Induction and Recursion

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Objectives

- Identify the parts of a proof by induction and their corresponding parts in a recursive function.
- Identify the requirements for a recursive function to terminate with a correct answer.

Induction

A proof by induction works by making two steps do the work of an infinite number of steps. It's really a way of being very lazy!

- Pick a property $P(n)$ which you'd like to prove for all n.
- **Base case:** Prove $P(n)$, for $n = 1$, or whatever n's smallest value should be.
- **Induction case:** You want to prove $P(n)$, for all n. To do that, *assume* that $P(n-1)$ is true, and use that information to prove that $P(n)$ has to be true.

The idea is that there are an infinite number of n such that $P(n)$ is true. But with this technique you only had to prove two cases.

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$$(1 + 3 + 5 + \cdots + 2n - 3) = (n-1)^2$$

So add $2n - 1$ to both sides ...

$$\Rightarrow (1 + 3 + 5 + \cdots + 2n - 3 + 2n - 1) = (n-1)^2 + 2n - 1$$

$$\Rightarrow n^2 - 2n + 1 + 2n - 1$$

$$\Rightarrow n^2$$

Recursion

A recursive routine has a similar structure. You have a base case, a recursive case, and a conditional to check which case is appropriate.

- Pick a function $f(n)$ which you'd like to compute for all n.
- **Base case:** Compute $f(n)$, for $n = 1$, or whatever n's smallest value should be.
- **Recursive case:** Assume that someone else already computed $f(n-1)$ for you. Use that information to compute $f(n)$, and then take all the credit.

Iterating Recursion Example

Suppose you want a recursive routine that computes the *n*th square.

```
nthsq 1 = 1
nthsq n = 2*n-1 + nthsq (n-1)
```

- The pattern matching checks which case is appropriate.
- Line 1 is the base case – it stops the recursion.
- Line 2 is the recursive case.

Important Things about Recursion

```
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```

- Your base case has to stop the computation.
- Your recursive case has to call the function with a *smaller* argument than the original call.
- Your `if` statement has to be able to tell when the base case is reached.
- Failure to do any of the above will cause an infinite loop.

History

(Discovered on Wikipedia)

- The proof that that the first n odd numbers sums to n^2 first appeared in *Arithmeticorum libri duo* by Francesco Maurolico in 1575.
- Wikipedia says it's the earliest known *explicit* use of proof by induction.
- Implicit uses of proof by induction can be found in the writings of Plato and Euclid in the 300's BCE.