Buckling of a Thin Plate

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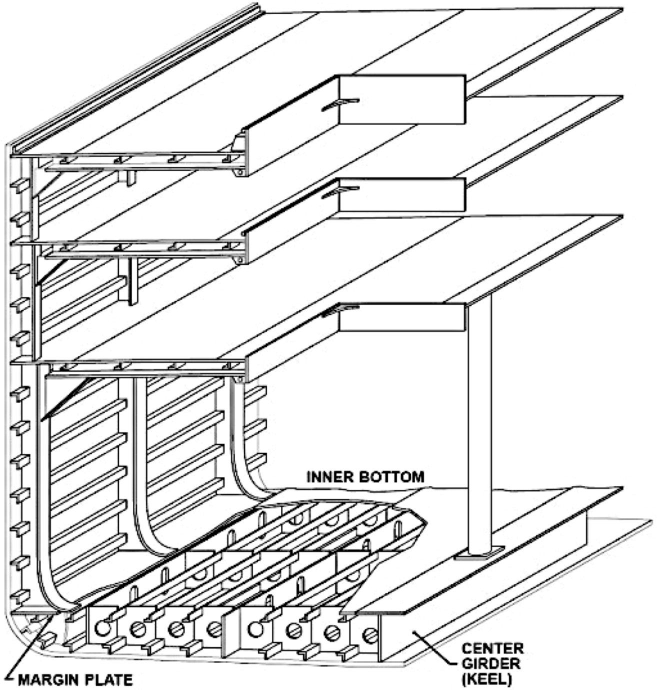
Content

- Introduction
- Real World Problems
- Governing Equations and Boundary Conditions
- Buckling of a Simply Supported Plate
- Effect of Boundary Conditions
- Buckling of Sections
- Post-buckling Response of Plates
- Ultimate Strength of Plates
- Effect of Initial Imperfection
- Reference



Introduction

Most of steel or aluminum structures are made of tubes or welded plates. Airplanes, ships and cars are assembled from metal plates pined by welling riveting or spot welding. Plated structures may fail by yielding fracture or buckling. This lecture deals with a brief introduction to the analysis of plate buckling.

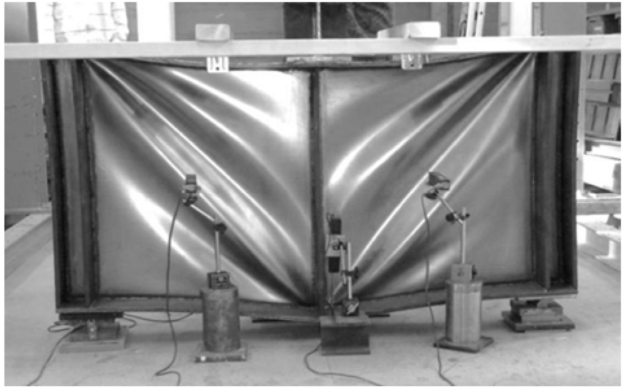
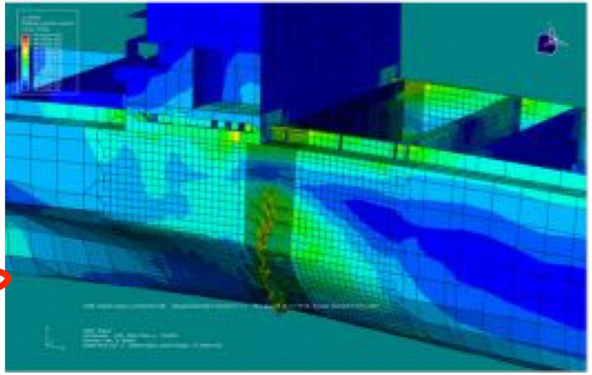


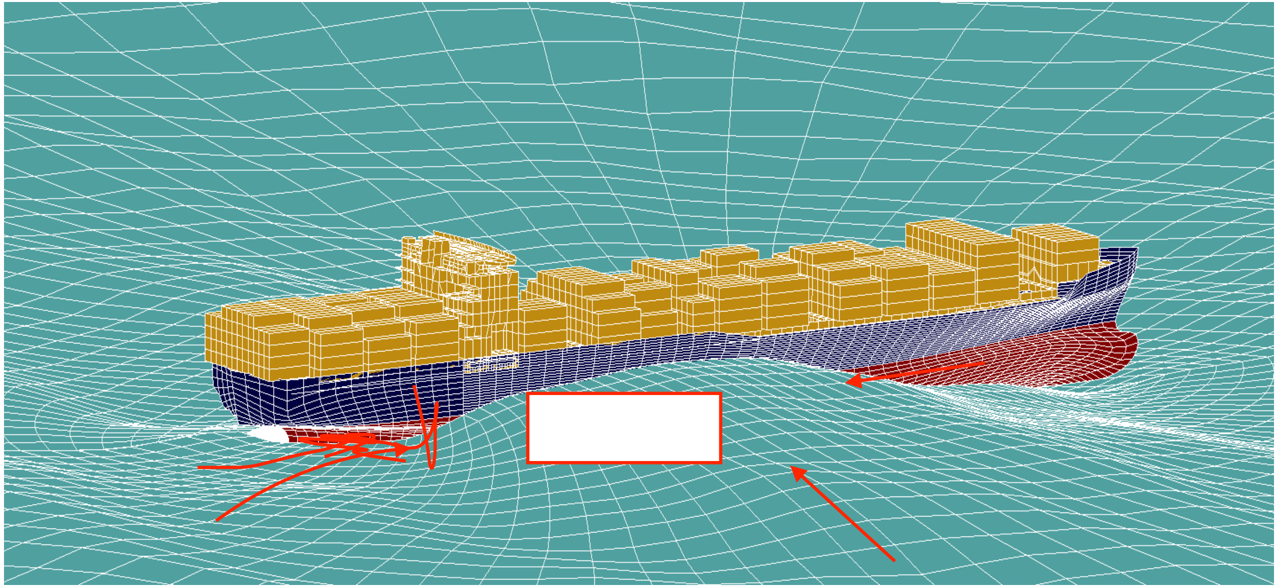
Figure. Post-buckling behavior of the tested plate girders.

Mamazizi, S. et al. Numerical and Experimental Investigation on the Post-Buckling Behavior of Steel Plate Girders Subjected to Shear.

Real world problems

- Direct wave load calculations
- Linear strength analysis
- Non-linear strength analysis
- Load and strength comparisons
- Simulation of crack propagation





1. Governing Equations and Boundary Conditions

Local equilibrium equation: $$EIw^{\mathrm{IV}} + Nw'' = 0 \quad ①$$

Plate buckling equation: $$D\nabla^4 w + \bar{N}_{\alpha\beta} w_{,\alpha\beta} = 0 \quad ②$$

$D\nabla^2\nabla^2 w =$

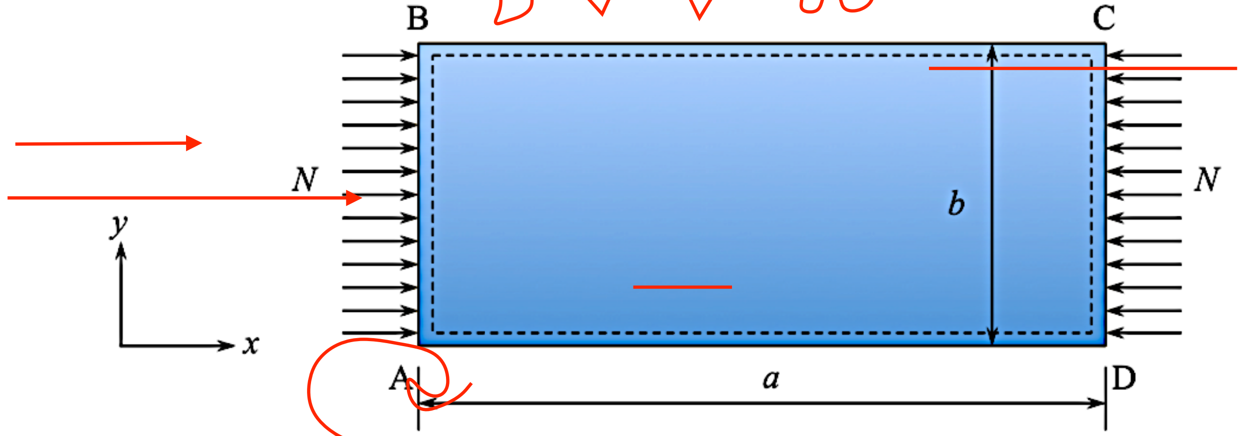


Figure 1: Geometry and loading of the classical plate buckling problem.

1. Governing Equations and Boundary Conditions

Boundary Condition:

$$w = 0 \quad \text{on ABCD}$$
$$M_{\mathrm{n}} = 0 \quad \text{on ABCD} \qquad (3)$$

In-plane direction in the normal and tangential direction:

$$(N_{\mathrm{n}} - \bar{N}_{\mathrm{n}})\delta u_{\mathrm{n}} = 0$$
$$(N_{\mathrm{t}} - \bar{N}_{\mathrm{t}})\delta u_{\mathrm{t}} = 0 \qquad (4)$$

Reduced equations for (3):

$$\left.\begin{aligned}(N_{xx} - \bar{N}_{xx})\delta u_x = 0\\ (N_{xy} - \bar{N}_{xy})\delta u_y = 0\end{aligned}\right\} \text{ on AB and CD}$$
$$\left.\begin{aligned}(N_{yy} - \bar{N}_{yy})\delta u_y = 0\\ (N_{xy} - \bar{N}_{xy})\delta u_x = 0\end{aligned}\right\} \text{ on AD and BC} \qquad (5)$$

1. Governing Equations and Boundary Conditions

The stress boundary conditions are applied and the tensor of external loading is:

$$\bar{N}_{\alpha\beta} = \begin{vmatrix} \bar{N} & 0 \\ 0 & 0 \end{vmatrix}, \quad N_{\alpha\beta} = \begin{vmatrix} N & 0 \\ 0 & 0 \end{vmatrix} \qquad ⑥$$

Constitutive equations:

$$N_{xx} = C(\epsilon^{\circ}_{xx} + \nu\epsilon^{\circ}_{yy})$$
$$0 = C(\epsilon^{\circ}_{yy} + \nu\epsilon^{\circ}_{xx}) \qquad ⑦$$

Displacement-strain relationship:

$$\epsilon^{\circ}_{xx} = \frac{\mathrm{d}u_x}{\mathrm{d}x}$$
$$\epsilon^{\circ}_{yy} = \frac{\mathrm{d}u_y}{\mathrm{d}y} \qquad ⑧$$

Solution: $$u_x = u_o\left(1 - \frac{x}{a}\right), \quad u_y = \nu u_o \frac{y}{a}, \quad N = \frac{Eh}{a}u_o \qquad ⑨$$

2. Buckling of a Simply Supported Plate

The expanded form of the governing equation: $$D\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right] + \bar{N}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 0 \qquad (10)$$

The solution is sought as a product of two harmonic functions:

$$w(x,y) = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad (11)$$

The bending moment M_n:

$$M_n = M_{xx} = D[\kappa_{xx} + \nu\kappa_{yy}] = -D\left[\left(\frac{m\pi}{a}\right)^2 + \nu\left(\frac{n\pi}{b}\right)^2\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad (12)$$

2. Buckling of a Simply Supported Plate

Substitute the condition on edges: (*x=0* and *x=a*; *y=0* and *y=b*: $M_n = 0$)

$$\left\{D\left[\left(\frac{m\pi}{a}\right)^4+2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2+\left(\frac{n\pi}{b}\right)^4\right]-\bar{N}\left(\frac{m\pi}{a}\right)^2\right\}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}=0 \quad (13)$$

The differential equation is satisfied for all values of (*x, y*) if the coefficients satisfy:

$$\bar{N}=D\left(\frac{\pi a}{m}\right)^2\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2 \quad (14)$$

Reduced equation for (12) :

$$\bar{N}_c=k_c\frac{\pi^2 D}{b^2} \quad (15)$$

Where $k_c=\left(\frac{mb}{a}+\frac{a}{mb}\right)^2$ (16)

2. Buckling of a Simply Supported Plate

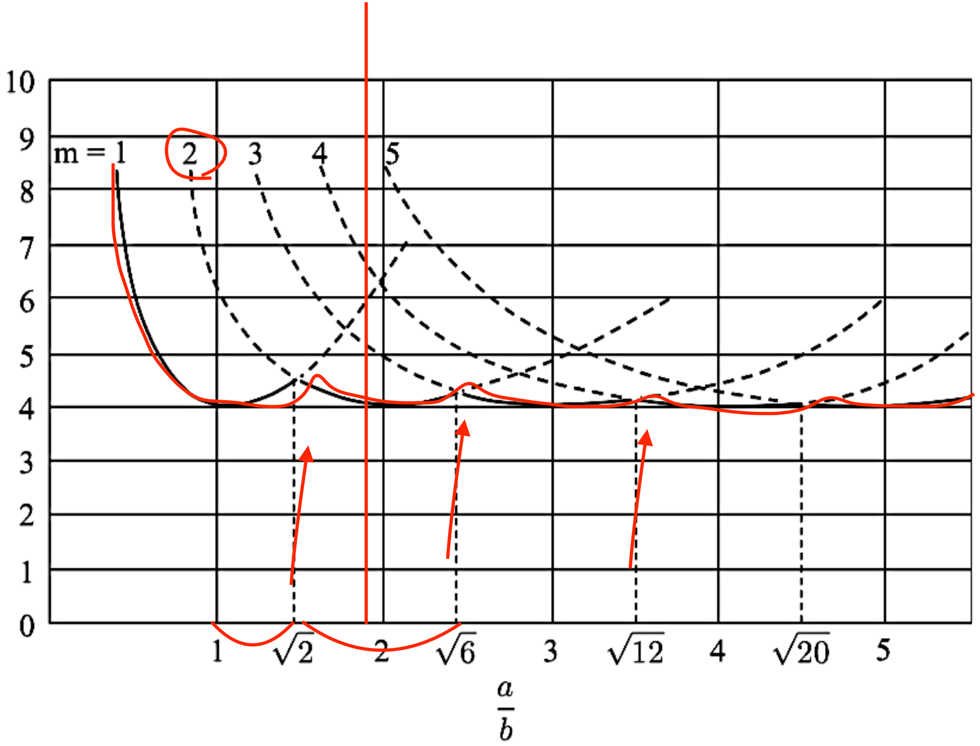


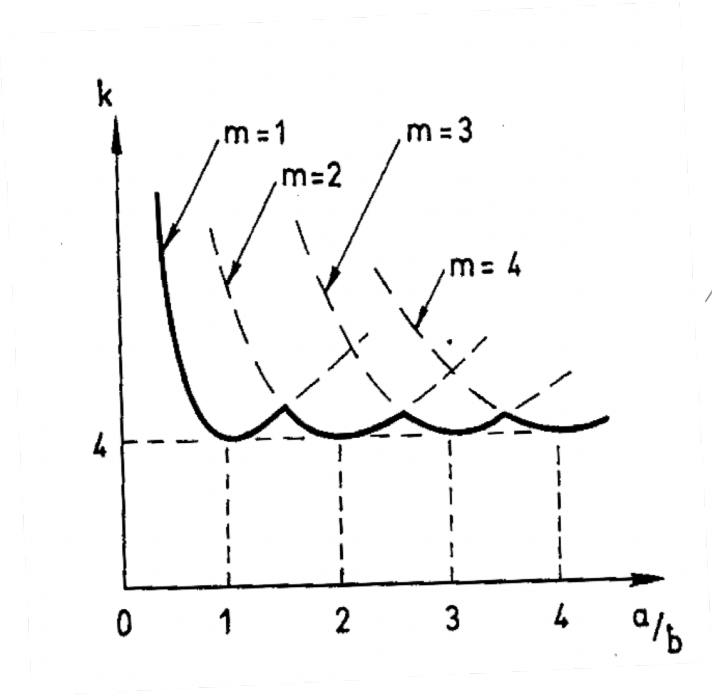
Figure 2: Plot of the buckling coefficient for a simply supported plate as a function of the plate aspect ratio a/b and different wave numbers.

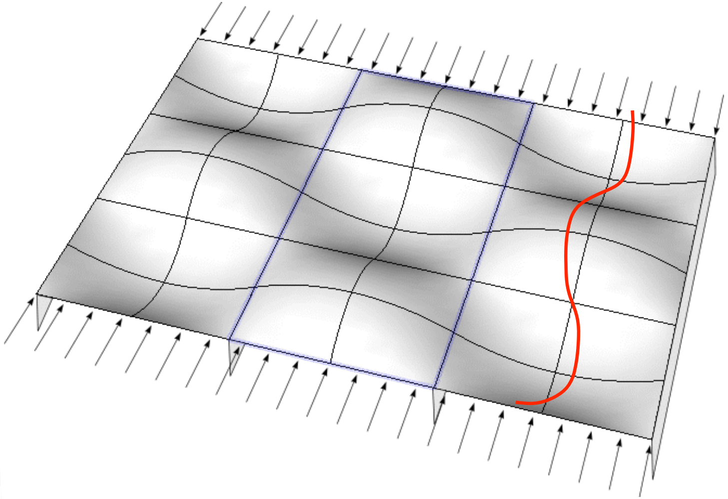
For example, the buckling coefficient corresponding to the first five buckling modes corresponding to $\frac{b}{a} = 2$

m	1	2	3	4	5
k_c	6.2	4	4.7	6.2	8.4

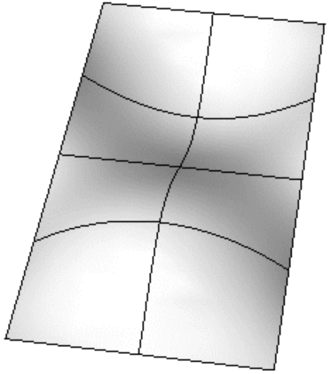
- The lowest buckling load $k_c = 4$ occurs when there are two half waves along the length of the plate, m = 2. The line separating the safe, shaded area in Fig.2 and the unsafe while area defines uniquely the buckling coefficient for all combination of a/b and m.

2. Buckling of a Simply Supported Plate

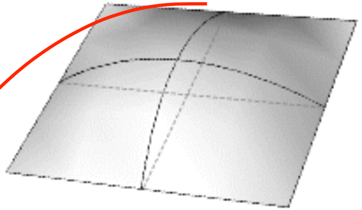




a/b=3, m=3



a/b=2, m=2



a/b=1, m=1

2. Buckling of a Simply Supported Plate

Consider now a long plate, $a \gg b$ for which the parameter m can be treated as a continuous variable.

$$\frac{\mathrm{d}k_\mathrm{c}}{\mathrm{d}m} = 0 \quad \rightarrow \quad a = mb \qquad (17)$$

What happens when the rectangular plate is restricted from lateral expansion：

$$u_y(y=0) = u_y(y=b) = 0 \qquad (18)$$

With no strain in the y-direction, $\varepsilon_{yy} = 0$, the constitutive equations (6) reduces to：

$$\left\{\begin{aligned} N_{xx} &= C\epsilon^{\circ}_{xx} \\ N_{yy} &= C\nu\epsilon^{\circ}_{xx} \end{aligned}\right. \qquad (19)$$

2. Buckling of a Simply Supported Plate

The new expression for the buckling coefficient is:

$$k_c = \frac{\left[\left(\frac{mb}{a}\right)^2 + n^2\right]^2}{\left(\frac{mb}{a}\right)^2 + \nu n^2} \quad (20)$$

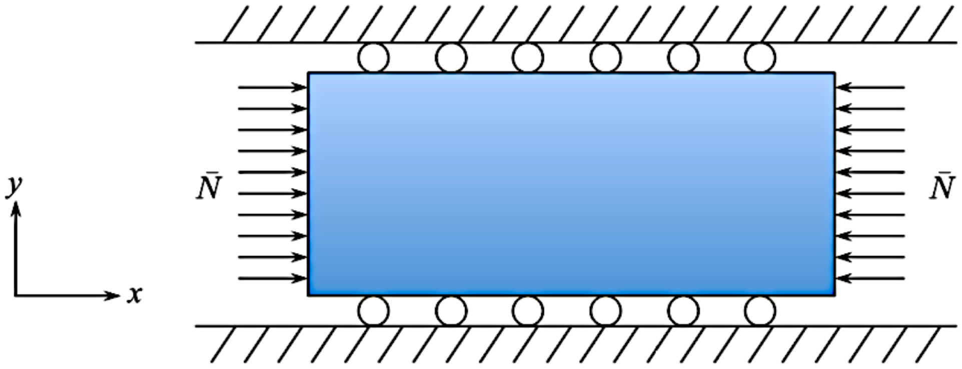


Figure 3: Constrained compression of the plate.

Taking again as an example a/b = 2, the values of the buckling coefficient corresponding to the nine first buckling modes are

n \ m	1	2	3
1	10.7	3	4.09
2	3.8	10.7	10.9
3	26	25	24.1

3. Effect of Boundary Conditions

The unloaded edges of rectangular plates can be either simply supported (ss), clamped (c) or free.

(The sliding boundary conditions will convert the eigenvalue problem into the equilibrium problem and therefore are not considered in the buckling analysis of plates).

An approximate analytical solution for the case "E" was derived by *Timoshenko* and *Gere* in the form

$$k_c = 0.456 + \left(\frac{b}{a}\right)^2 \tag{21}$$

An approximate solution for k_c to shear buckling, derived by *Timoshenko* and *Gere* has the form

$$k_c = 5.35 + 4\left(\frac{b}{a}\right)^2 \tag{22}$$

For a square plate the buckling coefficient is 9.35 while for an infinitely long plate, $a \gg b$ it reduces to 5.35.

4. Buckling of Sections

The buckling coefficient is plotted against the plate aspect ratio a/b for all these combinations in Fig.4. It is seen that the lowest buckling coefficient with m = 1 corresponds to a simply supported plate on three edges and free on the fourth edge.

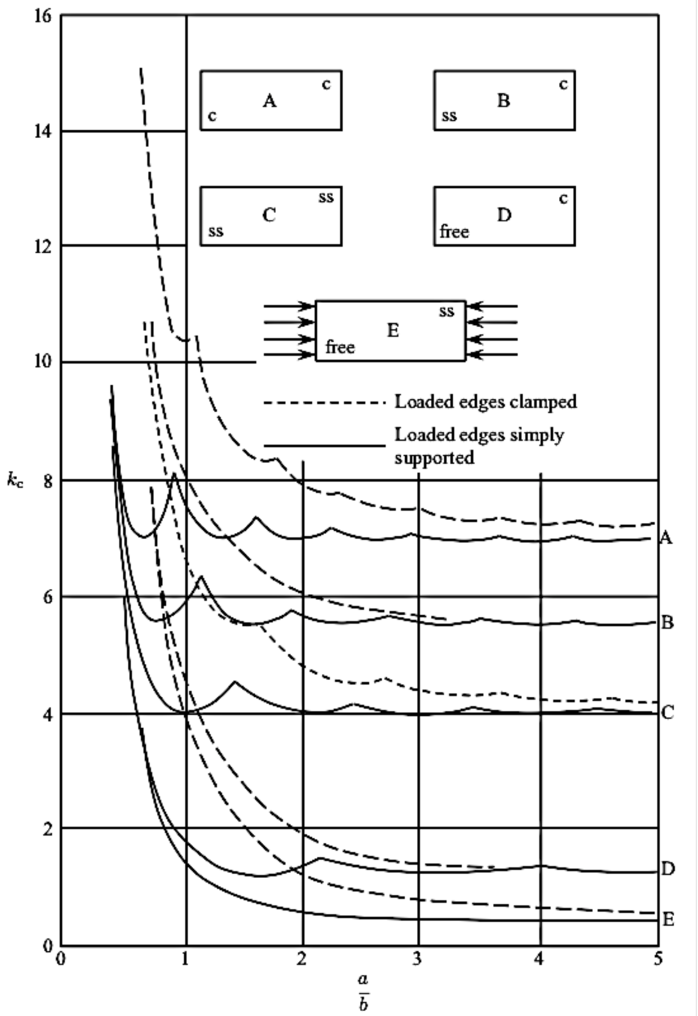


Figure 4: Effect of boundary conditions on the buckling coefficient of rectangular plates subjected to in-plane boundary conditions.

4. Buckling of Sections

An angle element, shown in Fig.5 is composed of two plates that are simply supported along the common edge and free on the either edges. Both plates rotate by the same amount at the common edges so that no edge restraining moment is developed. This corresponds to a simply supported boundary conditions.

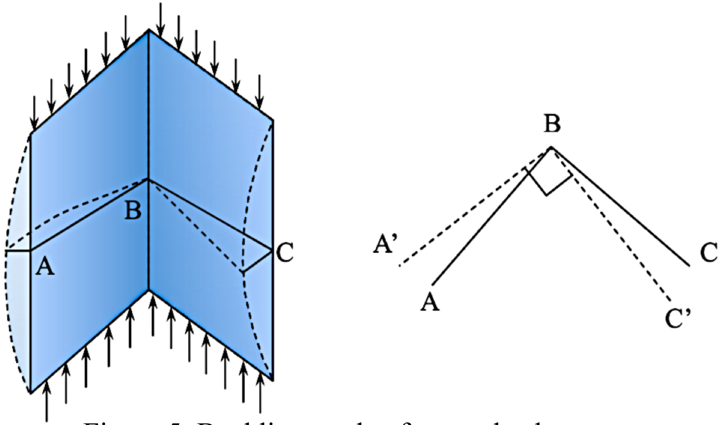


Figure 5: Buckling mode of an angle element.

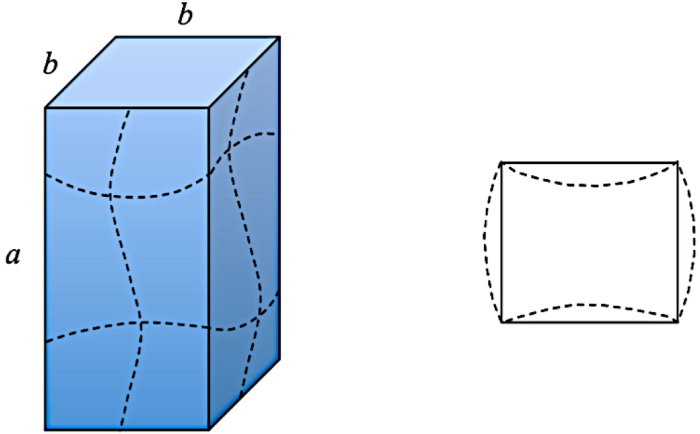


Figure 6: The buckling mode of a prismatic square column.

In a similar way it can be proved that the prismatic square column consists of four simply supported long rectangular plates. Upon compression, the buckling pattern has a form shown in Fig.6 Again, there are no relative rotations at the intersection line of any of the neighboring plates ensuring the simply supported boundary condition along four edges.

4. Buckling of Sections

Another very practical case is shear loading. For example “I” beams with a relatively high web or girders may fail by shear buckling, Fig.7, in the compressive side when subjected to bending.

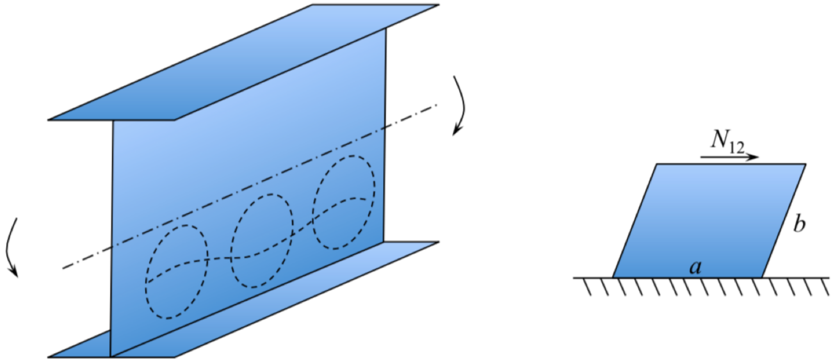


Figure 7: Buckling due to shear or bending.

Figure 8: A photograph of shear buckling of a plate representing the damage pattern on the ship’s hull inflicted upon grounding.

For a square plate the buckling coefficient is 9.35 while for an infinitely long plate, $a \gg b$ it reduces to 5.35. Loading the plate in the double shear experiment for beyond the elastic buckling load produces a set of regular skewed dimples seen in Fig.8.

4. Buckling of Sections

Figure 9: Some typical open and closed cross-sectional shape of prismatic members.

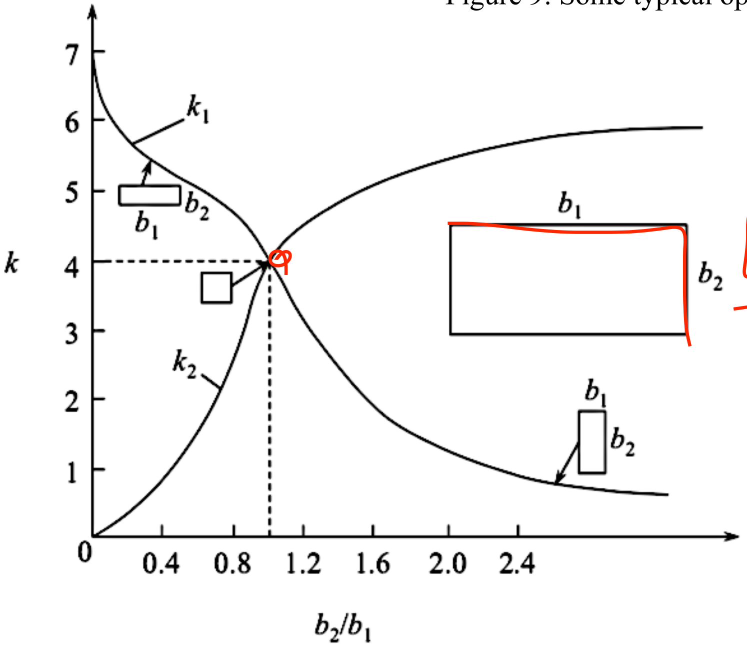


Figure 10: Buckling coefficients of a rectangular plate as a function of b2/b1.

Before buckling the strains and compressive stresses in the adjacent plates are the same:

Euler

$$\sigma_1 = \frac{N_1}{h_1} = \sigma_2 \frac{N_2}{h_2} \tag{23}$$

Where $N_1 = k_1 \frac{\pi^2 D_1}{b_1^2}, \quad N_2 = k_2 \frac{\pi^2 D_2}{b_2^2}$ (24)

From the above equation it follows that

$$k_2 = k_1 \left(\frac{b_2}{b_1}\right)^2 \tag{25}$$

4. Buckling of Sections

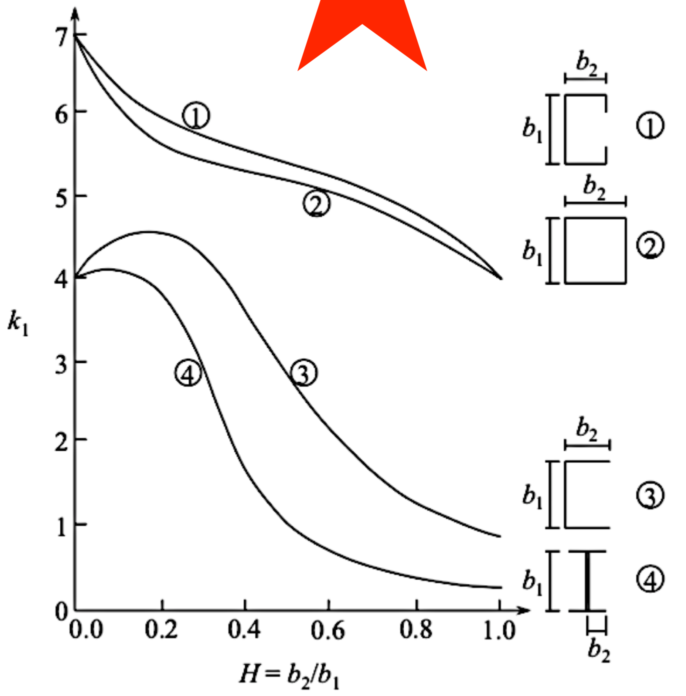


Figure 11: Buckling coefficients for four types of sections.

For more complex cross-sectional shape the buckling coefficient can be presented in a graphical form, as shown in Fig. 11. Knowing the buckling coefficient k_1 for a flange with the width b_1 and thickness h_1, the buckling coefficients of all other flanges is then calculated from:

$$k_i = k_1 \left(\frac{h_i b_1}{h_1 b_i} \right) \tag{26}$$

5. Post-buckling Response of Plates

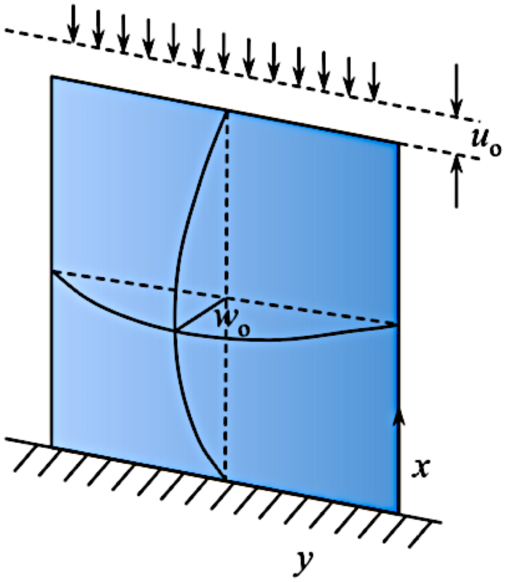


Figure 12: Two degree-of-freedom model of the buckled plate.

One more term should be added to the expression for u_y in order to satisfy zero traction at the unloaded edges.

$$u_x = u_o \left(1 - \frac{x}{a}\right)$$
$$u_y = \nu u_o \frac{y}{a} + f(x) \qquad (27)$$

The field of out-of-plane deformation is taken identical as in the buckling solution:

$$w = w_o \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \qquad (28)$$

5. Post-buckling Response of Plates

The total potential energy of the system is:

$$\Pi = U_{\mathrm{b}} + U_{\mathrm{m}} - PU_o \qquad (29)$$

The curvature tensor is defined by:

$$\kappa_{\alpha\beta} = -w_{,\alpha\beta} \qquad (30)$$

Where the assumed shape $\omega(x, y)$ has three components:

$$\kappa_{\alpha\beta} = w_o \left(\frac{\pi}{a}\right)^2 \begin{vmatrix} \sin\frac{\pi x}{a}\sin\frac{\pi y}{a} & -\cos\frac{\pi x}{a}\cos\frac{\pi y}{a} \\ -\cos\frac{\pi x}{a}\sin\frac{\pi y}{a} & \sin\frac{\pi x}{a}\sin\frac{\pi y}{a} \end{vmatrix} \qquad (31)$$

5. Post-buckling Response of Plates

The membrane strain:

$$\epsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) + \frac{1}{2}w_{,\alpha}w_{,\beta} \quad (32)$$

The components of the in-plane strain tensors are:

$$\left.\begin{aligned}\epsilon_{xx} &= -\frac{u_o}{a} + \frac{w_o^2}{2}\left(\frac{\pi}{a}\right)^2 \cos^2\frac{\pi x}{a}\sin^2\frac{\pi y}{a} \\ \epsilon_{yy} &= \nu\frac{u_o}{a} + f'(x) + \frac{w_o^2}{2}\left(\frac{\pi}{a}\right)^2 \sin^2\frac{\pi x}{a}\cos^2\frac{\pi y}{a}\end{aligned}\right\} \quad (33)$$

$$\epsilon_{xy} = \frac{w_o^2}{2}\left(\frac{\pi}{a}\right)^2 \cos^2\frac{\pi x}{a}\cos^2\frac{\pi y}{a} \quad (34)$$

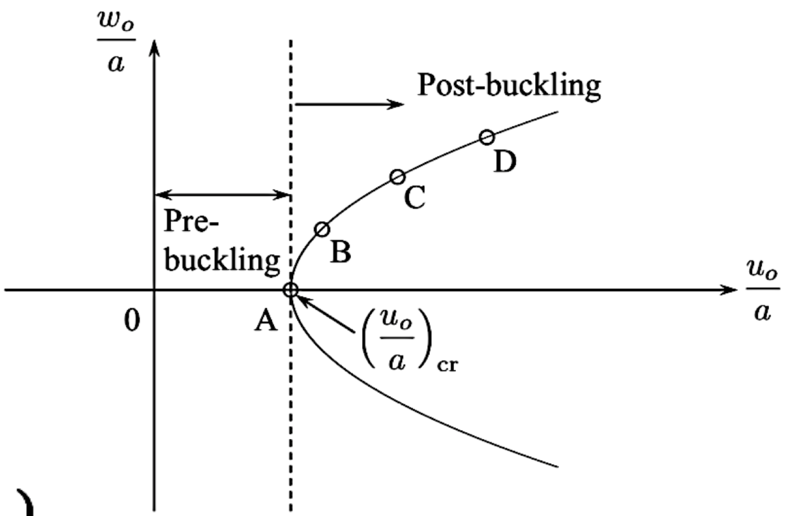


Figure 13: The out-of-plane displacement amplitude.

5. Post-buckling Response of Plates

The general expression for the bending energy of the plate:

$$U_b = \frac{D}{2}\int_0^a \int_0^a \{(\kappa_{xx} + \kappa_{yy})^2 - 2(1-\nu)\kappa_{\mathrm{G}}\}\,\mathrm{d}x\,\mathrm{d}y \tag{35}$$

The total bending energy of the plate is calculated to be:

$$U_b = \frac{1}{2} D w_o^2 \frac{\pi^4}{a^2} \tag{36}$$

The plane stress elasticity law is:

$$\begin{aligned} N_{xx} &= C(\epsilon_{xx} + \nu\epsilon_{yy}) \\ N_{yy} &= C(\epsilon_{yy} + \nu\epsilon_{xx}) \\ N_{xy} &= (1-\nu)C\epsilon_{xy} \end{aligned} \tag{37}$$

5. Post-buckling Response of Plates

From 19 , the in-plane membrane force in the y-direction is

$$N_{yy} = C\left[\nu\frac{u_o}{a} + \frac{1}{2}w_o^2\left(\frac{\pi}{a}\right)^2 \sin^2\frac{\pi x}{a}\cos^2\frac{\pi y}{a} + f' - \nu\frac{u_o}{a} + \frac{\nu}{2}w_o^2\left(\frac{\pi}{a}\right)^2\cos^2\frac{\pi x}{a}\sin^2\frac{\pi y}{a}\right] \quad (38)$$

The membrane force changes from point to point and at the unloaded edges $y = 0$ and $y = a$ is

$$N_{yy}(0,a) = C\left[\frac{1}{2}w_o^2\left(\frac{\pi}{a}\right)^2\sin^2\frac{\pi x}{a} + f'\right] \quad (39)$$

After lengthy algebra, the final expression of total membrane energy is

$$U_m = \frac{C}{2}\left[(1-\nu^2)u_o^2 - 2(1-\nu^2)\frac{\pi^2}{8}\frac{u_o}{a}w_o^2 + (3-2\nu)\frac{\pi^4}{64}\frac{w_o^4}{a^2}\right] \quad (40)$$

5. Post-buckling Response of Plates

The total potential energy of the system is: $\Pi(u_o, w_o) = U_\mathrm{b} + U_\mathrm{m} - Pu_o$ (41)

The equilibrium of the system requires that the first variation of the total potential energy vanishes $\delta\Pi(u_0, w_0) = 0$.

$$\frac{\partial \Pi}{\partial u_o} = 0 \quad \rightarrow \quad P = (1-\nu^2)C\left[u_o - \frac{\pi^2}{8}\frac{w_o^2}{a}\right] \qquad (42)$$

$$\frac{\partial \Pi}{\partial w_o} = 0 \quad \rightarrow \quad 64\left(\frac{\pi}{a}\right)^2 w_o\left[\frac{4\pi^2 D}{C} - (1-\nu^2)au_o + (3-2\nu)\frac{\pi^2}{8}w_o^2\right] = 0 \qquad (43)$$

The pre-buckling solution is recovered by setting $\omega_0 = 0$.

$$P = (1-\nu^2)Cu_o = (1-\nu^2)\frac{Eh}{1-\nu^2}u_o = Ehu_o \qquad (44)$$

5. Post-buckling Response of Plates

The relation between the in-plane and out-of-plane amplitude of the assumed displacement field

$$\frac{\pi^2}{8}\left(\frac{w_o}{a}\right)^2 = \frac{1-\nu^2}{3-2\nu}\frac{u_o}{a} - \frac{4D\pi^2}{C(3-2\nu)a^2} \quad (45)$$

Setting $\omega_0 = 0$ to critical displacement $(u_0)_c$: $$(u_o)_c = \frac{4\pi^2 D}{a}\frac{1}{C(1-\nu^2)} \quad (46)$$

Substitute (46) into Eq. (44)

$$P = \frac{13}{25}(1-\nu^2)Cu_o + \frac{1-\nu^2}{3-2\nu}\frac{4\pi^2 D}{a} \quad (47)$$

5. Post-buckling Response of Plates

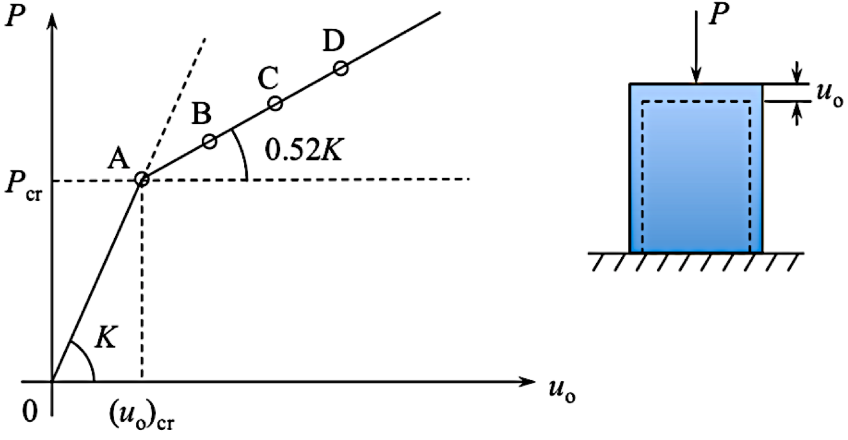


Figure 14: Pre and post-buckling response of a plate.

The post-buckling stiffness $K_{post} = \frac{dD}{du_0}$ is

$$K_{\text{post}} = \frac{13}{25}(1-\nu^2)C = 0.52K_{\text{pre}} \qquad (48)$$

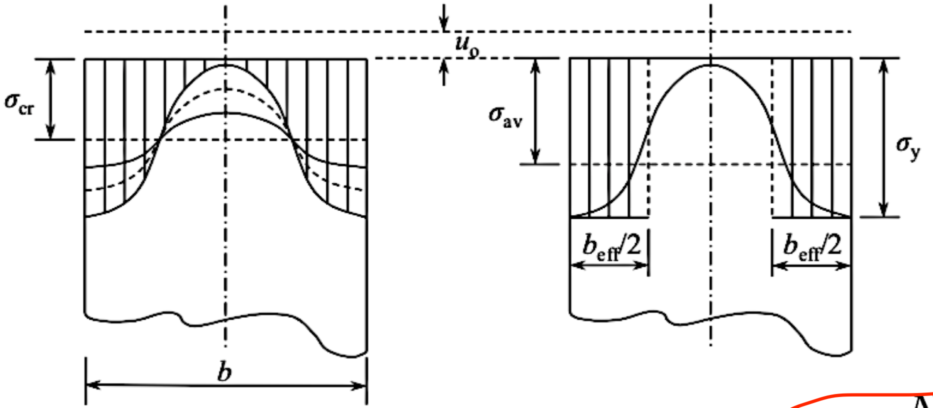
Substituting the expression for $(u_0)c$ into Eq. (44) the predicted buckling load is

$$P_c = \frac{4\pi^2 D}{a} \qquad (49)$$

6. Ultimate Strength of Plates

Let's examine the distribution of in-plane compressive stresses σ_{xx} at x = a. The components σ_{xx} is

$$\sigma_{xx}(y) = \frac{N_{xx}}{h} = \frac{E}{1-\nu^2}\left[-(1-\nu^2)\frac{u_o}{a} + \frac{\pi^2}{2}\left(\frac{w_o}{a}\right)^2 \sin^2\frac{\pi y}{a}\right] \quad (50)$$



von Karman used the expression for the critical buckling load N_c and looked at the relation between the stress at the loaded edge σ_e and the plate width b

$$\sigma_e = \frac{N_e}{h} = \frac{N_c}{h} = \frac{4\pi^2 D}{hb^2} = \frac{4\pi^2 E h^2}{12(1-\nu^2)b^2} = 1.9^2 E\left(\frac{h}{b}\right)^2 \quad (51)$$

Figure 15: Re-distribution of compressive stresses along the loaded edge and simple approximation by **von Karman.**

6. Ultimate Strength of Plates

von Karman asked what should be the width of the plate beff so that the edge stress reaches the yield stress.

Thus

$$\sigma_y = 1.9^2 E \left(\frac{h}{b_{\text{eff}}}\right)^2 \qquad (52)$$

Solving the above equation for b_{eff}

$$b_{\text{eff}} = 1.9h\sqrt{\frac{E}{\sigma_y}} \qquad (53)$$

Taking for example E = 200000 MPa, σ_y = 320 MPa, the effective width becomes

$$b_{\text{eff}} = 1.9h\sqrt{625} = 47.5h \qquad (54)$$

6. Ultimate Strength of Plates

The total load on the plate can be expressed in two ways

$$P_{\text{ult}} = b_{\text{eff}} \cdot \sigma_{\text{y}} = b \cdot \sigma_{\text{av}} \qquad (55)$$

where $\sigma_{av} = \sigma_{ult}$ is the average stress on the loaded edge at the point of ultimate strength,

$$\frac{\sigma_{\text{av}}}{\sigma_{\text{ult}}} = \frac{b_{\text{eff}}}{b} = 1.9\frac{h}{b}\sqrt{\frac{E}{\sigma_{\text{y}}}} \qquad (56)$$

The group of parameters

$$\beta = \frac{b}{h}\sqrt{\frac{\sigma_{\text{y}}}{E}} \qquad (57)$$

Using the parameter β, the ultimate strength of the plate normalized by the yield stress is

$$\frac{\sigma_{\text{ult}}}{\sigma_{\text{y}}} = \frac{1.9}{\beta} \qquad (58)$$

6. Ultimate Strength of Plates

Recall that the normalized buckling stress of the elastic plate is $\frac{\sigma_{cr}}{\sigma_y} = \left(\frac{1.9}{\beta}\right)^2$ (59)

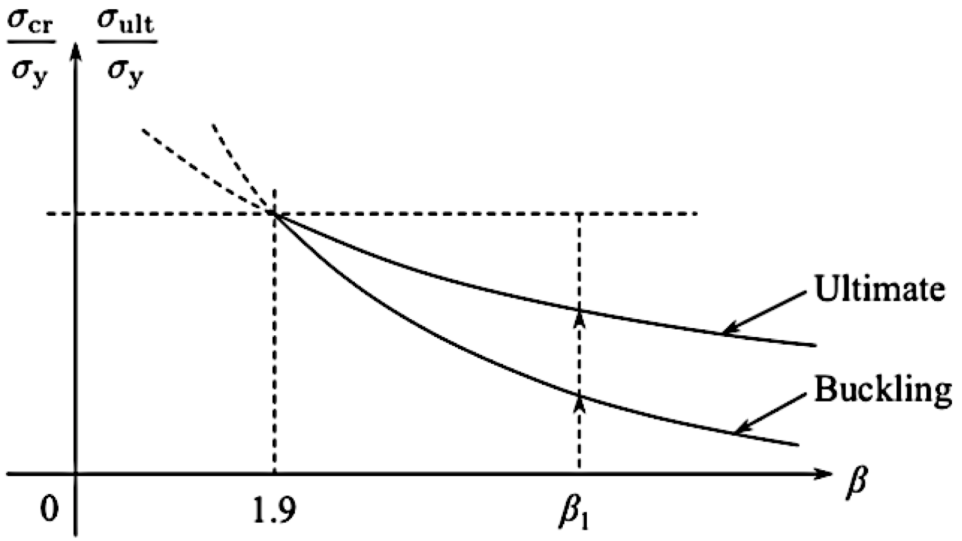


Figure 16: Dependence of the buckling stress and ultimate stress on the slenderness ratio.

From this figure one can identify the critical slenderness ratio

$$\beta_{cr} = 1.9 \qquad (60)$$

the ultimate stress

$$\sigma_{ult} = \sqrt{\sigma_{cr} \cdot \sigma_y} \qquad (61)$$

a small correction provides good fit of most of the test data for *von Karman*'s theory

$$\frac{\sigma_{ult}}{\sigma_y} = \frac{b_{eff}}{b} = \frac{1.9}{\beta} - \frac{0.9}{\beta^2} \qquad (62)$$

7. Effect of Initial Imperfection

The imperfections are distributed in the first mode $$\bar{w}(x,y) = \bar{w}_o \sin\frac{\pi x}{a}\sin\frac{\pi y}{a}$$ 63

With the initial imperfection the definition of the curvatures and membrane strains must be modified

$$\kappa_{\alpha\beta} = -(w - \bar{w})_{,\alpha\beta}$$ 64

$$\epsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) + \frac{1}{2}w_{,\alpha}w_{\beta} - \frac{1}{2}\bar{w}_{,\alpha}\bar{w}_{,\beta}$$

65

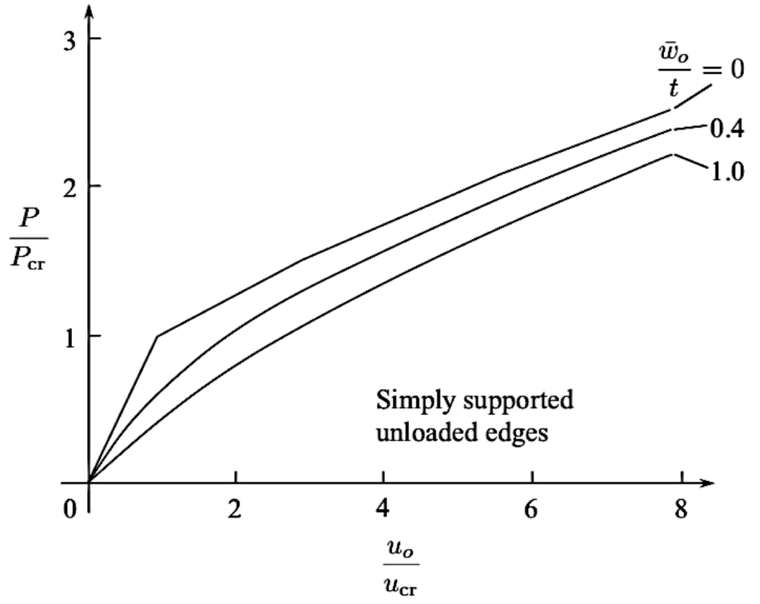


Figure 17: Load-displacement curves for imperfect simply supported plates.

Reference

[1]. 2.080 Structural Mechanics Lecture 11: Buckling of Plates and Sections
https://ocw.mit.edu/courses/mechanical-engineering/2-080j-structural-mechanics-fall-2013/course-notes/MIT2_080JF13_Lecture11.pdf

[2]. Lecture 04: Buckling and Ultimate Strength of Columns.pdf
http://ocw.snu.ac.kr/sites/default/files/NOTE/Lecture%2004%20Buckling_and_Ultimate_Strength_of_Columns.pdf

Thanks for watching!