Lambda Calculus Examples

Dr. Mattox Beckman

University of Illinois at Urbana-Champaign
Department of Computer Science

Objectives

You should be able to ...

Here are some examples!

- Perform a beta-reduction.
- Detect α-capture and use α-renaming to avoid it.
- Normalize any given λ-calculus term.

Examples

$$(\lambda x.x)\ a$$
$$(\lambda x.x\ x)\ a$$
$$(\lambda x.y\ x)\ a$$
$$(\lambda x.\lambda a.x)\ a$$
$$(\lambda x.\lambda x.x)\ a$$
$$(\lambda x.(\lambda y.y)\ x)\ a$$

Examples

$$
\begin{array}{lcl}
(\lambda x.x)\ a & \to_\beta & a \\
(\lambda x.x\ x)\ a & & \\
(\lambda x.y\ x)\ a & & \\
(\lambda x.\lambda a.x)\ a & & \\
(\lambda x.\lambda x.x)\ a & & \\
(\lambda x.(\lambda y.y)\ x)\ a & &
\end{array}
$$

$$
\begin{tikzcd}
 & @ \arrow[dl] \arrow[dr] & & \to_\beta & a \\
\lambda x \arrow[d] & & a & & \\
x & & & &
\end{tikzcd}
$$

Examples

$$
\begin{array}{lcl}
(\lambda x.x)\ a & \rightarrow_\beta & a \\
(\lambda x.x\ x)\ a & \rightarrow_\beta & a\ a \\
(\lambda x.y\ x)\ a & & \\
(\lambda x.\lambda a.x)\ a & & \\
(\lambda x.\lambda x.x)\ a & & \\
(\lambda x.(\lambda y.y)\ x)\ a & &
\end{array}
$$

$$
\begin{array}{ccccccccccccc}
 & & & & @ & & & \rightarrow_\beta & & & @ & & \\
 & & & \swarrow & & \searrow & & & & \swarrow & & \searrow & \\
 & & \lambda x & & & & a & & a & & & & a \\
 & & \downarrow & & & & & & & & & & \\
 & & @ & & & & & & & & & & \\
 & \swarrow & & \searrow & & & & & & & & & \\
x & & & & x & & & & & & & &
\end{array}
$$

Examples

$$
\begin{array}{lll}
(\lambda x.x)\ a & \rightarrow_\beta & a \\
(\lambda x.x\ x)\ a & \rightarrow_\beta & a\ a \\
(\lambda x.y\ x)\ a & \rightarrow_\beta & y\ a \\
(\lambda x.\lambda a.x)\ a & & \\
(\lambda x.\lambda x.x)\ a & & \\
(\lambda x.(\lambda y.y)\ x)\ a & &
\end{array}
$$

$$
\begin{tikzcd}
 & & @ \arrow[dl] \arrow[dr] & & \rightarrow_\beta & & @ \arrow[dl] \arrow[dr] & \\
 & \lambda x \arrow[d] & & a & & y & & a \\
 & @ \arrow[dl] \arrow[dr] & & & & & & \\
y & & x & & & & &
\end{tikzcd}
$$

Examples

$$
\begin{array}{llllll}
(\lambda x.x)\ a & \to_\beta & a & & \\
(\lambda x.x\ x)\ a & \to_\beta & a\ a & & \\
(\lambda x.y\ x)\ a & \to_\beta & y\ a & & \\
(\lambda x.\lambda a.x)\ a & \to_\alpha & (\lambda x.\lambda a'.x) & \to_\beta & \lambda a'.a \\
(\lambda x.\lambda x.x)\ a & & & & \\
(\lambda x.(\lambda y.y)\ x)\ a & & & &
\end{array}
$$

$$
\begin{tikzcd}
 & @ \arrow[dl] \arrow[dr] & & \not\to & \lambda a \arrow[d] & \to_{\alpha,\beta} & \lambda a' \arrow[d] \\
\lambda x \arrow[d] & & a & & a & & a \\
\lambda a \arrow[d] & & & & & & \\
x & & & & & &
\end{tikzcd}
$$

Examples

$$
\begin{array}{lllll}
(\lambda x.x)\ a & \rightarrow_\beta & a & & \\
(\lambda x.x\ x)\ a & \rightarrow_\beta & a\ a & & \\
(\lambda x.y\ x)\ a & \rightarrow_\beta & y\ a & & \\
(\lambda x.\lambda a.x)\ a & \rightarrow_\alpha & (\lambda x.\lambda a'.x) & \rightarrow_\beta & \lambda a'.a \\
(\lambda x.\lambda x.x)\ a & \rightarrow_\beta & \lambda x.x & & \\
(\lambda x.(\lambda y.y)\ x)\ a & & & &
\end{array}
$$

$$
\begin{array}{ccccc}
 & @ & & \rightarrow_\beta & \lambda x \\
\swarrow & & \searrow & & \downarrow \\
\lambda x & & a & & x \\
\downarrow & & & & \\
\lambda x & & & & \\
\downarrow & & & & \\
x & & & &
\end{array}
$$

Examples

$$
\begin{array}{llllll}
(\lambda x.x)\ a & \rightarrow_\beta & a & & & \\
(\lambda x.x\ x)\ a & \rightarrow_\beta & a\ a & & & \\
(\lambda x.y\ x)\ a & \rightarrow_\beta & y\ a & & & \\
(\lambda x.\lambda a.x)\ a & \rightarrow_\alpha & (\lambda x.\lambda a'.x) & \rightarrow_\beta & \lambda a'.a \\
(\lambda x.\lambda x.x)\ a & \rightarrow_\beta & \lambda x.x & & & \\
(\lambda x.(\lambda y.y)\ x)\ a & \rightarrow_\beta & (\lambda y.y)\ a & & &
\end{array}
$$

$$
\begin{tikzcd}
 & & @ \arrow[dl] \arrow[dr] \arrow[rrr, phantom, "\rightarrow_\beta"] & & & @ \arrow[dl] \arrow[dr] \arrow[rr, phantom, "\rightarrow_\beta"] & & a \\
 & \lambda x \arrow[d] & & a & \lambda y \arrow[d] & & a & \\
 & @ \arrow[dl] \arrow[dr] & & & y & & & \\
 \lambda y \arrow[d] & & x & & & & & \\
 y & & & & & & &
\end{tikzcd}
$$

Examples

$$
\begin{array}{llllll}
(\lambda x.x)\ a & \to_\beta & a & & & \\
(\lambda x.x\ x)\ a & \to_\beta & a\ a & & & \\
(\lambda x.y\ x)\ a & \to_\beta & y\ a & & & \\
(\lambda x.\lambda a.x)\ a & \to_\alpha & (\lambda x.\lambda a'.x) & \to_\beta & \lambda a'.a & \\
(\lambda x.\lambda x.x)\ a & \to_\beta & \lambda x.x & & & \\
(\lambda x.(\lambda y.y)\ x)\ a & \to_\beta & (\lambda y.y)\ a \to_\beta\ a & & &
\end{array}
$$

$$
\begin{tikzcd}
 & & @ \arrow[dl] \arrow[dr] & \to_\beta & & @ \arrow[dl] \arrow[dr] & \to_\beta & a \\
 & \lambda x \arrow[d] & & a & \lambda y \arrow[d] & & a & \\
 & @ \arrow[dl] \arrow[dr] & & & y & & & \\
\lambda y \arrow[d] & & x & & & & & \\
y & & & & & & &
\end{tikzcd}
$$

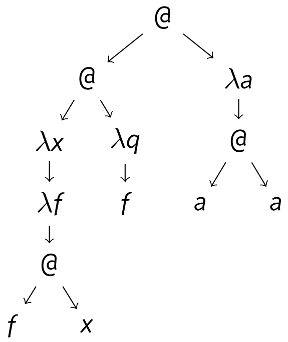
α capture

$$(\lambda x.\lambda a.x)\, a \to_\alpha \quad (\lambda x.\lambda a'.x) \quad \to_\beta \quad \lambda a'.a$$

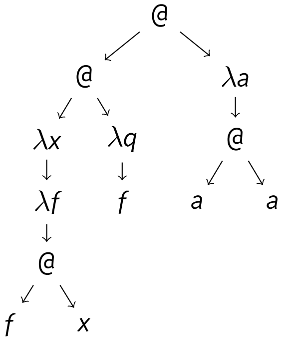
- If a free occurrence of a variable gets placed under a λ that binds it, this is called α capture.
- To resolve this, rename the *binder*.

Here's One for You to Try!

- Convert this tree into an equivalent λ term.
- Identify the free variables.
- Simplify it by performing as many β reductions (and necessary α renamings) as possible.



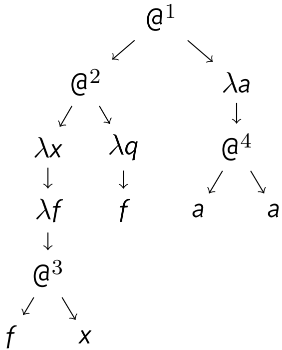
Solution

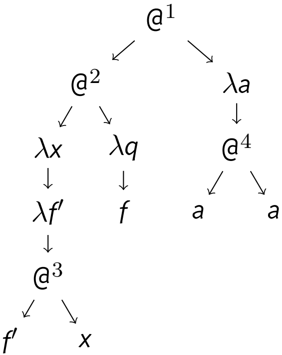


$$(\lambda xf.fx)(\lambda q.f)(\lambda a.aa)$$

- There is one free variable ....

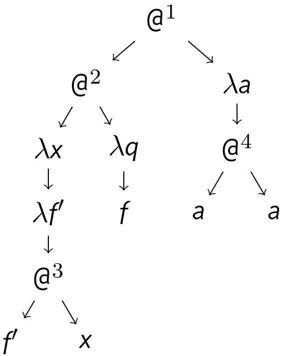
Solution, Step 1





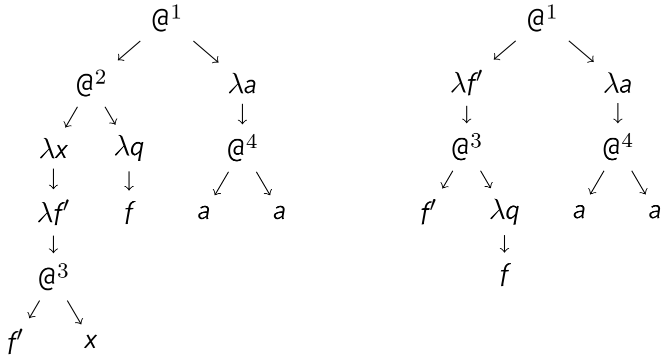
$$(\lambda xf.fx)(\lambda q.f)(\lambda a.aa) \quad \to_\alpha \quad (\lambda xf'.f'x)(\lambda q.f)(\lambda a.aa)$$

Solution, Step 2



$$(\lambda x f'.f'x)(\lambda q.f)(\lambda a.aa) \qquad \rightarrow_\beta$$

Solution, Step 2



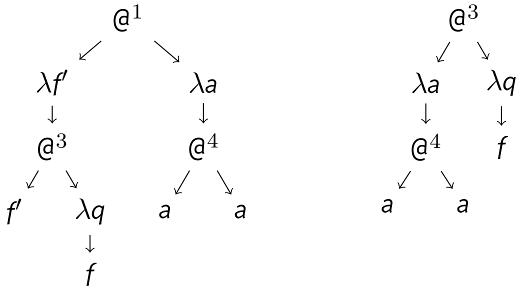
$$(\lambda x f'.f'x)(\lambda q.f)(\lambda a.aa) \quad \to_\beta \quad (\lambda f'.f'(\lambda q.f))(\lambda a.aa)$$

Solution, Step 3

$$
\begin{tikzcd}
 & & & @^1 \arrow[dll] \arrow[dr] & & \\
 & \lambda f' \arrow[d] & & & \lambda a \arrow[d] & \\
 & @^3 \arrow[dl] \arrow[dr] & & & @^4 \arrow[dl] \arrow[dr] & \\
f' & & \lambda q \arrow[d] & a & & a \\
 & & f & & &
\end{tikzcd}
$$

$$(\lambda f'.f'(\lambda q.f))(\lambda a.aa) \qquad \to_\beta$$

Solution, Step 3



$$(\lambda f'.f'(\lambda q.f))(\lambda a.aa) \quad \rightarrow_\beta \quad (\lambda a.aa)(\lambda q.f)$$

Solution, Step 4

$$\begin{array}{ccccccc}
 & & & & @^3 & & \\
 & & & \swarrow & & \searrow & \\
 & & \lambda a & & & & \lambda q \\
 & & \downarrow & & & & \downarrow \\
 & & @^4 & & & & f \\
 & \swarrow & & \searrow & & & \\
a & & & & a & &
\end{array}$$

$$(\lambda f'.f'(\lambda q.f))(\lambda a.aa) \quad \rightarrow_\beta$$

Solution, Step 4

$$
\begin{array}{ccccc}
 & & @^3 & & \\
 & \swarrow & & \searrow & \\
 & \lambda a & & \lambda q & \\
 & \downarrow & & \downarrow & \\
 & @^4 & & f & \\
\swarrow & & \searrow & & \\
a & & a & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & @^4 & \\
\swarrow & & \searrow \\
\lambda q & & \lambda q \\
\downarrow & & \downarrow \\
f & & f
\end{array}
$$

$$
(\lambda f'.f'(\lambda q.f))(\lambda a.aa) \quad \rightarrow_\beta \quad (\lambda q.f)(\lambda q.f)
$$

Solution, Step 5

$$
\begin{array}{ccc}
 & @^{4} & \\
\swarrow & & \searrow \\
\lambda q & & \lambda q \\
\downarrow & & \downarrow \\
f & & f
\end{array}
$$

$$(\lambda q.f)(\lambda q.f) \quad \rightarrow_\beta$$

Solution, Step 5

$$
\begin{array}{ccccc}
 & @^4 & & & f \\
\swarrow & & \searrow & & \\
\lambda q & & \lambda q & & \\
\downarrow & & \downarrow & & \\
f & & f & & \\
\end{array}
$$

$$(\lambda q.f)(\lambda q.f) \quad \to_\beta \quad f$$