Solving Partial

Abstract

This lecture presents a categorical framework for Green's functions, Stokes–Ostrogradsky theorems, and Fredholm and Volterra integral equations in the context of partial differential equations (PDEs). We incorporate the de Rham theorem, simplicial de Rham complexes, and synthetic differential geometry, drawing on insights from nLab to unify local differential and global integral structures within the category of smooth toposes and differential graded algebras.

1 Introduction: Mathematics as a Categorical Architecture

In the spirit of structural mathematics, partial differential equations (PDEs) are not merely computational tools but objects within a categorical framework, where morphisms reveal universal properties of local-to-global transitions. Green's functions, Stokes–Ostrogradsky theorems, and Fredholm and Volterra integral equations, together with the de Rham theorem and synthetic differential geometry, form a cohesive structure in the category of functional spaces and smooth toposes. This lecture elucidates these connections, emphasizing homological and operator-theoretic perspectives, and integrates insights from synthetic differential geometry to provide an axiomatic foundation for differential and integral structures [**?**].

2 Green's Functions: Kernels of Inverse Morphisms

Definition 2.1. *Let* $\mathbf{E}$ *and* $\mathbf{F}$ *be Hilbert spaces (e.g.,* $L_2(\Omega)$ *or Sobolev spaces* $H^k(\Omega)$*), and let* $L : \mathbf{E} \to \mathbf{F}$ *be a linear differential operator on a smooth manifold* Ω*. A* Green's function $G : \Omega \times \Omega \to \mathbb{R}$ *(or* $\mathbb{C}$*) for* L *is a morphism satisfying:*

$$L_x G(x, \xi) = \delta_\xi,$$

where $\delta_\xi \in \mathbf{F}^*$ *is the Dirac delta functional, defined by* $\langle \delta_\xi, f \rangle = f(\xi)$ *for all* $f \in \mathbf{F}$*.*

The Green's function induces an integral operator $K : \mathbf{F} \to \mathbf{E}$, given by:

$$(Kf)(x) = \int_\Omega G(x,\xi) f(\xi)\, d\mu(\xi),$$

which acts as a right inverse to L, i.e., $LK = \mathrm{id}_{\mathbf{F}}$ on a suitable subspace of $\mathbf{F}$. In the category of Hilbert spaces, G is a kernel in $\mathrm{Hom}(\mathbf{F}^*, \mathbf{E})$, embodying the duality between $\mathbf{F}$ and its dual $\mathbf{F}^*$.

3 Stokes–Ostrogradsky Theorems: Homological Isomorphisms

Theorem 3.1 (Generalized Stokes Theorem)**.** *Let M be an oriented smooth n-manifold with boundary ∂M, and let $\omega \in \Omega^{n-1}(M)$ be a differential $(n-1)$-form. Then:*

$$\int_M d\omega = \int_{\partial M} \omega,$$

where $d : \Omega^{n-1}(M) \to \Omega^n(M)$ is the exterior derivative [2].

- *Ostrogradsky–Gauss Theorem*: For a vector field $\mathbf{F}$ on $\Omega \subset \mathbb{R}^n$,

$$\int_\Omega \nabla \cdot \mathbf{F}\, dV = \oint_{\partial\Omega} \mathbf{F} \cdot \mathbf{n}\, dS.$$

- *Stokes Theorem*: For a vector field $\mathbf{F}$ on a surface $S \subset \mathbb{R}^3$,

$$\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{l}.$$

In the category of differential forms $\mathbf{\Omega}(M)$, the Stokes theorem is an isomorphism in the de Rham complex, reflecting the exactness of the sequence:

$$\cdots \to \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \to \cdots.$$

This isomorphism connects local differential structures to global integral structures, enabling the derivation of integral representations for PDE solutions.

4 de Rham Theorem and Complex

Definition 4.1 (de Rham Complex)**.** *Let M be a smooth manifold. The* de Rham complex *is the cochain complex of differential forms:*

$$\Omega^\bullet(M) : \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots,$$

where $\Omega^k(M)$ is the space of smooth k-forms, and d is the exterior derivative, satisfying $d \circ d = 0$. The de Rham cohomology *is the cohomology of this complex:*

$$H^k_{\mathrm{dR}}(M) = \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\mathrm{im}(d : \Omega^{k-1}(M) \to \Omega^k(M))}.$$

Theorem 4.1 (de Rham Theorem)**.** *For a smooth manifold M, the de Rham cohomology $H^k_{\mathrm{dR}}(M)$ is isomorphic to the singular cohomology $H^k(M;\mathbb{R})$ with real coefficients:*

$$H^k_{\mathrm{dR}}(M) \cong H^k(M;\mathbb{R}).$$

The isomorphism is induced by the integration map:

$$\omega \mapsto \left(\sigma \mapsto \int_\sigma \omega\right),$$

where $\sigma : \Delta^k \to M$ is a singular k-simplex [1].

In the category of sheaves on M, the de Rham complex is a complex of abelian sheaves:

$$\bar{\mathbf{B}}^k\mathbb{R} = (C^\infty(-,\mathbb{R}) \xrightarrow{d_{\mathrm{dR}}} \Omega^1(-) \xrightarrow{d_{\mathrm{dR}}} \cdots \xrightarrow{d_{\mathrm{dR}}} \Omega^k_{\mathrm{closed}}(-)).$$

The de Rham theorem establishes an equivalence in the derived category of sheaves, linking differential forms to topological invariants. In synthetic differential geometry, this complex is internalized in a smooth topos, where the Kock-Lawvere axiom ensures the existence of infinitesimals, facilitating intuitive reasoning about differentials [**?**].

5 Simplicial de Rham Complex

Definition 5.1 (Simplicial de Rham Complex)**.** *Let $X_\bullet : \Delta^{\mathrm{op}} \to$ **Diff** be a simplicial manifold in the category of smooth manifolds* **Diff***. The* simplicial de Rham complex *is the total complex of the double complex:*

$$\Omega^p(X_q) \xrightarrow{\sum_i (-1)^i \delta_i^*} \Omega^p(X_{q+1}),$$

$$\Omega^p(X_q) \xrightarrow{d_{\mathrm{dR}}} \Omega^{p+1}(X_q),$$

*where $\delta_i : X_{q+1} \to X_q$ are face maps, and d_{dR} is the de Rham differential [**?**].*

The simplicial de Rham complex is quasi-isomorphic to the Chevalley-Eilenberg algebra of the infinitesimal path ∞-groupoid $\mathbf{\Pi}_{\mathrm{inf}}(X_\bullet)$, reflecting the structure of a geometric ∞-Lie groupoid. This complex is crucial for equivariant de Rham cohomology and rational homotopy theory, where:

$$C^\infty(\mathbf{\Pi}_{\mathrm{inf}}(X_\bullet)) \simeq \mathrm{Tot}\,\Omega^\bullet(X_\bullet).$$

6 Synthetic Differential Geometry: Axiomatic Framework

Definition 6.1 (Smooth Topos)**.** *A* smooth topos $\mathbf{T}$ *is a topos equipped with a line object R satisfying the Kock-Lawvere axiom: for any $f : D \to R$, where*

$D = \{x \in R \mid x^2 = 0\}$ is the infinitesimal object, there exist unique $a, b \in R$ such that:

$$f(x) = a + bx \quad \text{for all } x \in D.$$

*A smooth topos models synthetic differential geometry, allowing intuitive reasoning with infinitesimals [**?**].*

In a smooth topos, the integration axiom posits that the shape modality $\int$ and flat modality $\flat$ define a cohesive structure:

$$\int \dashv \flat \dashv \sharp.$$

The de Rham complex in synthetic differential geometry is internalized as a complex of objects in $\mathbf{T}$, where differential forms are morphisms from infinitesimal thickenings. The integration map in this context is a morphism:

$$\int : \Omega^k(M) \to \Omega^k(\int M),$$

inducing a synthetic analogue of the de Rham theorem [**?**].

7 Integral Equations: Operator Algebras

Definition 7.1. *Let $\mathbf{E}$ be a Hilbert space, and let $K : \mathbf{E} \to \mathbf{E}$ be a compact linear operator.*

- *A* Fredholm integral equation of the second kind *is:*

  $$u = f + \lambda K u,$$

  where $f \in \mathbf{E}$, $\lambda \in \mathbb{C}$, and $u \in \mathbf{E}$.

- *A* Volterra integral equation of the second kind *is:*

  $$u(x) = f(x) + \lambda \int_a^x K(x, \xi) u(\xi) \, d\xi,$$

  where the integral has a variable upper bound.

The kernel $K(x, \xi)$ is often the Green's function $G(x, \xi)$ or its derivatives. For a PDE $Lu = f$ with boundary conditions, the solution may satisfy:

$$u(x) = \int_\Omega G(x, \xi) f(\xi) \, d\xi + \int_{\partial\Omega} K(x, \xi) u(\xi) \, dS.$$

In synthetic differential geometry, such equations are internalized as morphisms in $\mathbf{T}$, with K defined via the infinitesimal structure of R.

8 Structural Unity: Local-Global Duality

The categorical framework unifies these concepts:

- Green's functions are kernels in $\mathrm{Hom}(\mathbf{F}^*, \mathbf{E})$.
- Stokes–Ostrogradsky theorems are isomorphisms in the de Rham complex.
- The de Rham theorem links differential forms to singular cohomology via integration.
- Simplicial de Rham complexes extend this to ∞-groupoids.
- Synthetic differential geometry provides an axiomatic framework for infinitesimals and integration.
- Fredholm and Volterra equations are algebraic representations of these morphisms.

This reflects a Grothendieckian local-to-global duality, where PDE solutions are cohomology classes in the derived category of sheaves or modules over the ring of differential operators [1, **?**].

9 Conclusion

This lecture presents a categorical synthesis of PDE theory, integrating Green's functions, Stokes–Ostrogradsky theorems, de Rham complexes, synthetic differential geometry, and integral equations. Future lectures will:

1. Construct Green's functions and integral operators in smooth toposes.
2. Explore homological properties of the de Rham and simplicial de Rham complexes.
3. Analyze synthetic differential geometry and its integration axioms.
4. Synthesize these into a unified framework for PDEs.

References

[1] nLab, *de Rham theorem*, `https://ncatlab.org/nlab/show/de+Rham+theorem`, accessed 2024-11-21.

[2] nLab, *Stokes theorem*, `https://ncatlab.org/nlab/show/Stokes+theorem`, accessed 2025-05-11.

[3] nLab, *synthetic differential geometry*, `https://ncatlab.org/nlab/show/synthetic+differential+geometry`, accessed 2025-05-09.

[4] nLab, *simplicial de Rham complex*, `https://ncatlab.org/nlab/show/simplicial+de+Rham+complex`, accessed 2025-02-13.