MIT 6.042 - Assignment 2

Gabriel Chiong

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Problem 1

Part (a)

By only considering the position of a_3, we know that $a_3 < a_2$, otherwise a *3-chain* of a_1, a_2, a_3 would be formed. Considering the case of $a_3 < a_2$, a *3-chain* would result from a_4, regardless of its placement by either omitting a_2 or a_3 from the subsequence. Therefore, a_3 must be less than a_1.

Part (b)

If $a_4 > a_2$, a *3-chain* would form with a_1, a_2, a_4, so a_4 must be less than a_2. If $a_4 < a_3$, a *3-chain* would form with either a_1, a_3, a_4, or a_2, a_3, a_4. Therefore, a_4 must be greater than a_3. Therefore, we can only have $a_3 < a_4 < a_2$.

Part (c)

There are two considerations for the position of a_5. If $a_5 > a_4$, then a *3-chain* with a_3, a_4, a_5 is formed. If $a_5 < a_4$, then a *3-chain* forms from the subsequence a_2, a_4, a_5. Therefore, any value of a_5 results in a *3-chain*.

Part (d)

Proof. Suppose not. Assume that there exists some series of distinct values for a_1, a_2, a_3, a_4, a_5 that does not contain a *3-chain*. We proceed to construct this sequence of values.

From Part (a), the only placement of a_3 possible is if it has a value less than a_1. From Part (b), the only placement of a_4 that is possible without forming a *3-chain* is if it has a value of $a_3 < a_4 < a_2$. There are only two ways to place a_5 as outlined in Part (c). Either of these ways result in a *3-chain*.

This contradicts our original assumption, therefore any five-integer sequence must contain a *3-chain*. □

Problem 2

Proof. We proceed by induction on the value n. Let $P(n)$ be the proposition that the given equation holds.

Base case: For $n = 0$, the left side of the equation is $\sum_{i=0}^{0} i^3 = 0$. The right side of the equation gives $\left(\frac{0(1)}{2}\right)^2 = 0$. Therefore, $P(0)$ is true.

Inductive step: Assume that $P(n)$ is true, that $\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. Then we can prove that $P(n+1)$ is also true.

$$\begin{aligned}
\sum_{i=0}^{n+1} i^3 &= \sum_{i=0}^{n} i^3 + (n+1)^3 \\
&= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \\
&= \frac{n^2(n+1)^2}{2^2} + (n+1)^3 \\
&= \frac{(n+1)^2(n^2 + 2^2(n+1))}{2^2} \\
&= \frac{(n+1)^2(n+2)^2}{2^2} \\
&= \left(\frac{(n+1)(n+2)}{2}\right)^2
\end{aligned}$$

Therefore, $P(0)$ is true, and $P(n)$ implies $P(n+1)$ for all nonnegative integers. Thus, $P(n)$ is true for all nonnegative integers by the principle of induction. □

Problem 3

Proof. We define the *perimeter* of an infected set of students to be the number of edges with infection on exactly one side. Let I denote the perimeter of the initially-infected set of students.

We prove by induction on the number of time steps to prove that the perimeter of the infected region never increases. Let $P(k)$ be the proposition that after k time steps, the perimeter of the infected region is at most I.

Base case: $P(0)$ is true by definition; the perimeter of the infected region is at most I after 0 time steps, because I is defined to be the perimeter of the initially-infected region.

Inductive step: We must show that $P(k)$ implies $P(k+1)$ for all $k \geq 0$. Assuming that $P(k)$ is true, where $k \geq 0$; that is, the perimeter of the infected region is at most I after k steps. The perimeter can only change at step $k+1$ because some squares are newly infected. By the rules above, each newly-infected square is adjacent to at least two previously-infected squares. Thus, for each newly-infected square, at least two edges are removed from the perimeter of the infected region, and at most two edges are added to the perimeter. Therefore, the perimeter of the infected region cannot increase and is at most I after $k+1$ steps as well. This proves that $P(k)$ implies $P(k+1)$ for all $k \geq 0$.

By the principle of induction, $P(k)$ is true for all $k \geq 0$.

If an $n \times n$ grid is completely infected, then the perimeter of the infected region is $4n$. Thus, the whole grid can become infected only if the perimeter is initially at least $4n$. Since each square has perimeter 4, at least n squares must be infected initially for the whole grid to be infected. □

Problem 4

The flaw is in the *inductive step*, where the author implicitly assumes that $n \geq 1$ in the denominator when evaluating a^{n-1}, otherwise the exponent is not a nonnegative integer (and the inductive hypothesis becomes unusable). The base case was checked for $n = 0$, so it is invalid to assume that $n \geq 1$ when proving the statement for $n+1$ in the inductive step. The proposition breaks down when $n = 1$.

Problem 5

Proof. We prove the proposition $P(n)$ that $G_n = 3^2 - 2^n$ for all $n \in \mathbb{N}$ using strong induction on n.

Base case: For $n = 0$ and $n = 1$, $G_0 = 3^0 - 2^0 = 0$, and $G_1 = 3^1 - 2^1 = 1$. Therefore the base case holds.

Inductive step: Let $n \geq 2$ and that $P(k)$ is true for all $0 \leq k \leq n$. Thus, by the definition of the series, we have:

$$\begin{aligned}
G_{n+1} &= 5G_n - 6G_{n-1} \\
&= 5(3^n - 2^n) - 6(3^{n-1} - 2^{n-1}) \\
&= 5 \cdot 3^n - 5 \cdot 2^n - 2 \cdot 3^n + 3 \cdot 2^n \\
&= 3^{n+1} - 2^{n+1}
\end{aligned}$$

We have shown that $P(n)$ implies $P(n+1)$ and therefore the original statement is true by induction. □

Problem 6

Part (a)

Proof. No, a row move swaps a cell with a blank between the i-th and $(i+1)$-st position, which does not change its order relative to any other cell. Since no other cell changes position, the relative ordering of the current board state is preserved. □

Part (b)

Proof. A column move results in a change in the relative ordering of exactly three pairs of tiles. Sliding the i-th tile up puts the moved tile behind the $(i-1)$-st, $(i-2)$-nd, and $(i-3)$-rd tiles in relative ordering on the board. Sliding the i-th tile down puts the moved tile ahead of the $(i+1)$-st, $(i+2)$-nd, and $(i+3)$-rd tiles. Either way, the relative order of the moved tile changes relative order with exactly three tiles. □

Part (c)

Proof. By PART (A), there are no changes in the relative order of any tile on a row move. Therefore, the parity of the number of inversions does not change. □

Part (d)

Proof. By PART (B), a column move changes the relative order of three pairs of tiles. Since subtracting an odd number from any other number flips the parity of the resulting number from the original number, a column move flips the parity of the number of inversions either from odd to even, or vice versa. □

Part (e)

Proof. The proof is by induction on the number of moves, n. Let $P(n)$ be the proposition that after n moves, the parity of the number of inversions is different from the parity of the row containing the blank tile.

Base case: After zero moves, tiles N and O are out of order, therefore the number of inversions is one - an odd number. After zero moves, the blank square remains in row 4 - an even number. Therefore the base case holds.

Inductive step: We assume that $P(n)$ is true, that after n moves, the parity of the number of inversions is different from the parity of the row containing the blank tile. To prove that $P(n)$ implies $P(n+1)$, we consider the two moves possible.

For a row move, the parity of the number of inversions does not change as proven in PART (C). The row number of the blank tile also does not change in a row move, thereby keeping the same parity. Therefore if the $(n+1)$-st step is a column move, the parities remain different and $P(n+1)$ is true.

For a column move, the parity of the number of inversions is flipped, as proven in PART (D). The row of the blank tile also changes parity since it either slides up by one row, or down by one row. If the $(n+1)$-st step is a column move, the parities remain different and $P(n+1)$ is true.

In both cases, $P(n+1)$ is true for all $n \geq 0$, therefore the original statement is true by the principle of induction. □

Part (f)

Proof. The target state has an even parity of the number of inversions - zero. The blank tile is in row 4, which is also of even parity. From the lemma in PART (D), the target state is unreachable. $\square$

Problem 7

Proof. We prove that there are always at least as many B-lings as Z-lings; the claim that the number of Z-lings is always at most twice the number of B-lings is weaker and thus follows.

The proof is by induction on the generation number. Let $P(n)$ be the proposition that there are at least as many B-lings as Z-lings in generation n.

Base case: $P(1)$ is true because there are 800 B-lings and only 200 Z-lings in the first generation.

Inductive step: For $n \geq 1$ assume that there are at least as many B-lings as Z-lings in generation n to prove that there are at least as many B-lings as Z-lings in generation $n+1$. Let a be the number of Z-lings in generation n, and let b be the number of B-lings. Note that $b \geq a$ by the inductive hypothesis. Then there are no Z-Z pairs formed, there are a Z-B pairs formed, and there are $\lfloor \frac{b-a}{2} \rfloor$ B-B pairs formed. As a result, the number of Z-lings in generation $n+1$ is $a + \lfloor \frac{b-a}{2} \rfloor$, and the number of B-lings is $a + 2\lfloor \frac{b-a}{2} \rfloor$. Therefore, there are at least as many B-lings as Z-lings in generation $n+1$. Thus, for all $n \geq 1$, $P(n)$ implies $P(n+1)$ and the claim is proved by induction. $\square$