# The Solar System
# Project 3
# FYS3150

Ragnar Bruvoll Halvard Sutterud

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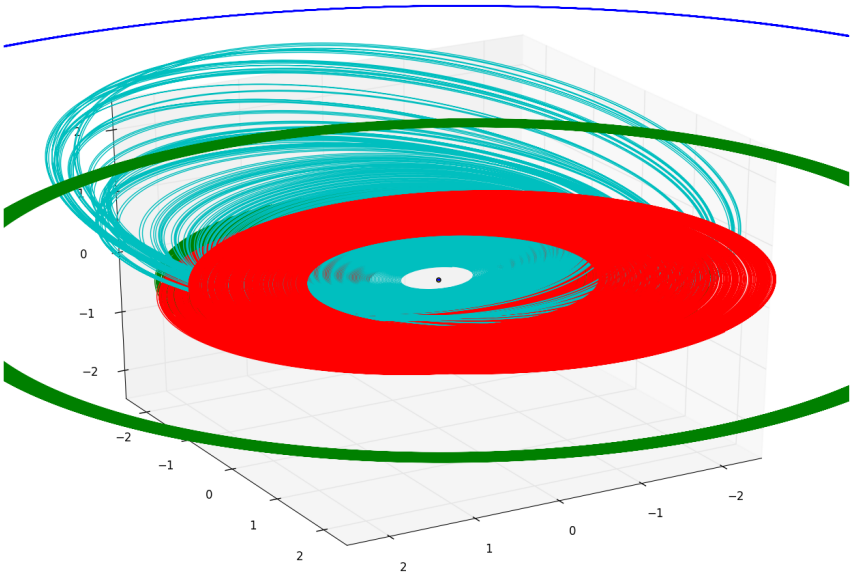


**Abstract**

We study a model of the solar system as gravitationally interacting celestial bodies. Using an object oriented approach, we implement classes in `c++` to solve the differential equations of the mechanics of the solar system in `c++`. We compare the effectiveness of two methods for numerical differentiation, namely Forward Euler and Velocity Verlet, and discuss their stabilities in terms of energy conservation. Some properties of the solar system are studied, including escape velocity for a planet in a two body system, a modified three body problem, and finally the phase precession of a Sun-Mercury system with relativistic corrections. The Forward Euler integration method is found to be less costly per iteration, but unlike the verlet method it doesn't conserve energy and is unstable over longer time periods (as seen in the front figure) .

# Contents



# 1 Introduction

The purpose of this project is to solve the differential equations of the mechanics of the solar system in an efficient way with precise results. There are several different algorithms to calculate the objects orbiting a star, each with their strengths and weaknesses. A couple of these will be showcased and compared to examine which one is preferable in our case.

Our entire simulation is based on Newton's second law of montion

$$F_G = \frac{M_{M\,Earth}}{r^2} \tag{1}$$

which turns out to be an adequate estimation for most objects in regular solar systems. In order to create a simulation with a desirable precission, a large number of iterations is necessary. The solar system is a complex one, with many factors and interactions. So our biggest challenge is to efficiently manipulate our computer with a program that exploits all its processing power without taking too much of our time. In other words, we want to make an as simple program as possible, without losing any of its precission. Different tricks and methods for simplifying the algorithm will therefore be explained and implemented.

We will gradually complicate our system in order to make it more realistic, and finally include a correction from of general relativity on objects significantly affected by it, where we prevously only used newtonian physics.

# 2 Theory

## 2.1 Units

Using Newton's second law, which states that the sum of all the forces equals mass times acceleration, $\sum F = ma$, we can calculate the acceleration of the planet in relation to the Sun. Assuming circular orbits, we know this acceleration to be the sentripetal acceleration, and the force is the gravitational force

$$F_G = M_a = M_{\frac{v^2}{r}} = G\frac{M_M}{r^2} \tag{2}$$

with the earth as an example. We will be using the astronomical symbols for the celestial bodies; $M$ is then the mass of the Earth and $M$ the mass of the sun. With this in mind, we can simplify the equations and calculations by using astronomical units and years as unit lengths. So instead of using $r = 1.5 \cdot 10^{11} m$ and , we'll simply assert $r = 1AU$. With simple multiplication, the expressions for sentripetal force and gravitational force can be written as

$$\Rightarrow v^2 r = GM_{(3)}$$

and with the new unit lengths applied we can simplify it further. We know that the radius from the earth to it's spin system's centre is one astronomical unit, 1 AU. The orbit speed is one round per year, that is $2\pi$AU/year. This means

$$GM_{=} v^2 r = 4\pi^2 AU^3/year^2 \tag{4}$$

We can now substitute this constant and get an expression for the general acceleration of any object with an circular orbit around the Sun

$$M_a = GM_{\frac{M}{r^2}} \tag{5}$$

$$a = \frac{4\pi^2}{r^2} \tag{6}$$

with the acceleration in units $[a] = \text{AU/yr}^2$.

The unit of energy is also redefined, with new units for mass, length and time:

$$\begin{aligned} E &= mass \cdot \frac{length^2}{time^2} = M\frac{AU^2}{year^2} \\ &= 1.989 \cdot 10^{30} kg \frac{(1.4959 \cdot 10^{11} m)^2}{(60 \cdot 60 \cdot 24 \cdot 365 s)^2} \\ &\equiv J, \end{aligned} \tag{7}$$

where we introduced a new energy unit, the Astrojoule $J_{\approx} 4.5 \cdot 10^{37} J$. Please note that these are the universal units for all equations and figures in this report.

## 2.2 Energy calculation

As all the forces involved are conservative, the energy of the system should be conserved. If this is not the case, it could be an indication that out program is not stable. For stability analysis, we will therefore make sure that the sum of the systems energies is constant. The energies are given by the following equations

$$T = \sum_{i=1}^{N} \frac{1}{2} m v_i^2 \qquad V = \sum_{i=1}^{N} \sum_{j=i+1}^{N} -r_{i,j} |F_{i,j}| \tag{8}$$

where $T$ is the total kinetic energy, found by summing over the kinetic energies of all $N$ objects, and $V$ is the potential energies due to gravitational force between bodies $F_{i,j}$, given by

$$F_{i,j} = G\frac{M_i M_j}{r_{i,j}^2} \tag{9}$$

with $M_i$ being the mass of object $i$ and $r_{i,j} = |\vec{r}_i - \vec{r}_j|$ as the distance between two objects.

### 2.2.1 Escape velocity

The escape velocity of an object in orbit is achieved when the kinetic energy equals the potential, in this case the gravitational potential;

$$\begin{aligned} \frac{1}{2} M_{v_e}^2 &= GM_{\frac{M}{r}} \\ v_e &= \sqrt{\frac{GM}{r}} \end{aligned} \tag{10}$$

### 2.2.2 Conservation of Angular Momentum

## 2.3 Integration

As the trajectories of all the objects in the solar system is being affected by all the other objects at all times, it is quite obvious that the acceleration is not constant as they would be in a circular orbit. We therefore need to integrate their position from Newton's equations for each time step. This can be done with different methods, in our case Euler's forward algorithm and velocity Verlet method. They're both derived from the Taylor series. As we cannot make the time steps of the integration infinitesimaly small, the methods are bound to be incorrect to a certain degree. The impact of this error will be scrutinized in a later section.

### 2.3.1 Taylor expantion

The Taylor series is an infinite sum of terms given by the function's derivative at each point. The series is used to express an unknown curve and it converges closer to the correct value with each degree.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{11}$$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \cdots \quad (12)$$

It is however impossible to add an infinite number of terms, and each term will further burden the calculating system. But as the value of the terms decrease for each degree, we can justifiably simplify the series to one of three terms. This will entail a certain degree of error, but as we will see it varies between methods, and is often so small we can ignore it.

### 2.3.2 Euler's forward algorithm

The Euler forward method is a first order method, meaning that the local error is proportional to the step size squared. This incentivizes ut to pick a step size as small as possible without making the program too slow. For this method we only use two of the taylor terms, knowing velocity derived to be acceleration.

$$v_{i+1} = v_i + ah \quad (13)$$

As for the position, we use the current velocity in the second term to increase the accuracy.

$$x_{i+1} = x_i + v_i h \quad (14)$$

With this algorithm we can calculate the position of the object at as many point as we like, with an increased error with bigger time steps. The algorithm requires an initial velocity, position and the acceleration at each point of the position, which we will derive from the forces action upon on the object. The Euler method has four FLOPS per time step.

This method is good for finding an approximated trajectory, but as we will see, it is far from perfect.

### 2.3.3 Velocity Verlet method

A more accurate method for finding the trajectory, the velocity and position that is, would be the velocity Verlet method. This too is based on the Taylor series, with every term from the fourth and on assumed to be negligible.

$$v_{i+1} = v_i + hv_i^{(1)} + \frac{h^2}{2}v_i^{(2)} + O(h^3) \quad (15)$$

$$x_{i+1} = x_i + hx_i^{(1)} + \frac{h^2}{2}x_i^{(2)} + O(h^3) \quad (16)$$

where $v_i^{(1)}$ is still the acceleration defined by the acting forces. We can remove second derivative by substituting it with the first derivative multiplied with the time step.

$$hv_i^{(2)} \approx v_{i+1}^{(1)} - v_i^{(1)}$$

This applied to our Taylor series gives us

$$\begin{aligned} v_{i+1} &= v_i + hv_i^{(1)} + \frac{h}{2}\left(v_{i+1}^{(1)} - v_i^{(1)}\right) \\ &= v_i + \frac{h}{2}\left(v_{i+1}^{(1)} + v_i^{(1)}\right) \end{aligned} \quad (17)$$

As for our position calculation, we will use the definition of motion derivatives, namely $v_i^{(1)} = x_i$.

$$\begin{aligned} x_{i+1} &= x_i + hx_i^{(1)} + \frac{h^2}{2}x_i^2 + O(h^3) \\ &= x_i + hv_i + \frac{h^2}{2}v_i^{(1)} \end{aligned} \quad (18)$$

As shown in figure **??** this method makes for a more precise calculation of the trajectories. But with 9 FLOPS per time step, this method is way more demanding in terms of computational calculation than the aforementioned.

## 2.4 Relativistic corrections

General relativity predicts corrections to the Newtonian force an object experiences in the vicinity of a massive object. For Mercury, the gravitational force is better approximated with the expression

$$F_G = \frac{GMM}{r^2}\left[1 + \frac{3l^2}{r^2c^2}\right], \quad (19)$$

where $M$ is the mass of the Sun, $M$ is the mass of Mercury, $l$ is the angular momentum of mercury around the sun center of mass and $c$ is the speed of light in vacuum.

## 2.5 FLOPS

FLOPS are floating point operations, meaning the effect of an operator on an expression. Examples are addition, multiplications etc. Every one of these operations will consume some processing power. Minimizing the total number of FLOPS is therefore essential in order to avoid a tedious program. By counting FLOPS we can attain an implication as to how much time the operations will consume.

#### 2.5.1 Euler:

$$\boxed{v_{i+1} = v_i + ah} \quad , \quad \boxed{x_{i+1} = x_i + v_i h} \tag{20}$$

With one iteration, that is one time step, the Euler method has 4 FLOPS.

#### 2.5.2 Verlet:

$$\boxed{v_{i+1} = v_i + \frac{h}{2}\left(v_{i+1}^{(1)} + v_i^{(1)}\right)}$$
$$\boxed{x_{i+1} = x_i + hv_i + \frac{h_2}{2}v_i^{(1)}} \tag{21}$$

By stating a new parameter $h_2 = h^2$ and calculating it before the loop, we will save one FLOP per iteration. This might not seem significant, but if the program runs 10 years with 10.000 iterations per year, every FLOP counts. This tallies up to a total of 8 FLOPS.

In conclusion, the Verlet method is slower than Euler's.

# 3 Methods

## 3.1 An object oriented approach

We implement three different classes in `c++`; `CelestialBodies` represents the individual bodies as point particles, storing their physical variables. These are then stored in `SolarSystem`, which represents the environment and the physical laws. Finally, `Integrator` is responsible for propagating the system forward in time, using the forces and an numerical integration method to find the object positions in the next time step.

The basis of this structure was laid by Anders Hafreager https://github.com/andeplane/solar-system, providing a basic forward euler and declaration of many of the member variables and functions needed for the project.

## 3.2 Two-body problem

We assume a circular orbit of the earth around the sun, and also that the acceleration of the sun due to the earth is negligible. The Sun-Earth-system is then initialized by setting the Sun to a fixed position at $\vec{r}_=0$, and the position of Earth to $\vec{r}_=1\,\text{AU}\hat{e}_x$ with velocity $\vec{v}_=2\pi\,\text{AU/yr}\hat{e}_y$.

The solution of the orbit of Earth is found using both Forward Euler and Velocity Verlet.

## 3.3

## 3.4 Stability analysis

Now that we have established a simple solar system with two bodies, we can test the stability of our algorithm. We do this by testing different time steps $\Delta t$. This is applied to both our Euler and our Verlet algorithm with time steps in $N \in \{5 \to 5 \times 2^{19}\}$ steps per year, increasing with steps of power of two.

## 3.5 Escape velocity

To study the conditions for a planet to be in a bound state (elliptical orbit), we plot a series of different initial velocities and integrating the force. Then we can observe on the plots what velocity is needed for the earth to leave the orbit. When this velocity is found, we can compare it to the analytical expression, given by expression (10).

## 3.6 Adding Jupiter

We analyse the stability of the solar system when including Jupiter () in our previous setup. The initial position of Jupiter is set to $\vec{r}_= - 5.2\,\text{AU}\hat{e}_x$, with velocity $\vec{v}_= - 1\,\text{AU/yr}\hat{e}_y$, found from inserting numbers in eq. (3). The mass of Earth is now relevant, as we will be calculating the gravitational effects from all bodies on all the others except on the Sun. The masses of the objects are listed in the appendix.

#### 3.6.1 Massive Jupiter

We then increase the mass of Jupiter to study the effects on the stability of the orbit of earth. The mass is increased to $10x$ and $1000x$ the original mass of Jupiter, but otherwise we use the same initial conditions.

#### 3.6.2 Three-body problem

Moving towards a more realistic model of the solar system, we let the Sun be affected by the gravitational effects of the planets. The initial conditions are now set from real data. NASA has an interface (`https://ssd.jpl.nasa.gov/horizons.cgi`) to celestial body initial conditions, which can be parsed by the preprocessor script `analyse/read_horizon.py` (Ephemeris type needs to be set to VECTORS). Consider that the velocities received by NASA are assumed to be around a common barycenter for the whole solar system; if one then excludes some planets, the total momentum will not be zero and the system will drift. Correcting for this is done with the `--fix_barycenter` argument of `read_horizon.py`. It

finds the total velocity of the barycenter and sets it to zero.

## 3.7 N-body problem

Our system now includes several objects that all interact with one another. We have made sure that our time step is a reasonable one and that the energy is conserved. The next step is adding all the other planets. We use the same strategy as with Jupiter, with the relevant data aquired from NASA.

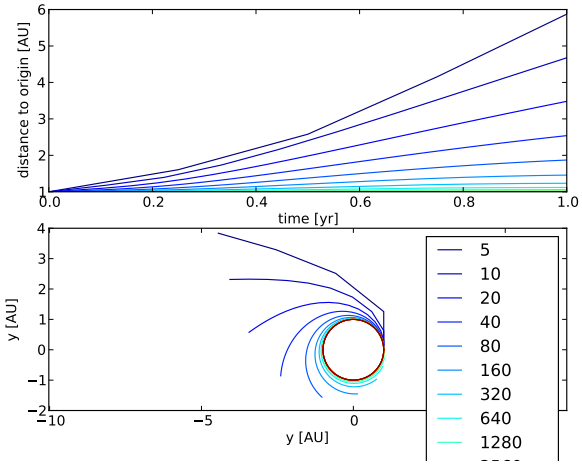



Figure 1: Stability of the Earth's orbit with different time steps $\Delta t$. Bigger time steps give bigger deviance.

## 3.8 Mercury perihelion precession

The final addition to our solar system is applying the non-newtonian general relativistic effects the sun has on Mercury, with equation (19)

## 3.9 Testing and algorithm analysis

### 3.9.1 Stability of $\Delta t$

### 3.9.2 Energy and angular momentum conservation

### 3.9.3 FLOPS

# 4 Results

## 4.1 Two body problem

### 4.1.1 Stability of $\Delta t$

As shown in fig. 1 below, the stability of the algorithm clearly depends on the size of the time step. The plots shows how the distance of the planet gradually increases in time.

The orbits are calculated for the earth over the course of one year, with the biggest time step resulting in a distance of almost $1.5AU$.

### 4.1.2 Energy conservation

By plotting the energies of the system, we expect the total energy to be constant independently of how long we run the simulation. Figure 2 confirms this with the velocity verlet method, showing the energies and the angular momentum of the Earth. Figure 3 shows the corresponding calculation for the Forward Euler method.

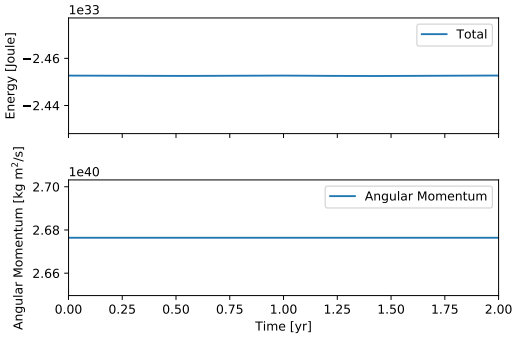



Figure 2: Conservation with the Verlet method. The energies and angular momentum of a two body system (the Earth and the Sun) in the reference frame of the largest object is plotted.

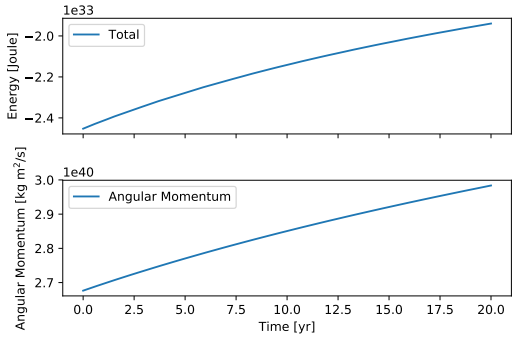



Figure 3: Conservation with the Euler method. The energies and angular momentum of a two body system (the Earth and the Sun) in the reference frame of the largest object is plotted.

#### 4.1.3 Escape velocity

The following figure shows how Earth leaves the orbit if the initial velocity is too large.

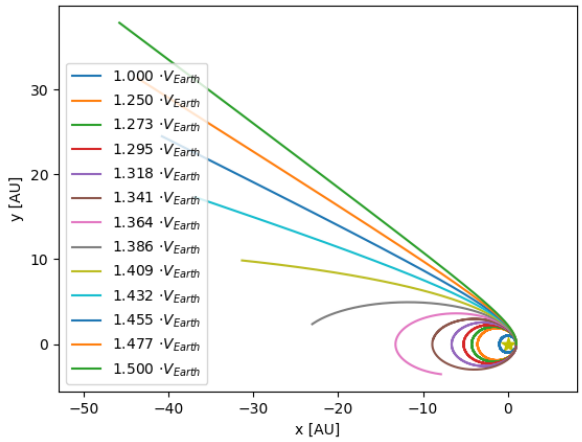



Figure 4: The trajectory of Earth with different initial velocities compared to the real one

The analytical escape velocity is found with equation (10) and results in

$$v_e = \sqrt{8\pi^2 AU^2/year^2} = \sqrt{2}2\pi AU/year \approx 1.414 V_{Earth}$$

### 4.2 Adding Jupiter

#### 4.2.1 Normal Jupiter

By adding Jupiter with the sun fixed in the centre

#### 4.2.2 Massive Jupiter

One we increase the mass of Jupiter to the mass of the sun, Earth immediately loses its track and gets violently tossed between the two objects (fig. 6). The energy is still conserved though, as seen in fig. 5. Changing the time step to a smaller one, we get the movement in fig. 9, showing an unstable system.

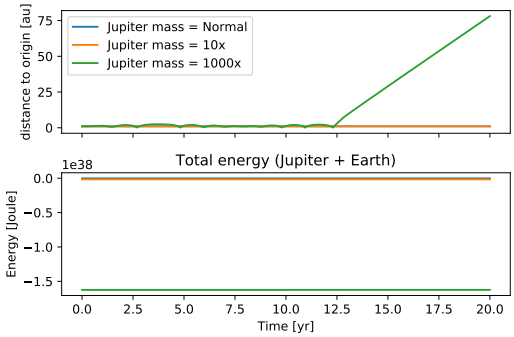



Figure 5: Energy of earth while jupiter is present, different masses for the planet jupiter, with a fixed sun.

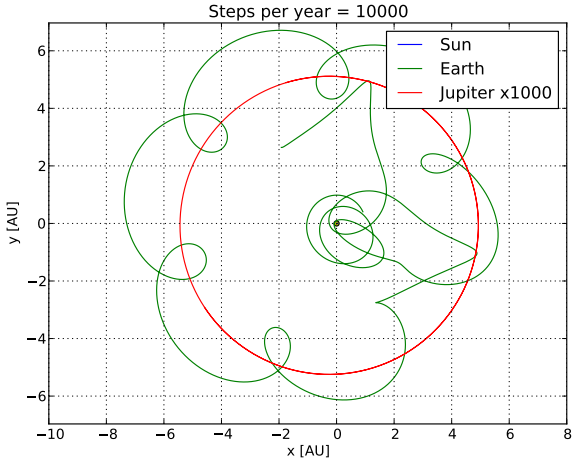



Figure 6: The Earth-Jupiter-Sun system where the mass of jupiter is increased by a factor 1000, with the Sun held in position

#### 4.2.3 Three-body problem

Letting the sun move as well, we get a plot similar to the one before in **??**. The corresponding orbits are shown in **??**, showing a binary system with a planet getting slung

around.

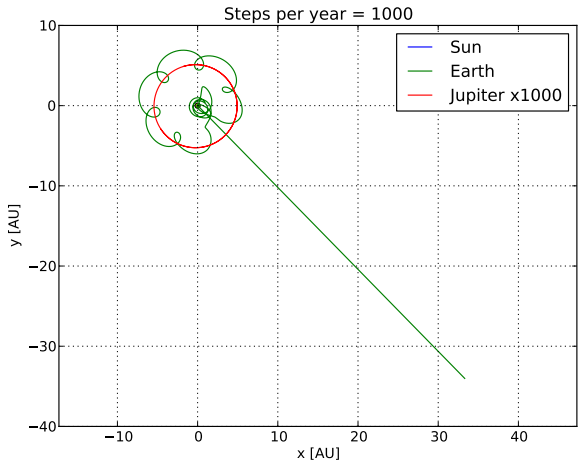



Figure 8: The Earth-Jupiter-Sun system where the mass of jupiter is increased by a factor 1000, with the Sun held in position

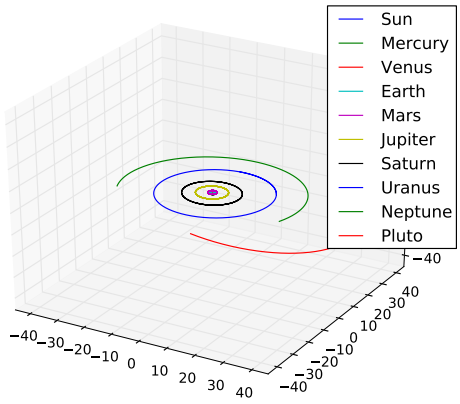



Figure 10: All ten objects of our solar system in three dimensions.

Zooming in, we can confirm that the inner planets also follow cyclical orbits

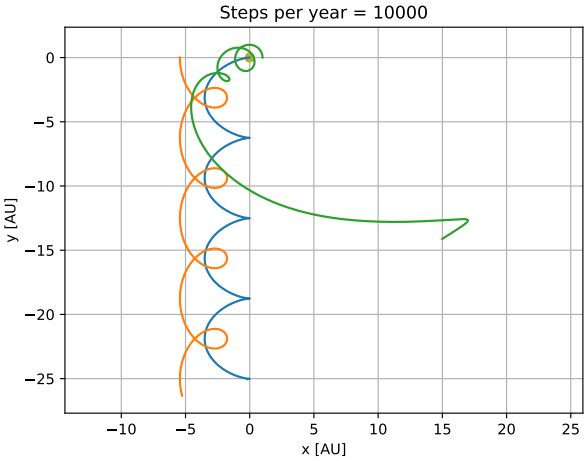
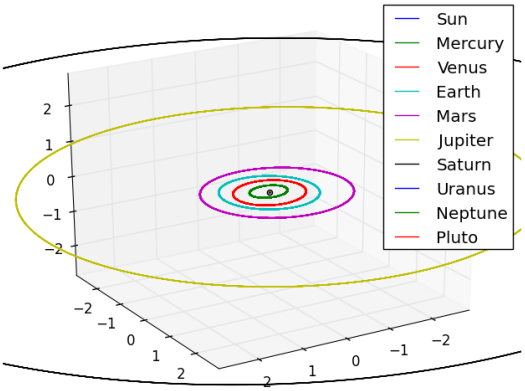



Figure 9: The Earth-Jupiter-Sun system where the mass of jupiter is increased by a factor 1000, with a moving Sun.

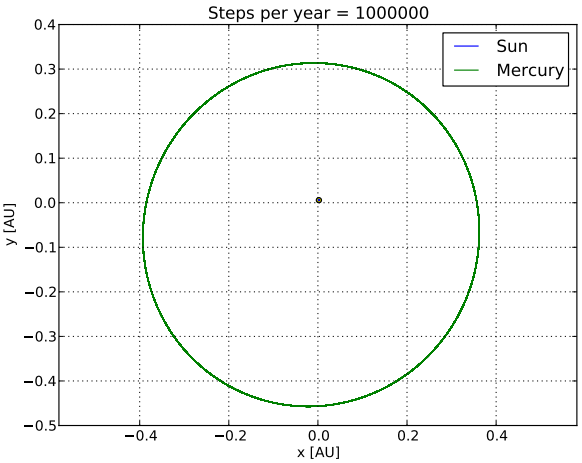



## 4.3 N-body problem

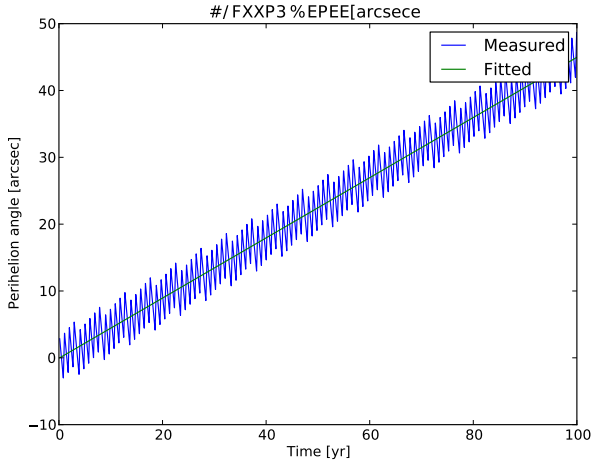
With all the planets inserted, our solar looks satisfyingly familiar

## 4.4 Mercury

With the relativistic effects applied to Mercury, we observe how the elliptical orbit slightly rotates around its elliptical orbit. The orbit precesses 44.5 arcseconds in 100 years, which isnt very far from the expected result of around 43 arcseconds.




The change is too small to recognized on the orbit, but by calculation, we can find. As we can see on the following plot, the angle of its orbit changes linearly.




# 5 Discussion and conclusions

## 5.1 Comparison of methods

As we predicted, the Verlet method produces far more precise results than Euler. With the Verlet method we were able to calculate the case of Mercury with $10^8$ time steps in 173.48 seconds. The program crashes due to overflow if it's run with Euler's method.

## 5.2 Testing of $\Delta t$

As seen in figure

## 5.3 Two-body system

### 5.3.1 Escape velocity

As figure — shows, the planet will remain in orbit for all initial velocities less than some value $v_e \in [1.409, 1.432]v_{Earth}$. This checks out with the analytical value we calculated to be $v_e = 1.414 \cdot v_{Earth}$.

## 5.4 Three-body system

### 5.4.1 Massive Jupiter

As we increased the mass of Jupiter, the Earth loses it's circular orbit as it starts getting pulled in different directions by the two massive objects. This is as expected as we have increased its mass to that of the sun, effectively making it a binary star system, except for the fact that the sun is fixed in the origin.

## 5.5 N-body system

With all the initial conditions collected from NASA, our solar system takes form. As we have have modelled our system in all three dimensions, it appears that not all planets move in the same plane. And sure enough, Pluto's $17^O$ inclination reveals itself.

# Appendix

See https://github.com/halvarsu/FYS3150/ for code.

From [?]:

Table 1: Masses of the bodies in the solar system

| Body | Mass $10 \times 10^{24}$ kg | Distance to sun in AU |
|---|---|---|
| Sun | $1.989 \times 10^6$ | 0 |
| Mercury | 0.3301 | 0.39 |
| Venus | 4.867 | 0.72 |
| Earth | 5.972 | 1 |
| Moon | 0.073 | 1 |
| Mars | 0.642 | 1.52 |
| Jupiter | 1898 | 5.3 |
| Saturn | 568 | 9.54 |
| Uranus | 86.8 | 19.19 |
| Neptune | 102 | 30.06 |
| Pluto | 0.0146 | 39.53 |

# References

[1] M.H.Jensen. Fys3150 lecture notes. fall 2015.