6 Indian and Islamic Mathematics

6.1 India, Hindu–Arabic Numerals & Zero

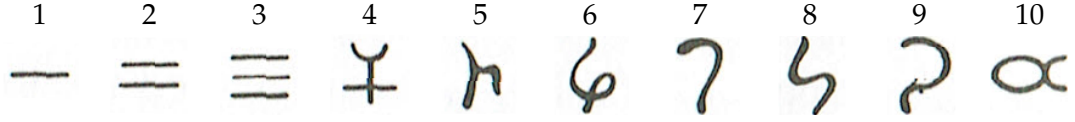
The Indian/South Asian subcontinent is bordered to the north by the Himalayas and the east by dense jungle. Its primary frontier has historically been the fertile Indus valley in the west, now the central corridor of Pakistan, where recorded civilization dates to at least 2500 BC. During the first millennium BC, Hinduism developed as an amalgamation of previous practices and beliefs; Buddhism and Jainism began to spread in the later part of this period, primarily in the Ganges valley.

Alexander the Great conquered as far east as the Indus in 326 BC, bringing Greek, Babylonian and Egyptian knowledge in his wake. The Greek overlords he left behind were rapidly overthrown and the subcontinent became largely unified under the Maurya Empire for the next 150 years. After this came 1000 years of shifting control with several invasions from the west by the nearby Persians. The expanding Islamic caliphate conquered the Indus around AD 1000, and the greater part of India was unified under the Islamic Mughal Empire by the 1500s. After its decline and fragmentation, the British conquered India in 1857.

The modern political situation reflects this complicated history. India gained its independence in 1947 after World War II, and was partitioned according to religion: the greater Indus valley and the lower Ganges/Brahmaputra comprise the Islamic states of Pakistan and Bangladesh, with the majority of the landmass becoming the nominally secular but majority Hindu *country* of India. The upper Indus valley (Kashmir) remains contested and has been the site of several military conflicts between India, Pakistan and China.

Ancient India is important not just for its separate contributions, but as a technological and cultural crossroads: it is estimated that it accounted for 25–30% of the world's economy during the 1st millennium AD! India's location made it well suited to absorb and synthesise ideas and technologies from both east and west; while some trade and knowledge passed north of the Himalayas directly between China and the Middle East/Europe, far more percolated slowly through India, being improved and given back in turn.

Brahmi Numerals & Numerical Naming Our primary focus is on possibly the most important practical mathematical development in history: the decimal positional system of enumeration, complete with fully-functional zero. The Brahmi numerals, one of the earliest antecedents of modern numerals, first appeared around the 3rd century BC.



The example dates from around 100 BC and was used in Mumbai/Bombay. Additional symbols denoted multiples of 10, 100, 1000, 10000, etc. As with Chinese characters, the system was partly positional (800 would be written by prefixing the symbol for 100 by that for 8) and there was no symbol or placeholder for zero.

Symbols are only part of the story. The modern approach to naming numbers and constructing large numbers can also be linked to the same period. The table below gives old Sanskrit names.

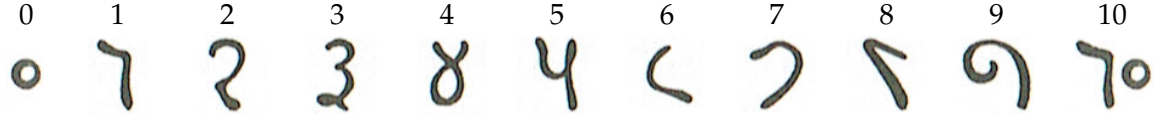
1	2	3	4	5	6	7	8	9
eka	dvi	tri	catur	pancha	sat	sapta	asta	nava
10	20	30	40	50	60	70	80	90
dasa	vimsati	trimsati	catvarimsat	panchasat	sasti	saptati	asiti	navati
100	1000	10000	100000	1000000	10^7	10^8	10^9	10^{10}
sata	sahasra	ayuta	niyuta	prayuta	arbuda	nyarbuda	samudra	madhya

You should recognize similarities with some of the numbers in European languages, many of which have Indian roots. The construction of larger numbers is also be familiar: for example *tri sahasra sat sata panchasat nava* is precisely how we read 3659.

There were also plenty of differences with modern verbiage. Sanskrit had distinct words for powers of 10 up to (at least!) 10^{62}. They also had a version of pre-subtraction: for example *ekanna-niyuta* meant 'one less than 100000,' or 99999.

Gwalior Numerals During the first few centuries AD, a fully positional decimal place system came into being. The earliest evidence comes from a manuscript found in Bakhshālī (Pakistan) in 1881, which has been carbon-dated to the 3rd or 4th century. The manuscript contains the earliest known version of the modern symbol for zero, a circular dot. It is conjectured that the decimal place system was inspired by the Chinese counting-board method, though convincing proof has yet to be uncovered. Regardless of attribution, Chinese mathematicians were copying the method by the 8th century.

The examples below are better understood than the Bakhshālī manuscript and come from Gwalior (northern India) around AD 876.



The similarity with modern numerals is clear; 0, 1, 2, 3, 4, 7, 9, 10 are very familiar. Zero has evolved from the Bakhshālī dot to a hollow circle. The symbols for 2 and 3 are conjectured to have developed in an attempt to write earlier versions (e.g. the Brahmi numerals) cursively; try writing three horizontal strokes quickly…

The system is fully positional. Below are the numbers 270 and 30984:

Sanskrit is written left-to-right, as are our modern numbers, with the leftmost digits representing the largest powers of 10. Note how zero is used as a placeholder to clarify position so that, e.g., 27, 207, and 270 are clearly distinguishable.

Zero On the right is a table of modern Sanskrit names and numerals; the digits and names are certainly similar to their Gwalior counterparts.

The Sanskrit *shuunyá* means *void* or *emptiness.* It is related to *svi* (hollow), which in turn derives from an ancient word meaning *to grow.* This reflects a major idea within religions of the area, with the void being the source of all things, of creation and creativity. Contemplation of the void (the doctrine of Shunyata) is recommended before composing music, creating art, etc. This contrasts with the Abrahamic religions where the void is something to be feared; an early conception of hell was the eternal absence of God.

The Gwalior numerals travelled westwards, with Europe eventually inheriting the system via Islam; as such they are today known the *Hindu–Arabic* numerals. Here is a short version of the etymological journey of zero into European languages.

- *Shunya* was transliterated to *sifr* in Arabic where the double-meaning persisted: *al-sifr* was the number zero, while *safira* meant *it was empty.*
- The term came to Europe in the 12$^{\text{th}}$-13$^{\text{th}}$centuries courtesy of Fibonacci where it became *cifra.* This was blended with *zephyrum* (*west wind*/*zephyr*) providing an alternate spelling.
- Cifra ultimately became the words *cipher* (English), *chiffre* (French) and *ziffer* (German), meaning a figure, digit, or code.
- Zephyrum became *zefiro* in Italian and *zero* in Venetian.

Zero and the Hindu–Arabic numerals also travelled eastwards, with Qin Jiushao introducing the zero symbol into China in the 13$^{\text{th}}$ century.

Our modern understanding of zero is a fusion of several concepts:

Numerical positioning For instance, to distinguish 101 from 11.

Absence of a quantity 101 contains no 10's.

Symbol First a dot (*bindu*), then a circle (*chidra/randhra* meaning *hole*). The relationship between *shunya* and a symbol was established by AD 2-300, as this quote from AD 400 (Vasavadatta) illustrates

> The stars shone forth, like zero dots [shunya-bindu] scattered as if on a blue rug. The Creator reckoned the total with a bit of the moon for chalk.

Mathematical operations By the time of Brahmagupta (7$^{\text{th}}$ C.), mathematical texts often contained a section called *shunya-gania,* with computations involving zero, including addition, multiplication, subtraction, effects on $\pm$-signs, division and the relationship with ∞ (*ananta*). In the 12$^{\text{th}}$ C., Bhaskaracharya stated:

> If you were to divide by zero you would get a number that was "as infinite as the god Vishnu."

Other ancient cultures had one or more of these aspects of zero, but the Indians were the first to put them all together.

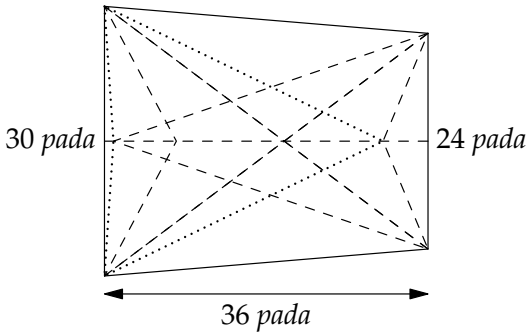
- The Egyptian hieroglyph *nfr* (beautiful/complete) indicated zero remainder in calculations as early as 1700 BC and was also used as a reference point/level in buildings.
- Very late in Babylonian times, a placeholder symbol was used to separate powers of 60. It was not used as a number.
- With the Chinese counting board, an empty space served as a placeholder.
- Various Mesoamerican cultures, such as the Maya, had a zero symbol that was used as a placeholder, particularly when writing dates.

'Real' Indian Mathematics

Indian mathematicians made great progress on several fronts, not merely the decimal place system.

Much ancient work was influenced by religion. The *sulbasutras* were written during pre-Hindu times and contained instructions for laying out altars using ruler-and-compass constructions. These could be quite complex, as the construction of the base of the *Mahavedi* (great altar) shows: The center line is divided left-to-right in the ratio

$$1 : 7 : 12 : 11 : 5$$



and the altar contains five distinct Pythagorean triples.

Of particular importance to our narrative is Indian work on trigonometry. Here are some highlights:

- The early 5th C. text *Paitāmahasiddhānta* is assumed to be an extension of Hipparchus' work, since it contains a table of chords based on a circle of radius 57,18; rather than Ptolemy's 60.
- Indian mathematicians instituted the use of *half-chords*, in line with our modern understanding of sine. Indeed the word *sine* is the result of a long sequence of (mis)translations and transliterations via Arabic and Latin from the Sanskrit *jyā-ardha* (*chord-half*).[29] The Indians also began to distinguish 'base sine' and 'perpendicular sine' (cosine).
- Created tables of sines/half-chords from 0° to 90° in steps of $3\frac{3}{4}°$, using linear interpolation to approximate values in between. By 650, Bhramagupta had much better approximations, using quadratic polynomials to interpolate. By 1530, Indian mathematicians had discovered cubic and higher approximations (essentially Taylor polynomials 130 years before Newton) for even greater accuracy of sine, cosine and arctangent.

Navigation was one of the drivers of this development. While Mediterranean sailors rarely strayed long out of sight of land, the Indians sailed the ocean and required accurate measurements to find their latitude.

[29] Amazingly this became related to the word *sinus* meaning *bay, gulf* or *bosom*!

Exercises 6.1. 1. The *Mahavedi* (pg. 56) contains five Pythagorean triples; find them.

2. To simplify square root expressions, Bhaskara used the formula

$$\sqrt{a+\sqrt{b}} = \sqrt{\frac{1}{2}\left(a+\sqrt{a^2-b}\right)} + \sqrt{\frac{1}{2}\left(a-\sqrt{a^2-b}\right)}$$

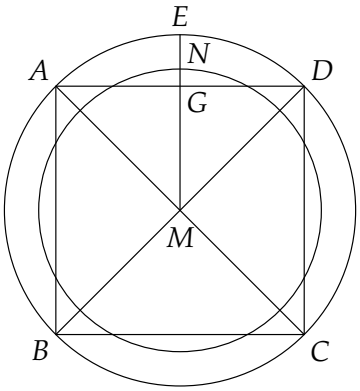
Prove Bhaskara's formula and use it to simplify $\sqrt{2+\sqrt{3}}$.

3. Here is an Indian method for 'finding' a circle whose area is equal to a given square.

In a square $ABCD$, let M be the intersection of the diagonals. Draw a circle with M as the center and MA the radius; let ME be the radius of the circle perpendicular to the side AD and cutting AD at G. Let $GN = \frac{1}{3}GE$. Then MN is the radius of the desired circle.
Show that if $AB = s$ and $MN = r$, then

$$\frac{r}{s} = \frac{2+\sqrt{2}}{6}$$

Show that this implies a value for π equal to 3.088311755.



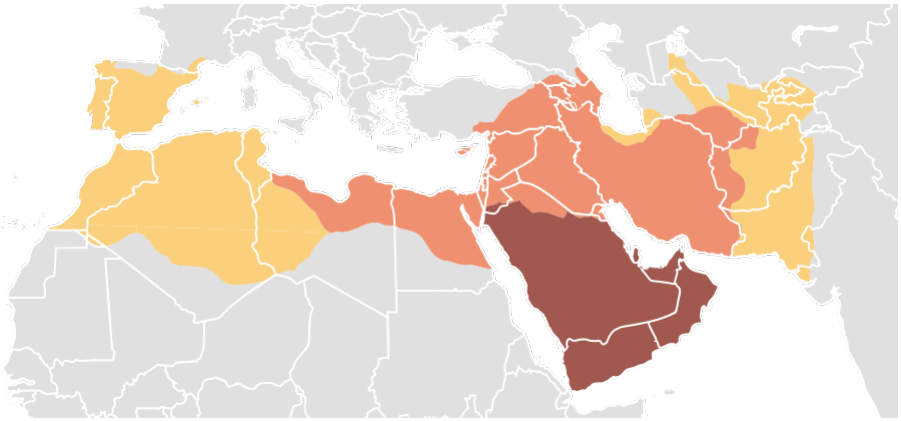
4. Solve the following problem of Mahāvīra.

> Of a collection of mango fruits, the king took 1/6; the queen took 1/5 of the remainder, and the three chief princes took 1/4, 1/3, 1/2 of what remained at each step. The youngest child took the remaining three mangoes. O you, who are clever in working miscellaneous problems on fractions, give out the measure of that collection of mangoes.

6.2 Islamic Mathematics and Algebra

Muhammad ibn Abdullah was born in Mecca (modern Saudi Arabia) in 570. Around 610 he began preaching *Islam* (*submission to the will of God*), his vision of Abrahamic religion (Judaism and Christianity). After years of persecution and exile, he returned with an army and conquered his hometown a few years before his death in 632.

Muhammad's successors expanded the caliphate through military conquest. The speed of the Muslim conquests is shown in the picture below. At the time of his death, the Arabian peninsula was Islamic. By 660 Islam had reached Libya and most of Persia, and by 750 extended from Iberia & Morocco to Afganistan & Pakistan. In later times there were serious schisms[30] and several successor empires emerged, the longest-lasting of which was the Ottoman Empire (c. 1300–1922). Even though centralized political control has ended, Islam remains dominant in the region shown (with the notable exceptions of Spain and Portugal) and over a greater region of Africa and south-east Asia.



As with the Romans, Muslim conquerors allowed the conquered to maintain their culture, especially if they were Jews or Christians (*people of the book*), provided they acknowledged their overlords and paid extra taxes. If conquered people converted to Islam they were welcomed as full citizens. Many of the great Islamic thinkers were just such people, born on the periphery and travelling to the great centers of learning. Islamic conquerors absorbed such knowledge as they found, including the wisdom of Alexandria and western India (Pakistan). In the mid-700s paper-making came to from China, greatly facilitating the dissemination and consolidation of knowledge. Schools (*madrassas*) reflected a strong cultural focus on learning.

During the European *dark ages* (c. 500–1200) following the fall of Rome, European philosophical development stagnated. The latter part of this period was, by contrast, Islam's *golden age.* By 1200, the crusades[31] were well underway and Islam was seen as the enemy of Christian Europe. Perhaps ironically, it was the infusion of knowledge that came to Europe from Islam starting around this time that helped spur the renaissance & scientific revolution. Among European scholars almost to the present day, it was fashionable to credit Islam merely with the *preservation* of ancient 'European knowledge,' a claim both fanciful and chauvinistic, but plainly stemming from medieval animosity.

[30] In particular between the Sunni and Shia branches of the faith. Much of the modern-day tension between Saudi Arabia and Iran stems from this rupture.

[31] A series of European religious–military campaigns 1096–1291 with the goal of wresting control of the Holy Land, particularly Jerusalem, from Islam.

Algebra & Algorithms

Concepts of proof and axiomatics were learned from Greek texts such as the *Elements.* Like the Greeks, Islamic scholars gave primacy to geometry and proved algebraic relations in a geometric manner.[32] Practical and accurate calculation was more important than it was to the Greeks, and great advances were made in this area. This included completing the development of the Indian decimal place system, hence the dual credit (*Hindu–Arabic* numerals).

The second most obvious legacy of Islamic mathematics is encountered daily in every mathematics classroom. *Algebra*[33] (*al-ğabr* means *restoring*) refers to moving a negative (*deficient*) quantity from one side of an equation to another. A second term *al-muqabala* (*comparing/balancing*) refers to subtracting the same positive quantity from both sides of an equation.

$$\begin{aligned} &\textit{Al-ğabr}: && x^2 + 7x = 4 - 2x^2 \implies 5x^2 + 7x = 4 \\ &\textit{Al-muqabala}: && x^2 + 7x = 4 + 5x \implies x^2 + 2x = 4 \end{aligned}$$

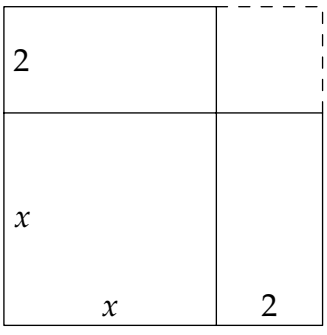
Islamic scholars did not use *symbols* or *equations* in a modern sense; statements were instead written out in sentences.

Muhammad ibn Mūsā al-Khwārizmī (780–850) Born near the Aral Sea in modern Uzbekistan, al-Khwārizmī eventually became chief librarian at the great school of learning, the *House of Wisdom,* in Baghdad. His *Compendious book on the calculation by restoring and balancing*[34] (820) is a synthesis of Babylonian methods with the Euclidean axiomatic approach; an algorithm demonstrated a solution, followed by a geometric proof. After being translated into Latin in the 1100s it became a standard textbook of European mathematics, displacing Euclid in places due to its greater emphasis on practical calculation. The word *algorithm* reflects its importance: the Latin *dixit algorismi* literally means *al-Kwārizmī says.* Here is his approach to the quadratic equation $x^2 + 4x = 60$, or, more properly:

> What must be the square which, when increased by four of its roots, amounts to sixty?

The algorithm may be applied to *any* equation of the form $x^2 + ax = b$ where $a, b > 0$: here a is the number of 'roots,' and b the total 'amount.'

- Halve the number of roots $\left(2 = \frac{1}{2}a\right)$
- Multiply by itself $\left(4 = \frac{1}{4}a^2\right)$
- Add to the total amount $\left(64 = \frac{1}{4}a^2 + b\right)$
- Take the root of this $\left(8 = \sqrt{\frac{1}{4}a^2 + b}\right)$
- Subtract half the number of roots $\left(6 = \sqrt{\frac{1}{4}a^2 + b} - \frac{a}{2}\right)$



Al-Kwārizmī essentially constructs the quadratic formula $= \frac{-a+\sqrt{a^2+4b}}{2}$, while the pictorial justification is Euclid's (*Elements*, Thm II. 4). Hopefully the geometry is obvious: the square has been increased by four of its roots and the algorithm is simply 'completing the square.'

[32] Like Book II of the *Elements.* Such Greek texts were venerated by Islamic scholars; recognizing the depth of Ptolemy's work on astronomy and trigonometry, they bestowed the name by which it is now known, the *Almagest* (*Great Work*).

[33] Many words beginning *al-* are of Arabic origin (alkali, albatross, etc.), as are others that have been latinized (elixir).

[34] *Al-kitāb al-mukhtasar fī hisāb al-ğabr wa'l-muqābala.*

Other mathematicians went far further. For instance, Abū Kāmil (Egypt 850–930) generalized Euclid's Book II geometric-algebra arguments to permit substitution, provided the resulting equation was quadratic; for instance

$$\text{If } y = \frac{1+x}{3+x} \text{ and } y^2 + y = 1 \implies y = \frac{\sqrt{5}-1}{2}, \text{ then}$$
$$\left(\frac{1+x}{3+x}\right)^2 + \frac{1+x}{3+x} = 1 \implies x = \sqrt{5}$$

In practice, he combined several algorithms of al-Khwārizmī, each of which was based on geometry, but when combined could no-longer be straightforwardly be justified geometrically. This method of substitution was an early step towards establishing the modern primacy of algebra and number over geometry and length.

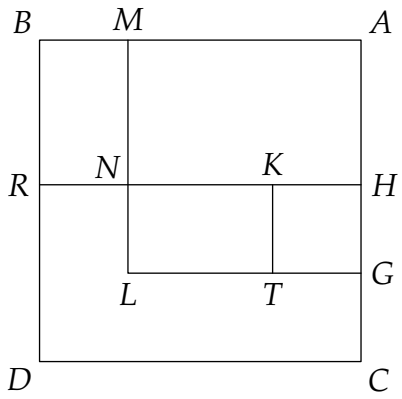
The most famous Islamic mathematician of the next several centuries was almost certainly Omar Khayyam (1048–1131), who produced ground-breaking work on the solution of cubic equations, astronomy, the binomial theorem, and irrational numbers.

Exercises 6.2. 1. Solve the equations $\frac{1}{2}x^2 + 5x = 28$ and $2x^2 + 10x = 48$ using al-Khwārizmī's methods (first multiply or divide by 2).

2. Al-Khwārizmī gives the following algorithm for solving the equation $bx + c = x^2$.

   - Halve the number of roots.
   - Multiply this by itself.
   - Add this square to the number.
   - Extract the square root.
   - Add this to half the roots.

   Translate this into a formula. Give a geometric argument for the validity of the approach using the picture: HC has length b where G is the midpoint, rectangle $ABRH$ has area c, $KHGT$ and $AMLG$ are squares, while the large square $ABDC$ has side length x.



3. Solve the following problems by Abū Kāmil (use modern algebra!).

   (a) Suppose 10 is divided into two parts and the product of one part by itself equals the product of the other part by the square root of 10. Find the parts.

   (b) Suppose 10 is divided into two parts, each of which is divided by the other, and the sum of the quotients equals the square-root of 5. Find the parts.

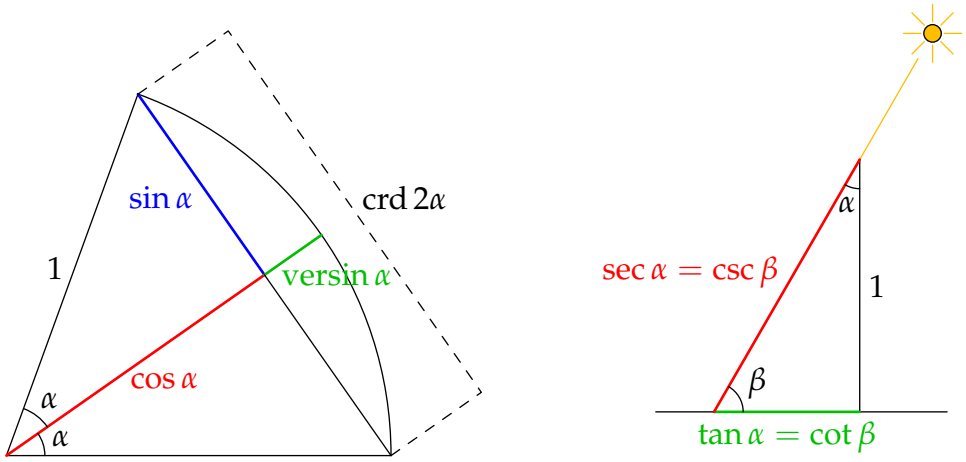
6.3 Spherical Trigonometry and the *Qibla*

Late 8th century Indian work on trigonometry, linking back to Hipparchus, was known in Baghdad, as was the work of Ptolemy. Islamic scholars were interested in trigonometry for reasons beyond astronomy. A primary requirement in Islam is to face the Ka'aba in the Great Mosque at Mecca when at prayer: this is the *qibla* (*direction* in Arabic). A mosque is typically built so that one wall faces Mecca for convenience. In Muhammad's time (when Muslims faced Jerusalem not Mecca), determining the *qibla* was relatively easy, though as Islam spread the curvature of the earth made determination more difficult. The resolution of this problem motivated Islamic mathematics for centuries.

Terminology and Trigonometric Tables Scholars worked with the Indian *half-chord* (sine), and with circles of various radii. Al-Battānī (c. 858–929) introduced an early version of *cosine* as the *complementary half-chord* for angles less than 90°. He also worked with a version of the modern function *versine*:[35]

$$\operatorname{versin}\theta = 1 - \cos\theta$$

Al-Bīrūnī (973–1048) defined versions of tangent, cotangent, secant and cosecant by projecting from a gnomon (sundial) onto either a horizontal or vertical plane. In the second picture below, the gnomon is the vertical stick of length 1. With this definition, al-Bīrūnī moves towards the modern consideration of trigonometry in terms of *triangles* rather than circles.



Trigonometric tables with improved accuracy over Ptolemy were created for all these 'functions.' By applying the half-angle formula many times and using the fact that

$$\sin(\alpha+\beta) - \sin\alpha < \sin\alpha - \sin(\alpha-\beta)$$

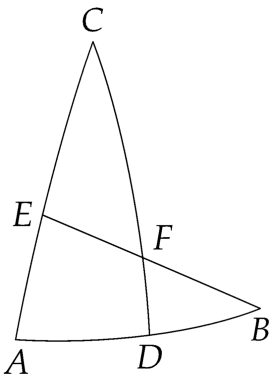
whenever[36] $0° < \alpha - \beta < \alpha + \beta < 90°$, Abū al-Wafā (940–998) and his descendants were able to compute sine & tangent values for every minute of arc accurate to five sexagesimal places!

[35] *Versed sine* refers to the measurement of a length in a re*versed* direction (perpendicular) to that of sine.
[36] This is simply the downwards concavity of the sine function.

Calculating the *Qibla* In what follows we observe several conventions:

- A single letter A refers to a *point* or to the *angle measure* in a triangle with vertex A.
- AB means the *segment* of the great-circle joining points A, B or its *arc-length*. A *spherical triangle* $\triangle ABC$ comprises three points on a sphere joined by segments of great-circles.
- $\overline{AB}$ means the *straight line* joining A, B with *length* $|AB|$.
- All results are modernized and applied to a unit sphere. The arc-length along a great-circle therefore equals the central angle subtended by that arc in radians: $AB = \measuredangle AOB$. To help visualize things, 3D movable versions of all pictures can be found by clicking on them…

Ptolemy and the Indians had already done some relevant work, though Ptolemy's approach relies heavily on Menelaus' Theorem (c. 100AD).



Theorem (Menelaus). *For the pictured configuration of spherical triangles on a sphere of radius 1,*

$$\frac{\sin CE}{\sin AE} = \frac{\sin CF}{\sin DF} \cdot \frac{\sin BD}{\sin AB}$$

Computations are difficult with Menelaus' Theorem, since one needs to create several new spherical triangles. Al-Wafā simplified things considerably with an alternative result.

Theorem (Al-Wafā). *If $\triangle ABC$ and $\triangle ADE$ are spherical triangles with right angles at B, D and a common acute angle at A, then*

$$\frac{\sin BC}{\sin AC} = \frac{\sin DE}{\sin AE}$$

In fact these ratios equal $\sin \alpha$ where α is the acute angle, though al-Wafā didn't say this.

Proof. Let O be the center of the sphere. Project C orthogonally to the plane containing O, A, B to produce K, then project K to $\overline{OA}$ to get L.

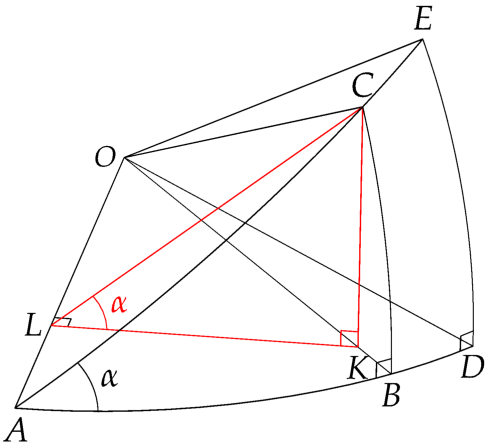
Consider the right-angled planar triangle CKL. Since α is the angle between two planes, we have $\alpha = \measuredangle CLK$. Moreover

$$|CK| = \sin \measuredangle COK = \sin \measuredangle COB = \sin BC$$
$$|CL| = \sin \measuredangle COL = \sin \measuredangle COA = \sin AC$$

The usual sine formula says

$$\sin \alpha = \frac{|CK|}{|CL|} = \frac{\sin BC}{\sin AC}$$

The same ratio is obtained for $\triangle ADE$.

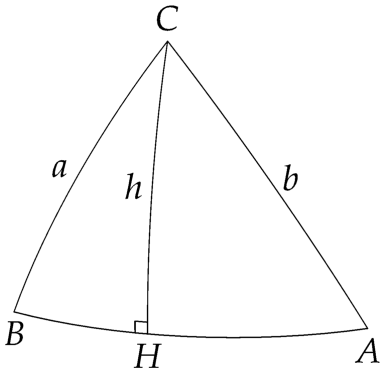


■

Solving triangles essentially means the sine and cosine rules, both of which follow by dropping perpendiculars. Al-Wafā's result quickly recovers the spherical sine rule.

Corollary (Sine rule). *If a, b, c are the side-lengths of a spherical triangle with angles A, B, C, then*

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$



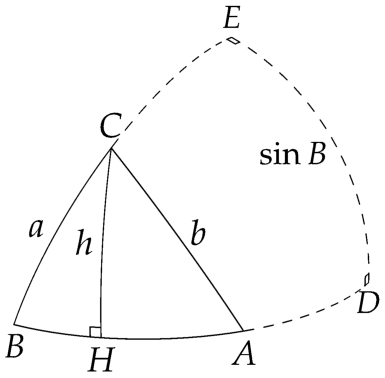
Proof. Drop a perpendicular to H from C. Al-Wafā says

$$\sin B = \frac{\sin h}{\sin a} \quad \text{and} \quad \sin A = \frac{\sin h}{\sin b}$$

Eliminate $\sin h$ for the first equality. The rest is symmetry. ∎

Al-Wafā's proof was similar, though a little more complicated. He extended AB and BC to quarter circles resulting in a spherical triangle with right angles at D and E. Since DE is an arc with central angle B, we have $DE = \sin B$. Al-Wafā's theorem implies

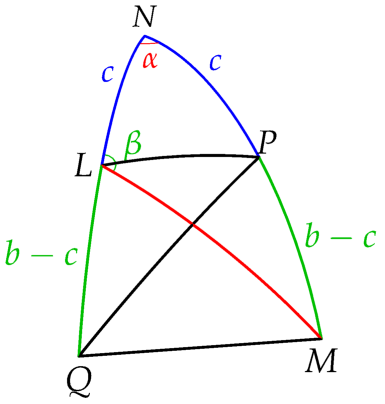
$$\frac{\sin h}{\sin a} = \frac{\sin B}{\sin 90°} \implies \sin h = \sin a \sin B$$



Mirroring this by extending AB past B and equating the $\sin h$ terms yields the result.

Armed with these results, al-Wafā could solve spherical triangles, though his method was rather complicated and required several auxiliary triangles. Al-Bīrūnī simplified matters by invoking the cosine rule. We apply his method to find the *qibla* from a location L on the Earth's surface which, for simplicity, we assume to have radius 1.

Let M be Mecca and N the north pole. We wish to compute β (the initial bearing from L to M). Our initial data are the latitudes and longitudes of L, M, specifically:



- α is the difference in the longitudes.
- b, c are the *colatitudes*[37] of M, L respectively.

The cosine rule follows from Ptolemy's Theorem (pg. 43). Extend NL to Q with the same latitude as M. Similarly let $P \in NM$ have the same latitude as L. By symmetry, L, P, Q, M are *coplanar,* whence the quadrilateral $LPQM$ lies on the intersection of a plane and a sphere: a circle! Measured as straight lines (chords) and using symmetry ($|PQ| = |LM|$ and $|LQ| = |PM|$), Ptolemy says

$$|LM|\,|PQ| = |LQ|\,|PM| + |LP|\,|QM| \implies |LM|^2 = |LQ|^2 + |LP|\,|QM|$$

[37] *Colatitude* is measured southwards from the north pole, adn thus equals 90°−latitude. Since the sphere has radius 1, the arc-lengths b, c equal the colatitudes in radians.

The great circle arc-lengths on the sphere are related to straight distances via the usual chord relations; e.g.,

$$|LM| = \operatorname{crd} LM = 2\sin\frac{LM}{2}$$

Thus Ptolemy's theorem becomes a relation between *arc-lengths*

$$\sin^2\frac{LM}{2} = \sin^2\frac{b-c}{2} + \sin\frac{LP}{2}\sin\frac{QM}{2}$$

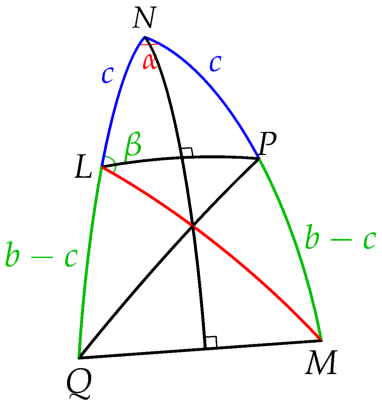
By bisecting α we obtain two pairs of right-triangles; al-Wafā tells us that

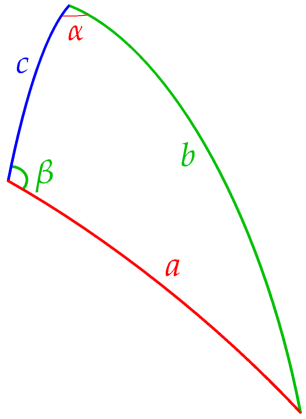
$$\sin\frac{\alpha}{2} = \frac{\sin\frac{LP}{2}}{\sin c} = \frac{\sin\frac{QM}{2}}{\sin b}$$

whence

$$\sin^2\frac{LM}{2} = \sin^2\frac{b-c}{2} + \sin^2\frac{\alpha}{2}\sin c\sin b \qquad (*)$$

For final simplifications, apply the multiple-angle formulæ ($\sin^2 x = \frac{1}{2}(1-\cos 2x)$ and $\cos(b-c) = \cos b\cos c + \sin b\sin c$).





Corollary (Cosine rule). *In a spherical triangle with sides a, b, c and angle α opposite a, we have*

$$\cos a = \cos b\cos c + \sin b\sin c\cos\alpha$$

Of course $a = LM$ in our triangle of interest. Given L, M, one uses the cosine rule to compute a and then the sine rule

$$\frac{\sin b}{\sin\beta} = \frac{\sin a}{\sin\alpha}$$

to compute the *qibla* β. Whew!

For fun, here is some real-world data: the co-ordinates of Mecca and London are 21°25′ N 39°49′ E and 51°30′ N 8′ W respectively. This corresponds to

$$\alpha = 39°57', \qquad b = 68°35', \qquad c = 38°30'$$

We therefore have

$$\cos a = \cos 68°35'\cos 38°30' + \sin 68°35'\sin 38°30'\cos 39°57' \implies a = 43.110°$$

Since Earth's circumference is 24,900 miles, the London → Mecca distance is $\frac{43.110\times 24{,}900}{360} = 2{,}981$ miles. Finally we find the *qibla*

$$\beta = 180° - \sin^{-1}\frac{\sin\alpha\sin b}{\sin a} = 118°59'$$

where we subtracted from 180° since London is north of Mecca. Check it yourself at the Great Circle Mapper (the website uses airports so the result uses slightly different initial data).

Spherical Trigonometry: Cheat Sheet!

Let $\triangle ABC$ be a spherical triange with side-lengths a, b, c on a sphere of radius 1.

Basic trigonometry: if $\triangle ABC$ is right-angled at C (Al-Wafā essentially proved the first of these),

$$\sin A = \frac{\sin a}{\sin c} \qquad \cos A = \frac{\tan b}{\tan c} \qquad \tan A = \frac{\tan a}{\sin b}$$

Sine rule (Al-Wafā)

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

Cosine rule (Al-Bīrūnī)

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

The spherical Pythagorean Theorem is $\cos c = \cos a \cos b$ $(C = 90°)$.

Combining these results allows one to 'solve' spherical triangles. If the sphere has radius r, simply divide all lengths by r before applying the results; e.g.,

$$\sin A = \frac{\sin(a/r)}{\sin(c/r)}$$

Note that as $r \to \infty$, we have $\sin(a/r) \approx \frac{a}{r}$ and $\cos(a/r) \approx 1 - \frac{a^2}{2r^2}$, which recovers the 'flat' versions of the above statements.

Examples

1. On a sphere of radius 1, an equilateral triangle has side length $\frac{\pi}{3}$. Splitting it in half creates two right-triangles with adjacent $\frac{\pi}{6}$ and hypotenuse $\frac{\pi}{3}$. The angles in the triangle are therefore

   $$\alpha = \cos^{-1} \frac{\tan \frac{\pi}{6}}{\tan \frac{\pi}{3}} = \cos^{-1} \frac{1}{3} \approx 70.53°$$

   The angle sum in the triangle is three times this: 211.59°.

2. On Earth's surface, an airfield is at position C and two planes are at A and B. The bearings and distances to the aircraft are 45°, 2000 miles, and 90°, 4000 miles respectively. Find the distance between the aircraft.

   This is just the cosine rule! We have a triangle with sides 2000 and 4000 and angle 45° between them. If $r = 4000$ miles is the radius of the Earth, then

   $$\cos \frac{c}{4000} = \cos \frac{1}{2} \cos 1 + \sin \frac{1}{2} \sin 1 \cos 45°$$
   $$\implies c = 2833 \text{ miles}$$

   Note that this is a little closer (as expected) than the value (2947 miles) one would obtain from assuming a flat Earth!

   Modern navigators use a slightly different approach to measuring the distance between airplanes, since the error in estimating cosine of small values tends to make for a high level of inaccuracy (look up the *haversine formula* if you're interested).

Exercises 6.3. 1. A right-isosceles triangle on the surface of a unit sphere has equal legs of length $\frac{\pi}{4}$. Find the length of the hypotenuse and the sum of the angles in the triangle.

2. Explain the observation on page 61 that

$$0° < \alpha - \beta < \alpha + \beta < 90° \implies \sin(\alpha + \beta) - \sin\alpha < \sin\alpha - \sin(\alpha - \beta)$$

is the downwards concavity of the sine function.

3. Determine the *qibla* for Rome (latitude 41°53′ N, longitude 12°30′ E).

4. Al-Bīrūnī devised a method for determining the radius r of the earth by sighting the horizon from the top of a mountain of known height h. He would measure α, the angle of depression from the horizontal to which one sights the apparent horizon. Show that

$$r = \frac{h\cos\alpha}{1 - \cos\alpha}$$

In a particular case, al-Bīrūnī measures $\alpha = 34'$ from a mountain of height $652;3,18$ cubits. Assuming that a cubit equals $18''$, convert your answer to miles and compare with the modern value. Discuss the efficacy of this method.