

# Issue XL: Modal Homotopy Type Theory

Максим Сохацький<sup>1</sup>

<sup>1</sup> Національний технічний університет України  
Київський політехнічний інститут імені Ігоря Сікорського  
15 травня 2025 р.

## Анотація

Formal definition of Cohesive Topos.

**Keywords:** Differential Geometry, Topos Theory, Modal HoTT

## Зміст

### 1 Modal Homotopy Type Theory

#### 1.1 Preliminaries

A **category**  $\mathcal{C}$  consists of:

- A class of **objects**,  $\text{Ob}(\mathcal{C})$ ,
- A class of **morphisms**,  $\text{Hom}_{\mathcal{C}}(X, Y)$ , for each pair  $X, Y \in \text{Ob}(\mathcal{C})$ ,
- Composition maps  $\circ : \text{Hom}(Y, Z) \times \text{Hom}(X, Y) \rightarrow \text{Hom}(X, Z)$ ,
- Identity morphisms  $\text{id}_X \in \text{Hom}(X, X)$  for each  $X$ ,

satisfying associativity and identity laws.

A **functor**  $F : \mathcal{C} \rightarrow \mathcal{D}$  assigns to each:

- Object  $X \in \mathcal{C}$  an object  $F(X) \in \mathcal{D}$ ,
- Morphism  $f : X \rightarrow Y$  a morphism  $F(f) : F(X) \rightarrow F(Y)$ ,

such that  $F(\text{id}_X) = \text{id}_{F(X)}$  and  $F(g \circ f) = F(g) \circ F(f)$ .

A **natural transformation**  $\eta : F \Rightarrow G$  between functors  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  consists of morphisms  $\eta_X : F(X) \rightarrow G(X)$  such that for every  $f : X \rightarrow Y$  in  $\mathcal{C}$ ,

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ F(f) \downarrow & & \downarrow G(f) \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

commutes.

An **adjunction** between categories  $\mathcal{C}$  and  $\mathcal{D}$  consists of functors

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

and natural transformations (unit  $\eta$  and counit  $\varepsilon$ )

$$\eta : \text{Id}_{\mathcal{C}} \Rightarrow G \circ F, \quad \varepsilon : F \circ G \Rightarrow \text{Id}_{\mathcal{D}}$$

satisfying the triangle identities.

## 1.2 Topos

A **topos**  $\mathcal{E}$  is a category that:

- Has all finite limits and colimits,
- Is Cartesian closed: has exponential objects  $[X, Y]$ ,
- Has a subobject classifier  $\Omega$ .

## 1.3 Geometric Morphism

A **geometric morphism**  $f : \mathcal{E} \rightarrow \mathcal{F}$  between topoi consists of an adjoint pair

$$f^* : \mathcal{F} \rightleftarrows \mathcal{E} : f_*$$

with  $f^* \dashv f_*$ , where  $f^*$  preserves finite limits (i.e., is left exact).

## 1.4 Cohesive Topos

A **cohesive topos** is a topos  $\mathcal{E}$  equipped with a quadruple of adjoint functors:

$$\Pi \dashv \Delta \dashv \Gamma \dashv \nabla : \mathcal{E} \rightleftarrows \mathbf{Set}$$

such that:

- $\Gamma$  is the global sections functor,
- $\Delta$  is the constant sheaf functor,
- $\nabla$  sends a set to a codiscrete object,
- $\Pi$  is the shape or fundamental groupoid functor,
- $\Delta$  and  $\nabla$  are fully faithful,
- $\Delta$  preserves finite limits,
- $\Pi$  preserves finite products (in some variants).

## 1.5 Cohesive Adjunction Diagram and Modalities

$$\begin{array}{ccc} \mathcal{E} & \begin{array}{c} \xrightarrow{\Pi} \\ \xleftarrow{\Gamma} \\ \xleftarrow{\Delta} \\ \xrightarrow{\nabla} \end{array} & \mathbf{Set} \end{array}$$
  

$$\begin{array}{ccc} & \downarrow \text{J} & \\ \mathcal{E} & \begin{array}{c} \xrightarrow{\quad} \\ \parallel \\ \downarrow \text{b} \\ \xrightarrow{\quad} \end{array} & \mathcal{E} \end{array}$$

## 1.6 Cohesive Modalities

The above adjoint quadruple canonically induces a triple of endofunctors on  $\mathcal{E}$ :

$$(\int \dashv \flat \dashv \sharp) : \mathcal{E} \rightarrow \mathcal{E}$$

defined as follows:

$$\begin{aligned}\int &:= \Delta \circ \Pi \\ \flat &:= \Delta \circ \Gamma \\ \sharp &:= \nabla \circ \Gamma\end{aligned}$$

This yields an **adjoint triple** of endofunctors on  $\mathcal{E}$ :

$$\int \dashv \flat \dashv \sharp$$

These are:

- $\int$  — the **shape modality**: captures the fundamental shape or homotopy type,
- $\flat$  — the **flat modality**: forgets cohesive structure while remembering discrete shape,
- $\sharp$  — the **sharp modality**: codiscretizes the structure, reflecting the full cohesion.

Each of these is an **idempotent** (co)monad, hence a *modality* in the internal language (type theory) of  $\mathcal{E}$ .

## 1.7 Differential Cohesion

A **differential cohesive topos** is a cohesive topos  $\mathcal{E}$  equipped with an additional adjoint triple of endofunctors:

$$(\mathfrak{R} \dashv \mathfrak{J} \dashv \&) : \mathcal{E} \rightarrow \mathcal{E}$$

These are:

- $\mathfrak{R}$ : the **reduction modality** — forgets nilpotents,
- $\mathfrak{J}$ : the **infinitesimal shape modality** — retains infinitesimal data,
- $\&$ : the **infinitesimal flat modality** — reflects formally smooth structure.

Important object classes:

- An object  $X$  is **reduced** if  $\mathfrak{R}(X) \cong X$ .
- It is **coreduced** if  $\&(X) \cong X$ .
- It is **formally smooth** if the unit map  $X \rightarrow \&X$  is an effective epimorphism.

**Formally étale maps** are those morphisms  $f : X \rightarrow Y$  such that the square

$$\begin{array}{ccc} X & \longrightarrow & \mathfrak{J}X \\ f \downarrow & & \downarrow \mathfrak{J}(f) \\ Y & \longrightarrow & \mathfrak{J}Y \end{array}$$

is a pullback.

## 1.8 Graded Differential Cohesion

In **graded differential cohesion**, such as used in synthetic supergeometry, one introduces an adjoint triple:

$$10) \Rightarrow \dashv \rightsquigarrow \dashv \mathbf{Rh}$$

$$(\Rightarrow \dashv \rightsquigarrow \dashv \mathbf{Rh}) : \mathcal{E} \rightarrow \mathcal{E}$$

These are:

- $\Rightarrow$ : the **fermionic modality** — captures anti-commuting directions,
- $\rightsquigarrow$ : the **bosonic modality** — filters out fermionic directions,
- $\mathbf{Rh}$ : the **rheonomic modality** — encodes constraint structures.

These modal operators form part of the internal logic of supergeometric or supersymmetric type theories.

## 1.9 Adjoint String of Identity Modalities

In Homotopy Type Theory (HoTT), identity systems (Contractible, Strict Id, Quotient, Isomorphism, Path = Equivalence) are modeled as modalities in the  $\infty$ -topos  $\mathcal{E} = \infty\text{Grp}$ . We construct an adjoint quadruple extending the Jacobs-Lawvere triple  $C \dashv \text{Id}_A \dashv Q(-/\sim)$ , incorporating Isomorphism and Path = Equivalence. The modalities are ordered by adjointness:  $\text{Contractible} \leq \text{Strict Id} \leq \text{Quotient} \leq \text{Isomorphism} \leq \text{Path} = \text{Equivalence}$ , reflecting their structure in HoTT, where Strict Id, Quotient, and Path = Equivalence are mere propositions for h-sets, while Isomorphism is not.

Homotopy Type Theory (HoTT) provides a framework for reasoning about equality via the identity type  $\text{Id}_A(x, y)$ . In the  $\infty$ -topos  $\mathcal{E} = \infty\text{Grp}$ , identity systems are modalities (monads), ordered by adjointness. The classical Jacobs-Lawvere adjunction triple  $C \dashv \text{Id}_A \dashv Q(-/\sim)$  captures **Contractible**, **Strict**, and **Quotient**. We extend this to a quadruple, including **Isomorphism** and **Path = Equiv**, respecting the HoTT equivalence of Path and Equivalence and the propositional nature of Strict Id, Quotient, and Path = Equivalence for h-sets.

**Definition 1.** In HoTT, the identity systems are:

- **Contractible:**  $(-1)$ -truncated types, mere propositions.
- **Strict:**  $\text{Id}_A(x, y)$  for h-sets (0-truncated), a mere proposition.
- **Quotient:** Set-quotients  $A/\sim$ , 0-truncated, equivalent to Strict Id.
- **Isomorphism:**  $\text{Iso}_A(x, y)$ , a triple  $(f, g, p)$ , not a mere proposition.
- **Path = Equiv:**  $\text{Path}_A(x, y) \simeq (x \simeq y)$ , equivalent in HoTT.

In  $\mathcal{E} = \infty\text{Grp}$ , we define categories:

- $\mathcal{E}_{\text{contr}} = \mathcal{E}_{\leq -1}$ : Mere propositions.
- $\mathcal{E}_{\text{strict}} = \mathcal{E}_{\leq 0} \cong \text{Set}$ : h-sets (Strict Id).
- $\mathcal{E}_{\text{quot}} = \mathcal{E}_{\leq 0} \cong \text{Set}$ : h-sets (Quotient).
- $\mathcal{E}_{\text{iso}} \cong \mathcal{E}$ :  $\infty$ -groupoids with isomorphisms.
- $\mathcal{E}_{\text{path/equiv}} \cong \mathcal{E}$ :  $\infty$ -groupoids with paths/equivalences.

The Jacobs-Lawvere triple  $\mathcal{C} \dashv \text{Id}_A \dashv Q(-/\sim)$  is extended to an adjoint quadruple:

$$\mathcal{E}_{\text{contr}} \xrightarrow{F_4} \mathcal{E}_{\text{strict}} \xrightarrow{F_3} \mathcal{E}_{\text{quot}} \xrightarrow{F_2} \mathcal{E}_{\text{iso}} \xrightarrow{F_1} \mathcal{E}_{\text{path/equiv}}$$

**Theorem 1.** The functors form an adjoint quadruple with adjunctions:

$$F_4 \dashv U_4, \quad F_3 \dashv U_3, \quad F_2 \dashv U_2, \quad F_1 \dashv U_1$$

- $F_4 : \mathcal{E}_{\text{contr}} \rightarrow \mathcal{E}_{\text{strict}}$ : Inclusion of  $(-1)$ -truncated objects into  $0$ -truncated objects. Right adjoint  $U_4$ :  $(-1)$ -truncation,  $U_4(X) = \|X\|_{-1}$ .
- $F_3 : \mathcal{E}_{\text{strict}} \rightarrow \mathcal{E}_{\text{quot}}$ : Canonical map to quotient structure, viewing h-sets as quotiented by trivial relations. Right adjoint  $U_3$ : Inverse map preserving h-set structure.
- $F_2 : \mathcal{E}_{\text{quot}} \rightarrow \mathcal{E}_{\text{iso}}$ : Inclusion of h-sets into  $\mathcal{E}$ ,  $\text{core}(X) \cong X$ . Right adjoint  $U_2$ :  $0$ -truncation,  $U_2(X) = \|X\|_0$ .
- $F_1 : \mathcal{E}_{\text{iso}} \rightarrow \mathcal{E}_{\text{path/equiv}}$ : Canonical inclusion of  $\infty$ -groupoids with isomorphisms into full  $\infty$ -groupoids with paths/equivalences. Right adjoint  $U_1$ : Core map, preserving isomorphism structure.

The adjunctions induce the ordering:

$$\text{Contractible} \leq \text{Strict Id} \leq \text{Quotient} \leq \text{Isomorphism} \leq \text{Path} = \text{Equivalence}$$

- **Contractible**: Coarsest, mere propositions ( $(-1)$ -truncated).
- **Strict**: h-sets,  $\text{Id}_A(x, y)$  is a mere proposition.
- **Quotient**: Equivalent to Strict Id,  $0$ -truncated set-quotients.
- **Isomorphism**:  $\text{Iso}_A(x, y)$  is not a mere proposition for general types.
- **Path = Equivalence**: Finest, full  $\infty$ -groupoid structure, equivalent via univalence.

The adjoint quadruple extends the Jacobs-Lawvere triple, capturing the structure of identity systems in HoTT. The ordering reflects their increasing complexity, with Strict Id, Quotient, and Path = Equivalence collapsing to mere propositions for h-sets, while Isomorphism retains higher structure. Future work could explore these adjunctions in other  $\infty$ -topoi or specific CTT models.