Topics in Algebraic Geometry (Mutiple View Geometry, 2020-1)
Homework 1

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April 5, 2020

Notation 0.1. When we write $H = A$ for a homography H and a matrix A, this means that A is contained in the equivalence class H of invertible matrices.

Therefore, $H = A$, $H = B$ doesn't imply $A = B$ but only that $A = \lambda B$ for nonzero scalar λ.

Notation 0.2. We will use extensively the *Einstein conventions* from physics: when an index appears two times in an expression, once as a subscript and once as a supercript, then the summation with respect to that index is omitted. For example,

$$x^i y_i = x^1 y_1 + x^2 y_2 + \cdots x^n y_n.$$

1 Problems

Problem 1. Show that a homography H preserves the dual absolute conic C^*_∞ if and only if it is a similarity.

Solution.
We have seen in the class that in the standard coordinates (together with the dual coordinates for the dual space)

$$C^*_\infty = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Also, the transformation rule for the dual conics were given by

$$H(C^*_\infty) = HC_\infty H^t.$$

Therefore, if we write

$$H = \begin{pmatrix} A & v \\ w & a \end{pmatrix}$$

then

$$HC^*_\infty H^t = \begin{pmatrix} A & v \\ w & a \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A^t & v^t \\ w^t & a \end{pmatrix} = \begin{pmatrix} AA^t & Aw^t \\ wA^t & a \end{pmatrix}.$$

This represents C^*_∞ iff $ww^t = 0, Aw^t = 0$ and AA^t the multiple of I_2. Since w is a row vector, $ww^t = 0$ implies $w = 0$. For AA^t to be multiple of the identity, one sees that after scaling that A is a scalar times an orthogonal matrix.

Since this conditions are also succifient to obtain $H(C^*_\infty) = C^*_\infty$, one sees that H preserves C^*_∞ if and only if $w = 0$, $A = sA'$ where $s \neq 0, A' \in O(2)$.

In this situations, since H is invertible, we can re-scale so we set $a = 1$. The conclusion then is that H preserves C^*_∞ if and only if

$$H = \begin{pmatrix} sA' & v \\ 0 & 1 \end{pmatrix}, A' \in O(2), s \in \mathbb{R} \setminus \{0\}.$$

The right-hand side represents a similarity.

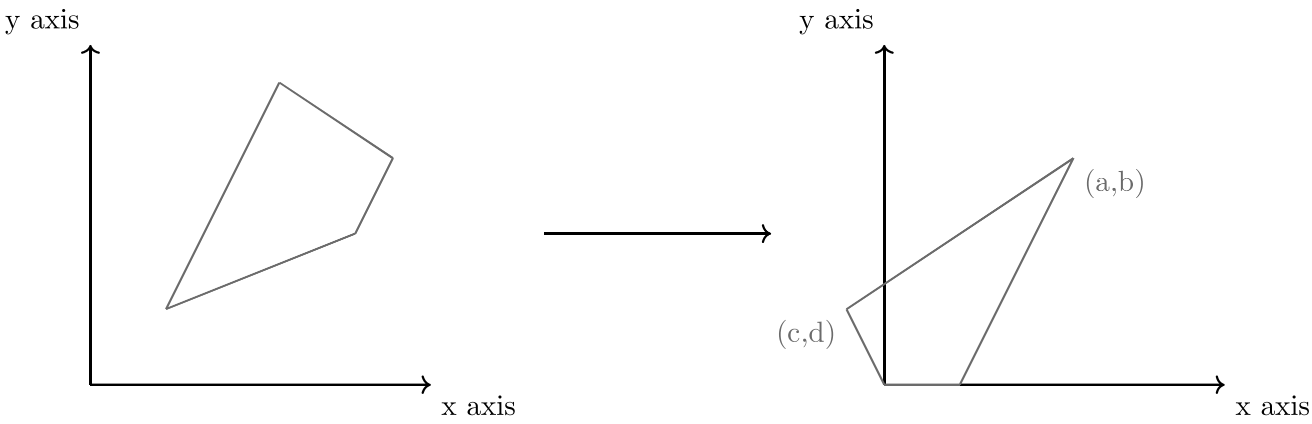
Therefore, we conclude that a homography H preserves C^*_∞ if and only if H is a similarity.

$\square$

Problem 2. Show that any convex quadrilateral in the real Euclidean plane is projectively equivalent to another one.

Solution.
Let Q be a convex quadrilateral in $\mathbb{R}^2$. Since the square $C = [0,1]^2$ is a conver quadrilateral, we will show that Q is projectively equivalent to C.

As the translations, rotations and the homotheties (multiplication by a scalar) are projectivities, one move a point of Q to $(0,0)$ then re-scale a side and rotate it so $[0,1] \times \{0\}$ is a side of Q:



Therfore, we may consider only Q's with points (0,0) – (1,0) – (a,b) – (c,d).

Now, let a homography $H = \begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix}$ (u, v, w are 2-dimensional column vectors and x, y, z are scalars). Then one asks that if Q is convex, can we establish

$$H(0,0) = (0,0), H(1,0) = (1,0), H(1,1) = (a,b), H(0,1) = (c,d) \ ?$$

The goal is to show the answer is *yes*.

By the first condition, one sees that $w = 0$. Therefore, we can normaliza $z = 1$. Then we are left with three equations

$$\begin{pmatrix} u & v & 0 \\ x & y & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} u & v & 0 \\ x & y & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mu \begin{pmatrix} a \\ b \\ 1 \end{pmatrix}, \begin{pmatrix} u & v & 0 \\ x & y & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \nu \begin{pmatrix} c \\ d \\ 1 \end{pmatrix}.$$

These can be re-written as

$$\begin{pmatrix} u \\ x \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} u \\ x \end{pmatrix} + \begin{pmatrix} v \\ y \end{pmatrix} = \mu \begin{pmatrix} a \\ b \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} v \\ y \end{pmatrix} = \nu \begin{pmatrix} c \\ d \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Solving this, one obtains

$$
\begin{aligned}
u &= \begin{pmatrix} \lambda \\ 0 \end{pmatrix}, \\
x &= \lambda - 1 \\
v &= \begin{pmatrix} \nu c \\ \nu d \end{pmatrix}, \\
y &= \nu - 1, \\
\mu &= \lambda + \nu - 1, \\
\lambda + \nu c &= \mu a, \\
\nu d &= \mu b.
\end{aligned}
$$

The four upper equations are just determining u, v, x, y, so what we are really interested is if we can find λ, μ, ν such that

$$\mu = \lambda + \nu - 1, \lambda + \nu c = \mu a, \nu d = \mu b.$$

This is equivalent to solving the matrix equation

$$\begin{pmatrix} a-1 & a-c \\ b & b-d \end{pmatrix} \begin{pmatrix} \lambda \\ \nu \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

As the determinant of the 2×2 matrix above is

$$(a-1)(b-d) - b(a-c) = b(c-1) + d(1-a)$$

which only vanishes iff $a - 1 : b = c - 1 : d$. This condition is equivalent to $(a, b), (c, d), (1, 0)$ lying in a single line. This is not the case for Q.

Therefore, there always exist H such that $H(Q) = C$. Explicitly,

$$H = \begin{pmatrix} \frac{bc-ad}{bc-ad-b+d} & \frac{-bc}{bc-ad-b+d} & 0 \\ 0 & \frac{-bd}{bc-ad-b+d} & 0 \\ \frac{b-d}{bc-ad-b+d} & \frac{ad-bc-d}{bc-ad-b+d} & 1 \end{pmatrix}.$$

$\square$

Remark 1.1. The convexity was never used in the proof. Therefore, both of the assumptions that the base field is $\mathbb{R}$ and Q is convex are unnecessary.

Problem 3. Let L be a line in $\mathbb{P}^3$ and let X, Y be two different points in L, P, Q two planes such that $L = P \cap Q$. Find a relation between the Plücker coordinates of L as an element of $\mathbb{G}r(1, \mathbb{P}^3)$ and the Plücker coordinates as an element of $\mathbb{G}r(1, (\mathbb{P}^3)^\vee)$.

Solution.
Let V be a 4-dimensional vector space over a field k. Let $X = [x], Y = [y], P = [p], Q = [q]$ where $x, y \in V, p, q \in V^\vee$. The line L as an element of $\mathbb{G}r(1, \mathbb{P}(V)) \subseteq \mathbb{P}(\bigwedge^2 V)$ is $[x \wedge y]$, and $[p \wedge q]$ as an element of $\mathbb{P}(\bigwedge^2 V^\vee)$.

Then the condition $X, Y \in L = P \cap Q$ reads as

$$p(x) = q(x) = p(y) = q(y) = 0.$$

There are few (multi)linear maps fundamental for the study of wedge linear algebra. First, the evaluation map

$$ev : \bigwedge^2 V \times \bigwedge^2 V^\vee \to k, ev(v \wedge w, \varphi \wedge \psi) = \varphi(v)\psi(w) - \varphi(w)\psi(v)$$

is a perfect pairing that identifies $\bigwedge^2 V^\vee \simeq (\bigwedge^2 V)^\vee, \bigwedge^2 V \simeq (\bigwedge^2 V^\vee)^\vee$.

Next, via taking the wedge product, one has

$$v \wedge w \wedge - : \bigwedge^2 V \to \bigwedge^4 V, \varphi \wedge \psi \wedge - : \bigwedge^2 V^\vee \to \bigwedge^4 V^\vee.$$

Therefore, when one fixes isomorphisms $\bigwedge^4 V \simeq k, \bigwedge^4 V^\vee \simeq k$, we have

$$v \wedge w \wedge - : \bigwedge^2 V \to k, \varphi \wedge \psi \wedge - : \bigwedge^2 V^\vee \to k.$$

The claim is that under the identifications above, we have

$$p \wedge q = \lambda x \wedge y \wedge - : \bigwedge^2 V \to k, x \wedge y = \lambda' p \wedge q - : \bigwedge^2 V^\vee \to k.$$

The proof of this is simple: if we extend x, y to the basis x, y, z, w of V, then both $p \wedge q$ and $x \wedge y$ vanish on elements $x \wedge y, x \wedge z, y \wedge z, x \wedge w, x \wedge z$ and not on $z \wedge w$.

For the second equality, extend p, q to a basis p, q, r, s of $V^\vee$ and do the same computations.

Therefore, the *Plücker coordinates* of L in two different ways coincide after taking the dual basis of one of the $\bigwedge^2 V, \bigwedge^2 V^\vee$.

When a basis $\{e_i\}$ for V and the dual basis $\{\varphi^i\}$ for $V^\vee$ are given, this means that if

$$L = [L^{12} : L^{13} : L^{14} : L^{34} : L^{42} : L^{23}]$$

with respect to $e_i \wedge e_j's$ and if

$$L = [L_{12} : L_{13} : L_{14} : L_{34} : L_{42} : L_{23}]$$

with respect to $\varphi^i \wedge \varphi^j$'s, then as elements of $\mathbb{P}^5$,

$$[L_{12} : L_{13} : L_{14} : L_{34} : L_{42} : L_{23}] = [L^{34} : L^{42} : L^{23} : L^{12} : L^{13} : L^{14}].$$

$\square$

Problem 4. Let L be a line through two different points X, Y and M a line through two different points Z, W in $\mathbb{P}^3$.

If $P \cap Q = L$ for two planes P, Q, show that

$$\det(XYZW) = 0 \Leftrightarrow P^t Z Q^t W = Q^t Z P^t W.$$

Solution.
Let V be a 4-dimensional vector space so $\mathbb{P}^3 = \mathbb{P}(V)$. Then, in view of Problem 3, the second equality is equivalent to

$$(P \wedge Q)(Z \wedge W) = 0$$

(notice that in this Problem, we abuse the notation so represent planes, points and the 1-dimensional subspace of $V, V^\vee$ representing it).

Since $P \wedge Q$ correspond to the element $X \wedge Y$ by Problem 3, the condition is equivalent to

$$X \wedge Y \wedge Z \wedge W = 0.$$

This is equivalent to the first equality, since after fixing a standard basis of V and the volume element ω corresponding to it, we have

$$X \wedge Y \wedge Z \wedge W = \det(XYZW)\omega.$$

$\square$