Assignment #1

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Question 1. Prove by induction that for $n \geq 1$, $\sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n}$

- Base case: For the base case we know that $n = 1$, then:
  $\sum_{i=1}^{1} \frac{1}{2^i} = \frac{1}{2}$; and also $1 - \frac{1}{2} = \frac{1}{2}$.
  Therefore the equation holds for the base case.

- Inductive hypothesis: Assume that the equation holds for $2 \leq n \leq k$ for some integer k

- Inductive step: For the case where $n = k + 1$, we have that:

$$\begin{aligned}\sum_{i=1}^{k+1} \frac{1}{2^i} &= \sum_{i=1}^{k} \frac{1}{2^i} + \frac{1}{2^{k+1}} \\ &= 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}} \text{ (by inductive hypothesis)} \\ &= 1 - (\frac{1}{2^k})(1 - \frac{1}{2}) \\ &= 1 - \frac{1}{2^{k+1}}\end{aligned}$$

  Then, if the equations holds for k it also holds for $k + 1$. Now by induction, using the base case, inductive hypothesis and inductive step, we have that for all $n \geq 1$ the equation holds, as we wanted to proof.

Question 2. Show that any integer postage greater than 7 cents can be formed by using only 3-cent and 5-cent stamps

Basically, what we want to demonstrate is that, for any $n > 7$, n can be written as

$$n = 3a + 5b \tag{1}$$

where a and b non negative numbers. Let k be an integer such that $k > 7$ and take $k \mod 3$:

- if $k \equiv 0 \mod 3$:
  Then, for some integer $q \geq 0$, $k = 3q$, so equation 1 holds.

- if $k \equiv 1 \mod 3$:
  We have that
  $$k - 10 \equiv 1 - 10 \mod 3$$
  $$k - 10 \equiv 0 \mod 3$$
  and, since $k > 7$ and $k \equiv 1 \mod 3$ we have that $k \geq 10$, hence $k - 10 \geq 0$ and there is $q \geq$ integer such that
  $$k - 10 = 3q$$
  $$k = 3(q) + 5(2)$$
  Therefore, equation 1 also holds for this case.

- if $k \equiv 2 \mod 3$:
  We have that
  $$k - 5 \equiv 2 - 5 \mod 3$$
  $$k - 5 \equiv 0 \mod 3$$
  and, since $k > 7$, there is $q \geq 0$ integer such that
  $$k - 5 = 3q$$
  $$k = 3(q) + 5(1)$$
  Therefore, equation 1 holds again.

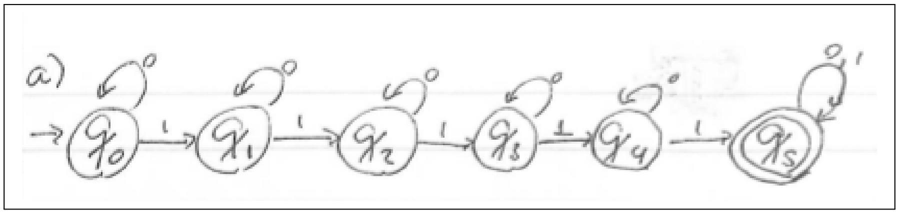
Since the equation 1 holds for all cases it holds for any case.

Question 3. Give a DFA for each of the following languages

(a) Binary strings that contais at least five 1s.

- $Q = \{q_0, q_1, q_2, q_3, q_4, q_5\}$,
- $\Sigma = \{0, 1\}$,
- $q = q_0$,
- $F = \{q_5\}$, and
- δ is defined as follows

	0	1
q_0	q_0	q_1
q_1	q_1	q_2
q_2	q_2	q_3
q_3	q_3	q_4
q_4	q_4	q_5
q_5	q_5	q_5

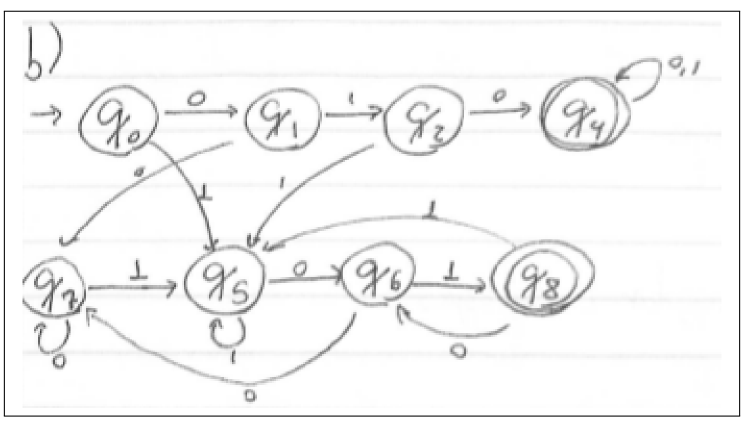


In this machine, the state q_i, $0 \leq i \leq 4$, represents that the string read has i 1s. The state q_5 represents that the string has 5 or more 1s.

(b) Binary strings that beggins with 010 or ends with 101

- $Q = \{q_0, q_1, q_2, q_4, q_5, q_6, q_7, q_8\}$,
- $\Sigma = \{0, 1\}$,
- $q = q_0$,
- $F = \{q_4, q_8\}$, and
- δ is defined as follows.

	0	1
q_0	q_1	q_5
q_1	q_7	q_2
q_2	q_4	q_5
q_4	q_4	q_4
q_5	q_6	q_5
q_6	q_7	q_8
q_7	q_7	q_5
q_8	q_6	q_5



The states q_i, $0 \leq i \leq 4$ are states that verify if the string starts with 010. If this fails, the machine goes to the states bellow which are responsible for verifying if the string ends with 101.

(c) Binary strings having the substring 010 but not having the substring 101

- $Q = \{q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_7\}$,
- $\Sigma = \{0, 1\}$,
- $q = q_0$,
- $F = \{q_4, q_5, q_8, q_9\}$, and
- δ is defined as follows.

	0	1
q_0	q_1	q_2
q_1	q_1	q_3
q_2	q_6	q_2
q_3	q_4	q_2
q_4	q_5	q_7
q_5	q_5	q_8
q_6	q_1	q_7
q_7	q_7	q_7
q_8	q_9	q_8
q_9	q_5	q_7

The states q_1 and q_3 are responsible for verifying if the string has substring 010, and once this is checked the machine stays in the states $q_4, q_5, q_8 and q_9$ as long as the string 101 is not read. If the substring 101 appears before 010 the machine goes to state q_7 through the states q_2 and q_6.