A SURVEY OF MORSE-SMALE SYSTEMS

Hai Nguyen Au

Advisor: Professor Slobodan Simić

Committee Members:

Professor Richard Kubelka

Professor Samih Obaid

April 25, 2016

Introduction

- The field of dynamical systems, which originated from classical Newtonian mechanics in the beginning of the 20$^{\text{th}}$ century, was first developed by Henri Poincaré (1854-1912).
- Started by his interest in studying the stability of the Solar System.
- *"What we should always try to do, is to solve the* **qualitative** *problem."*. We should not try to solve differential equations, because that is rarely possible.

- **Henri Poincaré**(1854-1912): analyzed the orbits, or trajectories, of these solutions.
- **Aleksandr Lyapunov**(1857-1918): studied the stability of dynamical systems, generalized the determination of the asymptotic behavior of these equilibria.
- **George Birkhoff**(1884-1944): proved Poincaré's "Last Geometric Theorem," a special case of the three-body problem.
- **Stephen Smale**(1930): developed the *Smale horseshoe*.

What is the behavior of dynamical systems as time goes to infinity? It is useful to investigate this behavior using some generic dynamical systems.

- **Kupka-Smale systems**
- **Morse-Smale systems**

This project:

- Morse-Smale systems in two-dimensional compact manifolds M^2 (the sphere S^2 and the torus T^2).
- Structural stability of Morse-Smale systems.
- Density of Morse-Smale systems.
- Approximating rational and irrational flows on a torus T^2 by Morse-Smale fields.

Preliminaries

Topology and Geometry

Definition

A **topological space** *is a pair* $(X, \mathcal{T})$ *where* X *is a set and* $\mathcal{T}$ *is a collection of subsets of* X *such that:*

- $\emptyset, X$ *are in* $\mathcal{T}$
- *The union of the elements of any subcollection of* $\mathcal{T}$ *is in* $\mathcal{T}$*.*
- *The intersection of the elements of any finite subcollection of* $\mathcal{T}$ *is in* $\mathcal{T}$*.*

$\mathcal{T}$ *is called the* **topology** *of the topological space* $(X, \mathcal{T})$*. A subset* U *of* X *is called an* **open set** *if* $U \in \mathcal{T}$*.*

Definition

A **topological manifold** *is a topological space M such that:*

- *M is Hausdorff (any two distinct points are contained in disjoint open sets)*
- *M is second countable (has a countable basis)*
- *M is locally Euclidean (every point has a neighborhood homeomorphic to an open set in $\mathbb{R}^n$).*

Definition

A map $F : U \to \mathbb{R}^n$, where U is an open set in $\mathbb{R}^n$, is called **smooth** *if it has partial derivatives of all orders at every point in U.*

Definition

Let $f : U \to \mathbb{R}$ be a smooth function on an open set $U \subset \mathbb{R}^n$. The C^r**-norm** *of f is defined by*

$$\|f\|_{C^r} = \max\{\|f\|, \left\|f^1\right\|, ..., \left\|f^{(r)}\right\|\}$$

Definition

A bijective map $F : M \to N$ is a **diffeomorphism** *if both F and F^{-1} are smooth.*

- *Example*: The *antipodal map*.

- A **tangent vector** at a point $p \in \mathbb{R}^n$ is a vector anchored at p.
- The set of all tangent vectors at p is call the **tangent space** of $\mathbb{R}^n$ at p.
- The tangent space at $p \in M$ is denoted as T_pM. The set of all tangent spaces at all points on M is called the **tangent bundle** TM.

Definition

Let U and V be submanifolds of M. U is **transverse** *to V if for all points $p \in U \cap V$, the tangent spaces T_pU and T_pV span T_pM.*

Example: x-axis and y-axis in $\mathbb{R}^2$.

Definition

Let $F : M \to N$ be a smooth map of smooth manifolds. Given a point $p \in M$, the derivative map:

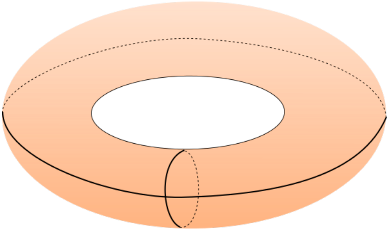
$$D_pF : T_pM \to T_{F(p)}N$$

is called the **pushforward** *map (F_*).*

Definition

*A **vector field** of class C^r on a manifold M is a C^r map $X : M \to TM$ that takes a point p on M to a vector $X_p \in T_pM$.*

Example: Two linearly independent, non-vanishing vector fields $\frac{\partial}{\partial\theta_1}$ and $\frac{\partial}{\partial\theta_2}$ on T^2.



Definition

*Let M, N be smooth manifolds and $F : M \to N$ a diffeomorphism. For every vector field X, there exists a unique vector field Y on N such that Y is F-**related** to X. That is, $F_*(X) = Y$.*

Dynamics

Definition

A **dynamical system** *(or a flow) on a manifold M is a smooth map $\phi : \mathbb{R} \times M \to M$ with the properties:*

1. $\phi_0(p) = p$.
2. $\phi_t \circ \phi_s = \phi_{t+s}$, *for* $t, s \in \mathbb{R}$.

Example: The first-order differential equation $\dot{x} = ax$ has the solution $\phi_t(x_0) = x_0 e^{at}$ $(t \in \mathbb{R})$.

Definition

The **orbit** *of a flow* ϕ *at a point* p *is the set*

$$\mathcal{O}(p) = \{\phi_t(p) : t \in \mathbb{R}\}.$$

Definition

Two flows ϕ *and* ψ *on* M *are called* **topologically conjugate** *if there exists a homeomorphism* $h : M \to M$ *such that*

$$h(\phi_t(p)) = \psi_t(h(p))$$

for all $t \in T$ *and* $p \in M$*.*

Definition

A vector field Y is ε-close to X in the C^r-topology if

$$\|Y - X\|_{C^r} < \varepsilon$$

.

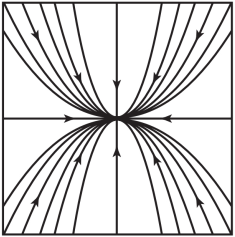
This means X and Y are ε-close together with their first r derivatives.

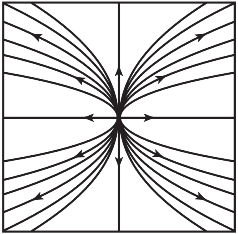
Definition

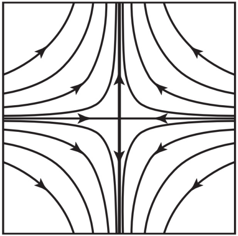
A point $p \in M$ is called a **singularity** *of a vector field X if*

$$X_p = 0.$$

Examples:



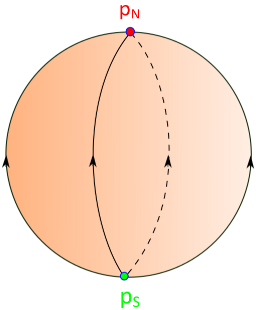




Definition

*Given a vector field X on M, an ω-***limit set** *of a point $p \in M$ is the destination of the orbit of p as $t \to \infty$. An α-***limit set** *of p is the origin of the orbit of p as $t \to -\infty$.*

Example:



For any point p that is not p_N or p_S, $\omega(p) = p_N$ and $\alpha(p) = p_S$

Definition

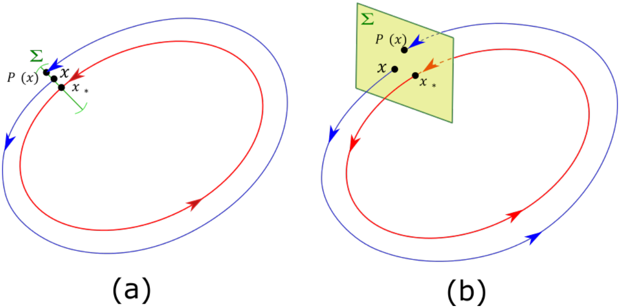
The **critical elements** *of a vector field X are all the singularities and closed orbits of X. A critical element may be an* **attractor** *or a* **repellor**.

Definition

A singularity p of X is called **hyperbolic** *if D_pX has eigenvalues with non-zero real parts, where D_pX is the derivative of X at p.*

The Poincaré Map

- $P : V \to \Sigma$
- $P(x)$: **first return point**.
- x_*: **fixed point**.



Definition

If $p \in M$ is a fixed point of a diffeomorphism f of M, then p is a **hyperbolic fixed point** *if the derivative matrix $D_pF : T_pM \to T_pM$ has no eigenvalue of modulus 1.*

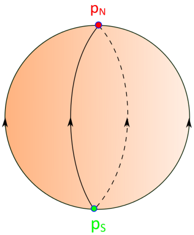
Definition

Let p be a hyperbolic fixed point of M. The **stable manifold** *$W^s(p)$ of p is the set of all points that have p as their ω-limit. The* **unstable manifold** *$W^u(p)$ of p is the set of all point that have p as their α-limit.*

$$W^s(p) = \{q \in M | \omega(q) = p\}$$

$$W^u(p) = \{q \in M | \alpha(q) = p\}$$

Example:

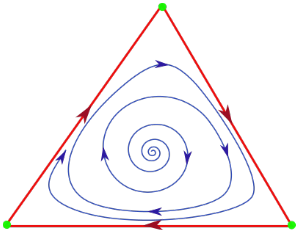


- $W^s(p_N) = S^2 - \{p_S\}$.
- $W^u(p_N) = \{p_N\}$.
- $W^s(p_S) = \{p_S\}$.
- $W^u(p_S) = S^2 - \{p_N\}$.

Theorem (Poincaré-Bendixson)

Let X be a vector field on the sphere S^2 with a finite number of singularities, and let p be a point on the sphere. Then exactly one of the following is true:

1. $\omega(p)$ *is a singularity;*
2. $\omega(p)$ *is a closed orbit;*
3. $\omega(p)$ *consists of singularities* $p_1, \ldots, p_n$ *and regular orbits such that if an orbit* $\gamma \subset \omega(p)$, *then* $\alpha(\gamma) = p_i$ *and* $\omega(\gamma) = p_j$ *for some* i, j *between* 1 *and* n.



The Poincaré-Bendixson suggests that in S^2, the asymptotic behavior of dynamical systems are predictable.

Definition

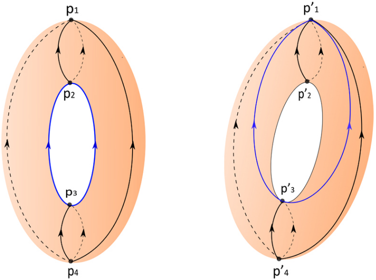
A vector field X is a **Kupka-Smale** *vector field if:*

1. *All critical elements of X are hyperbolic;*
2. *For any pair critical elements of X, their stable and unstable manifolds are transverse to each other.*

Theorem

Kupka-Smale vector fields are dense in $\mathfrak{X}^r(M)$.

Examples of Kupka-Smale and non-Kupka-Smale vector fields:



Morse-Smale Vector Fields

The concept of Morse-Smale systems comes from that of Kupka-Smale systems. In this section:

- The definition of Morse-Smale systems.
- Examples of Morse-Smale and non-Morse-Smale systems.

Definition

A point p in M is called a **wandering point** *for a vector field X if there exists a neighborhood V of p and a number $t_0 > 0$ such that $X_t(V) \cap V = \emptyset$ for $|t| > t_0$, where X_t is the flow of X. Otherwise, p is called a* **nonwandering point**.

The set of all nonwandering points of X is denoted by $\Omega(X)$.

Examples: In the North Pole-South Pole vector field, p_N and p_S are non-wandering, and other points are wandering.

Definition

A vector field X on a compact manifold M is a **Morse-Smale vector field** *if:*

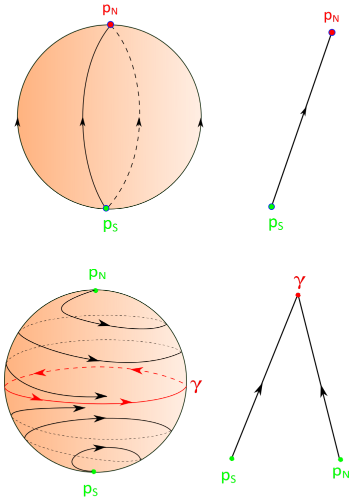
1. *There are only a finite number of critical elements on X and they are all hyperbolic.*
2. *For any two critical elements of X, their stable and unstable manifolds are transverse to each other*
3. *$\Omega(X)$ is the union of all the critical elements of X.*

Proposition

A vector field X on a compact manifold M is Morse Smale if and only if:

1. *There are only a finite number of critical elements on X and they are all hyperbolic.*
2. *There are no saddle-connections.*
3. *Each orbit on M has a unique critical element as its ω-limit and a unique critical element as its α-limit.*

Let X be a Morse-Smale vector field. The **phase diagram** of X is the simplest way to represent the qualitative behavior of X.



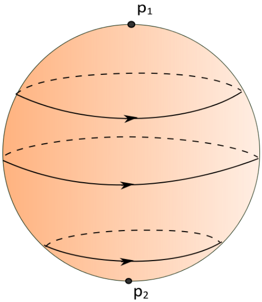


Figure: A non-Morse-Smale vector field with infinitely many critical elements

Structural Stability of Morse-Smale systems

Structural Stability of Morse-Smale systems

Definition

A vector field X is **structurally stable** *if there exists a C^r neighborhood U of X such that for every vector field Y in U, Y is topologically equivalent to X.*

To show that two vector fields X and Y are topologically equivalent to each other, we need to construct a homeomorphism that takes orbits of X to orbits of Y.

Theorem

Given a Morse-Smale vector field X on M, there exists a neighborhood U of X such that for every vector field $Y \in U$, Y is Morse-Smale and its phase diagram is isomorphic to that of X.

Theorem

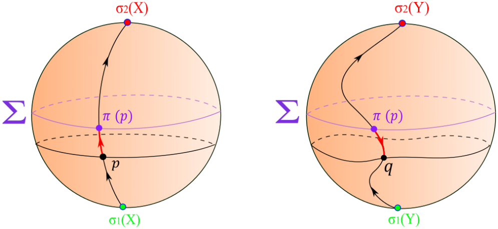
If X is a Morse-Smale vector field, then X is structurally stable.

Proof: (for the North Pole-South Pole vector field)

- By the previous theorem: there exists a neighborhood U of X such that for all $Y \in U$, Y is Morse-Smale.
- By the previous theorem: there exists an isomorphism σ that takes the phase diagram of X to the the phase diagram of Y.
- Let ψ_t be the flow by X and φ_t be the flow by Y.

Construct a homeomorphism $h : M \to M$ that takes orbits of X to orbits of Y

- Let h map $\sigma_1(X)$ to $\sigma_1(Y)$ and $\sigma_2(X)$ to $\sigma_2(Y)$.
- Define $h(p) = p$ for all $p \in \Sigma$.
- For a point p (neither a singularity nor a a point on Σ), $\pi(p) = \psi_t(p)$ and $h(p) = \varphi_{-t}(\psi_t(p))$.



Density of Morse-Smale systems

Density of Morse-Smale systems

- Morse-Smale systems are always structurally stable and open in the C^r-topology.
- Only in dimension 2, Morse-Smale systems are dense.

Lemma

All hyperbolic singularities are isolated.

Lemma

Hyperbolic closed orbits are isolated.

Theorem

Every Kupka-Smale field on S^2 is Morse-Smale.

It can be shown that every Kupka-Smale field X on S^2 satisfies:

1. There are finitely many critical elements and they are all hyperbolic.
2. There are no saddle-connections.
3. Each orbit has a unique critical element as its ω-limit and a unique critical element as its α-limit.

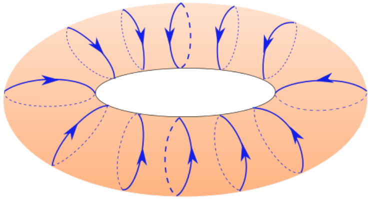
Thus, Morse-Smale fields are dense in S^2.

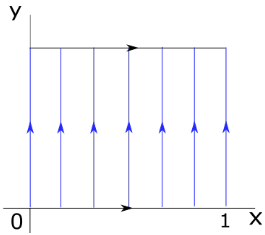
Approximating the Irrational Flow on a Torus by a Morse-Smale Field

- Since Morse-Smale systems are dense in M^2, they can be used to approximate a non-Morse-Smale vector field.
- We will show how to approximate a rational flow on T^2 by a Morse-Smale vector field.
- Irrational flow $\rightarrow$ rational flow $\rightarrow$ Morse-Smale field.

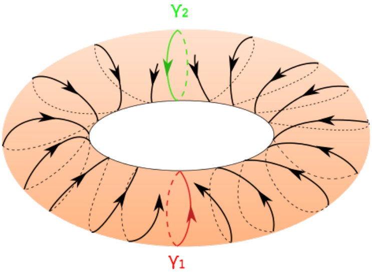
Approximating a Simple Rational Flow

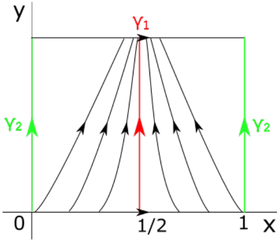
- Consider a rational flow induced by a vector field X on T^2 that has infinitely many closed orbits. This vector field is not Morse-Smale (has infinitely many non-hyperbolic critical elements).





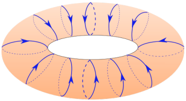
We will approximate this non-Morse Smale field by the Morse-Smale vector field Y with two closed orbits: an attractor γ_1 and a repellor γ_2.

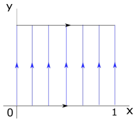




- Vector field X:

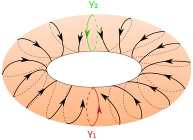
$$X = \frac{\partial}{\partial y}.$$

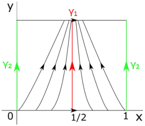




- Vector field Y: approximate by

$$X_\varepsilon = \frac{\partial}{\partial y} + f_\varepsilon(x)\frac{\partial}{\partial x},$$

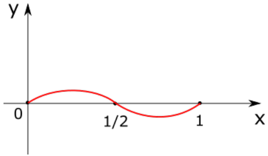




Find the function $f_\varepsilon(x)$:

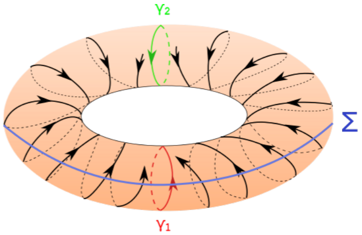
- $f_\varepsilon(x) = 0$ for $x = 0, \frac{1}{2}$ and 1.
- $f_\varepsilon(x) > 0$ for $x \in (0, \frac{1}{2})$.
- $f_\varepsilon(x) < 0$ for $x \in (\frac{1}{2}, 1)$

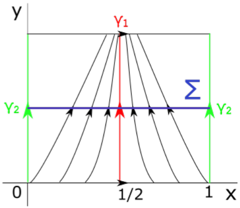
One example of $f_\varepsilon(x)$: $f_\varepsilon(x) = \varepsilon \sin(2\pi x)$ for $x \in [0, 1]$



Since the norm of this function is less than 1, $|f_\varepsilon(x)| < \varepsilon$ and $|f'_\varepsilon(x)|$ is small.

Let us investigate the projection of the approximated vector field to T^2.





γ_1 is an attractor and γ_2 is a repellor.

Approximating a General Rational Flow

- Any vector field that induces a rational flow on T^2 has an infinite number of closed orbits.
- If we can take each closed orbit and convert it into a closed orbit as in the last example, then the two vector fields are topologically equivalent.
- Approximate by a Morse-Smale vector field.

- The previous rational flow in $\mathbb{R}^2$: $\begin{cases} \dot{x} = 0 \\ \dot{y} = 1 \end{cases}$
- The general rational flow in $\mathbb{R}^2$: $\begin{cases} \dot{x} = 1 \\ \dot{y} = \alpha \end{cases}$, where $\alpha = \frac{m}{n}$ and $m, n \in \mathbb{Z}$ and are relatively prime.
- We can write these two vector fields on T^2 as $X = \frac{\partial}{\partial y}$ and $X_\alpha = \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial y}$ respectively.
- They both have infinitely many non-hyperbolic closed orbits on T^2.

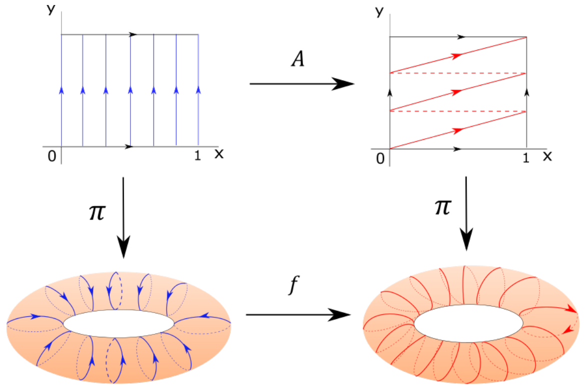
- Define the projection map $\pi : \mathbb{R}^2 \to T^2$ as a quotient map that takes a point $p \in \mathbb{R}^2$ to its equivalence class in T^2.
- Each orbit of X has period of 1. Each orbit of X_α has period of n and goes around the torus m times.
- Consider the matrix: $A = \begin{bmatrix} a & n \\ b & m \end{bmatrix}$, where a and b are integers such that $am - bn = 1$. Since $\det A = am - bn = 1$, A is invertible.

- $AX = \begin{bmatrix} a & n \\ b & m \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix} = n \begin{bmatrix} 1 \\ \frac{m}{n} \end{bmatrix} = nX_\alpha$
- For any vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$ where $x, y \in \mathbb{Z}$,

  $Av = \begin{bmatrix} a & n \\ b & m \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + ny \\ bx + my \end{bmatrix}$
- $A^{-1}v = A^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} m & -n \\ -b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} =$

  $\begin{bmatrix} m & -n \\ -b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} mx - ny \\ -bx + ay \end{bmatrix}$

- Both A and A^{-1} map integer vectors to an integer vectors.
- Define a map $f : T^2 \to T^2$ by $f(\pi(p)) = \pi(Ap)$
- It can be shown that f is a diffeomorphism.



Proposition

Let φ_t and ψ_t be the flows of X and Y on M respectively and $f : M \to M$ a diffeomorphism such that $f_*(X) = Y$, then

$$f(\varphi_t(p)) = \psi_t(f(p))$$

for all $p \in M$, i.e., f is a topological conjugacy.

Proposition

If ψ_t is the flow of Y, then $\psi_{\lambda t}$ is a flow of λY where $\lambda \in \mathbb{R}$. That is, if $\gamma(t)$ is an orbit of Y, then $\gamma(\lambda t)$ is an orbit of λY.

The process of converting Y to λY is called a **reparametrization** of Y.

- Since $f_* = A$, $f_*(X) = nX_\alpha$.
- Let φ_t and ψ_t^α be the flows of X and X_α respectively.
- By the two previous Propositions, the flow of nX_α is ψ_{nt}^α, and for all $p \in T^2$,

$$f(\varphi_t(p)) = \psi_{nt}^\alpha(f(p)).$$

- f takes the orbits of p in X to the orbits of $f(p)$ in X_α.
- f does not preserve the period. Therefore, f is a **topological equivalence**, not a **topological conjugacy**.

Find a Morse-Smale approximation of the general rational flow X_α

Let $\varepsilon > 0$ be arbitrary, there exists a Morse-Smale vector field Y such that

$$\|Y - X\|_{C^1} < \varepsilon.$$

$$\|f_*(Y) - f_*(X)\|_{C^1} < c\varepsilon,$$

for some number c depending only on m and n.

$$\left\|\frac{1}{n}f_*(Y) - \frac{1}{n}f_*(X)\right\|_{C^1} < \frac{c}{n}\varepsilon.$$

Since $f_*(X) = nX_\alpha$ and $\frac{1}{n} f_*(X) = X_\alpha$,

$$\left\| \frac{1}{n} f_*(Y) - X_\alpha \right\|_{C^1} < \frac{c}{n} \varepsilon,$$

- The vector field $Z = \frac{1}{n} f_*(Y)$ is C^1-close to X_α.
- Z is Morse-Smale because $f_*(Y)$ is the pushforward of the Morse-Smale field Y.
- Thus, Z is a Morse-Smale approximation of the general rational flow field X_α.

Conclusion

Conclusion

- Morse-Smale systems are structurally stable and form an open set in the space of dynamical systems in any dimension. In dimension two, they are also dense.
- We only showed a few simple examples in S^2 and T^2, but some of these properties can be generalized to more complicated cases.
- Besides S^2 and T^2, there are also studies that show the same results for the projective plane P^2 and the Klein bottle K^2.

- In higher dimensions, Morse-Smale systems are not dense. On these manifolds, there are also structurally stable systems that are not Morse-Smale.
- Morse-Smale systems cannot be used to approximate all vector fields in higher dimensions

References

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