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Introduction

Quadcopter is a rotor craft system with four identical propellers. It is become popular due to its accessibility, low weight, small size, lower risk at challenging environment and simplicity of mechanics. Quadcopter is a underactuated system since there are 6 DoF to be controlled with only 4 actuators. Thus, 2 DoF depends on other dynamics. The change in roll and pitch angle creates a horizontal component of thrust vector, which allows movement in the X and Y planes.

This paper focus on designing controllers for a quadcopter. Even though there are several control algorithms such as back-steering, model predictive, linear-quadratic regulator, PID controllers are implemented in this study due to its design simplicity and popularity.

To design controllers, firstly dynamics of quadcopter is modelled via Simulink. Then, simple control algorithms are implemented. For simplicity state estimator is not modelled and also linearized motor-system relationships are used instead of modelling the nonlinear motor dynamics. Different trajectories are given the closed loop system and responses of quadcopter are observed. Overall Simulink model will be clarified below.

Quadcopter Modelling

There are 2 quadcopter configurations commonly used: 'X' and '$+$'. In this study, the rotors are assigned a '+' configuration as the control of the roll and pitch dynamics is simpler. Configuration is given at Figure 1). The 1^{st} and 3^{rd} propellers rotate counter-clockwise, while 3^{rd} and 4^{th} rotate clockwise.

Basic principles of movement of quadcopter are:

- To prevent torque imbalance, 2 of the 4 rotors rotate in the same direction while the other 2 rotate in the opposite direction.
- All propellers rotate at nominal speed to compensate for gravitational acceleration.
- To raise the quadcopter, propellers' speed is increased by same amount.

- For acceleration in roll, while speed of 2^{nd} rotor is decreased speed of 4^{th} rotor is increased.
- For acceleration in pitch, speed of 3^{rd} rotor is increased and speed of 1^{st} rotor is decreased.
- Changing in yaw is provided by increasing $2^{nd}$and 4^{th} propellers' speed and by decreasing $1^{st}$and 3^{rd} propellers' speed.
- Motion along Y direction depends on roll and movement in X is associated with the pitch.

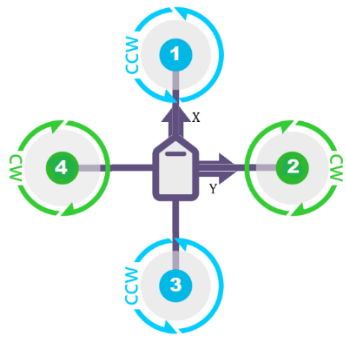

Figure 1. *Rotor Orientation for + Configuration*

Kinematics and Dynamics

Kinematics and dynamics equations describe the behavior of quadcopter. It is important to define and prediction of position according to the four propellers' speed. Also control algorithms are based on inverted dynamics.

To derive dynamics equation Newton's second law is written in Earth inertial frame while onboard measurement is taken mostly in body reference frame. Thus, transformation is necessary between these two frames.

Kinematics are,

$$\dot{\boldsymbol{\xi}} = \boldsymbol{J}_{\boldsymbol{\Theta}}\, \boldsymbol{\nu} \tag{1}$$

With

$$\boldsymbol{\xi} = [X \quad Y \quad Z \quad \phi \quad \theta \quad \psi]^T \tag{2}$$

Where $\boldsymbol{\xi}$ contains the positions in X, Y, Z planes and attitude angles ϕ, θ, ψ in Earth inertial reference frame. $\boldsymbol{J}_{\boldsymbol{\Theta}}$ consists of rotation $\boldsymbol{R}_{\boldsymbol{\Theta}}$, and transformation $\boldsymbol{T}_{\boldsymbol{\Theta}}$ matrices to Earth inertial frame from body reference.

Similarly, $\boldsymbol{\nu}$ is composed of linear and angular velocity vectors in body reference frame,

$$\boldsymbol{\nu} = [\boldsymbol{V}^B \quad \boldsymbol{\omega}^B]^T = [u \quad v \quad w \quad p \quad q \quad r]^T \tag{3}$$

Where $\boldsymbol{V}^B$is the linear velocity matrix consists of u, v, w. $\boldsymbol{\omega}^B$ is the angular velocity matrix with p, q, r.

Eq. (4) describes generalized rotation and transformation matrix

$$\boldsymbol{J_\Theta} = \begin{bmatrix} \boldsymbol{R_\Theta} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{T_\Theta} \end{bmatrix} \tag{4}$$

With

$$\boldsymbol{R_\Theta} = \begin{bmatrix} c_\psi c_\theta & -s_\psi c_\phi + c_\psi s_\theta s_\phi & s_\psi s_\phi + c_\psi s_\theta c_\phi \\ s_\psi c_\theta & c_\psi c_\phi + s_\psi s_\theta s_\phi & -c_\psi s_\phi + s_\psi s_\theta c_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix} \tag{5}$$

$$\boldsymbol{T_\Theta} = \begin{bmatrix} 1 & s_\phi t_\theta & c_\phi t_\theta \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi / c_\theta & c_\phi / c_\theta \end{bmatrix} \tag{6}$$

Where $c_k = \cos k$, $s_k = \sin k$, $t_k = \tan k$.

The assumptions listed below are used in deriving the equation of motion.

- Center of gravity is placed at the origin of body fixed frame.
- Quadcopter's structure, propellers and motors are rigid.
- Body frame coincide with the body principle axis, inertia matrix is diagonal.
- Thrust and drag forces are proportional to square of propeller's speed.

6 DoF rigid-body equation for quadcopter can be formulated as,

$$\boldsymbol{M}_B \dot{\boldsymbol{\nu}} + \boldsymbol{C}_B(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\Lambda} \tag{7}$$

$$\boldsymbol{\Lambda} = [\boldsymbol{F}^B \quad \boldsymbol{\tau}^B] = [F_x \quad F_y \quad F_z \quad \tau_x \quad \tau_y \quad \tau_z]^T \tag{8}$$

$$\boldsymbol{M}_B = \begin{bmatrix} m\boldsymbol{I}_{3x3} & \boldsymbol{0}_{3x3} \\ \boldsymbol{0}_{3x3} & \boldsymbol{I} \end{bmatrix} \tag{9}$$

$$\boldsymbol{C}_B(\boldsymbol{\nu}) = \begin{bmatrix} \boldsymbol{0}_{3\times3} & -m\,\boldsymbol{S}(\boldsymbol{V}^B) \\ \boldsymbol{0}_{3\times3} & -\boldsymbol{S}\,(\boldsymbol{I}\,\boldsymbol{\omega}^B) \end{bmatrix} \tag{10}$$

Where $\boldsymbol{M}_B$ is the system inertia matrix given in Eq. (9) and $\boldsymbol{C}_B$ is the Coriolis-centripetal matrix given with Eq. (10). $\boldsymbol{\Lambda}$ is the generalized force vector defined in Eq. (8). $\boldsymbol{F}^B$ is the quadrotor force vector in body reference frame and $\boldsymbol{\tau}^B$ is the quadrotor torques vector in body reference frame. m is the mass of the body and $\boldsymbol{I}$ is the inertia matrix. The notation $\boldsymbol{I}_{3\times3}$ means a 3x3 identity matrix and the notation $\boldsymbol{0}_{3\times3}$ means 3x3 zero matrix.

The skew-symmetric matrix $\boldsymbol{S}(\boldsymbol{k})$ for a three-dimension vector $\boldsymbol{k}$ is shown in Eq. (11).

$$\boldsymbol{S}(\boldsymbol{k}) = \begin{bmatrix} 0 & -k_3 & k_1 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}, \quad \boldsymbol{k} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \tag{11}$$

To define states, acting forces and moments on quadcopter must be analysed. There are three main force and moment sources: gravity, gyroscopic effect and movement inputs. Gravity acts on Z axis in Earth inertial frame and its effect can be shown with,

$$\boldsymbol{G}_B = \begin{bmatrix} \boldsymbol{R}_\Theta^T \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} \\ \boldsymbol{0}_{3\times1} \end{bmatrix} \tag{12}$$

Where $\boldsymbol{G}_B$ is the gravitational vector and g is the gravitational acceleration.

The rotation of the propeller causes gyroscopic effects. It affects roll and pitch movements.

$$\boldsymbol{O}_B(\boldsymbol{\nu})\,\boldsymbol{\Omega} = \begin{bmatrix} \boldsymbol{0}_{3\times1} \\ J_{TP} \begin{bmatrix} -q \\ p \\ 0 \end{bmatrix} \Omega \end{bmatrix} \tag{13}$$

Where $\boldsymbol{O}_B$ is the gyroscopic propeller matrix and J_{TP} is the total rotational moment of inertia around propeller axis. Ω is the overall propellers' speed and $\boldsymbol{\Omega}$ is the propellers' speed vector given with,

$$\Omega = -\Omega_1 + \Omega_2 - \Omega_3 + \Omega_4 \qquad \boldsymbol{\Omega} = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \end{bmatrix} \tag{14}$$

Subscripts 1,2,3 and 4 represent the propellers' number.

The third contribution produced by main movement inputs. Both aerodynamics' torques and moments are proportional to squared propellers' speed. More detail can be found in Ref. [1].

$$\boldsymbol{U}_B(\boldsymbol{\Omega}) = \boldsymbol{E}_B\,\boldsymbol{\Omega}^2 = \begin{bmatrix} 0 \\ 0 \\ b(\Omega_1^2 + \Omega_2^2 + \Omega_3^2 + \Omega_4^2) \\ b\,l\,(\Omega_4^2 - \Omega_2^2) \\ b\,l\,(\Omega_3^2 - \Omega_1^2) \\ d(-\Omega_1^2 + \Omega_2^2 - \Omega_3^2 + \Omega_4^2) \end{bmatrix} \tag{15}$$

Where $\boldsymbol{U}_B(\boldsymbol{\Omega})$ is the movement vector, $\boldsymbol{E}_B$ is the movement matrix. b is thrust coefficient, d is the drag coefficient and l is the distance between centre of quadrotor and the centre of propeller. Thrust and drag coefficients are estimated empirically.

Thus Eq. (7) becomes,

$$\boldsymbol{M}_B\dot{\boldsymbol{\nu}} + \boldsymbol{C}_B(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{G}_B(\boldsymbol{\xi}) + \boldsymbol{O}_B(\boldsymbol{\nu})\,\boldsymbol{\Omega} + \boldsymbol{E}_B\,\boldsymbol{\Omega}^2 \tag{16}$$

Eq. (17) and Eq. (18) define states of quadcopter.

$$\dot{\boldsymbol{\nu}} = \boldsymbol{M}_B^{-1}\,(-\boldsymbol{C}_B(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{G}_B(\boldsymbol{\xi}) + \boldsymbol{O}_B(\boldsymbol{\nu})\,\boldsymbol{\Omega} + \boldsymbol{E}_B\,\boldsymbol{\Omega}^2) \tag{17}$$

$$\begin{cases} \dot{u} &= (v\,r - w\,q) + g\,s_\theta \\ \dot{v} &= (w\,p - u\,r) - g\,c_\theta s_\phi \\ \dot{w} &= (u\,q - v\,p) - g\,c_\theta\,s_\phi + \dfrac{U_1}{m} \\ \dot{p} &= \dfrac{I_{yy} - I_{zz}}{I_{xx}}\,q\,r - \dfrac{J_{TP}}{I_{xx}}\,q\,\Omega + \dfrac{U_2}{I_{xx}} \\ \dot{q} &= \dfrac{I_{zz} - I_{xx}}{I_{yy}}\,p\,r + \dfrac{J_{TP}}{I_{yy}}\,p\,\Omega + \dfrac{U_3}{I_{xx}} \\ \dot{r} &= \dfrac{I_{xx} - I_{yy}}{I_{zz}}\,p\,q + \dfrac{U_4}{I_{xx}} \end{cases} \tag{18}$$

After 6 states are found in body frame, states in Earth inertial frame are found with Eqs. (1)-(6).

In his study, motor and dynamics are not modelled due to high nonlinearity. But, some relations must be defined. These relations can be found with linear regression approach.

Thrust and propellers' speed,

$$T = b\,\Omega^2 \tag{19}$$

Torque and propellers' speed,

$$\tau = d\,\Omega^2 \tag{20}$$

Percentage throttle and propellers' speed,

$$\Omega = (Throttle\ \%).\, C_r + C_b \tag{21}$$

Where C_r and C_b are coefficients of linear regression. Constants are estimated empirically. Instead of setting up a test, the results of the parameter identification study conducted in Ref. [2] are used. Also, motor delay is ignored.

Controller Design

Two sets of controllers were designed. One for attitude and altitude and another for position control in X and Y direction. Dynamics of quadcopter are highly coupled. Thus, to design a controller, dynamic expression is simplified by keeping the most dominant element. Thus, gyroscopic effects are ignored and small angles approach is used.

Control laws for yaw and altitude [3],

$$\psi_{cor} = k_p(\psi_d - \psi) + k_d(\dot{\psi}_d - \dot{\psi}) + k_i \int (\psi_d - \psi)dt \tag{22}$$

$$z_{cor} = k_p(z_d - z) + k_d(\dot{z}_d - \dot{z}) + k_i \int (z_d - z)dt - g_{offset} \tag{23}$$

g_{offset} is the minimum throttle command required for hover.

A nested control loop is designed for attitude stabilization to make angle control easier. While outer loop control attitude angle, inner loop controls the fast rate dynamics. PI controllers are selected for both control loops.

Control laws for attitude stabilization are implemented using,

$$\dot{\phi} \cong p, \qquad \dot{\theta} \cong q \tag{24}$$

$$\phi_{cor} = k_{p1}(\phi_d - \phi) + k_{i1} \int (\phi_d - \phi)\, dt + k_{p2}(\dot{\phi}_d - \dot{\phi}) + k_{i2} \int (\dot{\phi}_d - \dot{\phi})\, dt \tag{25}$$

$$\theta_{cor} = k_{p1}(\theta_d - \theta) + k_{i1} \int (\theta_d - \theta)\, dt + k_{p2}(\dot{\theta}_d - \dot{\theta}) + k_{i2} \int (\dot{\theta}_d - \dot{\theta})\, dt \tag{26}$$

Where k_{p1} and k_{i1} represent gains for roll and pitch angles, k_{p2} and k_{i2} stand for gains of rate of roll and pitch. Quadcopter assumed as axis-symmetric, so gains are kept the same for roll and pitch. Also, both angles are constrained to $\pm\, 4°$ to make small angle approximation is valid.

Position controller, i.e. outer loop controller is used to give commands for attitude and thus to follow the desired path. Control laws for the outer loop are [3],

$$\theta_d = k_p(u_d - u) - k_d(\dot{u}_d - \dot{u}), \qquad \phi_d = k_d(\dot{v}_d - \dot{v}_d) - k_p(v_d - v) \tag{27}$$

$$(u_d - u) = (X_d - X)\cos(\psi) + (Y_d - Y)\sin(\psi), \tag{28}$$

$$(v_d - v) = (Y_d - Y)\cos(\psi) - (X_d - X)\sin(\psi) \tag{29}$$

Corrected controller outputs, $\theta_{cor}, \phi_{cor}, \psi_{cor}\ and\ z_{cor}$, are used to produce throttle commands for actuators. Corrected values are taken and then mixed to send the throttle command to correct motor.

Commands for each motor according to mixing laws for '+' configuration are

$$\begin{cases} mc_1 = z_{cor} - \theta_{cor} - \psi_{cor}, \\ mc_2 = z_{cor} - \phi_{cor} + \psi_{cor}, \\ mc_3 = z_{cor} + \theta_{cor} - \psi_{cor}, \\ mc_4 = z_{cor} + \phi_{cor} + \psi_{cor} \end{cases} \tag{30}$$

Where $mc1, mc2, mc3, mc4$ are percentage throttle commands for motors. After receiving percentage throttle commands, each propeller rotates at the speed shown at Eq. (21). These processes satisfy Eq. (15).

Simulink Model Development

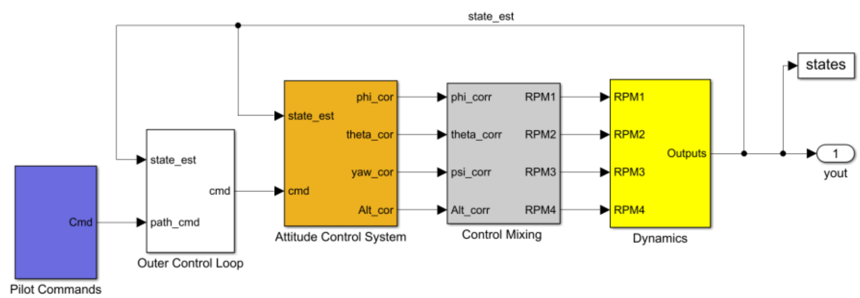

Figure 2 Simulink Block Diagram

Overall system is modelled by Simulink using quadcopter dynamics and control algorithms. Model is divided into 4 main subsystems: 'Outer Control Loop', 'Attitude Control System', 'Control Mixing' and 'Dynamics'.

'Pilot Commands' contains desired position in X, Y and Z axes and yaw angle, ψ. Different paths are defined to investigate the robustness of controllers.

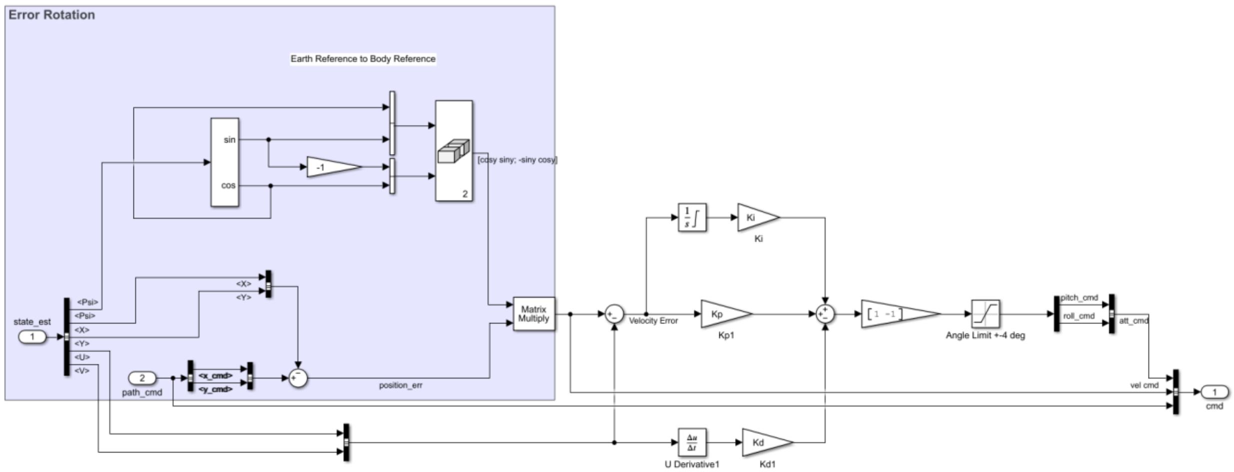

Figure 3 Outer Control Loop Subsystem

‘Outer Control Loop’ is designed according to Eqs. (27)-(29). The desired velocity command is found by rotating the position error along ψ axis. After error rotation velocity error is given into PID controller and desired pitch and roll angles (θ_d, ϕ_d) are obtained. Desired angles are limited with $\pm 4°$.

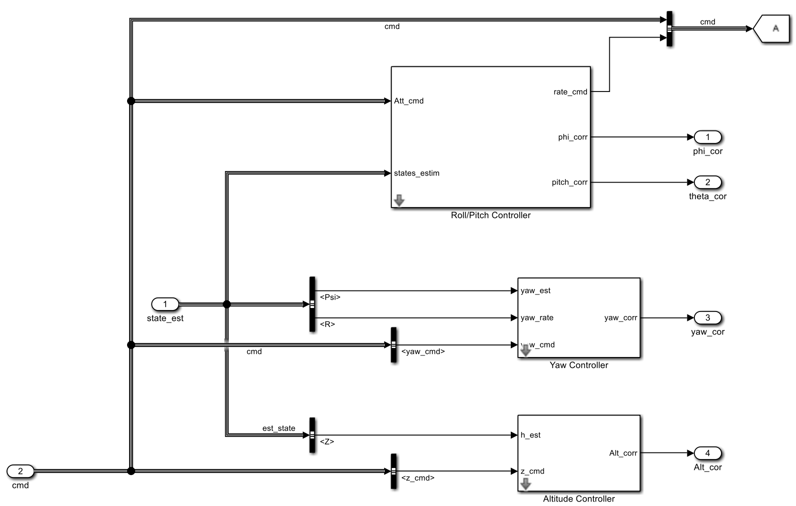

Figure 4 Attitude Controller Block

‘Attitude Control Loop’ contains controllers for ϕ, θ, ψ *and* Z. This subsystem represents Eqs.(22)-(26). It is the core of the Simulink model. It processes the task and instant states to estimate the throttle command for each motor. PID gains are found with trial-error approach. Firstly, each gains and commands are set to zero. Then, step input is given to attitude rate controller. According to response of system PID gains are tuned. Similarly, step input is used for ϕ and θ control. After tuning the attitude angles, Z and ψ are tuned according to pilot commands. Improvement in attitude response affect the position controller.

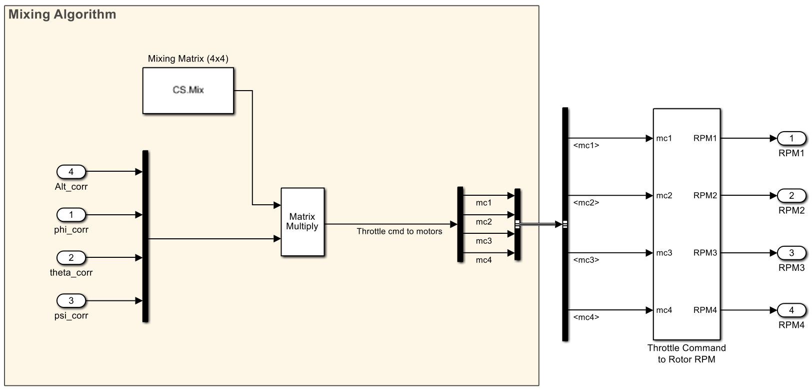

Figure 5 Control Mixing Block

‘Control Mixing’ block uses Eqs. (21),(30) to define speed of each motor with output of ‘Attitude Control’ subsystem. Mixing matrix is the coefficient matrix of Eq.(30) with,

$$Mix = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \tag{31}$$

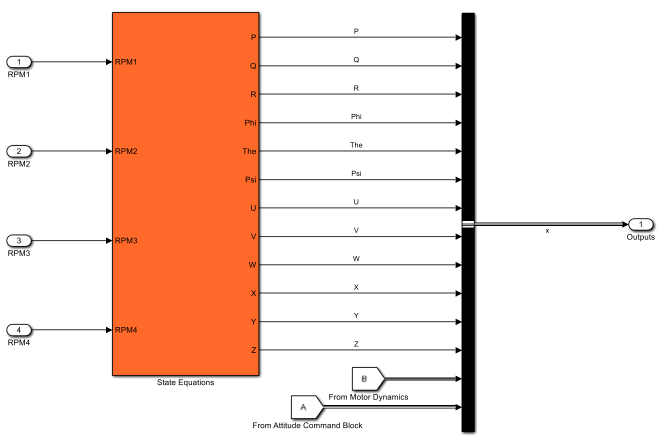

Figure 6 Dynamics Block

Figure 6) represents ‘Dynamics’ block. Each propellers’ speed is fed into this block as input to level 2 S-Function in MatLab Code. ‘State Equations’ contains Eqs. (1)-(18). Outputs of this block is states of quadcopter, which are fed back closed loop system.

Results and Discussion

A scenario is evaluated in Simulink. Each initial condition is set to zero. Pilot commands are given for $40\ s$. Also, the commands are chosen to allow observing the quadcopter's take-off, hover and landing modes. Pilot commands and response of quadcopter is given at Figure 7). Parameters taken from Ref. [2] are summarized in Table 1).

m	$0.4710\ kg$
I_{xx}, I_{yy}	$0.0108\ kg\ m^2$
I_{zz}	$0.0212\ kg\ m^2$
l	$0.22\ m$
b	$2 \times 10^{-7}\ \ N\ s^2\ RPM^{-1}$
d	$2 \times 10^{-9}\ N\ m\ s^2\ RPM^{-1}$
J_{TP}	$3.7882 \times 10^{-6}\ \ N\ m\ s^2$
C_r	107.65
C_b	1302.7

Table 1 Quadcopter Model Parameters

Quadcopter follows position commands in X and Y with a steady-state error of approximately 1 m due to lagging time. But in hover mode this error is compensated. Change in ψ is observed due to rate of attitude angle. If there is no change in either X or Y then change in ψ is significantly low. Relatively good response to altitude command are observed, lagging time is near $0.25\ s$. The maximum percentage overshoot is 2.2% for position controllers.

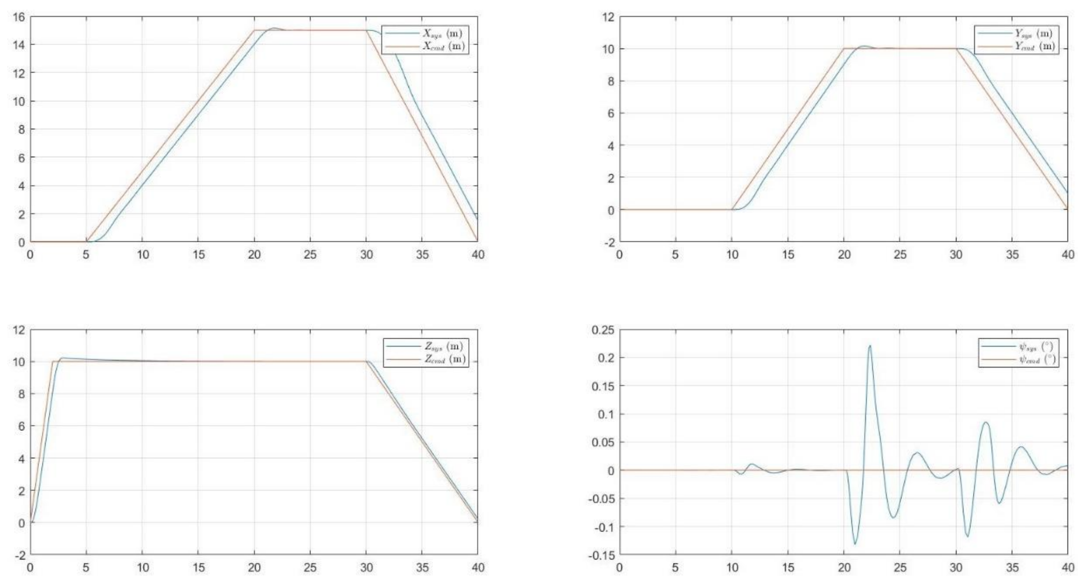

Figure 7 Response of Quadcopter to Given Commands

This paper covers a study conducted on quadcopter control during a student's internship. After modelling the quadcopter dynamics, a suitable PID controllers are designed to stabilize the quadcopter. For simplicity, empirical results are used for motor-system relationship and disturbance due to environment are ignored.

To improve this study, more accurate model can be used. An experimental setup can be created to obtain empirical parameters, such as drag and thrust coefficient. Also, motor and propeller's dynamics can be modelled considering physical constraints. Disturbance due to environment can be modelled. Since, dynamics of quadcopter is highly coupled in order to obtain a more stable system, control parameters need better tuning or other control methods can be used. Although PID controllers can be designed easily for linear systems, they are not enough for nonlinear systems.

References

[1] T. Bresciani, “Modelling, Identification and Control of a Quadrotor Helicopter,” Lund University, 2008.

[2] M. S., M. A., M. S. Z., M. M., S. A., A. R., M. K., I. M., Z. M., N. S. Ashraf, “Modelling and Parameters Identification of a Quadrotor Using a Custom Test Rig,” *International Journal of Power Electronics and Drive Systems (IJPEDS)*, vol. 9, no. 2, pp. 865–872, Jun. 2018.

[3] A. Tiwari, “Position Control of an Unmanned Aerial Vehicle from a Mobile Ground Vehicle,” Michigan Technological University, 2017. doi: 10.37099/mtu.dc.etdr/470.