Monads and Descent

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Abstract

Using category theory, we interpret descent data to determine, in very general settings, whether a morphism is a descent morphism or an effective descent morphism.

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1 Chevalley Bifibrations and Descent

Let $\mathrm{P} : \mathbf{M} \to \mathbf{A}$ denote a bifibrant functor [1]. For an object $A \in \mathbf{A}$, let $\mathbf{M}(A)$ denote the fibre over A. We assume that $\mathbf{A}$ has fibred products.

1.1 Monad Associated with an Arrow

Let $a : A_1 \to A_0$ be an arrow in $\mathbf{A}$. Denote by

$$a^* : \mathbf{M}(A_0) \to \mathbf{M}(A_1) \quad [\text{resp. } a_* : \mathbf{M}(A_1) \to \mathbf{M}(A_0)]$$

the inverse image functor (resp. direct image functor), and

$$\eta^a : \mathrm{Id}_{\mathbf{M}(A_1)} \to a^* a_*; \quad \varepsilon^a : a_* a^* \to \mathrm{Id}_{\mathbf{M}(A_0)}$$

the canonical natural transformations making a_* a left adjoint to a^*. This adjunction defines [2] on $\mathbf{M}(A_1)$ the monad $\mathbf{T}^a = (T^a, \mu^a, \eta^a)$, where

$$T^a = a^* a_* : \mathbf{M}(A_1) \to \mathbf{M}(A_1), \quad \mu^a = a^* \varepsilon^a a_* : T^a \circ T^a \to T^a.$$

Let $\mathbf{M}^a$ denote the category $\mathbf{M}(A_1)^{(\mathbf{T}^a)}$ of algebras over the monad $\mathbf{T}^a$, and let

$$\mathrm{U}^{\mathbf{T}^a} : \mathbf{M}^a \to \mathbf{M}(A_1), \quad \Phi^a : \mathbf{M}(A_0) \to \mathbf{M}^a$$

be the canonical functors.

1.2 Chevalley Property

Definition 1.1. *The functor* P *is a* Chevalley functor *if it satisfies the following property (C):*

(C) For every commutative diagram in $\mathbf{M}$

$$\begin{tikzcd} M_1 \arrow[r, "k_1"] \arrow[d, "\gamma"'] & M_2 \arrow[d, "\gamma'"] \\ M_3 \arrow[r, "k_0"'] & M_4 \end{tikzcd}$$

whose image under P *is a cartesian square in* $\mathbf{A}$*, if* γ *and* γ' *are cartesian and* k_0 *is cocartesian, then* k_1 *is cocartesian.*

1.3 Characterization of Descent Data

Assume henceforth that $\mathrm{P} : \mathbf{M} \to \mathbf{A}$ is a Chevalley functor. Let $a : A_1 \to A_0$ be an arrow in $\mathbf{A}$. Let A_2 be the fibred product $A_1 \times_{A_0} A_1$, with canonical projections $a_1, a_2 : A_2 \to A_1$. The property (C) defines, for every object $M_1 \in \mathbf{M}(A_1)$, a canonical bijection, natural in M_1,

$$\mathrm{Hom}_{\mathbf{M}(A_2)}(a_1^*(M_1), a_2^*(M_1)) \to \mathrm{Hom}_{\mathbf{M}(A_1)}(T^a(M_1), M_1),$$

denoted $\varphi \mapsto \mathrm{K}^a(\varphi)$.

Lemma 1.2. *An arrow* $\varphi : a_1^*(M_1) \to a_2^*(M_1)$ *such that* $\mathrm{P}(\varphi) = \mathrm{id}_{A_2}$ *is a descent datum if and only if* $\mathrm{K}^a(\varphi)$ *is an algebra over the monad* $\mathbf{T}^a$.

Let $\mathrm{D}(a)$ denote the category of descent data relative to a, and let

$$\Psi^a : \mathbf{M}(A_0) \to \mathrm{D}(a), \quad \mathrm{U}^a : \mathrm{D}(a) \to \mathbf{M}(A_1)$$

be the canonical functors.

Theorem 1.3. *The correspondence* $\varphi \mapsto \mathrm{K}^a(\varphi)$ *induces an equivalence of categories* $\mathrm{K}^a : \mathrm{D}(a) \to \mathbf{M}^a$*, making the following diagram commute:*

$$\begin{tikzcd} \mathbf{M}(A_0) \arrow[r, "\Psi^a"] & \mathrm{D}(a) \arrow[r, "\mathrm{K}^a"] \arrow[d, "\mathrm{U}^a"'] & \mathbf{M}^a \arrow[dl, "\mathrm{U}^{\mathbf{T}^a}"] \\ & \mathbf{M}(A_1) & \end{tikzcd}$$

Proposition 1.4. *The correspondence $\varphi \mapsto \mathrm{K}^a(\varphi)$ is universal. Precisely, for an arrow $b_0 : A'_0 \to A_0$ in $\mathbf{A}$, consider the change-of-base diagram in $\mathbf{A}$:*

$$\begin{tikzcd}
A'_2 \arrow[r, "a'_1\ a'_2"] & A'_1 \arrow[r, "b_2 \qquad\qquad b_1"] \arrow[dl, "a'"] & A_2 \arrow[r, "a_1\ a_2"] & A_1 \arrow[dl, "a"] \\
A'_0 \arrow[rr, "b_0"'] & & A_0 &
\end{tikzcd}$$

For $M_1 \in \mathbf{M}(A_1)$ and $\varphi : a_1^*(M_1) \to a_2^*(M_1)$ in $\mathbf{M}(A_2)$,

$$\mathrm{K}^{a'}(b_2^*(\varphi)) = b_1^*(\mathrm{K}^a(\varphi)).$$

In particular, taking $A'_0 = A_1$ and $b_0 = a$, if φ is a descent datum, then $b_2^*(\varphi)$ is an effective descent datum. The converse holds, yielding:

Corollary 1.5. *An arrow $\varphi : a_1^*(M_1) \to a_2^*(M_1) \in \mathbf{M}(A_2)$ is a descent datum if and only if its inverse image $b_2^*(\varphi)$ under the canonical change of base $b_0 = a : A'_0 = A_1 \to A_0$ is an effective descent datum.*

This eliminates the need for the "cocycle condition" in subsequent arguments.

2 First Applications

Using Theorem 1.3, Beck's criterion [2] provides necessary and sufficient conditions for Ψ^a to be faithful, fully faithful, or an equivalence of categories, in terms of commutation and reflection of certain cokernels by a^*.

Proposition 2.1. *If cokernels of pairs of arrows exist in $\mathbf{M}(A_0)$, then Ψ^a has a left adjoint.*

Proposition 2.2. *The functor Ψ^a is faithful if and only if a^* is faithful.*

Proposition 2.3. *If a^* reflects cokernels, then Ψ^a is fully faithful. In particular, if all fibres of $\mathbf{M}$ are abelian, then*

$$\Psi^a \textit{ faithful} \iff \Psi^a \textit{ fully faithful} \iff a^* \textit{ faithful.}$$

Definition 2.4. *An arrow $a : A_1 \to A_0$ is* faithfully flat *if a^* commutes with cokernels and reflects isomorphisms.*

Proposition 2.5. *If $a : A_1 \to A_0$ is faithfully flat and cokernels exist in $\mathbf{M}(A_0)$, then Ψ^a is an equivalence of categories.*

3 First Examples of Chevalley Functors

1. If $\mathbf{A}$ is the dual of the category of commutative rings and $\mathbf{M}$ is the dual of the category of modules over varying commutative rings, the obvious functor $\mathrm{P} : \mathbf{M} \to \mathbf{A}$ is Chevalley.

2. If $\mathbf{A}$ is a category with fibred products and $\mathbf{M} = \mathbf{Fl}(\mathbf{A})$ is the category of arrows in $\mathbf{A}$, the "target" functor $\mathrm{P} : \mathbf{M} \to \mathbf{A}$ is Chevalley.

3. If $\mathrm{P} : \mathbf{M} \to \mathbf{A}$ and $\mathrm{Q} : \mathbf{N} \to \mathbf{M}$ are Chevalley, their composite $\mathrm{P} \circ \mathrm{Q}$ is Chevalley.

4. If $\mathrm{P} : \mathbf{M} \to \mathbf{A}$ is Chevalley and $\mathbf{I}$ is any category, the functor $\mathrm{P}^{\mathbf{I}} : \mathbf{M}^{\mathbf{I}} \to \mathbf{A}^{\mathbf{I}}$ is Chevalley.

5. In a cartesian diagram of categories

$$\begin{tikzcd} \mathbf{X}' \arrow[r, "f^*"] \arrow[d, "g"'] & \mathbf{M} \arrow[d, "\mathrm{P}"] \\ \mathbf{X} \arrow[r, "f"'] & \mathbf{A} \end{tikzcd}$$

   if $\mathbf{X}$ has fibred products, f preserves fibred products, and P is Chevalley, then $f^*(\mathrm{P})$ is Chevalley.

In a future publication, we will provide further examples of Chevalley categories and more precise criteria for determining whether Ψ^a is faithful, fully faithful, or an equivalence when the fibres of $\mathbf{M}$ are algebraic categories (e.g., categories of modules).

References

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[2] F. E. J. Linton, Applied functorial semantics II, *Springer Lecture Notes in Mathematics*, No. 80, 1969.

[3] C. Chevalley, Séminaire sur la descente, 1964–1965 (unpublished).