3.** The functor  $\Psi^{\alpha}$  is faithful if and only if  $\alpha^*$  is faithful.

**Proposition 4.** If  $\alpha^*$  reflects cokernels, then  $\Psi^{\alpha}$  is fully faithful. In particular, if all fibres of M are abelian, then

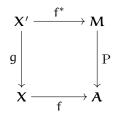
 $\Psi^{\alpha}$  faithful  $\iff \Psi^{\alpha}$  fully faithful  $\iff \alpha^*$  faithful.

**Definition 2.** An arrow  $a:A_1\to A_0$  is faithfully flat if  $\mathfrak{a}^*$  commutes with cokernels and reflects isomorphisms.

**Proposition 5.** If  $\alpha: A_1 \to A_0$  is faithfully flat and cokernels exist in  $\mathbf{M}(A_0)$ , then  $\Psi^{\alpha}$  is an equivalence of categories.

# 3 First Examples of Chevalley Functors

- 1. If **A** is the dual of the category of commutative rings and **M** is the dual of the category of modules over varying commutative rings, the obvious functor  $P: \mathbf{M} \to \mathbf{A}$  is Chevalley.
- 2. If **A** is a category with fibred products and  $\mathbf{M} = \mathsf{Fl}(\mathbf{A})$  is the category of arrows in **A**, the "target" functor  $P: \mathbf{M} \to \mathbf{A}$  is Chevalley.
- 3. If  $P: M \to A$  and  $Q: N \to M$  are Chevalley, their composite  $P \circ Q$  is Chevalley.
- 4. If  $P:M\to A$  is Chevalley and I is any category, the functor  $P^I:M^I\to A^I$  is Chevalley.
- 5. In a cartesian diagram of categories



if X has fibred products, f preserves fibred products, and P is Chevalley, then  $f^*(P)$  is Chevalley.

In a future publication, we will provide further examples of Chevalley categories and more precise criteria for determining whether  $\Psi^a$  is faithful, fully faithful, or an equivalence when the fibres of M are algebraic categories (e.g., categories of modules).

# Література

- [1] A. Grothendieck, Catégories fibrées et descente, Séminaire Bourbaki, 1959.
- [2] F. E. J. Linton, Applied functorial semantics II, Springer Lecture Notes in Mathematics, No. 80, 1969.
- [3] C. Chevalley, Séminaire sur la descente, 1964–1965 (unpublished).