First of all this is based on the book Using Mathematica for quantum mechanics by Roman Schmied, this is simply exercise 2.1

We describe a spin-1/2 system in the basis $\mathcal{B}$ containing the two states

$$|\Uparrow_{\vartheta,\varphi}\rangle = \cos\left(\frac{\vartheta}{2}\right)|\uparrow\rangle + e^{i\varphi}\sin\left(\frac{\vartheta}{2}\right)|\downarrow\rangle$$

$$|\Downarrow_{\vartheta,\varphi}\rangle = -e^{-i\varphi}\sin\left(\frac{\vartheta}{2}\right)|\uparrow\rangle + \cos\left(\frac{\vartheta}{2}\right)|\downarrow\rangle \quad (2.8)$$

1. Show that the basis $\mathcal{B} = \{|\Uparrow_{\vartheta,\varphi}\rangle, |\Downarrow_{\vartheta,\varphi}\rangle\}$ is orthonormal.
2. Show that the basis $\mathcal{B}$ is complete: $|\Uparrow_{\vartheta,\varphi}\rangle\langle\Uparrow_{\vartheta,\varphi}| + |\Downarrow_{\vartheta,\varphi}\rangle\langle\Downarrow_{\vartheta,\varphi}| = \mathbb{1}$.
3. Express the states $|\uparrow\rangle$ and $|\downarrow\rangle$ as vectors in the basis $\mathcal{B}$.
4. Express the Pauli operators $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$ as matrices in the basis $\mathcal{B}$.
5. Show that $|\Uparrow_{\vartheta,\varphi}\rangle$ and $|\Downarrow_{\vartheta,\varphi}\rangle$ are eigenvectors of $\hat{\sigma}(\vartheta,\varphi) = \hat{\sigma}_x \sin(\vartheta)\cos(\varphi) + \hat{\sigma}_y \sin(\vartheta)\sin(\varphi) + \hat{\sigma}_z \cos(\vartheta)$. What are the eigenvalues?

We start by defining the up and down basis

```
In[ ]:= up = {Cos[ν / 2], Exp[I ϕ] Sin[ν / 2]}
```

$$\text{Out[ ]=}\ \left\{\text{Cos}\left[\frac{\nu}{2}\right], e^{i\,\phi}\,\text{Sin}\left[\frac{\nu}{2}\right]\right\}$$

```
In[ ]:= MatrixForm[%]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \text{Cos}\left[\frac{\nu}{2}\right] \\ e^{i\,\phi}\,\text{Sin}\left[\frac{\nu}{2}\right] \end{pmatrix}$$

```
In[ ]:= down = {-Exp[-I ϕ] Sin[ν / 2], Cos[ν / 2]}
        MatrixForm[down[ϕ, ν]]
```

$$\text{Out[ ]=}\ \left\{-e^{-i\,\phi}\,\text{Sin}\left[\frac{\nu}{2}\right], \text{Cos}\left[\frac{\nu}{2}\right]\right\}$$

Out[ ]//MatrixForm=

$$\left\{-e^{-i\,\phi}\,\text{Sin}\left[\frac{\nu}{2}\right], \text{Cos}\left[\frac{\nu}{2}\right]\right\}[\phi, \nu]$$

To shown that this basis is orthonormal, we just do the internal product and we expect $\langle i \mid j \rangle = \delta_{ij}$ if it is an orthonormal basis. Note that we do not need to transpose as the Dot function in Mathematica does it for us

```
In[ ]:=
ComplexExpand[up. Conjugate[down] ]
```

```
Out[ ]= 0
```

Or alternative and way better looking

In[ ]:=
```
up. Conjugate[down]  // ComplexExpand
```
Out[ ]= 0

In[ ]:=
```
up. Conjugate[up] // ComplexExpand
```
Out[ ]= $\text{Cos}\left[\frac{\nu}{2}\right]^2 + \text{Sin}\left[\frac{\nu}{2}\right]^2$

That's obviously one, here we are in need of the full simplify

In[ ]:=
```
up. Conjugate[up] // ComplexExpand // FullSimplify
```
Out[ ]= 1

In[ ]:=
```
down.Conjugate[down] // ComplexExpand // FullSimplify
```
Out[ ]= 1

To show that the basis is complete we need to use the Kronecker product , to construct matrices from ket-bras

In[ ]:=
```
upup = KroneckerProduct[up, Conjugate[up]] // ComplexExpand // FullSimplify
```
Out[ ]= $\left\{\left\{\text{Cos}\left[\frac{\nu}{2}\right]^2, \frac{1}{2} e^{-i\phi}\,\text{Sin}[\nu]\right\}, \left\{\frac{1}{2} e^{i\phi}\,\text{Sin}[\nu], \text{Sin}\left[\frac{\nu}{2}\right]^2\right\}\right\}$

In[ ]:=
```
downdown = KroneckerProduct[down, Conjugate[down]] // ComplexExpand // FullSimplify
```
Out[ ]= $\left\{\left\{\text{Sin}\left[\frac{\nu}{2}\right]^2, -\frac{1}{2} e^{-i\phi}\,\text{Sin}[\nu]\right\}, \left\{-\frac{1}{2} e^{i\phi}\,\text{Sin}[\nu], \text{Cos}\left[\frac{\nu}{2}\right]^2\right\}\right\}$

In[ ]:=
```
upup + downdown // FullSimplify
```
Out[ ]= {{1, 0}, {0, 1}}

Now we are asked to express the two vectors that compose our basis in our basis, to to this we simply use $A = \sum_{ij} A_{ij} \left|i\right\rangle \left\langle j\right|$, where $A_{ij} = \left\langle i\right| A \left|j\right\rangle$ then

In[ ]:=
```
newup = Conjugate[up[[1]]] up + Conjugate[ down[[1]]] down // ComplexExpand // FullSimplify
```
Out[ ]= {1, 0}

In[ ]:=
```
newdown =
  Conjugate[up[[2]]] up + Conjugate[down[[2]]] down // ComplexExpand // FullSimplify
```
Out[ ]= {0, 1}

For the next section we are asked to express the Pauli operators as matrices in this basis

```
In[•]:= σx = PauliMatrix[1]
        σy = PauliMatrix[2]
        σz = PauliMatrix[3]
Out[•]= {{0, 1}, {1, 0}}
Out[•]= {{0, -ⅈ}, {ⅈ, 0}}
Out[•]= {{1, 0}, {0, -1}}
```

To express it in our basis we use the fact that the elements of an operator in a basis are $A_{ij} = \langle i | A | j \rangle$

```
In[•]:= E11[A_] = Conjugate[up].A.up // ComplexExpand // FullSimplify;
        E12[A_] = Conjugate[up].A.down // ComplexExpand // FullSimplify;
        E21[A_] = Conjugate[down].A.up // ComplexExpand // FullSimplify;
        E22[A_] = Conjugate[down].A.down // ComplexExpand // FullSimplify;
        newpauli[A_] = {{E11[A], E12[A]}, {E21[A], E22[A]}};
```

We can check that they are the same operator using $A = \sum_{ij} A_{ij} |i\rangle \langle j|$, for example

```
In[•]:= σx = newpauli[σx] // ComplexExpand // FullSimplify;
        MatrixForm[σx]
```

Out[•]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\phi]\,\text{Sin}[\nu] & e^{-i\phi}\,(\text{Cos}[\nu]\,\text{Cos}[\phi] + i\,\text{Sin}[\phi]) \\ e^{i\phi}\,(\text{Cos}[\nu]\,\text{Cos}[\phi] - i\,\text{Sin}[\phi]) & -\text{Cos}[\phi]\,\text{Sin}[\nu] \end{pmatrix}$$

```
In[•]:= σy = newpauli[σy] // ComplexExpand // FullSimplify;
        MatrixForm[σy]
```

Out[•]//MatrixForm=

$$\begin{pmatrix} \text{Sin}[\nu]\,\text{Sin}[\phi] & e^{-i\phi}\,(-i\,\text{Cos}[\phi] + \text{Cos}[\nu]\,\text{Sin}[\phi]) \\ e^{i\phi}\,(i\,\text{Cos}[\phi] + \text{Cos}[\nu]\,\text{Sin}[\phi]) & -\text{Sin}[\nu]\,\text{Sin}[\phi] \end{pmatrix}$$

```
In[•]:= σz = newpauli[σz] // ComplexExpand // FullSimplify;
        MatrixForm[σz]
```

Out[•]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\nu] & -e^{-i\phi}\,\text{Sin}[\nu] \\ -e^{i\phi}\,\text{Sin}[\nu] & -\text{Cos}[\nu] \end{pmatrix}$$

Let us see if they are the same using $A = \sum_{ij} A_{ij} |i\rangle \langle j|$

```
In[•]:= Compare[x_] := x[[1, 1]] × KroneckerProduct[up, Conjugate[up]] +
          x[[1, 2]] × KroneckerProduct[up, Conjugate[down]] +
          x[[2, 1]] × KroneckerProduct[down, Conjugate[up]] +
          x[[2, 2]] × KroneckerProduct[down, Conjugate[down]];
```

In[∘]:= σ_x == Compare[σx] // ComplexExpand // FullSimplify

Out[∘]= True

In[∘]:= σ_y == Compare[σy] // ComplexExpand // FullSimplify

Out[∘]= True

In[∘]:= σ_z == Compare[σz] // ComplexExpand // FullSimplify

Out[∘]= True

For the Last part we simply write this new operator

In[∘]:= σ = σ_x Sin[ν] Cos[ϕ] + σ_y Sin[ν] Sin[ϕ] + σ_z Cos[ν]

Out[∘]= {{Cos[ν], Cos[ϕ] Sin[ν] - i Sin[ν] Sin[ϕ]}, {Cos[ϕ] Sin[ν] + i Sin[ν] Sin[ϕ], -Cos[ν]}}

In[∘]:= σ // Eigenvalues

Out[∘]= {-1, 1}

In[∘]:= σ.up // ComplexExpand // FullSimplify

Out[∘]= $\left\{\mathrm{Cos}\left[\frac{\nu}{2}\right], e^{i\phi}\,\mathrm{Sin}\left[\frac{\nu}{2}\right]\right\}$

In[∘]:= up == σ.up // ComplexExpand // FullSimplify

Out[∘]= True

In[∘]:= σ.down // ComplexExpand // FullSimplify

Out[∘]= $\left\{e^{-i\phi}\,\mathrm{Sin}\left[\frac{\nu}{2}\right], -\mathrm{Cos}\left[\frac{\nu}{2}\right]\right\}$

In[∘]:= down = σ.down // ComplexExpand // FullSimplify

Out[∘]= $\left\{e^{-i\phi}\,\mathrm{Sin}\left[\frac{\nu}{2}\right], -\mathrm{Cos}\left[\frac{\nu}{2}\right]\right\}$