Monads and Descent

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Abstract

Using category theory, we interpret descent data to determine, in very general settings, whether a morphism is a descent morphism or an effective descent morphism.

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1 Chevalley Bifibrations and Descent

Let $P : \mathbf{M} \to \mathbf{A}$ denote a bifibrant functor [1]. For an object $A \in \mathbf{A}$, let $\mathbf{M}(A)$ denote the fibre over A. We assume that \mathbf{A} has fibred products.

1.1 Monad Associated with an Arrow

Let $a: A_1 \to A_0$ be an arrow in **A**. Denote by

$$a^*: \mathbf{M}(A_0) \to \mathbf{M}(A_1)$$
 [resp. $a_*: \mathbf{M}(A_1) \to \mathbf{M}(A_0)$]

the inverse image functor (resp. direct image functor), and

$$\eta^a: \mathrm{Id}_{\mathbf{M}(A_1)} \to a^* a_*; \quad \varepsilon^a: a_* a^* \to \mathrm{Id}_{\mathbf{M}(A_0)}$$

the canonical natural transformations making a_* a left adjoint to a^* . This adjunction defines [2] on $\mathbf{M}(A_1)$ the monad $\mathbf{T}^a = (T^a, \mu^a, \eta^a)$, where

$$T^a = a^* a_* : \mathbf{M}(A_1) \to \mathbf{M}(A_1), \quad \mu^a = a^* \varepsilon^a a_* : T^a \circ T^a \to T^a.$$

Let \mathbf{M}^a denote the category $\mathbf{M}(A_1)^{(\mathbf{T}^a)}$ of algebras over the monad \mathbf{T}^a , and let

$$U^{\mathbf{T}^a}: \mathbf{M}^a \to \mathbf{M}(A_1), \quad \Phi^a: \mathbf{M}(A_0) \to \mathbf{M}^a$$

be the canonical functors.

1.2 Chevalley Property

Definition 1.1. The functor P is a Chevalley functor if it satisfies the following property (C):

(C) For every commutative diagram in M

$$M_1 \xrightarrow{k_1} M_2$$

$$\uparrow \qquad \qquad \downarrow \gamma'$$

$$M_3 \xrightarrow{k_0} M_4$$

whose image under P is a cartesian square in \mathbf{A} , if γ and γ' are cartesian and k_0 is cocartesian, then k_1 is cocartesian.

1.3 Characterization of Descent Data

Assume henceforth that $P: \mathbf{M} \to \mathbf{A}$ is a Chevalley functor. Let $a: A_1 \to A_0$ be an arrow in \mathbf{A} . Let A_2 be the fibred product $A_1 \times_{A_0} A_1$, with canonical projections $a_1, a_2: A_2 \to A_1$. The property (C) defines, for every object $M_1 \in \mathbf{M}(A_1)$, a canonical bijection, natural in M_1 ,

$$\operatorname{Hom}_{\mathbf{M}(A_2)}(a_1^*(M_1), a_2^*(M_1)) \to \operatorname{Hom}_{\mathbf{M}(A_1)}(T^a(M_1), M_1),$$

denoted $\varphi \mapsto K^a(\varphi)$.

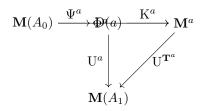
Lemma 1.2. An arrow $\varphi: a_1^*(M_1) \to a_2^*(M_1)$ such that $P(\varphi) = id_{A_2}$ is a descent datum if and only if $K^a(\varphi)$ is an algebra over the monad \mathbf{T}^a .

Let D(a) denote the category of descent data relative to a, and let

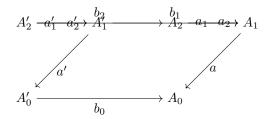
$$\Psi^a: \mathbf{M}(A_0) \to \mathbf{D}(a), \quad \mathbf{U}^a: \mathbf{D}(a) \to \mathbf{M}(A_1)$$

be the canonical functors.

Theorem 1.3. The correspondence $\varphi \mapsto K^a(\varphi)$ induces an equivalence of categories $K^a : D(a) \to M^a$, making the following diagram commute:



Proposition 1.4. The correspondence $\varphi \mapsto K^a(\varphi)$ is universal. Precisely, for an arrow $b_0: A'_0 \to A_0$ in \mathbf{A} , consider the change-of-base diagram in \mathbf{A} :



For $M_1 \in \mathbf{M}(A_1)$ and $\varphi : a_1^*(M_1) \to a_2^*(M_1)$ in $\mathbf{M}(A_2)$,

$$K^{a'}(b_2^*(\varphi)) = b_1^*(K^a(\varphi)).$$

In particular, taking $A'_0 = A_1$ and $b_0 = a$, if φ is a descent datum, then $b_2^*(\varphi)$ is an effective descent datum. The converse holds, yielding:

Corollary 1.5. An arrow $\varphi: a_1^*(M_1) \to a_2^*(M_1) \in \mathbf{M}(A_2)$ is a descent datum if and only if its inverse image $b_2^*(\varphi)$ under the canonical change of base $b_0 = a: A_0' = A_1 \to A_0$ is an effective descent datum.

This eliminates the need for the "cocycle condition" in subsequent arguments.

2 First Applications

Using Theorem 1.3, Beck's criterion [2] provides necessary and sufficient conditions for Ψ^a to be faithful, fully faithful, or an equivalence of categories, in terms of commutation and reflection of certain cokernels by a^* .

Proposition 2.1. If cokernels of pairs of arrows exist in $\mathbf{M}(A_0)$, then Ψ^a has a left adjoint.

Proposition 2.2. The functor Ψ^a is faithful if and only if a^* is faithful.

Proposition 2.3. If a^* reflects cokernels, then Ψ^a is fully faithful. In particular, if all fibres of \mathbf{M} are abelian, then

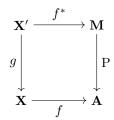
$$\Psi^a$$
 faithful $\iff \Psi^a$ fully faithful $\iff a^*$ faithful.

Definition 2.4. An arrow $a:A_1\to A_0$ is faithfully flat if a^* commutes with cokernels and reflects isomorphisms.

Proposition 2.5. If $a: A_1 \to A_0$ is faithfully flat and cokernels exist in $\mathbf{M}(A_0)$, then Ψ^a is an equivalence of categories.

3 First Examples of Chevalley Functors

- 1. If **A** is the dual of the category of commutative rings and **M** is the dual of the category of modules over varying commutative rings, the obvious functor $P: \mathbf{M} \to \mathbf{A}$ is Chevalley.
- 2. If **A** is a category with fibred products and $\mathbf{M} = \mathbf{Fl}(\mathbf{A})$ is the category of arrows in **A**, the "target" functor $P : \mathbf{M} \to \mathbf{A}$ is Chevalley.
- 3. If $P: \mathbf{M} \to \mathbf{A}$ and $Q: \mathbf{N} \to \mathbf{M}$ are Chevalley, their composite $P \circ Q$ is Chevalley.
- 4. If $P: M \to A$ is Chevalley and I is any category, the functor $P^I: M^I \to A^I$ is Chevalley.
- 5. In a cartesian diagram of categories



if **X** has fibred products, f preserves fibred products, and P is Chevalley, then $f^*(P)$ is Chevalley.

In a future publication, we will provide further examples of Chevalley categories and more precise criteria for determining whether Ψ^a is faithful, fully faithful, or an equivalence when the fibres of \mathbf{M} are algebraic categories (e.g., categories of modules).

References

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- [2] F. E. J. Linton, Applied functorial semantics II, Springer Lecture Notes in Mathematics, No. 80, 1969.
- [3] C. Chevalley, Séminaire sur la descente, 1964–1965 (unpublished).