2 Babylon/Mesopotamia

Babylon was an ancient city located near modern-day Baghdad, Iraq, though the term is shorthand for the many empires/civilizations dating back to before 3000 BC that arose in Mesopotamia[3]: Sumeria, Akkadia, Babylonia, etc.

Used *cuneiform* (wedge-shaped) script, typically indentations on clay tablets.

Most recovered tablets date from the time of Hammurabi (c. 1800 BC) or the Seleucid dynasty (c. 300 BC) which ruled after Alexander the Great's conquests.

Mathematical tablets are of two main types: tables of values (multiplication, reciprocals, measures) and worked problems.

Positional Enumeration The Babylonians used two symbols, roughly $\vee$ for 1 and $\triangleleft$ for 10, likely made by the same stylus. Any number up to 59 was written with combinations, e.g.

$$53 = \triangleleft\triangleleft\triangleleft\triangleleft\triangleleft\,\vee\vee\vee$$

with the picture showing a typical appearance as cuneiform. Arguably the greatest mathematical advance of the Babylonians was in evidence by 2000 BC; a *positional* number system. To understand what this means, consider our decimal (base-10) system where 3835 uses the same symbol 3 to represent two different concepts 3000 and 30. We might write this as

$$3835 = 3 \cdot 10^3 + 8 \cdot 10^2 + 3 \cdot 10 + 5$$

Babylonian enumeration did exactly the same thing base-60; we call it a *sexagesimal* system. The sexagesimal decomposition of 3835 is

$$3835 = 1 \cdot 60^2 + 3 \cdot 60 + 55$$

for which the Babylonians would write

$$\vee \qquad \vee\vee\vee \qquad \triangleleft\triangleleft\triangleleft\triangleleft\triangleleft\,\vee\vee\vee\vee\vee \tag{$*$}$$

In a positional system the meaning of a symbol depends on its position. For us, '3' might mean 30, 3000 or $\frac{3}{1000}$, while for the Babylonians $\vee$ could mean 1, 60, 3600, 216000, or fractions such as $\frac{1}{60}$, $\frac{1}{3600}$, etc. There was no symbol for zero until very late in Babylonian history, nor any *sexagesimal point*, so determining position on ancient tablets can be difficult. For instance, $(*)$ might instead have represented

$$60 + 3 + \tfrac{55}{60} = 63\tfrac{11}{12} \quad \text{or} \quad 60^3 + 3 \cdot 60^2 + 55 \cdot 60 = 230100$$

[3] 'Between two rivers,' namely the Tigris and Euphrates. As indicated on the map, these rivers formed the backbone of the *Fertile Crescent*, in which early civilization, farming, crop and animal domestication occurred.

To make things easier to read, we will write a sexagesimal number using commas to separate terms and, if necessary, a semicolon to denote the sexagesimal point. Thus

$$23, 12, 0; 15 = 23 \cdot 60^2 + 12 \cdot 60 + \frac{15}{60} = 83520\frac{1}{4}$$

Why base 60? There are many theories, but no-one is sure precisely why. Here are some ideas.

- 60 has many proper divisors (1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30). Many more numbers 'divide exactly' than with decimal arithmetic: e.g., as a terminating sexagesimal, $\frac{1}{3} = ;20$ is much simpler than the decimal $0.33333\ldots$
- A year has approximately 360 days (divisible by 60). The Babylonians were prolific astronomers and astrologers. Our modern usage of *degrees, minutes, seconds* for angle, *hours, minutes, seconds* for time, and the standard zodiac are all of Babylonian origin.
- The Babylonians possibly inherited two systems of counting (say base 10 and base 12) from older cultures and eventually combined them.

This is the sort of historical question that is rarely answerable in a satisfying way, particularly when discussing an ancient culture. Likely no-one 'decided' to use base 60; like most culture, it probably happened slowly and organically, without much fanfare.

It wasn't just counting that used 60s; units of Babylonian measure often used 60 when moving up and down the magnitude scale similarly to how we use multiples of 1000 (e.g. joules $\to$ kilojoules $\to$ megajoules).

Calculations Addition and subtraction of sexagesimals would have been as natural to the Babylonians as decimal calculations are to us. For instance, we might write

$$\begin{array}{rr} & 21{,}49 \\ + & 3{,}37 \\ \hline + & \overset{1}{2}4{,}26 \end{array} \qquad \text{(i.e. } 1309 + 217 = 1526)$$

Note how we carry 60s just like we are used to doing with 10s in decimal arithmetic: $49 + 37 = 1, 26$.

Multiplication is significantly harder. To mimic our decimal long-multiplication process would require knowing up to the 59 times table! For small products this might have been fine, but there is evidence that the Babylonians used two expressions to represent any product in terms of squares

$$xy = \frac{1}{2}\left[(x+y)^2 - x^2 - y^2\right] = \frac{1}{4}\left[(x+y)^2 - (x-y)^2\right]$$

Tablets consisting of tables of squares have been found, thus greatly aiding the computation of large products. For instance, to compute $31 \times 22 = 682$,

$$31 \times 22 = \frac{1}{4}\left[53^2 - 9^2\right] = \frac{1}{4}\left[46, 49 - 1, 21\right] = \frac{1}{4}\left[45, 28\right] = 11, 7 + 15 = 11, 22$$

This process would be combined with long-multiplication to multiply larger numbers.

Fractions/Division The Babylonians used sexagesimals rather than fractions. They produced tables of reciprocals $\frac{1}{n}$ which could be used to quickly evaluate divisions via $m \div n = m \times \frac{1}{n}$. For instance

$$\frac{1}{18} = 0;3,20 \implies \frac{23}{18} = 23(0;3,20) = 1;9 + 0;7,40 = 1;16,40$$

This works nicely provided the only primes dividing n are 2, 3 and 5, since any such $\frac{1}{n}$ will be an exact terminating sexagesimal.[4]

For reciprocals without terminating sexagesimals, approximations were used; a scribe would simply choose a nearby denominator with a exact sexagesimal and state that the answer was approximate

$$\frac{11}{29} \approx \frac{11}{30} = 11(0;2) = 0;22$$

More accuracy could be obtained by choosing a larger denominator. For instance, if a scribe wanted to divide by 11, they might observe that $11 \cdot 13 = 143 \approx 144$ and write[5]

$$\frac{1}{144} = 0;0,25 \implies \frac{1}{11} \approx \frac{13}{144} = 0;5,25$$

which is 99.3% accurate. Scribes were explicit in acknowledging that, say, "11 does not divide," and that the result is an approximation. Remember that a single digit in the second sexagesimal place means only $\frac{1}{3600}$, so even the most demanding application doesn't require many terms! The denominators in some of these tables were enormous, so far greater accuracy was often possible.

Another table listed all the ways an integer < 10 could be multiplied exactly to get 10.

1	10	5	2
2	5	6	1 40
3	3 20	8	1 15
4	2 30	9	1 6 40

We omit the commas for separation and the sexagesimal point, as they did not exist. Note also that 7 is missing since $\frac{1}{7}$ (and thus $\frac{10}{7}$) is not an exact sexagesimal. It should be clear from the table that

$$\frac{10}{6} = 1;40 \quad \text{and} \quad \frac{600}{9} = 1,6;40$$

In the latter case, note that $600 = 10 \cdot 60$ would be written the same as 10, so this amounts to moving the sexagesimal point in $\frac{10}{9} = 1;6,40$.

[4]This is analogous to the fact that $\frac{1}{n}$ has a terminating decimal if and only if the only primes dividing n are 2 and 5.

[5]In fact, being rational, $\frac{1}{11} = 0.09090909\ldots = 0;5,27,16,21,49,5,27,16,21,49,\ldots$ has a repeating sexagesimal expansion. This can be found by iteration, though it is time-consuming:

$$\frac{60}{11} = 5\frac{5}{11}, \quad \frac{5 \cdot 60}{11} = 27\frac{3}{11}, \quad \frac{60 \cdot 3}{11} = 16\frac{4}{11}, \ldots$$

Linear Systems These were solved by a mixture of the false position method (guess and modify as done by the Egyptians) and an approach modelled on homogeneous equations. For instance, here is a Babylonian approach to solving the following system

$$\begin{cases} 3x+2y=11 \\ 2x+y=7 \end{cases}$$

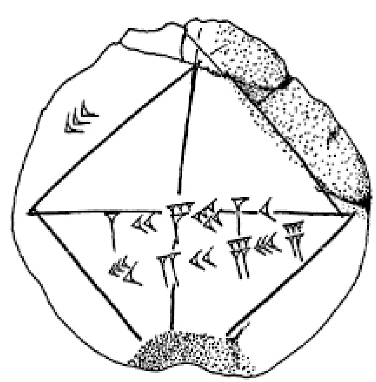
1. Choose an equation, say the second, and set $\hat{x}=\hat{y}$. Now solve, for instance using false position to obtain $\hat{x}=\frac{7}{3}=2;20$.
2. All solutions to the second equation have the form $x=\hat{x}+d$ and $y=\hat{y}-2d$, since $(d,-2d)$ is the general solution to the homogeneous equation $2x+y=0$. Substitute into the first equation:
$$11=3\left(\frac{7}{3}+d\right)+2\left(\frac{7}{3}-2d\right)=11+\frac{2}{3}-d \implies d=\frac{2}{3}$$
3. Now solve $x=\frac{7}{3}+\frac{2}{3}=3$, $y=\frac{7}{3}-\frac{4}{3}=1$.

Step 2 should should remind you of the 'nullspace' method from modern linear algebra: all solutions to the matrix equation $(2\ 1)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)=7$ have the form

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}+\mathbf{n}$$

where $\left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right)$ is some particular solution (here $x_0=y_0=\frac{7}{3}$) and $\mathbf{n}=\left(\begin{smallmatrix} d \\ -2d \end{smallmatrix}\right)$ lies in the nullspace of the (row) matrix $(2\ 1)$.

The Yale Tablet (YBC 7289) One of the most famous tablets concerns an approximation to $\sqrt{2}$. YBC stands for the *Yale Babylonian Collection* which contains over 45,000 objects. The tablet is shown below along with an enhanced representation of the numerals.



The tablet depicts a square of side 30 (or possibly $\frac{1}{2}=0;30$) and labels the diagonal in two ways:

- $1;24,51,10$ as an approximation to $\sqrt{2}$, an underestimate by roughly 1 part in 2.5 million!
- $42;25,35$ as an approximation to the diagonal when the side is 30.

The Babylonians more often used the simpler approximation $1;25 = 1.41666\ldots$ which is still very close. Given the impractical accuracy of this approximation, it is reasonable to ask how it was obtained. No-one knows for certain, but two methods are theorized since both were used to solve other problems. It should be stressed that no Babylonian *proofs* of these approaches are known.

1: Square root approximation $\sqrt{a^2 \pm b} \approx a \pm \frac{b}{2a}$. This is essentially the linear approximation from elementary calculus. The idea is to choose a rational number for a whose square is very close to 2, then the error should be very small. For instance:

- $\sqrt{2} = \sqrt{1+1} \approx 1 + \frac{1}{2} = 1;30$
- $\sqrt{2} = \sqrt{\left(\frac{4}{3}\right)^2 + \frac{2}{9}} \approx \frac{4}{3} + \frac{2/9}{8/3} = \frac{4}{3} + \frac{1}{12} = \frac{17}{12} = 1;25$
- $\sqrt{2} = \sqrt{\left(\frac{7}{5}\right)^2 + \frac{1}{25}} \approx \frac{7}{5} + \frac{1/25}{14/5} = \frac{99}{70} = 1;24,51,25,42,51,25,42,51,\ldots$

2: Method of the Mean It is easily checked (Exercise 11) that any sequence defined by the recurrence relation $a_{n+1} = \frac{a_n + 2/a_n}{2}$ converges to $\sqrt{2}$. Let us apply this to the sequence starting with $a_n = 1$.

$$a_1 = 1, \quad a_2 = \frac{3}{2} = 1;30, \quad a_3 = \frac{17}{12} = 1;25$$
$$a_4 = \frac{577}{408} = 1 + \frac{169}{408} = 1;24,51,10,35,17,\ldots$$
$$a_5 = \frac{665857}{470832} = 1;24,51,10,7,46\ldots$$

It seems incredible that any ancient culture would have bothered to go so far with these calculations to obtain the observed accuracy.

The same analysis can be used to approximate other roots. For example, we could start with with $a_1 = 3$ to approximate $\sqrt{11}$ via $a_{n+1} = \frac{1}{2}(a_n + \frac{11}{a_n})$:

$$a_2 = \frac{10}{3} = 3;20, \quad a_3 = \frac{199}{60} = 3;19, \quad a_4 = \frac{79201}{23880} = 3;18,59,50,57,17,\ldots$$

Quadratic Equations The above methods could be used to solve general quadratic equations. A question might be phrased as follows:

> I added twice the side to the square; the result is $2,51,40$. What is the side?

In modern language, we want the solution to $x^2 + 2x = 2 \cdot 60^2 + 51 \cdot 60 + 40 = 10300$.

Questions such as these were solved using templates. In the above example, the template is for solving $x(x+p) = q$ where $p, q > 0$. Other templates were required for the other types of quadratic equation ($x^2 = px + q$, etc.), since the Babylonians did not recognize negative numbers. Here is their algorithm applied to a simpler equation $x^2 + 4x = 2$:

Set $y = x + p$ ($y = x + 4$) then the equation can be decoupled:

$$\begin{cases} xy = q \\ y - x = p \end{cases} \qquad\qquad \begin{cases} xy = 2 \\ y - x = 4 \end{cases}$$

Use this to solve for $x+y$:

$$\begin{aligned} 4xy+(y-x)^2 &= p^2+4q \\ (y+x)^2 &= p^2+4q \\ x+y &= \sqrt{p^2+4q} \end{aligned} \qquad\qquad \begin{aligned} 4xy+(y-x)^2 &= 4^2+4\cdot 2 \\ (y+x)^2 &= 24 \\ x+y &= \sqrt{24} \approx 4;54 \end{aligned}$$

where the square-root was approximated using one of the earlier algorithms, e.g.

$$\sqrt{24}=\sqrt{5^2-1}\approx 5-\frac{1}{10}=4;54$$

Since $x+y$ and $x-y$ are now known, false position could then be used to find

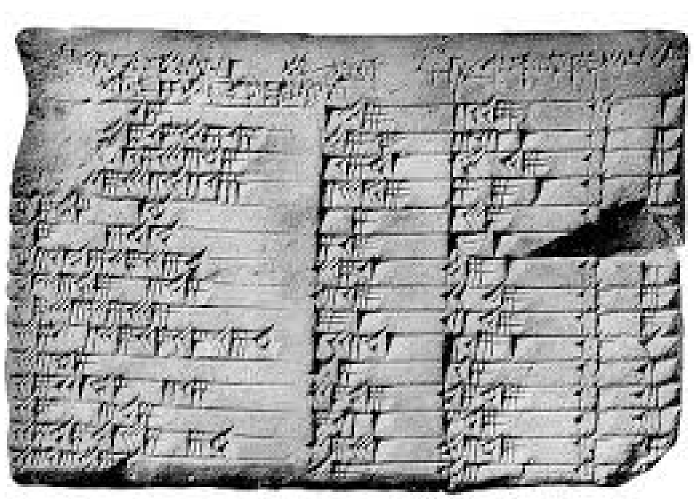
$$x=\frac{\sqrt{p^2+4q}-p}{2} \qquad\qquad x\approx 0;27$$

You should recognize the method of completing the square and the quadratic formula; this approach is at least 4000 years old!

While we've written this abstractly, in practice Babylonian scribes/students would be copying from a particular example of the same type. There were no abstract formulæ and everything was done without the benefit of any of our modern notation. We moreover have no written explanation from the Babylonians of what they were doing; typically all historians have to work with is the right-hand column of numbers and several examples like it!

Note that the template only found the positive solution; the Babylonians had no notion of negative numbers. Amazingly, the Babylonians were also able to address certain cubic equations similarly.

Pythagorean Triples Among the many tables of values created by the Babylonians are lists of Pythagorean triples. The Plimpton 322 tablet (also at Yale) has a large number of these (albeit with some mistakes).



Due to the strange way in which the triples were encoded, it took a long time for scholars realized what they had. The table is also broken on the left so some columns are probably missing.

As an example, line 15 of the table describes the Pythagorean triple $53^2 = 45^2 + 28^2$ as follows.

- The first entry is $(\frac{53}{45})^2 = 1;23,13,46,40$ (exact).
- The second entry is 28.
- The third entry is 53.
- The last two entries indicate line number 15.

The first three (interesting) entries are therefore $((\frac{c}{a})^2, b, c)$ where $c^2 = a^2 + b^2$. It is possible that a missing column of the tablet explicitly mentioned a.

It is not known how the table was completed, although the first column exhibits a descending pattern that provides clues to its construction. One theory is that a scribe found rational solutions to the equation $v^2 = 1 + u^2$ (equivalently $(v+u)(v-u) = 1$) by starting with a choice of $v+u$ and using a table of reciprocals to calculate $v - u$.

To revisit our example, if $v + u = \frac{9}{5} = 1;48$, then

$$v - u = \frac{1}{v+u} = \frac{5}{9} = 0;33,20$$

and we have a linear system of equations for u, v whose solutions are

$$v = 1;10,40 = \frac{53}{45}, \quad u = 0;37,20 = \frac{28}{45}$$

We investigate this further in Exercise 7. The Plimpton tablet has been the source of enormous scholarship; look it up!

Geometry The Babylonians also discusses many geometric problems. They used both $\pi \approx 3$ and $\pi \approx 3\frac{1}{8}$ to approximate areas of circles. They had calculations (both correct and erroneous) for the volume of a frustrum (truncated pyramid). They also knew that the altitude of an isosceles triangle bisects its base and that the angle in semicircle is a right angle. None of these statements were presented as theorems in a modern sense; we merely have computations that use these facts. We simply don't know how deep the Babylonian understanding of these principles was.

Summary

- Sexagesimal positional enumeration. No zero or fractions.
- More advanced than Egyptian mathematics but still practical/non-abstract. Perhaps only appears more advanced because we have much more evidence (1000s of tablets versus a handful of papyri). Like Egypt, we have worked examples without abstraction or any statement of general principles.
- Some distinction ('does not divide') between approximate and exact results.
- Limited geometry compared to algorithmic/numerical methods.

Exercises There is no single 'correct' way to do Babylonian calculations. The goal is simply to play with the ideas; use enough notation to get a feel for it without torturing yourself.

1. Convert the sexagesimal values $0;22,30$, $0;8,6$, $0;4,10$ and $0;5,33,20$ into ordinary fractions in lowest terms.
2. (a) Multiply 25 by $1,4$ (b) Multiply 18 by $1,21$.
   (*Either compute directly (long multiplication) or use the difference of squares method on page 8*)
3. (a) Use reciprocals to divide 50 by 18. (b) Repeat for $1,21$ divided by 32.
4. Use the Babylonian method of false position to solve the linear system $\begin{cases} 3x+5y=19 \\ 2x+3y=12 \end{cases}$
5. (a) Convert the approximation $\sqrt{2} \approx 1;24,51,10$ to a decimal and verify the accuracy of the approximation on page 10.
   (b) Multiply by 30 to check that the length of the diagonal is as claimed.
6. Babylonian notation is not required for this question.
   (a) Use the square root approximation (pg. 11) with $a = \frac{8}{3}$ to find an approximation to $\sqrt{7}$.
   (b) Taking $a_1 = 3$, apply the method of the mean to find the approximation a_3 to $\sqrt{7}$.
7. Recall that $v^2 = 1 + u^2$ in the construction of the Plimpton tablet.
   (a) If $v + u = \alpha$, show that $u = \frac{1}{2}(\alpha - \alpha^{-1})$ and $v = \frac{1}{2}(\alpha + \alpha^{-1})$.
   (b) Suppose $v + u = 1;30 = \frac{3}{2}$. Find u, v and the corresponding Pythagorean triple.
   (c) Repeat for $v + u = 1;52,30 = \frac{15}{8}$.
   (d) Repeat for $v + u = 2;05 = \frac{25}{12}$. This is line 9 of the tablet.
8. Solve the following problem from tablet YBC 4652. I found a stone, but did not weigh it; after I subtracted one-seventh, added one-eleventh (of the difference), and then subtracted one-thirteenth (of the previous total), it weighed 1 *mina* (= 60 *gin*). What was the stone's weight?
   (*Make your best guess as to the meaning of the problem, it might not be clear!*)
9. Solve the following problem from tablet YBC 6967. A number exceeds its reciprocal by 7. Find the number and the reciprocal.
   (*In this case, two numbers are reciprocals if their product is 60*)
10. (Hard) For this question it is helpful to think about the corresponding facts for decimals.
    (a) Explain the observation on page 9 regarding which reciprocals n have a terminating sexagesimal. Can you prove this?
    (b) Find the periodic sexagesimal representation of $\frac{1}{7}$ and use geometric series *prove* that you are correct.
11. For this question, look up the AM–GM inequality and remind yourself of some basic Analysis.
    (a) Suppose (a_n) is a sequence satisfying the recurrence $a_{n+1} = \frac{1}{2}(a_n + \frac{2}{a_n})$. Prove that $a_n \geq \sqrt{2}$ whenever $n \geq 2$.
    (b) Prove that $a_n \to \sqrt{2}$.