Issue XLIV: Adjoint String of Identity Modalities

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Abstract

In Homotopy Type Theory (HoTT), identity systems (Contractible, Strict Id, Quotient, Isomorphism, Path = Equivalence) are modeled as modalities in the ∞-topos $\mathcal{E} = \infty\text{Grp}$. We construct a non-degenerate adjoint quadruple extending the Jacobs-Lawvere triple $C \dashv \text{Id}_A \dashv Q(-/\sim)$, incorporating Isomorphism and Path = Equivalence. The modalities are ordered by adjointness: Contractible $\leq$ Strict Id $\leq$ Quotient $\leq$ Isomorphism $\leq$ Path = Equivalence, reflecting their structure in HoTT, where Strict Id, Quotient, and Path = Equivalence are mere propositions for h-sets, while Isomorphism is not.

1 Introduction

Homotopy Type Theory (HoTT) provides a framework for reasoning about equality via the identity type $\text{Id}_A(x, y)$. In the ∞-topos $\mathcal{E} = \infty\text{Grp}$, identity systems are modalities (monads), ordered by adjointness. The classical Jacobs-Lawvere adjunction triple $C \dashv \text{Id}_A \dashv Q(-/\sim)$ captures Contractible, Strict Id, and Quotient. We extend this to a quadruple, including Isomorphism and Path = Equivalence, respecting the HoTT equivalence of Path and Equivalence and the propositional nature of Strict Id, Quotient, and Path = Equivalence for h-sets.

2 Identity Systems as Modalities

Definition 1. In HoTT, the identity systems are:

- **Contractible**: (-1)-truncated types, mere propositions.
- **Strict Id**: $\text{Id}_A(x, y)$ for h-sets (0-truncated), a mere proposition.
- **Quotient**: Set-quotients $A/\sim$, 0-truncated, equivalent to Strict Id.
- **Isomorphism**: $\text{iso}_A(x, y)$, a triple (f, g, p), not a mere proposition.
- **Path = Equiv**: $\text{Id}_A(x, y) \simeq (x \simeq y)$, equivalent in HoTT.

In $\mathcal{E} = \infty\text{Grp}$, we define categories:

- $\mathcal{E}_{\text{contr}} = \mathcal{E}_{\leq -1}$: Mere propositions.
- $\mathcal{E}_{\text{strict}} = \mathcal{E}_{\leq 0} \cong \text{Set}$: h-sets (Strict Id).
- $\mathcal{E}_{\text{quot}} = \mathcal{E}_{\leq 0} \cong \text{Set}$: h-sets (Quotient).
- $\mathcal{E}_{\text{iso}} \cong \mathcal{E}$: ∞-groupoids with isomorphisms.
- $\mathcal{E}_{\text{path/equiv}} \cong \mathcal{E}$: ∞-groupoids with paths/equivalences.

3 Adjoint Quadruple

The Jacobs-Lawvere triple $C \dashv \mathrm{Id}_A \dashv Q(-/\sim)$ is extended to a non-degenerate adjoint quadruple:

$$\mathcal{E}_{\text{contr}} \xrightarrow{F_4} \mathcal{E}_{\text{strict}} \xrightarrow{F_3} \mathcal{E}_{\text{quot}} \xrightarrow{F_2} \mathcal{E}_{\text{iso}} \xrightarrow{F_1} \mathcal{E}_{\text{path/equiv}}$$

Theorem 1. The functors form an adjoint quadruple with non-degenerate adjunctions:

$$F_4 \dashv U_4, \quad F_3 \dashv U_3, \quad F_2 \dashv U_2, \quad F_1 \dashv U_1$$

- $F_4 : \mathcal{E}_{\textbf{contr}} \to \mathcal{E}_{\textbf{strict}}$: Inclusion of (-1)-truncated objects into 0-truncated objects. Right adjoint U_4: (-1)-truncation, $U_4(X) = \|X\|_{-1}$.
- $F_3 : \mathcal{E}_{\textbf{strict}} \to \mathcal{E}_{\textbf{quot}}$: Canonical map to quotient structure, viewing h-sets as quotiented by trivial relations. Right adjoint U_3: Inverse map preserving h-set structure.
- $F_2 : \mathcal{E}_{\textbf{quot}} \to \mathcal{E}_{\textbf{iso}}$: Inclusion of h-sets into $\mathcal{E}$, $\mathrm{core}(X) \cong X$. Right adjoint U_2: 0-truncation, $U_2(X) = \|X\|_0$.
- $F_1 : \mathcal{E}_{\textbf{iso}} \to \mathcal{E}_{\textbf{path/equiv}}$: Canonical inclusion of ∞-groupoids with isomorphisms into full ∞-groupoids with paths/equivalences. Right adjoint U_1: Core map, preserving isomorphism structure.

4 Ordering by Adjointness

The adjunctions induce the ordering:

$$\text{Contractible} \leq \text{Strict Id} \leq \text{Quotient} \leq \text{Isomorphism} \leq \text{Path} = \text{Equivalence}$$

- **Contractible**: Coarsest, mere propositions ((-1)-truncated).
- **Strict Id**: h-sets, $\mathrm{Id}_A(x, y)$ is a mere proposition.
- **Quotient**: Equivalent to Strict Id, 0-truncated set-quotients.
- **Isomorphism**: $\mathrm{iso}_A(x, y)$ is not a mere proposition for general types.
- **Path = Equivalence**: Finest, full ∞-groupoid structure, equivalent via univalence.

5 Conclusion

The adjoint quadruple extends the Jacobs-Lawvere triple, capturing the structure of identity systems in HoTT. The ordering reflects their increasing complexity, with Strict Id, Quotient, and Path = Equivalence collapsing to mere propositions for h-sets, while Isomorphism retains higher structure. Future work could explore these adjunctions in other ∞-topoi or specific CTT models.