A SURVEY OF MORSE-SMALE SYSTEMS

HAI NGUYEN AU

Advisor: Professor Slobodan Simić

Committee Members: Professors Richard Kubelka and Samih Obaid

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Abstract. This paper presents a brief survey of Morse-Smale systems. We focus on continuous-time dynamical systems, i.e., flows defined by smooth vector fields. A dynamical system is called Morse-Smale if: (1) it has finitely many equilibria and closed orbits, all hyperbolic; (2) the stable and unstable manifolds of the equilibria and closed orbits intersect transversely; (3) the nonwandering set consists only of equilibria and closed orbits. We provide the background material necessary to state the main properties of Morse-Smale systems; namely, that they form an open set in the space of all dynamical systems, that they are structurally stable, and, in dimension two, that they are also dense. Furthermore, on compact surfaces, a system is structurally stable if and only if it is Morse-Smale. We do not provide proofs of these results but instead focus on some specific but illustrative cases. In particular, by an explicit construction we show how to approximate an irrational flow on the torus by a Morse-Smale system.

1. Introduction

The field of dynamical systems, which originated from classical Newtonian mechanics in the beginning of the 20^{th} century, was first developed by Henri Poincaré (1854-1912). Before Poincaré, it was commonly believed that the Solar System was stable. Showing the stability of the Solar System was Poincaré's main interest at the time. Since the Solar System is modeled by differential equations defined by Newton's law of gravitation and his second law of motion, it is very complicated. As a result, Poincaré discovered what is called "chaotic" behavior. Since he could not rely on finding explicit solutions to these differential equations, Poincaré shifted his focus to studying the qualitative behavior of these systems. The methods he developed later became the field of qualitative analysis that we know today. In his plenary address at the 1908 International Congress of Mathematicians, Poincaré said:

> In the past, an equation was only considered to be solved when one had expressed the solution with the aid of a finite number of known functions; but this is hardly possible one time in a hundred. What we can always do, or rather what we should always try to do, is to solve the qualitative problem so to speak, that is, to try to find the general form of the curve representing the unknown function.

Instead of trying to find the solutions of these differential equations, Poincaré suggested that we should focus on the following questions:

- How many equilibrium solutions are there and what is their type?
- What is the limit of a "typical" solution as t goes to infinity?
- Is the given equation qualitatively the same as a simpler equation?

With this motivation in mind, Poincaré concentrated on analyzing the orbits, or trajectories, of these solutions. Later, mathematicians such as Aleksandr Lyapunov (1857-1918), George Birkhoff (1884-1944) and many others made important progress in the field. Lyapunov was famous for his studies of the stability of dynamical systems, which he developed in 1899. Determining the stability of an equilibrium point is straightforward if the point is hyperbolic. But in the case of non-hyperbolic equilibria, this could be problematic. Lyapunov's method helped to generalize the determination of the asymptotic behavior of these equilibria. His introduction of the *basin of attraction* of a stable equilibrium point was an important step toward addressing the stability of dynamical systems [4]. George Birkhoff, who proved Poincaré's "Last Geometric Theorem," a special case of the three-body problem, Stephen Smale, who developed the *Smale horseshoe*, and Oleksandr Sharkovsky, who developed *Sharkovsky's Theorem* for discrete dynamical systems, all made significant contributions to the field. Many mathematicians to this day are aiming toward finding dynamical systems that are simple, yet have significant applications to our everyday life. It should be noted that in the early days, without the help of computers, studying dynamical systems through the solutions of differential equations and their orbits was challenging. Therefore, sophisticated mathematical methods were required and not many classes of dynamical systems were discovered.

Later, Ivan Kupka and Stephen Smale proposed the concept of Kupka-Smale vector fields. A field of this type has only hyperbolic critical elements (i.e., equilibria and closed orbits) whose stable and unstable manifolds intersect transversely. This concept was refined even more when Smale applied Morse Theory to introduce the new Morse-Smale vector fields. Besides being a Kupka-Smale field, a Morse-Smale field has only a finite number of critical elements, and the set of all nonwandering points must be equal to the set of all critical elements. The focus of this project is on Morse-Smale vector fields in two-dimensional compact manifolds M^2 such as the sphere S^2 and the torus T^2. We will provide numerous examples of Morse-Smale as well as non-Morse-Smale fields. We will show that Morse-Smale fields are structurally stable and dense. These distinguishing characteristics of Morse-Smale systems have made them a field of study with many applications. For instance, Smale used Morse-Smale systems to generalize Poincaré's conjecture by decomposing a manifold M into the stable manifolds of the equilibria and closed orbits of the gradient flow of a Morse function f. In higher dimensions, there are other systems that are more significant than Morse-Smale systems. Nevertheless, in dimension two, Morse-Smale systems have been proven to have many useful characteristics that other systems do not have.

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2. Preliminaries

2.1. **Topology and Geometry.** Since we will discuss how dynamical systems work on manifolds, we need to define what manifolds are. The concept of a manifold is based on that of a topological space.

2.1. **Definition.** *A **topological space** is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a collection of subsets of X such that:*

- *$\emptyset, X$ are in $\mathcal{T}$*
- *The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.*
- *The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.*

*$\mathcal{T}$ is called the **topology** of the topological space $(X, \mathcal{T})$. A subset U of X is called an **open set** if $U \in \mathcal{T}$.*

Next, we will describe the properties of some special topological spaces.

2.2. **Definition.** *A topological space is called **Hausdorff** if any two distinct points are contained in disjoint open sets.*

Most important topological spaces are Hausdorff. The fact that in a Hausdorff topological space every two distinct points are contained in disjoint open sets implies the uniqueness of limits of sequences. There are also some spaces that are not Hausdorff, although the notion of "non-Hausdorff" is quite counterintuitive. Nevertheless, non-Hausdorff spaces can be useful in other branches of topology – for example, spaces with the Zariski topology in algebraic geometry [2].

2.3. **Definition.** *A topological space is* ***compact*** *if every open cover has a finite subcover.*

In this definition, a cover of a set X is a collection of sets whose union contains X. We can think of a subcover as a refined cover, which still contains X but has fewer elements than a cover. Some examples of compact spaces include a closed interval, a finite set of points, or a closed disc in two dimensions.

2.4. **Definition.** *Let $(X, \mathcal{T})$ be a topological space. A set of open sets B is called a* ***basis*** *for the topology if every open set is a union of sets in B.*

In other words, any open set can be constructed from some elements of B. For example, open balls in $\mathbb{R}^n$ (or open discs in $\mathbb{R}^2$) form a basis for the topology in $\mathbb{R}^n$. This is because we can take a union of open balls to form any given open set. Also, since rational numbers are countable, if the open balls in a basis have rational radii and rational center coordinates, we have an example of a **countable basis** for the topology of $\mathbb{R}^n$.

2.5. **Definition.** *A topological space is* ***second countable*** *if it has a countable basis for its topology.*

2.6. **Definition.** *A topological space is* ***locally Euclidean*** *of dimension n if for every point x in X, x has a neighborhood U homeomorphic to an open set $\hat{U}$ in $\mathbb{R}^n$.*

In other words, a locally Euclidean space locally looks like $\mathbb{R}^n$, even though globally it may be quite different from it.

2.7. **Definition.** *A* ***topological manifold*** *is a topological space M such that:*

- *M is Hausdorff,*
- *M is second countable,*
- *M is locally Euclidean.*

Of course since all Euclidean spaces $\mathbb{R}^n$ satisfy the conditions above, they are topological manifolds. Those manifolds will be our main focus in this project.

2.8. **Definition.** *Given a map* $f : \mathbb{R}^m \to \mathbb{R}^n$*, then* f *is* ***differentiable at point*** x_* *if there exists a linear transformation* $A : \mathbb{R}^m \to \mathbb{R}^n$ *such that:*

$$f(x) = f(x_*) + A(x - x_*) + R(x),$$

where $R(x)$ *is the remainder and*

$$\lim_{x \to x_*} \frac{R(x)}{|x - x_*|} = 0,$$

where $|.|$ *denotes the Euclidean norm.*

The definition above explains the concept of the differentiability of a map at a point in the regular calculus sense. This is applicable in Euclidean spaces, but can also be generalized to manifolds.

2.9. **Definition.** *A map* $F : U \to \mathbb{R}^n$*, where* U *is an open set in* $\mathbb{R}^n$*, is called* ***smooth*** *if it has partial derivatives of all orders at every point in* U*.*

If x is a point in $U \subset \mathbb{R}^n$, then $F(x) = (F_1(x), ..., F_n(x))$ is a point in $\mathbb{R}^n$, and F is smooth if for all $k \geq 0$ where $k = \alpha_1 + ... + \alpha_n = |\alpha| = |(\alpha_1, ..., \alpha_n)| \geq 0$, the partial derivatives of F can be defined by

$$\frac{\partial^k F_i}{\partial {x_1}^{\alpha_1} ... \partial {x_n}^{\alpha_n}}$$

as in regular multivariable calculus. In smooth manifolds other than $\mathbb{R}^n$, smooth maps are defined slightly differently by using coordinate charts.

2.10. **Definition.** *Let* $f : U \to \mathbb{R}$ *be a smooth function on an open set* $U \subset \mathbb{R}^n$*. The* C^r***-norm*** *of* f *is defined by*

$$\|f\|_{C^r} = \max\{\|f\|, \left\|f^1\right\|, ..., \left\|f^{(r)}\right\|\},$$

where $\|f\|, \|f^1\|, ..., \left\|f^{(r)}\right\|$ *denote the uniform norms of the derivatives of* f*.*

2.11. **Definition.** *A* ***coordinate chart*** *for a topological manifold* M *of dimension* n *is a pair* (U, φ)*, where* U *is an open subset of* M *and* $\varphi : U \to \hat{U}$ *is a homeomorphism from* U *to an open subset* $\hat{U} = \varphi(U) \subseteq \mathbb{R}^n$ *(FIGURE 1).*

By using coordinate charts, maps between manifolds can be defined more easily since we can always relate them to regular calculus definitions of maps in Euclidean spaces. If φ and ψ are two charts of a manifold M of dimension m, then the map $\psi \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^m$ is called a **transition map**. This map is useful because now we can use many types of calculus operations on it (FIGURE 2) [1]. Two charts are called **smoothly compatible** if the transition map is smooth. We call a topological manifold **smooth** if all its charts are smoothly compatible.

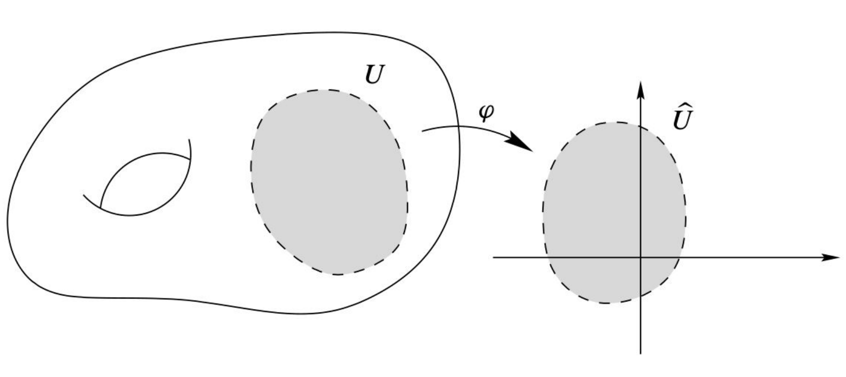

FIGURE 1. A Coordinate Chart of a Topological Manifold.

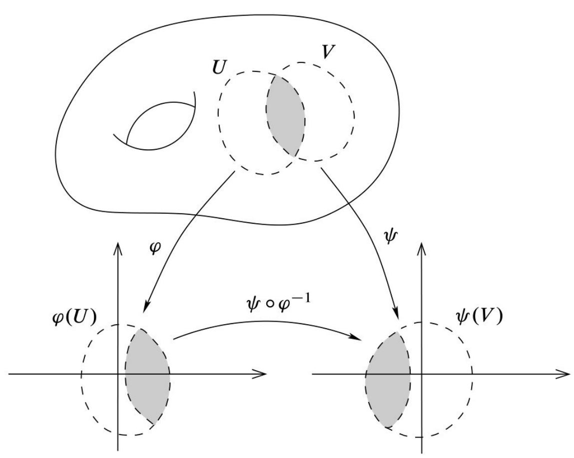

FIGURE 2. A Transition Map of a Topological Manifold.

Similarly, if $F : M \to N$ is a map between two manifolds, we define smoothness of F as follows.

[1]Figures 1 and 2 by courtesy of [3]

2.12. **Definition.** *The map $F : M \to N$ is **smooth** if the map $\psi \circ F \circ \varphi^{-1}$ is smooth in the regular calculus sense for all charts (U, φ) for M and (V, ψ) for N such that $F(U) \subset V$.*

2.13. **Definition.** *A bijective map $F : M \to N$ is a **diffeomorphism** if both F and F^{-1} are smooth.*

Note that smooth maps are continuous [3]. Some trivial examples of smooth maps include constant maps and identity maps. As a nontrivial example, take the *antipodal map*. This is the map $A : S^n \to S^n$ that takes a point $p \in S^n$ and reflects it through the origin to a point $-p \in S^n$.

We will now proceed to discuss the concept of tangent vectors. A **tangent vector** at a point $p \in \mathbb{R}^n$ can be thought of a vector anchored at p. The set of all such vectors is call the **tangent space** of $\mathbb{R}^n$ at p. Tangent vectors of a smooth manifold M at a point $p \in M$ can also be defined. We call v a tangent vector to M at p if v is a **derivation** at p, i.e., v is a linear map from $C^\infty(M)$ to $\mathbb{R}$ and satisfies the Leibniz rule:

$$v(fg) = v(f)g(p) + f(p)v(g),$$

for all $f, g \in C^\infty(M)$. The tangent space at $p \in M$ is denoted as T_pM, and the set of all tangent spaces at all points on M is called the **tangent bundle** TM.

2.14. **Definition.** *A **vector field** of class C^r on a manifold M is a C^r map $X : M \to TM$ that takes a point p on M to a vector $X_p \in T_pM$. The set of all C^r vector fields in M is written as $\mathfrak{X}^r(M)$.*

Vector fields are important in dynamical systems since they generate flows on manifolds. Consider the torus T^2; since $T^2 = S^1 \times S^1$, we can think of one unit circle as horizontal and the other as vertical. Note that we can define a vector field at a point on S^1 as $\frac{\partial}{\partial\theta}$ for some angle function θ. Thus, we get an example of two linearly independent, non-vanishing vector fields $\frac{\partial}{\partial\theta_1}$ and $\frac{\partial}{\partial\theta_2}$ on T^2.

2.15. **Definition.** *Let M, N be smooth manifolds and $F : M \to N$ a diffeomorphism. For every vector field $X \in \mathfrak{X}^r(M)$, there exists a unique vector field Y on N such that Y is F**-related** to X. That is, $F_*(X) = Y$ where the map F_* is called the **pushforward** of X [3].*

In particular, a pushforward maps a vector field X on M to a vector field Y on N by mapping each tangent vector X_p to a tangent vector $Y_{F(p)}$ where $p \in M$ and $F(p) \in N$.

2.2. Dynamics.

2.16. **Definition.** *A* ***dynamical system*** *on a manifold M consists of a set of times T and a continuously differentiable function $\phi : T \times M \to M$ such that $\phi(t, x) = \phi_t(x)$ where x is a point in M with the properties:*

(1) $\phi_0(p) = p$.
(2) $\phi_t \circ \phi_s = \phi_{t+s}$, for $t, s \in T$.

ϕ_t is also called a **flow** on M. The first property implies that at the initial time when $t = 0$, the flow starts at the initial point $p \in M$. The second property implies that the composition of flows for two different times is the same as the flow for the combined times. The flow is usually induced by a vector field on M. Namely, each vector field X locally defines a system of ordinary differential equations [4]. If X is smooth, then the system has a unique solution for each initial condition. It can be shown that the family of all solutions defines a flow.

As the simplest example, consider the first-order differential equation $\dot{x} = ax$. Solving this equation, we have the solution $\phi_t(x_0) = x_0 e^{at}$. This solution defines a flow for all times $t \in \mathbb{R}$. More generally, for a system of differential equations

$$\dot{X} = F(X)$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$, if any solution of this system is a function $\gamma : I \to \mathbb{R}^n$ where $I \subset \mathbb{R}$ is an interval then $\gamma(t)$ is a smooth curve in $\mathbb{R}^n$. In general, if M is a compact manifold and $X \in \mathfrak{X}^r(M)$ is vector field, then there exists a global C^r flow for X on M [4]. This flow $\phi : \mathbb{R} \times M \to M$ has the property:

$$\frac{\partial}{\partial t}\phi_t(p) = X(\phi_t(p))$$

for all $t \in \mathbb{R}$ and $p \in M$.

2.17. **Definition.** *The* ***orbit*** *of a flow ϕ at a point p is the set*

$$\mathcal{O}(p) = \{\phi_t(p) : t \in \mathbb{R}\}.$$

2.18. **Definition.** *Two flows ϕ and ψ on M are called* ***topologically conjugate*** *if there exists a homeomorphism $h : M \to M$ such that*

$$h(\phi_t(p)) = \psi_t(h(p))$$

for all $t \in T$ and $p \in M$.

In other words, h takes orbits of the flow ϕ to orbits of the flow ψ. If h preserves the time parameter, then h is called a **topological conjugacy**. Otherwise, h is called a **topological equivalence**. This issue will be discussed in more details later.

Next, we will discuss the idea of approximating vector fields on a compact manifold M. To determine if two vector fields are close to each other, we will need a way to measure the distance between them.

Let X be a vector field on a compact manifold M. By compactness, we can cover M by a finite set of coordinate charts $U_1, U_2, ..., U_k$. The vector field X can now be written in each chart $U_i \subset \mathbb{R}^n$ by

$$X = \sum_{j=1}^{n} X_j \frac{\partial}{\partial x_j}.$$

On each U_i, we define:

$$\|X\|_{C^r, U_i} = \max \|X_j\|_{C^r}.$$

Taking the maximum value of those $\|X\|_{C^r, U_i}$, we can define that value as the C^r**-norm** $\|X\|_{C^r}$ of X. This norm defines the C^r-topology on $\mathfrak{X}^r(M)$. We say that a vector field Y is ε-close to X in the C^r-topology if

$$\|Y - X\|_{C^r} < \varepsilon.$$

Now that we have defined dynamical systems on manifolds, we are ready to discuss their qualitative properties.

2.19. **Definition.** *A point $p \in M$ is called a* ***singularity*** *of a vector field X if*

$$X_p = 0.$$

A singlularity is a **sink** if nearby orbits tend toward it, is a **source** if nearby orbits tend away from it, and is a **saddle** if nearby orbits tend toward it in one direction and away from it in the other direction.

2.20. **Definition.** *Given a vector field X on M, an ω-**limit set** of a point $p \in M$, denoted as $\omega(p)$, is the set of points $q \in M$ such that $X^{t_n}(p) \to q$, for some sequence $t_n \to \infty$, where $\{X^t\}$ is the flow of X.*

*Similarly, an α-**limit set** of a point $p \in M$, $\alpha(p)$ is the set of points $q \in M$ such that $X^{t_n}(p) \to q$, for some sequence $t_n \to -\infty$.*

One can think of an orbit of a vector field on M as being born in an α-limit set and dying in an ω-limit set. These limit sets can be points or closed orbits.

2.1. **Example.** Consider the sphere $S^2 \subset \mathbb{R}^3$, where p_N and p_S are the North Pole and South Pole respectively (FIGURE 3). Let X be a vector field on S^2 such that X is tangent to the meridian and pointing upward from p_S towards p_N. Continuity of X requires that $X(p_N) = 0$ and $X(p_S) = 0$. Thus, p_N and p_S are the singularities of X. Since orbits that start from the singularities p_N and p_S always stay where they are, $\omega(p_N) = \alpha(p_N) = p_N$ and $\omega(p_S) = \alpha(p_S) = p_S$. For all points $p \in S^2$ that are not singularities, X will take them to p_N as $t \to \infty$, and to p_S as $t \to -\infty$. Therefore, $\omega(p) = p_N$, and $\alpha(p) = p_S$ for these points.

2.21. **Definition.** *The **critical elements** of a vector field X are all the singularities and closed orbits of X. A critical element x is an **attractor** (or **sink**) if there exists a neighborhood V of x such that for all points $p \in V$, $\omega(p) = x$. If $\alpha(p) = x$, we say x is a **repellor** (or **source**).*

2.22. **Definition.** *A singularity p of $X \in \mathfrak{X}^r(M)$ is called **hyperbolic** if D_pX has eigenvalues with non-zero real parts, where D_pX is the derivative of X at p [3].*

2.23. **Definition.** *Let γ be a closed orbit of a vector field X. At a point $x_0 \in \gamma$, let Σ be a cross section of γ (i.e., an arc that crosses γ at a non-zero angle) that is transverse to X. If V is a neighborhood of x_0 in Σ that is small enough, we can define the **Poincaré map** $P : V \to \Sigma$ that takes a point $x \in V$ to a point in Σ defined as follows: the first point*

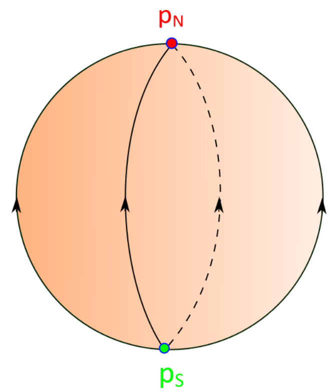

FIGURE 3. The North Pole - South Pole Vector Field.

*where the orbit of X intersects the cross section Σ is called the **first return point** $P(x)$. If $P(x) = x$, we call x a **fixed point** of γ (FIGURE 4).*

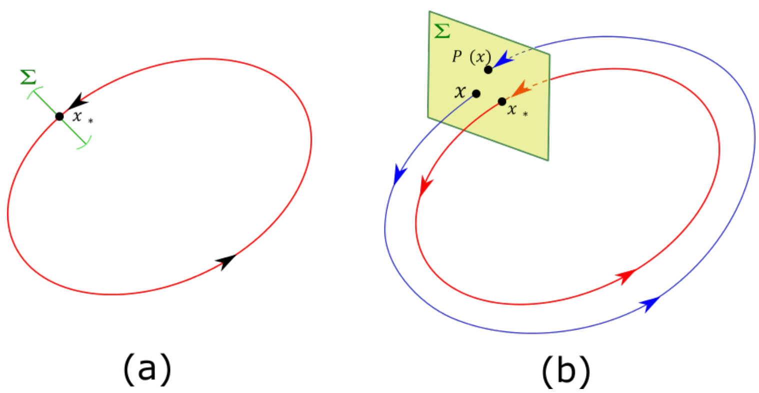

FIGURE 4. The Poincare maps in (a) dimension 2 and (b) dimension 3.

2.24. **Definition.** *If $p \in M$ is a fixed point of a diffeomorphism f of M, then p is a **hyperbolic fixed point** if the derivative matrix $D_pF : T_pM \to T_pM$ has no eigenvalue of modulus 1.*

Hyperbolic singularities and hyperbolic fixed points play an important role in establishing local stability of dynamical systems [5]. If γ is a closed orbit of X and Σ is a cross section transverse to X, we say that γ is a **hyperbolic closed orbit** of X if $x \in \gamma$ is a hyperbolic fixed point of the Poincaré map $P : V \subset \Sigma \to \Sigma$ (see FIGURE 4). The Poincaré map will also be used extensively in the proofs of this type of stability.

2.25. **Definition.** *Let p be a hyperbolic fixed point of M. The **stable manifold** $W^s(p)$ of p is the set of all points that have p as their ω-limit. The **unstable manifold** $W^u(p)$ of p is the set of all point that have p as their α-limit.*

$$W^s(p) = \{q \in M | \omega(q) = p\}$$

$$W^u(p) = \{q \in M | \alpha(q) = p\}$$

$W^s(p)$ and $W^u(p)$ are **immersed submanifolds** of M[3]. Again, let us consider the North Pole - South Pole vector field above. Since all points on S^2 except p_S have their ω-limit set as p_N, the stable manifold of p_N is $W^s(p_N) = S^2 - \{p_S\}$. p_N is the only point that has α-limit set as p_N, so the unstable manifold $W^u(p_N) = \{p_N\}$. Similarly for p_S, we have $W^s(p_S) = \{p_S\}$ and $W^u(p_S) = S^2 - \{p_N\}$.

2.26. **Definition.** *Let U and V be immersed submanifolds of M [3]. U is **transverse** to V if for all points $p \in U \cap V$, the tangent spaces T_pU and T_pV span T_pM.*

The simplest example of transversality is the x-axis and y-axis in $\mathbb{R}^2$. The intersection of these two submanifolds is the origin, and the two tangent spaces at the origin span the entire $\mathbb{R}^2$ plane.

Theorem 1 (Poincaré-Bendixson)**.** *Let X be a vector field on the sphere S^2 with a finite number of singularities, and let p be a point on the sphere. Then exactly one of the following is true:*

(a) $\omega(p)$ is a singularity;
(b) $\omega(p)$ is a closed orbit;
(c) $\omega(p)$ consists of singularities $p_1, ..., p_n$ and regular orbits such that if an orbit $\gamma \subset \omega(p)$, then $\alpha(\gamma) = p_i$ and $\omega(\gamma) = p_j$ for some i, j between 1 and n.

The proof of the Poincaré-Bendixson is discussed in [5]. Since S^2 is the simplest example of a compact manifold of dimension two, we would expect the ω and α-limit sets might behave nicely. We already discussed the case when $\omega(p)$ is a singularity in the North Pole - South Pole vector field. $\omega(p)$ can also be a closed orbit that all orbits nearby are attracted to as we will see in the next section. As an example of case (c), if we have three singularities and orbits that form a triangle as in FIGURE 5, then they are the ω-limit for all points p in the interior of the triangle. Let γ be any orbit in $\omega(p)$, then γ is a side of the triangle. It is clear from the picture that the ω- or α-limit of γ is a singularity.

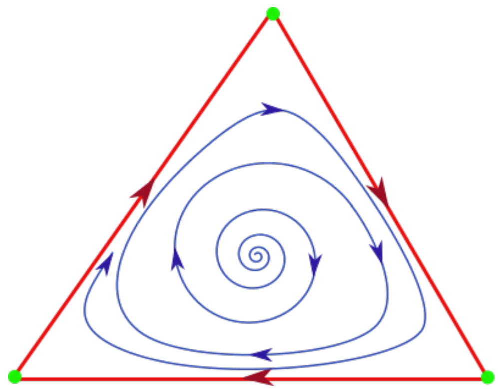

FIGURE 5. An Example of the Third Case of the Poincaré-Bendixson Theorem.

Since we will focus on some vector fields on S^2, the Poincaré-Bendixson theorem will become important for the proofs of some of the main results that come later.

2.27. **Definition.** *A vector field X is a **Kupka-Smale** vector field if:*

(a) All critical elements (singularities and closed orbits)of X are hyperbolic;

(b) For any pair critical elements σ_1 and σ_2 of X, the invariant manifolds $W^s(\sigma_1)$ and $W^u(\sigma_2)$ are transverse to each other.

This definition of Kupka-Smale vector fields is the crucial precursor to that of Morse-Smale vector fields. Examples of Kupka-Smale fields will be given when we are ready to compare between these two systems.

Theorem 2. *Kupka-Smale vector fields are dense in $\mathfrak{X}^r(M)$. In other words, for all $X \in \mathfrak{X}^r(M)$ and for each $\varepsilon > 0$, there exists a Kupka-Smale vector field $Y \in \mathfrak{X}^r(M)$ such that $\|Y - X\|_{C^r} < \varepsilon$.*

The proof of this theorem can be found in [5].

2.28. **Definition.** *Define a vector field X on $\mathbb{R}^2$ by*

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = \alpha \in \mathbb{R} \end{cases}$$

*We say X induces **rational flow** on $\mathbb{R}^2$ if $\alpha \in \mathbb{Q}$, and X induces **irrational flow** on $\mathbb{R}^2$ if $\alpha \notin \mathbb{Q}$. These flows project to flows on T^2, also called the rational and irrational flows on T^2.*

To illustrate the rational and irrational flows on T^2, we consider the map $\gamma : \mathbb{R} \to T^2$ that represents an orbit of this vector field:

$$\gamma(t) = (e^{2\pi it}, e^{2\pi i\alpha t}).$$

Note that $\gamma(0) = (1,1)$, and $\dot{\gamma}(t) = X_{\gamma(t)}$, so γ is the orbit of X starting at $(1,1) \in T^2$ (where we think of T^2 as $S^1 \times S^1$). If $\alpha = \frac{p}{q}$ is rational (assuming p, q are relatively prime integers), then all the orbits of X are closed. We can see that $\gamma(t)$ has the period of q since

$$\gamma(q) = (e^{2\pi iq}, e^{2\pi i\frac{p}{q}q}) = (1, e^{2\pi ip}) = (1,1) = \gamma(0).$$

It implies that after a period of q, every orbit comes back to its starting point, and is therefore closed.

If α is irrational, then every orbit of the flow induced by X is dense on T^2. Note that the orbits of this vector field on $\mathbb{R}^2$ are parallel lines with slopes of α. Thus, it suffices to show that the set the y-intercepts of these lines are dense in $\mathbb{R}$. Since a point $(e^{2\pi it}, e^{2\pi i\alpha t})$ in T^2 can be lifted to a point (x,y) in $\mathbb{R}^2$, it follows that $(x,y) = (t+m, \alpha t + n)$ for some $m, n \in \mathbb{Z}$. If the point (x,y) is a y-intercept, then $t + m = 0$ and these y-intercepts are in the form of $(0, n - \alpha m)$. To show that the set of $n - \alpha m$ is dense in $\mathbb{R}$, we will show that for arbitrary numbers $c \in \mathbb{R}$ and $\varepsilon > 0$, we can find two integers p and q such that $|c - (\alpha n - m)| < \varepsilon$. By **Dirichlet's Approximation Theorem** [3], we can find integers p and q such that $1 \le p \le N$ and

$$|q - \alpha p| < \frac{1}{N}$$

where N is an integer and $\frac{1}{N} < \varepsilon$. Then the number $\frac{c}{q - \alpha p}$ can be squeezed in between two consecutive integers k and $k+1$

$$k \le \frac{c}{q - \alpha p} < k + 1$$

$$\Rightarrow k(q-\alpha p) \leq c < (k+1)(q-\alpha p) \text{ (assumming } 0 < q-\alpha p)$$

$$\Rightarrow 0 \leq c - k(q-\alpha p) < q - \alpha p$$

$$\Rightarrow |c - k(q-\alpha p)| < |q-\alpha p| < \frac{1}{N} < \varepsilon.$$

Let $n = kq$ and $m = kp$, then we have found integers m, n such that c is close to $\alpha n - m$. Thus, the set of y-intercepts is dense and the flow of an irrational flow field on T^2 is dense.

3. Morse-Smale Vector Fields

We will investigate Morse-Smale vector fields — extensions of Kupka-Smale vector fields. First, we will define Morse-Smale vector fields, then we will show that Morse-Smale vector fields are structurally stable and dense on any differentiable manifold of dimension two.

Before we can define Morse-Smale vector fields, we will look at some examples of different types of vector fields in some common M^2 manifolds such as spheres and tori.

3.1. **Example.** In this example, we will look at X as the gradient vector field generated by the height function h on the torus T^2 as shown in FIGURE 6. The vector field X has four

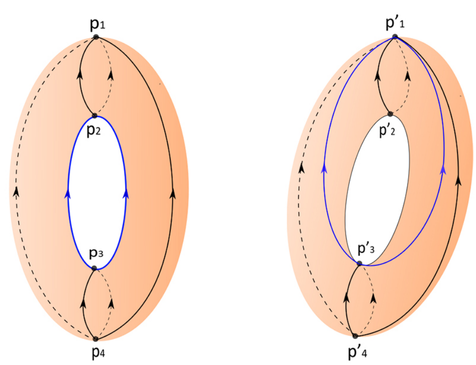

Figure 6. Gradient Vector Fields on Upright Torus and Tilted Torus

singularities: a source p_4, a sink p_1 and two saddles p_2 and p_3. The blue orbit represents both the stable manifold of p_2 and the unstable manifold of p_3. Since they are coincident, they do not intersect each other transversely, and they create an orbit connecting two saddles (a saddle connection). As a result, X fails to be a Kupka-Smale vector field. However, a small perturbation can eliminate this saddle connection and change X into a Kupka-Smale vector field X'. By tilting the torus slightly, the new singularities p_1', p_2', p_3' and p_4' will be very close to p_1, p_2, p_3 and p_4. However, we no longer have the old saddle connection as the unstable manifold of p_3' now flows to the sink p_1' (the red orbit). Thus, X' is now a Kupka-Smale field.

Morse-Smale vector fields will form a subclass of Kupka-Smale fields.

Before we look at the next example of a Kupka-Smale field, we will need a lemma about vector fields that induce rational and irrational flows.

3.2. **Proposition.** *A vector field that induces an irrational flow can be approximated by a vector field that induces a rational flow.*

Proof. Let X_α be a vector field given by the differential equations

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = \alpha \end{cases}$$

where α is an irrational number. Thus, X_α induces an irrational flow. From real analysis, for every $\epsilon > 0$, we can always find a rational number β such that if $|\alpha - \beta| < \delta$ for some $\delta > 0$, then $\|X_\alpha - X_\beta\| < \varepsilon$. So the vector field X_β defined by

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = \beta \end{cases}$$

induces an rational field and is an approximation of X_α. □

3.3. **Example.** Consider the vector field X that induces an irrational flow on the torus T^2 (FIGURE 7). As mentioned before, every orbit of the irrational flow is dense in T^2 and therefore, its ω-limit set is the entire torus. Since there are no critical elements (singularities or closed orbits) on the irrational flow, the field X does not have any element to fail the conditions of a Kupka-Smale vector field, and therefore is Kupka-Smale. Since the irrational flow is dense in the torus, the ω- and α-limit sets of the field are the entire torus.

However, being Kupka-Smale does not guarantee that X is structurally stable. By the previous Proposition, we can always approximate this irrational flow field by a rational flow field. Since an irrational flow field has no closed orbits whereas a rational flow field has only closed orbits, the irrational flow field is not structurally stable.

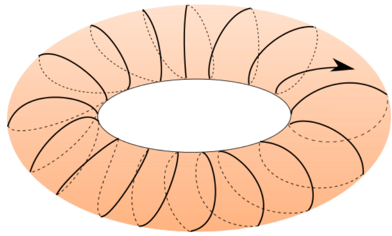

FIGURE 7. The vector field from an irrational flow on T^2

3.1. **Definition.** *Let $X \in \mathfrak{X}^r(M)$. $L_\alpha(X)$ is the union of all α-limit sets of X and $L_\omega(X)$ is the union of all ω-limit sets of X.*

3.2. **Definition.** *A point p in M is called a* ***wandering point*** *for a vector field X if there exists a neighborhood V of p and a number $t_0 > 0$ such that $X_t(V) \cap V = \emptyset$ for $|t| > t_0$, where X_t is the flow of X. Otherwise, p is called a* ***nonwandering point****.*

The set of all nonwandering points of X is denoted by $\Omega(X)$. It can be shown that $\Omega(X)$ contains the critical elements of X, is compact and invariant under the flow of X. Given two vector fields X and Y in $\mathfrak{X}^r(M)$ and a topological equivalence $h : M \to M$, then $h(\Omega(X)) = \Omega(Y)$.

Now we are ready to define Morse-Smale vector fields.

3.3. **Definition.** *A vector field $X \in \mathfrak{X}^r(M)$ on a compact manifold M is a* ***Morse-Smale vector field*** *if:*

(1) There are only a finite number of critical elements (singularities and closed orbits) on X and they are all hyperbolic.

(2) For any two critical elements σ_1 and σ_2 of X, their stable and unstable manifolds are transverse to each other

(3) $\Omega(X)$ is the union of all the critical elements of X.

3.4. **Proposition.** *A vector field $X \in \mathfrak{X}^r(M)$ on a compact manifold M is Morse Smale if and only if:*

(1) There are only a finite number of critical elements (singularities and closed orbits) on X and they are all hyperbolic.

(2) There are no saddle-connections.

(3) *Each orbit on M has a unique critical element as its ω-limit and a unique critical element as its α-limit.*

We will show that the three conditions of the definition of Morse-Smale fields are equivalent to the three conditions of the proposition when M is of dimension two.

Proof. The proof of this proposition is given in [5].

($\Rightarrow$) Suppose that a vector field X is Morse-Smale.

(1) X has a finite number of hyperbolic critical elements by definition.
(2) Suppose there is a saddle-connection between σ_1 and σ_2 in X (FIGURE 8). This orbit is both an unstable manifold of σ_1 ($W^u(\sigma_1)$) and a stable manifold of σ_2 ($W^s(\sigma_2)$). It follows that these two manifolds are not transverse to each other, making X fail to be Morse-Smale, a contradiction. Therefore, X cannot have any saddle-connections.

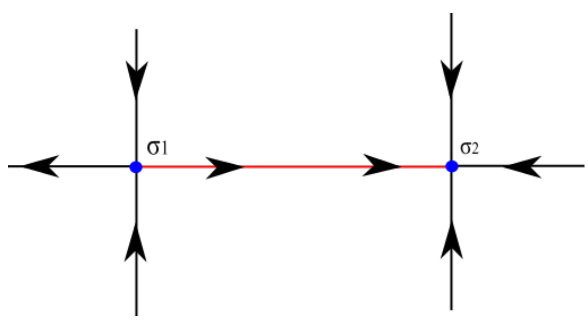

FIGURE 8. A saddle-connection in two dimension

(3) Since M is compact, for any increasing sequence of times t, the flow for that sequence of times starting at a point $p \in M$ must have a convergent subsequence. The limit of this subsequence, by definition, is the ω-limit of p. If there are two separate ω-limits for any orbit on M, then these two ω-limits are connected by an orbit [5]. Therefore, they are both saddles and we have a saddle-connection, which is not allowed. Thus, each orbit has a unique ω-limit. Similarly for the α-limit.

($\Rightarrow$) Now we will prove that if X satisfies all three conditions of the proposition, then X is Morse-Smale.

(1) This is one of our assumptions.

(2) In two dimensions, stable and unstable manifolds of two critical elements σ_1 and σ_2 are not transverse to each other only when they are connected saddles. Since there are no saddle-connections, it follows that $W^s(\sigma_1)$ is transverse to $W^u(\sigma_2)$ for all critical elements σ_1 and σ_2.

(3) We want to show that the set of nonwandering points $\Omega(X)$ is the set of all critical elements of X. Equivalently, we can prove that the set of points that are not critical elements are wandering.

In the case of a sink, if it is a singularity p, then there exists a disc D such that $D \subset W^s(p)$, $p \in D$ and the boundary of D is transverse to X. This is because near p, X is conjugate to its linearization $A = D_pX$ and a D exists for A. We will show that all the points on this boundary are wandering. First, we will look at two discs D_1 and D_{-1}, one inside and one outside of the boundary of D. In particular, since the boundary of D is transverse to the vector field X, we can define D_1 and D_1 such that $D_1 = X_1(D)$ and $D_{-1} = X_{-1}(D)$. This means at the time $t = -1$, the orbits of X intersect the boundary of D_{-1}, at the time $t = 0$, the orbits intersect the boundary of D, and at the time $t = 1$, the orbits intersect the boundary of D_1. Now let x be a point on the boundary C of D and consider a neighborhood V of x such that V is disjoint from D_1 and $M - D_{-1}$ (FIGURE 9a). Since for $|t| > 2$, the orbit through x will leave its neighborhood V to enter the disc D_1, and $X_t(V) \cap V = \emptyset$. Thus, x is a wandering point. Since x is an arbitrary point on the boundary of D, it follows that all points on the boundary of D are wandering. Also, because the set of wandering points is invariant under the flow, and $W^s(p) - \{p\} = \bigcup_{t\in\mathbb{R}} X_t(C)$, the stable manifold of p consists of all wandering points except p. Similarly, if the sink is a closed orbit γ, we can also find a neighborhood of γ such that its boundary is transverse to X. This neighborhood is no longer a disc, but is homeomorphic to an annulus if γ is orientable, and is homeomorphic to a Möbius strip if γ is not orientable [5].

If the critical element is a source, we can use the same argument to prove that all points x on the boundary of D are wandering.(FIGURE 9b)

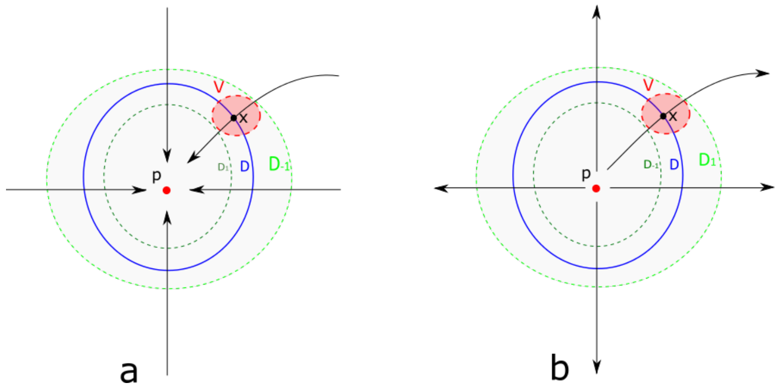

FIGURE 9. A wandering point x on the boundary of a disc D that leaves a neighborhood V in the case when a: point p is a sink; b: point p is a source.

Finally in the case when p is a saddle, then for any point x that is neither a singularity nor on a closed orbit, its ω-limit or α-limit cannot be a saddle since there are no saddle-connections. Therefore, either one of these limit sets must be a source (or closed orbit repellor). Then by the same argument earlier, this point x must be wandering.

Thus, a point x is wandering if it is not a critical element, and the set $\Omega(X)$ of nonwandering points consists of only critical elements.

□

3.4. **Definition.** *Let X be a Morse-Smale vector field. The* ***phase diagram*** *Γ of X is the set of critical elements of X with the partial order:*
For critical elements σ_1, $\sigma_2 \in \Gamma$, $\sigma_1 \leq \sigma_2$ if $W^u(\sigma_1) \cap W^s(\sigma_2) \neq \emptyset$. In other words, $\sigma_1 \leq \sigma_2$ if there exists an orbit that starts near σ_1 and ends at σ_2 [5].

Below are a few examples of some Morse-Smale vector fields.

3.5. **Example.** As the simplest example, we will revisit the North Pole-South Pole field on S^2 (FIGURE 10). As we recall, this vector field has two hyperbolic singularities, an attractor p_N and a repellor p_S. For any point that is not the North Pole or South Pole, its orbit moves away from p_S and toward p_N. In other words, the ω-limit set of the field is p_N and the α-limit set of the field is p_S. This vector field is indeed Morse-Smale. It has

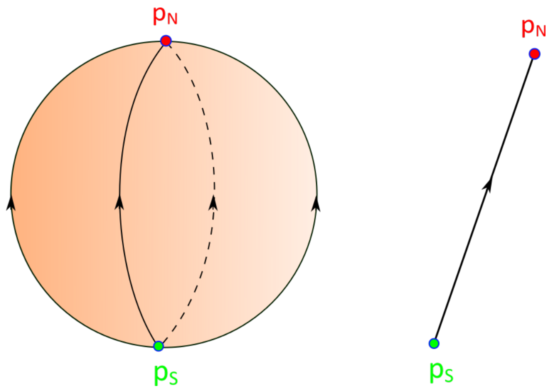

FIGURE 10. The North Pole - South Pole vector field on S^2 and its phase diagram

two hyperbolic critical elements p_N and p_S. There are no saddles, so there are no saddle connections in the field. All orbits of this vector field on the sphere have p_N as their ω-limit and p_S as their α-limit. The phase diagram shows that all orbits start at p_S and end at p_N.

3.6. **Example.** In this example, we consider X to be a vector field on S^2 with hyperbolic repellors p_N and p_S, and an attractor closed orbit γ as the equator of the sphere (FIGURE 11). The phase diagram shows three critical elements. Any orbit starting at the sources p_N and p_S will converge to the attracting closed orbit γ. There are three critical elements in X, so the number is finite. Once again, there are no saddles. For all orbits on the upper hemisphere, their α-limit set is the point p_N and their ω-limit set is the closed orbit γ. Similarly, all orbits on the lower hemisphere have p_S as their α-limit and γ as their ω-limit. Thus, this is a Morse-Smale vector field.

3.7. **Example.** Another Morse-Smale vector field on S^2 is a vector field with two hyperbolic attractors p_1 and p_2, two hyperbolic repellors r_1 and r_2, and two hyperbolic saddles s_1 and s_2 (FIGURE 12). The set of nonwandering points for this vector field is the set of singularities $\Omega(X) = \{p_1, p_2, r_1, r_2, s_1, s_2\}$ Its phase diagram describes the flow direction between the sources, sinks and saddles of the field. Note that there are orbits connecting different singularities, but there is no saddle-connection between s_1 and s_2. Sinks p_1 and p_2 are the ω-limits for all orbits on the upper hemisphere and lower hemisphere respectively (excluding points on the equator). Sources r_1 and r_2 are the α-limits for all orbits that do not contain

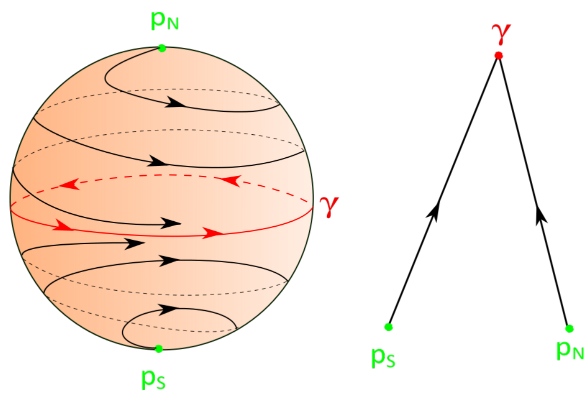

FIGURE 11. The vector field with repellor singularities and an attractor closed orbit on S^2

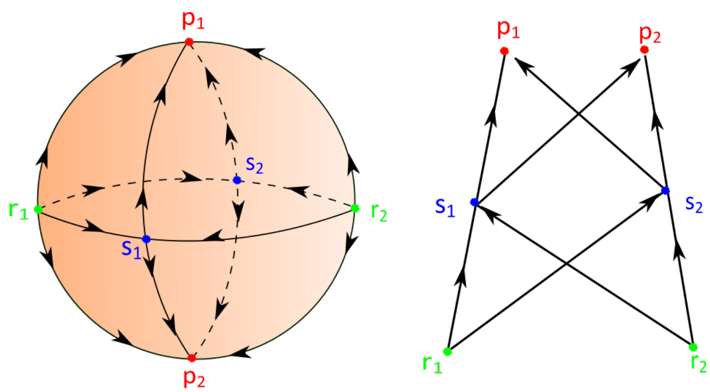

FIGURE 12. The vector field with two attractors, two repellors and two saddles on S^2 and its phase diagram.

s_1 or s_2. Saddles s_1 and s_2 are the ω-limits for the orbits on the equator and are the α-limits for the orbits connecting s_1 and s_2 to p_1 and p_2.

3.8. **Example.** As discussed before, we consider a vector field X induced by the gradient vector field from the height function on a tilted torus T^2. In that example, we have destroyed the saddle connection by slightly tilting the torus. Therefore, the transversality condition is satisfied. Since there are only four hyperbolic critical elements in X, the vector field is Morse-Smale. (FIGURE 13) Again in the phase diagram, there is no connection between two saddles p'_2 and p'_3. All orbits have either p'_2, p'_3 or p'_4 as the α-limit, and p'_1 as the ω-limit.

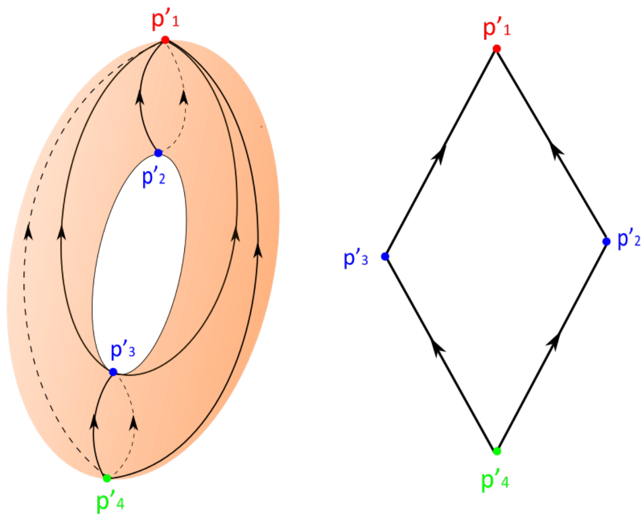

FIGURE 13. The gradient vector field on a tilted torus T^2 and its phase diagram.

3.9. **Example.** In another example of a Morse-Smale vector field on a torus T^2, we will consider a vector field with two hyperbolic closed orbits γ_1 and γ_2, which are an attractor and a repellor respectively (FIGURE 14). Its phase diagram is similar to the North Pole-South Pole vector field discussed earlier since there are also one attractor and one repellor, but in this field these are closed orbits rather than singularities. Here we have γ_1 as the α-limit and γ_2 as the ω-limit for all orbits.

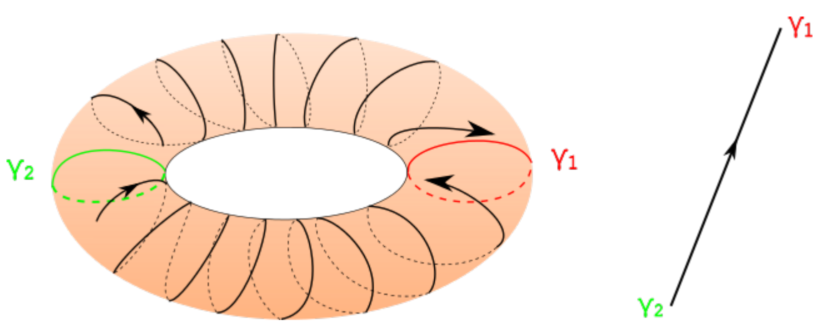

FIGURE 14. The vector field with repelling and attracting closed orbits on T^2 and its phase diagram.

It is also useful to discuss some examples of non-Morse-Smale fields. If a vector field violates any of the three conditions of the Morse-Smale field, it is not a Morse-Smale field.

We will give examples of vector fields that have infinitely many critical elements, a saddle-connection, or some orbits without an ω or α- limit. For the first example, we consider a vector field on S^2 that has two singularities at p_1 and p_2 and all other orbits are closed (FIGURE 15). Since the number of closed orbits (critical elements) is infinite, this vector field fails to be Morse-Smale.

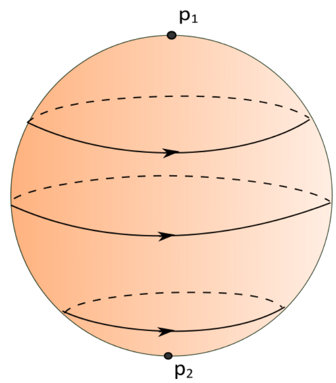

FIGURE 15. A non-Morse-Smale vector field with infinitely many critical elements

As we mentioned earlier, the gradient vector field on a upright torus has a saddle-connection, therefore is not Kupka-Smale or Morse-Smale. Finally, a vector field induced by the irrational flow on a torus fails the third condition because the orbits are dense in T^2 and do not have any α- or ω-limits.

4. Structural Stability of Morse-Smale Systems

In this section, we will look into an important property of Morse-Smale systems: structural stability.

4.1. **Definition.** *A vector field $X \in \mathfrak{X}^r(M)$ is* ***structually stable*** *if there exists a C^r neighborhood U of X such that for every vector field Y in U, Y is topologically equivalent to X.*

In other words, X is called structurally stable if the qualitative behavior of X (i.e., its flow) is stable under small perturbations. In order to show that two vector fields X and Y are topologically equivalent to each other, we need to construct a homeomorphism that takes orbits of X to orbits of Y. We recall that such a homeomorphism is a topological equivalence between X and Y. Note that a topological equivalence does not need to preserve the time parameter. But if it does, then it is a topological conjugacy, as we discussed earlier.

As an example of a structurally stable dynamical system, we consider a vector field defined by the system of differential equations in $\mathbb{R}^2$:

$$\begin{cases} \dot{x} = x \\ \dot{y} = -y. \end{cases}$$

This vector field has a hyperbolic saddle at the origin. It can be shown that this field is structurally stable using the Grobman-Hartman Theorem [5].

Theorem 3. *Given a Morse-Smale vector field X on M, there exists a neighborhood U of X such that for every vector field $Y \in U$, Y is Morse-Smale and its phase diagram is isomorphic to that of X.*

The proof of this theorem can be found in [5]. By this theorem, Morse-Smale fields form an open set. The field Y has all the same characteristics as the field X does, including the type and number of critical elements. This theorem is very important and will be used to show that every Morse-Smale vector field is structurally stable. Since small perturbations of a vector field give rise to a new Morse-Smale vector field with the same phase diagram, structural stability of Morse-Smale systems seems plausible. Indeed:

Theorem 4. *If X is a Morse-Smale vector field, then X is structurally stable.*

Proof. We will prove the theorem for the simplest case when X is the North Pole-South Pole vector field with a source at p_S and a sink at p_N. By the Theorem 3, there exists a neighborhood $U \in \mathfrak{X}^r(M)$ of X such that for all $Y \in U$, Y is Morse-Smale. The second part of the theorem also tells us that there exists an isomorphism $\sigma : \Gamma_1 \to \Gamma_2$ where Γ_1 and Γ_2 are the phase diagrams of X and Y. For the rest of the proof, we will call p_S $\sigma_1(X)$ and p_N $\sigma_2(X)$. Since the phase diagrams of X and Y are isomorphic, there are also a corresponding source $\sigma_1(Y)$ and a corresponding sink $\sigma_2(Y)$ for Y. We will now construct a homeomorphism $h : M \to M$ that takes orbits of X to orbits of Y.

First, let h map $\sigma_1(X)$ and $\sigma_2(X)$ to $\sigma_1(Y)$ and $\sigma_2(Y)$, respectively. Let Σ be the equator of the sphere S^2 and define $h(p) = p$, for all $p \in \Sigma$. Now, for a point p on the lower hemisphere that is not a singularity and not on the equator Σ, there exists a unique time $t > 0$ such that $\psi_t(p) \in \Sigma$. After a time t, this orbit will take p to a unique point $\pi(p)$ on the equator Σ. Now, to get the corresponding point induced by the flow of Y, we can take the point $\pi(p)$ and apply the flow by the vector field Y in $-t$ time. In particular, if ψ_t is the flow by X and φ_t is the flow by Y, then $\pi(p) = \psi_t(p)$ and

$$h(p) = \varphi_{-t}(\psi_t(p)).$$

(See FIGURE 16.) Since all flows are smooth and have smooth inverses, the map h that we

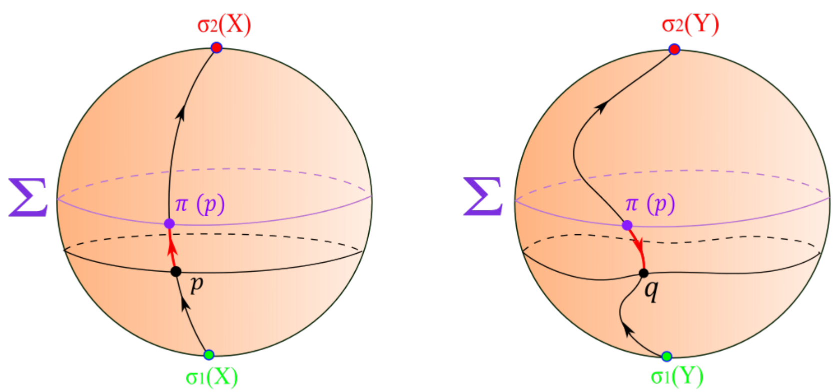

FIGURE 16. The topological equivalence between two Morse-Smale vector fields X and Y

have just constructed can be shown to be a topological equivalence between X an Y. □

5. Density of Morse-Smale Systems

So far we have shown that Morse-Smale vector fields are structurally stable under small perturbations. However, if Morse-Smale systems are scarce in M^2, studying them would not be very exciting. In this section, we will show that Morse-Smale systems form a dense subset in M^2. We will prove the claim on the sphere S^2. It is also possible to generalize to show that Morse-Smale systems are dense in any compact manifold of dimension two.

We will begin with an important proposition about singularities that will be useful for later proofs.

5.1. **Lemma.** *All hyperbolic singularities are isolated. That is, if p is a hyperbolic singularity, then there exists a neighborhood U of p that contains no other hyperbolic singularities other than p.*

Proof. Since the Lemma is local, it is enough to prove it for $M = \mathbb{R}^n$. Suppose x_* is a hyperbolic singularity of F, then $F(x_*) = 0$ and $\det DF(x_*) \neq 0$. We assume that x_* is not isolated. Thus, there exists a sequence of hyperbolic singularities $\{x_k\}$ that converges to x_*.

Let $A = \det DF(x_*)$ be the derivative matrix of F at (x_*); then by the definition

$$F(x) = F(x_*) + A(x - x_*) + o(\|x - x_*\|),$$

where $\lim\limits_{x \to x_*} \frac{o(\|x-x_*\|)}{\|x-x_*\|} = 0$. Thus, since $F(x_*) = F(x_k) = 0$,

$$F(x_k) = F(x_*) + A(x_k - x_*) + o(\|x_k - x_*\|)$$

$$\Rightarrow A(x_k - x_*) + o(\|x_k - x_*\|) = 0$$
$$\Rightarrow \frac{A(x_k - x_*)}{\|x_k - x_*\|} \to 0$$

as $k \to \infty$. Let the v_k be the unit vector $\frac{x_k - x_*}{\|x_k - x_*\|}$, then we have $Av_k = 0$. Since $\{v_k\}$ is an infinite sequence of unit vectors in a compact manifold S^{n-1}, it has a subsequence $\{v_{k_i}\}$ that converges to some unit vector v_*. Now since

$$0 = \lim_{i \to \infty} Av_{k_i} = A(\lim_{i \to \infty} v_{k_i}) = Av_*,$$

it follows that $Av_* = 0$. We know that A is invertible since $\det A \neq 0$. Therefore, v_* must be the 0 vector. This contradicts the assumption that v_* is a unit vector. So there cannot be a sequence of singularities x_k that converges to x_*, and x_* is therefore isolated.

□

The result of this lemma is not limited to hyperbolic singularities, but can be extended to hyperbolic closed orbits.

5.2. **Lemma.** *Hyperbolic closed orbits are isolated.*

Proof. Let X be a vector field and γ be a hyperbolic closed orbit in a compact manifold. Let Σ be a cross-section of γ that is transverse to X. Since γ is a closed orbit, the point of intersection x_* between γ and Σ is a fixed point of the Poincaré map $P : \Sigma \to \Sigma$. To show that γ is isolated, we will show that there are no other fixed points of P near x_*. In a compact manifold of dimension two, the hypersurface Σ is of dimension one, and therefore can be thought of as an interval on the real number line. So in this case, P can be viewed as a real function from $\mathbb{R}$ to $\mathbb{R}$. Since γ is hyperbolic, we then have $P(x) = x$ and $P'(x) \neq 0$. Suppose that there infinitely many fixed points that are arbitrarily close to x_*, then there exists a sequence of fixed points on Σ that converges to x_*. Analogously as in the proof of the previous lemma, this sequence cannot exist and can conclude that x_* is isolated. Therefore, hyperbolic closed orbits are isolated. □

We now claim:

Theorem 5. *Every Kupka-Smale field on S^2 is Morse-Smale.*

Proof. We need to show that every Kupka-Smale field X on S^2 satisfies all three conditions of Proposition 3.4, i.e.:

(1) There are finitely many critical elements (singularities or closed orbits) and they are all hyperbolic.
(2) There are no saddle-connections
(3) Each orbit has a unique critical element as its ω-limit and a unique critical element as its α-limit.

By the Poincaré-Bendixon theorem, an ω-limit set of any point on S^2 must be either a singularity, a closed orbit, or a set of saddles and saddle connections. Since there cannot be any saddle-connection in a Kupka-Smale field, an ω-limit set of any point must be either a singularity or a closed orbit, and so the third condition is satisfied. The second condition is also satisfied since in Kupka-Smale fields, for every pair of critical elements, their stable and unstable manifolds are transverse to each other. Since all critical elements in Kupka-Smale fields are hyperbolic, we only need to show that there are finitely many of them and the theorem will be proven. Since critical elements consist of singularities and closed orbits, we will show each case separately.

First, let us assume that X has infinitely many singularities. Then by Bolzano-Weierstrass theorem, there exists an infinite subsequence of singularities $\{x_n\}$ that converges to a hyperbolic singularity x in S^2. However, since all hyperbolic singularities in a compact manifold are isolated, $\{x_n\}$ does not exist. Therefore we have a contradiction, and there cannot be infinitely many hyperbolic singularities.

Next, we will consider all the hyperbolic closed orbits of X. Again, we assume that there are infinitely many of them. Then there exists a sequence $\{x_1, ..., x_n, ...\}$, with each point x_i lying on a unique closed orbit, which converges to some point x on the sphere. By the Poincaré-Bendixon theorem, $\omega(x)$ must be either a singularity or a closed orbit. In the case when $\omega(x)$ is a singularity, it can be a sink, a source, or a saddle. The cases when $\omega(x)$ is a sink or a source can be ruled out because there cannot be any sequence of closed orbits that are arbitrarily close to an attracting or repelling singularity. Thus, $\omega(x)$ may only be a saddle. Now, if $\omega(x)$ is a saddle, then $\alpha(x)$ is also a saddle for the same reason. This causes x itself to be a saddle because otherwise its orbit will be a saddle-connection. In fact, $x = \omega(x) = \alpha(x)$ is a saddle. Again, since there are no saddle-connections, the unstable manifold of x must go to a sink. If we look at the fate of the distinct closed orbits that $x_1, ..., x_n, ...$ are on, they must go to that sink and we will no longer have those closed orbits. Thus, we have a contradiction. In the case when $\omega(x)$ is a closed orbit, there cannot be a sequence of points lying on distinct closed orbits that converges to it since $\omega(x)$ is isolated. Thus, there can only be a finite number of hyperbolic closed orbits in X, and it follows that X is a Morse-Smale vector field. $\square$

Because Kupka-Smale vector fields are dense in S^2, it follows that Morse-Smale vector fields are also dense in S^2. The result can be extended to any compact manifold M^2.

6. Approximating the Irrational Flow on a Torus by a Morse-Smale Flow

Since Morse-Smale systems are dense in M^2, they can be used to approximate a non-Morse-Smale vector field. In this section, we will look at one example of how to do exactly that with the rational and irrational flows on a torus T^2. The idea is to find a way to approximate a rational flow on T^2 by a Morse-Smale vector field. Then since every irrational flow can be approximated by a rational flow, it follows that every irrational flow can be approximated by a Morse-Smale vector field by first approximating it by a rational flow.

6.1. **Approximating a Simple Rational Flow.** As in topology, we can think of a torus T^2 as a square on a Cartesian coordinate plane with each pair of opposite sides identified. We consider a rational flow induced by a vector field X on T^2 that has infinitely many closed orbits (FIGURE 17). This vector field is not Morse-Smale since there are infinitely many critical elements and these closed orbits are not hyperbolic (they neither attract nor repel). These orbits of the vector field X can be represented by vertical parallel orbits on the square. The question is whether we can approximate this non-Morse-Smale field by a Morse-Smale field. Since X has no singularities, we would like to use a Morse-Smale field with no singularities. One vector field that fits these criteria perfectly is the one in Example 3.9; let us call it Y. This vector field has two closed orbits: an attractor γ_1 and a repellor γ_2. Orbits in the vector field are born in γ_2, spiral along the torus and die in γ_1. We can then represent the orbits of Y in the Cartesian coordinate plane. The attractor γ_1 can be identified by the vertical line in the middle, and the repellor γ_2 can be identified by two vertical edges of the square. Orbits that start from either γ_1 or γ_2 should stay inside these critical elements, whereas orbits that start from points other than those on γ_1 and γ_2 are attracted to γ_1 (FIGURE 14).

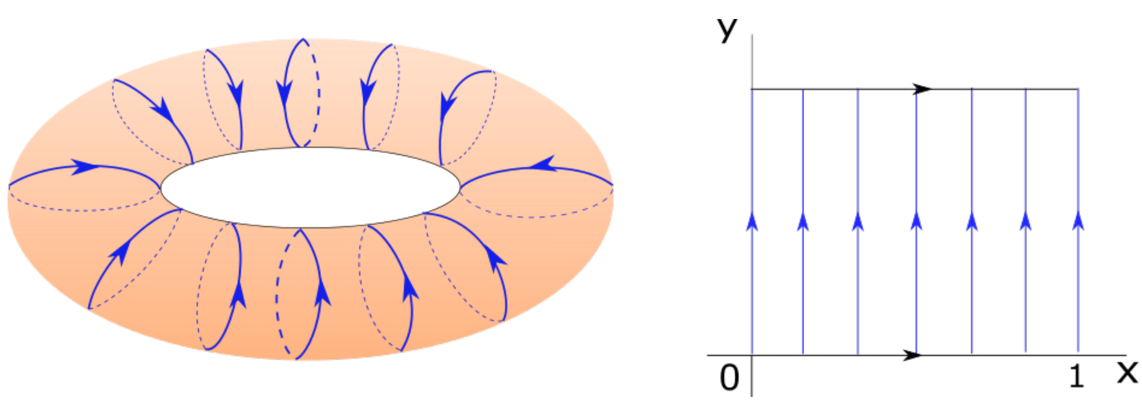

FIGURE 17. A non-Morse-Smale vector field X with infinitely many non-hyperbolic closed orbits on T^2 and its phase portrait

Now we will try to algebraically approximate the vector field X using Y. Since the orbits of X consist of vertical lines/circles and have only the y-component, we can represent X as

$$X = \frac{\partial}{\partial y}.$$

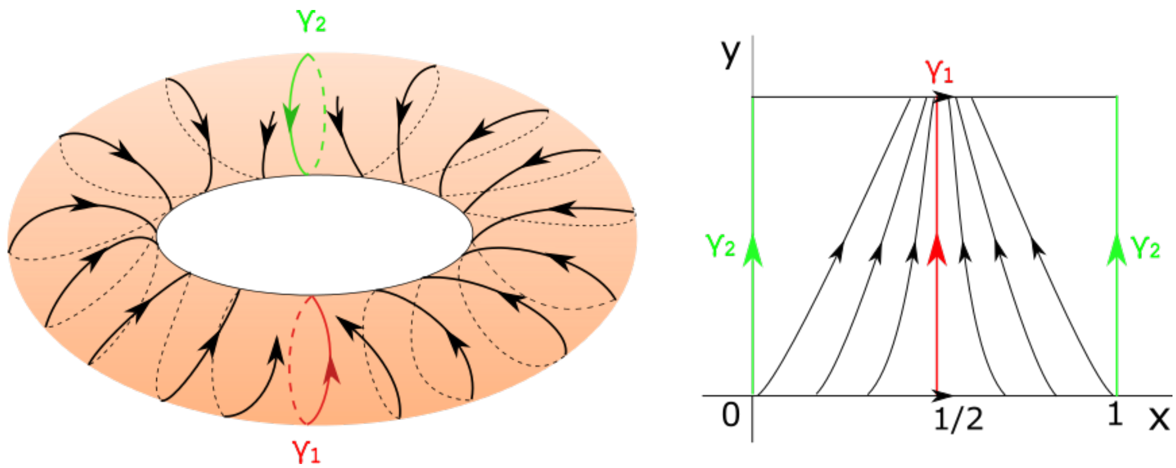

FIGURE 18. A Morse-Smale vector field Y with two hyperbolic closed orbits on T^2 and its phase portrait. γ_1 is an attractor and γ_2 is a repellor.

To introduce a small perturbation to convert X into Y, surely we want to have a new x-component so that the orbits can be tilted toward the attractor γ_1. So, the perturbation must have a non-constant component of $\frac{\partial}{\partial x}$ to account for the moving in the x direction. Also, since we would like to keep the y-component constant, the perturbation must have a constant $\frac{\partial}{\partial y}$ component. Let us call this perturbation X_ε, and

$$X_\varepsilon = \frac{\partial}{\partial y} + f_\varepsilon(x)\frac{\partial}{\partial x},$$

where $f_\varepsilon(x)$ is an ε-dependent function ($\varepsilon > 0$) such that $\|f_\varepsilon\|$ and $\|f'_\varepsilon\|$ are small. If we can choose ε to be arbitrarily small, we can then approximate Y by this vector field X_ε because $|f_\varepsilon(x)|$ and $|f'_\varepsilon(x)|$ can be as as small as we choose.

We will now proceed to find the function $f_\varepsilon(x)$. We want this function to be differentiable everywhere. In order to make the phase portrait, we will require $f_\varepsilon(x)$ to have certain properties. First, when $x = 0, \frac{1}{2}$, or 1, those are the positions of the two closed orbits γ_1 and γ_2. We need $f_\varepsilon(x) = 0$ for $x = 0, \frac{1}{2}$ and 1 in order to cancel out the effect of the x-component and make the flow stay inside those closed orbits. In the phase portrait, these orbits are vertical. Next, in the interval $(0, \frac{1}{2})$, we need $f_\varepsilon(x) > 0$ to have a positive effect on the x-component so that the orbits are tilted to the right toward γ_1. Similarly, we require

$f_\varepsilon(x) < 0$ in the interval $(\frac{1}{2}, 1)$ so there is negative effect on the orbits and push them to the left toward γ_1. The graph of $f_\varepsilon(x)$ should look similar to that in FIGURE 19.

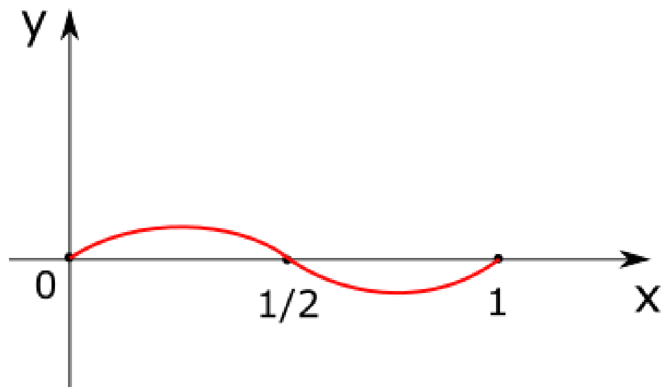

FIGURE 19. The graph of the ε-dependent function $f_\varepsilon(x)$.

There are several ways to have a function with those properties. One way is to use a sine function. If we let

$$f_\varepsilon(x) = \varepsilon \sin(2\pi x)$$

for $x \in [0, 1]$, then $f_\varepsilon(x)$ meets all the criteria described above. Since the norm of this function is less than 1, it is true that $|f_\varepsilon(x)| < \varepsilon$ and $|f'_\varepsilon(x)|$ is small. Using this function $f_\varepsilon(x)$ to plug into X_ϵ, then the difference

$$\|X_\varepsilon - X\|_{C^1} = \left\| f_\varepsilon(x) \frac{\partial}{\partial x} \right\|$$

is arbitrarily small depending on the magnitude of ε. Therefore, we have a vector field that is capable of approximating X using Y.

Now we need to show that the approximation that results in this phase portrait in FIGURE 18 is in fact the original vector field on T^2. In other words, the question here is whether the orbits actually are born in γ_2 and die in γ_1. Again, the Poincaré map becomes handy in this situation. We can consider the cross-section Σ that is the equator of the torus T^2. This cross-section Σ intersect all orbits on T^2. Let $P : \Sigma \to \Sigma$ be the return map of a point p on this vector field. For every point $x \in \Sigma$, the first return map will follow the orbit starting from x to a point $P^1(p) \in \Sigma$. In the phase portrait, the point $P^1(p)$ is identified by the point on the next orbit closer to the attractor γ_1. Thus, if we continue to iterate the next return maps $P^2(p), P^3(p), ..$, they will generate points on Σ that will eventually converge to $\Sigma \cap \gamma_1$. This is true for all points p on T^2 that do not belong to the two closed orbits. If we only look at the x-coordinates of these points, then

$$P^n(x) \to \frac{1}{2},$$

as $n \to +\infty$ for $x \in [0,1] - \{0, \frac{1}{2}, 1\}$. As a result, γ_1 is an attractor and γ_2 is a repellor. Consequently, X_ε is a good Morse-Smale approximation of this rational flow on T^2.

6.2. **Approximating a General Rational Flow.** In general, a rational flow on T^2 may not be simple like the previous example. Fortunately, there is a way to to reduce any rational flow to the one in this example. Any vector field that induces a rational flow on T^2 has an infinite number of closed orbits. Visually, if we can take each closed orbit and convert it into a closed orbit as in the last example, then the two vector fields are topologically equivalent, and the remaining task to approximate that field by a Morse-Smale field becomes simple.

The rational flow in the previous example can be written in $\mathbb{R}^2$ as the system of differential equations:

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 1 \end{cases}$$

since all orbits are vertical. On the other hand, a general rational flow in $\mathbb{R}^2$ can be written as:

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = \alpha \end{cases}$$

where $\alpha = \frac{m}{n}$ and $m, n \in \mathbb{Z}$ and are relatively prime. This vector field in $\mathbb{R}^2$ can be represented in the unit square by linear orbits of slope α. We can write these two vector fields on T^2 as $X = \frac{\partial}{\partial y}$ and $X_\alpha = \frac{\partial}{\partial x} + \alpha\frac{\partial}{\partial y}$ respectively. Both vector fields have infinitely many non-hyperbolic closed orbits on T^2. We define the projection map $\pi : \mathbb{R}^2 \to T^2$ as a quotient map that takes a point $p \in \mathbb{R}^2$ to its equivalence class in T^2. This equivalence class is consists of all vectors that differ by an integer vector. Each orbit of X has period of 1. Each orbit of X_α has period of n, and each orbit goes around the torus m times. Now, let us consider the matrix:

$$A = \begin{bmatrix} a & n \\ b & m \end{bmatrix},$$

where a and b are integers such that $am - bn = 1$. They exist since $\gcd(m, n) = 1$. Since $\det A = am - bn = 1$, A is invertible. Furthermore,

$$AX = \begin{bmatrix} a & n \\ b & m \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix} = n \begin{bmatrix} 1 \\ \frac{m}{n} \end{bmatrix} = nX_\alpha$$

Note that for any vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$ where $x, y \in \mathbb{Z}$, we have

$$Av = \begin{bmatrix} a & n \\ b & m \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + ny \\ bx + my \end{bmatrix},$$

which also has integer coordinates. Therefore, A maps any integer vector to another integer vector. Similarly, the matrix A^{-1} also takes integer vectors to integer vectors.

$$A^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \tfrac{1}{\det A} \begin{bmatrix} m & -n \\ -b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m & -n \\ -b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} mx - ny \\ -bx + ay \end{bmatrix}$$

Define a map $f : T^2 \to T^2$ by $f(\pi(p)) = \pi(Ap)$ (FIGURE 20). We claim that f is a diffeomorphism which maps orbits of X to orbits of X_α.

Note that we will slightly abuse the notations and use "X" and "X_α" to denote both the vector fields and their lifts to $\mathbb{R}^2$. To show that f is well-defined, note that that a point $\pi(p) \in T^2$ is an equivalence class of $p \in \mathbb{R}^2$, and two points are in $\mathbb{R}^2$ are of the same equivalence class if they differ by an integer vector. If two equivalence classes $\pi(p)$ and $\pi(q)$ of p and q in $\mathbb{R}^2$ respectively are equal, that is $\pi(p) = \pi(q)$, then p and q differ by an integer vector:

$$q = p + v,$$

for some integer vector $v \in \mathbb{Z}^2$. Applying the linear transformation A to both sides of this equation, we have

$$Aq = Ap + Av.$$

Since A takes integer vectors to integer vectors, Av must be an integer vector. That means the vectors Aq and Av differ by an integer vector, hence they belong to the same equivalence class. So $\pi(Ap) = \pi(Aq)$, and f is therefore well-defined. Since A is invertible and A^{-1} also takes integer vectors to integer vectors, we can define $f^{-1} : T^2 \to T^2$ similarly by projecting the map A^{-1} and show that f is a bijection. Finally, we know that that since A and A^{-1} are smooth, f and f^{-1} are unique and smooth by the Passing Smoothly to the Quotient theorem [3]. Hence, f is a diffeomorphism that takes orbits of X to orbits of nX_α. In fact, the orbits of X is the same of the orbits of X_α after being reparametrized, as we will see next.

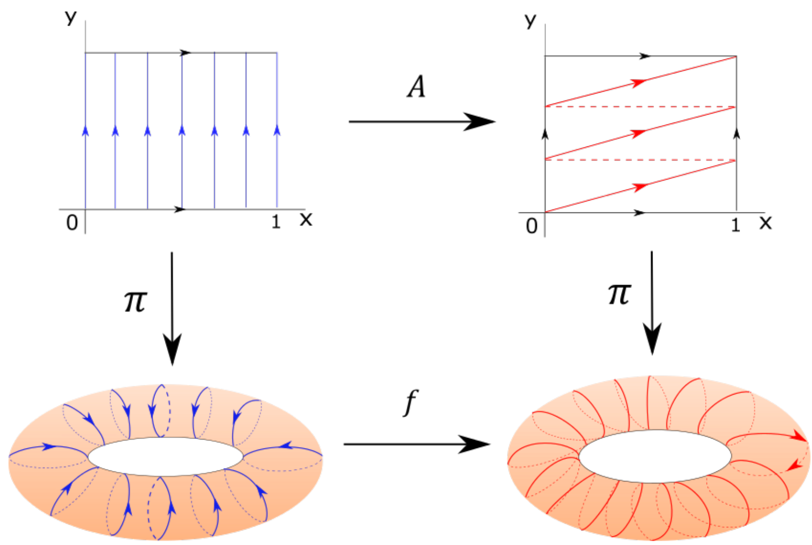

FIGURE 20. The maps between the nice rational flow X and the general rational flow X_α.

6.1. **Proposition.** *Let φ_t and ψ_t be the flows of X and Y on M respectively and $f : M \to M$ a diffeomorphism, then*

$$f(\varphi_t(p)) = \psi_t(f(p))$$

for all $p \in M$, i.e., f is a topological conjugacy.

Proof. Let $\gamma(t) = f(\varphi_t(p))$ be the image of the flow φ_t taken by f starting at p. Using chain rule to take the derivative of $\gamma(t)$, we have

$$\dot{\gamma}(t) = Df(\dot{\varphi}_t(p))$$

Since $\dot{\varphi}_t(p)$ is the derivative of the flow of X staring at the point p, it is equal to $X_{\varphi_t(p)}$. Furthermore, by the assumption that $f_*(X) = Df(X) = Y$, it follows that

$$\dot{\gamma}(t) = Df(X_{\varphi_t(p)}) = Y_{f(\varphi_t(p))}.$$

Because the velocity of $\gamma(t)$ equals Y, $\gamma(t)$ itself is an orbit of Y. In other words, f takes the flow of X to a flow of Y. At the time $t = 0$, $\varphi_0(p) = p$, so $\gamma(0) = f(p)$. By uniqueness, it follows that $\gamma(t)$ is the orbit of ψ_t that starts at $f(p)$, or

$$\gamma(t) = f(\varphi_t(p)) = \psi_t(f(p)).$$

□

6.2. **Proposition.** *If ψ_t is the flow of Y, then $\psi_{\lambda t}$ is a flow of λY where $\lambda \in \mathbb{R}$. That is, if $\gamma(t)$ is an orbit of Y, then $\gamma(\lambda t)$ is an orbit of λY.*

Proof. The process of converting Y to λY is called a *reparametrization* of Y. Using chain rule to take the derivative of $\gamma(\lambda t)$, we have

$$\dot{\gamma}(\lambda t) = \lambda Df(\dot{\psi}_t(p)) = \lambda Y$$

□

Since the pushforward f_* of f is the matrix $Df = A$, f_* maps the vector field X to the vector field nX_α and $f_*(X) = nX_\alpha$. Let φ_t and ψ_t^α be the flows of X and X_α respectively. By Proposition 6.2, the flow of nX_α is ψ_{nt}^α. Then by Proposition 6.1, for all $p \in T^2$,

$$f(\varphi_t(p)) = \psi_{nt}^\alpha(f(p)).$$

That means if p is a point on T^2, then f takes p to the point $f(p)$ and takes the orbit of p in X to the orbit of $f(p)$ in X_α. Note that f does not preserve the period t, but the new period is n times the old one. Therefore, f is not a topological conjugacy, but it is a topological equivalence.

Finally, we will find a Morse-Smale approximation of X_α. Let $\varepsilon > 0$ be arbitrary. We showed that there exists a Morse-Smale vector field Y such that

$$\|Y - X\|_{C^1} < \varepsilon.$$

It follows that $f_*(X)$ is C^1-close to $f_*(Y)$,

$$\|f_*(Y) - f_*(X)\|_{C^1} < c\varepsilon,$$

for some number c depending only on m and n. Thus,

$$\left\|\frac{1}{n}f_*(Y) - \frac{1}{n}f_*(X)\right\|_{C^1} < \frac{c}{n}\varepsilon.$$

Since $f_*(X) = nX_\alpha$ and $\frac{1}{n}f_*(X) = X_\alpha$,

$$\left\|\frac{1}{n}f_*(Y) - X_\alpha\right\|_{C^1} < \frac{c}{n}\varepsilon,$$

and the vector field $Z = \frac{1}{n}f_*(Y)$ is C^1-close to X_α. Z is clearly Morse-Smale because $f_*(Y)$ is the push-forward of the Morse-Smale field Y. Thus, Z is a Morse-Smale approximation of the general rational flow field X_α. This shows that given any non-Morse-Smale field that induces a rational flow on T^2, we can always approximate it by a simple rational field, and therefore can approximate this simple field by a Morse-Smale field. On the other hand, given an irrational flow field, we can also approximate it by a simple rational flow field, and again can approximate this simple rational flow field by a Morse-Smale field.

7. Conclusion

In dimension two, Morse-Smale systems are structurally stable and form a dense open set as we have seen in the previous sections. With those qualities, we can always approximate a non-Morse-Smale field by a Morse-Smale field. Although we only showed a few simple examples in S^2 and T^2, some of these properties can be generalized to more complicated cases. For example, the **Closing Lemma** shows that Morse-Smale systems are dense in $\mathfrak{X}^1(M^2)$, whether M is orientable or not [6]. Besides S^2 and T^2, there are also studies that show the same results for the projective plane P^2 and the Klein bottle K^2 [5]. However, in dimension three or higher, Morse-Smale systems are not dense [5]. In these manifolds, there are also structurally stable systems that are not Morse-Smale. Some of these systems have infinitely many closed orbits [5]. This implies that Morse-Smale systems cannot be used to approximate all vector fields in higher dimensions. In discrete dynamical systems, there are also non-Morse-Smale systems that cannot be approximated by Morse-Smale ones. In recent years, Morse-Smale systems have been generalized to a discrete setting and used in the field of topological data analysis [1].

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Department of Mathematics and Statistics, San José State University, San José, CA 95192-0103