

# Issue XXXIV: Grothendieck Schemes

Namdak Tonpa

May 19, 2025

## Abstract

We present Grothendieck's functorial definition of schemes as sheaves on the category of affine schemes, structured according to the functor of points perspective. We also outline a path toward formalizing these objects within Homotopy Type Theory (HoTT).

## 1 Grothendieck Schemes

We view schemes as **sheaves on the category of affine schemes**, satisfying a gluing condition analogous to the usual descent condition in topology.

### 1.1 Affine Schemes

Let:

$$\mathbf{Aff} := (\mathbf{CRing})^{\mathrm{op}}$$

denote the category of affine schemes, i.e., the opposite of the category of commutative rings.

An affine scheme is of the form  $\mathrm{Spec}(A)$ , for a commutative ring  $A$ .

### 1.2 Zariski Covers

A **presheaf of sets** on  $\mathbf{Aff}$  is a functor:

$$F : \mathbf{Aff}^{\mathrm{op}} \rightarrow \mathbf{Set}.$$

This is the *functor of points* perspective: each affine scheme  $\mathrm{Spec}(A)$  represents the "test ring"  $A$ , and  $F(\mathrm{Spec}(A))$  can be thought of as the  $A$ -points of  $F$ .

A **Zariski sheaf** is a presheaf that satisfies descent for Zariski covers: if  $\{\mathrm{Spec}(A_{f_i}) \rightarrow \mathrm{Spec}(A)\}$  is a Zariski open affine cover, then the diagram

$$F(\mathrm{Spec}(A)) \rightarrow \mathrm{Eq} \left( \prod_i F(\mathrm{Spec}(A_{f_i})) \rightrightarrows \prod_{i,j} F(\mathrm{Spec}(A_{f_i f_j})) \right)$$

is an equalizer diagram.

### 1.3 Grothendieck Scheme

A **scheme** is a Zariski sheaf

$$F : \mathbf{Aff}^{\mathrm{op}} \rightarrow \mathbf{Set}$$

such that:

- There exists a Zariski cover  $\{U_i \rightarrow F\}$  where each  $U_i$  is **representable**, i.e.,  $U_i \cong \mathrm{Spec}(A_i)$  for some ring  $A_i$ .
- Each morphism  $U_i \rightarrow F$  is an **open immersion** (in the sheaf-theoretic sense).

This means  $F$  is **locally isomorphic to affine schemes** and satisfies Zariski descent.

**Equivalently:** Schemes are Zariski sheaves on  $\mathbf{Aff}$  that are **locally representable by affine schemes**.

### 1.4 Formalization in HoTT

#### Categories and Presheaves in HoTT

In HoTT, a category can be defined as a type of objects together with types of morphisms and operations satisfying associativity and identity laws up to higher homotopies. A presheaf is then a functor:

$$F : \mathcal{C}^{\mathrm{op}} \rightarrow \mathcal{U}_0$$

where  $\mathcal{U}_0$  is the universe of 0-types (sets). For  $\mathcal{C} = \mathbf{Aff}$ , this gives us the functor-of-points view.

#### Sheaf Conditions in HoTT

A sheaf in HoTT is a presheaf that satisfies a descent condition with respect to a Grothendieck topology, formalized via homotopy limits or truncations, depending on the level of the types involved.

#### Defining Schemes in HoTT

Within HoTT, a scheme is a sheaf  $F : \mathbf{Aff}^{\mathrm{op}} \rightarrow \mathcal{U}_0$  satisfying:

- A Zariski descent condition.
- Local representability: there exists a family of open immersions  $\{\mathrm{Spec}(A_i) \rightarrow F\}$  covering  $F$ .

This mirrors the classical definition but is grounded in type-theoretic and higher-categorical constructions.

## 1.5 Conclusion

Grothendieck's functorial approach to schemes provides a clean and general definition that is well-suited for formalization in Homotopy Type Theory. This opens the way for a synthetic and structured foundation for algebraic geometry in type-theoretic settings.