MAE 6110: HW #1

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1. Let $\mathbf{v} = \mathbf{a} \times \mathbf{b}$. Using indicial notation, show

(a) $\mathbf{v} \cdot \mathbf{v} = |\mathbf{a}|^{\mathbf{2}}|\mathbf{b}|^{\mathbf{2}} sin^2\theta$, where θ is the angle between $\mathbf{a}, \mathbf{b}$

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{v} &= v_i \mathbf{e}_i \cdot v_j \mathbf{e}_j \\
&= (a_k \mathbf{e}_k \times b_l \mathbf{e}_l) \cdot (a_m \mathbf{e}_m \times b_n \mathbf{e}_n) \\
&= (a_k b_l sin\theta \mathbf{n}) \cdot (a_m b_n sin\theta \mathbf{n}) \\
&= a_k b_l a_m b_n sin^2\theta \mathbf{n} \cdot \mathbf{n} \\
&= |\mathbf{a}|^2 |\mathbf{b}|^2 sin^2\theta
\end{aligned}$$

(b) $\mathbf{v} \cdot \mathbf{a} = 0$

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{a} &= (a_j \mathbf{e}_j \times b_k \mathbf{e}_k) \cdot a_m \mathbf{e}_m \\
&= (a_j b_k \mathbf{e}_j \times \mathbf{e}_k) \cdot a_m \mathbf{e}_m \\
&= a_j b_k a_m (\mathbf{e}_j \times \mathbf{e}_k) \cdot \mathbf{e}_m \\
&= a_j b_k a_m \epsilon_{jkm}
\end{aligned}$$

Here, since a_j and a_m stands for the same vector $\mathbf{a}$, we have $j = m$. Therefore we obtain $\epsilon_{jkm} = 0$. Hence, $\mathbf{v} \cdot \mathbf{a} = 0$.

2. Expand and simplify the following expressions where possible

a. $\delta_{ij}\delta_{ij} = \delta_{ii}$

b. $\delta_{ij}\delta_{jk}\delta_{ki} = \delta_{ii}$

c. $\delta_{ij}A_{ik} = A_{jk}$

3. Determine the simplest form of

(a)

$$\epsilon_{3jk} a_j a_k = \epsilon_{312} a_1 a_2 + \epsilon_{321} a_2 a_1 = a_1 a_2 - a_2 a_1 = 0$$

(b)

$$\begin{aligned}
\epsilon_{ijk}\delta_{jk} &= \epsilon_{123}\delta_{23} + \epsilon_{132}\delta_{32} + \epsilon_{213}\delta_{13} + \epsilon_{231}\delta_{31} + \epsilon_{312}\delta_{12} + \epsilon_{321}\delta_{21} \\
&= \delta_{23} - \delta_{32} - \delta_{13} + \delta_{31} + \delta_{12} - \delta_{21} = 0
\end{aligned}$$

(c)

$$\begin{aligned}
\epsilon_{1jk} a_2 T_{kj} &= \epsilon_{123} a_2 T_{32} + \epsilon_{132} a_2 T_{23} \\
&= a_2 T_{32} - a_2 T_{23}
\end{aligned}$$

4a. Show that the definition of skew symmetric is independent of basis, that is, if $B_{ik} = -B_{ki}$ in one basis, then $B'_{ik} = -B'_{ki}$ for a different basis.

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$$\mathbf{B} = B_{ik}\mathbf{e}_i\mathbf{e}_k = B'_{ik}\mathbf{e}'_i\mathbf{e}'_k$$

Given that $B_{ik} = -B_{ki}$, we have

$$\mathbf{B} = -B_{ki}\mathbf{e}_i\mathbf{e}_k = B'_{ik}\mathbf{e}'_i\mathbf{e}'_k$$

By the vector transformation rule, we have $\mathbf{e}'_i = p_{ij}\mathbf{e}_j$, then the previous equation takes the form:

$$\begin{aligned} -B'_{ik}\mathbf{e}'_i\mathbf{e}'_k &= B'_{ki}p_{km}\mathbf{e}_m p_{in}\mathbf{e}_n \\ &= B'_{ki}p_{km}p_{in}\mathbf{e}_m\mathbf{e}_n \\ &= B'_{ki}\mathbf{e}'_k\mathbf{e}'_i \end{aligned}$$

Therefore, $B'_{ik} = B'_{ki}$.

b. A tensor $\mathbf{B}$ has components $B_{ik} = \epsilon_{ijk}v_j$. show that it is skew symmetric.
From $B_{ik} = \epsilon_{ijk}v_j$, we got to know that:

$$\begin{gathered} B_{11} = 0,\ B_{12} = \epsilon_{132}v_3,\ B_{13} = \epsilon_{123}v_2, \\ B_{21} = \epsilon_{231}v_3,\ B_{22} = 0,\ B_{23} = \epsilon_{213}v_1, \\ B_{31} = \epsilon_{321}v_2,\ B_{32} = \epsilon_{312}v_1,\ B_{33} = 0 \end{gathered}$$

Therefore, tensor $\mathbf{B}$ writes: $\mathbf{B} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$

Then we can write $\mathbf{B}^{\mathrm{T}} = \begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{bmatrix}$ and $-\mathbf{B} = \begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{bmatrix}$

Therefore we have - $\mathbf{B} = \mathbf{B}^{\mathrm{T}}$

c. Let $\mathbf{B}$ be skew symmetric, define the vector $\mathbf{v} \equiv \epsilon_{ijk}B_{jk}\mathbf{e}_i$. Show $B_{mq} = \frac{1}{2}\epsilon_{mqi}v_i$
To show $B_{mq} = \frac{1}{2}\epsilon_{mqi}v_i$, we first substitute the given term $v_i = \epsilon_{ijk}B_{jk}\mathbf{e}_i$ into the equation:

$$\begin{aligned} B_{mq} &= \frac{1}{2}\epsilon_{mqi}\epsilon_{ijk}B_{jk}\mathbf{e}_i \\ &= \frac{1}{2}(\delta_{mj}\delta_{qk} - \delta_{mk}\delta_{qj})B_{jk}\mathbf{e}_i \\ &= \frac{1}{2}(B_{mq} - B_{qm})\mathbf{e}_i \\ &= \frac{1}{2}B_{mq} - \frac{1}{2}B_{qm} \\ \frac{1}{2}B_{mq} &= -\frac{1}{2}B_{qm} \end{aligned}$$

Since $\mathbf{B}$ be skew symmetric, we know $B_{mq} = -B_{qm}$. Then the preceding equation is automatically satisfied.

5. Solve the equation $A_{ij} = cB_{ij} + B_{kk}\delta_{jj}$ for B_{ij} where c is a given scalar constant.

$$A_{ij} = cB_{ij} + B_{kk}$$

Let $A_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $B_{ij} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$

We can further expand the given equation:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} cb_{11} & cb_{12} & cb_{13} \\ cb_{21} & cb_{22} & cb_{23} \\ cb_{31} & cb_{32} & cb_{33} \end{bmatrix} + \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} (c+1)b_{11} & cb_{12} & cb_{13} \\ cb_{21} & (c+1)b_{22} & cb_{23} \\ cb_{31} & cb_{32} & (c+1)b_{33} \end{bmatrix}$$

Therefore, we can write the solution of the equation as:

$$\begin{cases} B_{ij} = \dfrac{1}{c}A_{ij}, & i \neq j \\ B_{ij} = \dfrac{1}{c+1}A_{ij}, & i = j \end{cases}$$

6. (a) Let $\mathbf{A} = a_{ij}\mathbf{e}_i\mathbf{e}_j, a_{ij} = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) Let $\mathbf{B} = b_{ij}\mathbf{e}_i\mathbf{e}_j, b_{ij} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \lambda_i > 0$

Give a geometrical description of these linear transformations, hint: what does it do to a vector? What about the tensor $\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$?

(a) If we set a vector $\mathbf{D} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$, then the linear transformation of $\mathbf{A}(\mathbf{D}) = a_{ij}\mathbf{e}_i\mathbf{e}_j \cdot \mathbf{D}$

$$= \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} d_1cos\theta - d_2sin\theta \\ d_1sin\theta + d_2cos\theta \\ d_3 \end{bmatrix}$$

The geometric description is: tensor $\mathbf{A}$ acts as a rotational transformation of angle θ along the x_3 axis.

(b) With the same vector $\mathbf{D}$, we have $\mathbf{B}(\mathbf{D}) = b_{ij}\mathbf{e}_i\mathbf{e}_j \cdot \mathbf{D}$

$$= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 d_1 \\ \lambda_2 d_2 \\ \lambda_3 d_3 \end{bmatrix}$$

The geometric description is: each axial distance of the vector multiplies by λ_i

(c) With $\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$, applying to vector $\mathbf{D}$, we have $\mathbf{C}(\mathbf{D}) = c_{ij}\mathbf{e}_i\mathbf{e}_j \cdot \mathbf{D}$

$$= \begin{bmatrix} \lambda_1 cos\theta & -\lambda_2 sin\theta & 0 \\ \lambda_1 sin\theta & \lambda_2 cos\theta & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 d_1 cos\theta - \lambda_2 d_2 sin\theta \\ \lambda_1 d_1 sin\theta + \lambda_2 d_2 cos\theta \\ \lambda_3 d_3 \end{bmatrix}$$

The geometric description of $\mathbf{C}$ is the it rotate a vector along the x_3 with each axial distance multiplies by λ_i.