Making all equalities equal

Guilherme Horta Alvares da Silva

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Abstract

1 Equalities

There are multiple ways of defining equalities in a theorem prover. In the next sections, they will be defined.

1.1 Imports

First, it will be necessary to give some agda arguments:

```
{-# OPTIONS --cubical --cumulativity #-}
module paper where
```

The cubical flag is necessary because we are using cubical equality, and the cumulativiy flag is also necessary for level subtyping,

```
open import Agda.Primitive.Cubical using (I; i0; i1)
```

This library loads Cubical Agda Primitives as the equality interval.

1.2 Martin-Löf Equality

At the begin of Agda and in most theorems proves, equality is given by Martin-Löf's definition:

```
module Martin-Löf {ℓ} {A : Set ℓ} where

  data _≡_ (x : A) : A → Set ℓ where
    refl : x ≡ x
```

This equality is very convenient in proof assistances like Agda because it is possible to pattern match using them:

```
private variable
  x y z : A

sym : x ≡ y → y ≡ x
sym refl = refl

trans : x ≡ y → y ≡ z → x ≡ z
trans refl refl = refl
```

But the problem of this equality is that it does not handle extensionality and other axioms very well.

```
module FunExt {ℓ ℓ'} {A : Set ℓ} {B : Set ℓ'} where
  open Martin-Löf

  funExt-Type = {f g : A → B}
    → ((x : A) → f x ≡ g x) → f ≡ g
```

1.3 Cubical Equality

To solve this problem, Agda adopted cubical type theory that equality is a function from the path to type:

```
module CubicalEquality {ℓ} {A : Set ℓ} where
  postulate
    PathP : (A : I → Set ℓ) → A i0 → A i1 → Set ℓ

  _≡_ : A → A → Set ℓ
  _≡_ = PathP λ _ → A
```

From this equality, I will define reflection, symmetry and extensionality:

```
module CubicalResults {ℓ ℓ'} {A : Set ℓ} {B : Set ℓ'} where
  open import Cubical.Core.Primitives

  private variable
    x y z : A

  refl : x ≡ x
  refl {x = x} = λ _ → x

  sym : x ≡ y → y ≡ x
  sym p i = p (˜ i)

  funExt : {f g : A → B}
    → ((x : A) → f x ≡ g x) → f ≡ g
  funExt p i x = p x i
```

The operator ∼ invert the interval. If the interval i goes from *i0* to *i1*, the interval ∼ i goes from *i1* to *i0*.

1.4 Leibniz equality

Leibniz equality is defined in this way: If a is equal to b, then for every propositional P, if P a, then P b. The main idea is that if both values are equal, then they are seen equal for every angle.

```
module LeibnizEquality {A : Set} where
  _≐_ : A → A → Set₁
  a ≐ b = (P : A → Set) → P a → P b
```

2 Joining all equalities

All equalities have something in common. They are all equal to each other. So it will be defined as a common record that all equalities should have. In the next definition, all equalities are equal to cubical equality:

```
open import Cubical.Foundations.Prelude
open import Cubical.Foundations.Isomorphism
open import Cubical.Foundations.Equiv
open import Cubical.Foundations.Univalence
open import Cubical.Foundations.Function
open import Cubical.Data.Equality

module _ {a ℓ} {A : Set a} where
  ≙-Type = A → A → Set ℓ
  private
    ℓ₁ = ℓ-max a ℓ

  private variable
    x y z : A

  record IsEquality (_≙_ : ≙-Type) : Set (ℓ-suc (ℓ-max a ℓ)) where
    constructor eq
    field
      ≙-≡-≡ : let
        x≡y : Type ℓ₁
        x≡y = x ≡ y

        x≙y : Type ℓ₁
        x≙y = x ≙ y

        in _≡_ {ℓ-suc ℓ₁} x≙y x≡y

    ≡-≡-≙ : let
      x≡y : Type (ℓ-max a ℓ)
```

```
      x≡y = x ≡ y
      x≜y = x ≜ y
      in x≡y ≡ x≜y
    ≡-≡-≜ = sym ≜-≡-≡

  module _ {_≜_ : ≜-Type} where
    sym-Equality : (≡-≡-≜ : {x y : A} → let
      x≡y : Type (ℓ-max a ℓ)
      x≡y = x ≡ y
      x≜y = x ≜ y
      in x≡y ≡ x≜y)
      → IsEquality _≜_
    sym-Equality ≡-≡-≜ = eq (sym ≡-≡-≜)

  record Equality : Set (ℓ-suc (ℓ-max a ℓ)) where
    constructor eqC
    field
      _≜_ : ≜-Type
      { isEquality } : IsEquality _≜_

  EqFromInstance : {≜ : ≜-Type} → IsEquality ≜ → Equality
  EqFromInstance inst = eqC _ { inst }

  eqsEqual : (_≜1_ _≜2_ : ≜-Type)
    { ≜1-eq : IsEquality _≜1_ }
    { ≜2-eq : IsEquality _≜2_ }
    → ∀ {x y} → let
      x≜1y : Type ℓ1
      x≜1y = x ≜1 y

      x≜2y : Type ℓ1
      x≜2y = x ≜2 y

      in _≡_ {ℓ-suc ℓ1} x≜1y x≜2y
  eqsEqual _ _ { eq ≜-≡-≡1 } { eq ≜-≡-≡2 } = ≜-≡-≡1 ∙ sym ≜-≡-≡2
```

It will be defined for each equality, its instance:

2.1 Cubical Equality

The simplest example is the cubical equality hence this equality is already equal to itself.

```
module _ {a} {A : Set a} where
  instance
    ≡-IsEquality : IsEquality {A = A} _≡_
    ≡-IsEquality = eq refl
  ≡-Equality : Equality {ℓ = a}
  ≡-Equality = eqC _≡_
```

2.2 Martin-Löf equality

The proof of Martin-Löf equality is more difficult, but it is already in Cubical library as p-c.

```
instance
  ≡p-IsEquality : IsEquality {A = A} _≡p_
  ≡p-IsEquality = sym-Equality p-c
≡p-Equality : Equality {ℓ = a}
≡p-Equality = eqC _≡p_
```

2.3 Isomorphism

The isomorphism is an equality between types.

```
module _ {ℓ} where
  univalencePath' : {A B : Type ℓ} → (A ≡ B) ≡ (A ≃ B)
  univalencePath' {A} {B} =
    ua {ℓ-suc ℓ} {A ≡ B} {A ≃ B} (compEquiv (univalence {ℓ} {A} {B})
    (isoToEquiv (iso {ℓ} {ℓ-suc ℓ}
    (λ x → x) (λ x → x) (λ b i → b) λ a i → a)))
```

univalencePath is already defined in Agda library, but with $A \simeq B$ instead of *Lifted (A ≃ B)*. This change can be done because of the cumulativity flag.

```
instance
  ≃-IsEquality : IsEquality
    {A = Type ℓ} _≃_
  ≃-IsEquality = sym-Equality univalencePath'
≃-Equality : Equality {ℓ = ℓ}
≃-Equality = eqC _≃_
```

2.4 Leibniz Equality

The hardest equality to proof that is equalty is the Leibniz Equality.

```
liftIso : ∀ {a b} {A : Type a} {B : Type b}
  → Iso {a} {b} A B → Iso {ℓ-max a b} {ℓ-max a b} A B
liftIso {a} f = iso fun inv
  (λ x i → rightInv x i) (λ x i → leftInv x i)
```

This liftIso will be used to lift the Isomorphism to types of the same maximum level of both.

```
where open Iso f
```

```
open import leibniz
open Leibniz
```

It is importing the definition of Leibniz equality made by [?]. In this work, there is already a proof of the isomorphism between the Leibniz and the Martin-Löf equality.

```
module FinalEquality {A : Set} where
  open MainResult A

  ≐≅≡ : ∀ {a b} → Iso (a ≐ b) (a ≡p b)
  ≐≅≡ = iso j i (ptoc ∘ ji) (ptoc ∘ ij)
```

In Cubical Library, the definition of isomorphism uses cubical equality instead of Martin-Löf equality when we have to proof that ∀ x → *from (to x)* ≡ x and ∀ x → *to (from x)* ≡ x. ptoc is necessary to do this convertion from these equalities.

```
≐≡≡ : ∀ {a b} → (a ≐ b) ≡c (a ≡p b)
≐≡≡ = let lifted = liftIso ≐≅≡ in isoToPath lifted
```

Using the univalence and liftIso defined previously, it is possible to transform the isormorphism into an equality.

```
open IsEquality

instance
  ≐-IsEquality : IsEquality {A = A} _≐_
  ≐-IsEquality = eq λ {x} {y} → ≐≡≡ •
    λ i → ≡p-IsEquality {ℓ-zero} .≜-≡-≡ {x} {y} i

≐-Equality : Equality {ℓ = ℓ-suc ℓ-zero}
≐-Equality = eqC _≐_
```

The last pass is to join the three equalities between equalities: Leibniz to Martin-Löf to cubical equality.

3 New Equalities types

The equalities used previously were defined using the cubical equality. Now I will define them using other equalities.

```
module Equalities {a ℓ} {A : Set a} where
  private
    ≜-Type' = ≜-Type {a} {ℓ} {A}
    ℓ₁ = ℓ-max a ℓ

  private variable
    x y z : A
```

Loaded the modules using the levels to be more generic.

```
module _
  (Eq₁ : Equality {_} {ℓ} {A})
  where

  open Equality Eq₁ renaming (_≙_ to _≡₁_; isEquality to eq₁)
```

I am importing a generic equality to use it to define a more generic equality.

```
record IsEquality₂ (_≙_ : ≙-Type') : Set (ℓ-suc ℓ₁) where
  constructor eq
  field
    ≙-≡-≡ : let
      x≡y : Type ℓ₁
      x≡y = x ≡₁ y
```

Different from previously definition of IsEquality, the cubical equality defined in the line above was substituted by the more generic equality ≡₁.

```
x≙y : Type ℓ₁
x≙y = x ≙ y

in _≡_ {ℓ-suc ℓ₁} x≙y x≡y

≡-≡-≙ : let
  x≡y : Type ℓ₁
  x≡y = x ≡₁ y
  x≙y = x ≙ y
  in x≡y ≡ x≙y
≡-≡-≙ = sym ≙-≡-≡
```

The rest of the definition is the same.

```
instance
  ≙-isEquality : IsEquality _≙_
  ≙-isEquality = eq (≙-≡-≡ • IsEquality.≙-≡-≡ eq₁)
```

From a more generic defintion of equality, it is easily possible to return to the less generic definition.

```
module _
  (Eq₂ : Equality {_} {ℓ} {A})
  where

  open Equality Eq₂ renaming (_≙_ to _≡₂_; isEquality to eq₂)
```

I am defining a new generic equality to prove that it is an equality of type 2:

```
eqsEqual₂ : let
  x≜₁y : Type ℓ₁
  x≜₁y = x ≡₁ y

  x≜₂y : Type ℓ₁
  x≜₂y = x ≡₂ y

  in _≡_ {ℓ-suc ℓ₁} x≜₁y x≜₂y
eqsEqual₂ = eqsEqual _≡₁_ _≡₂_

instance
  ≡₂-Equality₂ : IsEquality₂ _≡₂_
  ≡₂-Equality₂ = eq (sym eqsEqual₂)
    where open IsEquality

module _ {_≜_ : ≜-Type} where
  sym-Equality₂ : (≡-≡-≜ : {x y : A} → let
    x≡y : Type ℓ₁
    x≡y = x ≡₁ y
    x≜y = x ≜ y
    in x≡y ≡ x≜y)
    → IsEquality₂ _≜_
  sym-Equality₂ ≡-≡-≜ = eq (sym ≡-≡-≜)
```

Given a symmetric definition of the previous equality, it is easy to prove that it is also an equality of type 2.

3.1 Everything is an equality

In this part, a relation is an equality when it is equal (using a general equality) to a cubical equality.

```
module _
  (Eq₃ : Equality {A = Set ℓ₁})
  where

  open Equality Eq₃ renaming (_≜_ to _≡₃_; isEquality to eq₃)

  record IsEquality₃ (_≜_ : ≜-Type') : Set (ℓ-suc ℓ₁) where
    constructor eq
    field
      ≜-≡-≡ : (x ≜ y) ≡₃ (x ≡₁ y)
```

With this definition of equality, it is possible to prove that if an equality is equal to a cubical equality, so it is equal (using the general or cubical equality) to the cubical equality.

```
instance
  ≜-isEquality₂ : IsEquality₂ _≜_
  ≜-isEquality₂ = eq (transport (IsEquality.≜-≡-≡ eq₃) ≜-≡-≡)

  ≜-isEquality : IsEquality _≜_
  ≜-isEquality = eq (IsEquality₂.≜-≡-≡ ≜-isEquality₂ • IsEquality.≜-≡-≡ eq₁)
```

This is the proof that the symmetric definition of equality is also valid.

```
≡-≡-≜ : (x ≡₁ y) ≡₃ (x ≜ y)
≡-≡-≜ = let
  α₁ = IsEquality.≡-≡-≜ eq₃
  α₂ = IsEquality.≜-≡-≡ eq₁
  α₃ = IsEquality.≡-≡-≜ ≜-isEquality
  in transport α₁ (α₂ • α₃)
```

It is possible to proof that a general equality is an equality from this definition:

```
module _
  (Eq₂ : Equality {_} {ℓ} {A})
  where

  open Equality Eq₂ renaming (_≜_ to _≡₂_; isEquality to eq₂)

  instance
    ≡₂-Equality₃ : IsEquality₃ _≡₂_
    ≡₂-Equality₃ = eq α
      where
        open IsEquality eq₃
        α : (x ≡₂ y) ≡₃ (x ≡₁ y)
        α = transport ≡-≡-≜ (IsEquality₂.≜-≡-≡ (≡₂-Equality₂ (eqC _≡₂_)))
```

If there is a proof of the symmetrical equality, so it is also an equality from this definition:

```
module _ {_≜_ : ≜-Type'} where
  sym-Equality₃ :
    (≡-≡-≜ : ∀ {x y} → (x ≡₁ y) ≡₃ (x ≜ y))
    → IsEquality₃ _≜_
  sym-Equality₃ ≡-≡-≜ = eq (let
    α₁ = IsEquality.≡-≡-≜ eq₃
    α₂ = transport (sym α₁) ≡-≡-≜
    in transport α₁ (sym α₂))
```

4 Using the definitions

The best part of defining all of these stuffs is that it is now easy to proof that Leibniz equality is an equality.

```
module LeibnizFromPEquality {A : Set} where
  open Equalities {ℓ-zero} {ℓ-suc ℓ-zero}

  _≡p ₁_ : A → A → Set ₁
  x ≡p ₁ y = x ≡p y
```

I redefined this equality because it must be a set of universe one. And because of that, I have to prove again that this is an equality:

```
instance
  ≡p ₁-isEquality : IsEquality _≡p ₁_
  ≡p ₁-isEquality = eq λ {x y} → (sym λ i → let
    α : Type ₁
    α = p-c {ℓ-zero} {x = x} {y = y} i
    in α)
```

With just one line of code, it is possible now to prove that Leibniz equality is an equality from Martin-Löf Equality.

```
leibniz : IsEquality {A = A} _≐_
leibniz = IsEquality ₂.≜-isEquality {Eq ₁ = eqC _≡p_} (eq FinalEquality.≐≡≡)
```

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