MIT 6.042 - Problem Set 1

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Problem 1

The domain is the set of people X.

$$\begin{aligned} S(x) &\coloneqq x \textit{ has been a student of 6.042.} \\ A(x) &\coloneqq x \textit{ has gotten an `A' in 6.042.} \\ T(x) &\coloneqq x \textit{ is a TA of 6.042.} \\ E(x,y) &\coloneqq x \textit{ and } y \textit{ are the same person.} \end{aligned}$$

Part (a)

There are people who have taken 6.042 and have gotten A's in 6.042.

$$\exists x \in X : S(x) \land A(x)$$

Part (b)

All people who are 6.042 TA's and have taken 6.042 got A's in 6.042.

$$\forall x \in X : T(x) \land S(x) \implies A(x)$$

Part (c)

There are no people who are 6.042 TA's who did not get A's in 6.042.

$$\neg \exists x \in X : T(x) \land \neg A(x)$$

Part (d)

There are at least three people who are TA's in 6.042 and have not taken 6.042.

$$\begin{aligned} \exists x, y, z \in X : \neg E(x,y) \land \neg E(x,z) \land \neg E(y,z) \\ \land T(x) \land T(y) \land T(z) \land \neg S(x) \land \neg S(y) \land \neg S(z) \end{aligned}$$

Problem 2

Part (a)

$$\neg(P \vee (Q \wedge R)) \equiv (\neg P) \wedge (\neg Q \vee \neg R)$$

P	Q	R	Q $\wedge R$	$\neg Q \vee \neg R$	$\neg(P \vee (Q \wedge R))$	$(\neg P) \wedge (\neg Q \vee \neg R)$
T	T	T	T	F	F	F
T	T	F	F	T	F	F
T	F	T	F	T	F	F
T	F	F	F	T	F	F
F	T	T	T	F	F	F
F	T	F	F	T	T	T
F	F	T	F	T	T	T
F	F	F	F	T	T	T

Proven by truth table.

Part (b)

$$\neg(P \wedge (Q \vee R)) \equiv \neg P \vee (\neg Q \vee \neg R)$$

P	Q	R	Q $\vee R$	$\neg Q \vee \neg R$	$\neg(P \wedge (Q \vee R))$	$\neg P \vee (\neg Q \vee \neg R)$
T	T	T	T	F	F	F
T	T	F	T	T	F	T
T	F	T	T	T	F	T
T	F	F	F	T	T	T
F	T	T	T	F	T	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T
F	F	F	F	T	T	T

Disproven by truth table.

Problem 3

Part (a)

We know that A `nand` B is equivalent to $\neg(A \wedge B)$.

Sub-part (i)

$$A \wedge B \equiv \neg(A \texttt{ nand } B)$$

Sub-part (ii)

$$A \vee B \equiv \neg A \texttt{ nand } \neg B$$

Sub-part (iii)

$$A \implies B \equiv A \texttt{ nand } (A \texttt{ nand } B)$$

Part (b)

$$\neg A \equiv A \texttt{ nand } A$$

Part (c)

$$\begin{aligned} \texttt{true} &\equiv A \texttt{ nand } A \texttt{ nand } A \\ \texttt{false} &\equiv (A \texttt{ nand } A \texttt{ nand } A) \texttt{ nand } (A \texttt{ nand } A \texttt{ nand } A) \end{aligned}$$

Problem 4

First, divide the 12 coins arbitrarily into two subgroups of six. Place each subgroup of coins on either side of the balance scale. Remove the heavier subgroup from consideration as we know that since the number of coins in each subgroup is the same, the lighter subgroup contains the fake coin within it.

Now consider the lighter subgroup of six coins. Again, divide the group of six arbitrarily into two smaller subgroups of three coins each and place each subgroup of three coins onto either side of the balance scale. Remove the heavier subgroup of three coins from consideration for the same reason as we removed the heavier subgroup of six coins in the previous step.

Consider the lighter subgroup of three coins. Since we know the previously removed coins from consideration must all be genuine, arbitrarily select one coin from the previously removed pile and place it into the current subgroup of three coins under consideration. We now have four coins in total, one of which is fake. Arbitrarily divide the group of four coins into two subgroups of two coins each and place them on either side of the balance scale. Again, remove the heavier subgroup of two coins as in the previous steps.

Now we are left with two coins, one of which must be fake since we have always removed the heavier subgroups from consideration in previous steps. Arbitrarily divide the group of two coins into two subgroups of one coin each and place them on either side of the balance scale. The lighter coin is the fake coin.

Problem 5

Prove the following statement: if r is irrational, then $r^{1/5}$ is irrational.

Proof. We prove this by the contrapositive: if $r^{1/5}$ is rational, then r is rational.

If $r^{1/5}$ is rational, then it can be written in terms of a fraction in lowest terms, $r^{1/5} = \frac{m}{n}$, for any integers m and n.

Taking the 5^{th} power on either side of the equation for $r^{1/5}$, we get $(r^{(1/5)})^5 = (\frac{m}{n})^5$, which simplifies to

$$r = \frac{m^5}{n^5},$$

where m^5 and n^5 are integers (since m and n are integers, and $b^5 \neq 0$ since b is a positive integer), thus $\frac{m^5}{n^5}$ must be rational by definition. Therefore, this means that r must be rational if $r^{1/5}$ is rational. This proves the contrapositive, so the original statement must also be true. □

Problem 6

Suppose that $w^2 + x^2 + y^2 = z^2$, where w, x, y, z always denote positive integers. Prove the proposition: z is even if and only if w, x, y are even.

Proof. We proceed by case analysis. There are four cases to consider, either all w, x, y are even, one of w, x, y is odd, two of w, x, y are odd, or all of w, x, y are odd.

Case 1: w, x, y are all even. We can rewrite them as $w = 2i$, $x = 2j$, and $y = 2k$ for some positive integers i, j, k. Substituting into our original equation, we have $z^2 = (2i)^2 + (2j)^2 + (2k)^2 = 4(i^2 + j^2 + k^2)$. Since i, j, k are integers, the sum of their square is an integer as well, say l. We can simplify the substituted equation to $z^2 = 4l$. In this case, z must be an even integer.

Case 2: One of w, x, y is odd. Without any loss of generality, we can assume that w is odd since addition is commutative, and the variable names are just labels which can be ordered in any way. Thus, we can write $w = 2i + 1$, $x = 2j$, and $y = 2k$. Substituting into our original equation, we have $z^2 = (2i + 1)^2 + (2j)^2 + (2k)^2 = (4i^2 + 4i + 1) + 4j^2 + 4k^2$. Collecting terms and simplifying, we get $z^2 = 4(i^2 + i + j^2 + k^2) + 1$. We know the $4(i^2 + i + j^2 + k^2)$ portion of the equation must be even, and adding one gives an odd number. However, if z is even, z^2 is also even. Therefore, since z^2 is odd in this case, z cannot be even.

Case 3: Two of w, x, y are odd. Without any loss of generality, we can assume that w and x are odd for the same reasons as in Case 1. Thus, we can write $w = 2i + 1$, $x = 2j + 1$, and $y = 2k$. Substituting into our original equation, we have $z^2 = (2i+1)^2 + (2j+1)^2 + (2k)^2 = 4(i^2 + i + j^2 + j + k^2) + 2$. Note that 4 does not divide this sum. However, if z was an even integer, then z^2 would be a multiple of 4. So z cannot be an even integer in this case.

Case 4: w, x, y are all odd. We can write $w = 2i + 1$, $x = 2j + 1$, and $y = 2k + 1$. Substituting into our original equation, we have $z^2 = (2i + 1)^2 + (2j + 1)^2 + (2k + 1)^2 = 4(i^2 + i + j^2 + j + k^2 + k) + 3$. By the same argument as Case 2, since z^2 is odd, z cannot be even in this case.

With this, we have covered all the cases for the parities of w, x, y and have proven that z is even if and only if w, x, y are all even. □