The Church-Rosser Property

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Objectives

You should be able to ...

- Describe the Church-Rosser property.
- Explain the advantages it confers when a language has that property.

Other Arrow Notations

Notations

$$
\begin{array}{lcl}
\rightarrow^0 & \equiv & \text{The identity} \\
\rightarrow^1 & \equiv & \rightarrow \\
\rightarrow^n & \equiv & \rightarrow \cdot \rightarrow^{n-1} \\
\rightarrow^* & \equiv & \bigcup_{i=0}^{\infty} \rightarrow^i \\
\rightarrow^+ & \equiv & \bigcup_{i=1}^{\infty} \rightarrow^i \\
a \leftarrow b & \equiv & b \rightarrow a \\
\leftrightarrow & \equiv & \rightarrow \cup \leftarrow \\
\leftrightarrow^* & \equiv & (\rightarrow \cup \leftarrow)^*
\end{array}
$$

Example

$3 \rightarrow^* 3$, and `if 3 > 2 then 5 + 9 else 2 * 4` $\rightarrow^*$ `14`

Be Careful with $\leftrightarrow^*$

$$a \leftrightarrow^* b \not\equiv a \leftarrow^* b \cup a \rightarrow^* b$$

For example $a \leftrightarrow^* b$ when

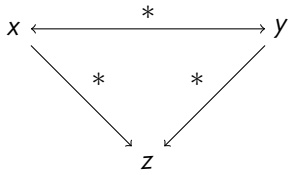
$$a \leftarrow a_1 \rightarrow a_2 \rightarrow a_3 \leftarrow b_2 \leftarrow b_1 \rightarrow b$$

Term Rewriting Systems

Transition semantics can be thought of as a *term rewriting system*. Common questions:

- Does an expression always terminate?
- Can we tell if two expressions are equal?

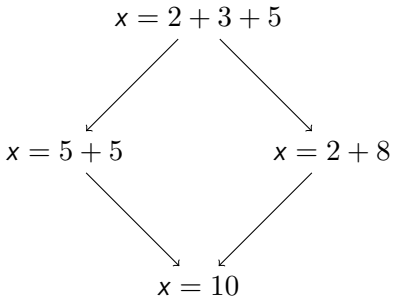
Church-Rosser property: if $x \leftrightarrow^* y$ then x and y normalize to the same value.



Example

Confluence

If $x \to y_1$ and $x \to y_2$ then y_1 and y_2 normalize to the same value. (Confluence and the Church-Rosser property coincide.)



This is also known as the "diamond property."

Who Has It?

- Alonzo Church and J. Barkley Rosser proved that the λ-calculus has these properties in 1936.
- Very important for theorem provers
- Most programming languages have this property (some of the time).
- One Benefit: you can check for equality of x and y by evaluating them.