Issue XXXIV: Grothendieck Schemes

Namdak Tonpa

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Abstract

We present Grothendieck's functorial definition of schemes as sheaves on the category of affine schemes, structured according to the functor of points perspective. We also outline a path toward formalizing these objects within Homotopy Type Theory (HoTT).

1 Grothendieck Schemes

We view schemes as **sheaves on the category of affine schemes**, satisfying a gluing condition analogous to the usual descent condition in topology.

1.1 Affine Schemes

Let:

$$\mathbf{Aff} := (\mathbf{CRing})^{\mathrm{op}}$$

denote the category of affine schemes, i.e., the opposite of the category of commutative rings.

An affine scheme is of the form $\mathrm{Spec}(A)$, for a commutative ring A.

1.2 Zariski Covers

A **presheaf of sets** on **Aff** is a functor:

$$F : \mathbf{Aff}^{\mathrm{op}} \to \mathbf{Set}.$$

This is the *functor of points* perspective: each affine scheme $\mathrm{Spec}(A)$ represents the "test ring" A, and $F(\mathrm{Spec}(A))$ can be thought of as the A-points of F.

A **Zariski sheaf** is a presheaf that satisfies descent for Zariski covers: if $\{\mathrm{Spec}(A_{f_i}) \to \mathrm{Spec}(A)\}$ is a Zariski open affine cover, then the diagram

$$F(\mathrm{Spec}(A)) \to \mathrm{Eq}\left(\prod_i F(\mathrm{Spec}(A_{f_i})) \rightrightarrows \prod_{i,j} F(\mathrm{Spec}(A_{f_i f_j}))\right)$$

is an equalizer diagram.

1.3 Grothendieck Scheme

A **scheme** is a Zariski sheaf

$$F : \mathbf{Aff}^{\mathrm{op}} \to \mathbf{Set}$$

such that:

- There exists a Zariski cover $\{U_i \to F\}$ where each U_i is **representable**, i.e., $U_i \cong \mathrm{Spec}(A_i)$ for some ring A_i.
- Each morphism $U_i \to F$ is an **open immersion** (in the sheaf-theoretic sense).

This means F is **locally isomorphic to affine schemes** and satisfies Zariski descent.

Equivalently: Schemes are Zariski sheaves on **Aff** that are **locally representable by affine schemes**.

1.4 Formalization in HoTT

Categories and Presheaves in HoTT

In HoTT, a category can be defined as a type of objects together with types of morphisms and operations satisfying associativity and identity laws up to higher homotopies. A presheaf is then a functor:

$$F : \mathcal{C}^{\mathrm{op}} \to \mathcal{U}_0$$

where $\mathcal{U}_0$ is the universe of 0-types (sets). For $\mathcal{C} = \mathbf{Aff}$, this gives us the functor-of-points view.

Sheaf Conditions in HoTT

A sheaf in HoTT is a presheaf that satisfies a descent condition with respect to a Grothendieck topology, formalized via homotopy limits or truncations, depending on the level of the types involved.

Defining Schemes in HoTT

Within HoTT, a scheme is a sheaf $F : \mathbf{Aff}^{\mathrm{op}} \to \mathcal{U}_0$ satisfying:

- A Zariski descent condition.
- Local representability: there exists a family of open immersions $\{\mathrm{Spec}(A_i) \to F\}$ covering F.

This mirrors the classical definition but is grounded in type-theoretic and higher-categorical constructions.

1.5 Conclusion

Grothendieck's functorial approach to schemes provides a clean and general definition that is well-suited for formalization in Homotopy Type Theory. This opens the way for a synthetic and structured foundation for algebraic geometry in type-theoretic settings.