MIT 6.042 - Problem Set 2

Gabriel Chiong

June 2021

Problem 1

Part (a)

Assuming $a_1 < a_2$, we show that if there is no *3-chain* in our sequence of a_1, a_2, a_3, a_4, a_5, then a_3 must be less than a_1. Note that subsequences by deleting in between elements count too.

Since we assume in the problem that $a_1 < a_2$ and that all integers in the sequence are distinct, a_3 must be less than a_2 at a minimum (without even considering a_4, a_5 yet), since if $a_3 > a_2$, then we have a 3-chain by $a_1 < a_2 < a_3$.

So we know $a_2 > a_3$, but what is the relation between a_3 and a_1? First we consider if $a_3 > a_1$. Then a_4 will form a 3-chain within the sequence regardless of its relationship to a_3. Say $a_4 > a_3$. Then by taking the subsequence $a_1 < a_2 < a_4$, we have a 3-chain. Alternatively, if $a_4 < a_3$, we have a 3-chain by $a_2 > a_3 > a_4$.

Therefore, a_3 must be less than a_1 for there not to be a 3-chain in the sequence.

Part (b)

Since our assumption states that $a_1 < a_2$, we know that a_3 must be less than a_2 (and we also know by part (a) that it is less than a_1), otherwise the 3-chain $a_1 < a_2 < a_3$ would form.

Since the relationship between a_2 and a_3 has to be $a_2 > a_3$, a_4 cannot be less than a_3, otherwise a 3-chain would form: $a_2 > a_3 > a_4$. We also cannot allow a_4 to be greater than a_2, otherwise that would result in a 3-chain with $a_1 < a_2 < a_4$.

Therefore, the relationship under the constraints above require that $a_3 < a_2$, $a_3 < a_4$, and $a_4 < a_2$. Putting this together, we have shown that in order for the original claim to hold, we must have $a_3 < a_4 < a_2$.

Part (c)

We first consider the case in which a_5 is less than a_4. Taking our assumptions above, this means that we can form a 3-chain with $a_2 > a_3 > a_5$ (removing a_4

from the sequence).

Next, we consider the case where a_5 is greater than a_4. However, this results in the 3-chain forming: $a_3 < a_4 < a_5$.

These are the only two cases, since we know that all integers are distinct. Therefore, we have shown that whatever the value of a_5 is, there is a 3-chain subsequence.

Part (d)

Proof. We proceed by contradiction. Suppose not; that is, suppose that any sequence of five distinct integers does not contain a 3-chain. Assuming the above proposition, let us construct a sequence of five distinct integers which does not contain a 3-chain.

Taking Part 1(a) to be Lemma 1, in order to create the first 3 subsequence of a total 5-integer sequence without a 3-chain, we must have $a_1 < a_2$ and $a_3 < a_1$.

Taking Part 2(b) to be Lemma 2, in order to add a fourth element to the sequence we require that a_4 be greater than a_3 and less than a_2.

Lemma 1 and Lemma 2 guarantee that the 4-element subsequence a_1,a_2,a_3, a_4 have no 3-chain. Taking Part 1(c) to be Lemma 3, we see that whichever value we select for a_5 (be it greater or less than a_4), we obtain a sequence with a 3-chain.

Lemmas 1, 2, 3 all initially assume $a_1 < a_2$. In the case that $a_1 > a_2$, this is just the case where each $>$ is replaced by $<$ and vice versa in every Lemma. Therefore, any choice of a_5 also results in a 5 integer sequence with a 3-chain.

This contradicts our initial assumption and we therefore prove that any sequence of five distinct integers must contain a 3-chain. $\square$

Problem 2

Proof. We prove this by induction on variable n, where $n \geq 0$. Let the predicate $P(n)$ be $\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Base Case: $n = 0$. Then on the left-hand side of the predicate, we have $P(0) = \sum_{i=0}^{0} i^3$, which is equal to 0 by convention of summations. The right-hand side of the predicate is $\left(\frac{0(0+1)}{2}\right)^2$, which also equals 0. Therefore the base case holds.

Inductive Step: Assuming $P(n)$ is true, to show that $P(n) \implies P(n+1)$; that is, $\sum_{i=0}^{n+1} i^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2$, we start with the equation for $P(n)$, and add $(n+1)^3$ on both sides.

$$\sum_{i=0}^{n} i^3 + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$$

Simplifying the left-hand side, we obtain $\sum_{i=0}^{n+1} i^3$, and the right-hand side simplifies to

$$\begin{aligned}
\left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 &= \frac{n^2(n+1)^2 + 2^2(n+1)^3}{2^2} \\
&= \frac{(n+1)^2(n^2+4n+4)}{2^2} \\
&= \left(\frac{(n+1)(n+2)}{2}\right)^2
\end{aligned}$$

Thus, we have shown that $P(0)$ is true, and by the inductive hypothesis, we have shown that $P(n) \implies P(n+1)$ for all nonnegative integers. Therefore, $P(n)$ is true for all nonnegative integers by the principle of induction. $\square$

Proof. We proceed by Contradiction. Assume that the claim is false. Then some nonnegative integers serve as counterexamples to it. Let us define the set of counterexamples in a set:

$$C \coloneqq \left\{ n \in \mathbb{N} : \sum_{i=0}^{n} i^3 \neq \left(\frac{n(n+1)}{2}\right)^2 \right\}$$

Assuming there are counterexamples, the set C is a nonempty set of nonnegative integers. So, by WOP, C has a minimum element, which we will call c. Since c is the smallest counterexample, we know that $P(n)$ is false for $n = c$ but true for all nonnegative integers $n < c$. But $P(n)$ is true for $n = 0$, so $c > 0$. This means $c - 1$ is a nonnegative number which satisfies $P(n)$. We can write $P(c-1)$ as:

$$\sum_{i=0}^{c-1} i^3 = \left(\frac{(c-1)(c-1+1)}{2}\right)^2 = \frac{c^4 - 2c^3 + c^2}{4}$$

Adding c^3 to both sides results in:

$$\sum_{i=0}^{c} i^3 = \left(\frac{c^4 - 2c^3 + c^2 + 4c}{4}\right) = \left(\frac{c^4 - 2c^3 + c^2 + 4c^3}{4}\right) = \left(\frac{c(c+1)}{2}\right)^2$$

This means that $P(n)$ does hold for c, after all. This is a contradiction, and we are done. $\square$

Problem 3

We begin by proving a lemma on an invariant of the system; that is, this particular property of the system does not change in the initial state, and all stages reachable from this initial state.

We first define a property of the system in which we will prove is an invariant in order to prove a lemma to aid in the proof. The *perimeter*, denoted I, is the number of edges surrounding the initial set of infected squares.

Proof. Lemma 1 states: Given I of any system's initial state, all states reachable from this initial state has a perimeter of at most I. We prove this Lemma by induction on the time-step of the system, t.

Base Case: At time-step $t = 0$, this is the initial stage and I is the perimeter of the system by definition.

Inductive Step: Assuming the inductive hypothesis holds for all t such that $0 \leq t$, on step $t+1$, in order for any infection to spread, there will be a minimum of two edges removed from the perimeter (forming a new infected square), and at most two edges added (a new infected square shares a border with another unrelated infected square). By this reasoning, I can either remain the same at the $t+1$ step (if the number of new edges equal the number of edges removed), or decrease (surrounded by more than two infected squares).

Thus, we have shown that $P(0)$ is true, and by the inductive hypothesis, we have shown that $P(t) \implies P(t+1)$ for all $t \geq 0$. Therefore the invariant holds for all reachable states from the initial state. □

The original theorem that states: For a $n \times n$ grid, if fewer than n students are infected, the whole class will never be infected.

Proof. We proceed by Case Analysis. For a fully infected class of $n \times n$ students, the perimeter I would have to be $4n$ (the four sides of the square-grid). We consider the cases where the number of students, k is less than n, and the case where k is greater than or equal to n.

Case 1: $k < n$. From Lemma 1, we know that $I \geq 4n$ since $4n$ is the only perimeter for which the grid is fully infected. Therefore, the initial state must also have $I \geq 4n$, since I can never increase. If $k < n$, it is impossible for I to maintain this invariant, since each infected square has only four sides. Therefore k cannot be less than n in order to fully infect the entire grid.

Case 2: $k \geq n$. By similar argument as Case 1, $I \geq 4n$ is the invariant which must hold in order for the grid to become fully infected. This can only be true if the number of infected squares in the initial state is greater than or equal to n since each square has four sides. Therefore, if $k \geq n$ there exists initial states of certain configurations that would result in a fully infected board.

We have shown by case analysis that if the number of infected students is less than n for a $n \times n$ grid, the entire grid cannot become fully infected. □

Problem 4

The inductive step uses $P(n) \implies P(n+1)$. An assumption of the inductive step is that $n \geq 1$, otherwise the denominator's exponent would be a negative integer - when it was explicitly defined as nonnegative. However the base case is $P(0)$, where $n = 0$. Therefore, the inductive step cannot assume that $n \geq 1$.

Problem 5

Proof. Let $P(n)$ be the proposition that $G_n = 3^n - 2^n$. We prove this by strong induction on variable n, where $n \in \mathbb{N}$ and $n \geq 0$.

Base Case: $n = 0$. We have $P(0) = 3^0 - 2^0 = 0$, which holds. When $n = 1$, we have $P(1) = 3^1 - 2^1 = 1$, which also holds.

Inductive Step: Assuming $P(n)$ holds for all $n \geq 0$, we show that $P(n+1) = 3^{n+1} - 2^{n+1}$. By definition, we can write G_{n+1} in terms of G_n and G_{n-1} using the inductive hypothesis as

$$\begin{aligned}
5G_{n-1} - 6G_{n-2} &= 5 \cdot (3^n - 2^n) - 6 \cdot (3^{n-1} - 2^{n-1}) \\
&= 5 \cdot 3^n - 5 \cdot 2^n - \frac{6 \cdot 3^n}{3} + \frac{6 \cdot 2^n}{2} \\
&= 3 \cdot 3^n - 2 \cdot 2^n \\
&= 3^{n+1} - 2^{n+1}
\end{aligned}$$

Thus, we have shown that $P(0)$ is true, and by the inductive hypothesis, we have shown that $P(n) \implies P(n+1)$ for all nonnegative integers. Therefore, $P(n)$ is true for all nonnegative integers by the principle of induction. □

Problem 6

Part (a)

Proof. We proceed by Case Analysis. Enumerate the order of the flattened structure starting at the 0-th tile to the $(n \times n - 1)$-th tile. The only allowable row moves in any puzzle are at the positions $i + 1$ and $i - 1$, for an arbitrary location of the blank square, i. We consider two possible move cases: switching the positions of the i and $i + 1$ tiles, or switching the positions of the i and $i - 1$ tiles.

Case 1: Switch the blank tile i with the tile at position $i + 1$. Thus, the order goes from $..., i, i+1, ...$ to $..., i+1, i, ...$ Relative to the subsequences prior to $i + 1$, there is no change. Likewise, the subsequences following $i + 1$, there is no change either. Therefore, when there is a row move where i is exchanged with the tile at $i + 1$, the order of the flattened structure is not changed.

Case 2: Switch the blank tile i with the tile at position $i - 1$. By a similar argument to Case 1, a row move where i is exchanged with the tile at $i - 1$ does not change the order of the structure (going from $..., i - 1, i, ...$ to $..., i, i - 1, ...$).

We have shown that whichever allowable row move is made (whether the blank square i is exchanged with the $i+1$ or $i-1$ tile), the order of the flattened structure is not changed. □

Part (b)

Proof. We proceed by Case Analysis. Similar to Part (a) we enumerate the tiles in the same way. There are two cases to consider: switching a blank tile with the tile directly above it, or switching a blank tile with the tile directly below it.

Case 1: Switching a blank tile with the tile directly above it. In a 4×4 square grid, this is essentially switching the tile above the blank tile with the next three tiles, one-by one, until it is located at the blank tile. More formally, for a sequence of tiles $..., L_1, L_2, L_3, L_4, L_5, ...$ where L_5 is the blank tile and L_1 is the tile directly above L_5, the relative orders of $L_1 - L_2$, $L_1 - L_3$, $L_1 - L_4$ are flipped (into $..., L_5, L_2, L_3, L_4, L_1, ...$). Therefore, a column move to the tile above the blank square results in three relative orders being changed.

Case 2: Switching a blank tile with the tile directly below it. By a similar argument to Case 1, this is essentially switching the tile below the blank tile with its previous three tiles, one-by-one, until it is located at the blank tile. More formally, a sequence of tiles $..., L_1, L_2, L_3, L_4, L_5, ...$ where L_1 is the blank tile and L_5 is the tile directly above L_1, the relative orders of $L_5 - L_4$, $L_5 - L_3$, $L_5 - L_2$ are flipped (into $..., L_5, L_2, L_3, L_4, L_1, ...$). Therefore, a column move to the tile below the blank square results in three relative orders being changed.

We have thus shown that any allowable column move (either with the tile above, or the tile below) results in exactly three relative orders being changed within the grid. □

Part (c)

Proof. An inversion can only be reversed by switching the relative ordering between the two inverted tiles, since by definition, an inversion is an out-of-order pairing.

Let Part (a) be denoted as Lemma 1. By Lemma 1, any row move does not change the number of inversions since any row move does not affect the relative ordering of the structure. □

Part (d)

Proof. We proceed by Case Analysis. Let Part (b) be denoted as Lemma 2. We consider all four cases of changing the relative ordering of three pairs by a column move (each denoted by an ordered pair where the first element is the number of inversions caused and the second element is the number of inversions fixed): $\langle 3, 0\rangle$, $\langle 2, 1\rangle$, $\langle 1, 2\rangle$, and $\langle 0, 3\rangle$.

Case 1: Causing 3 inversions, fixing 0 inversions. An odd number added to a number of parity flips the number's parity. Therefore, causing three inversions results in a different parity.

Case 2: Causing 2 inversions, fixing 1 inversion. Simplifying, $2 - 1 = 1$, therefore the number of inversions is increased by one. By similar reasoning to Case 1, the parity of the number of inversions is changed in this case.

Case 3: Causing 1 inversion, fixing 2 inversions. Similar to Case 2, except $1 - 2 = -1$. This is also a change in the number of inversions by one, where the parity of the number of inversions is changed.

Case 4: Causing 0 inversions, fixing 3 inversions. The parity of the number of inversions is also flipped. This is Case 1 in disguise.

Thus, we have shown that for all column moves, the parity of the number of inversions is flipped at each move. □

Part (e)

Proof. We prove this using Induction on the variable $n \in \mathbb{N}$, which represents each time-step from the initial stage, $n = 0$.

Base Case: The parity of the number of inversions is odd (one inverted pair $O - N$), and the parity of the row number containing the blank square is even (row 4). Therefore the parities are different, and the base case holds.

Inductive Step: We assume $P(n)$ holds for all $n \geq 0$. At the $(n+1)$-th step, we can either make a row move, or a column move. We consider these cases separately.

Case 1: The player makes a row move. By Part (c), we know that a row move does not change the number of inversions. Neither does a row move change the parity of the row number containing the blank square since the row move slides "horizontally" and not vertically. Since neither parity changes from the $P(n)$ stage, they remain different for the $(n+1)$-th stage, and $P(n+1)$ holds.

Case 2: The player makes a column move. By Part (d), we know that any column move flips the parity of the number of inversions. A column move by definition either adds one or subtracts one from the row number. Therefore, the parity of the row number is also flipped during any column move. Since both parities change from the $P(n)$ stage, they remain different for the $(n+1)$-th stage, and $P(n+1)$ holds.

Thus, we have shown that $P(0)$ is true, and by the inductive hypothesis, we have shown that $P(n) \implies P(n+1)$ for $n \geq 0$. Therefore, $P(n)$ is true for all nonnegative integers by the principle of induction. □

Part (f)

Proof. In the goal stage where all tiles are in order, the number of inversions is 0, which is even by convention. The row number containing the blank tile is 4, which is also even.

However, in the initial stage, the number of inversions is one, since $O - N$ has an inverted order, therefore its parity is odd. The blank square is in the fourth row, therefore the parity of the row number is even. Since by Part (e), we know that any reachable state from the initial stage always has different parities between the number of inversions and row number, it is impossible to reach the goal stage where both parities are even.

This proves that the goal state is not reachable from the initial state, which is the claim we originally wanted to prove. □

Problem 7

Proof. Rather than using induction on the claim that the number of Z-lings will always be at most twice the number of B-lings, we use a stronger hypothesis. We proceed by Induction on the variable n, which represents the time-step which begins at 0 and is a nonnegative integer. Let $P(n)$ denote that the number of Z-lings will always be at most equal to the number of B-lings at all time-steps.

Base Case: $P(0)$. At time $n = 0$, the system begins with 200 Z-lings and 800 B-lings. Since $200 \leq 800$, the base case holds.

Inductive Step: We assume $P(n)$ holds for all $n \geq 0$. Let z_n and b_n denote the number of Z-lings and B-lings at the previous time-step n. Since the inductive hypothesis holds for $P(n)$, we know that $z_n \leq b_n$. Therefore, at time-step $n + 1$, all z_n's will be matched with b_n's as that is the priority matching and that results in z_n number of Z-lings and also z_n number of B-lings. This also means that no $Z - Z$ pairs are possible since there are no Z-lings left over. The remaining B-lings are then paired up with each other (if there are more than 1 B-lings left over), and form 2 B-lings and 1 Z-ling for each pair. If there are 1 or less B-lings left over, $P(n+1)$ holds since there are equal numbers of Z-lings and B-lings (which is z_n numbers of both). If there are more than 2 B-lings left over, they produce double the number of B-lings as Z-lings, therefore $P(n+1)$ holds as well.

Thus, we have shown that $P(0)$ is true, and by the inductive hypothesis, we have shown that $P(n) \implies P(n+1)$ for $n \geq 0$. Therefore, $P(n)$ is true for all nonnegative integers by the principle of induction. $\square$