MIT 6.042 - Homework 3

Gabriel Chiong

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Problem 4.8

Proof. We begin on the RHS in propositional form:

$$(A \cap \overline{B}) \cup (B \cap \overline{C}) \cup (C \cap \overline{A}) \cup (A \cap B \cap C)$$

Considering each disjunctive element in turn converting the entire RHS into propositional form, we first rewrite $A \cap \overline{B} = (A \cap \overline{B}) \cap (C \cup \overline{C}) = (A \cap \overline{B} \cap C) \cup (A \cap \overline{B} \cap \overline{C})$.

Similarly, we can rewrite $B \cap \overline{C} = (A \cap B \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C})$. And finally, we can also rewrite $C \cap \overline{A} = (\overline{A} \cap B \cap C) \cup (\overline{A} \cap \overline{B} \cap C)$.

Now we can proceed by chain of iff's:

$$\begin{aligned}
&(A \cap \overline{B}) \cup (B \cap \overline{C}) \cup (C \cap \overline{A}) \cup (A \cap B \cap C) \\
&\text{IFF } (A \cap \overline{B} \cap C) \cup (A \cap \overline{B} \cap \overline{C}) \cup (A \cap B \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \\
&\cup (\overline{A} \cap B \cap C) \cup (\overline{A} \cap \overline{B} \cap C) \cup (A \cap B \cap C)
\end{aligned}$$

The DNF form of the RHS considers every combination of sets between the three sets A, B, C, since all the cases of 0, 1 and 2 sets included are considered. Since these disjunctive elements together cover all elements in $A \cup B \cup C$, the claimed set equality is proven. $\square$

Problem 4.13

Proof. We prove the proposition by contradiction; that is, if $A \times B$ and $C \times D$ are disjoint sets, then both A and C, and B and D have common elements.

Two sets are disjoint when their intersection is the empty set (no elements shared). Suppose some arbitrary elements $a \in A$, $b \in B$, $c \in C$, and $d \in D$. For the sets to be disjoint, the elements of each Cartesian product must be strictly not equal in at least one position within the ordered pair: either $a \neq c$ or $b \neq d$. Since a, b, c, d are arbitrary elements, this must mean that there can be no common elements either between A and C, or B and D. This contradicts our earlier assumption that both A and C, and B and D have common elements. Therefore, either A and C, or B and D must be disjoint. $\square$

Problem 4.14

Part (a)

First consider predicate L. Let b, c be any arbitrary elements such that $b \in B$ and $c \in C$. Then we know the ordered pair $(b, c) \in L$ since $L ::= (A \cup B) \times (C \cup D)$.

Now consider R. The ordered pair (b, c) cannot be in R since $(b, c) \notin (A \times C)$ or $(b, c) \notin (B \times D)$. This counterexample shows $L \neq R$.

Part (b)

The fourth "iff" statement does not distribute the elements over the OR in the previous "iff" correctly.

Part (c)

Continuing the proof after the third "iff" (note the use of the disjunctive form):

IFF $(A \cap C) \cup (B \cap D)$

IFF $[((A \cap C) \cap (B \cup \overline{B})) \cap (D \cup \overline{D})] \cup [((B \cap D) \cap (A \cup \overline{A})) \cap (C \cup \overline{C})]$

IFF $(A \cap B \cap C \cap D) \cup (A \cap \overline{B} \cap C \cap \overline{D}) \cup (A \cap B \cap C \cap D) \cup (\overline{A} \cap B \cap \overline{C} \cap D)$

IFF $(A \cap B \cap C \cap D) \cup (A \cap \overline{B} \cap C \cap \overline{D}) \cup (\overline{A} \cap B \cap \overline{C} \cap D)$

IFF $((x, y) \in (A \cap C) \times (B \cap D)) \cup ((x, y) \in (A \times C) \cap (x, y) \notin (B \times D))$
$\cup ((x, y) \in (B \times D) \cap (x, y) \notin (A \times C))$

These disjunctive elements are all in the set $L ::= (A \cup B) \times (C \cup D)$. Since $(x, y) \in (B \times C)$ and $(x, y) \in (A \times D)$ are also both subsets of L, but not in included in R, we can conclude that $R \subseteq L$.

Problem 4.18

Note, these are real-valued functions on the real numbers, $\mathbb{R}$.

(a) $x \to x + 1$: Bijection.

(b) $x \to 2x$: Bijection.

(c) $x \to x^2$: Neither an Injection nor a Surjection.

(d) $x \to x^3$: Bijection.

(e) $x \to \sin x$: Neither an Injection nor a Surjection.

(f) $x \to x \sin x$: Neither an Injection nor a Surjection.

(g) $x \to e^x$: Injection, but not a Surjection.

Problem 4.25

Part (a)

Proof. We proceed by contradiction. Suppose not; that is, suppose that if h is surjective and f is total and injective, then g is injective. By definition, if h is surjective, then $|A| \geq |C|$. Also, since f is total and injective, $|B| \leq |C|$. For g to be injective, this means that $|A| \leq |B|$. But this cannot be true since we established previously that for the antecedent to hold, $|A| \geq |C| \geq |B|$. Therefore, our initial proposition must be false, and we have proven that g must be surjective. □

Part (b)

Proof. We proceed by contradiction. Suppose not; that is, suppose that if h is injective and f is total, then g is surjective. By the definition of a surjective function, this means that every point in B has at least one arrow incoming from A. Since f is total, each point in B has at least one arrow mapping to an element in C. This means that there are at least the same or greater number of elements in C than A. This contradicts our earlier assumption which stated that h is injective ($|A| \geq |C|$). Therefore, our initial proposition must be false, and we have proven that g must be injective. □

An example of a counterexample that this claim fails when f is not total, is the case when the number of out arrows from B are less than the elements in C. This means that $|A| \geq |C|$, which turns h into a surjective function.

Problem 4.31

1. Injective (definition of injective, simplified).
2. Not injective.
3. Injective (contrapositive).
4. Injective (definition of injective).
5. Injective (definition of injective, simplified).
6. Not injective.
7. Injective (has inverse).
8. Not injective.

Problem 4.37

Labelling each side of the ordered pair $(i, j) \in A \times B$, where $i \in A$ and $j \in B$, we enumerate each combination using the bijective function $(j \cdot n + i)$ to map the ordered pair to a unique nonnegative integer from 0 to $mn-1$. For example, the first pair $(0, 0) \mapsto 0$ and the last pair $(n-1, m-1) \mapsto mn-1$.