Dependent Types in GHC

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References

This talk:

- https://github.com/halfaya/BayHac

References:

- https://github.com/halfaya/BayHac/blob/master/references.md

What are Dependent Types?

```
data ℕ : Set where
  zero : ℕ
  suc : ℕ → ℕ

data Vec (A : Set) : ℕ → Set where
  [] : Vec A zero
  _::_ : {n : ℕ} → A → Vec A n → Vec A (suc n)
```

Vector Append

$$
\begin{aligned}
&\_+\_ : \mathbb{N} \to \mathbb{N} \to \mathbb{N} \\
&\mathsf{zero} + n = n \\
&\mathsf{suc}\ m + n = \mathsf{suc}\ (m + n)
\end{aligned}
$$

$$
\begin{aligned}
&\_{+}{+}\_ : \{A : \mathsf{Set}\} \to \{m\ n : \mathbb{N}\} \to \\
&\quad \mathsf{Vec}\ A\ m \to \mathsf{Vec}\ A\ n \to \mathsf{Vec}\ A\ (m + n) \\
&[\,]\ {+}{+}\ y = y \\
&(x :: xs)\ {+}{+}\ y = x :: (xs\ {+}{+}\ y)
\end{aligned}
$$

Vector Lookup

```
data Fin : ℕ → Set where
  zero : {n : ℕ} → Fin (suc n)
  suc : {n : ℕ} → Fin n → Fin (suc n)

lookup : {A : Set} → {n : ℕ} → Fin n → Vec A n → A
lookup ()    [] -- can be omitted
lookup zero (x :: _ ) = x
lookup (suc n) (_ :: xs) = lookup n xs
```

Some Basic Types

```
data Bool : Set where
  true : Bool
  false : Bool
```

```
data _≡_ {A : Set} : A → A → Set where
  refl : {a : A} → a ≡ a
```

Vector Lookup 2

```
_<_ : ℕ → ℕ → Bool
_     < zero = false
zero < suc n = true
suc m < suc n = m < n
```

```
lookup' : {A : Set} → {n : ℕ} →
  (m : ℕ) → m < n ≡ true → Vec A n → A
lookup' _     ()     []
lookup' zero refl (x :: _ ) = x
lookup' (suc m) p (_ :: xs) = lookup' m p xs
```

Why Use Dependent Types?

- More expressive and precise
- Propositions as Types (PAT, Curry-Howard Correspondence):
  - Universal Quantification ($\forall$) corresponds to Π types.
    $\Pi_{x:A} B(x)$ or $(x : A) \to B_x$.
  - Existential Quantification ($\exists$) corresponds to Σ types.
    $\Sigma_{x:A} B(x)$ or $(x : A) \times B_x$.

Dependent Types are Not New

- 1971 System F
- 1971 Martin-Löf Type Theory
- 1972 LCF/ML
- 1978 Hindley-Milner (H 1969)
- 1979 Constructive math and computer programming
- 1982 Damas-Milner
- 1983 Standard ML
- 1984 Calculus of Constructions
- 1984 NuPrl
- 1985 Miranda
- 1987 Caml
- 1988 CiC
- 1989 Coq
- 1990 Haskell
- 1990 Nordström, et al, ALF
- 1998 Cayenne
- 1999 Agda 1
- 1991 Caml Light
- 1996 OCaml
- 2007 Agda 2
- 2011 Idris
- 2013 Homotopy Type Theory
- 2013 Lean
- 2015 Cubical Type Theory

The Golden Age is Now

- Increased use of FP in industry.
  Big Data, Finance, Security
- Better correctness guarantees for software.
  CompCert, DeepSpec, etc.
- Mechanical verification of mathematics.
  Four Color Theorem, Feit-Thompson, BigProof
- Natural Language Processing.
  Grammatical Framework
- Theoretical work.
  HoTT, Cubical Type Theory, Category Theory and FP, etc.

Robert Harper

> *Eventually all the arbitrary programming languages are going to be just swept away with the oceans, and we will have the permanence of constructive, intuitionistic type theory as the master theory of computation—without doubt, in my mind, no question. So, from my point of view—this is a personal statement—working in anything else is a waste of time.*

CMU Homotopy Type Theory lecture 1, 52:56–53:20.

Dependent Types in Haskell

Richard Eisenberg's PhD Thesis

1. Introduction
2. Preliminaries
3. Motivation
4. Dependent Haskell
5. PICO: The Intermediate Language
6. Type Inference and Elaboration, or how to BAKE a PICO
7. Implementation
8. Related and Future Work

Time Line

From Richard Eisenberg's Blog:

When can we expect dependent types in GHC?

The short answer:
GHC 8.4 (2018) at the very earliest.
More likely 8.6 or 8.8 (2019-20).

Nat

```
{-# LANGUAGE ExplicitForAll, GADTs, TypeFamilies,
             TypeOperators, TypeInType #-}
import Data.Kind (Type)

data Nat :: Type where
  Zero :: Nat
  Succ :: Nat -> Nat
```

```
data ℕ : Set where
  zero : ℕ
  suc : ℕ → ℕ
```

Vector

```
data Vec :: Type -> Nat -> Type where
  VNil  :: ∀ (a :: Type). Vec a 'Zero
  VCons :: ∀ (a :: Type)(n :: Nat).
    a -> Vec a n -> Vec a ('Succ n)
```

```
data Vec (A : Set) : ℕ → Set where
  [] : Vec A zero
  _::_ : {n : ℕ} → A → Vec A n → Vec A (suc n)
```

Plus

```
type family (m :: Nat) + (n :: Nat) where
  Zero    + n = n
  'Succ m + n = 'Succ (m + n)
```

$$\_+\_ : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$$
$$\mathsf{zero} + n = n$$
$$\mathsf{suc}\ m + n = \mathsf{suc}\ (m + n)$$

Append

```
(++) :: ∀ (a :: Type)(m :: Nat)(n :: Nat).
        Vec a m -> Vec a n -> Vec a (m + n)
VNil      ++ v = v
VCons x u ++ v = VCons x (u ++ v)
```

$$
\begin{aligned}
&\_{++}\_ : \{A : \mathsf{Set}\} \to \{m\ n : \mathbb{N}\} \to \\
&\quad \mathsf{Vec}\ A\ m \to \mathsf{Vec}\ A\ n \to \mathsf{Vec}\ A\ (m + n) \\
&[]\ {++}\ y = y \\
&(x :: xs)\ {++}\ y = x :: (xs\ {++}\ y)
\end{aligned}
$$

Fin

```
data Fin :: Nat -> Type where
  FZero :: ∀ (n :: Nat).          Fin (Succ n)
  FSucc :: ∀ (n :: Nat). Fin n -> Fin (Succ n)
```

```
data Fin : ℕ → Set where
  zero : {n : ℕ} → Fin (suc n)
  suc : {n : ℕ} → Fin n → Fin (suc n)
```

Lookup

```
lookup :: ∀ (a :: Type)(n :: Nat). Fin n -> Vec a n -> a
lookup FZero      (VCons x _)  = x
lookup (FSucc n) (VCons _ xs) = lookup n xs
```

```
lookup : {A : Set} → {n : ℕ} → Fin n → Vec A n → A
lookup ()   [] -- can be omitted
lookup zero (x :: _ ) = x
lookup (suc n) (_ :: xs) = lookup n xs
```

$\equiv$ and $<$

```
data (a :: k) :~: (b :: k) where
  Refl :: ∀ (k :: Type)(a :: k). a :~: a

type family (m :: Nat) < (n :: Nat) where
  _          < Zero       = False
  'Zero      < ('Succ _)  = True
  ('Succ m)  < ('Succ n)  = m < n
```

$$\mathsf{data}\ \_{\equiv}\_\ \{A : \mathsf{Set}\} : A \to A \to \mathsf{Set}\ \mathsf{where}$$
$$\quad \mathsf{refl} : \{a : A\} \to a \equiv a$$

$$\_{<}\_ : \mathbb{N} \to \mathbb{N} \to \mathsf{Bool}$$
$$\_ \quad < \mathsf{zero} = \mathsf{false}$$
$$\mathsf{zero} < \mathsf{suc}\ n = \mathsf{true}$$
$$\mathsf{suc}\ m < \mathsf{suc}\ n = m < n$$

Attempt at Lookup'

```
lookupBad :: ∀ (a :: Type)(m :: Nat)(n :: Nat).
             m -> m < n :~: True -> Vec a n -> a
```

```
  • Expected a type, but ‘m’ has kind ‘Nat’
  • In the type signature:
      lookupBad :: ∀ (a :: Type) (m :: Nat) (n :: Nat).
                   m -> (m < n) :~: True -> Vec a n -> a
```

```
> :k (->)
(->) :: Type -> Type -> Type
```

```
lookup' : {A : Set} → {n : ℕ} →
  (m : ℕ) → m < n ≡ true → Vec A n → A
lookup' _      ()     []
lookup' zero refl (x :: _ ) = x
lookup' (suc m) p (_ :: xs) = lookup' m p xs
```

Singleton Nat

```
data SNat :: Nat -> Type where
  SZero :: SNat 'Zero
  SSucc :: ∀ (n :: Nat). SNat n -> SNat ('Succ n)
```

Lookup'

```
lookup' :: ∀ (a :: Type)(m :: Nat)(n :: Nat).
           SNat m -> m < n :~: True -> Vec a n -> a
lookup' SZero     Refl (VCons x _)  = x
lookup' (SSucc m) Refl (VCons _ xs) = lookup' m Refl xs
```

```
lookup' : {A : Set} → {n : ℕ} →
  (m : ℕ) → m < n ≡ true → Vec A n → A
lookup' _    ()    []
lookup' zero refl (x :: _ ) = x
lookup' (suc m) p (_ :: xs) = lookup' m p xs
```

Another Variation

```
nth :: ∀ (a :: Type)(m :: Nat)(n :: Nat).
       (m < n) ~ 'True => SNat m -> Vec a n -> a
nth SZero     (VCons a _)  = a
nth (SSucc m) (VCons _ as) = nth m as
```

Lookup' vs Nth

```
lookup' :: ∀ (a :: Type)(m :: Nat)(n :: Nat).
           SNat m -> m < n :~: True -> Vec a n -> a
lookup' _          _    VNil          = undefined
lookup' SZero      Refl (VCons x _)   = x
lookup' (SSucc m) Refl (VCons _ xs) = lookup' m Refl xs

nth :: ∀ (a :: Type)(m :: Nat)(n :: Nat).
       (m < n) ~ 'True => SNat m -> Vec a n -> a
nth _          VNil          =  undefined
nth SZero      (VCons a _)   = a
nth (SSucc m) (VCons _ as) = nth m as
```

- Couldn't match type ''True' with ''False'
  Inaccessible code in
  a pattern with constructor: VNil :: ∀ a. Vec a 'Zero,
  in an equation for 'nth'

Vectors in Dependent Haskell

```
type family (m :: Nat) + (n :: Nat) where
  Zero    + n = n
  'Succ m + n = 'Succ (m + n)

(+) :: Nat -> Nat -> Nat
Zero   + m = m
Succ n + m = Succ (n + m)
```

$$
\begin{aligned}
&\_+\_ : \mathbb{N} \to \mathbb{N} \to \mathbb{N} \\
&\mathsf{zero} + n = n \\
&\mathsf{suc}\ m + n = \mathsf{suc}\ (m + n)
\end{aligned}
$$

Append

```
(++) :: ∀ (a :: Type)(m :: Nat)(n :: Nat).
        Vec a m -> Vec a n -> Vec a (m + n)
VNil       ++ v = v
VCons x u ++ v = VCons x (u ++ v)
```

```
(++) :: ∀ (a :: Type)(m :: Nat)(n :: Nat).
        Vec a m -> Vec a n -> Vec a (m '+ n)
VNil       ++ v = v
VCons x u ++ v = VCons x (u ++ v)
```

$$
\begin{array}{l}
\_{+}{+}\_ : \{A : \mathsf{Set}\} \to \{m\ n : \mathbb{N}\} \to \\
\quad \mathsf{Vec}\ A\ m \to \mathsf{Vec}\ A\ n \to \mathsf{Vec}\ A\ (m + n) \\
[]\ {+}{+}\ y = y \\
(x :: xs)\ {+}{+}\ y = x :: (xs\ {+}{+}\ y)
\end{array}
$$

Lookup

```
lookup' :: ∀ (a :: Type)(m :: Nat)(n :: Nat).
           SNat m -> m < n :~: True -> Vec a n -> a
lookup' SZero     Refl (VCons x _)  = x
lookup' (SSucc m) Refl (VCons _ xs) = lookup' m Refl xs

lookup' :: ∀ (a :: Type)(n :: Nat). PI (m :: Nat) ->
           m -> m < n :~: True -> Vec a n -> a
lookup' Zero     Refl (VCons x _)  = x
lookup' (Succ m) Refl (VCons _ xs) = lookup' m Refl xs
```

```
lookup' : {A : Set} → {n : ℕ} →
  (m : ℕ) → m < n ≡ true → Vec A n → A
lookup' _    ()    []
lookup' zero refl (x :: _ ) = x
lookup' (suc m) p (_ :: xs) = lookup' m p xs
```

Conclusion

This is just a tiny taste.

See Richard's thesis and the other references for much more.

Try playing with a proof assistant such as Coq or Agda.
Software Foundations is a great place to start.