Maximally Entangled Qubits and some Applications

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Abstract

Entanglement is a bizarre property of the physical world, according to which certain entities affect each other at a distance, with no communication. It was first described in the famous Einstein-Podolsky-Rosen (EPR) paper [7], where it was presented almost as a paradox, to the point Einstein himself described entanglement as "Spooky action at a distance"[1]. Since then, many experiments have been conducted, and the general consensus it that entanglement is real, to the point that in recent times quantum computers have been implemented that exploit entanglement to perform computations. In this report we present entanglement, focusing on its strongest form observable in a quantum computer, maximal entanglement. We present physical implications and showcase Quantum Teleportation and Superdense Coding and provide with an example of implementation with IBM's Q Experience platform.

1 Overview and Applications – Task 1

1.1 Quantum Computing Representation

Using the notion of a qubit it is possible to express physical states that exhibit entanglement. From the qubit register representation of multiple qubits, we show how measurement probabilities imply entangled behaviour. Let us begin by showing some examples of entangled qubits in the typical notation of Quantum Computing. The first state we wish to consider is,

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\big(\,|00\rangle + |11\rangle\,\big). \tag{1}$$

This is known as the *first Bell state*, or sometimes just as the *Bell state*. We will keep referring it, because it very markedly displays entanglement and because of it's applications. A pair of qubits in this state is sometimes referred to as a *Bell pair* or an *EPR pair*. Applying Born's rule we get that the probabilities of measuring 00 and 11, from a system of two qubits in state $|\Psi^+\rangle$, are both equal to $\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$. The remaining 2 outcomes, 01 and 10. Thus consider what happens if we measure the qubits sequentially. The measurement of the first has equal probability of being 1 or 0 while,

[1]J.S. Bell further expands on the EPR views on entanglement in p.143 of [3].

- if the first one reads 0, we immediately know that the second will read 0, because we know that we are in state $|00\rangle$;
- if the first one reads 1, we immediately know that the second will read 1, because we know that we are in state $|11\rangle$;

The above argument shows that the second qubit measured has to "know" about what happened to the first measured one. This is a manifestation of entanglement. In the Bell state example the impact on measurement was categorical, the first measurement uniquely determined the second. Such is the defining property of *maximal entanglement*. But entanglement doesn't only manifest in such form, for example applying Born's rule shows a state such as the following also entangled but not maximally[2],

$$|\chi\rangle = \frac{1}{\sqrt{6}}\big(\sqrt{2}\,|00\rangle + |01\rangle + |10\rangle + \sqrt{2}\,|11\rangle\,\big). \tag{2}$$

1.2 Physical Implications

In the previous section we saw how measuring a qubit, can tells us something about another qubit when entangled with the first. This interaction is instantaneous. For example, given a pair a qubits in the Bell state ($|\Psi^+\rangle$), if the qubits are separated in space (preserving coherence) and measured at times very close together, they still present entangled behaviour. This is true even if the time between measurements is less that that it would take light to travel from the first measured to the second. If the qubits were communicating with some kind of emission, it would be in violation of a fundamental principle of Special Relativity, which states that nothing travels faster than the speed of light in a vacuum. Because of this, when entanglement was first described in the EPR paper, it was proposed that the model of quantum mechanics had to be incomplete and that there had to be some *local hidden variables*, details of the system not encompassed by the theory, which provided the additional information quantum mechanical entities needed to make their measurements follow entangled patterns. However Bell's Theorem [2; 1] shows that EPR local hidden variables are mathematically inconsistent. It therefore appears that entangled qubits violate the principle of locality, which states that an object is influenced only by its immediate surroundings, since qubits as far a part in space as we like, can instantly give information about each other [6].

From here, it is reasonable to wonder whether this violation of locality enables faster than light sharing of information, *super-luminal communication*. Let us consider an example to reason about why super-luminal communication through entangled states is not possible. Alice creates some pairs of maximally entangled qubits, say in the Bell state, and shares half of each pair with Bob. When Alice measures one of her qubits she immediately knows what Bob will measure from its paired qubit, even if Alice and Bob have moved very far from each other. Though Alice instantly "determines" what Bob will read from the qubits paired with ones she measured, Bob has no way of distinguishing between states that had the other side of the pair measured by Alice and states that hadn't. Under the assumptions of quantum mechanics, this argument is formalized in the No-communication theorem, to qubits states of any degree of entanglement and coherence.

[2]Presented for illustrative purposes, obtaining such a state in a quantum computer may be involved.

Precisely the No-communication theorem proves that quantum mechanical measurement in no way allows super-luminal communication [8].

1.3 Communication with maximally entangled pairs qubits

Even though sharing entangled pairs of qubits might not give us faster than light communication, it does give us some ways of communication. In this section we look at ways of transferring information that uses the impact of the "spooky action at a distance" characteristic of entanglement. We look at a way of transferring a quantum state, known as quantum teleportation, and a way of transferring classical bits known as super-dense coding.

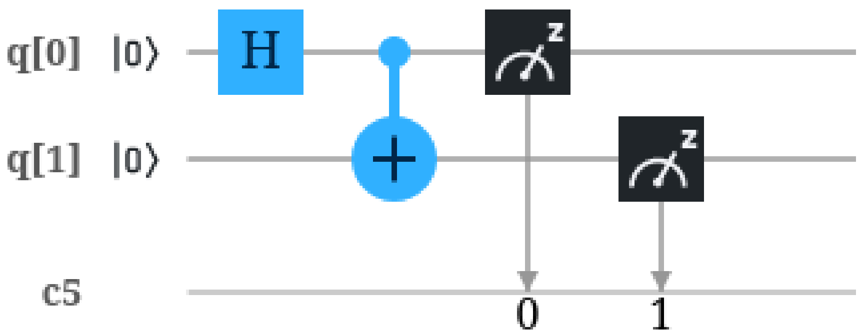
Imagine that, similarly to the previous example, Alice and Bob have shared a pair of qubits in the Bell state. Alice somehow obtains a new qubit in some state $|\psi\rangle$, and wants to send it to Bob. Thanks to entanglement, she can do so without sending any qubits across, with quantum teleportation [4]. All she has to do is apply certain quantum gates to the qubit in state $|\psi\rangle$ and her qubit of the entangled pair and a measurement, then send two bits of classical information. Bob will then be able to recover $|\psi\rangle$ from his entangled qubit. Implementation details are discussed in Section 2.2. With Quantum Teleportation something sensibly different from anything classical is happening. One might initially be skeptical of its applications, because we still need to share qubits (ahead of time, moreover) and send classical information. Nevertheless ideas for applications of Quantum teleportation have been explored. For example, quantum teleportation could constitute a secure form of communication, because the shared classical information carries no meaning without the entangled pair, hence only who is in possession of one of the entangled qubits can read the quantum state being transferred. Protocols for moving qubits have been proposed and demonstrated on physical hardware [11] and could be necessary for building large scale modular quantum computers. Quantum networks based on quantum teleportation have even been discussed [13; 15]

The next form of communication we examine is Superdense Coding, which is a secure method of transmitting bits classical information over a quantum channel (i.e. by sending physical qubits) [5]. A single qubit is sent across to transfer two classical bits, hence the "Superdense" in the name. In this scheme as well, we start with a Bell pair shared between Alice and Bob. Depending on the choice of classical bits to be transmitted, Alice conditionally applies an X gate and/or a Z gate to her side of the Bell pair. Then sends her qubit to Bob, over a potentially insecure quantum channel. Bob can recover the two bits of information by applying two more quantum gates and measuring. The strength of Superdense Coding is in the fact that the single qubit sent by Alice to Bob is not sufficient to recover the classical bits that were encoded in it. Therefore an eavesdropper cannot recover those initial two bits of information. Although as the original authors themselves [5] note, this communication scheme is vulnerable to manipulation of the channel, verification techniques are required to mitigate the issue. By using more entangled pairs, more advanced secure protocols based on superdense coding were developed [14; 10]. Superdense coding techniques have even been extended to transfer quantum states themselves [9].

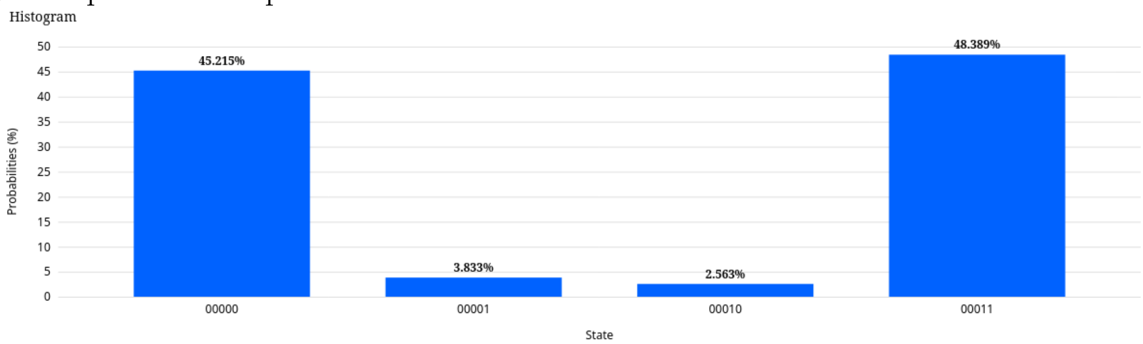
2 Implementation – Task 2

2.1 Entangling qubits

In this second part of the report we demonstrate quantum teleportation with the IBM Q Experience platform. We build an experiment step by step based on the schematic given in [12]. First, let us consider the Quantum Circuit that we will use to produce a Bell Pair, the heart of our communication techniques.



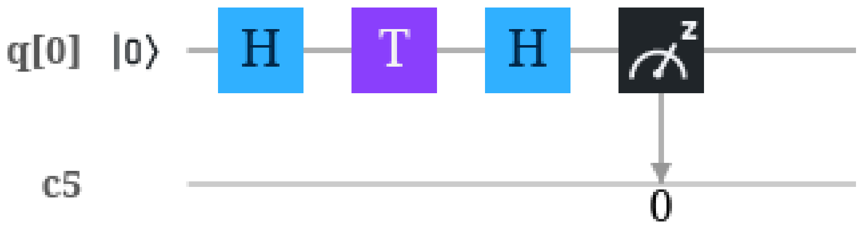
Note that the last two operations are measurements and do not contribute to the creation the pair. We convince ourselves that the circuit is producing Bell pairs by showing the histogram of measurement outcomes obtained by running the circuit 4096 times, on IBM's physical quantum computer in Essex.

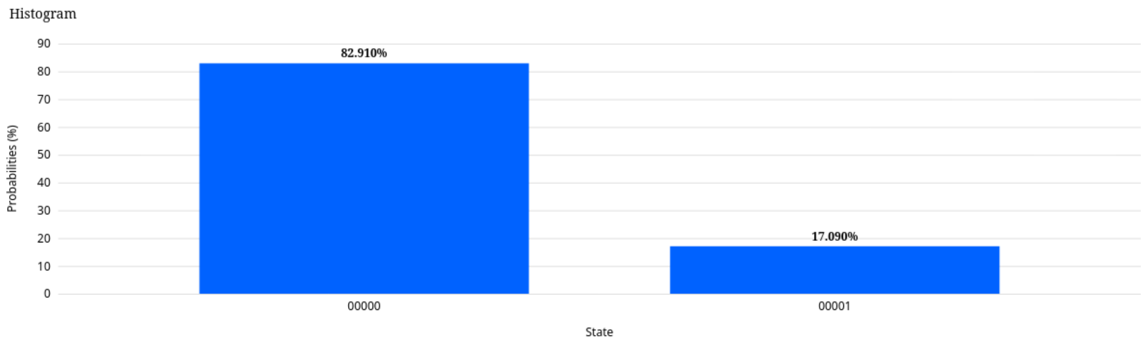


It presents the expected distribution, dominated, with nearly equal probability by the outcomes 00 and 11, as derived in Section 1.1.

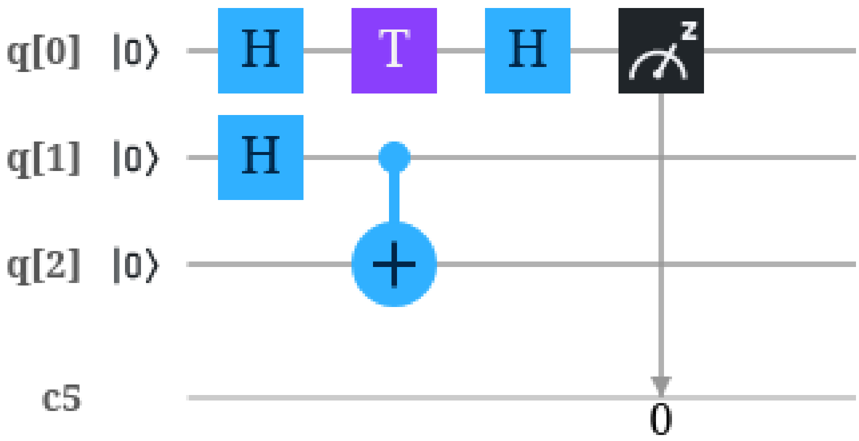
2.2 Quantum Teleportation

The purpose of Quantum teleportation is moving a quantum state. In this section we teleport a state from one qubit to another, inside a physical quantum computer. To guide us in implementing Quantum Teleportation we will pretend that Alice wants to send a quantum state to Bob. To make the experiment interesting, our first step is obtain a qubit in a non trivial state. The following circuit, as demonstrated by its measurement histogram of 4096 executions on IBM's machine in Essex, is not trivial.

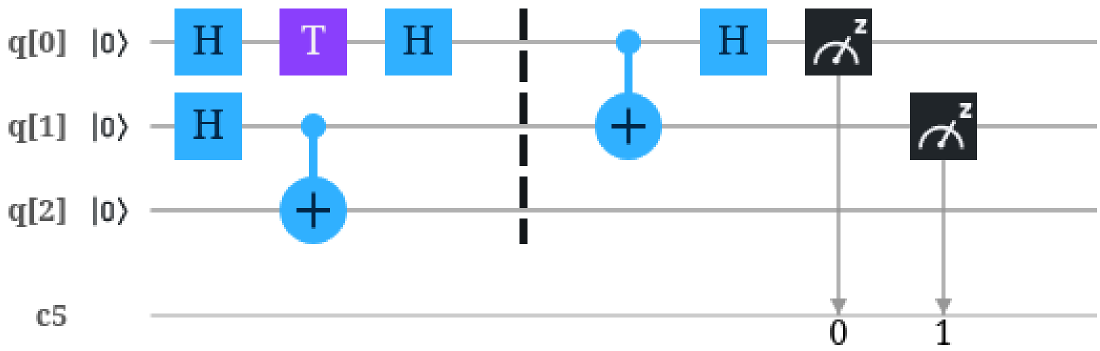




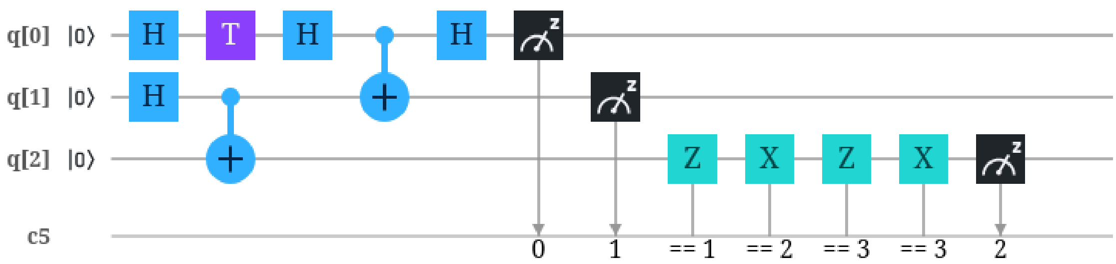
We will assume that this state, in $q[0]$, is the one Alice wants to send, calling it $|\psi\rangle$. The objective is to teleport $|\psi\rangle$ to Bob. We add two more qubits to the circuit, $q[1]$ and $q[2]$ which will be, respectively, Alice's and Bob's parts of a Bell pair.



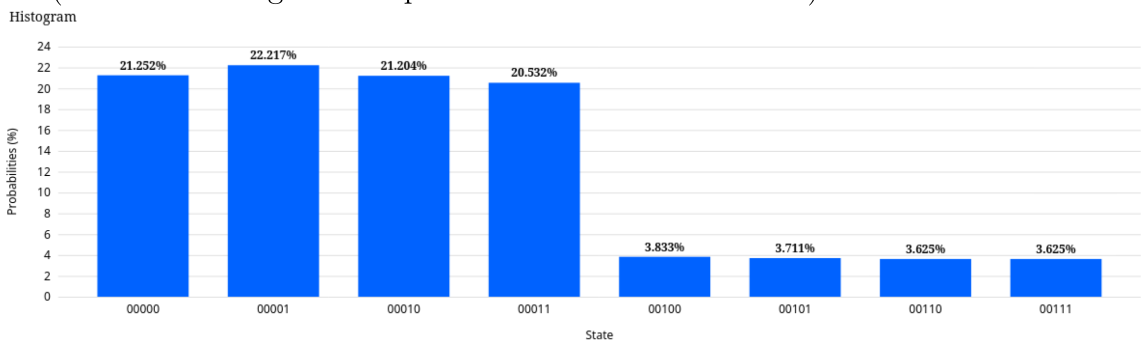
Alice performs a CNOT and a Hadamard and measures her qubits, as specified by the protocol. The measurement readouts are stored in the classical register drawn as c5 in the circuit. In this example c5 served as the classical channel in which the two bits are shared.



On Bob's side, following with the protocol, we use Alice's readouts from the classical register to determine which gates to apply. The conditional application of gates is represented by the equality conditions at the bottom of the figure. It must be noted that the integers correspond to the decimal representation of the readout sequence of the measurements. Thus the application of gates is in accordance with the table given in [12].



The circuit is complete, so now can run it and analyze the results. At the time time of writing none of IBM's free physical computers that could run the circuit were available, so we give the results of a simulator execution. The readout sequence reads from right to left (i.e. the most significant qubit is the one measured last).



These results match our expectation: the ratio between the number of outcomes with a 0 read from $q[2]$ and a 1 resembles the ratio obtained when measuring $|\psi\rangle$ directly (as we did at the beginning, when we only had $q[0]$ in state $|\psi\rangle$). This execution suggests that the circuit can perform quantum teleportation. We make a final note by observing how the relative readout distributions of $q[0]$ and $q[1]$ are very similar for fixed values read from $q[2]$. This reflects the fact that the readouts of $q[0]$ and $q[1]$, which is the information sent over the classical channel, tells nothing about $|\psi\rangle$.

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