

Statistics of Signal Detection Models

usings GLMs in R

Guillermo Aguilar

Technische Universität Berlin

August 27th, 2017

Goal

Learn to analyze the most common experimental designs
in the framework of Signal Detection Theory using GLMs in R

- 1 Signal detection theory
 - Yes/No detection experiment
 - Equal-variance, Gaussian SDT
- 2 Generalized Linear Models
 - GLMs and SDT
 - Conditions and model comparison with GLMs
- 3 Discrimination Experiment: 2-AFC
- 4 MLDS
- 5 Further Exercises
- 6 Appendix: some mathematical derivations

Knoblauch & Maloney (2012). Modeling Psychophysical Data in R

Signal detection theory

- ▶ Wickens (2002). Elementary Signal Detection Theory.
- ▶ McNicol (2005). A primer of Signal Detection Theory
- ▶ Macmillan and Creelman (2005). Detection Theory. A User's Guide
- ▶ Green and Swets (1966). Signal Detection Theory and Psychophysics.

R

- ▶ any statistical textbook that uses R

GLMs

- ▶ Wood (2006). Generalized Additive Models. An Introduction in R.

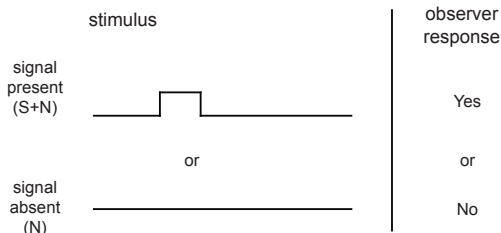
Repository

All slides and code are available in the following *git* repository:

https://github.com/guillermoaguilar/sdt_tutorial.git

Yes/No detection experiment

On each trial, one stimulus is presented



Signal			
		present	absent
Response	Yes	Hit	False Alarm
	No	Miss	Correct rejection
		N_s	N_n

- ▶ Hit rate

$$pH = \frac{\#H}{\#N_s}$$
- ▶ False Alarm rate

$$pFA = \frac{\#FA}{\#N_n}$$

Example

		Signal	
		present	absent
Response	Yes	180	50
	No	20	150
		200	200

- Hit rate

$$pH = \frac{\#H}{\#N_s} = \frac{180}{200} = 0.9$$

- False Alarm rate

$$pFA = \frac{\#FA}{\#N_n} = \frac{50}{200} = 0.25$$

Exercise 1: Analyze a detection experiment

Datafile: *data1.csv*

Description: columns are:

Resp (response, 1: Yes, 0: No)

Stim (type of stimulus, 'S': signal-present, 'A': signal-absent).

Steps:

1. Load *data1.csv* as a dataframe in R using

```
read.csv()
```

2. Examine its contents with

```
head() summary()
```

3. Calculate Hit and False Alarm Rate (p_H and p_{FA})

Hints: slicing the data in R can be done with square brackets ([]) and selecting a column is done with the dollar sign (\$)

Exercise 1: Analyze a detection experiment

Datafile: *data1.csv*

Description: columns are:

Resp (response, 1: Yes, 0: No)

Stim (type of stimulus, 'S': signal-present, 'A': signal-absent).

Steps:

1. Load *data1.csv* as a dataframe in R using

```
read.csv()
```

2. Examine its contents with

```
head() summary()
```

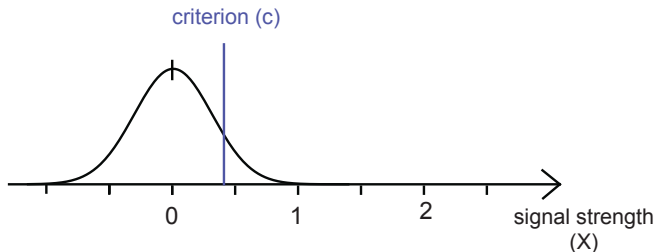
3. Calculate Hit and False Alarm Rate (pH and pFA)

Hints: slicing the data in R can be done with square brackets ([]) and selecting a column is done with the dollar sign (\$)

$$pH = 0.936$$

$$pFA = 0.288$$

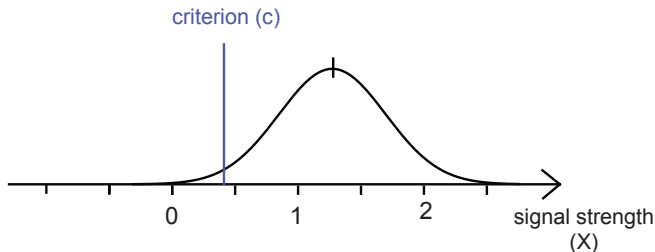
Signal detection theory



Assumptions:

- ▶ Internal dimension representing some (sensory) evidence, X
- ▶ subjected to fluctuation (noise), X is a random var.
- ▶ simple decision rule: if $X > c$ then respond *YES*
otherwise respond *NO*

Signal detection theory



Assumptions:

- ▶ Internal dimension representing some (sensory) evidence, X
- ▶ subjected to fluctuation (noise), X is a random var.
- ▶ simple decision rule: if $X > c$ then respond *YES*
otherwise respond *NO*

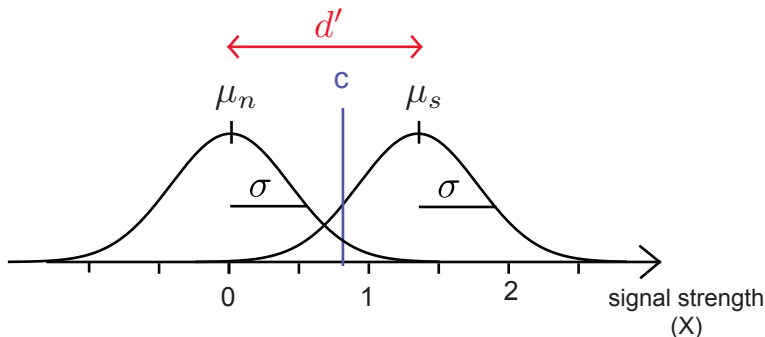
SDT with some added assumptions

1. Assumption of Gaussian distribution

$$X_n \sim \mathcal{N}(0, 1) \text{ and } X_s \sim \mathcal{N}(\mu_s, \sigma_s^2)$$

2. Equal-variance assumption

even further simplification $\sigma_s^2 = 1$

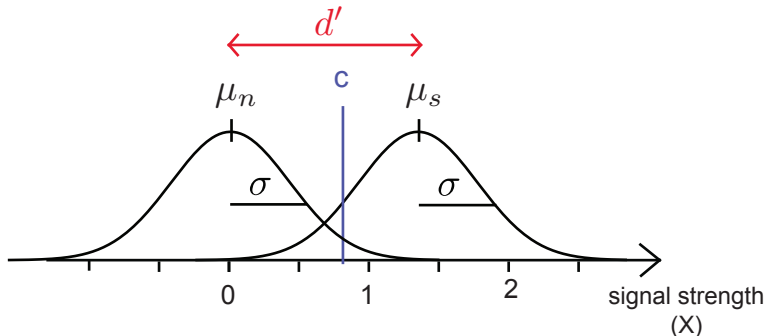


Equal-variance, Gaussian-distributed signal detection

It is then defined

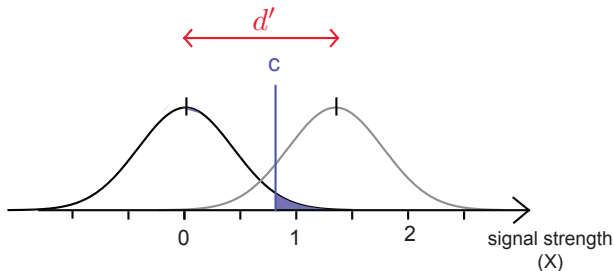
d' , a *measure of sensitivity* defined as the distance between the two distributions

c is the sensory criterion and a *measure of bias*.



Equal-variance, Gaussian-distributed signal detection

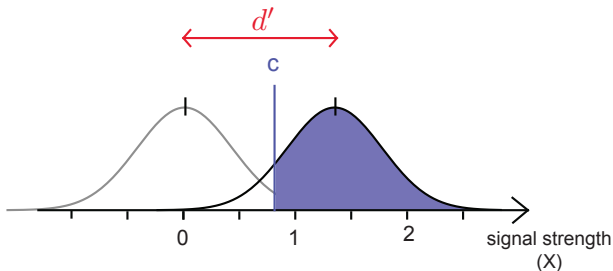
Parameters d' and c can be calculated directly from pH and pFA .



$$pFA = 1 - \Phi(c)$$

Equal-variance, Gaussian-distributed signal detection

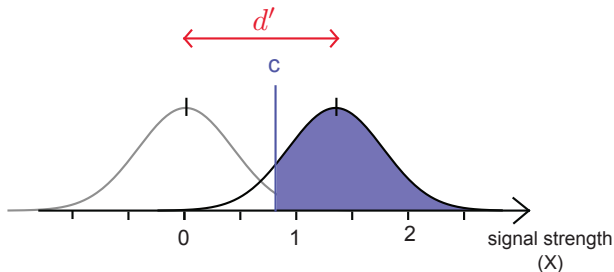
Parameters d' and c can be calculated directly from pH and pFA .



$$pH = 1 - \Phi(c - d')$$

Equal-variance, Gaussian-distributed signal detection

Parameters d' and c can be calculated directly from pH and pFA .



$$pFA = 1 - \Phi(c) \quad pH = 1 - \Phi(c - d')$$

it can be solved for d' and c [full derivation in extra slides *1]

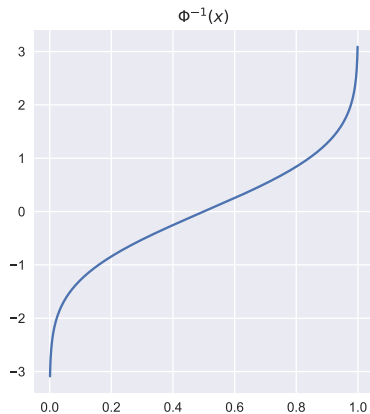
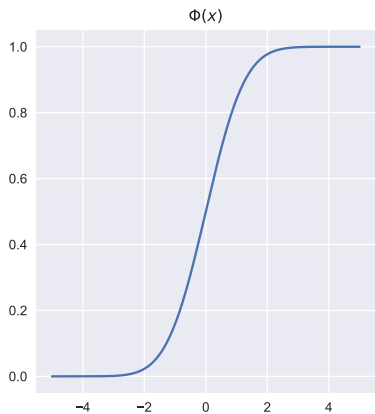
$$c = -\Phi^{-1}(pFA) \tag{1}$$

$$d' = \Phi^{-1}(pH) - \Phi^{-1}(pFA) \tag{2}$$

Exercise 2: explore the normal distribution

1. Draw 1000 random samples from a (standard) normal distribution ($\mu_0, \sigma^2 = 1$). Save in a vector x
2. Plot the histogram of x
3. Plot a cumulative histogram: plot the cumulative sum against the histogram bins.
4. Plot the inverse of it
5. Relate the functions $qnorm()$ and $pnorm()$ with these obtained plots

$\Phi()$ is the cumulative normal dist. function
and $\Phi^{-1}()$ its inverse, or *quantile* function



Exercise 3: Analyze a detection experiment

1. Use the Eqs. to calculate d' and c .

$$c = -\Phi^{-1}(pFA)$$

$$d' = \Phi^{-1}(pH) - \Phi^{-1}(pFA)$$

Exercise 3: Analyze a detection experiment

1. Use the Eqs. to calculate d' and c .

$$c = -\Phi^{-1}(pFA)$$

$$d' = \Phi^{-1}(pH) - \Phi^{-1}(pFA)$$

```
> dp_hat <- qnorm(pH) - qnorm(pFA)
2.081273
> c_hat <- -qnorm(pFA)
0.559237
```

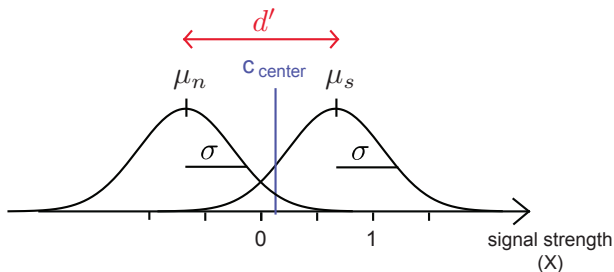
$$\hat{d}' = 2.081273$$

$$\hat{c} = 0.559237$$

Exercise 3': Analyze a detection experiment

Some textbooks use the calculation of a modified criterion $c_{center} = C$ that is midway between the two distributions

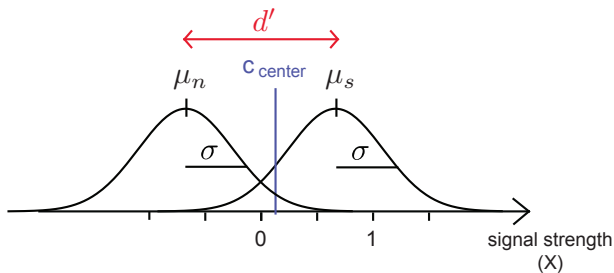
$$\begin{aligned} C = c_{center} &= -\Phi^{-1}(pH) + d'/2 = -\Phi^{-1}(pFA) - d'/2 \\ &= -[\Phi^{-1}(pH) + \Phi^{-1}(pFA)]/2 \end{aligned}$$



Exercise 3': Analyze a detection experiment

Some textbooks use the calculation of a modified criterion $c_{center} = C$ that is midway between the two distributions

$$\begin{aligned} C = c_{center} &= -\Phi^{-1}(pH) + d'/2 = -\Phi^{-1}(pFA) - d'/2 \\ &= -[\Phi^{-1}(pH) + \Phi^{-1}(pFA)]/2 \end{aligned}$$



```
> c_center <- -(qnorm(pH) + qnorm(pFA)) / 2.0
-0.4813996
```

Generalized Linear Models (GLMs)

Another approach to estimate \hat{d}' and \hat{c} is using GLMs.

A GLM is a linear model generalized for response variables that are not necessarily continuous.

Let's consider a linear model

$$y = \beta_1 \cdot x + \beta_0$$

GLMs made linear models **general** by having y replaced by a probability value (the expectation of Y , $E[Y]$), and adding a function $g()$ that 'links' the linear predictors (right-hand side) with the probability in the response variable.

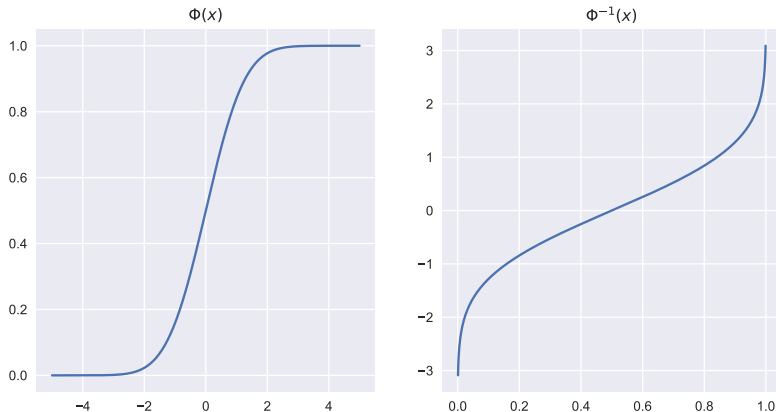
$$g(E[y]) = \beta_1 \cdot x + \beta_0$$

There could be explanatory variables $(\beta_0, \beta_1, \beta_2 \dots)$, therefore GLMs are annotated in a vector form

$$g(E[\mathbf{Y}]) = \mathbf{X} \cdot \boldsymbol{\beta}$$

Link function $g()$

For this case an appropriate choice of the link function is the quantile normal function $g() = \Phi^{-1}()$



GLM and SDT

Note that the False Alarm and Hit Rates are probabilities

$$pFA = P[R = YES | S = absent] = P[Y = 1 | X = 0]$$

$$pH = P[R = YES | S = present] = P[Y = 1 | X = 1]$$

Thus, one could write

$$g(P[Y = 1]) = \beta_1 \cdot X + \beta_0$$

where
 $X = 1$ for signal-present trials and
 $X = 0$ for signal-absent trials.

GLM and SDT

Let's consider our detection experiment. The GLM for a detection experiment can be arranged

$$g(E[\mathbf{Y}]) = \mathbf{X} \cdot \boldsymbol{\beta}$$

$$g\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ \vdots & \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

The first set of responses correspond to signal-absent trials, and the second set to signal-present trials.

This GLM can be then solved using maximum likelihood estimation (MLE).

The procedure returns estimates for β_0 and β_1 .

Exercise 4: Analyze a detection experiment using a GLM

1. Use the *GLM method* to analyse the detection experiment data

Exercise 4: Analyze a detection experiment using a GLM

1. Use the *GLM method* to analyse the detection experiment data

```
> fit <- glm(Resp ~ Stim, data=df,  
             family=binomial('probit'))
```

```

> summary(fit)
Call:
glm(formula = Resp ~ Stim, family = binomial("probit"), data = df)
Deviance Residuals:
    Min       1Q   Median       3Q      Max
-2.3447  -0.8242   0.3637   0.3637   1.5778
Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -0.55924     0.05935  -9.422  <2e-16 ***
StimS        2.08127     0.10563  19.704  <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
(Dispersion parameter for binomial family taken to be 1)
    Null deviance: 1335.69  on 999  degrees of freedom
Residual deviance:  838.19  on 998  degrees of freedom
AIC: 842.19
Number of Fisher Scoring iterations: 5

```

Do these numbers look familiar?

$$\beta_1 = 2.08127 \text{ and } \beta_0 = -0.55924$$

Explanation to the meaning of β_0 and β_1

Remember from before

$$\begin{aligned}d' &= \Phi^{-1}(pH) - \Phi^{-1}(pFA) \\ c &= -\Phi^{-1}(pFA)\end{aligned}$$

For **signal-absent** trials, $X = 0$ and the expanded model is

$$\begin{aligned}g(P[Y = 1|X = 0]) &= \beta_1 \cdot 0 + \beta_0 \\ \Phi^{-1}(P[Y = 1|X = 0]) &= \beta_0 \\ \Phi^{-1}(pFA) &= \beta_0 \\ -c &= \beta_0\end{aligned}$$

Explanation to the meaning of β_0 and β_1

Remember from before

$$\begin{aligned}d' &= \Phi^{-1}(pH) - \Phi^{-1}(pFA) \\c &= -\Phi^{-1}(pFA)\end{aligned}$$

For **signal-present** trials, $X = 1$ and the expanded model is

$$\begin{aligned}g(P[Y = 1|X = 1]) &= \beta_1 \cdot 1 + \beta_0 \\ \Phi^{-1}(P[Y = 1|X = 1]) &= \beta_1 \cdot 1 + \Phi^{-1}(pFA) \\ \Phi^{-1}(pH) &= \beta_1 + \Phi^{-1}(pFA) \\ \Phi^{-1}(pH) - \Phi^{-1}(pFA) &= \beta_1 \\ d' &= \beta_1\end{aligned}$$

Therefore, the coefficients obtained from GLM have a direct correspondence to the parameters d' and c .

The power of GLMs

Confidence intervals

```
> confint(fit)
Waiting for profiling to be done...
              2.5 %      97.5 %
(Intercept) -0.6761901 -0.4434893
StimS        1.8773729  2.2917211
```


Exercise 5: Detection exp. in two conditions

To include different conditions in the GLM framework

Reading data from two exp. conditions

```
> df1 <- read.csv("data1.csv")  
> df1$Cond <- "A"  
  
> df2 <- read.csv("data2.csv")  
> df2$Cond <- "B"  
  
> df <- rbind(df1, df2)
```

Exercise 5: Detection exp. in two conditions

Fitting different models

(a) *Unique model*

```
> fit1 <- glm(Resp ~ Stim,  
             family=binomial('probit'), data=df)
```

(b) *Model with different criteria, same sensitivity (d')*

```
> fit2 <- glm(Resp ~ Stim + Cond,  
             family=binomial('probit'), data=df)
```

(c) *Model with different criteria and sensitivity (d')*

```
> fit3 <- glm(Resp ~ Stim + Cond + Stim:Cond,  
             family=binomial('probit'), data=df)
```

Design matrix

$$g(E[\mathbf{Y}]) = \mathbf{X} \cdot \boldsymbol{\beta}$$

- ▶ The design matrix X is automatically generated by R .
- ▶ It can always be obtained using the function `model.matrix()`

Understanding the design matrix

For the unique model (a), the GLM looks like

$$g(E[\mathbf{Y}]) = \mathbf{X} \cdot \boldsymbol{\beta}$$

$$g\left(\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ \vdots & \\ 1 & 1 \\ 1 & 1 \\ \vdots & \\ 1 & 0 \\ 1 & 0 \\ \vdots & \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

Understanding the design matrix

For the model(b): different intercept (*fit2*), the GLM looks like

$$g(E[\mathbf{Y}]) = \mathbf{X} \cdot \boldsymbol{\beta}$$

$$g\left(\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ \vdots & & \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ \vdots & & \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ \vdots & & \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}$$

see [*2] for full derivation of parameters

Understanding the design matrix

For the model (c): different intercept and different d' (*fit3*), the GLM looks like

$$g(E[\mathbf{Y}]) = \mathbf{X} \cdot \boldsymbol{\beta}$$

$$g\left(\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \vdots & & & \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \vdots & & & \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ \vdots & & & \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

see [*3] for full derivation of parameters

Exercise 5: Detection exp. in two conditions

Model selection

```
> anova(fit1, fit2, fit3, test='Chisq')
```

```
Analysis of Deviance Table
```

```
Model 1: Resp ~ Stim
```

```
Model 2: Resp ~ Stim + Cond
```

```
Model 3: Resp ~ Stim + Cond + Stim:Cond
```

	Resid. Df	Resid. Dev	Df	Deviance	Pr(>Chi)
1	1998	1735.5			
2	1997	1701.8	1	33.698	6.435e-09 ***
3	1996	1701.7	1	0.140	0.7081

```
---
```

```
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 1
```

fit2 explains the data sig. better than fit1, but fit3 does not improve it further

Exercise 5: Detection experiment on two conditions

Parameter extraction

```
> coef(fit2)
(Intercept)      StimS      CondB
-0.5500044    2.0521900   -0.3999372
```

How are these parameters encoded?

Exercise 5: Detection experiment on two conditions

Parameter extraction

```
> coef(fit2)
(Intercept)      StimS      CondB
-0.5500044      2.0521900     -0.3999372
```

How are these parameters encoded?

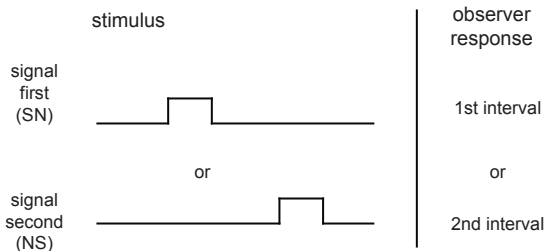
Remember from earlier that $\beta_0 = -c$ and $\beta_1 = d'$

- ▶ The intercept applies for the first condition (A), thus $\beta_0 = -c_A$.
- ▶ $\beta_1 = d'$, assumed equal for both conditions A and B.
- ▶ The new parameter β_2 is the difference with respect to β_0 . Thus, $c_B = -(\beta_0 + \beta_2)$.

Thus,

```
> c_b <- -(coef(fit2)[[1]]+coef(fit2)[[3]])
0.9499416
```

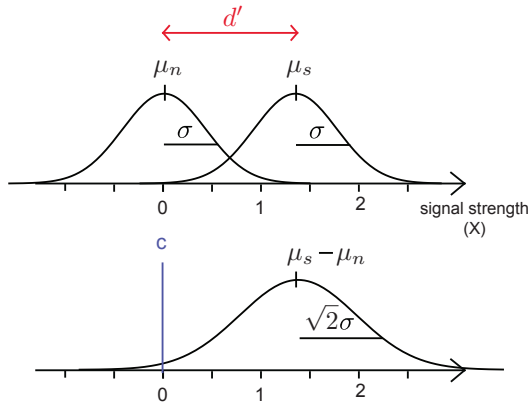
Discrimination Experiment: 2-AFC



- ▶ Statistically similar as in a Yes/No detection experiment
- ▶ Signal-present or -absent can be relabeled to 'SN'/'NS' and responses as '1st interval'/'2nd interval', or '1st position'/'2nd position' depending on the exp. design
- ▶ Extra step of differencing

Discrimination Experiment: 2-AFC

On each trial, two stimuli are presented. Thus, two draws are taken, one from each distribution



$$\Delta = X_s - X_n \quad \Delta \sim \mathcal{N}(\mu_s - \mu_n, 2\sigma^2)$$

Discrimination Experiment: 2-AFC

$$\Delta = X_s - X_n \quad \Delta \sim \mathcal{N}(\mu_s - \mu_n, 2\sigma^2)$$

As $\mu_n = 0$, $\mu_s = d'$, and $\sigma = 1$, we have

$$\Delta \sim \mathcal{N}(d', 2)$$

and the probability of correct when $\Delta > 0$,

$$Pc = 1 - \Phi\left(\frac{-d'}{\sqrt{2}}\right) = \Phi\left(\frac{d'}{\sqrt{2}}\right)$$

Thus, d' can be calculated from the percentage correct in a 2-AFC experiment

$$d' = \Phi^{-1}(Pc)\sqrt{2} \quad (3)$$

Discrimination Experiment: 2-AFC

Estimates of d' from 2AFC and Yes/No related to each other by

$$d'_{AFC} = d'_{YesNo} \sqrt{2}$$

(see derivation in Wickens 2002, or in McNicol 1972)

However, this relationship has been called into question recently (Yeshurun, Carrasco & Maloney, 2008)

Exercise 6: Discrimination experiment: 2-AFC

Analyze the data from a 2-AFC discrimination experiment using a GLM

Datafile: *data2afc.csv*

Description: columns are:

Resp (response, 1: 1st interval, 0: 2nd interval)

Stim (type of stimulus, 'SN': stimulus in 1st interval,
or 'NS': stimulus in 2nd interval).

Steps:

1. Load data
2. Calculate percentage correct P_c and from it d' using Eq. 3
3. Calculate d' and c using a GLM
4. Compare these two ways of obtaining d'

Exercise 6: Discrimination experiment: 2-AFC

Analyze the data from a 2-AFC discrimination experiment using a GLM

Datafile: *data2afc.csv*

Description: columns are:

Resp (response, 1: 1st interval, 0: 2nd interval)

Stim (type of stimulus, 'SN': stimulus in 1st interval,
or 'NS': stimulus in 2nd interval).

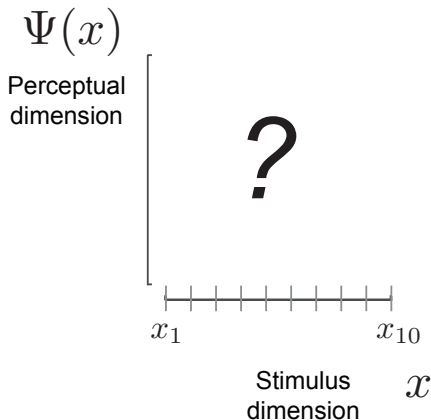
Steps:

1. Load data
2. Calculate percentage correct P_c and from it d' using Eq. 3
3. Calculate d' and c using a GLM
4. Compare these two ways of obtaining d'

Solution: $d' = 1.53$, $c = 0.77$, $c_{center} = 0.01$

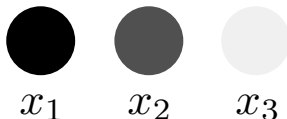
Maximum Likelihood Difference Scaling

Scaling method aimed to find $\Psi(x)$



Maximum Likelihood Difference Scaling (MLDS)

Task: *Which pair is more different?*



Decision model

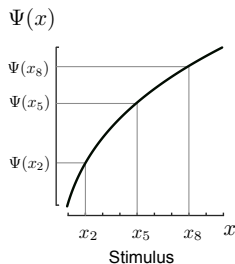
$$\begin{aligned}\Delta &= [\Psi(x_3) - \Psi(x_2)] - [\Psi(x_2) - \Psi(x_1)] + \epsilon \\ &= \Psi(x_3) - 2\Psi(x_2) + \Psi(x_1) + \epsilon\end{aligned}$$

$$\epsilon \sim \mathcal{N}(0, \sigma^2)$$

if $\Delta > 0 \rightarrow (x_2, x_3)$
otherwise $\rightarrow (x_1, x_2)$

Maximum Likelihood Difference Scaling (MLDS)

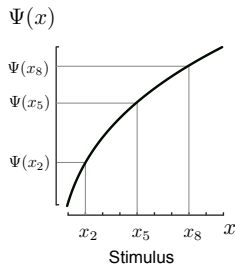
MLDS uses a Generalized Linear Model (GLM) to find $\Psi(x)$



$$\Delta = \Psi(x_2) - 2\Psi(x_5) + \Psi(x_8)$$

Maximum Likelihood Difference Scaling (MLDS)

MLDS uses a Generalized Linear Model (GLM) to find $\Psi(x)$



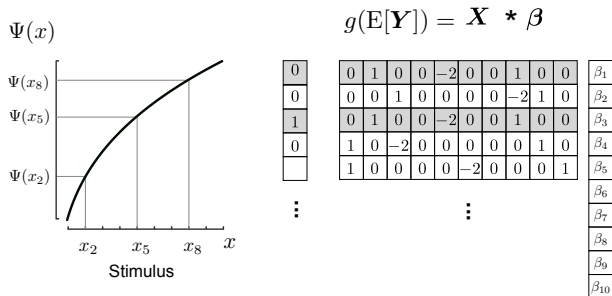
$$\Delta = \Psi(x_2) - 2\Psi(x_5) + \Psi(x_8) + \epsilon$$

$$g(E[Y]) = \mathbf{1} \cdot \beta_2 - \mathbf{2} \cdot \beta_5 + \mathbf{1} \cdot \beta_8 + \epsilon$$

$$\dots$$

Maximum Likelihood Difference Scaling (MLDS)

MLDS uses a Generalized Linear Model (GLM) to find $\Psi(x)$



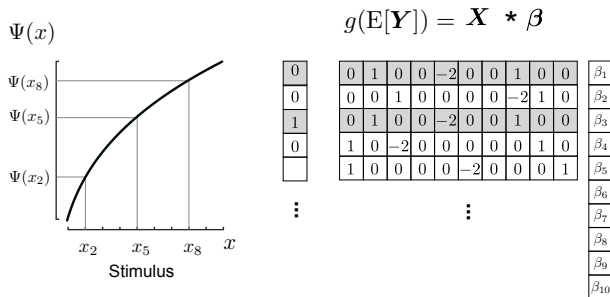
$$\Delta = \Psi(x_2) - 2\Psi(x_5) + \Psi(x_8) + \epsilon$$

$$g(E[Y]) = \mathbf{1} \cdot \beta_2 - \mathbf{2} \cdot \beta_5 + \mathbf{1} \cdot \beta_8 + \epsilon$$

...

Maximum Likelihood Difference Scaling (MLDS)

MLDS uses a Generalized Linear Model (GLM) to find $\Psi(x)$



\mathbf{Y} : vector of observer responses (binary)

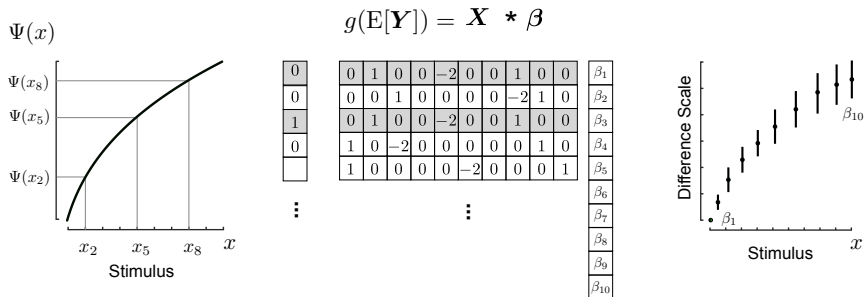
\mathbf{X} : design matrix

$\boldsymbol{\beta}$: GLM coefficients \rightarrow perceptual scale

$g()$: link function

Maximum Likelihood Difference Scaling (MLDS)

MLDS uses a Generalized Linear Model (GLM) to find $\Psi(x)$



\mathbf{Y} : vector of observer responses (binary)

\mathbf{X} : design matrix

$\boldsymbol{\beta}$: GLM coefficients \rightarrow perceptual scale

$g()$: link function

Exercise 7: Estimate a perceptual scale using MLDS

Analyze the data from a MLDS scaling experiment

Datafile: *dataml.ds.csv*

Description: columns are:

Resp (response, 1: 2nd pair (x2, x3), 0: 1st pair (x1, x2))

s1, s2, s3: Stimulus values on each triad

Steps:

1. Load data
2. Explore the data with *head()* and *summary()*
3. Calculate the perceptual scale using the function *mlds()*, from package *MLDS* (package needs to be installed).

Exercise 7: Estimate a perceptual scale using MLDS

Analyze the data from a MLDS scaling experiment

Datafile: *dataml.ds.csv*

Description: columns are:

Resp (response, 1: 2nd pair (x2, x3), 0: 1st pair (x1, x2))

s1, s2, s3: Stimulus values on each triad

Steps:

1. Load data
2. Explore the data with *head()* and *summary()*
3. Calculate the perceptual scale using the function *mlds()*, from package *MLDS* (package needs to be installed).

Solution

Further Exercises

1. To the analysis of a Yes/No experiment in Exercise 5, add other two conditions that were also measured: *data3.csv* and *data4.csv*. Does d' changes across conditions?
2. Analyze the data of a 2-AFC experiment in other condition: *dataafc-2.csv*. Test a model for a change in sensitivity

Thank you

*1: Derivation 1

Given $pFA = 1 - \Phi(c)$ and $pH = 1 - \Phi(c - d')$

$$\Phi(c) = 1 - pFA$$

$$c = \Phi^{-1}(1 - pFA)$$

$$c = -\Phi^{-1}(pFA)$$

$$\Phi(c - d') = 1 - pH$$

$$c - d' = \Phi^{-1}(1 - pH)$$

$$d' = c - \Phi^{-1}(1 - pH)$$

$$d' = -\Phi^{-1}(pFA) - \Phi^{-1}(1 - pH)$$

$$d' = \Phi^{-1}(pH) - \Phi^{-1}(pFA)$$

Note the symmetry of the quantile function

$$\Phi^{-1}(1 - x) = -\Phi^{-1}(x)$$

*2: Derivation 2

For the model(b): different intercept (*fit2*), the GLM looks like

$$g(E[\mathbf{Y}]) = \mathbf{X} \cdot \boldsymbol{\beta}$$

$$g\left(\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ \vdots & & \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ \vdots & & \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ \vdots & & \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}$$

*2: Derivation 2

Writing each case we have

$$g(E[Y = 1|X = 0, C = A]) = 1 * \beta_0 + 0 * \beta_1 + 0 * \beta_2$$

$$g(E[Y = 1|X = 1, C = A]) = 1 * \beta_0 + 1 * \beta_1 + 0 * \beta_2$$

$$g(E[Y = 1|X = 0, C = B]) = 1 * \beta_0 + 0 * \beta_1 + 1 * \beta_2$$

$$g(E[Y = 1|X = 1, C = B]) = 1 * \beta_0 + 1 * \beta_1 + 1 * \beta_2$$

by replacing expectations for FA and Hit Rates, and by replacing the link function for the quantile function $g() = \Phi^{-1}()$, we have

$$\Phi^{-1}(pFA_A) = \beta_0 \tag{4}$$

$$\Phi^{-1}(pH_A) = \beta_0 + \beta_1 \tag{5}$$

$$\Phi^{-1}(pFA_B) = \beta_0 + \beta_2 \tag{6}$$

$$\Phi^{-1}(pH_B) = \beta_0 + \beta_1 + \beta_2 \tag{7}$$

*2: Derivation 2

Simplifying Eq. 4,

$$\beta_0 = \Phi^{-1}(pFA_A) = -c_A \quad c_A = -\beta_0$$

inserting β_0 in Eq. 5...

$$\begin{aligned}\Phi^{-1}(pH_A) &= \Phi^{-1}(pFA_A) + \beta_1 \\ \Phi^{-1}(pH_A) - \Phi^{-1}(pFA_A) &= \beta_1 \\ d' &= \beta_1\end{aligned}$$

Then, Eq. 6..

$$\begin{aligned}\Phi^{-1}(pFA_B) &= \beta_0 + \beta_2 \\ \Phi^{-1}(pFA_B) - \beta_0 &= \beta_2 \\ -c_B + c_A &= \beta_2 \\ -(c_B - c_A) &= \beta_2\end{aligned}$$

*2: Derivation 2

β_2 is the difference in intercept between conditions B and A

Finally inserting $\beta_0, \beta_1, \beta_2$ in Eq. 7 ..

$$\Phi^{-1}(pH_B) = \beta_0 + \beta_1 + \beta_2$$

$$\begin{aligned}\Phi^{-1}(pH_B) &= \Phi^{-1}(pFA_A) + \Phi^{-1}(pH_A) - \Phi^{-1}(pFA_A) \\ &\quad + \Phi^{-1}(pFA_B) - \Phi^{-1}(pFA_A)\end{aligned}$$

$$\Phi^{-1}(pH_B) = \Phi^{-1}(pH_A) + \Phi^{-1}(pFA_B) - \Phi^{-1}(pFA_A)$$

$$\Phi^{-1}(pH_B) - \Phi^{-1}(pFA_B) = \Phi^{-1}(pH_A) - \Phi^{-1}(pFA_A)$$

$$d'_B = d'_A$$

*3: Derivation 3

Writing each case we have

$$g(E[Y = 1|X = 0, C = A]) = 1 * \beta_0 + 0 * \beta_1 + 0 * \beta_2 + 0 * \beta_3$$

$$g(E[Y = 1|X = 1, C = A]) = 1 * \beta_0 + 1 * \beta_1 + 0 * \beta_2 + 0 * \beta_3$$

$$g(E[Y = 1|X = 0, C = B]) = 1 * \beta_0 + 0 * \beta_1 + 1 * \beta_2 + 0 * \beta_3$$

$$g(E[Y = 1|X = 1, C = B]) = 1 * \beta_0 + 1 * \beta_1 + 1 * \beta_2 + 1 * \beta_3$$

by replacing expectations for FA and Hit Rates, and by replacing the link function for the quantile function $g() = \Phi^{-1}()$, we have

$$\Phi^{-1}(pFA_A) = \beta_0 \tag{8}$$

$$\Phi^{-1}(pH_A) = \beta_0 + \beta_1 \tag{9}$$

$$\Phi^{-1}(pFA_B) = \beta_0 + \beta_2 \tag{10}$$

$$\Phi^{-1}(pH_B) = \beta_0 + \beta_1 + \beta_2 + \beta_3 \tag{11}$$

*3: Derivation 3

Simplifying Eq. 8,

$$\beta_0 = \Phi^{-1}(pFA_A) = -c_A$$

inserting β_0 in Eq. 9...

$$\begin{aligned}\Phi^{-1}(pH_A) &= \Phi^{-1}(pFA_A) + \beta_1 \\ \Phi^{-1}(pH_A) - \Phi^{-1}(pFA_A) &= \beta_1 \\ d'_A &= \beta_1\end{aligned}$$

inserting β_0 in Eq. 10 ..

$$\begin{aligned}\Phi^{-1}(pFA_B) &= \Phi^{-1}(pFA_A) + \beta_2 \\ \Phi^{-1}(pFA_B) - \Phi^{-1}(pFA_A) &= \beta_2 \\ -(c_B - c_A) &= \beta_2\end{aligned}$$

β_2 is the difference in intercept between conditions B and A

*2: Derivation 2

Finally inserting $\beta_0, \beta_1, \beta_2$ in Eq. 11 ..

$$\Phi^{-1}(pH_B) = \beta_0 + \beta_1 + \beta_2 + \beta_3$$

$$\begin{aligned}\Phi^{-1}(pH_B) &= \Phi^{-1}(pFA_A) + \Phi^{-1}(pH_A) - \Phi^{-1}(pFA_A) \\ &\quad + \Phi^{-1}(pFA_B) - \Phi^{-1}(pFA_A) + \beta_3\end{aligned}$$

$$\Phi^{-1}(pH_B) = \Phi^{-1}(pH_A) + \Phi^{-1}(pFA_B) - \Phi^{-1}(pFA_A) + \beta_3$$

$$\Phi^{-1}(pH_B) - \Phi^{-1}(pFA_B) = \Phi^{-1}(pH_A) - \Phi^{-1}(pFA_A) + \beta_3$$

$$d'_B = d'_A + \beta_3$$

$$d'_B - d'_A = \beta_3$$

β_3 is the difference in d' between conditions B and A