MIT 6.042 - Assignment 1

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Problem 1

The domain is the set of people X.

$$
\begin{aligned}
S(x) &\coloneqq \textit{x has been a student of 6.042.} \\
A(x) &\coloneqq \textit{x has gotten an `A' in 6.042.} \\
T(x) &\coloneqq \textit{x is a TA of 6.042.} \\
E(x, y) &\coloneqq \textit{x and y are the same person.}
\end{aligned}
$$

Part (a)

There are people who have taken 6.042 and have gotten A's in 6.042.

$$\exists x \in X : S(x) \land A(x)$$

Part (b)

All people who are 6.042 TA's and have taken 6.042 got A's in 6.042.

$$\forall x \in X : T(x) \land S(x) \implies A(x)$$

Part (c)

There are no people who are 6.042 TA's who did not get A's in 6.042.

$$\neg \exists x \in X : T(x) \land \neg A(x)$$

Part (d)

There are at least three people who are TA's in 6.042 and have not taken 6.042.

$$
\begin{aligned}
&\exists x, y, z \in X : (\neg E(x, y) \land \neg E(x, z) \land \neg E(y, z)) \\
&\qquad\qquad\qquad \land T(x) \land T(y) \land T(z) \land \neg S(x) \land \neg S(y) \land \neg S(z)
\end{aligned}
$$

Problem 2

Part (a)

$$\neg(P \lor (Q \land R)) \equiv (\neg P) \land (\neg Q \lor \neg R)$$

P	Q	R	Q $\land R$	$\neg Q \lor \neg R$	$\neg(P \lor (Q \land R))$	$(\neg P) \land (\neg Q \lor \neg R)$
T	T	T	T	F	F	F
T	T	F	F	T	F	F
T	F	T	F	T	F	F
T	F	F	F	T	F	F
F	T	T	T	F	F	F
F	T	F	F	T	T	T
F	F	T	F	T	T	T
F	F	F	F	T	T	T

Proven by truth table.

Part (b)

$$\neg(P \land (Q \lor R)) \equiv \neg P \lor (\neg Q \lor \neg R)$$

P	Q	R	Q $\lor R$	$\neg Q \lor \neg R$	$\neg(P \land (Q \lor R))$	$\neg P \lor (\neg Q \lor \neg R)$
T	T	T	T	F	F	F
T	T	F	T	T	F	T
T	F	T	T	T	F	T
T	F	F	F	T	T	T
F	T	T	T	F	T	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T
F	F	F	F	T	T	T

Disproven by truth table.

Problem 3

Part (a)

We know that A `nand` B is equivalent to $\neg(A \land B)$.

Sub-part (i)

$$A \land B \equiv \neg(A \texttt{ nand } B)$$

Sub-part (ii)

$$A \lor B \equiv \neg A \texttt{ nand } \neg B$$

Sub-part (iii)

$$A \implies B \equiv A \texttt{ nand } (A \texttt{ nand } B)$$

Part (b)

$$\neg A \equiv A \texttt{ nand } A$$

Part (c)

$$\begin{aligned} \texttt{true} &\equiv A \texttt{ nand } A \texttt{ nand } A \\ \texttt{false} &\equiv (A \texttt{ nand } A \texttt{ nand } A) \texttt{ nand } (A \texttt{ nand } A \texttt{ nand } A) \end{aligned}$$

Problem 4

Split the coins into three groups of four and denote the arbitrarily A, B, and C. First weigh A against B. If they balance on the scale, then the fake coin must be in pile C. Otherwise it is in the lighter pile of A and B. Now that there are only four coins left in consideration, weigh any two coins against each other from this pile. If one is lighter, that is the fake coin. Otherwise if the first two coins are balanced on the scale, weigh the other two against each other and the lighter coin is the fake. In the worst case, this strategy takes 3 weightings.

Problem 5

Prove the following statement: if r is irrational, then $r^{1/5}$ is irrational.

Proof. We prove the given statement by proving its contrapositive: if $r^{1/5}$ is rational, then r is rational.

Since we assume $r^{1/5}$ is rational, there exists an integer m and a positive integer n such that:

$$r^{1/5} = \frac{a}{b}$$

Since 5 is a positive integer, taking both sides of the expression to the fifth power yields:

$$r = \frac{m^5}{n^5}$$

Since both m^5 and n^5 are integers, this implies that r is rational. Note that $n^5 \neq 0$, since n is positive.

This proves the contrapositive, so the original statement is also true. $\square$

Problem 6

Suppose that $w^2+x^2+y^2 = z^2$, where w, x, y, z always denote positive integers. Prove the proposition: z is even if and only if w, x, y are even.

Proof. We prove the proposition by cases.

Case 1: w, x, y are all even. We can rewrite w, x, y as $2i$, $2j$, $2k$. Substituting into the original equation, we have $4i^2 + 4j^2 + 4k^2 = 4(i^2 + j^2 + k^2)$. Since z^2 is a multiple of 4, z must be an even integer.

Case 2: Exactly one of w, x, y is odd. Arbitrarily assume w as the odd integer. The sum of two squares of even integers x and y is even. The square of an odd integer, w is odd. Therefore, z^2 must be odd, as the sum of an odd integer, w^2 and an even integer, $x^2 + y^2$ is odd.

Case 3: Exactly two of w, x, y are odd. Arbitrarily assume w and x are odd integers. We can rewrite the original equation as $w^2+x^2+y^2 = (2i+1)^2+(2j+1)^2+(2k)^2$. Simplifying, this expression becomes $4i^2+4i+1+4j^2+4j+1+4k^2$, and eventually $4(i^2 + j^2 + k^2 + i + j) + 2$. As this sum is not a multiple of 4, z^2 is therefore not a multiple of 4, and z cannot be even.

Case 4: w, x, y are all odd. The square of an odd integer is odd, and the sum of any two odd integers are odd. Therefore, if w, x, y are odd, the square of their sum, z^2 is odd as well.

With this, we have covered all the cases for the parities of w, x, y, and the only case where z is even if is all of w, x, and y are even. This proves that the original statement is true. □