Dependent Types in GHC

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References

This talk:

- https://github.com/halfaya/BayHac

References:

- https://github.com/halfaya/BayHac/blob/master/references.md

Outline

- Vectors in Agda
- Big Picture
- Dependent Types in Haskell
- Vectors in today's Haskell
- Vectors in Dependent Haskell

What are Dependent Types?

```
data ℕ : Set where
  zero : ℕ
  suc  : ℕ → ℕ

data Vec (A : Set) : ℕ → Set where
  []    : Vec A zero
  _::_ : {n : ℕ} → A → Vec A n → Vec A (suc n)

v : Vec ℕ (suc (suc zero))
v = zero :: suc zero :: []
```

Vector Append

$$
\begin{aligned}
&\_+\_ : \mathbb{N} \to \mathbb{N} \to \mathbb{N} \\
&\mathrm{zero} \quad + n = n \\
&\mathrm{suc}\ m + n = \mathrm{suc}\ (m + n)
\end{aligned}
$$

$$
\begin{aligned}
&\_{++}\_ : \{A : \mathrm{Set}\} \to \{m\ n : \mathbb{N}\} \to \\
&\qquad\quad \mathrm{Vec}\ A\ m \to \mathrm{Vec}\ A\ n \to \mathrm{Vec}\ A\ (m + n) \\
&[] \qquad\quad ++\ y = y \\
&(x :: xs)\ ++\ y = x :: (xs\ ++\ y)
\end{aligned}
$$

Vector Lookup

```
data Fin : ℕ → Set where
  zero : {n : ℕ}          → Fin (suc n)
  suc  : {n : ℕ} → Fin n → Fin (suc n)

lookup : {A : Set} → {n : ℕ} → Fin n → Vec A n → A
lookup ()        []
lookup zero      (x :: _ ) = x
lookup (suc n) (_ :: xs) = lookup n xs
```

Some Basic Types

```
data Bool : Set where
  true  : Bool
  false : Bool
```

```
data _≡_ {A : Set} : A → A → Set where
  refl : {a : A} → a ≡ a
```

Vector Lookup 2

$$
\begin{aligned}
&\_<\_ : \mathbb{N} \to \mathbb{N} \to \mathrm{Bool}\\
&\_ \quad\quad\;\, < \mathrm{zero} \;\;\, = \mathrm{false}\\
&\mathrm{zero} \;\;\, < \mathrm{suc}\ n = \mathrm{true}\\
&\mathrm{suc}\ m < \mathrm{suc}\ n = m < n
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{lookup}' : \{A : \mathrm{Set}\} \to \{n : \mathbb{N}\} \to\\
&\qquad\qquad (m : \mathbb{N}) \to m < n \equiv \mathrm{true} \to \mathrm{Vec}\ A\ n \to A\\
&\mathrm{lookup}'\ \_ \qquad\qquad ()\quad\ [\,]\\
&\mathrm{lookup}'\ \mathrm{zero} \qquad \mathrm{refl}\ (x :: \_\ ) = x\\
&\mathrm{lookup}'\ (\mathrm{suc}\ m)\ p \quad (\_ :: xs) = \mathrm{lookup}'\ m\ p\ xs
\end{aligned}
$$

Why Use Dependent Types?

- More expressive and precise
- Propositions as Types (PAT, Curry-Howard Correspondence):
  - Universal Quantification ($\forall$) corresponds to Π types.
    $\Pi_{x:A} B(x)$ or $(x : A) \to B_x$.
  - Existential Quantification ($\exists$) corresponds to Σ types.
    $\Sigma_{x:A} B(x)$ or $(x : A) \times B_x$.

Dependent Types are Not New

1971 System F
1971 Martin-Löf Type Theory
1972 LCF/ML
1978 Hindley-Milner (H 1969)
1979 Constructive math and computer programming
1982 Damas-Milner
1983 Standard ML
1984 Calculus of Constructions
1984 NuPrl
1985 Miranda
1987 Caml
1988 CiC
1989 Coq
1990 Haskell
1990 Nordström, et al, ALF
1991 Caml Light
1996 OCaml
1998 Cayenne
1999 Agda 1
2005 Haskell GADTs
2007 Agda 2
2011 Idris
2013 Homotopy Type Theory
2013 Lean
2015 Cubical Type Theory

The Golden Age is Now

- Increased use of FP in industry.
  Big Data, Finance, Security
- Better correctness guarantees for software.
  CompCert, DeepSpec, etc.
- Mechanical verification of mathematics.
  Four Color Theorem, Feit-Thompson, BigProof
- Natural Language Processing.
  Grammatical Framework
- Theoretical work.
  HoTT, Cubical Type Theory, Category Theory and FP, etc.

Robert Harper

> *Eventually all the arbitrary programming languages are going to be just swept away with the oceans, and we will have the permanence of constructive, intuitionistic type theory as the master theory of computation—without doubt, in my mind, no question. So, from my point of view—this is a personal statement—working in anything else is a waste of time.*

CMU Homotopy Type Theory lecture 1, 52:56–53:20.

Dependent Types in Haskell

Richard Eisenberg's PhD Thesis

1. Introduction
2. Preliminaries
3. Motivation
4. Dependent Haskell
5. PICO: The Intermediate Language
6. Type Inference and Elaboration, or how to BAKE a PICO
7. Implementation
8. Related and Future Work

Time Line

From Richard Eisenberg's Blog:

When can we expect dependent types in GHC?

The short answer:
GHC 8.4 (2018) at the very earliest.
More likely 8.6 or 8.8 (2019-20).

Nat

```
{-# LANGUAGE UnicodeSyntax, ExplicitForAll, GADTs,
             TypeFamilies, TypeOperators, TypeInType #-}
import Data.Kind (Type)

data Nat :: Type where
  Zero :: Nat
  Succ :: Nat → Nat
```

```
data ℕ : Set where
  zero : ℕ
  suc  : ℕ → ℕ
```

Vector

```
data Vec :: Type → Nat → Type where
  Nil  :: ∀ (a :: Type). Vec a 'Zero
  (:>) :: ∀ (a :: Type)(n :: Nat).
    a → Vec a n → Vec a ('Succ n)
```

```
data Vec (A : Set) : ℕ → Set where
  []  : Vec A zero
  _::_ : {n : ℕ} → A → Vec A n → Vec A (suc n)
```

Plus

```
type family (m :: Nat) + (n :: Nat) :: Nat where
  'Zero   + n = n
  'Succ m + n = 'Succ (m + n)
```

$$
\begin{aligned}
&\_+\_ : \mathbb{N} \to \mathbb{N} \to \mathbb{N} \\
&\mathrm{zero} \;\; + n = n \\
&\mathrm{suc}\ m + n = \mathrm{suc}\ (m + n)
\end{aligned}
$$

Append

```
(++) :: ∀ (a :: Type) (m :: Nat) (n :: Nat) .
        Vec a m → Vec a n → Vec a (m + n)
Nil     ++ v = v
x :> u ++ v = x :> (u ++ v)
```

$$
\begin{array}{ll}
\_{+}{+}\_ : \{A : \mathrm{Set}\} \to \{m\ n : \mathbb{N}\} \to & \\
\qquad \mathrm{Vec}\ A\ m \to \mathrm{Vec}\ A\ n \to \mathrm{Vec}\ A\ (m + n) & \\
[] \quad\quad\ {+}{+}\ y = y & \\
(x :: xs)\ {+}{+}\ y = x :: (xs\ {+}{+}\ y) &
\end{array}
$$

Fin

```
data Fin :: Nat → Type where
  FZero :: ∀ (n :: Nat).          Fin ('Succ n)
  FSucc :: ∀ (n :: Nat). Fin n → Fin ('Succ n)
```

```
data Fin : ℕ → Set where
  zero : {n : ℕ}         → Fin (suc n)
  suc  : {n : ℕ} → Fin n → Fin (suc n)
```

Lookup

```
lookup :: ∀ (a :: Type)(n :: Nat). Fin n → Vec a n → a
lookup FZero      (x :> _)  = x
lookup (FSucc n) (_ :> xs) = lookup n xs
```

```
lookup : {A : Set} → {n : ℕ} → Fin n → Vec A n → A
lookup ()       []
lookup zero     (x :: _ ) = x
lookup (suc n) (_ :: xs) = lookup n xs
```

$\equiv$ and $<$

```
data (a :: k) ≡ (b :: k) where
  Refl :: ∀ (k :: Type)(a :: k). a ≡ a

type family (m :: Nat) < (n :: Nat) :: Bool where
  _          < 'Zero     = 'False
  'Zero      < ('Succ _) = 'True
  ('Succ m) < ('Succ n) = m < n
```

```
data _≡_ {A : Set} : A → A → Set where
  refl : {a : A} → a ≡ a

_<_ : ℕ → ℕ → Bool
_     < zero  = false
zero  < suc n = true
suc m < suc n = m < n
```

Attempt at Lookup'

```
lookupBad :: ∀ (a :: Type)(m :: Nat)(n :: Nat).
             m → m < n ≡ True → Vec a n → a

  • Expected a type, but 'm' has kind 'Nat'
  • In the type signature:
      lookupBad :: ∀ (a :: Type) (m :: Nat) (n :: Nat).
                   m → (m < n) ≡ True → Vec a n → a
> :k (→)
(→) :: Type → Type → Type
```

$$
\begin{array}{l}
\mathrm{lookup}' : \{A : \mathrm{Set}\} \to \{n : \mathbb{N}\} \to \\
\qquad (m : \mathbb{N}) \to m < n \equiv \mathrm{true} \to \mathrm{Vec}\ A\ n \to A \\
\mathrm{lookup}'\ \_ \qquad ()\ \ [] \\
\mathrm{lookup}'\ \mathrm{zero} \qquad \mathrm{refl}\ (x :: \_\ ) = x \\
\mathrm{lookup}'\ (\mathrm{suc}\ m)\ p \quad (\_ :: xs) = \mathrm{lookup}'\ m\ p\ xs
\end{array}
$$

Another Attempt

```
type family LookupUp (a :: Type) (m :: Nat) (n :: Nat)
                     (p :: m < n ≡ 'True)
                     (v :: Vec a n) :: a where

LookupUp a 'Zero      ('Succ n) 'Refl (x ':> _)  = x
LookupUp a ('Succ m) ('Succ n) p      (_ ':> xs) =
  LookupUp a m n p xs
```

```
lookup′ : {A : Set} → {n : ℕ} →
          (m : ℕ) → m < n ≡ true → Vec A n → A
lookup′ _        ()   []
lookup′ zero     refl (x :: _ ) = x
lookup′ (suc m) p     (_ :: xs) = lookup′ m p xs
```

Singleton Nat

```
data SNat :: Nat → Type where
  SZero :: SNat 'Zero
  SSucc :: ∀ (n :: Nat). SNat n → SNat ('Succ n)
```

Lookup’

```
lookup’ :: ∀ (a :: Type)(m :: Nat)(n :: Nat).
             SNat m → m < n ≡ True → Vec a n → a
lookup’ SZero      Refl (x :> _)  = x
lookup’ (SSucc m) Refl (_ :> xs) = lookup’ m Refl xs
```

```
lookup′ : {A : Set} → {n : ℕ} →
          (m : ℕ) → m < n ≡ true → Vec A n → A
lookup′ _        ()   []
lookup′ zero     refl (x :: _ ) = x
lookup′ (suc m) p    (_ :: xs) = lookup′ m p xs
```

Vectors in Dependent Haskell

```
type family (m :: Nat) + (n :: Nat) :: Nat where
  'Zero   + n = n
  'Succ m + n = 'Succ (m + n)

(+) :: Nat → Nat → Nat
Zero   + m = m
Succ n + m = Succ (n + m)
```

$$\_+\_ : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$$
$$\mathrm{zero} \quad + n = n$$
$$\mathrm{suc}\ m + n = \mathrm{suc}\ (m + n)$$

Append

```
(++) :: ∀ (a :: Type) (m :: Nat) (n :: Nat) .
        Vec a m → Vec a n → Vec a (m + n)
Nil     ++ v = v
x :> u ++ v = x :> (u ++ v)
```

```
(++) :: ∀ (a :: Type) (m :: Nat) (n :: Nat) .
        Vec a m → Vec a n → Vec a (m '+ n)
Nil     ++ v = v
x :> u ++ v = x :> (u ++ v)
```

$$
\begin{array}{ll}
\_{+}{+}\_ : \{A : \mathrm{Set}\} \to \{m\, n : \mathbb{N}\} \to & \\
\quad \mathrm{Vec}\, A\, m \to \mathrm{Vec}\, A\, n \to \mathrm{Vec}\, A\, (m + n) & \\
[\,] \quad {+}{+}\; y = y & \\
(x :: xs)\; {+}{+}\; y = x :: (xs\; {+}{+}\; y) &
\end{array}
$$

Lookup'

```
lookup' :: ∀ (a :: Type)(m :: Nat)(n :: Nat).
           SNat m → m < n ≡ True → Vec a n → a
lookup' SZero     Refl (x :> _)  = x
lookup' (SSucc m) Refl (_ :> xs) = lookup' m Refl xs

lookup' :: ∀ (a :: Type)(n :: Nat). Π (m :: Nat) →
           m < n ≡ True → Vec a n → a
lookup' Zero     Refl (x :> _)  = x
lookup' (Succ m) Refl (_ :> xs) = lookup' m Refl xs
```

```
lookup′ : {A : Set} → {n : ℕ} →
          (m : ℕ) → m < n ≡ true → Vec A n → A
lookup′ _         ()   []
lookup′ zero      refl (x :: _ ) = x
lookup′ (suc m) p      (_ :: xs) = lookup′ m p xs
```

Conclusion

This is just a tiny taste.

See Richard's thesis and the other references for much more.

Try playing with a proof assistant such as Coq or Agda.
Software Foundations is a great place to start.