Identification and Estimation of Spillover Effects in Randomized Experiments: Supplemental Appendix

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Abstract

This supplemental appendix provides the proofs of the results in the paper and additional discussions and results not included in the paper to conserve space.

Contents

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A1 Endogenous effects and structural models

Consider the structural model:

$$Y_{ig} = \phi(D_{ig}, \mathbf{D}_{(i)g}) + \gamma \bar{Y}_g^{(i)} + u_{ig}.$$

which assumes additive separability between the functions that depend on treatment assignments and on outcomes. In this model, $\phi(1, \mathbf{d}_g) - \phi(0, \mathbf{d}_g)$, measures the direct effect of the treatment, $\phi(d, \mathbf{d}_g) - \phi(d, \tilde{\mathbf{d}}_g)$ measures the spillover effects of peers' treatments, commonly known as exogenous or contextual effects, and γ measures the endogenous effect.

Suppose that the assumptions from Corollary 1 hold, so that the treatment vector is randomly assigned, peers are exchangeable and spillover effects are linear. By exchangeability,

$$\begin{aligned}
\phi(D_{ig}, \mathbf{D}_{(i)g}) = \phi(D_{ig}, S_{ig}) &= \sum_{s=0}^{n_g} \tilde{\beta}_s \mathbb{1}(S_{ig} = s)(1 - D_{ig}) + \sum_{s=0}^{n_g} \tilde{\delta}_s \mathbb{1}(S_{ig} = s) D_{ig} \\
&= \tilde{\beta}_0 + (\tilde{\delta}_0 - \tilde{\beta}_0) D_{ig} + \sum_{s=1}^{n_g} (\tilde{\beta}_s - \tilde{\beta}_0) \mathbb{1}(S_{ig} = s)(1 - D_{ig}) \\
&+ \sum_{s=1}^{n_g} (\tilde{\delta}_s - \tilde{\delta}_0) \mathbb{1}(S_{ig} = s) D_{ig}
\end{aligned}$$

where the second equality is without loss of generality because all the variables are discrete, and where

$$\tilde{\beta}_s = \phi(0, s), \quad \tilde{\delta}_s = \phi(1, s).$$

Let $\alpha = \tilde{\beta}_0$, $\beta = \tilde{\delta}_0 - \tilde{\beta}_0$, $\gamma_s^0 = \tilde{\beta}_s - \tilde{\beta}_0$, $\gamma_s^1 = \tilde{\delta}_s - \tilde{\delta}_0$ and rewrite the above model as:

$$\phi(D_{ig}, S_{ig}) = \alpha + \beta D_{ig} + \sum_{s=1}^{n_g} \gamma_s^0 \mathbb{1}(S_{ig} = s)(1 - D_{ig}) + \sum_{s=1}^{n_g} \gamma_s^1 \mathbb{1}(S_{ig} = s) D_{ig}.$$

Next, by linearity of spillover effects, $\gamma_0^d = \kappa_d s$ and

$$\sum_{s=1}^{n_g} \gamma_s^d \mathbb{1}(S_{ig} = s) = \kappa_d \sum_{s=1}^{n_g} s \mathbb{1}(S_{ig} = s) = \kappa_d S_{ig}$$

Therefore,

$$Y_{ig} = \alpha + \beta D_{ig} + \kappa_0 S_{ig}(1 - D_{ig}) + \kappa_1 S_{ig} D_{ig} + \gamma \bar{Y}_g^{(i)} + u_{ig}$$

In addition, suppose that contextual effects are equal between treated and controls so that $\kappa_0 = \kappa_1 = \kappa$. The model then reduces to:

$$\begin{aligned}
Y_{ig} &= \alpha + \beta D_{ig} + \kappa S_{ig} + \gamma \bar{Y}_g^{(i)} + u_{ig} \\
&= \alpha + \beta D_{ig} + \kappa n_g \bar{D}_g^{(i)} + \gamma \bar{Y}_g^{(i)} + u_{ig}
\end{aligned}$$

Noting that κ can be a function of n_g, $\kappa = \kappa(n_g)$, let $\theta = \kappa(n_g)n_g$ where the dependence on n_g is left implicit, so that:

$$Y_{ig} = \alpha + \beta D_{ig} + \theta \bar{D}_g^{(i)} + \gamma \bar{Y}_g^{(i)} + u_{ig}$$

which is a standard LIM model where β is the direct effect of the treatment, θ is the exogenous or contextual effect and γ is the endogenous effect.

Next, note that $\bar{Y}_g^{(i)} = \frac{n_g+1}{n_g}\bar{Y}_g - \frac{Y_{ig}}{n_g}$ which implies that:

$$Y_{ig}\left(1 + \frac{\gamma}{n_g}\right) = \alpha + \beta D_{ig} + \theta \bar{D}_g^{(i)} + \gamma \left(\frac{n_g+1}{n_g}\right)\bar{Y}_g + u_{ig}$$

and

$$\bar{Y}_g = \alpha + \beta \bar{D}_g + \theta \bar{D}_g + \gamma \bar{Y}_g + \bar{u}_g.$$

The last equation implies that, as long as $\gamma \neq 1$,

$$\begin{aligned}
\bar{Y}_g &= \frac{\alpha}{1-\gamma} + \frac{\beta+\theta}{1-\gamma}\bar{D}_g + \frac{\bar{u}_g}{1-\gamma} \\
&= \frac{\alpha}{1-\gamma} + \frac{\beta+\theta}{1-\gamma}\left(\frac{1}{n_g+1}\right) D_{ig} + \frac{\beta+\theta}{1-\gamma}\left(\frac{n_g}{n_g+1}\right)\bar{D}_g^{(i)} + \frac{\bar{u}_g}{1-\gamma}
\end{aligned}$$

so plugging back:

$$\begin{aligned}
Y_{ig}\left(1 + \frac{\gamma}{n_g}\right) = \alpha &+ \gamma\left(\frac{n_g+1}{n_g}\right)\frac{\alpha}{1-\gamma} \\
&+ \beta D_{ig} + \gamma\left(\frac{n_g+1}{n_g}\right)\frac{\beta+\theta}{1-\gamma}\left(\frac{1}{n_g+1}\right) D_{ig} \\
&+ \theta \bar{D}_g^{(i)} + \gamma\left(\frac{n_g+1}{n_g}\right)\frac{\beta+\theta}{1-\gamma}\left(\frac{n_g}{n_g+1}\right)\bar{D}_g^{(i)} \\
&+ u_{ig} + \gamma\left(\frac{n_g+1}{n_g}\right)\frac{\bar{u}_g}{1-\gamma}
\end{aligned}$$

After some simplifications,

$$\begin{aligned}
Y_{ig}\left(1 + \frac{\gamma}{n_g}\right) = &\left[1 + \left(\frac{n_g+1}{n_g}\right)\frac{\gamma}{1-\gamma}\right]\alpha + \left[\beta + \frac{\gamma}{1-\gamma}\cdot\frac{\beta+\theta}{n_g}\right] D_{ig} \\
&+ \left[\theta + \gamma\cdot\frac{\beta+\theta}{1-\gamma}\right]\bar{D}_g^{(i)} + u_{ig} + \gamma\left(\frac{n_g+1}{n_g}\right)\frac{\bar{u}_g}{1-\gamma}
\end{aligned}$$

and thus

$$Y_{ig} = \alpha^* + \beta^* D_{ig} + \theta^* \bar{D}_g^{(i)} + u_{ig}^*$$

where

$$
\begin{aligned}
\alpha^* &= \left[1 + \left(\frac{n_g+1}{n_g}\right)\frac{\gamma}{1-\gamma}\right]\left(1+\frac{\gamma}{n_g}\right)^{-1}\alpha \\
\beta^* &= \left[\beta + \frac{\gamma}{1-\gamma}\cdot\frac{\beta+\theta}{n_g}\right]\left(1+\frac{\gamma}{n_g}\right)^{-1} \\
\theta^* &= \left[\theta + \gamma\cdot\frac{\beta+\theta}{1-\gamma}\right]\left(1+\frac{\gamma}{n_g}\right)^{-1} \\
u_{ig}^* &= u_{ig}\left(1+\frac{\gamma}{n_g}\right)^{-1} + \gamma\left(\frac{n_g+1}{n_g}\right)\left(1+\frac{\gamma}{n_g}\right)^{-1}\frac{\bar{u}_g}{1-\gamma}.
\end{aligned}
$$

In this context, random assignment of the treatment implies that $\mathbb{E}[u_{ig}|D_{ig}, \mathbf{D}_{(i)g}] = 0$ and hence the reduced-form parameters $(\alpha^*, \beta^*, \theta^*)$ are identified. As in any structural LIM model, however, the structural parameters $(\alpha, \beta, \theta, \gamma)$ are not identified without further assumptions.

A2 Implications for experimental design

Theorem 4 shows that the accuracy of the standard normal to approximate the distribution of the standardized statistic depends on the treatment assignment mechanism through $\underline{\pi}_n$. The intuition behind this result is that the amount of information to estimate each $\mu(\mathbf{a})$ depends on the number of observations facing assignment $\mathbf{a}$, and this number depends on $\pi(\mathbf{a})$. When the goal is to estimate all the $\mu(\mathbf{a})$ simultaneously, the binding factor will be the number of observations in the smallest cell, controlled by $\underline{\pi}_n$. When an assignment sets a value of $\underline{\pi}_n$ that is very close to zero, the normal distribution may provide a poor approximation to the distribution of the estimators.

When designing an experiment to estimate spillover effects, the researcher can choose distribution of treatment assignments $\pi(\cdot)$. Theorem 4 provides a way to rank different assignment mechanisms based on their rate of the approximation, which gives a principled way to choose between different assignment mechanisms.

To illustrate these issues, consider the case of an exchangeable exposure mapping $\mathcal{A}_n = \{(d, s) : d = 0, 1, s = 0, 1, \ldots, n\}$. The results below compare two treatment assignment mechanisms. The first one, *simple random assignment (SR)*, consists in assigning the treatment independently at the individual level with probability $\mathbb{P}[D_{ig} = 1] = p$. The second mechanism is *two-stage randomization with fixed margins (2SR-FM)*. See Section A3 for further details on this design.

Corollary A1 (SR) *Under simple random assignment, condition (5) holds whenever:*

$$
\frac{n+1}{\log G} \to 0. \tag{1}
$$

Corollary A2 (2SR-FM) *Under a 2SR-FM mechanism, condition (5) holds whenever:*

$$\frac{\log(n+1)}{\log G} \to 0. \tag{2}$$

In qualitative terms, both results imply that estimation and inference for spillover effects requires group size to be small relative to the total number of groups. Thus, these results formalize the requirement of "many small groups" that is commonly invoked, for example, when estimating LIM models (see e.g. Davezies, D'Haultfoeuille, and Fougère, 2009; Kline and Tamer, 2019).

Corollary A1 shows that when the treatment is assigned using simple random assignment, group size has to be small relative to $\log G$. Given the concavity of the log function, this is a strong requirement. Hence, groups have to be very small relative to the sample size for inference to be asymptotically valid. The intuition behind this result is that under a SR, the probability of the tail assignments $(0,0)$ and $(1,n)$ decreases exponentially fast with group size.

On the other hand, Corollary A2 shows that a 2SR-FM mechanism reduces the requirement to $\log(n+1)/\log G \approx 0$, so now the log of group size has to be small compared to the log of the number of groups. This condition is much more easily satisfied, which in practical terms implies that a 2SR-FM mechanism can handle larger groups compared to SR. The intuition behind this result is that, by fixing the number of treated units in each group, a 2SR-FM design has better control on how small the probabilities of each assignment can be, hence facilitating the estimation of the tail assignments.

A3 Assignment mechanism for 2SR-FM

In a 2SR-FM assignment mechanism, given a group size $n+1$ groups are assigned to receive $0, 1, 2, \ldots, n+1$ treated units with probabilities $q_0, q_1, \ldots, q_{n+1}$. Treatment assignments in this case are given by $\mathbf{A}_{ig} = (D_{ig}, T_g)$ where $D_{ig} \in \{0,1\}$ and $T_g \in \{0, 1, \ldots, n+1\}$, and $\pi(\mathbf{a}) = \mathbb{P}[D_{ig} = d | T_g = t] q_t = q_t \left(\frac{t}{n+1}\right)^d \left(1 - \frac{t}{n+1}\right)^{1-d}$. Assume for simplicity that $n+1$ is odd. The choice of q_t is determined by the following system of equations:

$$\begin{aligned} q_j = q_{n+1-j}, \quad & j \leq \frac{n}{2} \\ q_j = \frac{(n+1)q_0}{j}, \quad & j \leq \frac{n}{2} \\ \sum_j q_j = 1. & \end{aligned}$$

The first set of equations imposes symmetry, that is, $\mathbb{P}[T_g = 0] = \mathbb{P}[T_g = n+1]$ and so on. The second set of equations makes the expected sample size in the smallest assignment in each group (untreated units in high-intensity treatment groups and vice versa) equal to the expected sample size of pure controls. The solution to this system is given by:

$$q_0 \left(1 + (n+1) \sum_{j=1}^{\frac{n}{2}} \frac{1}{j} \right) = \frac{1}{2}$$

and the remaining probabilities are obtained from the previous relationships. If $n+1$ is even,

$$q_0 \left(2 + (n+1) \sum_{j=1}^{\frac{n+1}{2}-1} \frac{1}{j} \right) = \frac{1}{2}.$$

A4 Unequally-sized groups

To explicitly account for different group sizes, let n (the total number of peers in each group) take values in $\mathcal{N} = \{n_1, n_2, \ldots, n_K\}$ where $n_k \geq 1$ for all k and $n_1 < n_2 < \ldots < n_K$. Let the potential outcome be $Y_{ig}(n, d, s(n))$ where $n \in \mathcal{N}$ and $s(n) \in \{0, 1, 2, \ldots, n\}$. Let N_g be the observed value of n_g, $S_{ig}(n) = \sum_{j \neq i}^{n} D_{jg}$ and $S_{ig} = \sum_{k=1}^{K} S_{ig}(n_k) \mathbb{1}(N_g = n_k)$. The independence assumption can be modified to hold conditional on group size:

$$\{Y_{ig}(n, d, s(n)) : d = 0, 1, s(n) = 0, 1, \ldots, n\}_{i=1}^{n} \perp\!\!\!\perp \mathbf{D}_g(n))|N_g = n$$

where $\mathbf{D}_g(n)$ is the vector of all treatment assignments when the group size is $n+1$.

Under this assumption, we have that for $n \in \mathcal{N}$ and $s \leq n$,

$$\mathbb{E}[Y_{ig}|D_{ig} = d, S_{ig} = s, N_g = n] = \mathbb{E}[Y_{ig}(n, d, s)].$$

The average observed outcome conditional on $N_g = n$ can be written as:

$$\begin{aligned}
\mathbb{E}[Y_{ig}|D_{ig}, S_{ig}, N_g = n] = \mathbb{E}[Y_{ig}(n, 0, 0)] &+ \tau_0(n) D_{ig} \\
&+ \sum_{s=1}^{n} \theta_0(s, n) \mathbb{1}(S_{ig} = s)(1 - D_{ig}) \\
&+ \sum_{s=1}^{n} \theta_1(s, n) \mathbb{1}(S_{ig} = s) D_{ig}
\end{aligned}$$

The easiest approach is to simply run separate analyses for each group size and estimate all the effects separately. In this case, it is possible to test whether spillover effects are different in groups with different sizes. The total number of parameters in this case is given by $\sum_{k=1}^{K}(n_k + 1)$.

In practice, however, there may be cases in which group size has a rich support with only a few groups at each value n_g, so separate analyses may not be feasible. In such a setting, a possible solution is to impose an additivity assumption on group size. According to this assumption, the average direct and spillover effects do not change with group size. For example, the spillover effect of having one treated neighbor is the same in a group with two or three units. Under this assumption,

$$\begin{aligned}\mathbb{E}[Y_{ig}|D_{ig},S_{ig},N_g] = &\sum_{n_g\in\mathcal{N}_g}\alpha(n_g)\mathbb{1}(N_g=n_g)+\tau_0 D_{ig}\\ &+\sum_{s=1}^{N_g}\theta_0(s)\mathbb{1}(S_{ig}=s)(1-D_{ig})\\ &+\sum_{s=1}^{N_g}\theta_1(s)\mathbb{1}(S_{ig}=s)D_{ig}\end{aligned}$$

where the first sum can be seen in practice as adding group-size fixed effects. Then, the identification results and estimation strategies in the paper are valid after controlling for group-size fixed effects. Note that in this case the total number of parameters to estimate is n_K+K-1 where n_K is the size of the largest group and K is the total number of different group sizes.

Another possibility is to assume that for any constant $c\in\mathbb{N}$, $Y_{ig}(c\cdot n,d,c\cdot s)=Y_{ig}(n,d,s)$. This assumption allows us to rewrite the potential outcomes as a function of the ratio of treated peers, $Y_{ig}(d,s/n)$. Letting $P_{ig}=S_{ig}/N_g$, all the parameters can be estimated by running a regression including D_{ig}, $\mathbb{1}(P_{ig}=p)$ for all possible values of $p>0$ (excluding $p=0$ to avoid perfect collinearity) and interactions. In this case, the total number of parameters can be bounded by $n_1+\sum_{k=2}^{K}(n_k-1)$. Note that assuming that the potential outcomes depend only on the proportion of treated siblings does not justify in any way including the variable P_{ig} linearly, as commonly done in linear-in-means models.

A5 Including covariates

There are several reasons why one may want to include covariates when estimating direct and spillover effects. First, pre-treatment characteristics may help reduce the variability of the estimators and decrease small-sample bias, which is standard practice when analyzing randomly assigned programs. Covariates can also help get valid inference when the assignment mechanisms stratifies on baseline covariates. This can be done by simply augmenting Equation (8) with a vector of covariates $\boldsymbol{\gamma}'\mathbf{x}_{ig}$ which can vary at the unit or at the group level. The covariates can also be interacted with the treatment assignment indicators to explore effect heterogeneity across observable characteristics (for example, by separately estimating effects for males and females.

Second, exogenous covariates can be used to relax the mean-independence assumption in observational studies. More precisely, if $\mathbf{X}_g$ is a matrix of covariates, a conditional mean-independence assumption would be

$$\mathbb{E}[Y_{ig}(d,\mathbf{d}_g)|\mathbf{X}_g,\mathbf{D}_g]=\mathbb{E}[Y_{ig}(d,\mathbf{d}_g)|\mathbf{X}_g]$$

which is a version of the standard unconfoundeness condition. The vector of covariates can include both individual-level and group-level characteristics.

Third, covariates can be included to make an exposure mapping more likely to be correctly specified. For instance, the exchangeability assumption can be relaxed by assuming it holds after conditioning on covariates, so that for any pair of treatment assignments $\mathbf{d}_g$ and $\tilde{\mathbf{d}}_g$ with the same number of ones,

$$\mathbb{E}[Y_{ig}(d, \mathbf{d}_g)|\mathbf{X}_g] = \mathbb{E}[Y_{ig}(d, \tilde{\mathbf{d}}_g)|\mathbf{X}_g]$$

As an example, exchangeability can be assumed to hold for all siblings with the same age, gender or going to the same school.

All the identification results in the paper can be adapted to hold after conditioning on covariates. In terms of implementation, when the covariates are discrete the parameters of interest can be estimated at each possible value of the matrix $\mathbf{X}_g$, although this strategy can worsen the dimensionality problem. Alternatively, covariates can be included in a regression framework after imposing parametric assumptions, for example, assuming the covariates enter linearly.

A6 Additional theoretical results

Lemma A1 (Assignment probabilities) *Let* $\hat{\pi}(\mathbf{a}) := \sum_g \sum_i \mathbb{1}_{ig}(\mathbf{a})/G(n+1)$. *Then under the assumptions of Lemma 2,*

$$\max_{\mathbf{a}\in\mathcal{A}_n} \left| \frac{\hat{\pi}(\mathbf{a})}{\pi(\mathbf{a})} - 1 \right| \to_{\mathbb{P}} 0.$$

A7 Proofs of additional results

Proof of Lemma A1 Take $\varepsilon > 0$, then

$$\begin{aligned}
\mathbb{P}\left[\max_{\mathbf{a}\in\mathcal{A}_n} \left| \frac{\hat{\pi}(\mathbf{a})}{\pi(\mathbf{a})} - 1 \right| > \varepsilon\right] &\leq \sum_{\mathbf{a}\in\mathcal{A}_n} \mathbb{P}\left[\left| \frac{\hat{\pi}(\mathbf{a})}{\pi(\mathbf{a})} - 1 \right| > \varepsilon\right] \leq |\mathcal{A}_n| \max_{\mathbf{a}\in\mathcal{A}_n} \mathbb{P}\left[|\hat{\pi}(\mathbf{a}) - \pi(\mathbf{a})| > \varepsilon\pi(\mathbf{a})\right] \\
&= |\mathcal{A}_n| \max_{\mathbf{a}\in\mathcal{A}_n} \mathbb{P}\left[|N(\mathbf{a}) - \mathbb{E}[N(\mathbf{a})]| > \varepsilon\mathbb{E}[N(\mathbf{a})]\right]
\end{aligned}$$

Now,

$$N(\mathbf{a}) - \mathbb{E}[N(\mathbf{a})] = \sum_g \sum_i \mathbb{1}_{ig}(\mathbf{a}) - G(n+1)\pi(\mathbf{a}) = \sum_g W_g$$

where $W_g = \sum_i \mathbb{1}_{ig}(\mathbf{a}) - (n+1)\pi(\mathbf{a}) = N_g(\mathbf{a}) - \mathbb{E}[N_g(\mathbf{a})]$. Note that the W_g are independent and:

$$\begin{aligned}
\mathbb{E}[W_g] &= 0 \\
|W_g| &\le (n+1)\max\{\pi(\mathbf{a}), 1-\pi(\mathbf{a})\} \\
\mathbb{V}[W_g] &= \mathbb{V}\left[\sum_i \mathbb{1}_{ig}(\mathbf{a})\right] = \sum_i \mathbb{V}[\mathbb{1}_{ig}(\mathbf{a})] + 2\sum_i \sum_{j>i} \mathbb{C}\mathrm{ov}(\mathbb{1}_{ig}(\mathbf{a}), \mathbb{1}_{jg}(\mathbf{a})) \\
&= (n+1)\pi(\mathbf{a})(1-\pi(\mathbf{a})) + (n+1)(n+2)\{\mathbb{E}[\mathbb{1}_{ig}(\mathbf{a})\mathbb{1}_{jg(\mathbf{a})}] - \pi(\mathbf{a})^2\} \\
&\le (n+1)\pi(\mathbf{a})(1-\pi(\mathbf{a})) + (n+1)(n+2)\pi(\mathbf{a})(1-\pi(\mathbf{a})) \\
&= (n+1)(n+3)\pi(\mathbf{a})(1-\pi(\mathbf{a}))
\end{aligned}$$

Then, by Bernstein's inequality,

$$\begin{aligned}
\mathbb{P}\left[|W_g|| > \varepsilon\mathbb{E}[N(\mathbf{a})]\right] &\le 2\exp\left\{-\frac{\mathbb{E}[N(\mathbf{a})]^2\varepsilon^2}{\sum_g \mathbb{V}[W_g] + \frac{1}{3}(n+1)\max\{\pi(\mathbf{a}), 1-\pi(\mathbf{a})\}\mathbb{E}[N(\mathbf{a})]\varepsilon}\right\} \\
&= 2\exp\left\{-\frac{\frac{1}{2}G^2(n+1)^2\pi(\mathbf{a})^2\varepsilon^2}{G(n+1)(n+3)\pi(\mathbf{a})(1-\pi(\mathbf{a})) + \frac{1}{3}G(n+1)^2\pi(\mathbf{a})\max\{\pi(\mathbf{a}), 1-\pi(\mathbf{a})\}\varepsilon}\right\} \\
&= 2\exp\left\{-\frac{\frac{1}{2}G\pi(\mathbf{a})\varepsilon^2}{\frac{n+3}{n+1}(1-\pi(\mathbf{a})) + \frac{1}{3}\max\{\pi(\mathbf{a}), 1-\pi(\mathbf{a})\varepsilon\}}\right\} \\
&\le 2\exp\left\{-\frac{\frac{1}{2}G\pi(\mathbf{a})\varepsilon^2}{\frac{n+3}{n+1} + \frac{\varepsilon}{3}}\right\}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{P}\left[\max_{\mathbf{a}\in\mathcal{A}_n}\left|\frac{\hat{\pi}(\mathbf{a})}{\pi(\mathbf{a})} - 1\right| > \varepsilon\right] &\le |\mathcal{A}_n| \max_{\mathbf{a}\in\mathcal{A}_n} \mathbb{P}\left[|N(\mathbf{a}) - \mathbb{E}[N(\mathbf{a})]| > \varepsilon\mathbb{E}[N(\mathbf{a})]\right] \\
&\le 2\exp\left\{-G\underline{\pi}_n\left(\frac{\frac{1}{2}\varepsilon^2}{\frac{n+3}{n+1} + \frac{\varepsilon}{3}} - \frac{\log|\mathcal{A}_n|}{G\underline{\pi}_n}\right)\right\} \to 0.
\end{aligned}$$

as required. □

Proof of Corollary A1 Under exchangeability $\pi(\mathbf{a}) = \pi(d,s) = p^d(1-p)^{1-d}\binom{n}{s}p^s(1-p)^{n-s} = \binom{n}{s}p^{s+d}(1-p)^{n+1-s-d}$. This function is minimized at $\underline{\pi}_n = \underline{p}^{n+1} \propto \underline{p}^n$ where $\underline{p} = \min\{p, 1-p\}$. Thus,

$$\frac{\log|\mathcal{A}_n|}{G\underline{p}^n} = \exp\left\{-\log G\left(1 - \frac{n+1}{\log G}2\log\underline{p} - \frac{\log\log|\mathcal{A}_n|}{\log G}\right)\right\}$$

and since $|\mathcal{A}_n| = 2(n+1)$, this term converges to zero when $(n+1)/\log G \to 0$. □

Proof of Corollary A2 Under exchangeability, $\pi(\mathbf{a}) = \pi(d,s) = q_{d+s}\left(\frac{s+1}{n+1}\right)^d \times \left(1 - \frac{s}{n+1}\right)^{1-d}$. Under the assignment mechanism in Section A3, $\underline{\pi}_n = q_0$. Suppose for simplicity that $n+1$

is odd. Then,

$$q_0 = \frac{1}{2\left(1+(n+1)\sum_{j=1}^{\frac{n}{2}}\frac{1}{j}\right)} \geq \frac{1}{2\,(n+2)}$$

and thus:

$$\frac{\log|\mathcal{A}_n|}{G\underline{\pi}_n} \leq \exp\left\{-\log G\left(1-\frac{\log 2}{\log G}-\frac{\log(n+2)}{\log G}-\frac{\log\log 2(n+1)}{\log G}\right)\right\} \to 0$$

if $\log(n+1)/\log G \to 0$. $\square$

A8 Proofs of main results

Proof of Lemma 1 If $\mathbb{P}[D_{ig}=d, \mathbf{H}_{ig}=\mathbf{h}]>0$,

$$\begin{aligned}
\mathbb{E}[Y_{ig}|D_{ig}=d,\mathbf{H}_{ig}=\mathbf{h}] &= \sum_{\mathbf{h}_0}\mathbb{E}[Y_{ig}|D_{ig}=d,\mathbf{H}_{ig}=\mathbf{h},\mathbf{H}^0_{ig}=\mathbf{h}_0] \\
&\quad\times\mathbb{P}[\mathbf{H}^0_{ig}=\mathbf{h}_0|D_{ig}=d,\mathbf{H}_{ig}=\mathbf{h}] \\
&= \sum_{\mathbf{h}_0}\mathbb{E}[Y_{ig}(d,\mathbf{h}_0)|D_{ig}=d,\mathbf{H}_{ig}=\mathbf{h},\mathbf{H}^0_{ig}=\mathbf{h}_0] \\
&\quad\times\mathbb{P}[\mathbf{H}^0_{ig}=\mathbf{h}^0|D_{ig}=d,\mathbf{H}_{ig}=\mathbf{h}] \\
&= \sum_{\mathbf{h}_0}\mathbb{E}[Y_{ig}(d,\mathbf{h}_0)]\mathbb{P}[\mathbf{H}^0_{ig}=\mathbf{h}_0|D_{ig}=d,\mathbf{H}_{ig}=\mathbf{h}]
\end{aligned}$$

where the first equality follows from the law if iterated expectations, the second equality follows by definition of the observed outcomes and the third equality follows from random assignment of the treatment vector given that both $\mathbf{H}^0_{ig}$ and $\mathbf{H}_{ig}$ are deterministic functions of $\mathbf{D}_g$. Finally, if $h_0(\cdot)$ is coarser than $h(\cdot)$, then $\mathbf{H}_{ig}=\mathbf{h}$ uniquely determines the value of $\mathbf{H}^0_{ig}$ and the result follows. $\square$

Proof of Theorem 1 Follows from Lemma 1 letting $h(\cdot)$ be a constant function, using the fact that by construction $\beta_{\mathsf{D}}=\mathbb{E}[Y_{ig}|D_{ig}=1]-\mathbb{E}[Y_{ig}|D_{ig}=0]$. $\square$

Proof of Theorem 2 The coefficients from Equation (2) are characterized by the minimization problem:

$$\min_{(\alpha_\ell,\beta_\ell,\gamma_\ell)}\mathbb{E}\left[\left(Y_{ig}-\alpha_\ell-\beta_\ell D_{ig}-\gamma_\ell\bar{D}_g^{(i)}\right)^2\right].$$

The objective function can be rewritten as:

$$\mathbb{E}\left[\left(Y_{ig}-\alpha_\ell-\gamma_\ell\bar{D}_g^{(i)}\right)^2(1-D_{ig})\right]+\mathbb{E}\left[\left(Y_{ig}-\alpha_\ell-\beta_\ell-\gamma_\ell\bar{D}_g^{(i)}\right)^2 D_{ig}\right]$$

which can be reparameterized as

$$\mathbb{E}\left[\left(Y_{ig}-\alpha_0-\gamma_\ell\bar{D}_g^{(i)}\right)^2(1-D_{ig})\right]+\mathbb{E}\left[\left(Y_{ig}-\alpha_1-\gamma_\ell\bar{D}_g^{(i)}\right)^2 D_{ig}\right]$$

where $\alpha_0=\alpha_\ell$ and $\alpha_1=\alpha_\ell+\beta_\ell$. The first-order condition for α_0 and α_1 are:

$$\begin{aligned}
0&=\mathbb{E}\left[\left(Y_{ig}-\alpha_0-\gamma_\ell\bar{D}_g^{(i)}\right)(1-D_{ig})\right]\\
0&=\mathbb{E}\left[\left(Y_{ig}-\alpha_1-\gamma_\ell\bar{D}_g^{(i)}\right)D_{ig}\right]
\end{aligned}$$

from which

$$\begin{aligned}
\alpha_0&=\mathbb{E}[Y_{ig}|D_{ig}=0]-\gamma_\ell\mathbb{E}[\bar{D}_g^{(i)}|D_{ig}=0]\\
\alpha_1&=\mathbb{E}[Y_{ig}|D_{ig}=1]-\gamma_\ell\mathbb{E}[\bar{D}_g^{(i)}|D_{ig}=1].
\end{aligned}$$

But $\beta_\ell=\alpha_1-\alpha_0$ and thus:

$$\begin{aligned}
\beta_\ell&=\mathbb{E}[Y_{ig}|D_{ig}=1]-\mathbb{E}[Y_{ig}|D_{ig}=0]-\gamma_\ell(\mathbb{E}[\bar{D}_g^{(i)}|D_{ig}=1]-\mathbb{E}[\bar{D}_g^{(i)}|D_{ig}=0])\\
&=\mathbb{E}[Y_{ig}|D_{ig}=1]-\mathbb{E}[Y_{ig}|D_{ig}=0]-\frac{\gamma_\ell}{n_g}(\mathbb{E}[S_{ig}|D_{ig}=1]-\mathbb{E}[S_{ig}|D_{ig}=0]).
\end{aligned}$$

The first-order condition for γ_ℓ is:

$$\begin{aligned}
0&=\mathbb{E}\left[\left(Y_{ig}-\alpha_0-\gamma_\ell\bar{D}_g^{(i)}\right)\bar{D}_g^{(i)}(1-D_{ig})\right]+\mathbb{E}\left[\left(Y_{ig}-\alpha_1-\gamma_\ell\bar{D}_g^{(i)}\right)\bar{D}_g^{(i)}D_{ig}\right]\\
&=\mathbb{E}\left[\left(Y_{ig}-\mathbb{E}[Y_{ig}|D_{ig}=0]-\gamma_\ell(\bar{D}_g^{(i)}-\mathbb{E}[\bar{D}_g^{(i)}|D_{ig}=0])\right)\bar{D}_g^{(i)}(1-D_{ig})\right]\\
&+\mathbb{E}\left[\left(Y_{ig}-\mathbb{E}[Y_{ig}|D_{ig}=1]-\gamma_\ell(\bar{D}_g^{(i)}-\mathbb{E}[\bar{D}_g^{(i)}|D_{ig}=1])\right)\bar{D}_g^{(i)}D_{ig}\right]\\
&=\mathbb{C}\mathrm{ov}(Y_{ig},\bar{D}_g^{(i)}|D_{ig}=0)\mathbb{P}[D_{ig}=0]+\mathbb{C}\mathrm{ov}(Y_{ig},\bar{D}_g^{(i)}|D_{ig}=1)\mathbb{P}[D_{ig}=1]\\
&-\gamma_\ell\left(\mathbb{V}[\bar{D}_g^{(i)}|D_{ig}=0]\mathbb{P}[D_{ig}=0]+\mathbb{V}[\bar{D}_g^{(i)}|D_{ig}=1]\mathbb{P}[D_{ig}=1]\right)
\end{aligned}$$

from which:

$$\gamma_\ell=\frac{\mathbb{C}\mathrm{ov}(Y_{ig},\bar{D}_g^{(i)}|D_{ig}=0)\mathbb{P}[D_{ig}=0]+\mathbb{C}\mathrm{ov}(Y_{ig},\bar{D}_g^{(i)}|D_{ig}=1)\mathbb{P}[D_{ig}=1]}{\mathbb{V}[\bar{D}_g^{(i)}|D_{ig}=0]\mathbb{P}[D_{ig}=0]+\mathbb{V}[\bar{D}_g^{(i)}|D_{ig}=1]\mathbb{P}[D_{ig}=1]}.$$

Next,

$$\mathbb{V}[\bar{D}_g^{(i)}|D_{ig}=d]=\frac{1}{n_g^2}\mathbb{V}[S_{ig}|D_{ig}=d]$$

and

$$
\begin{aligned}
\mathbb{C}\mathrm{ov}(Y_{ig}, \bar{D}_g^{(i)}|D_{ig}=d) &= \frac{1}{n_g}\mathbb{C}\mathrm{ov}(Y_{ig}, S_{ig}|D_{ig}=d) \\
&= \frac{1}{n_g}\mathbb{C}\mathrm{ov}(\mathbb{E}[Y_{ig}|D_{ig}=d, S_{ig}], S_{ig}|D_{ig}=d) \\
&= \frac{1}{n_g}\sum_{s=0}^{n_g}\mathbb{E}[Y_{ig}|D_{ig}=d, S_{ig}=s]\mathbb{C}\mathrm{ov}(\mathbb{1}(S_{ig}=s), S_{ig}|D_{ig}=d).
\end{aligned}
$$

But

$$
\begin{aligned}
\mathbb{C}\mathrm{ov}(\mathbb{1}(S_{ig}=s), S_{ig}|D_{ig}=d) &= \mathbb{E}[\mathbb{1}(S_{ig}=s)S_{ig}|D_{ig}=d] - \mathbb{E}[\mathbb{1}(S_{ig}=s)|D_{ig}=d]\mathbb{E}[S_{ig}|D_{ig}=d] \\
&= (s-\mathbb{E}[S_{ig}|D_{ig}=d])\mathbb{P}[S_{ig}=s|D_{ig}=d].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\gamma_\ell &= \frac{\frac{1}{n_g}\sum_{s=0}^{n_g}\mathbb{E}[Y_{ig}|D_{ig}=0, S_{ig}=s](s-\mathbb{E}[S_{ig}|D_{ig}=0])\mathbb{P}[S_{ig}=s|D_{ig}=0]\mathbb{P}[D_{ig}=0]}{\frac{1}{n_g^2}\mathbb{V}[S_{ig}|D_{ig}=0]\mathbb{P}[D_{ig}=0]+\frac{1}{n_g^2}\mathbb{V}[S_{ig}|D_{ig}=1]\mathbb{P}[D_{ig}=1]} \\
&+ \frac{\frac{1}{n_g}\sum_{s=0}^{n_g}\mathbb{E}[Y_{ig}|D_{ig}=1, S_{ig}=s](s-\mathbb{E}[S_{ig}|D_{ig}=1])\mathbb{P}[S_{ig}=s|D_{ig}=1]\mathbb{P}[D_{ig}=1]}{\frac{1}{n_g^2}\mathbb{V}[S_{ig}|D_{ig}=0]\mathbb{P}[D_{ig}=0]+\frac{1}{n_g^2}\mathbb{V}[S_{ig}|D_{ig}=1]\mathbb{P}[D_{ig}=1]} \\
&= \frac{\sum_{s=0}^{n_g}\mathbb{E}[Y_{ig}|D_{ig}=0, S_{ig}=s]n_g\mathbb{P}[D_{ig}=0](s-\mathbb{E}[S_{ig}|D_{ig}=0])\mathbb{P}[S_{ig}=s|D_{ig}=0]}{\mathbb{V}[S_{ig}|D_{ig}=0]\mathbb{P}[D_{ig}=0]+\mathbb{V}[S_{ig}|D_{ig}=1]\mathbb{P}[D_{ig}=1]} \\
&+ \frac{\sum_{s=0}^{n_g}\mathbb{E}[Y_{ig}|D_{ig}=1, S_{ig}=s]n_g\mathbb{P}[D_{ig}=1](s-\mathbb{E}[S_{ig}|D_{ig}=1])\mathbb{P}[S_{ig}=s|D_{ig}=1]}{\mathbb{V}[S_{ig}|D_{ig}=0]\mathbb{P}[D_{ig}=0]+\mathbb{V}[S_{ig}|D_{ig}=1]\mathbb{P}[D_{ig}=1]} \\
&= \sum_{s=0}^{n_g}\phi_0(s)\mathbb{E}[Y_{ig}|D_{ig}=0, S_{ig}=s] + \sum_{s=0}^{n_g}\phi_1(s)\mathbb{E}[Y_{ig}|D_{ig}=1, S_{ig}=s]
\end{aligned}
$$

where

$$
\phi_d(s) = \frac{n_g\mathbb{P}[D_{ig}=d]\mathbb{P}[S_{ig}=s|D_{ig}=d]}{\mathbb{V}[S_{ig}|D_{ig}=0]\mathbb{P}[D_{ig}=0]+\mathbb{V}[S_{ig}|D_{ig}=1]\mathbb{P}[D_{ig}=1]}\cdot(s-\mathbb{E}[S_{ig}|D_{ig}=d]).
$$

Also note that:

$$
\begin{aligned}
\sum_{s=0}^{n_g}\phi_d(s) &= n_g\mathbb{P}[D_{ig}=d]\sum_{s=0}^{n_g}(s-\mathbb{E}[S_{ig}|D_{ig}=d])\mathbb{P}[S_{ig}=s|D_{ig}=d] \\
&= n_g\mathbb{P}[D_{ig}=d]\mathbb{E}\left[S_{ig}-\mathbb{E}[S_{ig}|D_{ig}=d]|D_{ig}=d\right] \\
&= 0.
\end{aligned}
$$

This implies that:

$$
\begin{aligned}
\sum_{s=0}^{n_g} \phi_d(s)\mathbb{E}[Y_{ig}|D_{ig}=d, S_{ig}=s] &= \sum_{s=0}^{n_g} \phi_d(s)(\mathbb{E}[Y_{ig}|D_{ig}=d, S_{ig}=s] - \mathbb{E}[Y_{ig}|D_{ig}=d, S_{ig}=0]) \\
&+ \sum_{s=0}^{n_g} \phi_d(s)\mathbb{E}[Y_{ig}|D_{ig}=d, S_{ig}=0] \\
&= \sum_{s=0}^{n_g} \phi_d(s)(\mathbb{E}[Y_{ig}|D_{ig}=d, S_{ig}=s] - \mathbb{E}[Y_{ig}|D_{ig}=d, S_{ig}=0]) \\
&+ \mathbb{E}[Y_{ig}|D_{ig}=d, S_{ig}=0] \sum_{s=0}^{n_g} \phi_d(s) \\
&= \sum_{s=1}^{n_g} \phi_d(s)(\mathbb{E}[Y_{ig}|D_{ig}=d, S_{ig}=s] - \mathbb{E}[Y_{ig}|D_{ig}=d, S_{ig}=0])
\end{aligned}
$$

which gives the required result. □

Proof of Corollary 1 When the true model satisfies exchangeability, potential outcomes have the form $Y_{ig}(d, s)$. From Theorem 2,

$$
\begin{aligned}
\gamma_\ell &= \sum_{s=1}^{n_g} \phi_0(s)(\mathbb{E}[Y_{ig}|D_{ig}=0, S_{ig}=s] - \mathbb{E}[Y_{ig}|D_{ig}=0, S_{ig}=0]) \\
&+ \sum_{s=1}^{n_g} \phi_1(s)(\mathbb{E}[Y_{ig}|D_{ig}=1, S_{ig}=s] - \mathbb{E}[Y_{ig}|D_{ig}=1, S_{ig}=0]) \\
&= \sum_{s=1}^{n_g} \phi_0(s)\mathbb{E}[Y_{ig}(0,s) - Y_{ig}(0,0)] + \sum_{s=1}^{n_g} \phi_1(s)\mathbb{E}[Y_{ig}(1,s) - Y_{ig}(1,0)]
\end{aligned}
$$

By linearity, $\mathbb{E}[Y_{ig}(d,s) - Y_{ig}(d,0)] = s\kappa_d$ and thus:

$$
\begin{aligned}
\gamma_\ell &= \sum_{s=1}^{n_g} \phi_0(s)\mathbb{E}[Y_{ig}(0,s) - Y_{ig}(0,0)] + \sum_{s=1}^{n_g} \phi_1(s)\mathbb{E}[Y_{ig}(1,s) - Y_{ig}(1,0)] \\
&= \kappa_0 \sum_{s=1}^{n_g} s\phi_0(s) + \kappa_1 \sum_{s=1}^{n_g} s\phi_1(s).
\end{aligned}
$$

But

$$
\begin{aligned}
\sum_{s=1}^{n_g} s\phi_d(s) &= \frac{n_g \mathbb{P}[D_{ig}=d] \sum_{s=1}^{n_g} s(s - \mathbb{E}[S_{ig}|D_{ig}=d])\mathbb{P}[S_{ig}=s|D_{ig}=d]}{\mathbb{V}[S_{ig}|D_{ig}=0]\mathbb{P}[D_{ig}=0] + \mathbb{V}[S_{ig}|D_{ig}=1]\mathbb{P}[D_{ig}=1]} \\
&= \frac{n_g \mathbb{P}[D_{ig}=d]\mathbb{V}[S_{ig}|D_{ig}=d]}{\mathbb{V}[S_{ig}|D_{ig}=0]\mathbb{P}[D_{ig}=0] + \mathbb{V}[S_{ig}|D_{ig}=1]\mathbb{P}[D_{ig}=1]}.
\end{aligned}
$$

Hence,

$$\begin{aligned}\gamma_\ell &= \kappa_0 \sum_{s=1}^{n_g} s\phi_0(s) + \kappa_1 \sum_{s=1}^{n_g} s\phi_1(s) \\ &= n_g\kappa_1\lambda + n_g\kappa_0(1-\lambda)\end{aligned}$$

where

$$\lambda = \frac{\mathbb{P}[D_{ig}=d]\mathbb{V}[S_{ig}|D_{ig}=d]}{\mathbb{V}[S_{ig}|D_{ig}=0]\mathbb{P}[D_{ig}=0] + \mathbb{V}[S_{ig}|D_{ig}=1]\mathbb{P}[D_{ig}=1]}.$$

But $n_g\kappa_d = \mathbb{E}[Y_{ig}(d, n_g) - Y_{ig}(d, 0)]$ which gives the result for γ_ℓ.

On the other hand, from Theorem 1, the difference in means is:

$$\begin{aligned}\beta_{\mathsf{D}} &= \mathbb{E}[Y_{ig}|D_{ig}=1] - \mathbb{E}[Y_{ig}|D_{ig}=0] \\ &= \mathbb{E}[Y_{ig}(1,0) - Y_{ig}(0,0)] + \sum_{s=1}^{n_g} \mathbb{E}[Y_{ig}(1,s) - Y_{ig}(1,0)]\mathbb{P}[S_{ig}=s|D_{ig}=1] \\ &- \sum_{s=1}^{n_g} \mathbb{E}[Y_{ig}(0,s) - Y_{ig}(0,0)]\mathbb{P}[S_{ig}=s|D_{ig}=0].\end{aligned}$$

By linearity,

$$\begin{aligned}\beta_{\mathsf{D}} &= \mathbb{E}[Y_{ig}(1,0) - Y_{ig}(0,0)] + \kappa_1 \sum_{s=1}^{n_g} s\mathbb{P}[S_{ig}=s|D_{ig}=1] \\ &- \kappa_0 \sum_{s=1}^{n_g} s\mathbb{P}[S_{ig}=s|D_{ig}=0] \\ &= \mathbb{E}[Y_{ig}(1,0) - Y_{ig}(0,0)] + \kappa_1\mathbb{E}[S_{ig}|D_{ig}=1] - \kappa_0\mathbb{E}[S_{ig}|D_{ig}=0].\end{aligned}$$

But

$$\begin{aligned}\beta_\ell &= \beta_{\mathsf{D}} - \frac{\gamma_\ell}{n_g}(\mathbb{E}[S_{ig}|D_{ig}=1] - \mathbb{E}[S_{ig}|D_{ig}=0]) \\ &= \mathbb{E}[Y_{ig}(1,0) - Y_{ig}(0,0)] + (\kappa_1-\kappa_0)\{(1-\lambda)\mathbb{E}[S_{ig}|D_{ig}=1] + \lambda\mathbb{E}[S_{ig}|D_{ig}=0]\}\end{aligned}$$

which gives the required result. □

Proof of Theorem 3 The coefficients from Equation (3) are characterized by the minimization problem:

$$\min_{\tilde{\alpha}_\ell, \tilde{\beta}_\ell, \gamma_\ell^0, \gamma_\ell^1} \mathbb{E}\left[\left(Y_{ig} - \tilde{\alpha}_\ell - \tilde{\beta}_\ell D_{ig} - \gamma_\ell^0 \bar{D}_g^{(i)}(1-D_{ig}) - \gamma_\ell^1 \bar{D}_g^{(i)} D_{ig}\right)^2\right]$$

The objective function can be rewritten as:

$$\mathbb{E}\left[\left(Y_{ig}-\tilde{\alpha}_\ell-\gamma_\ell^0\bar{D}_g^{(i)}\right)^2(1-D_{ig})\right]+\mathbb{E}\left[\left(Y_{ig}-\tilde{\alpha}_\ell-\tilde{\beta}_\ell-\gamma_\ell^1\bar{D}_g^{(i)}\right)^2 D_{ig}\right]$$

which can be reparameterized as:

$$\mathbb{E}\left[\left(Y_{ig}-\alpha_0-\gamma_\ell^0\bar{D}_g^{(i)}\right)^2(1-D_{ig})\right]+\mathbb{E}\left[\left(Y_{ig}-\alpha_1-\gamma_\ell^1\bar{D}_g^{(i)}\right)^2 D_{ig}\right]$$

where $\alpha_0=\tilde{\alpha}_\ell$ and $\alpha_1=\tilde{\alpha}_\ell+\tilde{\beta}_\ell$. The first-order conditions for α_0 and α_1 imply:

$$\begin{aligned}\alpha_0&=\mathbb{E}[Y_{ig}|D_{ig}=0]-\gamma_\ell^0\mathbb{E}[\bar{D}_g^{(i)}|D_{ig}=0]\\ \alpha_1&=\mathbb{E}[Y_{ig}|D_{ig}=1]-\gamma_\ell^1\mathbb{E}[\bar{D}_g^{(i)}|D_{ig}=1]\end{aligned}$$

and since $\tilde{\beta}_\ell=\alpha_1-\alpha_0$,

$$\begin{aligned}\tilde{\beta}_\ell&=\mathbb{E}[Y_{ig}|D_{ig}=1]-\mathbb{E}[Y_{ig}|D_{ig}=0]-\left(\gamma_\ell^1\mathbb{E}[\bar{D}_g^{(i)}|D_{ig}=1]-\gamma_\ell^0\mathbb{E}[\bar{D}_g^{(i)}|D_{ig}=0]\right)\\ &=\mathbb{E}[Y_{ig}|D_{ig}=1]-\mathbb{E}[Y_{ig}|D_{ig}=0]-\left(\frac{\gamma_\ell^1}{n_g}\mathbb{E}[S_{ig}|D_{ig}=1]-\frac{\gamma_\ell^0}{n_g}\mathbb{E}[S_{ig}|D_{ig}=0]\right).\end{aligned}$$

On the other hand, the first-order condition for each γ_ℓ^d implies:

$$\begin{aligned}0&=\mathbb{E}\left[\left(Y_{ig}-\alpha_d-\gamma_\ell^d\bar{D}_g^{(i)}\right)\bar{D}_g^{(i)}\mathbb{1}(D_{ig}=d)\right]\\ &=\mathbb{E}\left[\left(Y_{ig}-\mathbb{E}[Y_{ig}|D_{ig}=d]-\gamma_\ell^d(\bar{D}_g^{(i)}-\mathbb{E}[\bar{D}_g^{(i)}|D_{ig}=d])\right)\bar{D}_g^{(i)}\mathbb{1}(D_{ig}=d)\right]\end{aligned}$$

Thus:

$$\gamma_\ell^d=\frac{\mathbb{C}\mathrm{ov}(Y_{ig},\bar{D}_g^{(i)}|D_{ig}=d)}{\mathbb{V}[\bar{D}_g^{(i)}|D_{ig}=d]}$$

which gives the required result by calculations shown in the proof of Theorem 2. □

Proof of Corollary 2 When the true model satisfies exchangeability, potential outcomes have the form $Y_{ig}(d,s)$. From Theorem 3,

$$\begin{aligned}\gamma_\ell^d&=\sum_{s=1}^{n_g}\omega_d(s)(\mathbb{E}[Y_{ig}|D_{ig}=d,S_{ig}=s]-\mathbb{E}[Y_{ig}|D_{ig}=d,S_{ig}=0])\\ &=\sum_{s=1}^{n_g}\omega_d(s)\mathbb{E}[Y_{ig}(d,s)-Y_{ig}(d,0)].\end{aligned}$$

By linearity, $\mathbb{E}[Y_{ig}(d,s)-Y_{ig}(d,0)] = s\kappa_d$ and thus:

$$\begin{aligned}\gamma_\ell^d &= \frac{\kappa_d n_g}{\mathbb{V}[S_{ig}|D_{ig}=d]}\sum_{s=1}^{n_g} s(s-\mathbb{E}[S_{ig}|D_{ig}=d])\mathbb{P}[S_{ig}|D_{ig}=d] \\ &= \kappa_d n_g \\ &= \mathbb{E}[Y_{ig}(d,n_g)-Y_{ig}(d,0)].\end{aligned}$$

On the other hand,

$$\begin{aligned}\tilde{\beta}_\ell &= \beta_{\mathsf{D}} - \left(\frac{\gamma_\ell^1}{n_g}\mathbb{E}[S_{ig}|D_{ig}=1] - \frac{\gamma_\ell^0}{n_g}\mathbb{E}[S_{ig}|D_{ig}=0]\right) \\ &= \mathbb{E}[Y_{ig}(1,0)-Y_{ig}(0,0)] + \kappa_1\mathbb{E}[S_{ig}|D_{ig}=1] - \kappa_0\mathbb{E}[S_{ig}|D_{ig}=0] \\ &- \kappa_1\mathbb{E}[S_{ig}|D_{ig}=1] + \kappa_0\mathbb{E}[S_{ig}|D_{ig}=0] \\ &= \mathbb{E}[Y_{ig}(1,0)-Y_{ig}(0,0)]\end{aligned}$$

as required. □

Proof of Lemma 2 Take a constant $c \in \mathbb{R}$. Then

$$\mathbb{P}\left[\min_{\mathbf{a}\in\mathcal{A}_n} N(\mathbf{a}) \le c\right] \le |\mathcal{A}_n| \max_{\mathbf{a}\in\mathcal{A}_n} \mathbb{P}[N(\mathbf{a}) \le c].$$

Now, for any $\delta > 0$,

$$\begin{aligned}\mathbb{P}[N(\mathbf{a}) \le c] &= \mathbb{P}\left[N(\mathbf{a}) \le c, \left|\frac{\hat{\pi}(\mathbf{a})}{\pi(\mathbf{a})}-1\right| > \delta\right] + \mathbb{P}\left[N(\mathbf{a}) \le c, \left|\frac{\hat{\pi}(\mathbf{a})}{\pi(\mathbf{a})}-1\right| \le \delta\right] \\ &\le \mathbb{P}\left[\left|\frac{\hat{\pi}(\mathbf{a})}{\pi(\mathbf{a})}-1\right| > \delta\right] + \mathbb{P}[N(\mathbf{a}) \le c, G(n+1)\pi(\mathbf{a})(1-\delta) \le N(\mathbf{a}) \le \pi(\mathbf{a})G(n+1)(1+\delta)] \\ &\le \mathbb{P}\left[\left|\frac{\hat{\pi}(\mathbf{a})}{\pi(\mathbf{a})}-1\right| > \delta\right] + \mathbb{1}(G(n+1)\pi(\mathbf{a}) \le c/(1-\delta)) \\ &\le \mathbb{P}\left[\left|\frac{\hat{\pi}(\mathbf{a})}{\pi(\mathbf{a})}-1\right| > \delta\right] + \mathbb{1}(G(n+1)\underline{\pi}_n \le c/(1-\delta))\end{aligned}$$

which implies

$$|\mathcal{A}_n| \max_{\mathbf{a}\in\mathcal{A}_n} \mathbb{P}[N(\mathbf{a}) \le c] \le |\mathcal{A}_n| \max_{\mathbf{a}\in\mathcal{A}_n} \mathbb{P}\left[\left|\frac{\hat{\pi}(\mathbf{a})}{\pi(\mathbf{a})}-1\right| > \delta\right] + |\mathcal{A}_n|\mathbb{1}(G(n+1)\underline{\pi}_n \le c/(1-\delta))$$

which converges to zero under condition 5 and using Lemma A1. □

Proof of Theorem 4 All the estimators below are only defined when $\mathbb{1}(N(\mathbf{a}) > 0)$. Because under the conditions for Lemma 2 this event occurs with probability approaching one, the indicator will be omitted to simplify the notation. Let $\varepsilon_{ig}(\mathbf{a}) = Y_{ig} - \mathbb{E}[Y_{ig}|\mathbf{A}_{ig} = \mathbf{a}]$.

For the consistency part, we have that

$$\frac{\sum_g \sum_i \varepsilon_{ig}(\mathbf{a})\mathbb{1}_{ig(\mathbf{a})}}{N(\mathbf{a})} = \frac{\sum_g \sum_i (\varepsilon_{ig}(\mathbf{a})\mathbb{1}(|\varepsilon_{ig}| > \xi_n) - \mathbb{E}[\varepsilon_{ig}(\mathbf{a})\mathbb{1}(|\varepsilon_{ig}| > \xi_n)])\mathbb{1}_{ig(\mathbf{a})}}{N(\mathbf{a})} + \frac{\sum_g \sum_i (\varepsilon_{ig}(\mathbf{a})\mathbb{1}(|\varepsilon_{ig}| \le \xi_n) - \mathbb{E}[\varepsilon_{ig}(\mathbf{a})\mathbb{1}(|\varepsilon_{ig}| \le \xi_n)])\mathbb{1}_{ig(\mathbf{a})}}{N(\mathbf{a})}$$

for some increasing sequence of constants ξ_n whose rate will be determined along the proof. Let

$$\underline{\varepsilon}_{ig}(\mathbf{a}) = \varepsilon_{ig}(\mathbf{a})\mathbb{1}(|\varepsilon_{ig}(\mathbf{a})| \le \xi_n) - \mathbb{E}[\varepsilon_{ig}(\mathbf{a})\mathbb{1}(|\varepsilon_{ig}(\mathbf{a})| \le \xi_n)]$$

and

$$\bar{\varepsilon}_{ig}(\mathbf{a}) = \varepsilon_{ig}(\mathbf{a})\mathbb{1}(|\varepsilon_{ig}(\mathbf{a})| > \xi_n) - \mathbb{E}[\varepsilon_{ig}(\mathbf{a})\mathbb{1}(|\varepsilon_{ig}(\mathbf{a})| > \xi_n)]$$

For the first term,

$$\mathbb{P}\left[\max_{\mathbf{a}\in\mathcal{A}_n}\left|\frac{\sum_g \sum_i \underline{\varepsilon}_{ig}(\mathbf{a})\mathbb{1}_{ig}(\mathbf{a})}{N(\mathbf{a})}\right| > Mr_n\middle|\mathbf{A}\right] \le |\mathcal{A}_n| \max_{\mathbf{a}\in\mathcal{A}_n}\mathbb{P}\left[\left|\frac{\sum_g \sum_i \underline{\varepsilon}_{ig}(\mathbf{a})\mathbb{1}_{ig}(\mathbf{a})}{N(\mathbf{a})}\right| > Mr_n\middle|\mathbf{A}\right]$$

For the right-hand side, by Bernstein's inequality

$$\begin{aligned}\mathbb{P}\left[\left|\sum_g \sum_i \underline{\varepsilon}_{ig}(\mathbf{a})\mathbb{1}_{ig}(\mathbf{a})\right| > N(\mathbf{a})Mr_n\middle|\mathbf{A}\right] &\le 2\exp\left\{-\frac{1}{2}\frac{M^2r_n^2N(\mathbf{a})^2}{\sigma^2(\mathbf{a})N(\mathbf{a}) + 2\xi_n Mr_nN(\mathbf{a})/3}\right\} \\ &= 2\exp\left\{-\frac{1}{2}\frac{M^2r_n^2N(\mathbf{a})}{\sigma^2(\mathbf{a}) + 2M\xi_n r_n/3}\right\} \\ &\le 2\exp\left\{-\frac{1}{2}\frac{M^2r_n^2\min_{\mathbf{a}\in\mathcal{A}_n}N(\mathbf{a})}{\bar{\sigma}^2 + 2M\xi_n r_n/3}\right\}\end{aligned}$$

Set

$$r_n = \sqrt{\frac{\log|\mathcal{A}_n|}{G(n+1)\underline{\pi}_n}}$$

Next, use the fact that

$$\frac{\min_{\mathbf{a}\in\mathcal{A}_n}N(\mathbf{a})}{G(n+1)\underline{\pi}_n} \to_{\mathbb{P}} 1$$

which follows from Lemma A1, since

$$\mathbb{P}[\pi(\mathbf{a})(1-\varepsilon) \le \hat{\pi}(\mathbf{a}) \le \pi(\mathbf{a})(1+\varepsilon),\, \forall\, \mathbf{a}] \to 1$$

for any $\varepsilon > 0$ and

$$\begin{aligned}\mathbb{P}[\pi(\mathbf{a})(1-\varepsilon) \le \hat{\pi}(\mathbf{a}) \le \pi(\mathbf{a})(1+\varepsilon),\, \forall\, \mathbf{a}] &\le \mathbb{P}[\underline{\pi}_n(1-\varepsilon) \le \min_{\mathbf{a}}\hat{\pi}(\mathbf{a}) \le \underline{\pi}_n(1+\varepsilon)] \\ &= \mathbb{P}\left[\left|\frac{\min_{\mathbf{a}}\hat{\pi}(\mathbf{a})}{\underline{\pi}_n} - 1\right| \le \varepsilon\right].\end{aligned}$$

Then,

$$\mathbb{P}\left[\left|\sum_g\sum_i \varepsilon_{ig}(\mathbf{a})\mathbb{1}_{ig}(\mathbf{a})\right| > N(\mathbf{a})Mr_n\middle|\mathbf{A}\right] \leq 2\exp\left\{-\frac{1}{2}\frac{M^2\log|\mathcal{A}_n|(1+o_{\mathbb{P}}(1))}{\bar{\sigma}^2+2M\xi_n r_n/3}\right\}$$

and therefore

$$\mathbb{P}\left[\max_{\mathbf{a}\in\mathcal{A}_n}\left|\frac{\sum_g\sum_i \varepsilon_{ig}(\mathbf{a})\mathbb{1}_{ig}(\mathbf{a})}{N(\mathbf{a})}\right| > Mr_n\middle|\mathbf{A}\right] \leq$$

$$2\exp\left\{\log|\mathcal{A}_n|\left(1-\frac{1}{2}\frac{M^2(1+o_{\mathbb{P}}(1))}{\bar{\sigma}^2+2Mr_n\xi_n/3}\right)\right\}$$

which can be made arbitrarily small for sufficiently large M as long as $r_n\xi_n = O(1)$.

For the second term, by Markov's inequality

$$\begin{aligned}\mathbb{P}\left[\left|\sum_g\sum_i \bar{\varepsilon}_{ig}(\mathbf{a})\mathbb{1}_{ig}(\mathbf{a})\right| > N(\mathbf{a})Mr_n\middle|\mathbf{A}\right] &\leq \frac{\mathbb{E}[\varepsilon^2_{ig}(\mathbf{a})\mathbb{1}(|\varepsilon_{ig}(\mathbf{a})|>\xi_n)]N(\mathbf{a})}{M^2r_n^2N(\mathbf{a})^2}\\ &\leq \frac{b^{2+\delta}}{M^2\xi_n^\delta}\frac{1}{r_n^2\min_{\mathbf{a}\in\mathcal{A}_n}N(\mathbf{a})}\\ &= \frac{b^{2+\delta}}{M^2}\frac{1}{\xi_n^\delta\log|\mathcal{A}_n|(1+o_{\mathbb{P}}(1))}\end{aligned}$$

so that

$$\mathbb{P}\left[\max_{\mathbf{a}\in\mathcal{A}_n}\left|\sum_g\sum_i \bar{\varepsilon}_{ig}(\mathbf{a})\mathbb{1}_{ig}(\mathbf{a})\right| > N(\mathbf{a})Mr_n\middle|\mathbf{A}\right] \leq \frac{b^{2+\delta}}{M^2}\frac{|\mathcal{A}_n|}{\xi_n^\delta\log|\mathcal{A}_n|(1+o_{\mathbb{P}}(1))}$$

Finally, set $\xi_n = r_n^{-1}$. Then, the above term can be made arbitrarily small for M sufficiently large, as long as

$$\frac{|\mathcal{A}_n|}{\log|\mathcal{A}_n|}\left(\frac{\log|\mathcal{A}_n|}{G(n+1)\underline{\pi}_n}\right)^{\delta/2} = O(1)$$

Setting $\delta = 2$, this condition reduces to:

$$\frac{|\mathcal{A}_n|}{G(n+1)\underline{\pi}_n} = O(1)$$

Therefore,

$$\max_{\mathbf{a}\in\mathcal{A}_n}|\hat{\mu}(\mathbf{a})-\mu(\mathbf{a})| = O_{\mathbb{P}}\left(\sqrt{\frac{\log|\mathcal{A}_n|}{G(n+1)\underline{\pi}_n}}\right)$$

The proof for the standard error estimator uses the same reasoning after replacing $\varepsilon_{ig}(\mathbf{a})$ by $\hat{\varepsilon}^2_{ig}(\mathbf{a})$ and using consistency of $\hat{\mu}(\mathbf{a})$.

For the second part, we want to bound

$$\Delta = \max_{\mathbf{a}\in\mathcal{A}_\mathbf{n}}\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left[\frac{\hat{\mu}(\mathbf{a})-\mu(\mathbf{a})}{\sqrt{\mathbb{V}[\hat{\mu}(\mathbf{a})|\mathbf{A}]}}\leq x\right]-\Phi(x)\right|$$

$$\begin{aligned}
\Delta &= \max_{\mathbf{a}\in\mathcal{A}_\mathbf{n}} \sup_{x\in\mathbb{R}} \left| \mathbb{P}\left[\frac{\hat{\mu}(\mathbf{a}) - \mu(\mathbf{a})}{\sqrt{\mathbb{V}[\hat{\mu}(\mathbf{a})|\mathbf{A}]}} \leq x \right] - \Phi(x) \right| \\
&= \max_{\mathbf{a}\in\mathcal{A}_\mathbf{n}} \sup_{x\in\mathbb{R}} \left| \mathbb{E}\left\{ \mathbb{P}\left[\frac{\hat{\mu}(\mathbf{a}) - \mu(\mathbf{a})}{\sqrt{\mathbb{V}[\hat{\mu}(\mathbf{a})|\mathbf{A}]}} \leq x \middle| \mathbf{A} \right] - \Phi(x) \right\} \right| \\
&\leq \mathbb{E}\left\{ \max_{\mathbf{a}\in\mathcal{A}_\mathbf{n}} \sup_{x\in\mathbb{R}} \left| \mathbb{P}\left[\frac{\hat{\mu}(\mathbf{a}) - \mu(\mathbf{a})}{\sqrt{\mathbb{V}[\hat{\mu}(\mathbf{a})|\mathbf{A}]}} \leq x \middle| \mathbf{A} \right] - \Phi(x) \right| \right\}
\end{aligned}$$

Then,

$$\left| \mathbb{P}\left[\frac{\hat{\mu}(\mathbf{a}) - \mu(\mathbf{a})}{\sqrt{\mathbb{V}[\hat{\mu}(\mathbf{a})|\mathbf{A}]}} \leq x \middle| \mathbf{A} \right] - \Phi(x) \right| = \left| \mathbb{P}\left[\frac{\sum_g \sum_i \varepsilon_{ig} \mathbb{1}_{ig}(\mathbf{a})}{\sigma(\mathbf{a})\sqrt{N(\mathbf{a})}} \leq x \middle| \mathbf{A} \right] - \Phi(x) \right|$$

By the Berry-Esseen bound,

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left[\frac{\sum_g \sum_i \varepsilon_{ig} \mathbb{1}_{ig}(\mathbf{a})}{\sigma(\mathbf{a})\sqrt{N(\mathbf{a})}} \leq x \middle| \mathbf{A} \right] - \Phi(x) \right| \leq \frac{Cb^3}{\underline{\sigma}^3} \cdot \frac{1}{\sqrt{N(\mathbf{a})}}$$

But

$$\frac{1}{N(\mathbf{a})} = O_{\mathbb{P}}\left(\frac{1}{G(n+1)\pi(\mathbf{a})} \right)$$

Therefore,

$$\max_{\mathbf{a}\in\mathcal{A}_n} \sup_{x\in\mathbb{R}} \left| \mathbb{P}\left[\frac{\sum_g \sum_i \varepsilon_{ig} \mathbb{1}_{ig}(\mathbf{a})}{\sigma(\mathbf{a})\sqrt{N(\mathbf{a})}} \leq x \middle| \mathbf{A} \right] - \Phi(x) \right| \leq \frac{Cb^3}{\underline{\sigma}^3} \cdot O_{\mathbb{P}}\left(\frac{1}{\sqrt{G(n+1)\underline{\pi}_n}} \right)$$

and the result follows. □

Proof of Theorem 5 We want to bound:

$$\Delta^*(\mathbf{a}) = \sup_x \left| \mathbb{P}^*\left[\frac{\hat{\mu}^*(\mathbf{a}) - \hat{\mu}(\mathbf{a})}{\sqrt{\mathbb{V}^*[\hat{\mu}(\mathbf{a})]}} \leq x \right] - \Phi(x) \right|$$

uniformly over $\mathbf{a}$, where

$$\hat{\mu}^*(\mathbf{a}) = \frac{\sum_g \sum_i Y_{ig}^* \mathbb{1}_{ig}(\mathbf{a})}{N(\mathbf{a})}$$

if the denominator is non-zero, and zero otherwise, and where

$$Y_{ig}^* \mathbb{1}_{ig}(\mathbf{a}) = (\bar{Y}(\mathbf{a}) + (Y_{ig} - \bar{Y}(\mathbf{a}))w_{ig})\mathbb{1}_{ig}(\mathbf{a}) = (\bar{Y}(\mathbf{a}) + \hat{\varepsilon}_{ig} w_{ig})\mathbb{1}_{ig}(\mathbf{a})$$

Then, if $N(\mathbf{a}) > 0$,

$$\mathbb{E}^*[\hat{\mu}^*(\mathbf{a})] = \hat{\mu}(\mathbf{a})$$
$$\mathbb{V}^*[\hat{\mu}^*(\mathbf{a})] = \frac{\sum_g \sum_i \hat{\varepsilon}_{ig}^2 \mathbb{1}_{ig}(\mathbf{a})}{N(\mathbf{a})^2}$$

The centered and scaled statistic is given by:

$$\frac{\sum_g \sum_i \hat{\varepsilon}_{ig} \mathbb{1}_{ig}(\mathbf{a}) w_{ig}}{\sqrt{\sum_g \sum_i \hat{\varepsilon}_{ig}^2 \mathbb{1}_{ig}(\mathbf{a})}}$$

By Berry-Esseen,

$$\sup_x \left| \mathbb{P}^* \left[\frac{\sum_g \sum_i \hat{\varepsilon}_{ig} \mathbb{1}_{ig}(\mathbf{a}) w_{ig}}{\sqrt{\sum_g \sum_i \hat{\varepsilon}_{ig}^2 \mathbb{1}_{ig}(\mathbf{a})}} \leq x \right] - \Phi(x) \right| \leq C \frac{\sum_g \sum_i |\hat{\varepsilon}_{ig}|^3 \mathbb{1}_{ig}(\mathbf{a}) / N(\mathbf{a})}{\left(\sum_g \sum_i \hat{\varepsilon}_{ig}^2 \mathbb{1}_{ig}(\mathbf{a}) / N(\mathbf{a}) \right)^{3/2}} \cdot \frac{1}{\sqrt{N(\mathbf{a})}}$$

We also have that

$$\begin{aligned}
\frac{\sum_g \sum_i |\hat{\varepsilon}_{ig}|^3 \mathbb{1}_{ig}(\mathbf{a})}{N(\mathbf{a})} &\leq \frac{\sum_g \sum_i |Y_{ig} - \mu(\mathbf{a})|^3 \mathbb{1}_{ig}(\mathbf{a})}{N(\mathbf{a})} + |\bar{Y}(\mathbf{a}) - \mu(\mathbf{a})|^3 + O_{\mathbb{P}}(N(\mathbf{a})^{-2}) \\
&= \mathbb{E}[|Y_{ig} - \mu(\mathbf{a})|^3] + O_{\mathbb{P}}(N(\mathbf{a})^{-1})
\end{aligned}$$

and

$$\begin{aligned}
\frac{\sum_g \sum_i \hat{\varepsilon}_{ig}^2 \mathbb{1}_{ig}(\mathbf{a})}{N(\mathbf{a})} &= \frac{\sum_g \sum_i (Y_{ig} - \mu(\mathbf{a}))^2 \mathbb{1}_{ig}(\mathbf{a})}{N(\mathbf{a})} + (\bar{Y}(\mathbf{a}) - \mu(\mathbf{a}))^2 \\
&= \sigma^2(\mathbf{a}) + O_{\mathbb{P}}(N(\mathbf{a})^{-1})
\end{aligned}$$

Then,

$$\begin{aligned}
\Delta^*(\mathbf{a}) &\leq \sup_x \left| \mathbb{P}^* \left[\frac{\sum_g \sum_i \hat{\varepsilon}_{ig} \mathbb{1}_{ig}(\mathbf{a}) w_{ig}}{\sqrt{\sum_g \sum_i \hat{\varepsilon}_{ig}^2 \mathbb{1}_{ig}(\mathbf{a})}} \leq x \right] - \Phi(x) \right| \mathbb{1}(N(\mathbf{a}) > 0) + 2\mathbb{1}(N(\mathbf{a}) = 0) \\
&= C \frac{\mathbb{E}[|Y_{ig} - \mu(\mathbf{a})|^3] + O_{\mathbb{P}}(N(\mathbf{a})^{-1})}{[\sigma^2(\mathbf{a}) + O_{\mathbb{P}}(N(\mathbf{a})^{-1}))]^{3/2}} \cdot \frac{\mathbb{1}(N(\mathbf{a}) > 0)}{\sqrt{N(\mathbf{a})}} + 2\mathbb{1}(N(\mathbf{a}) = 0)
\end{aligned}$$

and the result follows from the facts that $\mathbb{P}[\min_{\mathbf{a}} N(\mathbf{a}) = 0] \to 0$ and by Lemma A1. $\square$

References

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