

Volume I: Foundations

Introduction to Homotopy Type Theory

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Issue I: Type Theory

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Анотація

Martin-Löf Type Theory (MLTT), introduced by Per Martin-Löf in 1972 as MLTT-72, is a cornerstone of constructive mathematics, providing a foundation for formalizing mathematical proofs and programming languages. Its 1975 variant, MLTT-75, incorporates dependent types (Π , Σ) and identity types, with the J eliminator as a key construct for reasoning about equality. Historically, internalizing MLTT in a type checker while constructively proving the J eliminator has been challenging due to limitations in pure functional systems. This article presents a canonical formalization of MLTT-75 without disjoint union types and natural numbers \mathbb{N} , denoted as MLTT-73 and its complete internalization in **Per**, a minimal dependent type theory language equipped with cubical type primitives. Using presented type theory, we constructively prove all MLTT-73 inference rules, including the J eliminator, and demonstrate suitability as a robust type checker. We also provide logical, categorical, and homotopical interpretations of MLTT to contextualize its significance. This work advances the mechanization of constructive mathematics and offers a blueprint for future type-theoretic explorations.

Keywords: Martin-Löf Type Theory, Cubical Type Theory.

Introduction to MLTT-73

For decades, type theorists have sought to fully internalize Martin-Löf Type Theory (MLTT) within a type checker, a task akin to building a self-verifying blueprint for mathematics. Introduced by Per Martin-Löf in 1972 [3] and refined in 1975 [4], MLTT-75 is a foundational system that combines dependent types (Π for universal quantification, Σ for existential quantification) with identity types, enabling rigorous reasoning about equality. Central to MLTT-75 is the J eliminator, a rule that governs how identity proofs are used, but its constructive derivation has long eluded pure functional type checkers due to the complexity of equality types.

The MLTT-75 without Disjoint Union Types (+) and Natural Numbers (\mathbb{N}) we will denote MLTT-73. This article addresses this challenge by presenting a

canonical formalization of MLTT-73 and its complete internalization in **Per**, a novel type theory language designed for constructive proofs. Leveraging cubical type theory [17], **Per** incorporates Path types and universe polymorphism to faithfully embed MLTT-73 rules, achieving a constructive proof of the J eliminator. This internalization serves as an ultimate test of a type checker’s robustness, verifying its ability to fuse introduction and elimination rules through beta and eta equalities.

To make MLTT accessible, we provide intuitive interpretations of its types: logical (as quantifiers), categorical (as functors), and homotopical (as spaces). These perspectives highlight MLTT’s role as a bridge between mathematics and computation. Our work builds on Martin-Löf’s vision of constructive mathematics, offering a minimal yet powerful framework for mechanized reasoning. We aim to inspire researchers and practitioners to explore type theory’s potential in formalizing mathematics and designing reliable software.

Syntax of Per

The BNF notation of type checker language used in code samples consists of: i) telescopes (contexts or sigma chains) and definitions; ii) pure dependent type theory syntax; iii) inductive data definitions (sum chains) and split eliminator; iv) cubical face system; v) module system. It is slightly based on cubicaltt.

```

F = module I where L
L = ∅ | import I | def I T : O := O
T = ∅ | ( I : O ) T
O = I | ( O ) | U | O → O | O O
    | Π ( I : O ) , O | λ ( I : O ) , O
    | ∃ O O O O O | ⟨ O ⟩ O | O @ O | transp O O
    | 0 | 1 | ¬O | O ∧ O | O ∨ O | □
    | Σ ( I : O ) , O | O .1 | O .2 | O , O

```

Here = (definition), ∅ (empty set), |, are parts of BNF language and ⟨, ⟩, (,), :=, ∨, ∧, ¬, →, 0, 1, @, □, **module**, **import**, **where**, **transp**, **.1**, **.2**, and , are terminals of type checker language.

1 Interpretations

Type-Theoretical Interpretation

Martin-Löf Type Theory (MLTT), introduced by Per Martin-Löf in 1972 [3] and refined in 1975 [4], is a foundational system for constructive mathematics, blending logical rigor with computational expressiveness. Its 1975 variant, MLTT-75, centers on dependent types (Π , Σ) and identity types (Id), which underpin its ability to formalize mathematical reasoning and type checking. This section explores four interpretations of MLTT-75: logical, categorical, homotopical, and set-theoretical; — to illuminate its versatility and contextualize its internalization in the **Per** language. These perspectives reveal MLTT-75 as a unifying framework bridging logic, category theory, homotopy theory, and set

Type Theory	Logic	Category Theory	Homotopy Theory
A type	class	object	space
$\text{isProp } A$	proposition	(-1) -truncated object	space
$a:A$ program	proof	generalized element	point
$B(x)$	predicate	indexed object	fibration
$b(x) : B(x)$	conditional proof	indexed elements	section
$\mathbf{0}$	\perp false	initial object	empty space
$\mathbf{1}$	\top true	terminal object	singleton
$\mathbf{2}$	boolean	subobject classifier	\mathbb{S}^0
$A + B$	$A \vee B$ disjunction	coproduct	coproduct space
$A \times B$	$A \wedge B$ conjunction	product	product space
$A \rightarrow B$	$A \Rightarrow B$	internal hom	function space
$\sum x : A, B(x)$	$\exists_{x:A} B(x)$	dependent sum	total space
$\prod x : A, B(x)$	$\forall_{x:A} B(x)$	dependent product	space of sections
\mathbf{Path}_A	equivalence $=_A$	path space object	path space A^I
quotient	equivalence class	quotient	quotient
W-type	induction	colimit	complex
type of types	universe	object classifier	universe
quantum circuit	proof net	string diagram	

theory, with each interpretation highlighting distinct aspects of its types and rules.

In MLTT, types are defined by five classes of rules: (1) *formation*, specifying the type’s signature; (2) *introduction*, defining constructors for its elements; (3) *elimination*, providing a dependent induction principle; (4) *computation* (beta-equality), governing reduction; and (5) *uniqueness* (eta-equality), ensuring canonical forms, though the latter is absent for identity types in homotopical settings.

For MLTT-73, we focus on Π (dependent function types), Σ (dependent pair types), and Id (identity types), with the latter replaced by Path types in cubical type theory to enable constructive proofs, such as the J eliminator, in **Per**. The identity type, introduced in MLTT-75 [4], is particularly significant, enabling reasoning about equality constructively. Unlike MLTT-72, which included only Π and Σ types, MLTT-75’s Id types originally enforced uniqueness of identity proofs (UIP) via an eta-rule.

However, modern homotopical interpretations, pioneered by Hofmann and Streicher [7], refute UIP, adopting Path types that model equality as paths in a space, aligning with cubical type theory’s constructive framework. This shift is crucial for **Per**, as Path types facilitate the internalization of MLTT-75’s rules.

Type checkers operate within contexts, binding variables to indexed universes, built-in types, or user-defined types via de Bruijn indices or names. These contexts enable queries about type derivability and code extraction, forming the core of **Per**’s type checker. By encoding MLTT-75’s syntax and rules, **Per** supports multiple interpretations, each offering unique insights into its structure and applications.

1.1 Logical Interpretation

The logical interpretation casts MLTT-75 as a system for intuitionistic higher-order logic, where types correspond to propositions and terms to proofs, embodying the Curry-Howard correspondence. In this view, a type A represents a proposition, and a term $a : A$ is a proof of A . The Π -type, $\prod_{x:A} B(x)$, encodes universal quantification ($\forall x : A, B(x)$), while the Σ -type, $\sum_{x:A} B(x)$, represents existential quantification ($\exists x : A, B(x)$). The identity type, $\text{Id}_A(a, b)$, captures propositional equality ($a =_A b$), with the J eliminator providing a constructive means to reason about equalities.

Each type's five rules (formation, introduction, elimination, computation, and uniqueness, except for Id in cubical settings) mirror the structure of logical inference rules. For instance, the introduction rule for Π constructs a lambda term (proof of a universal statement), while its elimination rule applies the term to an argument (using the universal statement).

MLTT-75 is not standalone framework for constructive mathematics but rather the extended foundational core on top of MLTT-72. Adding **0** (Empty), **1** (Unit), **2** (Bool) types allows resulting type system to internalize intuitionistic propositional logic (IPL), via Gödel's double-negation translation, classical logic can be encoded within IPL [13]. In **Per**, this logical framework underpins the type checker's ability to verify MLTT-75's rules, ensuring constructive consistency.

1.2 Categorical Interpretation

The categorical interpretation models MLTT-75 within category theory, where types are objects, terms are morphisms, and type constructions are functors. This perspective, formalized by Cartmell and Seely [16], views MLTT-75 with **0**, **1**, **2** types as a locally cartesian closed category (LCCC). Here, Π -types correspond to dependent products (right adjoints to base change functors), and Σ -types to dependent sums (left adjoints). The identity type, Id_A , is modeled as a path space object, reflecting equality as a morphism.

For example, given a morphism $f : A \rightarrow B$ in a category, the Π_f functor maps a dependent type over B to one over A , generalizing function spaces, while Σ_f constructs the total space of a fibration. In **Per**, this interpretation informs the type checker's handling of dependent types, with cubical primitives enabling precise categorical semantics for Path types. Topos-theoretical models, such as presheaves, further enrich this interpretation by treating fibrations as functors, aligning with MLTT-75's expressive power [11].

1.3 Homotopical Interpretation

The homotopical interpretation, a breakthrough in modern type theory, views MLTT-75's types as spaces and terms as points, with identity types as paths. Introduced by Hofmann and Streicher's groupoid model [7], this perspective refutes the uniqueness of identity proofs (UIP) in classical MLTT-75, replacing

Id with Path types that model equality as continuous paths in a space. In cubical type theory, Path types are functions from an interval $[0, 1]$ to a type, enabling constructive proofs of MLTT-75's rules, including the J eliminator, in **Per**.

Here, Π -types represent spaces of sections, Σ -types denote total spaces of fibrations, and Path types form path spaces (A'). This interpretation connects MLTT-75 to homotopy theory, where types are ∞ -groupoids, and fibrations (dependent types) are studied geometrically. For instance, a Π -type can be seen as a trivial fiber bundle, with its introduction rule constructing a section [1]. In **Per**, cubical primitives like connections and compositions support this interpretation, making MLTT-75's internalization homotopically robust.

1.4 Set-Theoretical Interpretation

The set-theoretical interpretation models MLTT-75's types as sets and terms as elements, aligning with classical first-order logic. In this view, a type A is a set, and a term $a : A$ is an element. The Π -type represents a set of functions, Σ -type a disjoint union of sets, and $\text{Id}_A(a, b)$ an equality relation. However, this interpretation is limited, as it cannot capture higher equalities (e.g., paths between paths) or inductive types directly, due to its 0-truncated nature [1].

2 Dependent Type Theory

2.1 Dependent Product (Π)

Π is a dependent product type, the generalization of functions. As a function it can serve the wide range of mathematical constructions as its domain and codomain, which are in general: objects, types, or spaces; and could have as its instance: sets, functions, polynomial functors, infinitesimals, ∞ -groupoids, topological ∞ -groupoid, CW-complexes, categories, languages, etc.

At this light there could be many interpretation of Π types from different areas of mathematics. We give here three: i) logical interpretation of Π as \forall quantifier from higher order logic that forms a ground of type theory; ii) geometric interpretation of Π as fiber bundle; iii) categorical interpretation of functions as functors.

Type-theoretical interpretation

As a logical system dependent type theory could correspond to higher order logic. However here only type-theoretical model is given completely.

Definition 1 (Π -Formation). Π -types represents the way we create the spaces of dependent functions $f : \Pi(x : A), B(x)$ with domain in A and codomain in type family $B : A \rightarrow U$ over A .

$$\Pi : U =_{def} \prod_{A:U} \prod_{x:A} B(x).$$

def Pi (A : U) (B : A → U) : U := Π (x : A), B x

Definition 2 (Π-Introduction). Lambda constructor defines a new lambda function in the space of dependent functions. It is called lambda abstraction and displayed as $\lambda x.b(x)$ or $x \mapsto b(x)$.

$$\backslash (x : A) \rightarrow b : \Pi(A, B) =_{def}$$

$$\prod_{A:U} \prod_{B:A \rightarrow U} \prod_{b:B(a)} \lambda x.b_x.$$

def lambda (A: U) (B: A → U) (b: Π A B) : Π A B := λ (x : A), b x
def lam (A B: U) (f: A → B) : A → B := λ (x : A), f x

When codomain is not dependent on valude from domain the function $f : A \rightarrow B$ is studied in System F_ω , dependent case is studied in System P_ω or Calculus of Construction (CoC).

Definition 3 (Π-Induction Principle). States that if predicate holds for lambda function then there is a function from function space to the space of predicate.

def Π-ind (A : U) (B : A → U) (C : Π A B → U) (g: Π (x: Π A B), C x)
: Π (p: Π A B), C p := λ (p: Π A B), g p

Definition 4 (Π-Elimination). Application reduces the term by using recursive substitution.

$$f a : B(a) =_{def} \prod_{A:U} \prod_{B:A \rightarrow U} \prod_{a:A} \prod_{f:\prod_{x:A} B(a)} f(a).$$

def apply (A: U) (B: A → U) (f: Π A B) (a: A) : B a := f a
def app (A B: U) (f: A → B) (x: A) : B := f x

Theorem 1 (Π-Composition). Composition is using application of appropriate singnatures.

$$f(a) =_{B(a)} (\lambda(x : A) \rightarrow f(a))(a).$$

def o[⊤] (α β γ: U) : U
:= (β → γ) → (α → β) → (α → γ)

def o (α β γ : U) : o[⊤] α β γ
:= λ (g: β → γ) (f: α → β) (x: α), g (f x)

Theorem 2 (Π-Computation). β -rule shows that composition $\text{lam} \circ \text{app}$ could be fused.

$$f(a) =_{B(a)} (\lambda(x : A) \rightarrow f(a))(a).$$

```

def  $\Pi$ - $\beta$  (A : U) (B : A  $\rightarrow$  U) (a : A) (f :  $\Pi$  A B)
  : Path (B a) (apply A B (lambda A B f) a) (f a)
:= idp (B a) (f a)

```

Theorem 3. (Π -Uniqueness). η -rule shows that composition $\text{app} \circ \text{lam}$ could be fused.

$$f =_{(x:A) \rightarrow B(a)} (\lambda(y : A) \rightarrow f(y)).$$

```

def  $\Pi$ - $\eta$  (A : U) (B : A  $\rightarrow$  U) (a : A) (f :  $\Pi$  A B)
  : Path ( $\Pi$  A B) f ( $\lambda$  (x : A), f x)
:= idp ( $\Pi$  A B) f

```

Categorical interpretation

The adjoints Π and Σ is not the only adjoints could be presented in type system. Axiomatic cohesions could contain a set of adjoint pairs as a core type checker operations.

Definition 5 (Dependent Product). The dependent product along morphism $g : B \rightarrow A$ in category \mathcal{C} is the right adjoint $\Pi_g : \mathcal{C}_{/B} \rightarrow \mathcal{C}_{/A}$ of the base change functor.

Definition 6 (Space of Sections). Let \mathbf{H} be a $(\infty, 1)$ -topos, and let $E \rightarrow B : \mathbf{H}_{/B}$ a bundle in \mathbf{H} , object in the slice topos. Then the space of sections $\Gamma_\Sigma(E)$ of this bundle is the Dependent Product:

$$\Gamma_\Sigma(E) = \Pi_\Sigma(E) \in \mathbf{H}.$$

Theorem 4 (Homotopy Equivalence). If fiber space is set for all base, and there are two functions $f, g : (x : A) \rightarrow B(x)$ and two homotopies between them, then these homotopies are equal.

```

def setPi (A: U) (B: A  $\rightarrow$  U)
  (h:  $\Pi$  (x: A), isSet (B x)) (f g:  $\Pi$  A B)
  (p q: Path ( $\Pi$  A B) f g)
  : Path (Path ( $\Pi$  A B) f g) p q

```

Theorem 5 (Contractability). If domain and codomain is contractible then the space of sections is contractible.

```

def piIsContr (A: U) (B: A  $\rightarrow$  U) (u: isContr A)
  (q:  $\Pi$  (x: A), isContr (B x))
  : isContr ( $\Pi$  A B)

```

Definition 7 (Section). A section of morphism $f : A \rightarrow B$ in some category is the morphism $g : B \rightarrow A$ such that $f \circ g : B \xrightarrow{g} A \xrightarrow{f} B$ equals the identity morphism on B .

Homotopical interpretation

Geometrically, Π type is a space of sections, while the dependent codomain is a space of fibrations. Lambda functions are sections or points in these spaces, while the function result is a fibration. Π type also represents the cartesian family of sets, generalizing the cartesian product of sets.

Definition 8. (Fiber). The fiber of the map $p : E \rightarrow B$ in a point $y : B$ is all points $x : E$ such that $p(x) = y$.

Definition 9. (Fiber Bundle). The fiber bundle $F \rightarrow E \xrightarrow{p} B$ on a total space E with fiber layer F and base B is a structure (F, E, p, B) where $p : E \rightarrow B$ is a surjective map with following property: for any point $y : B$ exists a neighborhood U_b for which a homeomorphism $f : p^{-1}(U_b) \rightarrow U_b \times F$ making the following diagram commute.

$$\begin{array}{ccc} p^{-1}(U_b) & \xrightarrow{f} & U_b \times F \\ p \downarrow & \swarrow pr_1 & \\ U_b & & \end{array}$$

Definition 10. (Cartesian Product of Family over B). Is a set F of sections of the bundle with elimination map $app : F \times B \rightarrow E$ such that

$$F \times B \xrightarrow{app} E \xrightarrow{pr_1} B \quad (1)$$

pr_1 is a product projection, so pr_1, app are morphisms of slice category Set/B . The universal mapping property of F : for all A and morphism $A \times B \rightarrow E$ in Set/B exists unique map $A \rightarrow F$ such that everything commute. So a category with all dependent products is necessarily a category with all pullbacks.

Definition 11 (Trivial Fiber Bundle). When total space E is cartesian product $\Sigma(B, F)$ and $p = pr_1$ then such bundle is called trivial $(F, \Sigma(B, F), pr_1, B)$.

Theorem 6 (Functions Preserve Paths). For a function $f : (x : A) \rightarrow B(x)$ there is an $ap_f : x =_A y \rightarrow f(x) =_{B(x)} f(y)$. This is called application of f to path or congruence property (for non-dependent case — *cong* function). This property behaves functorially as if paths are groupoid morphisms and types are objects.

Theorem 7 (Trivial Fiber Bundle equals Family of Sets). Inverse image (fiber) of fiber bundle $(F, B * F, pr_1, B)$ in point $y : B$ equals $F(y)$.

```
def Family (B : U) : U1 := B → U
def Fibration (B : U) : U1 := Σ (X : U), X → B

def encode-Pi (B : U) (F : B → U) (y : B)
  : fiber (Sigma B F) B (pr1 B F) y → F y
```

```

:= λ (x : fiber (Sigma B F) B (pr₁ B F) y),
    subst B F x.1.1 y (<i> x.2 @ -i) x.1.2

def decode-Pi (B : U) (F : B → U) (y : B)
  : F y → fiber (Sigma B F) B (pr₁ B F) y
:= λ (x : F y), ((y, x), idp B y)

def decode-encode-Pi (B : U) (F : B → U) (y : B) (x : F y)
  : Path (F y) (transp (<i> F (idp B y @ i)) 0 x) x
:= <j> transp (<i> F y) j x

def encode-decode-Pi (B : U) (F : B → U) (y : B)
  (x : fiber (Sigma B F) B (pr₁ B F) y)
  : Path (fiber (Sigma B F) B (pr₁ B F) y)
    ((y, encode-Pi B F y x), idp B y) x
:= <i> ( (x.2 @ i, transp (<j> F (x.2 @ i ∨ -j)) i x.1.2),
    <j> x.2 @ i ∧ j )

def Bundle=Pi (B : U) (F : B → U) (y : B)
  : PathP (<_> U) (fiber (Sigma B F) B (pr₁ B F) y) (F y)
:= iso→Path (fiber (Sigma B F) B (pr₁ B F) y) (F y)
  (encode-Pi B F y) (decode-Pi B F y)
  (decode-encode-Pi B F y) (encode-decode-Pi B F y)

```

2.2 Dependent Sum (Σ)

Σ -type is a space that contains dependent pairs where type of the second element depends on the value of the first element. As only one point of fiber domain present in every defined pair, Σ -type is also a dependent sum, where fiber base is a disjoint union.

Σ is a dependent sum type, the generalization of products. Σ type is a total space of fibration. Element of total space is formed as a pair of basepoint and fibration.

Spaces of dependent pairs are using in type theory to model cartesian products, disjoint sums, fiber bundles, vector spaces, telescopes, lenses, contexts, objects, algebras, \exists -type, etc.

Type-theoretical interpretation

Definition 12 (Σ -Formation). The dependent sum type is indexed over type A in the sense of coproduct or disjoint union, where only one fiber codomain $B(x)$ is present in pair.

$$\Sigma : U =_{def} \sum_{x:A} B(x).$$

def Sigma (A: U) (B: A \rightarrow U) : U := Σ (x: A), B(x)

Definition 13 (Σ -Introduction). The dependent pair constructor is a way to create indexed pair over type A in the sense of coproduct or disjoint union.

$$\mathbf{pair} : \Sigma(A, B) =_{def} \prod_{A:U} \prod_{B:A \rightarrow U} \prod_{a:A} \prod_{b:B(a)} (a, b).$$

def pair (A: U) (B: A \rightarrow U) (a: A) (b: B a) : Sigma A B := (a, b)

Definition 14 (Σ -Elimination). The dependent projections $pr_1 : \Sigma(A, B) \rightarrow A$ and $pr_2 : \prod_{x:\Sigma(A, B)} B(pr_1(x))$ are pair deconstructors.

$$\begin{aligned} \mathbf{pr}_1 &: \prod_{A:U} \prod_{B:A \rightarrow U} \prod_{x:\Sigma(A, B)} A \\ &=_{def} .1 =_{def} (a, b) \mapsto a. \\ \mathbf{pr}_2 &: \prod_{A:U} \prod_{B:A \rightarrow U} \prod_{x:\Sigma(A, B)} B(x.1) \\ &=_{def} .2 =_{def} (a, b) \mapsto b. \end{aligned}$$

def pr₁ (A: U) (B: A \rightarrow U) (x: Sigma A B) : A := x.1
def pr₂ (A: U) (B: A \rightarrow U) (x: Sigma A B) : B (pr₁ A B x) := x.2

Definition 15 (Σ -Induction). States that if predicate holds for two projections then predicate holds for total space.

```
def  $\Sigma$ -ind (A : U) (B : A  $\rightarrow$  U)
  (C :  $\Pi$  (s :  $\Sigma$  (x : A), B x), U)
  (g :  $\Pi$  (x : A) (y : B x), C (x, y))
  (p :  $\Sigma$  (x : A), B x)
  : C p := g p.1 p.2
```

Theorem 8 (Σ -Computation). $\text{def } \Sigma\text{-}\beta_1$ (A : U) (B : A \rightarrow U) (a : A) (b : B a) : Path A a (pr₁ A B (a, b)) := idp A a

```
def  $\Sigma\text{-}\beta_2$  (A : U) (B : A  $\rightarrow$  U) (a : A) (b : B a)
  : Path (B a) b (pr2 A B (a, b)) := idp (B a) b
```

Theorem 9 (Σ -Uniqueness). $\text{def } \Sigma\text{-}\eta$ (A : U) (B : A \rightarrow U) (p : Sigma A B) : Path (Sigma A B) p (pr₁ A B p, pr₂ A B p) := idp (Sigma A B) p

Categorical interpretation

Definition 16. (Dependent Sum). The dependent sum along the morphism $f : A \rightarrow B$ in category \mathcal{C} is the left adjoint $\Sigma_f : \mathcal{C}/_A \rightarrow \mathcal{C}/_B$ of the base change functor.

Set-theoretical interpretation

Theorem 10. (Axiom of Choice). If for all $x : A$ there is $y : B$ such that $R(x, y)$, then there is a function $f : A \rightarrow B$ such that for all $x : A$ there is a witness of $R(x, f(x))$.

```
ac (A B : U) (R : A  $\rightarrow$  B  $\rightarrow$  U)
  : (p : (x : A)  $\rightarrow$  (y : B) * (R x y))
   $\rightarrow$  (f : A  $\rightarrow$  B) * ((x : A)  $\rightarrow$  R(x) (f x))
```

Theorem 11. (Total). If fiber over base implies another fiber over the same base then we can construct total space of section over that base with another fiber.

```
total (A : U) (B C : A  $\rightarrow$  U)
  (f : (x : A)  $\rightarrow$  B x  $\rightarrow$  C x) (w : Sigma A B)
  : Sigma A C = (w.1, f (w.1) (w.2))
```

Theorem 12. (Σ -Contractability). If the fiber is set then the Σ is set.

```
setSig (A : U) (B : A  $\rightarrow$  U) (sA : isSet A)
  (sB : (x : A)  $\rightarrow$  isSet (B x))
  : isSet (Sigma A B)
```

Theorem 13. (Path Between Sigmas). Path between two sigmas $t, u : \Sigma(A, B)$ could be decomposed to sigma of two paths $p : t_1 =_A u_1$ and $(t_2 =_{B(p@i)} u_2)$.

```

pathSig (A:U) (B : A → U) (t u : Sigma A B)
  : Path U (Path (Sigma A B) t u)
    ((p: Path A t.1 u.1)
     * PathP (<i>B(p@i)) t.2 u.2)

```


2.3 Path Space (Ξ)

The Path identity type or Ξ defines a Path space with elements and values. Elements of that space are functions from interval $[0, 1]$ to a values of that path space. This ctt file reflects ¹CCHM cubicaltt model with connections. For ²ABCFHL yacctt model with variables please refer to ytt file. You may also want to read ³BCH, ⁴AFH. There is a ⁵PO paper about CCHM axiomatic in a topos.

Cubical interpretation

Cubical interpretation was first given by Simon Huber [18] and later was written first constructive type checker in the world by Anders Mörtberg [17].

Definition 17. (Path Formation).

```
Hetero (A B: U) (a: A) (b: B) (P: Path U A B)
  : U = PathP P a b
Path (A: U) (a b: A)
  : U = PathP (<i>A) a b
```

Definition 18. (Path Reflexivity). Returns an element of reflexivity path space for a given value of the type. The inhabitant of that path space is the lambda on the homotopy interval $[0, 1]$ that returns a constant value a . Written in syntax as $|<i>a|$ which equals to $\lambda (i : I) \rightarrow a$.

```
refl (A: U) (a: A) : Path A a a
```

Definition 19. (Path Application). You can apply face to path.

```
app1 (A: U) (a b:A) (p:Path A a b):A=p@0
app2 (A: U) (a b:A) (p:Path A a b):A=p@1
```

Definition 20. (Path Composition). Composition operation allows to build a new path by given to paths in a connected point.

$$\begin{array}{ccc}
 & a & \xrightarrow{\text{comp}} c \\
 \lambda(i : I) \rightarrow a & \uparrow & \uparrow q \\
 & a & \xrightarrow{p@i} b
 \end{array}$$

¹Cyril Cohen, Thierry Coquand, Simon Huber, Anders Mörtberg. Cubical Type Theory: a constructive interpretation of the univalence axiom. 2015. <https://5ht.co/cubicaltt.pdf>

²Carlo Angiuli, Brunerie, Coquand, Kuen-Bang Hou (Favonia), Robert Harper, Dan Licata. Cartesian Cubical Type Theory. 2017. <https://5ht.co/ccctt.pdf>

³Marc Bezem, Thierry Coquand, Simon Huber. A model of type theory in cubical sets. 2014. <http://www.cse.chalmers.se/~coquand/mod1.pdf>

⁴Carlo Angiuli, Kuen-Bang Hou (Favonia), Robert Harper. Cartesian Cubical Computational Type Theory: Constructive Reasoning with Paths and Equalities. 2018. <https://www.cs.cmu.edu/~cangiuli/papers/ccctt.pdf>

⁵Andrew Pitts, Ian Orton. Axioms for Modelling Cubical Type Theory in a Topos. 2016. <https://arxiv.org/pdf/1712.04864.pdf>

```

composition
  (A: U) (a b c: A)
  (p: Path A a b) (q: Path A b c)
  : Path A a c
  = comp (<i>Path A a (q@i)) p []

```

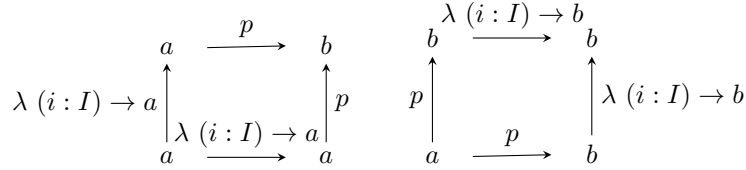
Theorem 14. (Path Inversion).

```

inv (A: U) (a b: A) (p: Path A a b)
  : Path A b a = <i>p @ -i

```

Definition 21. (Connections). Connections allows you to build square with given only one element of path: i) $\lambda (i, j : I) \rightarrow p @ \min(i, j)$; ii) $\lambda (i, j : I) \rightarrow p @ \max(i, j)$.



```

meet (A: U) (a b: A) (p: Path A a b)
  : PathP (<x> Path A a (p@x)) (<i>a) p
  = <x y> p @ (x /\ y)

```

```

join (A: U) (a b: A) (p: Path A a b)
  : PathP (<x> Path A (p@x) b) p (<i>b)
  = <y x> p @ (x \/ y)

```

Theorem 15. (Congruence). Is a map between values of one type to path space of another type by an encode function between types. Implemented as lambda defined on $[0, 1]$ that returns application of encode function to path application of the given path to lambda argument $|\lambda (i:I) \rightarrow f (p @ i)|$ for both cases.

```

ap (A B: U) (f: A -> B)
  (a b: A) (p: Path A a b)
  : Path B (f a) (f b)

apd (A: U) (a x:A) (B: A -> U) (f: A -> B a)
  (b: B a) (p: Path A a x)
  : Path (B a) (f a) (f x)

```

Theorem 16. (Transport). Transports a value of the domain type to the value of the codomain type by a given path element of the path space between domain and codomain types. Defined as path composition with $|||$ of a over a path $p - |comp p a |||$.

```

trans (A B: U) (p: Path U A B) (a: A) : B

```

Type-theoretical interpretation

Definition 22. (Singleton).

$\text{singl } (A: U) (a: A): U = (x: A) * \text{Path } A \ a \ x$

Theorem 17. (Singleton Instance).

$\text{eta } (A: U) (a: A): \text{singl } A \ a = (a, \text{refl } A \ a)$

Theorem 18. (Singleton Contractability).

$\text{contr } (A: U) (a \ b: A) (p: \text{Path } A \ a \ b)$
: $\text{Path } (\text{singl } A \ a) (\text{eta } A \ a) (b, p)$
= $\langle i \rangle (p @ i, \langle j \rangle p @ i / \setminus j)$

Theorem 19. (Path Elimination, Paulin-Mohring). J is formulated in a form of Paulin-Mohring and implemented using two facts that singleton are contractible and dependent function transport.

$J (A: U) (a \ b: A)$
($P: \text{singl } A \ a \rightarrow U$)
($u: P (a, \text{refl } A \ a)$)
($p: \text{Path } A \ a \ b$) : $P (b, p)$

Theorem 20. (Path Elimination, HoTT). J from HoTT book.

$J (A: U) (a \ b: A)$
($C: (x: A) \rightarrow \text{Path } A \ a \ x \rightarrow U$)
($d: C \ a \ (\text{refl } A \ a)$)
($p: \text{Path } A \ a \ b$) : $C \ b \ p$

Theorem 21. (Path Computation).

$\text{trans_comp } (A: U) (a: A)$
: $\text{Path } A \ a \ (\text{trans } A \ A \ (\langle _ \rangle) \ a)$
= $\text{fill } (\langle i \rangle) \ A \ a \ []$
 $\text{subst_comp } (A: U) (P: A \rightarrow U) (a: A) (e: P \ a)$
: $\text{Path } (P \ a) \ e \ (\text{subst } A \ P \ a \ a \ (\text{refl } A \ a) \ e)$
= $\text{trans_comp } (P \ a) \ e$
 $J_comp (A: U) (a: A) (C: (x: A)$
 $\rightarrow \text{Path } A \ a \ x \rightarrow U) (d: C \ a \ (\text{refl } A \ a))$
: $\text{Path } (C \ a \ (\text{refl } A \ a)) \ d$
($J \ A \ a \ C \ d \ a \ (\text{refl } A \ a)$)
= $\text{subst_comp } (\text{singl } A \ a) \ T \ (\text{eta } A \ a) \ d$
where $T (z: \text{singl } A \ a)$
: $U = C \ a \ (z.1) \ (z.2)$

Note that Path type has no Eta rule due to groupoid interpretation.

Groupoid interpretation

The groupoid interpretation of type theory is well known article by Martin Hofmann and Thomas Streicher, more specific interpretation of identity type as infinity groupoid.

2.4 Contexts and Universes

Speaking of type checker execution, we introduce context or dictionary with types and terms, from which we can derive typed variables. This chain could be implemented as nested sigma types (due to R.A.G.Seely) or list types (due to Voevodsky). Categorically dependent type theory is built upon categories of contexts.

Definition 23. (Empty Context).

$$\gamma_0 : \Gamma =_{def} \star.$$

Definition 24. (Context Comprehension).

$$\Gamma ; A =_{def} \sum_{\gamma:\Gamma} A(\gamma).$$

Definition 25. (Context Derivability).

$$\Gamma \vdash A =_{def} \prod_{\gamma:\Gamma} A(\gamma).$$

Definition 26. (Terms). Point in initial object of language AST inductive definition is called a term. If type theory or language is defined as an inductive type (AST) then the term is defined as its instance.

Definition 27. (Sorts). N-indexed set of universes $U_{n \in \mathbb{N}}$. Could have any number of elements which defines different type systems. All built-in types as long as user defined types are landed usually by default in U_0 universe. Sorts represented in type checker as a separate constructor.

Definition 28. (Axioms). The inclusion rules $U_i : U_j, i, j \in \mathbb{N}$, that define which universe is element of another given universe. You may attach any rules that joins i, j in some way. Axioms with sorts define universe hierarchy.

Definition 29. (Rules). The set of landings $U_i \rightarrow U_j : U_{\lambda(i,j)}, i, j \in \mathbb{N}$, where $\lambda : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$. These rules define term dependence or how we land (in which universe) formation rules in definitions.

Definition 30. (Predicative hierarchy). If λ in Rules is an uncurried function $\max : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ then such universe hierarchy is called predicative.

Definition 31. (Impredicative hierarchy). If λ in Rules is a second projection of a tuple $\text{snd} : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ then such universe hierarchy is called impredicative.

Definition 32. (Definitional Equality). For any $U_i, i \in \mathbb{N}$ there is defined an equality between its members and between its instances. For all $x, y \in A$, there is defined a $x=y$. Definitional equality compares normalized term instances.

Definition 33. (SAR). The universum space is configured with a triple of: i) sorts, a set of universes $U_{n \in \mathbb{N}}$ indexed over set \mathbb{N} ; ii) axioms, a set of inclusions $U_i : U_j, i, j \in \mathbb{N}$; iii) rules of term dependence universe landing, a set of landings $U_i \rightarrow U_j : U_{\lambda(i,j), i, j \in \mathbb{N}}$, where λ could be function *max* (predicative) or *snd* (impredicative).

Example 1. (CoC). $\text{SAR} = \{\{\star, \square\}, \{\star : \square\}, \{i \rightarrow j : j; i, j \in \{\star, \square\}\}\}$. Terms live in universe \star , and types live in universe \square . In CoC $\lambda = \text{snd}$.

Example 2. ($\text{PTS}^\infty, \text{MLTT}^\infty$).

$\text{SAR} = \{U_{i \in \mathbb{N}}, U_i : U_{j; i < j; i, j \in \mathbb{N}}, U_i \rightarrow U_j : U_{\lambda(i,j), i, j \in \mathbb{N}}\}$. Where U_i is a universe of i -level or i -category in categorical interpretation.

2.5 MLTT-73

Here is given formal model of type-theoretical interpretation of Martin-Löf Type Theory. It combines 4 Path rules (no eta), 5 Π rules, and 6 Σ rules (two elims). The proof is provided by direct embedding (internalizing) the model into the model of type checker which is even more powerful.

Definition 34. (MLTT-73). The MLTT as a Type is defined by taking all rules for Π , Σ and Path types into one Σ telescope or context.

```

MLTT (A: U): U
= (Pi_Former: (A → U) → U)
* (Pi_Intro: (B: A → U) (a: A) → B a → (A → B a))
* (Pi_Elim: (B: A → U) (a: A) → (A → B a) → B a)
* (Pi_Comp1: (B: A → U) (a: A)
  (f: A → B a) → Path (B a)
  (Pi_Elim B a (Pi_Intro B a (f a))) (f a))
* (Pi_Comp2: (B: A → U) (a: A)
  (f: A → B a) → Path (A → B a) f (\(x:A) → f x))
* (Sigma_Former: (A → U) → U)
* (Sigma_Intro: (B: A → U) (a: A) (b: B a) → Sigma A B)
* (Sigma_Elim1: (B: A → U)
  (_: Sigma A B) → A)
* (Sigma_Elim2: (B: A → U)
  (x: Sigma A B) → B (pr1 A B x))
* (Sigma_Comp1: (B: A → U) (a: A) (b: B a)
  → Path A a (Sigma_Elim1 B (Sigma_Intro B a b)))
* (Sigma_Comp2: (B: A → U) (a: A)
  (b: B a) → Path (B a) b
  (Sigma_Elim2 B (a, b)))
* (Sigma_Comp3: (B: A → U) (p: Sigma A B)
  → Path (Sigma A B) p (pr1 A B p, pr2 A B p))
* (Id_Former: A → A → U)
* (Id_Intro: (a: A) → Path A a a)
* (Id_Elim: (x: A) (C: D A)
  (d: C x x (Id_Intro x))
  (y: A) (p: Path A x y) → C x y p)
* (Id_Comp: (a:A) (C: D A)
  (d: C a a (Id_Intro a)) →
  Path (C a a (Id_Intro a))
  d (Id_Elim a C d a (Id_Intro a))) * U

```

Theorem 22. (Model Check). There is an instance of MLTT.

```
instance (A: U) : MLTT A
= ( ( Pi A, lam A, app A, Beta A, Eta A,
    Sigma A, dpair A, pr1 A, pr2 A, Beta1 A, Beta2 A, Eta2 A,
    Path A, refl A, J A, J_comp A, A
  )
```

The result of the work is a `mltt.ctt` file which can be runned using `cubicaltt`. Note that MLTT-73 internalization includes only eliminator and computational rule (without uniqueness rule) for identity system, as cubical Path spaces refute uniqueness of identity proofs.

Conclusions

This article presents a landmark achievement in type theory: the constructive internalization of Martin-Löf Type Theory (MLTT-73) within the **Per** language, a minimal type system equipped with cubical type theory primitives. By embedding MLTT-73’s core types — dependent function types (Π), dependent pair types (Σ), and identity types (Id , modeled as Path (Ξ) types in cubical syntax) — we have constructively verified all inference rules, including the pivotal J eliminator, using the cubical type checker [17].

This internalization, formalized in the `mltt.ctt` file and validated with `cubicaltt`, marks a significant advance in mechanized reasoning, as it provides a rigorous test of a type checker’s ability to fuse introduction and elimination rules through beta and eta equalities.

Language	U^n	Π	Σ	Id	Ξ	\mathbb{N}	0/1/2	W	Ind
System P_ω (CoC-88)		x							
MLTT-72		x	x						
Henk (ECC)	x	x							
Errett (LCCC/IPL)	x	x	x				x		
MLTT-73	x	x	x	x					
Per	x	x	x		x				
MLTT-75	x	x	x	x		x	x		
MLTT-80	x	x	x	x			x	x	
Anders (HTS)	x	x	x	x	x		x	x	
Frank (CoC+CIC)	x	x							x
Christine (Coq)	x	x	x	x					x
cubicaltt		x	x		x				x
Agda	x	x	x	x	x				x
Lean	x	x	x	x					x
NuPRL		x	x	x					x

The significance of this work lies in its constructive approach to the J eliminator, a cornerstone of MLTT-75’s identity type, which previous internalization attempts failed to derive constructively [4, 13]. By leveraging cubical type theory’s Path types and operations (e.g., connections, compositions), **Per** achieves a

compact and robust framework for encoding MLTT-75, as demonstrated by the type-theoretical model presented herein. The article also elucidates MLTT-75’s versatility through logical, categorical, homotopical, and set-theoretical interpretations, offering a comprehensive landscape for researchers and newcomers to type theory.

The internalization of MLTT-75 in **Per** advances the mechanization of constructive mathematics by providing a verified foundation for type checkers. **Per**’s cubical framework ensures that all inference rules, except for Path types refuting UIP are derivable, fulfilling a core objective of constructive reasoning: verifiable mathematical proofs.

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Issue II: Inductive Types

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Анотація

Impredicative Encoding of Inductive Types in HoTT.

Keywords: Formal Methods, Type Theory, Programming Languages, Theoretical Computer Science, Applied Mathematics, Cubical Type Theory, Martin-Löf Type Theory

3 Inductive Encodings

3.1 Church Encoding

You know Church encoding which also has its dependent analogue in CoC, however in Coq it is impossible to derive Inductive Principle as type system lacks fixpoint and functional extensionality. The example of working compiler of PTS languages are Om and Morte. Assume we have Church encoded NAT:

$$\text{nat} = (X:U) \rightarrow (X \rightarrow X) \rightarrow X \rightarrow X$$

where first parameter $(X \rightarrow X)$ is a *succ*, the second parameter X is *zero*, and the result of encoding is landed in X . Even if we encode the parameter

$$\text{list } (A: U) = (X:U) \rightarrow X \rightarrow (A \rightarrow X) \rightarrow X$$

and parameter A let's say live in 42 universe and X live in 2 universe, then by the signature of encoding the term will be landed in X , thus 2 universe. In other words such dependency is called impredicative displaying that landed term is not a predicate over parameters. This means that Church encoding is incompatible with predicative type checkers with predicative of predicative-cumulative hierarchies.

3.2 Scott Encoding

3.3 Parigot Encoding

3.4 CPS Encoding

3.5 Interaction Networks Encoding

3.6 Impredicative Encoding

In HoTT n -types is encoded as n -groupoids, thus we need to add a predicate in which n -type we would like to land the encoding:

$$\text{NAT } (A: U) = (X:U) \rightarrow \text{isSet } X \rightarrow X \rightarrow (A \rightarrow X) \rightarrow X$$

Here we added *isSet* predicate. With this motto we can implement propositional truncation by landing term in *isProp* or even HIT by landing in *isGroupoid*:

$$\begin{aligned} \text{TRUN } (A:U) \text{ type} &= (X: U) \rightarrow \text{isProp } X \rightarrow (A \rightarrow X) \rightarrow X \\ \text{S1} &= (X:U) \rightarrow \text{isGroupoid } X \rightarrow ((x:X) \rightarrow \text{Path } X \ x \ x) \rightarrow X \\ \text{MONOPLE } (A:U) &= (X:U) \rightarrow \text{isSet } X \rightarrow (A \rightarrow X) \rightarrow X \\ \text{NAT} &= (X:U) \rightarrow \text{isSet } X \rightarrow X \rightarrow (A \rightarrow X) \rightarrow X \end{aligned}$$

The main publication on this topic could be found at [11] and [10].

The Unit Example

Here we have the implementation of Unit impredicative encoding in HoTT.

```

upPath      (X Y:U) (f:X→Y) (a:X→X): X → Y = o X X Y f a
downPath    (X Y:U) (f:X→Y) (b:Y→Y): X → Y = o X Y Y b f
naturality  (X Y:U) (f:X→Y) (a:X→X) (b:Y→Y): U
  = Path (X→Y) (upPath X Y f a) (downPath X Y f b)

unitEnc': U = (X: U) → isSet X → X → X
isUnitEnc (one: unitEnc'): U
  = (X Y:U) (x: isSet X) (y: isSet Y) (f:X→Y) →
    naturality X Y f (one X x) (one Y y)

unitEnc: U = (x: unitEnc') * isUnitEnc x
unitEncStar: unitEnc = (\(X:U) (\_ : isSet X) →
  idfun X, \ (X Y: U) (\_ : isSet X) (\_ : isSet Y) → refl (X→Y))
unitEncRec  (C: U) (s: isSet C) (c: C): unitEnc → C
  = \ (z: unitEnc) → z.1 C s c
unitEncBeta (C: U) (s: isSet C) (c: C)
  : Path C (unitEncRec C s c unitEncStar) c = refl C c
unitEncEta  (z: unitEnc): Path unitEnc unitEncStar z = undefined
unitEncInd  (P: unitEnc → U) (a: unitEnc): P unitEncStar → P a
  = subst unitEnc P unitEncStar a (unitEncEta a)
unitEncCondition (n: unitEnc'): isProp (isUnitEnc n)
  = \ (f g: isUnitEnc n) →
    <h> \ (x y: U) → \ (X: isSet x) → \ (Y: isSet y)
    → \ (F: x → y) → <i> \ (R: x → Y (F (n x X R))) (n y Y (F R))
    (<j> f x y X Y F @ j R) (<j> g x y X Y F @ j R) @ h @ i

```

3.7 Lambek Encoding: Homotopy Initial Algebras

4 Inductive Types

4.1 W

Well-founded trees without mutual recursion represented as W-types.

Definition 35. (W-Formation). For $A : \mathcal{U}$ and $B : A \rightarrow \mathcal{U}$, type W is defined as $W(A, B) : \mathcal{U}$ or

$$W_{(x:A)}B(x) : \mathcal{U}.$$

$\text{def } W \ (A : \mathcal{U}) \ (B : A \rightarrow \mathcal{U}) : \mathcal{U} := W \ (x : A), B \ x$

Definition 36. (W-Introduction). Elements of $W_{(x:A)}B(x)$ are called well-founded trees and created with single sup constructor:

$$\text{sup} : W_{(x:A)}B(x).$$

$\text{def } \text{sup}\$'\$' \ (A : \mathcal{U}) \ (B : A \rightarrow \mathcal{U}) \ (x : A) \ (f : B \ x \rightarrow W^* \ A \ B)$
 $\quad : W^* \ A \ B$
 $\quad := \text{sup} \ A \ B \ x \ f$

Theorem 23. (Induction Principle ind_W). The induction principle states that for any types $A : \mathcal{U}$ and $B : A \rightarrow \mathcal{U}$ and type family C over $W(A, B)$ and the function $g : G$, where

$$G = \prod_{x:A} \prod_{f:B(x)} \prod_{b:B(x)} C(f(b)) \beta C(\text{sup}(x, f))$$

there is a dependent function:

$$\text{ind}_W : \prod_{C:W(A,B) \rightarrow \mathcal{U}} \prod_{g:A \rightarrow C} \prod_{a:A} \prod_{f:B(a)} \prod_{b:B(a)} C(f(b)).$$

$\text{def } W\text{-ind} \ (A : \mathcal{U}) \ (B : A \rightarrow \mathcal{U})$
 $\quad (C : (W \ (x : A), B \ x) \rightarrow \mathcal{U})$
 $\quad (g : \prod \ (x : A) \ (f : B \ x \rightarrow (W \ (x : A), B \ x)),$
 $\quad \quad (\prod \ (b : B \ x), C \ (f \ b)) \rightarrow C \ (\text{sup} \ A \ B \ x \ f))$
 $\quad (a : A) \ (f : B \ a \rightarrow (W \ (x : A), B \ x)) \ (b : B \ a)$
 $\quad : C \ (f \ b) := \text{ind}^W \ A \ B \ C \ g \ (f \ b)$

Theorem 24. (ind_W Computes). The induction principle ind^W satisfies the equation:

$$\text{ind}_W\text{-}\beta : g(a, f, \lambda b. \text{ind}^W(g, f(b)))$$

$$=_{\text{def}} \text{ind}_W(g, \text{sup}(a, f)).$$

$\text{def } \text{ind}^W\text{-}\beta \ (A : \mathcal{U}) \ (B : A \rightarrow \mathcal{U})$
 $\quad (C : (W \ (x : A), B \ x) \rightarrow \mathcal{U}) \ (g : \prod \ (x : A)$
 $\quad \quad (f : B \ x \rightarrow (W \ (x : A), B \ x)), \ (\prod \ (b : B \ x), C \ (f \ b)) \rightarrow C \ (\text{sup} \ A \ B \ x \ f))$
 $\quad (a : A) \ (f : B \ a \rightarrow (W \ (x : A), B \ x))$
 $\quad : \text{PathP} \ (<_> \ C \ (\text{sup} \ A \ B \ a \ f))$
 $\quad \quad (\text{ind}^W \ A \ B \ C \ g \ (\text{sup} \ A \ B \ a \ f))$
 $\quad \quad (g \ a \ f \ (\lambda \ (b : B \ a), \text{ind}^W \ A \ B \ C \ g \ (f \ b))))$
 $\quad := <_> \ g \ a \ f \ (\lambda \ (b : B \ a), \text{ind}^W \ A \ B \ C \ g \ (f \ b))$

4.2 M

4.3 Empty

The Empty type represents False-type logical $\mathbf{0}$, type without inhabitants, void or \perp (Bottom). As it has not inhabitants it lacks both constructors and eliminators, however, it has induction.

Definition 37. (Formation). Empty-type is defined as built-in $\mathbf{0}$ -type:

$$\mathbf{0} : \mathcal{U}.$$

Theorem 25. (Induction Principle ind_0). $\mathbf{0}$ -type is satisfying the induction principle:

$$\text{ind}_0 : \prod_{C : \mathbf{0} \rightarrow \mathcal{U}} \prod_{z : \mathbf{0}} C(z).$$

`def Empty-ind (C: $\mathbf{0} \rightarrow \mathcal{U}$) (z: $\mathbf{0}$) : C z := ind0 (C z) z`

Definition 38. (Negation or isEmpty). For any type A negation of A is defined as arrow from A to $\mathbf{0}$:

$$\neg A := A \rightarrow \mathbf{0}.$$

`def isEmpty (A: \mathcal{U}): \mathcal{U} := A $\rightarrow \mathbf{0}$`

The witness of $\neg A$ is obtained by assuming A and deriving a contradiction. This techniques is called proof of negation and is applicable to any types in constrast to proof by contradiction which implies $\neg\neg A \rightarrow A$ (double negation elimination) and is applicable only to decidable types with $\neg A + A$ property.

4.4 Unit

Unit type is the simplest type equipped with full set of MLTT inference rules. It contains single inhabitant \star (star).

- 4.5 Bool
- 4.6 Maybe
- 4.7 Either
- 4.8 Nat
- 4.9 List
- 4.10 Vector
- 4.11 Stream
- 4.12 Interpreter

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Issue III: Homotopy Type Theory

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Анотація

Here is presented distinctive points of Homotopy Type Theory as an extension of Martin-Löf Type Theory but without higher inductive types which will be given in the next issue. The study of identity system is given. Groupoid (categorical) interpretation is presented as categories of spaces and paths between them as invertible morphisms. At last constructive proof $\Omega(S^1) = \mathbb{Z}$ is given through helix.

Keywords: Homotopy Theory, Type Theory

5 Groupoid Interpretation

5.1 Introduction: Type Theory

Type theory is a universal programming language for pure mathematics, designed for theorem proving. It supports an arbitrary number of consistent axioms, structured as pseudo-isomorphisms consisting of *encode* functions (methods for constructing type elements), *decode* functions (dependent eliminators of the universal induction principle), and their equations—beta and eta rules governing computability and uniqueness.

As a programming language, type theory includes basic primitives (axioms as built-in types) and accompanying documentation, such as lecture notes or textbooks, explaining their applications, including:

- Function (**II**)
- Context (**Σ**)
- Identification (=)
- Polynomial (**W**)
- Path (**Ξ**)
- Gluing (**Glue**)
- Infinitesimal (**ℑ**)
- Complex (**HIT**)

Students (10) are tasked with applying type theory to prove an initial but non-trivial result addressing an open problem in one of the following areas offered by the Department of Pure Mathematics (KM-111):

$$\text{Mathematics} := \left\{ \begin{array}{l} \text{Homotopy Theory} \\ \text{Homological Algebra} \\ \text{Category Theory} \\ \text{Functional Analysis} \\ \text{Differential Geometry} \end{array} \right. .$$

5.2 Motivation: Homotopy Type Theory

The primary motivation of homotopy type theory is to provide computational semantics for homotopic types and CW-complexes. The central idea, as described in, is to combine function spaces (Π), context spaces (Σ), and path spaces (Ξ) to form a fiber bundle, proven within HoTT to coincide with the Π type itself.

Key definitions include:

```
def contr (A: U) : U :=  $\Sigma$  (x: A),  $\Pi$  (y: A),  $\Xi$  A x y
def fiber (A B: U) (f: A  $\rightarrow$  B) (y: B): U :=  $\Sigma$  (x: A), Path B y (f x)
def isEquiv (A B: U) (f: A  $\rightarrow$  B): U :=  $\Pi$  (y: B), contr (fiber A B f y)
def equiv (X Y: U): U :=  $\Sigma$  (f: X  $\rightarrow$  Y), isEquiv X Y f
def ua (A B: U) (p :  $\Xi$  U A B) : equiv A B
:= transp (<i>equiv A (p @ i)) 0 (idEquiv A)
```

The absence of an eta-rule for equality implies that not all proofs of the same path space are equal, resulting in a multidimensional ∞ -groupoid structure for path spaces. Further definitions include:

```
def isProp (A : U) : U
:=  $\Pi$  (a b : A),  $\Xi$  A a b

def isSet (A : U) : U
:=  $\Pi$  (a b : A) (x y :  $\Xi$  A a b),  $\Xi$  ( $\Xi$  A a b) x y

def isGroupoid (A : U) : U
:=  $\Pi$  (a b : A) (x y :  $\Xi$  A a b) (i j :  $\Xi$  ( $\Xi$  A a b) x y),
 $\Xi$  ( $\Xi$  ( $\Xi$  A a b) x y) i j
```

The groupoid interpretation raises questions about the existence of a language for mechanically proving all properties of the categorical definition of a groupoid:

```
def CatGroupoid (X : U) (G : isGroupoid X)
: isCatGroupoid (PathCat X)
:= ( idp X,
    comp-Path X,
    G,
    sym X,
    comp-inv-Path-1 X,
    comp-inv-Path X,
    comp-Path-left X,
    comp-Path-right X,
    comp-Path-assoc X,
    *
  )
```

5.3 Metatheory: Adjunction Triples

The course is divided into four parts, each exploring type-axioms and their meta-theoretical adjunctions.

5.3.1 Fibrational Proofs

$$\Sigma \dashv f_{\star} \dashv \Pi$$

Fibrational proofs are modeled by primitive axioms, which are type-theoretic representations of categorical meta-theoretical models of adjunctions of three Cockett-Reit functors, giving rise to function spaces (Π) and pair spaces (Σ). These proof methods enable direct analysis of fibrations.

5.3.2 Equality Proofs

$$Q \dashv \Xi \dashv C$$

In intensional type theory, the equality type is embedded as type-theoretic primitives of categorical meta-theoretical models of adjunctions of three Jacobs-Lambek functors: quotient space (Q), identification system (Ξ), and contractible space (C). These methods allow direct manipulation of identification systems, strict for set theory and homotopic for homotopy theory.

5.3.3 Inductive Proofs

$$W \dashv \odot \dashv M$$

Inductive types in type theory can be embedded as polynomial functors (W , M) or general inductive type schemes (Calculus of Inductive Constructions), with properties including: 1) Verification of program finiteness; 2) Verification of strict positivity of parameters; 3) Verification of mutual recursion.

In this course, induction and coinduction are introduced as type-theoretic primitives of categorical meta-theoretical models of adjunctions of polynomial functors (Lambek-Bohm), enabling manipulation of initial and terminal algebras, algebraic recursive data types, and infinite processes. Higher inductive proofs, where constructors include path spaces, are modeled by polynomial functors using monad-algebras and comonad-coalgebras (Lumsdaine-Shulman).

Historical Notes

Homotopy Type Theory takes its origins in 1996 from groupoid interpretation by Hofmann and Streicher's, and later (in 10 years) was formalized by Awodey, Warren and Voevodsky. Voevodsky constructed Kan simplicial sets interpretation of type theory and discovered the property of this model, that was named univalence. This property allows to identify isomorphic structures in terms of type theory.

Homotopy type theory to classical homotopy theory is like Euclidian syntethic geometry (points, lines, axioms and deduction rules) to analytical geometry with cartesian coordinates on \mathbb{R}^n (geometric and algebraic)¹.

In the same way as inductive types extends MLTT for inductive programming, the higher inductive types (HIT) extend homotopy type theory for geometry programming. You can directly encode CW-complexes by using HIT. The definition of HIT syntax will be given in the next **Issue IV: Higher Inductive Types**.

Cubical with HITs has very lightweight core and syntax, and is an internal language of $(\infty, 1)$ -topos. Cubical with $[0, 1]$ Path types but without HITs is an internal language of $(\infty, 1)$ -categories, while MLTT is an internal language of locally cartesian closed categories.

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¹We will denote geometric, type theoretical and homotopy constants bold font **R** while analitical will be denoted with double lined letters \mathbb{R} .

6 Homotopy Type Theory

6.1 Identity Systems

Definition 39. (Identity System). An identity system over type A in universe X_i is a family $R : A \rightarrow A \rightarrow X_i$ with a function $r_0 : \prod_{a:A} R(a, a)$ such that any type family $D : \prod_{a,b:A} R(a, b) \rightarrow X_i$ and $d : \prod_{a:A} D(a, a, r_0(a))$, there exists a function $f : \prod_{a,b:A} \prod_{r:R(a,b)} D(a, b, r)$ such that $f(a, a, r_0(a)) = d(a)$ for all $a : A$.

```
def IdentitySystem (A : U) : U
:=  $\Sigma$  ( $\text{--form} : A \rightarrow A \rightarrow U$ )
    ( $\text{--ctor} : \prod (a : A), \text{--form } a \ a$ )
    ( $\text{--elim} : \prod (a : A) (C : \prod (x \ y : A)
        (p : \text{--form } x \ y), U)
        (d : C \ a \ a (\text{--ctor } a)) (y : A)
        (p : \text{--form } a \ y), C \ a \ y \ p)$ )
    ( $\text{--comp} : \prod (a : A) (C : \prod (x \ y : A)
        (p : \text{--form } x \ y), U)
        (d : C \ a \ a (\text{--ctor } a)),
        \exists (C \ a \ a (\text{--ctor } a)) \ d
        (\text{--elim } a \ C \ d \ a (\text{--ctor } a))) , 1$ 
```

Example 3. There are number of equality signs used in this tutorial, all of them listed in the following table of identity systems:

Sign	Meaning
--_{def}	Definition
$=$	Id
\equiv	Path
\simeq	Equivalence
\cong	Isomorphism
\sim	Homotopy
\approx	Bisimulation

Theorem 26. (Fundamental Theorem of Identity System).

Definition 40. (Strict Identity System). An identity system over type A and universe of pretypes V_i is called strict identity system ($=$), which respects UIP.

Definition 41. (Homotopy Identity System). An identity system over type A and universe of homotopy types U_i is called homotopy identity system (\equiv), which models discrete infinity groupoid.

6.2 Path (Ξ)

The homotopy identity system defines a **Path** space indexed over type A with elements as functions from interval $[0, 1]$ to values of that path space $[0, 1] \rightarrow A$. HoTT book defines two induction principles for identity types: path induction and based path induction.

Definition 42. (Path Formation).

$$\equiv : U =_{def} \prod_{A:U} \prod_{x,y:A} \mathbf{Path}_A(x, y).$$

```
def  $\Xi$  (A : U) (x y : A) : U
:= PathP (<_> A) x y
```

```
def  $\Xi'$  (A : U) (x y : A)
:=  $\Pi$  (i : I),
    A [  $\partial$  i ] -> [ (i = 0)  $\rightarrow$  x ,
                  (i = 1)  $\rightarrow$  y ]]
```

Definition 43. (Path Introduction). Returns a reflexivity path space for a given value of the type. The inhabitant of that path space is the lambda on the homotopy interval $[0, 1]$ that returns a constant value x . Written in syntax as $[i]x$.

$$\text{id}_{\equiv} : x \equiv_A x =_{def} \prod_{A:U} \prod_{x:A} [i]x$$

```
def idp (A: U) (x: A)
:  $\Xi$  A x x := <_> x
```

Definition 44. (Path Application).

```
def at0 (A: U) (a b: A)
(p: Path A a b) : A := p @ 0
```

```
def at1 (A: U) (a b: A)
(p: Path A a b): A := p @ 1
```

Definition 45. (Path Connections). Connections allow you to build a square with only one element of path: i) $[i, j]p @ \min(i, j)$; ii) $[i, j]p @ \max(i, j)$.

$$\begin{array}{ccc} b & \xrightarrow{[i]b} & b \\ \uparrow p & & \uparrow [i]b \\ a & \xrightarrow{p} & b \end{array} \quad \begin{array}{ccc} a & \xrightarrow{p} & b \\ \uparrow [i]a & & \uparrow p \\ a & \xrightarrow{[i]a} & a \end{array}$$

```
def join (A: U) (a b: A) (p: Path A a b)
  : PathP (<x> Path A (p@x) b) p (<i> b)
:= <y x> p @ (x /\ y)
```

```
def meet (A: U) (a b: A) (p: Path A a b)
  : PathP (<x> Path A a (p@x)) (<i> a) p
:= <x y> p @ (x /\ y)
```

Definition 46. (Path Inversion).

Theorem 27. (Congruence).

$$\text{ap} : f(a) \equiv f(b) =_{def}$$

$$\prod_{A:U} \prod_{a,x:A} \prod_{B:A \rightarrow U} \prod_{f:\Pi(A,B)} \prod_{p:a \equiv_A x} [i]f(p@i).$$

```
def ap (A B: U) (f: A -> B)
  (a b: A) (p: Path A a b)
  : Path B (f a) (f b)
```

```
def apd (A: U) (a x: A) (B: A -> U)
  (f: A -> B a) (b: B a) (p: Path A a x)
  : Path (B a) (f a) (f x)
```

Maps a given path space between values of one type to path space of another type using an encode function between types. Implemented as a lambda defined on $[0, 1]$ that returns application of encode function to path application of the given path to lamda argument $[i]f(p@i)$ in both cases.

Definition 47. (Generalized Transport Kan Operation). Transports a value of the left type to the value of the right type by a given path element of the path space between left and right types.

$$\text{transport} : A(0) \rightarrow A(1) =_{def}$$

$$\prod_{A:I \rightarrow U} \prod_{r:I} \lambda x, \mathbf{transp}([i]A(i), 0, x).$$

```
def transp' (A: U) (x y: A) (p : PathP (<_>A) x y) (i: I)
:= transp (<_> (\(_:A), A) (p @ i)) i x
```

```
def transpU (A B: U) (p : PathP (<_>U) A B) (i: I)
:= transp (<_> (\(_:U), U) (p @ i)) i A
```

Definition 48. (Partial Elements).

$$\text{Partial} : V =_{def} \prod_{A:U} \prod_{i:I} \mathbf{Partial}(A, i).$$

```
def Partial' (A : U) (i : I)
: V := Partial A i
```

Definition 49. (Cubical Subtypes).

$$\text{Subtype} : V =_{def}$$

$$\prod_{A:U} \prod_{i:I} \prod_{u:\mathbf{Partial}(A,i)} A[i \mapsto u].$$

```
def sub (A : U) (i : I) (u : Partial A i)
: V := A [i ↦ u]
```

Definition 50. (Cubical Elements).

$$\text{inS} : A [(i = 1) \mapsto a] =_{def}$$

$$\prod_{A:U} \prod_{i:I} \prod_{a:A} \mathbf{inc}(A, i, a).$$

$$\text{outS} : A [i \mapsto u] \rightarrow A =_{def}$$

$$\prod_{A:U} \prod_{i:I} \prod_{u:\mathbf{Partial}(A,i)} \mathbf{ouc}(a).$$

```
def inS (A : U) (i : I) (a : A)
: sub A i [(i = 1) → a] := inc A i a
```

```
def outS (A : U) (i : I) (u : Partial A i)
: A [i ↦ u] → A := λ (a: A[i ↦ u]), ouc a
```

Theorem 28. (Heterogeneous Composition Kan Operation).

$$\text{comp}_{\text{CCHM}} : A(0) [r \mapsto u(0)] \rightarrow A(1) =_{\text{def}}$$

$$\prod_{A:U} \prod_{r:I} \prod_{u:\Pi_{i:I} \mathbf{Partial}(A(i),r)} \prod_{\lambda u_0, \mathbf{hcomp}(A(1), r, \lambda i. [(r=1) \rightarrow \mathbf{transp}([j]A(i/j), i, u(i, 1=1))], \mathbf{transp}([i]A(i), 0, \mathbf{ouc}(u_0)))}.$$

```
def compCCHM (A : I → U) (r : I)
  (u : Π (i : I), Partial (A i) r)
  (u₀ : (A 0)[r ↦ u 0]) : A 1
:= hcomp (A 1) r (λ (i : I),
  [(r = 1) → transp (<j> A (i ∨ j)) i (u i 1=1)])
  (transp (<i> A i) 0 (ouc u₀))
```

Theorem 29. (Homogeneous Composition Kan Operation).

$$\text{comp}_{\text{CHM}} : A [r \mapsto u(0)] \rightarrow A =_{\text{def}}$$

$$\prod_{A:U} \prod_{r:I} \prod_{u:I \rightarrow \mathbf{Partial}(A,r)} \prod_{\lambda u_0, \mathbf{hcomp}(A, r, u, \mathbf{ouc}(u_0))}.$$

```
def compCHM (A : U) (r : I)
  (u : I → Partial A r) (u₀ : A[r ↦ u 0]) : A
:= hcomp A r u (ouc u₀)
```

Theorem 30. (Substitution).

$$\text{subst} : P(x) \rightarrow P(y) =_{\text{def}}$$

$$\prod_{A:U} \prod_{P:A \rightarrow U} \prod_{x,y:A} \prod_{p:x=y} \prod_{\lambda e. \mathbf{transp}([i]P(p@i), 0, e)}.$$

```
def subst (A: U) (P: A → U) (x y: A) (p: Path A x y)
  : P x → P y
:= λ (e: P x), transp (<i> P (p @ i)) 0 e
```

Other synonyms are `mapOnPath` and `cong`.

Theorem 31. (Path Composition).

$$\begin{array}{ccc} a & \xrightarrow{pcomp} & c \\ [i]a \uparrow & & \uparrow q \\ a & \xrightarrow{p @ i} & b \end{array}$$

```
def pcomp (A: U) (a b c: A)
  (p: Path A a b) (q: Path A b c)
  : Path A a c := subst A (Path A a) b c q p
```

Composition operation allows building a new path from two given paths in a connected point. The proof term is **comp**($[i]\mathbf{Path}_A(a, q@i), p, []$).

Theorem 32. (J by Paulin-Mohring).

```
def J (A: U) (a b: A)
  (P: singl A a -> U)
  (u: P (a, refl A a))
  : Π (p: Path A a b), P (b, p)
```

J is formulated in a form of Paulin-Mohring and implemented using two facts that singletons are contractible and dependent function transport.

Theorem 33. (Contractability of Singleton).

```
def singl (A: U) (a: A) : U
:= Σ (x: A), Path A a x

def contr (A: U) (a b: A) (p: Path A a b)
  : Path (singl A a) (a, <_>a) (b, p)
```

Proof that singleton is contractible space. Implemented as $[i](p@i, [j]p@(i \wedge j))$.

Theorem 34. (HoTT Dependent Eliminator).

```
def J (A: U) (a: A)
  (C: (x: A) -> Path A a x -> U)
  (d: C a (refl A a)) (x: A)
  : Π (p: Path A a x) : C x p
```

Theorem 35. (Diagonal Path Induction).

```
def D (A: U) : U
:= Π (x y: A), Path A x y -> U

def J (A: U) (x: A) (C: D A)
  (d: C x x (refl A x))
  (y: A)
  : Π (p: Path A x y), C x y p
```

Theorem 36. (Path Computation).

```

def trans_comp (A: U) (a: A)
  : Path A a (trans A A (<_>) a)

def subst_comp (A: U) (P: A → U) (a: A) (e: P a)
  : Path (P a) e (subst A P a a (refl A a) e)

def J_comp (A: U) (a: A)
  (C: (x: A) → Path A a x → U)
  (d: C a (refl A a))
  : Path (C a (refl A a)) d
    (J A a C d a (refl A a))

```

Note that in HoTT there is no Eta rule, otherwise Path between element would requested to be unique applying UIP at any Path level which is prohibited. UIP in HoTT is defined only as instance of n-groupoid, see the PROP type.

6.3 Glue

Glue types defines composition structure for fibrant universes that allows partial elements to be extended to fibrant types. In other words it turns equivalences in the multidimensional cubes to path spaces. Unlike ABCHFL, CCHM needn't another universe for that purpose.

Definition 51. (Glue Formation). The Glue types take a partial family of types A that are equivalent to the base type B . These types are then “glued” onto B and the equivalence data gets packaged up into a new type.

$$\mathbf{Glue}(A, \varphi, e) : U.$$

```
def Glue' (A : U) (φ : I)
  (e : Partial (Σ (T : U), equiv T A) φ) : U
:= Glue A φ e
```

Definition 52. (Glue Introduction).

$$\mathbf{glue} \ \varphi \ u \ (\mathbf{ouc} \ a) : \mathbf{Glue} \ A \ [\varphi=1 \mapsto (T, f)].$$

```
def glue' (A : U) (φ : I)
  (u : Partial (Σ (T : U), equiv T A × T) φ)
  (a : A [φ ↦ [(φ = 1) → (u 1=1).2.1.1 (u 1=1).2.2]])
:= glue φ u (ouc a)
```

Definition 53. (Glue Elimination).

$$\mathbf{unglue}(b) : A \ [\varphi \mapsto f(b)].$$

```
def unglue' (A : U) (φ : I)
  (e : Partial (Σ (T : U), equiv T A) φ)
  (a : Glue A φ e) : A
:= unglue φ e a
```

Theorem 37. (Glue Computation).

$$b = \mathbf{glue} \ [\varphi \mapsto b] \ (\mathbf{unglue} \ b).$$

Theorem 38. (Glue Uniqueness).

$$\mathbf{unglue} \ (\mathbf{glue} \ [\varphi \mapsto t] \ a) = a : A.$$

6.4 Fibration

Definition 54 (Fiber). The fiber of the map $p : E \rightarrow B$ at a point $y : B$ is the set of all points $x : E$ such that $p(x) = y$.

```

fiber (E B: U) (p: E -> B) (y: B): U
  = (x: E) *  $\Xi$  B y (p x)

```

Definition 55 (Fiber Bundle). The fiber bundle $F \rightarrow E \xrightarrow{p} B$ on a total space E with fiber layer F and base B is a structure (F, E, p, B) , where $p : E \rightarrow B$ is a surjective map with the following property: for any point $y : B$ there exists a neighborhood U_b for which there is a homeomorphism

$$f : p^{-1}(U_b) \rightarrow U_b \times F$$

making the following diagram commute:

$$\begin{array}{ccc}
 p^{-1}(U_b) & \xrightarrow{f} & U_b \times F \\
 p \downarrow & \swarrow pr_1 & \\
 U_b & &
 \end{array}$$

Definition 56 (Trivial Fiber Bundle). When the total space E is the cartesian product $\Sigma(B, F)$ and $p = pr_1$, then such a bundle is called trivial: $(F, \Sigma(B, F), pr_1, B)$.

```

Family (B: U): U = B -> U

```

```

total (B: U) (F: Family B): U = Sigma B F
trivial (B: U) (F: Family B): total B F -> B = \ (x: total B F) -> x.1
homeo (B E: U) (F: Family B) (p: E -> B) (y: B):
  fiber E B p y -> total B F

```

Theorem 39 (Fiber Bundle $\equiv \Pi$). The inverse image (fiber) of the trivial bundle $(F, B \times F, pr_1, B)$ at a point $y : B$ equals $F(y)$. Proof sketch:

```
F y = ( _ : isContr B ) * ( F y )
      = ( x y : B ) * ( _ :  $\Xi$  B x y ) * ( F y )
      = ( z : B ) * ( k : F z ) *  $\Xi$  B z y
      = ( z : E ) *  $\Xi$  B z.1 y
      = fiber (total B F) B (trivial B F) y
```

The equality is shown using the *isoPath* lemma and *encode/decode* functions.

```
def Family (B : U) : U1 := B → U
def Fibration (B : U) : U1 :=  $\Sigma$  (X : U), X → B

def encode-Pi (B : U) (F : B → U) (y : B)
  : fiber (Sigma B F) B (pr1 B F) y → F y
:= \ (x : fiber (Sigma B F) B (pr1 B F) y),
    subst B F x.1.1 y (<i> x.2 @ -i) x.1.2

def decode-Pi (B : U) (F : B → U) (y : B)
  : F y → fiber (Sigma B F) B (pr1 B F) y
:= \ (x : F y), ((y, x), idp B y)

def decode-encode-Pi (B : U) (F : B → U) (y : B) (x : F y)
  :  $\Xi$  (F y) (transp (<i> F (idp B y @ i)) 0 x) x
:= <j> transp (<i> F y) j x

def encode-decode-Pi (B : U) (F : B → U) (y : B)
  (x : fiber (Sigma B F) B (pr1 B F) y)
  :  $\Xi$  (fiber (Sigma B F) B (pr1 B F) y)
    ((y, encode-Pi B F y x), idp B y) x
:= <i> ( (x.2 @ i, transp (<j> F (x.2 @ i  $\vee$  -j)) i x.1.2),
    <j> x.2 @ i  $\wedge$  j )

def Bundle=Pi (B : U) (F : B → U) (y : B)
  : PathP (<> U) (fiber (Sigma B F) B (pr1 B F) y) (F y)
:= iso→Path (fiber (Sigma B F) B (pr1 B F) y) (F y)
  (encode-Pi B F y) (decode-Pi B F y)
  (decode-encode-Pi B F y) (encode-decode-Pi B F y)
```

Definition 57. (Fibration-1) Dependent fiber bundle derived from Ξ contractability.

```
def isFBundle1 (B: U) (p: B → U) (F: U): U1
:= Σ ( _ : Π (b: B), isContr (PathP (<_>U) (p b) F)), (Π (x: Sigma B p), B)
```

Definition 58. (Fibration-2). Dependent fiber bundle derived from surjective function.

```
def isFBundle2 (B: U) (p: B → U) (F: U): U
:= Σ (v: U) (w: surjective v B), (Π (x: v), PathP (<_>U) (p (w.1 x)) F)
```

Definition 59. (Fibration-3). Non-dependent fiber bundle derived from fiber truncation.

```
def im1 (A B: U) (f: A → B): U
:= Σ (b: B), ||_||-1 (Π (a : A), Path B (f a) b)

def BAut (F: U): U := im1 1 U (λ (x: 1), F)

def 1-Im1 (A B: U) (f: A → B): im1 A B f → B
:= λ (x : im1 A B f), x.1

def 1-BAut (F: U): BAut F → U := 1-Im1 1 U (λ (x: 1), F)

def classify (E: U) (A' A: U) (E': A' → U) (E: A → U)
(f: A' → A): U := Π(x: A'), Ξ U (E' (x)) (E(f(x)))

def isFBundle3 (E B: U) (p: E → B) (F: U): U1
:= Σ (X: B → BAut F),
    classify E B (BAut F) (λ (b: B), fiber E B p b)
    (1-BAut F) X
```

Definition 60. (Fibration-4). Non-dependen fiber bundle derived as pullback square.

```
def isFBundle4 (E B: U) (p: E → B) (F: U): U1
:= Σ (X: U) (v: surjective X B)
    (v': prod X F → E),
    pullbackSq (prod X F) E X B p v.1 v' (λ (x: prod X F), x.1)
```

6.5 Equivalence

Definition 61. (Fiberwise Equivalence). Fiberwise equivalence \simeq or **Equiv** of function $f : A \rightarrow B$ represents internal equality of types A and B in the universe U as contractible fibers of f over base B .

$$A \simeq B : U =_{def} \mathbf{Equiv}(A, B) : U =_{def} \sum_{f:A \rightarrow B} \prod_{y:B} \sum_{x:\Sigma_{x:A} y=f(x)} \sum_{w:\Sigma_{x:A} y=f(x)} x =_{\Sigma_{x:A} y=f(x)} w.$$

```
def isContr (A: U) : U
:= Σ (x: A), Π (y: A), ∃ A x y

def fiber (A B : U) (f: A → B) (y : B): U
:= Σ (x : A), ∃ B y (f x)

def isEquiv (A B : U) (f : A → B) : U
:= Π (y : B), isContr (fiber A B f y)

def equiv (A B : U) : U
:= Σ (f : A → B), isEquiv A B f
```

Definition 62. (Fiberwise Reflection). There is a fiberwise instance id_{\simeq} of $A \simeq A$ that is derived as $(\text{id}(A), \text{isContrSingl}(A))$:

$$\text{id}_{\simeq} : \mathbf{Equiv}(A, A).$$

```
def singl (A: U) (a: A): U
:= Σ (x: A), ∃ A a x

def contr (A : U) (a b : A) (p : ∃ A a b)
: ∃ (singl A a) (eta A a) (b, p)
:= <i> (p @ i, <j> p @ i /\ j)

def isContrSingl (A : U) (a : A) : isContr (singl A a)
:= ((a, idp A a), (\(z: singl A a), contr A a z.1 z.2))

def idEquiv (A : U) : equiv A A
:= (\(a:A) -> a, isContrSingl A)
```

Theorem 40. (Fiberwise Induction Principle). For any $P : A \rightarrow B \rightarrow A \simeq B \rightarrow U$ and it's evidence d at $(B, B, \text{id}_{\simeq}(B))$ there is a function \mathbf{Ind}_{\simeq} . HoTT 5.8.5

$$\mathbf{Ind}_{\simeq}(P, d) : (p : A \simeq B) \rightarrow P(A, B, p).$$

```
def J-equiv (A B: U)
(P: Π (A B: U), equiv A B → U)
(d: P B B (idEquiv B))
: Π (e: equiv A B), P A B e
:= λ (e: equiv A B),
  subst (single B) (\ (z: single B), P z.1 B z.2)
  (B, idEquiv B) (A, e)
  (contrSinglEquiv A B e) d
```

Theorem 41. (Fiberwise Computation of Induction Principle).

```
def compute-Equiv (A : U)
  (C :  $\Pi$  (A B: U), equiv A B  $\rightarrow$  U)
  (d : C A A (idEquiv A))
  :  $\Xi$  (C A A (idEquiv A)) d
    (ind-Equiv A A C d (idEquiv A))
```

Definition 63. (Surjective).

```
isSurjective (A B: U) (f: A  $\rightarrow$  B): U
  = (b: B) * pTrunc (fiber A B f b)

surjective (A B: U): U
  = (f: A  $\rightarrow$  B)
    * isSurjective A B f
```

Definition 64. (Injective).

```
isInjective' (A B: U) (f: A  $\rightarrow$  B): U
  = (b: B)  $\rightarrow$  isProp (fiber A B f b)

injective (A B: U): U
  = (f: A  $\rightarrow$  B)
    * isInjective A B f
```

Definition 65. (Embedding).

```
isEmbedding (A B: U) (f: A  $\rightarrow$  B) : U
  = (x y: A)  $\rightarrow$  isEquiv ( $\Xi$  A x y) ( $\Xi$  B (f x) (f y)) (cong A B f x y)

embedding (A B: U): U
  = (f: A  $\rightarrow$  B)
    * isEmbedding A B f
```

Definition 66. (Half-adjoint Equivalence).

```
isHae (A B: U) (f: A  $\rightarrow$  B): U
  = (g: B  $\rightarrow$  A)
    * ( $\eta$ _:  $\Xi$  (id A) (o A B A g f) (idfun A))
    * ( $\epsilon$ _:  $\Xi$  (id B) (o B A B f g) (idfun B))
    * ((x: A)  $\rightarrow$   $\Xi$  B (f (( $\eta$ _ @ 0) x)) (( $\epsilon$ _ @ 0) (f x)))

hae (A B: U): U
  = (f: A  $\rightarrow$  B)
    * isHae A B f
```

6.6 Homotopy

The first higher equality we meet in homotopy theory is a notion of homotopy, where we compare two functions or two path spaces (which is sort of dependent families). The homotopy interval $I = [0, 1]$ is the perfect foundation for definition of homotopy.

Definition 67. (Interval). Compact interval.

```
def I : U := inductive { i0 | i1 | seg : i0 ≡ i1 }
```

You can think of \mathbf{I} as isomorphism of equality type, disregarding carriers on the edges. By mapping $i0, i1 : \mathbf{I}$ to $x, y : A$ one can obtain identity or equality type from classic type theory.

Definition 68. (Interval Split). The conversion function from I to a type of comparison is a direct eliminator of interval. The interval is also known as one of primitive higher inductive types which will be given in the next **Issue IV: Higher Inductive Types**.

```
def pathToHtpy (A: U) (x y: A) (p: ≡ A x y) : I → A
:= split { i0 → x | i1 → y | seg @ i → p @ i }
```

Definition 69. (Homotopy). The homotopy between two function $f, g : X \rightarrow Y$ is a continuous map of cylinder $H : X \times \mathbf{I} \rightarrow Y$ such that

$$\begin{cases} H(x, 0) = f(x), \\ H(x, 1) = g(x). \end{cases}$$

```
homotopy (X Y: U) (f g: X → Y)
(p: (x: X) → ≡ Y (f x) (g x))
(x: X): I → Y = pathToHtpy Y (f x) (g x) (p x)
```

Definition 70. (funExt-Formation)

$$\text{funext_form } (A B: U) (f g: A \rightarrow B): U \\ = \Xi (A \rightarrow B) f g$$

Definition 71. (funExt-Introduction)

$$\text{funext } (A B: U) (f g: A \rightarrow B) (p: (x:A) \rightarrow \Xi B (f x) (g x)) \\ : \text{funext_form } A B f g \\ = \langle i \rangle \backslash (a: A) \rightarrow p a @ i$$

Definition 72. (funExt-Elimination)

$$\text{happly } (A B: U) (f g: A \rightarrow B) (p: \text{funext_form } A B f g) (x: A) \\ : \Xi B (f x) (g x) \\ = \text{cong } (A \rightarrow B) B (\backslash (h: A \rightarrow B) \rightarrow \text{apply } A B h x) f g p$$

Definition 73. (funExt-Computation)

$$\text{funext_Beta } (A B: U) (f g: A \rightarrow B) (p: (x:A) \rightarrow \Xi B (f x) (g x)) \\ : (x:A) \rightarrow \Xi B (f x) (g x) \\ = \backslash (x:A) \rightarrow \text{happly } A B f g (\text{funext } A B f g p) x$$

Definition 74. (funExt-Uniqueness)

$$\text{funext_Eta } (A B: U) (f g: A \rightarrow B) (p: \Xi (A \rightarrow B) f g) \\ : \Xi (\Xi (A \rightarrow B) f g) (\text{funext } A B f g (\text{happly } A B f g p)) p \\ = \text{refl } (\Xi (A \rightarrow B) f g) p$$

6.7 Isomorphism

Definition 75. (iso-Formation)

$\text{iso_Form } (A \ B : U) : U = \text{isIso } A \ B \rightarrow \exists U \ A \ B$

Definition 76. (iso-Introduction)

$\text{iso_Intro } (A \ B : U) : \text{iso_Form } A \ B$

Definition 77. (iso-Elimination)

$\text{iso_Elim } (A \ B : U) : \exists U \ A \ B \rightarrow \text{isIso } A \ B$

Definition 78. (iso-Computation)

$\text{iso_Comp } (A \ B : U) \ (p : \exists U \ A \ B)$
 $: \exists (\exists U \ A \ B) \ (\text{iso_Intro } A \ B \ (\text{iso_Elim } A \ B \ p)) \ p$

Definition 79. (iso-Uniqueness)

$\text{iso_Uniq } (A \ B : U) \ (p : \text{isIso } A \ B)$
 $: \exists (\text{isIso } A \ B) \ (\text{iso_Elim } A \ B \ (\text{iso_Intro } A \ B \ p)) \ p$

6.8 Univalence

Definition 80. (uni-Formation)

$\text{univ_Formation } (A B : U) : U = \text{equiv } A B \rightarrow \Xi U A B$

Definition 81. (uni-Introduction)

$\text{equivTo}\Xi (A B : U) : \text{univ_Formation } A B$
 $= \backslash (p : \text{equiv } A B) \rightarrow \langle i \rangle \text{ Glue } B \text{ [(i=0) } \rightarrow (A, p),$
 $(i=1) \rightarrow (B, \text{subst } U (\text{equiv } B) B B (\langle _ \rangle B) (\text{idEquiv } B))]$

Definition 82. (uni-Elimination)

$\text{pathToEquiv } (A B : U) (p : \Xi U A B) : \text{equiv } A B$
 $= \text{subst } U (\text{equiv } A) A B p (\text{idEquiv } A)$

Definition 83. (uni-Computation)

$\text{eqToEq } (A B : U) (p : \Xi U A B)$
 $: \Xi (\Xi U A B) (\text{equivToPath } A B (\text{pathToEquiv } A B p)) p$
 $= \langle j \ i \rangle \text{ let } Ai : U = p@i \text{ in Glue } B$
 $\text{ [(i=0) } \rightarrow (A, \text{pathToEquiv } A B p),$
 $(i=1) \rightarrow (B, \text{pathToEquiv } B B (\langle k \rangle B)),$
 $(j=1) \rightarrow (p@i, \text{pathToEquiv } Ai B (\langle k \rangle p @ (i \setminus k)))]$

Definition 84. (uni-Uniqueness)

$\text{transPathFun } (A B : U) (w : \text{equiv } A B)$
 $: \Xi (A \rightarrow B) w.1 (\text{pathToEquiv } A B (\text{equivToPath } A B w)).1$

6.9 Loop

Definition 85. (Pointed Space). A pointed type (A, a) is a type $A : U$ together with a point $a : A$, called its basepoint.

```
pointed : U = (A : U) * A
point (A : pointed) : A.1 = A.2
space (A : pointed) : U = A.1
```

Definition 86. (Loop Space).

$$\Omega(A, a) =_{def} ((a =_A a), refl_A(a)).$$

```
omega1 (A : pointed) : pointed
= (Ξ (space A) (point A) (point A), refl A.1 (point A))
```

Definition 87. (n-Loop Space).

$$\begin{cases} \Omega^0(A, a) =_{def} (A, a) \\ \Omega^{n+1}(A, a) =_{def} \Omega^n(\Omega(A, a)) \end{cases}$$

```
omega : nat -> pointed -> pointed = split
zero -> idfun pointed
succ n -> \ (A : pointed) -> omega n (omega1 A)
```

6.10 Groupoid

The first text about groupoid interpretation of type theory can be found in Francois Lamarche: A proposal about Foundations². Then Martin Hofmann and Thomas Streicher wrote the initial document on groupoid interpretation of type theory³.

Equality	Homotopy	∞ -Groupoid
reflexivity	constant path	identity morphism
symmetry	inversion of path	inverse morphism
transitivity	concatenation of paths	composition of morphisms

There is a deep connection between higher-dimensional groupoids in category theory and spaces in homotopy theory, equipped with some topology. The category or groupoid could be built where the objects are particular spaces or types, and morphisms are path types between these types, composition operation is a path concatenation. We can write this groupoid here recalling that it should be category with inverted morphisms.

```

cat : U = (A : U) * (A → A → U)
groupoid : U = (X : cat) * isCatGroupoid X
PathCat (X : U) : cat = (X, \ (x y : X) → Path X x y)

def isCatGroupoid (C : cat) : U := Σ
  (id :      Π (x : C.ob), C.hom x x)
  (c :       Π (x y z : C.ob), C.hom x y → C.hom y z → C.hom x z)
  (HomSet :  Π (x y : C.ob), isSet (C.hom x y))
  (inv :     Π (x y : C.ob), C.hom x y → C.hom y x)
  (inv-left : Π (x y : C.ob) (p : C.hom x y),
    ≡ (C.hom x x) (c x y x p (inv x y p)) (id x))
  (inv-right : Π (x y : C.ob) (p : C.hom x y),
    ≡ (C.hom y y) (c y x y (inv x y p) p) (id y))
  (left :     Π (x y : C.ob) (f : C.hom x y),
    ≡ (C.hom x y) f (c x x y (id x) f))
  (right :    Π (x y : C.ob) (f : C.hom x y),
    ≡ (C.hom x y) f (c x y y f (id y)))
  (assoc :    Π (x y z w : C.ob) (f : C.hom x y)
    (g : C.hom y z) (h : C.hom z w),
    ≡ (C.hom x w) (c x z w (c x y z f g) h)
    (c x y w f (c y z w g h))), *

```

²<http://www.cse.chalmers.se/~coquand/Proposal.pdf>

³Martin Hofmann and Thomas Streicher. The Groupoid Interpretation of Type Theory. 1996.

```

def isProp (A : U) : U
:=  $\Pi$  (a b : A),  $\Xi$  A a b

def isSet (A : U) : U
:=  $\Pi$  (a b : A) (x y :  $\Xi$  A a b),
 $\Xi$  ( $\Xi$  A a b) x y

def isGroupoid (A : U) : U
:=  $\Pi$  (a b : A) (x y :  $\Xi$  A a b)
(i j :  $\Xi$  ( $\Xi$  A a b) x y),
 $\Xi$  ( $\Xi$  ( $\Xi$  A a b) x y) i j

def CatGroupoid (X : U) (G : isGroupoid X)
: isCatGroupoid (PathCat X)
:= ( idp X,
comp-Path X,
G,
sym X,
comp-inv-Path-1 X,
comp-inv-Path X,
comp-Path-left X,
comp-Path-right X,
comp-Path-assoc X,
 $\star$ 
)

def comp- $\Xi$  (A : U) (a b c : A) (p :  $\Xi$  A a b) (q :  $\Xi$  A b c) :  $\Xi$  A a c
:= <i> hcomp A ( $\partial$  i)
( $\lambda$  (j : I), [(i = 0)  $\rightarrow$  a,
(i = 1)  $\rightarrow$  q @ j]) (p @ i)

def comp-inv- $\Xi$ -1 (A : U) (a b : A) (p :  $\Xi$  A a b)
:  $\Xi$  ( $\Xi$  A a a) (comp- $\Xi$  A a b a p (<i> p @ -i)) (<_> a)
:= <k j> hcomp A ( $\partial$  j  $\vee$  k)
( $\lambda$  (i : I), [(j = 0)  $\rightarrow$  a,
(j = 1)  $\rightarrow$  p @ -i  $\wedge$  -k,
(k = 1)  $\rightarrow$  a]) (p @ j  $\wedge$  -k)

def comp-inv- $\Xi$  (A : U) (a b : A) (p :  $\Xi$  A a b)
:  $\Xi$  ( $\Xi$  A b b) (comp- $\Xi$  A b a b (<i> p @ -i) p) (<_> b)
:= <j i> hcomp A ( $\partial$  i  $\vee$  j)
( $\lambda$  (k : I), [(i = 0)  $\rightarrow$  b,
(j = 1)  $\rightarrow$  b,
(i = 1)  $\rightarrow$  p @ j  $\wedge$  k]) (p @ -i  $\vee$  j)

def comp- $\Xi$ -left (A : U) (a b : A) (p :  $\Xi$  A a b)
:  $\Xi$  ( $\Xi$  A a b) p (comp- $\Xi$  A a a b (<_> a) p)
:= <j i> hcomp A ( $\partial$  i  $\vee$  -j)
( $\lambda$  (k : I), [(i = 0)  $\rightarrow$  a,
(i = 1)  $\rightarrow$  p @ k,
(j = 0)  $\rightarrow$  p @ i  $\wedge$  k]) a

def comp- $\Xi$ -right (A : U) (a b : A) (p :  $\Xi$  A a b)
:  $\Xi$  ( $\Xi$  A a b) p (comp- $\Xi$  A a b b p (<_> b))
:= <j i> hcomp A ( $\partial$  i  $\vee$  -j)
( $\lambda$  (k : I), [(i = 0)  $\rightarrow$  a,
(i = 1)  $\rightarrow$  b,
(j = 0)  $\rightarrow$  p @ i]) (p @ i)

```

```

def comp-≡-assoc (A : U) (a b c d : A)
  (f : ≡ A a b) (g : ≡ A b c) (h : ≡ A c d)
  : ≡ (≡ A a d) (comp-≡ A a c d (comp-≡ A a b c f g) h)
    (comp-≡ A a b d f (comp-≡ A b c d g h))
:= J A a (λ (a : A) (b : A) (f : ≡ A a b),
  Π (c d : A) (g : ≡ A b c) (h : ≡ A c d),
  ≡ (≡ A a d) (comp-≡ A a c d (comp-≡ A a b c f g) h)
    (comp-≡ A a b d f (comp-≡ A b c d g h))))
  (λ (c d : A) (g : ≡ A a c) (h : ≡ A c d),
  comp-≡ (≡ A a d)
    (comp-≡ A a c d (comp-≡ A a a c (<_> a) g) h)
    (comp-≡ A a c d g h)
    (comp-≡ A a a d (<_> a) (comp-≡ A a c d g h))
    (<i> comp-≡ A a c d (comp-≡-left A a c g @ -i) h)
    (comp-≡-left A a d (comp-≡ A a c d g h))) b f c d g h

```

6.11 Homotopy Groups

Definition 88. (n-th Homotopy Group of m-Sphere).

$$\pi_n S^m = ||\Omega^n(S^m)||_0.$$

```
piS (n: nat): (m: nat) -> U = split
  zero  -> sTrunc (space (omega n (bool, false)))
  succ x -> sTrunc (space (omega n (Sn (succ x), north)))
```

Theorem 42. $(\Omega(S^1) = \mathbb{Z})$.

```
data S1 = base
  | loop <i> [ (i=0) -> base ,
              (i=1) -> base ]

loopS1 : U =  $\Xi$  S1 base base

encode (x:S1) (p: $\Xi$  S1 base x)
  : helix x
  = subst S1 helix base x p zeroZ

decode : (x:S1) -> helix x ->  $\Xi$  S1 base x = split
  base -> loopIt
  loop @ i -> rem @ i where
    p :  $\Xi$  U (Z -> loopS1) (Z -> loopS1)
    = <j> helix (loop1@j) ->  $\Xi$  S1 base (loop1@j)
  rem : PathP p loopIt loopIt
    = corFib1 S1 helix (\(x:S1)-> $\Xi$  S1 base x) base
      loopIt loopIt loop1 (\(n:Z) ->
        comp (<i>  $\Xi$  loopS1 (oneTurn (loopIt n))
          (loopIt (testIsoPath Z Z sucZ predZ
            sucpredZ predsucZ n @ i)))
          (<i>(lem1It n)@-i) [])
```

```
loopS1eqZ :  $\Xi$  U Z loopS1
  = isoPath Z loopS1 (decode base) (encode base)
  sectionZ retractZ
```

6.12 Hopf Fibrations

Example 4. ($S^1 \mathbb{R}$ Hopf Fiber).

```

data bool = false | true

negBool : bool -> bool
  = split { false -> true ; true -> false }

negBoolK : (b : bool) ->  $\Xi$  bool (negBool (negBool b)) b
  = split { false -> false ; true -> true }

negBoolEquiv : equiv bool bool
  = (negBool, gradLemma bool bool negBool negBool negBoolK negBoolK)

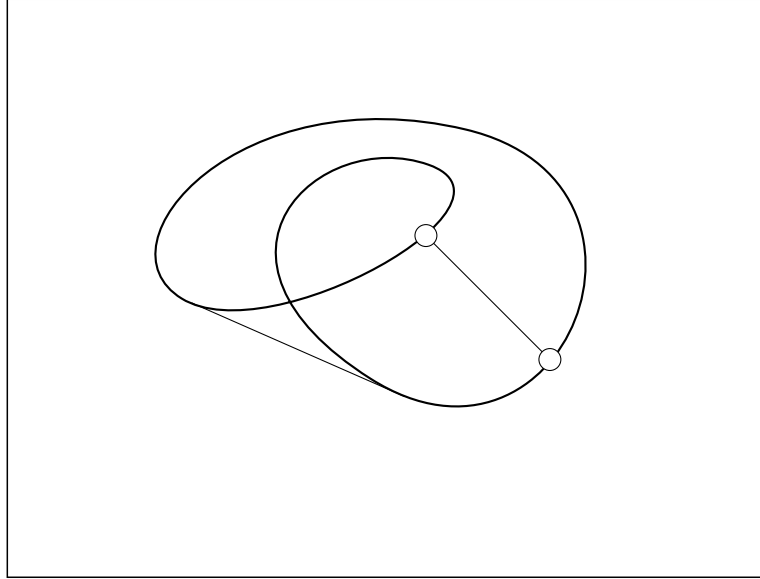
S2 : U = susp S1
S3 : U = susp S2

ua (A B : U) (e : equiv A B) :  $\Xi$  U A B =
  <i> Glue B [ (i = 0) -> (A,e),
              (i = 1) -> (B,idEquiv B) ]

moebius : S1 -> U = split
  base -> bool
  loop @ i -> ua bool bool negBoolEquiv @ i

TH0 : U = (c : S1) * moebius c

```



Example 5. ($S^3 \mathbb{C}$ Hopf Fiber). S^3 Fibration was peconeered by Guillaume Brunerie.

```

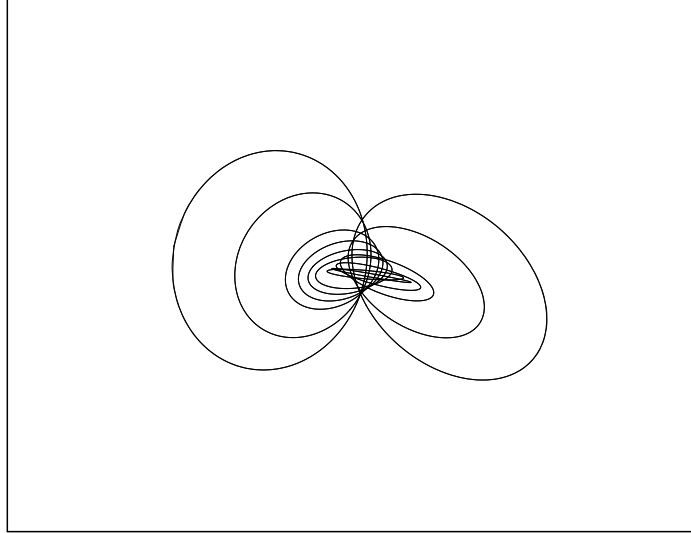
rot : (x : S1) → ∃ S1 x x = split
      base → loop1
      loop @ i → constSquare S1 base loop1 @ i

mu : S1 → equiv S1 S1 = split
      base → idEquiv S1
      loop @ i → equivPath S1 S1 (idEquiv S1)
                  (idEquiv S1) (<j> \ (x : S1) → rot x @ j) @ i

H : S2 → U = split
      north → S1
      south → S1
      merid x @ i → ua S1 S1 (mu x) @ i

total : U = (c : S2) * H c

```



Definition 89. (H-space). H-space over a carrier A is a tuple

$$H_A = \begin{cases} A : U \\ e : A \\ \mu : A \rightarrow A \rightarrow A \\ \beta : \Pi(a : A), \mu(e, a) = a \times \mu(a, e) = a \end{cases}$$

.

Theorem 43. (Hopf Invariant). Let $\phi : S^{2n-1} \rightarrow S^n$ a continuous map. Then homotopy pushout (cofiber) of ϕ is $\text{cofib}(\phi) = S^n \bigcup_{\phi} \mathbb{D}^{2n}$ has ordinary cohomology

$$H^k(\text{cofib}(\phi), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } k = n, 2n \\ 0 & \text{otherwise} \end{cases}$$

Theorem 44. (Four). There are fiber bundles: (S^0, S^1, p, S^1) , (S^1, S^3, p, S^2) , (S^3, S^7, p, S^4) , (S^7, S^{15}, p, S^8) .

Hence for α, β generators of the cohomology groups in degree n and $2n$, respectively, there exists an integer $h(\phi)$ that expresses the **cup product** square of α as a multiple of $\beta - \alpha \sqcup \alpha = h(\phi) \cdot \beta$. This integer $h(\phi)$ is called Hopf invariant of ϕ .

Theorem 45. (Adams, Atiyah). Hopf Fibrations are only maps that have Hopf invariant 1.

Issue IV: Higher Inductive Types

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Анотація

CW-complexes are central to both homotopy theory and homotopy type theory (HoTT) and are encoded in cubical theorem-proving systems as higher inductive types (HIT), similar to recursive trees for (co)inductive types. We explore the basic primitives of homotopy theory, which are considered as a foundational basis in theorem-proving systems.

Keywords: Homotopy Theory, Type Theory

7 CW-Complexes

CW-complexes are spaces constructed by attaching cells of various dimensions. In HoTT, they are encoded as higher inductive types (HIT), where cells are constructors for points and paths.

Definition 90. (Cell Attachment). The attachment of an n -cell to a space X along $f : S^{n-1} \rightarrow X$ is a pushout:

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{f} & X \\ \downarrow \iota & & \downarrow j \\ D^n & \xrightarrow{g} & X \cup_f D^n \end{array}$$

Here, $\iota : S^{n-1} \hookrightarrow D^n$ is the boundary inclusion, and $X \cup_f D^n$ is the pushout that attaches an n -cell to X via f . The result depends on the homotopy class of f .

Definition 91. (CW-Complex). A CW-complex is a space X , constructed inductively by attaching cells, with a skeletal filtration:

- (-1) -skeleton: $X_{-1} = \emptyset$.

- For $n \geq 0$, the n -skeleton X_n is obtained by attaching n -cells to X_{n-1} . For indices J_n and maps $\{f_j : S^{n-1} \rightarrow X_{n-1}\}_{j \in J_n}$, X_n is the pushout:

$$\begin{array}{ccc} \coprod_{j \in J_n} S^{n-1} & \xrightarrow{\coprod f_j} & X_{n-1} \\ \downarrow \coprod \iota_j & & \downarrow i_n \\ \coprod_{j \in J_n} D^n & \xrightarrow{\coprod g_j} & X_n \end{array}$$

where $\coprod_{j \in J_n} S^{n-1}$, $\coprod_{j \in J_n} D^n$ are disjoint unions, and $i_n : X_{n-1} \hookrightarrow X_n$ is the inclusion.

- X is the colimit:

$$\emptyset = X_{-1} \hookrightarrow X_0 \hookrightarrow X_1 \hookrightarrow \dots \hookrightarrow X,$$

where X_n is the n -skeleton, and $X = \operatorname{colim}_{n \rightarrow \infty} X_n$. The sequence is the skeletal filtration.

In HoTT, CW-complexes are higher inductive types (HIT) with constructors for cells and paths for attachment.

7.1 Introduction: Countable Constructors

Some HITs require an infinite number of constructors for spaces, such as Eilenberg-MacLane spaces or the infinite sphere S^∞ .

```
def S∞ : U
:= inductive { base
              | loop (n: ℕ) : base ≡ base
              }
```

Challenges include type checking, computation, and expressiveness.

Agda Cubical uses cubical primitives to handle HITs, supporting infinite constructors via HITs indexed by natural numbers, as colimits.

7.2 Motivation: Higher Inductive Types

HITs in HoTT enable direct encoding of topological spaces, such as CW-complexes. In homotopy theory, spaces are constructed by attaching cells via attaching maps. HoTT views types as spaces, elements as points, and equalities as paths, making HITs a natural choice. Standard inductive types cannot capture higher homotopies, but HITs allow constructors for points and paths. For example, the circle S^1 (Definition 2) has a base point and a loop, encoding its fundamental group \mathbb{Z} . HITs avoid the use of multiple quotient spaces, preserving the synthetic nature of HoTT. In cubical type theory, paths are intervals (e.g., $\langle i \rangle$) with computational content, unlike propositional equalities, enabling efficient type checking in tools such as Agda Cubical.

7.3 Metatheory: Cohesive Topoi

7.3.1 Geometric Proofs

$$\mathfrak{R} \dashv \mathfrak{S} \dashv \&$$

For differential geometry, type theory incorporates primitive axioms of categorical meta-theoretical models of three Schreiber-Shulman functors: infinitesimal neighborhood (\mathfrak{S}), reduced modality (\mathfrak{R}), and infinitesimal discrete neighborhood ($\&$).

7.3.2 Flat Proofs

7.3.3 Sharp Proofs

7.3.4 Bose Proofs

7.3.5 Fermi Proofs

7.3.6 Linear Proofs

$$\otimes \dashv x \dashv \multimap$$

For engineering applications (e.g., Milner’s π -calculus, quantum computing) and linear type theory, type theory embeds linear proofs based on the adjunction

of the tensor and linear function spaces: $(A \otimes B) \multimap A \simeq A \multimap (B \multimap C)$, represented in a symmetric monoidal category \mathbf{D} for a functor $[A, B]$ as: $\mathbf{D}(A \otimes B, C) \simeq \mathbf{D}(A, [B, C])$.

8 Higher Inductive Types

CW-complexes are central to HoTT and appear in cubical type checkers as HITs. Unlike inductive types (recursive trees), HITs encode CW-complexes, capturing points (0-cells) and higher paths (n-cells). The definition of an HIT specifies a CW-complex through cubical composition, an initial algebra in the cubical model.

8.1 Suspension

The suspension ΣA of a type A is a higher inductive type that constructs a new type by adding two points, called poles, and paths connecting each point of A to these poles. It is a fundamental construction in homotopy theory, often used to shift homotopy groups, e.g., obtaining S^{n+1} from S^n .

Definition 92. (Formation). For any type $A : \mathcal{U}$, there exists a suspension type $\Sigma A : \mathcal{U}$.

Definition 93. (Constructors). For a type $A : \mathcal{U}$, the suspension $\Sigma A : \mathcal{U}$ is generated by the following higher inductive compositional structure:

$$\Sigma := \begin{cases} \text{north} \\ \text{south} \\ \text{merid} : (a : A) \rightarrow \text{north} \equiv \text{south} \end{cases}$$

```
def Σ (A: U) : U
:= inductive {
  | north
  | south
  | merid (a: A) : north ≡ south
}
```

Theorem 46. (Elimination). For a family of types $B : \Sigma A \rightarrow \mathcal{U}$, points $n : B(\text{north})$, $s : B(\text{south})$, and a family of dependent paths

$$m : \Pi(a : A), \text{PathOver}(B, \text{merid}(a), n, s),$$

there exists a dependent map $\text{Ind}_{\Sigma A} : (x : \Sigma A) \rightarrow B(x)$, such that:

$$\begin{cases} \text{Ind}_{\Sigma A}(\text{north}) = n \\ \text{Ind}_{\Sigma A}(\text{south}) = s \\ \text{Ind}_{\Sigma A}(\text{merid}(a, i)) = m(a, i) \end{cases}$$

```
def PathOver (B: Σ A → U) (a: A) (n: B north) (s: B south) : U
:= PathP (λ i , B (merid a @ i)) n s
```

```
def Ind_ΣA (A: U) (B: Σ A → U) (n: B north) (s: B south)
(m: (a: A) → PathOver B (merid a) n s) : (x: Σ A) → B x
:= split { north → n | south → s | merid a @ i → m a @ i }
```

Theorem 47. (Computation).

$$\text{Ind}_{\Sigma A}(\text{north}) = n, \text{Ind}_{\Sigma A}(\text{south}) = s, \text{Ind}_{\Sigma A}(\text{merid}(a, i)) = m(a, i)$$

```
def Σ-β (A: U) (B: Σ A → U) (n: B north) (s: B south)
(m: (a: A) → PathOver B (merid a) n s) (x: Σ A)
: Path (B x) (Σ-I A B n s m x)
split { north → n | south → s | merid a @ i → m a @ i }
```

Theorem 48. (Uniqueness). Any two maps $h_1, h_2 : (x : \Sigma A) \rightarrow B(x)$ are homotopic if they agree on north, south, and merid, i.e., if $h_1(\text{north}) = h_2(\text{north})$, $h_1(\text{south}) = h_2(\text{south})$, and $h_1(\text{merid } a) = h_2(\text{merid } a)$ for all $a : A$.

8.2 Pushout

The pushout (amalgamation) is a higher inductive type that constructs a type by gluing two types A and B along a common type C via maps $f : C \rightarrow A$ and $g : C \rightarrow B$. It is a fundamental construction in homotopy theory, used to model cell attachment and cofibrant objects, generalizing the topological notion of a pushout.

Definition 94. (Formation). For types $A, B, C : \mathcal{U}$ and maps $f : C \rightarrow A$, $g : C \rightarrow B$, there exists a pushout $\sqcup(A, B, C, f, g) : \mathcal{U}$.

Definition 95. (Constructors). The pushout is generated by the following higher inductive compositional structure:

$$\sqcup := \begin{cases} \text{po}_1 : A \rightarrow \sqcup(A, B, C, f, g) \\ \text{po}_2 : B \rightarrow \sqcup(A, B, C, f, g) \\ \text{po}_3 : (c : C) \rightarrow \text{po}_1(f(c)) \equiv \text{po}_2(g(c)) \end{cases}$$

```
def  $\sqcup$  (A B C : U) (f : C  $\rightarrow$  A) (g : C  $\rightarrow$  B) : U
:= inductive {
  | po1 (a : A)
  | po2 (b : B)
  | po3 (c : C) : po1(f(c))  $\equiv$  po2(g(c))
}
```

Theorem 49. (Elimination). For a type $D : \mathcal{U}$, maps $u : A \rightarrow D$, $v : B \rightarrow D$, and a family of paths $p : (c : C) \rightarrow u(f(c)) \equiv v(g(c))$, there exists a map $\text{Ind}_{\sqcup} : \sqcup(A, B, C, f, g) \rightarrow D$, such that:

$$\begin{cases} \text{Ind}_{\sqcup}(\text{po}_1(a)) = u(a) \\ \text{Ind}_{\sqcup}(\text{po}_2(b)) = v(b) \\ \text{Ind}_{\sqcup}(\text{po}_3(c, i)) = p(c, i) \end{cases}$$

```
def PathOver (A B C : U) (f : C  $\rightarrow$  A) (g : C  $\rightarrow$  B)
  (D :  $\sqcup$  A B C f g  $\rightarrow$  U)
  (c : C) (u : D (po1 (f c))) (v : D (po2 (g c))) : U
:= PathP ( $\lambda$  i, D (po3 c i)) u v

def Ind $\sqcup$  : (A B C : U) (f : C  $\rightarrow$  A) (g : C  $\rightarrow$  B)
  (D :  $\sqcup$  A B C f g  $\rightarrow$  U)
  (u : (a : A)  $\rightarrow$  D (po1 a))
  (v : (b : B)  $\rightarrow$  D (po2 b))
  (p : (c : C)  $\rightarrow$  PathOver D c (u (f c)) (v (g c)))
  : (x :  $\sqcup$  A B C f g)  $\rightarrow$  D x
:= split { po1 a  $\rightarrow$  u a | po2 b  $\rightarrow$  v b | po3 c @ i  $\rightarrow$  p c @ i }
```

Theorem 50. (Computation). For $x : \sqcup(A, B, C, f, g)$,

$$\begin{cases} \text{Ind}_{\sqcup}(\text{po}_1(a)) \equiv u(a) \\ \text{Ind}_{\sqcup}(\text{po}_2(b)) \equiv v(b) \\ \text{Ind}_{\sqcup}(\text{po}_3(c, i)) \equiv p(c, i) \end{cases}$$

Theorem 51. (Uniqueness). Any two maps $u, v : \sqcup(A, B, C, f, g) \rightarrow D$ are homotopic if they agree on po_1 , po_2 , and po_3 , i.e., if $u(\text{po}_1(a)) = v(\text{po}_1(a))$ for all $a : A$, $u(\text{po}_2(b)) = v(\text{po}_2(b))$ for all $b : B$, and $u(\text{po}_3(c)) = v(\text{po}_3(c))$ for all $c : C$.

Example 6. (Cell Attachment) The pushout models the attachment of an n -cell to a space X . Given $f : S^{n-1} \rightarrow X$ and inclusion $g : S^{n-1} \rightarrow D^n$, the pushout $\sqcup(X, D^n, S^{n-1}, f, g)$ is the space $X \cup_f D^n$, attaching an n -disk to X along f .

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{f} & X \\ \downarrow g & & \downarrow \\ D^n & \longrightarrow & X \cup_f D^n \end{array}$$

8.3 Spheres

Spheres are higher inductive types with higher-dimensional paths, representing fundamental topological spaces.

Definition 96. (Pointed n-Spheres) The n -sphere S^n is defined recursively as a type in the universe \mathcal{U} using general recursion over dimensions:

$$S^n := \begin{cases} \text{point} : \mathbb{S}^n, \\ \text{surface} : \langle i_1, \dots, i_n \rangle [(i_1 = 0) \rightarrow \text{point}, (i_1 = 1) \rightarrow \text{point}, \dots \\ (i_n = 0) \rightarrow \text{point}, (i_n = 1) \rightarrow \text{point}] \end{cases}$$

Definition 97. (n-Spheres via Suspension) The n -sphere S^n is defined recursively as a type in the universe \mathcal{U} using general recursion over natural numbers \mathbb{N} . For each $n \in \mathbb{N}$, the type $S^n : \mathcal{U}$ is defined as:

$$\mathbb{S}^n := \begin{cases} S^0 = \mathbf{2}, \\ S^{n+1} = \Sigma(S^n). \end{cases}$$

`def sphere : $\mathbb{N} \rightarrow \mathcal{U} := \text{N-iter } \mathbf{U} \ \mathbf{2} \ \Sigma$`

This iterative definition applies the suspension functor Σ to the base type $\mathbf{2}$ (0-sphere) n times to obtain S^n .

Example 7. (Sphere as CW-Complex) The n -sphere S^n can be constructed as a CW-complex with one 0-cell and one n -cell:

$$\begin{cases} X_0 = \{\text{base}\}, \text{ one point} \\ X_k = X_0 \text{ for } 0 < k < n, \text{ no additional cells} \\ X_n : \text{Attachment of an } n\text{-cell to } X_{n-1} = \{\text{base}\} \text{ along } f : S^{n-1} \rightarrow \{\text{base}\} \end{cases}$$

The constructor cell attaches the boundary of the n -cell to the base point, yielding the type S^n .

8.4 Hub and Spokes

The hub and spokes construction \odot defines an n -truncation, ensuring that the type has no non-trivial homotopy groups above dimension n . It models the type as a CW-complex with a hub (central point) and spokes (paths to points).

Definition 98. (Formation). For types $S, A : \mathcal{U}$, there exists a hub and spokes type $\odot (S, A) : \mathcal{U}$.

Definition 99. (Constructors). The hub and spokes type is freely generated by the following higher inductive compositional structure:

$$\odot := \begin{cases} \text{base} : A \rightarrow \odot (S, A) \\ \text{hub} : (S \rightarrow \odot (S, A)) \rightarrow \odot (S, A) \\ \text{spoke} : (f : S \rightarrow \odot (S, A)) \rightarrow (s : S) \rightarrow \text{hub}(f) \equiv f(s) \end{cases}$$

```
def  $\odot$  (S A: U) : U
:= inductive { base (x: A)
              | hub (f: S  $\rightarrow$   $\odot$  S A)
              | spoke (f: S  $\rightarrow$   $\odot$  S A) (s:S) : hub f  $\equiv$  f s
              }
```

Theorem 52. (Elimination). For a family of types $P : \text{HubSpokes } S \ A \rightarrow \mathcal{U}$, maps $\text{pbase} : (x : A) \rightarrow P(\text{base } x)$, $\text{phub} : (f : S \rightarrow \text{HubSpokes } S \ A) \rightarrow P(\text{hub } f)$, and a family of paths $\text{pspoke} : (f : S \rightarrow \text{HubSpokes } S \ A) \rightarrow (s : S) \rightarrow \text{PathP}(< i > P(\text{spoke } f \ s \ @ \ i)) (\text{phub } f) (P(f \ s))$, there exists a map $\text{hubSpokesInd} : (z : \text{HubSpokes } S \ A) \rightarrow P(z)$, such that:

$$\begin{cases} \text{Ind}_{\odot} (\text{base } x) = \text{pbase } x \\ \text{Ind}_{\odot} (\text{hub } f) = \text{phub } f \\ \text{Ind}_{\odot} (\text{spoke } f \ s \ @ \ i) = \text{pspoke } f \ s \ @ \ i \end{cases}$$

8.5 Truncation

Set Truncation

Definition 100. (Formation). Set truncation (0-truncation), denoted $\|A\|_0$, ensures that the type is a set, with homotopy groups vanishing above dimension 0.

Definition 101. (Constructors). For $A : \mathcal{U}$, $\|A\|_0 : \mathcal{U}$ is defined by the following higher inductive compositional structure:

$$\|_-\|_0 := \begin{cases} \text{inc} : A \rightarrow \|A\|_0 \\ \text{squash} : (a, b : \|A\|_0) \rightarrow (p, q : a \equiv b) \rightarrow p \equiv q \end{cases}$$

```
def \|_ \|_0 (A: U) : U
:= inductive { inc (a: A)
              | squash (a b: \|A\|_0) (p q: Path (\|A\|_0) a b)
                <i j> [ (i = 0) -> p @ j, (i = 1) -> q @ j,
                      (j = 0) -> a,      (j = 1) -> b ]
              }
```

Theorem 53. (Elimination $\|A\|_0$) For a set $B : \mathcal{U}$ (i.e., $\text{isSet}(B)$), and a map $f : A \rightarrow B$, there exists $\text{setTruncRec} : \|A\|_0 \rightarrow B$, such that $\text{Ind}_{\|A\|_0}(\text{inc}(a)) = f(a)$.

Groupoid Truncation

Definition 102. (Formation). Groupoid truncation (1-truncation), denoted $\|A\|_1$, ensures that the type is a 1-groupoid, with homotopy groups vanishing above dimension 1.

Definition 103. (Constructors). For $A : \mathcal{U}$, $\|A\|_1 : \mathcal{U}$ is defined by the following higher inductive compositional structure:

$$\|_-\|_1 := \begin{cases} \text{inc} : A \rightarrow \|A\|_1 \\ \text{squash} : (a, b : \|A\|_1) \rightarrow (p, q : a \equiv b) \rightarrow (r, s : p \equiv q) \rightarrow r \equiv s \end{cases}$$

```
def \|_ \|_1 (A: U) : U
:= inductive { inc (a: A)
              | squash (a b: \|A\|_1) (p q: Path (\|A\|_1) a b)
                (r s: Path (Path (\|A\|_1) a b) p q) <i j k>
                [ (i = 0) -> r @ j @ k, (i = 1) -> s @ j @ k,
                  (j = 0) -> p @ k,      (j = 1) -> q @ k,
                  (k = 0) -> a,          (k = 1) -> b ]
              }
```

Theorem 54. (Elimination $\|A\|_1$) For a 1-groupoid $B : \mathcal{U}$ (i.e., $\text{isGroupoid}(B)$), and a map $f : A \rightarrow B$, there exists $\text{Ind}_{\|A\|_1} : \|A\|_1 \rightarrow B$, such that $\text{Ind}_{\|A\|_1}(\text{inc}(a)) = f(a)$.

8.6 Quotients

Set Quotient Spaces

Quotient spaces are a powerful computational tool in type theory, embedded in the core of Lean.

Definition 104. (Formation). Set quotient spaces construct a type A , quotiented by a relation $R : A \rightarrow A \rightarrow \mathcal{U}$, ensuring that the result is a set.

Definition 105. (Constructors). For a type $A : \mathcal{U}$ and a relation $R : A \rightarrow A \rightarrow \mathcal{U}$, the set quotient space $A/R : \mathcal{U}$ is freely generated by the following higher inductive compositional structure:

$$A/R := \begin{cases} \text{quot} : A \rightarrow A/R \\ \text{ident} : (a, b : A) \rightarrow R(a, b) \rightarrow \text{quot}(a) \equiv \text{quot}(b) \\ \text{trunc} : (a, b : A/R) \rightarrow (p, q : a \equiv b) \rightarrow p \equiv q \end{cases}$$

```
def / (A : U) (R : A → A → U) : U
:= inductive { quot (a : A)
| ident (a b : A) (r : R a b) : quot(a) ≡ quot(b)
| trunc (a b : / A R) (p q : Path (/ A R) a b)
  <i j> [ (i = 0) → p @ j , (i = 1) → q @ j ,
        (j = 0) → a , (j = 1) → b ]
}
```

Theorem 55. (Elimination). For a family of types $B : A/R \rightarrow \mathcal{U}$ with $\text{isSet}(Bx)$, and maps $f : (x : A) \rightarrow B(\text{quot}(x))$, $g : (a, b : A) \rightarrow (r : R(a, b)) \rightarrow \text{PathP}(< i > B(\text{ident}(a, b, r) @ i))(f(a))(f(b))$, there exists $\text{Ind}_{A/R} : \Pi(x : A/R), B(x)$, such that $\text{Ind}_{A/R}(\text{quot}(a)) = f(a)$.

Groupoid Quotient Spaces

Definition 106. (Formation). Groupoid quotient spaces extend set quotient spaces to produce a 1-groupoid, including constructors for higher paths. Groupoid quotient spaces construct a type A , quotiented by a relation $R : A \rightarrow A \rightarrow \mathcal{U}$, ensuring that the result is a groupoid.

Definition 107. (Constructors). For a type $A : \mathcal{U}$ and a relation $R : A \rightarrow A \rightarrow \mathcal{U}$, the groupoid quotient space $A//R : \mathcal{U}$ includes constructors for points, paths, and higher paths, ensuring a 1-groupoid structure.

8.7 Wedge

The wedge of two pointed types A and B , denoted $A \vee B$, is a higher inductive type representing the union of A and B with identified base points. Topologically, it corresponds to $A \times \{y_0\} \cup \{x_0\} \times B$, where x_0 and y_0 are the base points of A and B , respectively.

Definition 108. (Formation). For pointed types $A, B : \text{pointed}$, the wedge $A \vee B : \mathcal{U}$.

Definition 109. (Constructors). The wedge is generated by the following higher inductive compositional structure:

$$\vee := \begin{cases} \text{winl} : A.1 \rightarrow A \vee B \\ \text{winr} : B.1 \rightarrow A \vee B \\ \text{wglue} : \text{winl}(A.2) \equiv \text{winr}(B.2) \end{cases}$$

```
def ∨ (A : pointed) (B : pointed) : U
:= inductive { winl (a : A.1)
              | winr (b : B.1)
              | wglue : winl(A.2) ≡ winr(B.2)
              }
```

Theorem 56. (Elimination). For a type $P : A \vee B \rightarrow \mathcal{U}$, maps $f : A.1 \rightarrow C$, $g : B.1 \rightarrow C$, and a path $p : \text{PathOverlue}(P, f(A.2), g(B.2))$, there exists a map $\text{Ind}_\vee : A \vee B \rightarrow C$, such that:

$$\begin{cases} \text{Ind}(\text{winl}(a)) = f(a) \\ \text{Ind}(\text{winr}(b)) = g(b) \\ \text{Ind}(\text{wglue}(x)) = p(x) \end{cases}$$

```
def PathOverGlue : (P : A ∨ B → U)
  (p : P (inl (A.2))) (q : P (inr (B.2))) : U
:= PathP (λ i → P (wglue i)) p q

def Ind_∨ (A B : pointed) (C : U) (f : A.1 → C) (g : B.1 → C)
  (p : Path C (f A.2) (g B.2))
  : A ∨ B → C
:= split { winl a → f a | winr b → g b | wglue @ x → p @ x }
```

Theorem 57. (Computation). For $z : \text{Wedge } AB$,

$$\begin{cases} \text{Ind}_\vee(\text{winl } a) \equiv f(a) \\ \text{Ind}_\vee(\text{winr } b) \equiv g(b) \\ \text{Ind}_\vee(\text{wglue } @ x) \equiv p @ x \end{cases}$$

Theorem 58. (Uniqueness). Any two maps $h_1, h_2 : \text{Wedge } AB \rightarrow C$ are homotopic if they agree on winl , winr , and wglue , i.e., if $h_1(\text{winl } a) = h_2(\text{winl } a)$ for all $a : A.1$, $h_1(\text{winr } b) = h_2(\text{winr } b)$ for all $b : B.1$, and $h_1(\text{wglue}) = h_2(\text{wglue})$.

8.8 Smash Product

The smash product of two pointed types A and B , denoted $A \wedge B$, is a higher inductive type that quotients the product $A \times B$ by the pushout $A \sqcup B$. It represents the space $A \times B / (A \times \{y_0\} \cup \{x_0\} \times B)$, collapsing the wedge to a single point.

Definition 110. (Formation). For pointed types $A, B : \text{pointed}$, the smash product $A \wedge B : \mathcal{U}$.

Definition 111. (Constructors). The smash product is generated by the following higher inductive compositional structure:

$$A \wedge B := \begin{cases} \text{basel} : A \wedge B \\ \text{baser} : A \wedge B \\ \text{proj}(x : A.1)(y : B.1) : A \wedge B \\ \text{gluel}(a : A.2) : \text{proj}(a, B.2) \equiv \text{basel} \\ \text{gluer}(b : B.2) : \text{proj}(A.2, b) \equiv \text{baser} \end{cases}$$

```
def  $\wedge$  (A : pointed) (B : pointed) : U
:= inductive {
  | basel
  | baser
  | proj (a : A.1) (b : B.1)
  | gluel (a : A.2) : proj(a, B.2)  $\equiv$  basel
  | gluer (a : B.2) : proj(A.2, b)  $\equiv$  baser
}
```

Theorem 59. (Elimination). For a family of types $P : \text{Smash } A B \rightarrow \mathcal{U}$, points $\text{pbasel} : P(\text{basel})$, $\text{pbaser} : P(\text{baser})$, maps $\text{pproj} : (x : A.1) \rightarrow (y : B.1) \rightarrow P(\text{proj } x y)$, and a family of paths $\text{pgluel} : (a : A.1) \rightarrow \text{pproj}(a, B.2) \equiv \text{pbasel}$, $\text{pgluer} : (b : B.1) \rightarrow \text{pproj}(A.2, b) \equiv \text{pbaser}$, there exists a map $\text{Ind}_\wedge : (z : A \wedge B) \rightarrow P(z)$, such that:

$$\begin{cases} \text{Ind}_\wedge(\text{basel}) = \text{pbasel} \\ \text{Ind}_\wedge(\text{baser}) = \text{pbaser} \\ \text{Ind}_\wedge(\text{proj } x y) = \text{pproj } x y \\ \text{Ind}_\wedge(\text{gluel } a @ i) = \text{pgluel } a @ i \\ \text{Ind}_\wedge(\text{gluer } b @ i) = \text{pgluer } b @ i \end{cases}$$

```
def Ind $\wedge$  (A B : pointed) (P : A  $\wedge$  B  $\rightarrow$  U)
  (pbasel : P basel) (pbaser : P baser)
  (pproj : (x : A.1)  $\rightarrow$  (y : B.1)  $\rightarrow$  P (proj x y))
  (pgluel : (a : A.1)  $\rightarrow$  PathP (<i> P (gluel a @ i)) (pproj a B.2) pbasel)
  (pgluer : (b : B.1)  $\rightarrow$  PathP (<i> P (gluer b @ i)) (pproj A.2 b) pbaser)
  : (z : A  $\wedge$  B)  $\rightarrow$  P z
:= split {
  | basel  $\rightarrow$  pbasel
  | baser  $\rightarrow$  pbaser
  | proj x y  $\rightarrow$  pproj x y
  | gluel a @ i  $\rightarrow$  pgluel a @ i
  | gluer b @ i  $\rightarrow$  pgluer b @ i
}
```

Theorem 60. (Computation). For a family of types $P : A \wedge B \rightarrow \mathcal{U}$, points $\text{pbase} : P(\text{base})$, $\text{pbaser} : P(\text{baser})$, map $\text{pproj} : (x : A.1) \rightarrow (y : B.1) \rightarrow P(\text{proj } x y)$, and families of paths $\text{pgluel} : (a : A.1) \rightarrow \text{PathP } (< i > P(\text{gluel } a @ i)) (\text{pproj } a B.2) \text{pbase}$, $\text{pgluer} : (b : B.1) \rightarrow \text{PathP } (< i > P(\text{gluer } b @ i)) (\text{pproj } A.2 b) \text{pbaser}$, the map $\text{Ind}_\wedge : (z : A \wedge B) \rightarrow P(z)$ satisfies all equations for all variants of the predicate P :

$$\left\{ \begin{array}{l} \text{Ind}_\wedge (\text{base}) \equiv \text{pbase} \\ \text{Ind}_\wedge (\text{baser}) \equiv \text{pbaser} \\ \text{Ind}_\wedge (\text{proj } x y) \equiv \text{pproj } x y \\ \text{Ind}_\wedge (\text{gluel } a @ i) \equiv \text{pgluel } a @ i \\ \text{Ind}_\wedge (\text{gluer } b @ i) \equiv \text{pgluer } b @ i \end{array} \right.$$

Theorem 61. (Uniqueness). For a family of types $P : A \wedge B \rightarrow \mathcal{U}$, and maps $h_1, h_2 : (z : A \wedge B) \rightarrow P(z)$, if there exist paths $e_{\text{base}} : h_1(\text{base}) \equiv h_2(\text{base})$, $e_{\text{baser}} : h_1(\text{baser}) \equiv h_2(\text{baser})$, $e_{\text{proj}} : (x : A.1) \rightarrow (y : B.1) \rightarrow h_1(\text{proj } x y) \equiv h_2(\text{proj } x y)$, $e_{\text{gluel}} : (a : A.1) \rightarrow \text{PathP } (< i > h_1(\text{gluel } a @ i) \equiv h_2(\text{gluel } a @ i)) (e_{\text{proj } a B.2} e_{\text{base}})$, $e_{\text{gluer}} : (b : B.1) \rightarrow \text{PathP } (< i > h_1(\text{gluer } b @ i) \equiv h_2(\text{gluer } b @ i)) (e_{\text{proj } A.2 b} e_{\text{baser}})$, then $h_1 \equiv h_2$, i.e., there exists a path $(z : A \wedge B) \rightarrow h_1(z) \equiv h_2(z)$.

8.9 Join

The join of two types A and B , denoted $A \vee B$, is a higher inductive type that constructs a type by joining each point of A to each point of B via a path. Topologically, it corresponds to the join of spaces, forming a space that interpolates between A and B .

Definition 112. (Formation). For types $A, B : \mathcal{U}$, the join $A * B : \mathcal{U}$.

Definition 113. (Constructors). The join is generated by the following higher inductive compositional structure:

$$A \vee B := \begin{cases} \text{joinl} : A \rightarrow A \vee B \\ \text{joinr} : B \rightarrow A \vee B \\ \text{join}(a : A)(b : B) : \text{joinl}(a) \equiv \text{joinr}(b) \end{cases}$$

```
def ∨ (A : U) (B : U) : U
:= inductive { joinl (a : A)
              | joinr (b : B)
              | join (a : A) (b : B) : joinl(a) ≡ joinr(b)
            }
```

Theorem 62. (Elimination). For a type $C : \mathcal{U}$, maps $f : A \rightarrow C$, $g : B \rightarrow C$, and a family of paths $h : (a : A) \rightarrow (b : B) \rightarrow f(a) \equiv g(b)$, there exists a map $\text{Ind}_\vee : A \vee B \rightarrow C$, such that:

$$\begin{cases} \text{Ind}_\vee(\text{joinl}(a)) = f(a) \\ \text{Ind}_\vee(\text{joinr}(b)) = g(b) \\ \text{Ind}_\vee(\text{join}(a, b, i)) = h(a, b, i) \end{cases}$$

```
def Ind_∨ (A B C : U) (f : A → C) (g : B → C)
  (h : (a : A) → (b : B) → Path C (f a) (g b))
  : A ∨ B → C
:= split { joinl a → f a
          | joinr b → g b
          | join a b @ i → h a b @ i
        }
```

Theorem 63. (Computation). For all $z : A \vee B$, and predicate P , the rules of Ind_\vee hold for all parameters of the predicate P .

Theorem 64. (Uniqueness). Any two maps $h_1, h_2 : A \vee B \rightarrow C$ are homotopic if they agree on joinl , joinr , and join .

8.10 Colimit

Colimits construct the limit of a sequence of types, connected by maps, e.g., propositional truncations.

Definition 114. (Colimit) For a sequence of types $A : \mathbb{N} \rightarrow \mathcal{U}$ and maps $f : (n : \mathbb{N}) \rightarrow A n \rightarrow A(\text{succ}(n))$, the colimit type $\text{colimit}(A, f) : \mathcal{U}$.

$$\text{colim} := \begin{cases} \text{ix} : (n : \mathbb{N}) \rightarrow A n \rightarrow \text{colimit}(A, f) \\ \text{gx} : (n : \mathbb{N}) \rightarrow (a : A(n)) \rightarrow \text{ix}(\text{succ}(n), f(n, a)) \equiv \text{ix}(n, a) \end{cases}$$

```
def colimit (A : nat -> U) (f : (n : nat) -> A n -> A (succ n)) : U
:= inductive {
  | ix (n : nat) (x : A n)
  | gx (n : nat) (a : A n)
    <i> [ (i=0) -> ix (succ n) (f n a),
        (i=1) -> ix n a ]
}
```

Theorem 65. (Elimination colimit) For a type $P : \text{colimit } A f \rightarrow \mathcal{U}$, with $p : (n : \mathbb{N}) \rightarrow (x : A n) \rightarrow P(\text{ix}(n, x))$ and $q : (n : \mathbb{N}) \rightarrow (a : A n) \rightarrow \text{PathP}(\langle i \rangle P(\text{gx}(n, a) @ i))(p(\text{succ } n)(f n a))(p n a)$, there exists $i : \prod_{x : \text{colimit } A f} P(x)$, such that $i(\text{ix}(n, x)) = p n x$.

8.11 Coequalizers

Coequalizer

The coequalizer of two maps $f, g : A \rightarrow B$ is a higher inductive type (HIT) that constructs a type consisting of elements in B , where f and g agree, along with paths ensuring this equality. It is a fundamental construction in homotopy theory, capturing the subspace of B where $f(a) = g(a)$ for $a : A$.

Definition 115. (Formation). For types $A, B : \mathcal{U}$ and maps $f, g : A \rightarrow B$, the coequalizer $\text{coeq } ABfg : \mathcal{U}$.

Definition 116. (Constructors). The coequalizer is generated by the following higher inductive compositional structure:

$$\text{Coeq} := \begin{cases} \text{inC} : B \rightarrow \text{Coeq}(A, B, f, g) \\ \text{glueC} : (a : A) \rightarrow \text{inC}(f(a)) \equiv \text{inC}(g(a)) \end{cases}$$

```
def Coeq (A B: U) (f g: A → B) : U
:= inductive { inC (b: B)
              | glueC (a: A) : inC (f a) ≡ inC (g a)
            }
```

Theorem 66. (Elimination). For a type $C : \mathcal{U}$, map $h : B \rightarrow C$, and a family of paths $y : (x : A) \rightarrow \text{Path}_C(h(fx), h(gx))$, there exists a map $\text{coequRec} : \text{coeq } ABfg \rightarrow C$, such that:

$$\begin{cases} \text{coequRec}(\text{inC}(x)) = h(x) \\ \text{coequRec}(\text{glueC}(x, i)) = y(x, i) \end{cases}$$

```
def coequRec (A B C : U) (f g : A → B) (h : B → C)
  (y : (x : A) → Path C (h (f x)) (h (g x)))
  : (z : coeq A B f g) → C
:= split { inC x → h x | glueC x @ i → y x @ i }
```

Theorem 67. (Computation). For $z : \text{coeq } ABfg$,

$$\begin{cases} \text{coequRec}(\text{inC } x) \equiv h(x) \\ \text{coequRec}(\text{glueC } x @ i) \equiv y(x) @ i \end{cases}$$

Theorem 68. (Uniqueness). Any two maps $h_1, h_2 : \text{coeq } ABfg \rightarrow C$ are homotopic if they agree on inC and glueC , i.e., if $h_1(\text{inC } x) = h_2(\text{inC } x)$ for all $x : B$ and $h_1(\text{glueC } a) = h_2(\text{glueC } a)$ for all $a : A$.

Example 8. (Coequalizer as Subspace) The coequalizer $\text{coeq } ABfg$ represents the subspace of B , where $f(a) = g(a)$. For example, if $A = B = \mathbb{R}$ and $f(x) = x^2$, $g(x) = x$, the coequalizer captures the points where $x^2 = x$, i.e., $\{0, 1\}$.

Path Coequalizer

The path coequalizer is a higher inductive type that generalizes the coequalizer to handle pairs of paths in B . Given a map $p : A \rightarrow (b_1, b_2 : B) \times (\text{Path}_B(b_1, b_2)) \times (\text{Path}_B(b_1, b_2))$, it constructs a type where elements of A generate pairs of paths between points in B , with paths connecting the endpoints of these paths.

Definition 117. (Formation). For types $A, B : \mathcal{U}$ and a map $p : A \rightarrow (b_1, b_2 : B) \times (b_1 \equiv b_2) \times (b_1 \equiv b_2)$, there exists a path coequalizer $\text{Coeq}_{\equiv}(A, B, p) : \mathcal{U}$.

Definition 118. (Constructors). The path coequalizer is generated by the following higher inductive compositional structure:

$$\text{Coeq}_{\equiv} := \begin{cases} \text{inP} : B \rightarrow \text{Coeq}_{\equiv}(A, B, p) \\ \text{glueP} : (a : A) \rightarrow \text{inP}(p(a).2.2.1 @ 0) \equiv \text{inP}(p(a).2.2.2 @ 1) \end{cases}$$

```
data Coeq≡ (A B : U) (p : A → Σ (b1 b2 : B), b1 ≡ b2 × b1 ≡ b2)
  = inP (b : B)
  | glueP (a : A) <i> [(i=0) → inP ((p a).2.2.1 @ 0),
                     (i=1) → inP ((p a).2.2.2 @ 1)]
```

Theorem 69. (Elimination). For a type $C : \mathcal{U}$, map $h : B \rightarrow C$, and a family of paths $y : (a : A) \rightarrow h(p(a).2.2.1 @ 0) \equiv h(p(a).2.2.2 @ 1)$, there exists a map $\text{Ind-Coeq}_{\equiv} : \text{Coeq}_{\equiv}(A, B, p) \rightarrow C$, such that:

$$\begin{cases} \text{coeqPRec}(\text{inP}(b)) = h(b) \\ \text{coeqPRec}(\text{glueP}(a, i)) = y(a, i) \end{cases}$$

```
def Ind-Coeq≡ (A B C : U)
  (p : A → Σ (b1 b2 : B) (x : Path B b1 b2), Path B b1 b2)
  (h : B → C) (y : (a : A) → Path C (h ((p a).2.2.1 @ 0)) (h ((p a).2.2.2 @ 1)))
  : (z : coeqP A B p) → C
:= split { inP b → h b | glueP a @ i → y a @ i }
```

Theorem 70. (Computation). For $z : \text{coeqP } ABp$,

$$\begin{cases} \text{coeqPRec}(\text{inP } b) \equiv h(b) \\ \text{coeqPRec}(\text{glueP } a @ i) \equiv y(a) @ i \end{cases}$$

Theorem 71. (Uniqueness). Any two maps $h_1, h_2 : \text{coeqP } ABp \rightarrow C$ are homotopic if they agree on inP and glueP , i.e., if $h_1(\text{inP } b) = h_2(\text{inP } b)$ for all $b : B$ and $h_1(\text{glueP } a) = h_2(\text{glueP } a)$ for all $a : A$.

8.12 K(G,n)

Eilenberg-MacLane spaces $K(G, n)$ have a single non-trivial homotopy group $\pi_n(K(G, n)) = G$. They are defined using truncations and suspensions.

Definition 119. ($K(G, n)$) For an abelian group $G : \text{abgroup}$, the type $KGn(G) : \text{nat} \rightarrow \mathcal{U}$.

$$K(G, n) := \begin{cases} n = 0 \rightsquigarrow \text{discreteTopology}(G) \\ n \geq 1 \rightsquigarrow \|\Sigma^{n-1}(K1'(G.1, G.2.1))\|_n \end{cases}$$

```
def KGn (G: abgroup) : N -> U
:= split { zero -> discreteTopology G
          | succ n -> nTrunc (Σ (K1' (G.1, G.2.1)) n) (succ n)
          }
```

Theorem 72. (Elimination KGn) For $n \geq 1$, a type $B : \mathcal{U}$ with $\text{isNGroupoid}(B, \text{succ } n)$, and a map $f : \text{suspension}(K1'G) \rightarrow B$, there exists $\text{rec}_{KGn} : KGnG(\text{succ } n) \rightarrow B$, defined via nTruncRec .

8.13 Localization

Localization constructs an F -local type from a type X , with respect to a family of maps $F_A : S(a) \rightarrow T(a)$.

Definition 120. (Localization Modality) For a family of maps $F_A : S(a) \rightarrow T(a)$, the F -localization $L_F^{AST}(X) : \mathcal{U}$.

$$L_F^A(X) := \begin{cases} \text{center} : X \rightarrow L_{F_A}(X) \\ \text{ext}(a : A) \rightarrow (S(a) \rightarrow L_{F_A}(X)) : T(a) \rightarrow L_{F_A}(X) \\ \text{isExt}(a : A)(f : S(a) \rightarrow L_{F_A}(X)) \rightarrow (s : S(a)) : \text{ext}(a, f, F(a, s)) \equiv f(s) \\ \text{extEq}(a : A)(g, h : T(a) \rightarrow L_{F_A}(X)) \\ \quad (p : (s : S(a)) \rightarrow g(F(a, s)) \equiv h(F(a, s))) \\ \quad (t : T(a)) : g(t) \equiv h(t) \\ \text{isExtEq} : (a : A)(g, h : T(a) \rightarrow L_{F_A}(X)) \\ \quad (p : (s : S(a)) \rightarrow g(F(a, s)) \equiv h(F(a, s))) \\ \quad (s : S(a)) : \text{extEq}(a, g, h, p, F(a, s)) \equiv p(s) \end{cases}$$

```
data Localize (A X: U) (S T: A → U) (F : (x:A) → S x → T x)
= center (x: X)
| ext (a: A) (f: S a → Localize A X S T F) (t: T a)
| isExt (a: A) (f: S a → Localize A X S T F) (s: S a) <i>
  [ (i=0) → ext a f (F a s) , (i=1) → f s ]
| extEq (a: A) (g h: T a → Localize A X S T F)
  (p: (s : S a) → Path (Localize A X S T F) (g (F a s)) (h (F a s)))
  (t : T a) <i> [ (i=0) → g t , (i=1) → h t ]
| isExtEq (a: A) (g h : T a → Localize A X S T F)
  (p: (s : S a) → Path (T a → Localize A X S T F) (g (F a s)) (h (F a s)))
  (s : S a) <i> [ (i=0) → extEq a g h p (F a s) , (i=1) → p s ]
```

Theorem 73. (Localization Induction) For any $P : \Pi_{X:U} L_{F_A}(X) \rightarrow U$ with $\{n, r, s\}$, satisfying coherence conditions, there exists $i : \Pi_{x:L_{F_A}(X)} P(x)$, such that $i \cdot \text{center}_X = n$.

Conclusion

HITs directly encode CW-complexes in HoTT, bridging topology and type theory. They enable the analysis and manipulation of homotopical types.

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Issue V: Modalities and Identity Systems

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Анотація

This article explores the interplay between modalities, identity systems, and homologies in the framework of Homotopy Type Theory (HoTT). We formalize modalities and identity systems as structures within $(\infty,1)$ -categories and investigate the homological properties arising when their functor compositions are treated as groups. Special attention is given to topological structures, such as the Möbius strip, that emerge from non-trivial compositions, and their role in generating non-trivial fundamental groups. A classification of generators is provided, highlighting their categorical and homotopical properties.

9 Modalities and Identity Systems

Homotopy Type Theory (HoTT) provides a powerful framework for studying categorical structures through the lens of types, paths, and higher homotopies. In this context, *modalities* and *identity systems* serve as fundamental constructs that encode localization and identification properties, respectively. When compositions of their associated functors are interpreted as groups, they give rise to homological structures, such as fundamental groups, that can model complex topological spaces like the Möbius strip. This article formalizes these concepts and explores their implications in $(\infty,1)$ -toposes, with a focus on the emergence of CW-complexes and homologies.

9.1 Modality

Definition 121 (Modality). A modality in HoTT is a structure comprising:

```
def Modality :=
  Σ (modality : U → U)
    (isModal : U → U)
    (eta : Π (A : U), A → modality A)
    (elim : Π (A : U) (B : modality A → U)
      (B-Modal : Π (x : modality A), isModal (B x))
      (f : Π (x : A), (B (eta A x))),
      (Π (x : modality A), B x))
    (elim-β : Π (A : U) (B : modality A → U)
      (B-Modal : Π (x : modality A), isModal (B x))
      (f : Π (x : A), (B (eta A x))) (a : A),
      PathP (<_>B (eta A a)) (elim A B B-Modal f (eta A a)) (f a))
    (modalityIsModal : Π (A : U), isModal (modality A))
    (propIsModal : Π (A : U), Π (a b : isModal A),
      PathP (<_>isModal A) a b)
    (=Modal : Π (A : U) (x y : modality A),
      isModal (PathP (<_>modality A) x y)), 1
```

where \mathcal{U} is a universe of types, η is a natural inclusion, and `elim` provides a universal property for modal types (see [1] for details).

Modalities act as localization functors, projecting types onto subcategories of modal types. For instance, the *discrete modality* (\flat) trivializes higher homotopies, while the *codiscrete modality* (\sharp) makes types contractible.

9.2 Identity Systems

Definition 122 (Identity System). For a type $A : \mathcal{U}$, an identity system is defined as:

```
def IdentitySystem (A : U) : U :=
  Σ (=-form : A → A → U)
    (=-ctor : Π (a : A), =-form a a)
    (=-elim : Π (a : A) (C: Π (x y : A)
      (p : =-form x y), U)
      (d : C a a (=-ctor a)) (y : A)
      (p : =-form a y), C a y p)
    (=-comp : Π (a : A) (C: Π (x y : A)
      (p : =-form x y), U)
      (d : C a a (=-ctor a)),
      Path (C a a (=-ctor a)) d
      (=-elim a C d a (=-ctor a))), 1
```

where $=$ -form generalizes the identity type, and $=$ -ctor ensures reflexivity.

Identity systems generalize paths in HoTT, allowing the construction of types with non-trivial fundamental groups, such as the Möbius strip, where identifications generate \mathbb{Z} .

9.3 Classification of Generators

The following table classifies key generators, including modalities and identity systems, based on their categorical and homotopical properties.

Табл. 1: Classification of Generators in Homotopy Type Theory

Generator	Notation	Type	Adjunction
Discrete	\flat	Modality	$\flat \dashv \sharp$
Codiscrete	\sharp	Comodality	$\flat \dashv \sharp$
Bosonic	\bigcirc	Modality	$\bigcirc \dashv \bigcirc^+$
Fermionic/Infinitesimal	\Im	Modality	$\Im \dashv \Im^+$
Rheonomic	Rh	Modality	—
Reduced	R	Modality	—
Polynomial	W	Inductive	—
Polynomial	M	Coinductive	—
Higher Inductive	HIT	Inductive	$\mathsf{HIT} \dashv \mathsf{Path}$
Higher Coinductive	CoHIT	Coinductive	$\mathsf{Path} \dashv \mathsf{CoHIT}$
Path Spaces	Path	Identification	$\mathsf{HIT} \dashv \Im$
Identity	$=, \simeq, \cong$	Identification	—
Isomorphism	$=, \simeq, \cong$	Identification	—
Equality	$=, \simeq, \cong$	Identification	—

9.4 Homologies from Functor Compositions

When functor compositions of modalities and identity systems are treated as groups, they generate homological structures, such as fundamental groups or homology groups. For example, consider the composition $\flat \circ \sharp \circ \flat$. In a topological context, this may correspond to a localization that preserves certain homotopical features, potentially yielding a CW-complex like the Möbius strip.

Theorem 74. Let \mathcal{C} be an $(\infty, 1)$ -topos, and let $F = \flat \circ \sharp \circ \flat$ be a functor composition treated as a group action. The resulting structure induces a fundamental group isomorphic to \mathbb{Z} for types modeling the Möbius strip.

Sketch. The Möbius strip can be constructed as a higher inductive type (HIT) with an identity system generating \mathbb{Z} . The functor \flat discretizes the type, \sharp contracts it, and the second \flat reintroduces discrete structure, preserving the non-trivial loop in the identification system. The resulting type has a fundamental group $\pi_1 \cong \mathbb{Z}$. \square

9.5 Topological Interpretation

The Möbius strip, as a CW-complex, arises naturally in this framework. Its non-trivial fundamental group is generated by an identity system, while modalities like \Im or \bigcirc introduce twisting or orientation properties. This connects to topological quantum field theories (TQFTs), where surfaces like the Möbius strip encode non-trivial symmetries.

9.6 Conclusion

Modalities and identity systems in HoTT provide a rich framework for modeling categorical and topological phenomena. By treating functor compositions as groups, we uncover homological structures that bridge type theory and topology. Future work may explore applications in TQFT and synthetic differential geometry.

Література

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