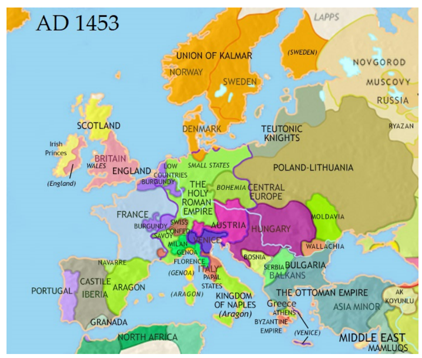
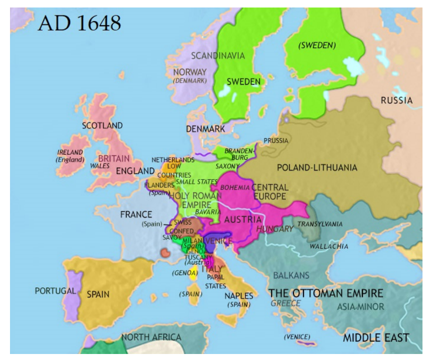
7 The Renaissance in Europe

7.1 The Communication of Knowledge into Europe

Between the fall of Rome in 476 and the early renaissance[38] around 1100, came Europe's dark ages. The one-dimensional view is that this was a time of little learning, or technological progress, though the reality was more complex. Prior to the 1100's, learning continued in European monasteries, though monks were unlikely to publish and widely promote their understanding. By 1100 the small shifting kingdoms that had characterised Europe were coming together as more stable nation states.[39] While occasional war is certainly a stimulus for development, the existence of large stable states has generally been more of boon for education.[40] By the 1700s, the borders of western Europe are largely recognizable; political and social organization had expanded so that a much larger proportion of the population, still tiny in comparison to today, were able to take advantage of and contribute to the growth of knowledge. The maps below show feudal societies (France, England, Holy Roman Empire, Poland, Austria) consolidating. The European renaissance is often contrasted with a decline in Islamic power, but again the story is more complex; Islam retreats from Spain, while the Ottoman Empire becomes dominant in south-eastern Europe.





Between 900 and 1300, European population roughly tripled to around 100 million. Trade increased, along with which came knowledge.[41] Land and knowledge also came from Islamic areas via war (the Crusades, Spain, etc.). It helped that Islamic scholars so venerated the Greeks; Europeans could tell themselves that they were merely 'reclaiming' ancient knowledge which had been 'stolen' by their cultural and religious enemies.[42] The fall of Constantinople to Mehmet the Conqueror in 1453 marks both the high point of Islamic conquest and the start of the decline of Islamic scientific dominance. Many intellectuals fled Constantinople (which, under the Byzantines, had preserved the last European vestiges of Alexandria's knowledge) for Rome helping to further fuel learning. With powerful enemies to the east, Europeans began travelling greater distances by sea,[43] beginning the period of colonization and global European empires.

[38] Literally *rebirth.* The timing of the renaissance varies depending on geography and discipline (Italy vs. France, art vs. philosophy, etc.) but a very wide net would encompass the 12$^{\text{th}}$ to 17$^{\text{th}}$ centuries.

[39] This is civilization in the non-pejorative sense: a system of rules and norms allowing *large* numbers of people to live together. For the peasantry, its arrival often meant little more than the overthrow of a small local by a larger regional one.

[40] For much of history, non-religious education was restricted to an elite with the time, money and inclination to devote to it. Large states support a larger elite population and are more able to fund universities and libraries.

[41] Venice was a particularly important trading hub; the Venetian Marco Polo (1254–1324) is perhaps the most famous trader of the time, travelling the silk road to China.

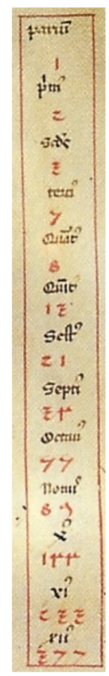
[42] A chauvinistic attitude that persisted well into the 1900s and, some would argue, is still alive and well.

[43] Christopher Columbus (born Genoa 1451) famously 'discovered' America in 1492 while looking for sea routes to Asia.

Scientific and philosophical progress was spurred by the translation of works from Arabic and ancient Greek into Latin; the first universities were formed to teach this canon: Bologna 1088, Paris 1150 and Oxford 1167. The typical student was a young man of wealth who had been privately tutored in grammar, logic & rhetoric (the *trivium*). At university he would study the *quadrivium* (geometry, astronomy, arithmetic & music). While Islamic improvements were incorporated, scholars gave pre-eminence to the Greeks: Euclid for geometry, Aristotle for logic/physics, Hippocrates/Galen for medicine, Ptolemy for astronomy. Early universities were often funded by the Church and 'research' was more likely to involve the justification of biblical passages using Aristotle than the conduct of experiments.

Leonardo Pisano (Fibonacci c. 1175–1250) Fibonacci[44] likely first encountered the Hindu–Arabic numerals while trading with his father in North Africa. He was impressed by the ease of calculation they afforded and is the first European known to use them; *Liber Abaci* (1202) was written to instruct traders in their use. In the first picture below, Fibonacci explains how to compute with decimal fractions; the two columns at the bottom of the page show how to repeatedly multiply 100 (and then 10) by the fraction $\frac{9}{10}$. Thus:

$$100, \quad 90, \quad 81, \quad \frac{9}{10}72\ (= 72.9), \quad \frac{1}{10}\frac{6}{10}65\ (= 65.61), \quad \frac{9}{10}\frac{4}{10}\frac{0}{10}59\ (= 59.049), \quad \text{etc.}$$



Note how the decimal part is written backwards on the left, using fractions with a bar to separate numerator and denominator; the Indians wrote fractions without a bar and it is thought Islamic scholars inserted it for clarity in the 1100s. The second picture is of Fibonacci's famous sequence: read top-to-bottom 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377. Amongst other inheritances from the Hindu–Arabic tradition, Fibonacci is also the first known European to work with negative numbers, provided these represented deficiencies or debts in accounting. Over the following centuries, Hindu–Arabic numerals slowly came to replace Roman numerals in Europe.

[44] The name was given to Leonardo by French scholars of the 1800s: *filius Bonacci* means *son of Bonacci.*

7.2 Algebraic Notation and Development

To a modern reader, the most obvious mathematical development of the renaissance is notational. Fibonacci's fractional notation was cutting-edge for the 1200s but essentially everything else except numbers and fractions was written in sentences. The slow mathematical revolution of the next 500 years is in many ways the story of how notational improvements eventually allowed algebra to eclipse geometry as the primary language of reasoning. Here is a brief summary.

Italian Abacists, 14th C. This group continued Fibonacci's advocacy for the Hindu–Arabic system against the traditional use of Roman numerals, and also for the use of accompanying algorithms. Their approach was highly practical and largely for the purposes of conducting trade. Here is a typical problem described by the group:

> The *lira* earns three *denarii* a month in interest. How much will sixty *lire* earn in eight months?

A trader would use their texts to find a worked solution to a problem similar to the one they needed. The Abacists also introduced the use of shorthands and symbols for unknowns and mathematical operations. *Cosa* ('thing') was used for an unknown, while *censo, cubo* and *radice* meant, respectively, square, cube and (square-)root. These expressions could be combined, for example 'ce cu' (read *censo di cubo*) referred to the sixth power of an unknown $(x^3)^2 = x^6$.

Luca Pacioli, Italy late 1400s. Introduced $\overline{p}, \overline{m}$ (*piu, meno*) for plus and minus. For example $8\overline{m}2$ denoted eight minus two.

Nicolas Chuquet, France 1484. *Triparty en la science des nombres* borrowed the Pacioli's $\overline{p}, \overline{m}$ notation and introduced an R-notation for roots. For instance $R^4 7$ meant $\sqrt[4]{7}$ while

$$\sqrt[5]{4-3\sqrt{2}} \quad \text{would be written} \quad R^5\underline{4\overline{m}3R2}$$

where the underline indicates grouping (essentially parentheses).

Christoff Rudolff, Vienna 1520s. Introduced symbols similar to x and ζ for an unknown and its square. He had other symbols for odd powers and produced tables showing how to multiply these. The words he used show Italian and French influence: algebra in the German-speaking world was known as the art of the *coss* (German for *thing*). Rudolff also introduced $\pm$ symbols as *algebraic operations.* These had been used for around 30 years as a prefix denoting an excess or deficiency in a quantity (profit/loss in accounting). A period denoted equals and he is also credited with the first use of the square-root sign $\sqrt{\ }$, which is nothing more than a stylized r. This would be written *in front* of a number to denote a root, e.g. $\sqrt{\ }2$ rather than $\sqrt{2}$.

Robert Recorde, England 1557. Introduced the equals sign in *The Whetstone of Witt*, asserting, 'No two things are more equal than a pair of parallel lines.'

Francois Viète, France 1540–1603. Before Viète, mathematicians typically described how to solve particular equations algorithmically via examples (e.g. $x^3 + 3x = 14$ rather than $x^3 + bx = c$), expecting readers to change numbers to fit any required example. Viète pioneered the use of abstract constants, Used letters to represent unknowns and constants, pioneering a more modern approach to algebra.

Simon Stevin, Holland 1548–1620. *De Thiende* (The Tenths) demonstrated how to calculate using decimals rather than fractions. Stevin arguably completed the journey whereby the concept of *number* subsumed that of *magnitude,* asserting that every ratio is a number.

William Oughtred, England 1575–1660. Introduced $\times$ for multiplication, though he often simply used juxtaposition. Oughtred combined Viète's general approach (abstract constants) with symbolic algebra. For instance, to solve a quadratic equation $A_q + BA + C = 0$, where A_q means 'A-squared' (A-*quadratum* in Latin) he'd write the quadratic formula as

$$A = \sqrt{} : \frac{1}{4}B_q - C : -\frac{1}{2}B$$

In Oughtred's notation, colons were parentheses.

Thomas Harriot, England 1560–1621. Made several steps towards modern notation including juxtaposition for multiplication and a modern *encompassing* root-sign. For example,

$$\sqrt[4]{cccc + 27aa\sqrt[3]{2+b}} \quad \text{meant} \quad \sqrt[4]{c^4 + 27a^2\sqrt[3]{2+b}}$$

René Descartes, France/Holland 1596–1650. Uses exponents for powers (a^2, a^3) and starts the convention of using letters at the end of the alphabet (x, y, z) for unknowns and those at the beginning (a, b, c) for constants.

While modern mathematics uses many specialized symbols (e.g. $\emptyset$, $\Rightarrow$, e, π), basic notation is essentially unchanged from Descartes' time, though some mathematicians were still writing equations in sentences through the late 1800s. It is also worth mentioning in this context Gutenberg's invention of the printing press around 1439, which naturally aided the dissemination of all learning. The sudden relative ease of production meant a great increase in the availability of written material, but also in the rejection of some texts: if a new edition was not produced an older text might not be preserved.

Cardano and the Cubic

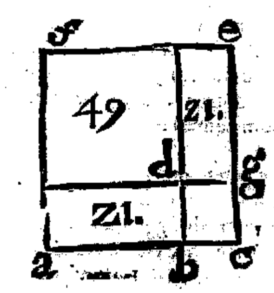
As an example of contemporary algebraic notation, we consider Girolamo Cardano's 1545 *Ars Magna* (*Great Art or the Rules of Algebra*), in which he describes how to solve quadratic and cubic equations.

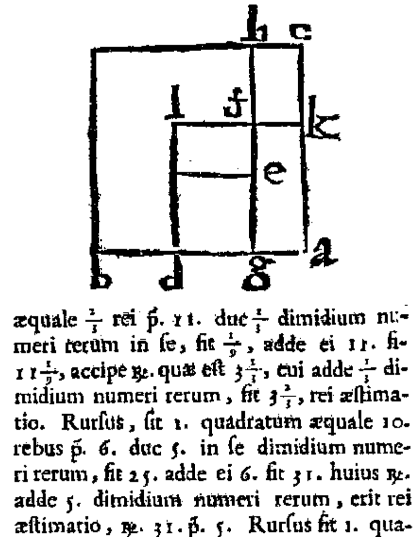
The example on the right (start of *Caput V,* page 9 of the linked pdf), is the beginning of Cardano's description of how to solve $x^2 + 6x = 91$, *quadratum & 6 res aequalie 91* (square and 6 things equals 91), by completing the square. He employs several single-letter abbreviations but still writes in sentences and provides a pictorial justification. In modern algebra, the argument is essentially,

$$\begin{aligned} x^2 + 6x = 91 &\implies x^2 + 2 \cdot 3x + 3^2 = 91 + 3^2 \\ &\implies (x+3)^2 = 100 \\ &\implies x + 3 = 10 \\ &\implies x = 7 \end{aligned}$$

where the picture essentially justifies $7^2 + 2 \cdot 21 + 3^2 = 10^2$.

SIT quadratum f d & 6. res (gratiâ exempli) æquale 91. tunc producam d b & d g, quæ ſint 3. dimidium 6. numeri rerum, & complebo quadratum d g b c, inq́ue productis c g & c b perficiam quadratum a f e c, prout in quarta ſecundi Elementorum, quia igitur d b ducta in a b ex diffinitione ſecundi Elementorum producit a d, & ex numero quolibet in rei æſtimatio-





æquale $\frac{2}{3}$ rei p̃. 11. duc $\frac{1}{3}$ dimidium numeri rerum in se, fit $\frac{1}{9}$, adde ei 11. fit 11$\frac{1}{9}$, accipe ℞. quæ est 3$\frac{1}{3}$, cui adde $\frac{1}{3}$ dimidium numeri rerum, fit 3$\frac{2}{3}$, rei æstimatio. Rursus, sit 1. quadratum æquale 10. rebus p̃. 6. duc 5. in se dimidium numeri rerum, fit 25. adde ei 6. fit 31. huius ℞. adde 5. dimidium numeri rerum, erit rei æstimatio, ℞. 31. p̃. 5. Rursus fit 1. qua-

The quadratic algorithms were well-established by this time; on the following page of his text, we find a picture from the work of al-Khwārizmī (Exercise 6.2.2), after which comes more obvious mathematical notation. Even though Rudolff's x, $\pm$ and $\surd$ were in use, Cardano wrote almost everything in words augmented by fractional notation and Pacioli's $\overline{p}$ and $\overline{m}$.

As was typical for the time, Cardano describes negative solutions to equations as fictitious, though he even writes the square root of -15 in one 'solution,' if only to mention its absurdity. He also follows the Islamic approach of solving a concrete problem of each type, rather than proceeding abstractly.

It is for the solution of cubic (and quartic) equations that Cardano is most famous. Below we describe, in modern notation, Cardano's method for solving the cubic equation $x^3 + bx = c$ where $b, c > 0$, though we stress (again) that Cardano only gave *examples* not a general formula.

Let u, v satisfy $u^3 - v^3 = c$ and $uv = \frac{b}{3}$. Then $x = u - v$ is seen to solve the cubic.

$$\begin{aligned} x^3 + bx &= (u-v)^3 + b(u-v) = u^3 - 3u^2v + 3uv^2 - v^3 + b(u-v) \\ &= (u^3 - v^3) + (u-v)(b - 3uv) = c \end{aligned}$$

However u and v also satisfy

$$(u^3+v^3)^2 = (u^3-v^3)^2 + 4(uv)^3 = c^2 + 4\left(\frac{b}{3}\right)^3$$

so we obtain a system of linear equations in the unknowns u^3, v^3 which are easily solved:

$$\begin{cases} u^3 + v^3 = \sqrt{c^2 + 4\left(\frac{b}{3}\right)^3} \\ u^3 - v^3 = c \end{cases} \implies \begin{cases} u = \sqrt[3]{\sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3} + \frac{c}{2}} \\ v = \sqrt[3]{\sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3} - \frac{c}{2}} \end{cases}$$

$$\implies x = u - v = \sqrt[3]{\frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3}} + \sqrt[3]{\frac{c}{2} - \sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3}}$$

Example Consider $x^3 + 3x = 14$: we have

$$\begin{cases} u^3 - v^3 = 14 \\ uv = \frac{3}{3} = 1 \end{cases} \implies (u^3+v^3)^2 = 14^2 + 4 \cdot 1^3 = 200 \implies u^3 + v^3 = 10\sqrt{2}$$

$$\implies u^3 = \frac{1}{2}(10\sqrt{2} + 14) = 5\sqrt{2} + 7, \qquad v^3 = 5\sqrt{2} - 7$$

$$\implies x = u - v = \sqrt[3]{7 + 5\sqrt{2}} + \sqrt[3]{7 - 5\sqrt{2}} = (1 + \sqrt{2}) + (1 - \sqrt{2}) = 2$$

As the last step shows, Cardano's formula might produce an ugly expression for a simple answer!

Being unable or unwilling to work directly with negative numbers, Cardano modified his method to solve other cubics such as $x^3 + c = bx$, and moreover described how to remove a quadratic term from a cubic using what is now known as the *Tschirnhaus substitution* ($x = y - \frac{a}{3}$):

$$x^3 + ax^2 + bx + c = \left(y - \frac{a}{3}\right)^3 + a\left(y - \frac{a}{3}\right)^2 + b\left(y - \frac{a}{3}\right) + c = y^3 - \frac{a^2}{3}y + \cdots \qquad (*)$$

Lodovico Ferrari (Cardano's student) extended the method to solve quartic equations in terms of the solution of a resultant cubic.

Negative Solutions and Complex Numbers By the late 1500s, mathematicians were mentioning negative solutions to equations; these were usually described as *fictitious*, or *false roots*, but this didn't stop them from being investigated. Rafael Bombelli (1526–1572, Rome) introduced a notation for complex numbers, described their algebra, and showed how they could be used to find solutions to any quadratic or cubic equation. For example, he would write $4 + 3i$ and $4 - 3i$ as follows:

4 *p di m* 3, read '*quattro piu di meno tre*' (four plus of minus three), and,
4 *m di m* 3, '*quattro meno di meno tre.*'

Given Bombelli's stated belief in the fictitiousness of complex numbers, the effort he expended in their honor is extraordinary: this is a prime example of pure abstraction; math for the sake of math!

Using modern language, the three roots of the cubic $x^3 + bx = c$ are

$$u - v, \quad \zeta u - \zeta^2 v, \quad \zeta^2 u - \zeta v$$

where $\zeta = e^{2\pi i/3} = \frac{-1+\sqrt{3}i}{2}$ is a primitive cube root of unity. Together with the Tschirnhaus substitution $(*)$, Cardano's formula therefore solves all cubic equations.

Examples 1. Returning to our previous example, if $x^3 + 3x = 14$, then $u = \sqrt{2} + 1$ and $v = \sqrt{2} - 1$, from which the three solutions are

$$u - v = 2$$

$$\zeta u - \zeta^2 v = (\sqrt{2}+1)\frac{-1+\sqrt{3}i}{2} - (\sqrt{2}-1)\frac{-1-\sqrt{3}i}{2} = -1 + \sqrt{6}i$$

$$\zeta^2 u - \zeta v = (\sqrt{2}+1)\frac{-1-\sqrt{3}i}{2} - (\sqrt{2}-1)\frac{-1+\sqrt{3}i}{2} = -1 - \sqrt{6}i$$

2. To find a solution to $x^3 + 3x^2 = 3$, we perform the substitution $x = y - 1$ before applying Cardano's method:

$$y^3 - 3y^2 + 3y - 1 + 3(y^2 - 2y + 1) = 3 \implies y^3 - 3y = 1 \qquad (b = -3, c = 1)$$

$$\implies \begin{cases} u^3 - v^3 = 1 \\ uv = -\frac{3}{3} = -1 \end{cases} \implies (u^3 + v^3)^2 = 1^2 + 4(-1)^3 = -3$$

$$\implies u^3 + v^3 = \sqrt{3}i$$

$$\implies x = u - v - 1 = \sqrt[3]{\frac{1+\sqrt{3}i}{2}} + \sqrt[3]{\frac{1-\sqrt{3}i}{2}} - 1$$

This is ugly, but is in fact a real number, being the sum of complex conjugates. If you know Euler's formula, you can check that $x = 2\cos 20° - 1 \approx 0.8794$.

Factorization & the Fundamental Theorem of Algebra

By the late 1500s, Viète's abstraction allowed him to improve and generalize Cardano's methods. He also investigated the relationship between the coefficients of a polynomial and its roots; for Viète the roots had to be positive, but later improvements by Thomas Harriot and Albert Girard (1629) applied this to all polynomials. For instance, if $ax^2+bx+c=0$ has roots r_1, r_2, then

$$(r_1-r_2)\big(a(r_1+r_2)+b\big)=a(r_1^2-r_2^2)+b(r_1-r_2)=\big(ar_1^2+br_1+c\big)-\big(ar_2^2+br_2+c\big)=0$$

Provided the roots are distinct, we conclude that

$$\frac{b}{a}=-(r_1+r_2), \quad \frac{c}{a}=-\big(r_1^2-(r_1+r_2)r_1\big)=r_1r_2$$

These are the simplest version of what are known as Viète's formulas. Their use amounts to an early form of factorization.

Example To solve $3x^2-2x-1=0$, first spot that $r_1=1$ is a root. By Viète's formulas,

$$r_1+r_2=-\frac{b}{a}=\frac{2}{3} \implies r_2=-\frac{1}{3} \quad \text{or alternatively} \quad r_1r_2=\frac{c}{a}=-\frac{1}{3} \implies r_2=-\frac{1}{3}$$

Think about the relationship between this approach and factorization!

A nice side-effect is a way to obtain the quadratic formula analogous to Cardano's cubic approach.

$$\begin{cases} r_1+r_2=-\frac{b}{a} \\ r_1-r_2=\sqrt{(r_1-r_2)^2}=\sqrt{\left(\frac{b}{a}\right)^2-4\frac{c}{a}} \end{cases} \implies r_1, r_2=-\frac{b}{2a}\pm\frac{1}{2}\sqrt{\left(\frac{b}{a}\right)^2-4\frac{c}{a}}$$

Viète's formulas were understood for polynomials of arbitrary degree. These relationships were central to the later development of Galois Theory (1830) and the Abel–Ruffini theorem regarding the insolubility of quintic and higher-degree polynomials.

The remaining key results regarding solutions of polynomial also appeared around this time:

Fundamental Theorem of Algebra Up to multiplicity, a degree n polynomial has n complex roots. Girard offered the first version in 1629, though a complete proof wasn't given until the work of Argand, Cauchy and Gauss in the early 1800s.

Factor Theorem In 1637, Descartes proved that $p(r)=0 \iff p(x)$ is divisible by $x-r$.

For instance, to find the roots of the polynomial $p(x)=x^3+2x^2-13x+10$, Descartes would observe:

- $p(1)=0 \implies x-1$ is a factor, so divide to get $p(x)=(x-1)(x^2+3x-10)$.
- $q(x)=x^2+3x-10$ has $q(2)=0$; divide to get $q(x)=(x-2)(x+5)$.
- The roots of $p(x)$ are therefore 1, 2 and -5 (in keeping with the times, Descartes called this last a *false root*).

Exercises 7.2. 1. I am owed 3240 *florins.* The debtor pays me 1 *florin* the first day, 2 the second day, 3 the third say, etc. How many days does it take to pay off the debt?

2. (A problem of Antonio de Mazzinghi) Find two numbers such that multiplying one by the other makes 8 and the sum of their squares is 27.

   (*Hint: let the numbers be* $x \pm \sqrt{y}$)

3. (a) Find Viète's formulas for the polynomial $p(x) = x^3 + ax^2 + bx + c$ with roots x_1, x_2, x_3; that is, find the coefficients a, b, c in terms of the roots.

   (*Hint: the simplest way is to multiply out; pretend you know how to factorize!*)

   (b) Solve $x^3 - 6x^2 + 9x - 4 = 0$ using Girard's method: first, determine one solution by inspection, then use Viète's formulas to investigate the relationship between the remaining roots.

4. (a) Apply Cardano's method to the equation $x^3 + 6x = 20$.

   (*Hint: to finish, compute* $(1+\sqrt{3})^3$)

   (b) If $b, c > 0$, Cardano's method finds a single positive solution to $x^3 + bx = c$. Explain why such an equation always has exactly one real solution which is moreover positive.

5. Prove that if t is a root of $x^3 = cx + d$, then

$$r_1 = \frac{t}{2} + \sqrt{c - \frac{3t^2}{4}} \quad \text{and} \quad r_2 = \frac{t}{2} - \sqrt{c - \frac{3t^2}{4}}$$

   are both roots of $x^3 + d = cx$. Use this to solve $x^3 + 3 = 8x$.

6. Consider the cubic equation $aaa - 3raa + ppa = 2xxx$ (as written by Harriot). Show that the substitution $a = e + r$ reduces this to an equation without a square term.

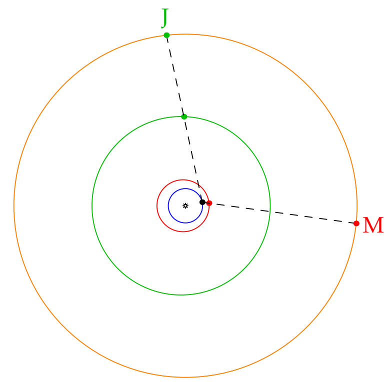
   As an example, reduce the equation $aaa - 18aa + 87a = 110$ to a cubic in e without a square term. Find all three solutions in e and therefore find the solutions to the original equation in a.

7. (If you are very comfortable with complex numbers) Use Euler's formula to verify that the equation $x^3 + 3x^2 = 3$ has the positive solution $x = 2\cos 20° - 1$. It also has two negative real solutions; find them!

7.3 Astronomy and Trigonometry in the Renaissance

Genuinely European progress in trigonometry came courtesy of Johannes Müller (Regiomontanus,[45] 1436–1476). *De Triangulis Omnimodius* (*Of all kinds of triangles,* 1463) provided an axiomatic update of Ptolemy's *Almagest* and its Islamic improvements. Though the title refers to triangles, his approach remains circle-based (chords and half-chords). Regiomontanus was a renowned astronomer; his tracking of a comet from late 1471 to spring 1472 provided controversial evidence that objects could move between the, supposedly fixed, heavenly spheres of ancient Greek theory.

Domenico Novara (1454–1504), one of Regiomontanus' students, inherited much of his unpublished work. He became a student of Luca Pacioli in Florence and an astronomer at the University of Bologna, though he is now perhaps best known as adviser to a young Pole, Nicolaus Copernicus (1473–1543), who studied in Bologna from 1496 with the ostensible intent of joining the priesthood...



Copernicus concluded that Ptolemy's geocentric model could not be reconciled with astronomical observation. *De revolutionibus orbium celestium* (*On the revolutions of the heavenly spheres*), published a year after his death, describes how to compute within a *heliocentric* (*sun-centered*) model. This, Copernicus believed, was the obvious solution to the problem of *retrograde motion* that had plagued the ancient Greeks.[46]

The animation demonstrates Copernicus' solution; with the sun at the center, the retrograde motion of Mars and Jupiter are easily explained. The outer circle represents the 'fixed stars' against which the motion of the planets are observed.

Copernicus' work is now described as a revolution, but at the time it was not perceived as such. *De revolutionibus* was dedicated to the Pope, welcomed by the Church and used by Vatican astronomers to aid in calculation. The difficulty and narrow readership of his work made it unthreatening to contemporary dogma. Copernicus did not present heliocentrism as reality nor advocate for overturning long-held beliefs. Within a century, however, the Copernican theory had found its bulldog in Galileo and conflict between science and the Church became unavoidable.

Trigonometry is finally about triangles!

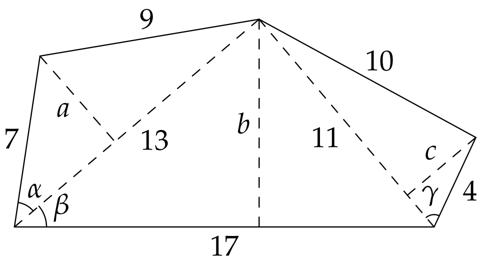
Georg Rheticus (1514–1574) defined trigonometric functions purely in terms of triangles, referring to the *perpendiculum* (sine) and *basis* (cosine) of a right-triangle with fixed hypotenuse. Rheticus was a student of Copernicus and helped to posthumously publish his work.

In 1595, Bartholomew Pitiscus finally introduced the modern term with his book *Trigonometriæ*, in which he purposefully sets out to solve problems related to triangles. The picture is the title page from the second edition (1600—MDC in Roman numerals). Both Rheticus and Pitiscus had problems which look very familiar to modern readers, such as solving for unknown sides of triangles.

[45] His grand-sounding name is a latinization of his birthplace Königsberg, literally *King's Mountain* (Bavaria, Germany).

[46] Copernicus wasn't the first to suggest such a model. Several ancient Greek scholars embraced heliocentrism, with Aristarchus of Samos (c. 310–230 BC) credited with its first presentation. However, Aristarchus' views were rejected by the Greeks, and it is likely Copernicus never encountered his work.

Example (Pitiscus) A field has five straight edges of lengths 7, 9, 10, 4 and 17 in order. The distance from the first to third vertex is 13 and from the third to fifth is 11. What is the area of the field?



The problem can be solved easily using Heron's formula, but Pitiscus opts for trigonometry; we give a modernized version that depends on applying the law of cosines to the three large triangles.

$$\cos\alpha = \frac{7^2+13^2-9^2}{2\cdot 7\cdot 13} = \frac{137}{182} \qquad \cos\beta = \frac{17^2+13^2-11^2}{2\cdot 17\cdot 13} = \frac{337}{442}$$
$$\cos\gamma = \frac{4^2+11^2-10^2}{2\cdot 4\cdot 11} = \frac{37}{88}$$

The values of α,β,γ and therefore the altitudes a,b,c of the three major triangles could be read off a table, or found exactly using Pythagoras':

$$a = 7\sin\alpha = 7\sqrt{1-\cos^2\alpha} = \frac{7}{182}\sqrt{182^2-137^2} = \frac{3}{26}\sqrt{1595},$$
$$b = 13\sin\beta = \frac{1}{34}\sqrt{81795}, \qquad c = 4\sin\gamma = \frac{5}{22}\sqrt{255}$$

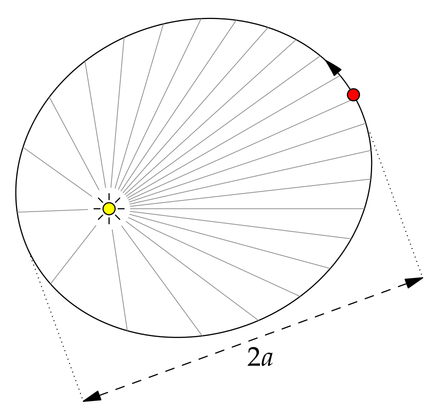
The total area is then easily computed:

$$A = \frac{1}{2}(13a+17b+11c) = \frac{1}{4}\left(3\sqrt{1595}+\sqrt{81795}+5\sqrt{255}\right) \approx 121.4$$

Kepler's Laws

Johannes Kepler (1571–1630) was a student of Tycho Brahe[47] for the last two years of Brahe's life. He inherited Brahe's position, his decades of accurate data and his philosophy on the importance of theory based on observation. Kepler also embraced the mystical Pythagorean view that nature reflects harmony; a belief that partly drove his scientific pursuits. To Kepler, any observation of a natural ratio was something of great import. For example, in observing that the daily movement of Saturn at its furthest point from the sun was roughly 4/5 of that at its nearest point, his temptation was to assume that 'roughly' must be 'exactly.'

Thanks to Brahe, Kepler had data on roughly thirteen orbits of Mars and two of Jupiter. From these data, he posited three laws.



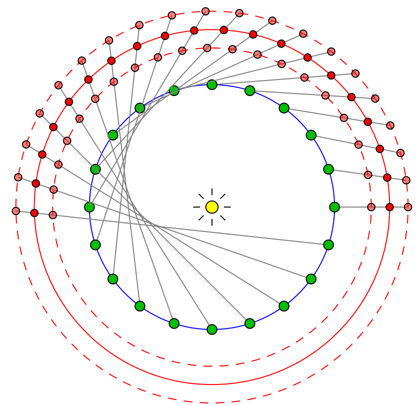
1. Planets move in ellipses with the sun at one focus.
2. The orbital radius sweeps out equal areas in equal times. In the picture, the sectors all have the same area: the planet therefore moves more slowly the further it is from the sun.
3. The square of the orbital period is proportional to the cube of the semi-major axis of the ellipse: $T^2 \propto a^3$.

[47]Tycho Brahe (1546–1601) was a Danish astronomer who worked for Austro-Hungarian Emperor Rudolph II in Prague for 25 years, producing a wealth of extremely accurate astronomical measurements. While these helped burnish the Copernican theory, he is better known for his 1572 observation of a *nova* (a new star that later disappeared, now known to be the death of a star) and then a comet in 1577, which provided yet more evidence of the changeability of the heavens.

Kepler's laws are empirical observations rather than the result of mathematical proof. The process of their discovery however demonstrates Kepler's tremendous mathematical ability.

Starting point Kepler began by assuming the essential correctness of the Copernican model in that all planets exhibit uniform circular motion round the sun.

Orbital Estimation Kepler's data told him the *direction* to each planet, but not the *distance,* though his Copernican assumption allowed him to estimate the *relative distance*.[48] For example, using the direction from Earth to Mars at equally spaced times and by drawing circles of different radii for possible orbits of Mars, he was able estimate an orbit where Mars' location was also equally spaced. This required an enormous number of trigonometric calculations; each measurement of planetary longitude/latitude *relative to Prague* had to be converted to measurements relative to the sun. Everything was subject to errors of estimation.



Modifying the model Kepler altered his model to reflect Earth's slightly non-circular orbit. He first tried an equant model(!), offsetting the center of the orbit slightly from the sun. Despite this, he failed to fit his data for Mars to pure circular motion.

The First Law Kepler now permitted planets to move in ovals. He decided to approximate Mars with an ellipse and set out to calculate its parameters, stumbling on an almost perfect match when the sun was placed at a focus. The *geometric* significance of the focus provided exactly the natural beauty Kepler sought. Having now established the first law for Mars, he repeated the exercise for the other known planets (Mercury, Venus, Earth, Jupiter, Saturn) as well as possible given his inferior data.

The Second Law Kepler's second law followed an infinitesimal argument based on inspired guesswork. Using his elliptical model, planetary velocity was non-constant, appearing inversely proportional to the distance from the sun ($v = \frac{k}{r}$). Kepler used this to approximate the area of a sector swept out by the radius vector: over a small time-interval Δt, a planet travels a distance $v\Delta t$ and thus sweeps out an approximate triangle of area $\frac{1}{2}rv\Delta t = \frac{1}{2}k\Delta t$. In modern language, this is simply the conservation of angular momentum. Kepler had no justification beyond the fact that it seemed to fit the data. In particular, he did not know why a planet should move more slowly when further from the sun.

The Third Law This was stated without analysis: given the relative sizes of each of the planetary orbits and their periods, some inspired guesswork allowed him to observe that $\frac{T^2}{a^3}$ is approximately the same value for each.

Kepler's discoveries were revealed over many years in several texts, with his magnum opus *Epitome astronomiæ Copernicanæ* published in 1621. Within a century Issac Newton had provided a mathematical justification of Kepler's laws based on the theory of calculus and his own axioms: an inverse square law for gravitational acceleration and his own three laws of motion.

[48] Following Ptolemy, Brahe thought the Earth-Sun distance was around 1/10 of the true value. Kepler thought this an underestimate by at least a factor of three. In 1659, Christiaan Huygens found the distance to an accuracy of 3%.

A Religious Interlude: Protestantism, the Counter-Reformation and Calendar Reform

In 1563, Pope Gregory began the Catholic Church's push-back against the spread of Protestantism,[49] the *counter-reformation.* Of particular interest to science and mathematics was the newly created *Index Librorum Prohibitorum,* a list of books contradicting Church doctrine. Kepler's book was banned immediately upon publication in 1621. His distance from Rome, however, meant that Kepler and his ideas were relatively safe. The ultimate result Gregory's crackdown was the slow ceding of scientific power to northern (Protestant) Europe where papal diktat had no effect.

In contrast to the anti-science book-banning fervor of the counter-reformation, Pope Gregory is also famous for shepherding an astounding scientific achievement: calendar reform. By 1500, astronomers knew the solar year to be roughly $11\frac{1}{4}$ minutes shorter than the $365\frac{1}{4}$ days of the Julian calendar.[50] The Church based the date of Easter on the vernal equinox;[51] by 1500, the accumulated error in this date had grown to 10–11 days. The impetus to correct the date of Easter meant that calendar reform became an important church project.

Over a century of effort[52] resulted in the *Gregorian calendar.* Gregory imposed the new calendar in all Catholic countries in 1582. It corrected the 10 day deficit by deleting October 5$^{\text{th}}$–14$^{\text{th}}$ 1582 and reforming the computation of leap-years: centuries are leap-years only if they are divisible by 400, thus 1600 was a leap year, but 1900 was not. The Gregorian calendar is astonishingly accurate, losing only one day every 3000 years. Since it emanated from Rome, many Protestant parts of Europe took decades if not centuries to adopt the new calendar. The Eastern Orthodox Church still computes Easter using the Julian calendar.

Galileo Galilei (1564–1642)

Based in northern Italy, Galileo was close to the center of Church power; unlike Copernicus and Kepler, he openly challenged its orthodoxy. While undoubtedly a great mathematician, he is more importantly considered the father of the scientific revolution for his reliance on experiment and observation. He famously observed Jupiter's moons with a telescope of his own invention, and noted that the presence of other objects orbiting an alien body was counter to Ptolemaic theory. Skeptics, when shown this image, preferred to assert that it must be somewhere *inside* the telescope!

In 1632 Galileo published *Dialogue Concerning the Two Chief World Systems,* a Socratic discussion between three characters: Salviati argued for Copernicus, Simplicio argued for Ptolemy, and Sagredo was an independent questioner. The character of Simplicio was provocatively modeled on conservative philosophers who refused to consider experiments and moreover bore a notable resemblance to the Pope. Salviati almost always came out on top and Simplicio was made to appear foolish. The text resulted in Galileo's conviction for heresy; all his publications, past and future, were banned, and he spent the remainder of his life under house arrest.

[49] Martin Luther's publication of his *Ninety-five Theses* in 1517 is generally considered the start of the Protestant Reformation: over the next 150 years Europe saw several religious wars as various countries broke away from Catholicism.

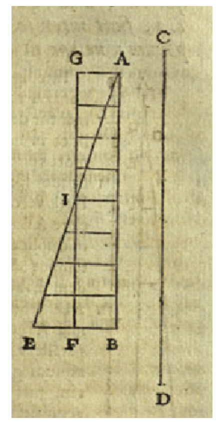
[50] Named for Julius Caesar, the Julian year has 365 days with a leap-day every four years.

[51] For roughly 1200 years, the Church decreed Easter to be the Sunday after the first full moon after the vernal equinox (March 20$^{\text{th}}$/21$^{\text{st}}$). By 1500 the equinox could be determined with great accuracy, yet it was happening roughly 10 days early (March 10$^{\text{th}}$).

[52] Pope Sixtus IV tried to recruit Regiomontanus to the cause in 1475, though the mathematician died first. Copernicus was among those invited to consider proposals in the early 1500s, though he distanced himself, perhaps because he knew that his developing heliocentric ideas would not be accepted.

Despite Church efforts, Galileo's works continued to be distributed by his supporters and he continued working. His most important scientific text, *Discourses Concerning Two New Sciences* (materials science and kinematics) was smuggled out of Italy to be published in Holland in 1638. In this book he resurrects his characters from *Two Chief World Systems*, and famously refutes Aristotle's claim that heavier objects fall more rapidly than lighter ones. Here are two mathematical results from this text.

Theorem. *If acceleration is uniform, then the average speed is the average of the initial and terminal speeds.*



Proof. Galileo argues pictorially; we augment with a little algebra.

$\overline{CD}$ is the time axis, increasing downward. Velocity is measured horizontally as the distance from $\overline{CD}$ to the uniformly sloped line $\overline{AE}$.

As an object passes point A it has speed v_A. Its speed increases to v_B uniformly over the time interval $t = |AB|$. Clearly

$$\text{area}(\triangle ABE) = \frac{1}{2}t(v_B - v_A) = \text{area}(\square ABFG)$$

But then

$$\begin{aligned} t \cdot v_{\text{av}} &= \text{distance travelled under motion} \\ &= t \cdot v_A + \text{area}(\triangle ABE) = \frac{1}{2}(v_B + v_A) \\ &= t \cdot \text{average of initial and final speeds} \end{aligned}$$

■

Corollary. *A falling object dropped from rest will traverse distance in proportion to time-squared,*

$$d_1 : d_2 = t_1^2 : t_2^2$$

This is Galileo's version of the well-known kinematics formula $d = \frac{1}{2}gt^2$.

Proof. Let d_1, d_2, v_1, v_2 represent the distances travelled and the speeds of the dropped body at times t_1 and t_2. Since acceleration is uniform,

$$v_1 : v_2 = t_1 : t_2$$

By the Theorem,

$$d_1 = \frac{0 + v_1}{2}t_1 = \frac{1}{2}v_1t_1, \quad \text{and} \quad d_2 = \frac{1}{2}v_2t_2$$

whence

$$d_1 : d_2 = v_1t_1 : v_2t_2 = t_1^2 : t_2^2$$

■

Galileo follows this by decomposing the motion of a projectile into horizontal (uniform speed) and vertical (uniform acceleration) components, thereby proving that projectiles follow parabolic paths.

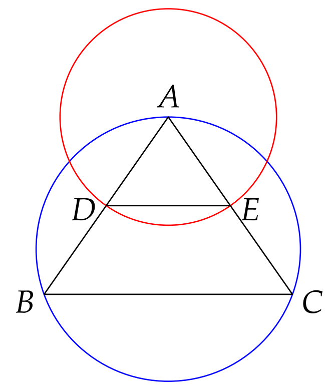
Galileo covered several other important mathematical topics, some of which we'll mention when we discuss calculus. While his mathematical ideas were cutting-edge for the time, it is his insistence on testing theory against data that makes him a true revolutionary. By 1600 very few of Aristotle's easy-to-refute claims had been rejected due to experimental testing; the hostility Galileo provoked by doing so perhaps explains why. This is the core of the scientific revolution: primacy is given to experiment and observation over ancient 'wisdom,' whatever the source.

Galileo was finally cleared of heresy by the Catholic Church in 1992.

Exercises 7.3. 1. *Compute* today's date in the Julian calendar and explain your calculation.

2. (A problem of Copernicus) Given the three sides of an isosceles triangle, to find the angles.

   Suppose $\overline{AB}$ and $\overline{AC}$ are the equal legs of the triangle. Circumscribe a circle around the triangle and draw another with center A and radius $\overline{AD} = \frac{1}{2}\overline{AB}$.



   (a) Why is Copernicus introducing the second circle?

   (b) Explain why the ratio of each of the equal sides to the base of $\triangle ABC$ equals that of the radius $\overline{AD}$ to the chord $\overline{DE}$.

   (c) If $|AB| = |AC| = 10$ and $|BC| = 6$, use modern trigonometry to find the three angles of the original triangle.

3. Verify that Heron's formula gives the same solution to Pitiscus' problem (pg. 76).

4. Given that Earth's orbital period is 1 year, and that the mean distance of Mars from the sun is 1.524 times that of Earth, use Kepler's third law to determine the orbital period of Mars.

5. According to Kepler's second law, at what point in a planet's orbit will it be moving fastest?

6. Galileo states the following.

   > A projectile fired at an angle $\alpha = 45°$ above the horizontal at a given initial speed reaches a distance of 20,000. Then, with the *same* initial speed it will reach a distance of 17,318 when $\alpha = 60°$, or $\alpha = 30°$.

   Check this statement: if you want a challenge, try not to use the standard Physics formulæ and do it all using ratios!