PERSONAL NOTES

Nonlinear F.E.A.

Hanfeng Zhai

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2024

FEM (Nonlinear)

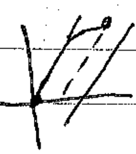
- What is FEA? -
  Method to solve PDE.

- Why Nonlinear FEA important?
  large deformation.
  (linearize the strain).
  ↓
  plastic deformation.
  ↳ material response.
  ↳ hyperelastic, etc. . .

- tools from linear FEA . . - relevant?.

Strong form → weak form.
↓
discretize.
↓

* Key difference: the solution.



shape function.

* ~~Ab~~ local support.

↓

prescribe the behavior within the element ← (interpolation)
↳ derivatives.
↓

solution ← assembly. ← Gauss Quadrature
↳ postprocessing.
(integration).

Sources of nonlinearities in solids.

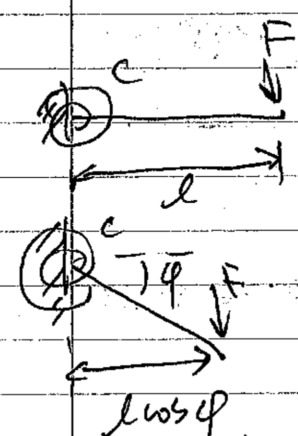
- geometric nonlinearities.
- deformation
- material responses
- instabilities
- BCs.
- coupled problem.
  - ← fluid, multiphysics

- large deformation (disp.) of rigid beam

⇒ Example:

a rigid beam, rotational spring stiffness c.

$\sum M = 0 \rightarrow Fl \cos\varphi = c\varphi$.



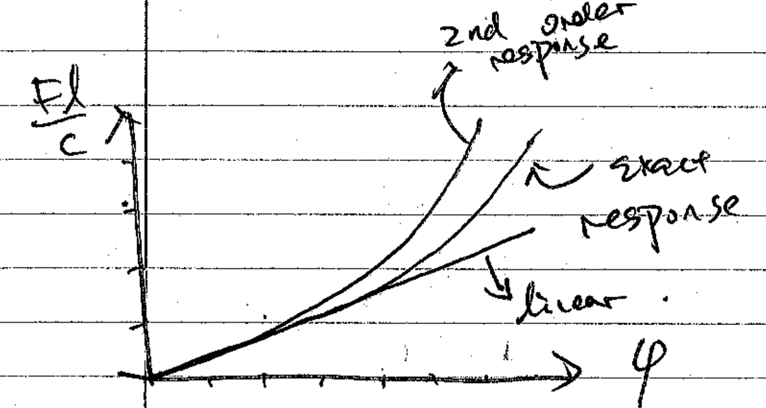
for small rotations $\cos\varphi \rightarrow 1$; ↳ $\varphi \rightarrow 0$

$F = c\varphi / l$

to capture the nonlinearity, → 2nd order theory

expand the function into Taylor series:

$$\cos\varphi \approx 1 - \frac{\hat{\varphi}^2}{2}.$$



$$F = \frac{c\hat{\varphi}}{l\left(1 - \frac{\hat{\varphi}^2}{2}\right)}$$

"2nd order"

★ geometric nonlinearities.

general motion.

+ this is not deformation - no strain tensor involved.

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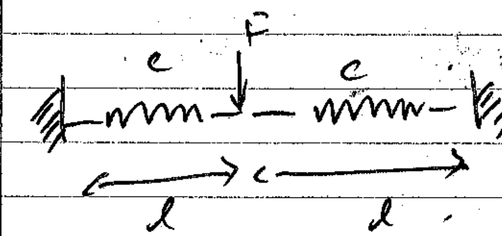
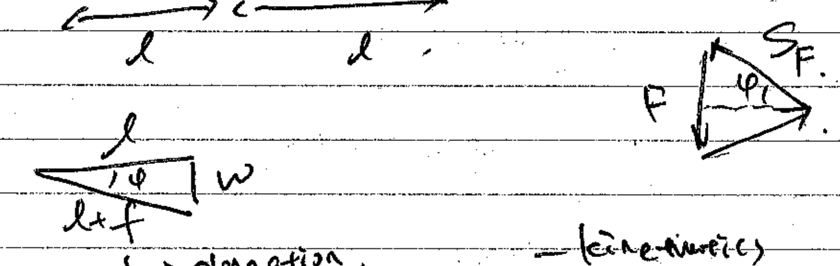
large deformation of elastic system



$\sum \vec{F} = 0$.



- kinematics
- equilibrium.
- constitutive law

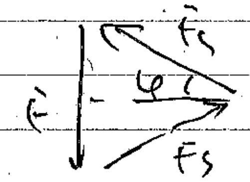
Kinematics.

$$w^2 + l^2 = (l+f)^2$$

↳ $$\sin\varphi = \frac{w}{l+f}$$

Equilibrium.

$$2F_s \sin\varphi = F$$



Constitutive law:

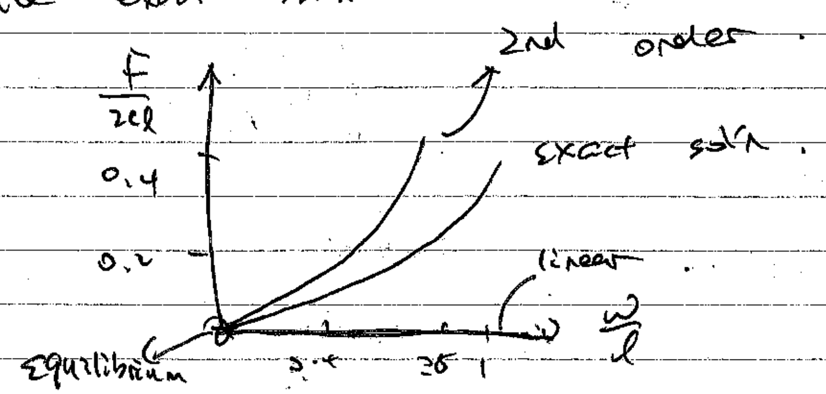
$F_s = cf$ ← elongation

↖ spring constant

exact sol'n:

$$\frac{w}{l}\left[1 - \frac{1}{\sqrt{1+\left(\frac{w}{l}\right)^2}}\right] = \frac{F}{2cl}$$

the exact sol'n:



Taylor expansion:

$$1/\sqrt{1+\left(\frac{w}{l}\right)^2} \approx 1 - \frac{1}{2}\left(\frac{\hat{w}}{l}\right)^2$$

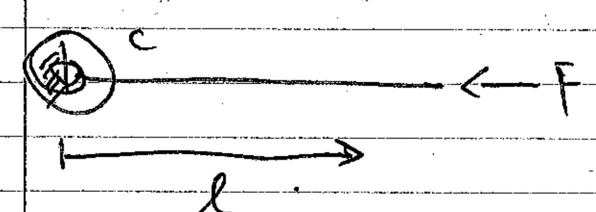
small values of w, → $\frac{w}{l} \ll 1$.

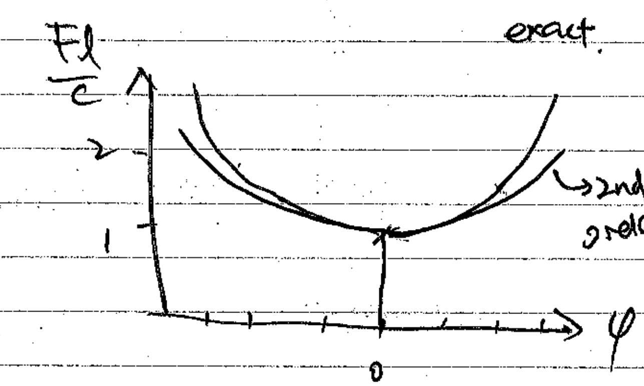
$$\frac{\hat{w}}{l}\left[\frac{1}{2}\left(\frac{\hat{w}}{l}\right)^2\right] = \frac{F}{2cl}$$

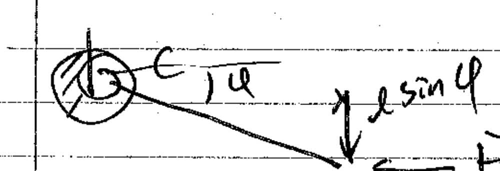
Source of nonlinearities: deformation & rotation.

Bifurcation.

$$Fl\sin\varphi = c\varphi.$$







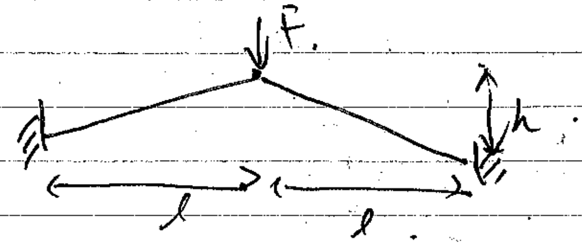
Approximation:

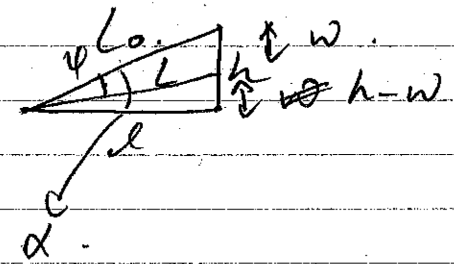
taylor expansion: $\sin\varphi \approx \hat{\varphi} - \hat{\varphi}^3/6$

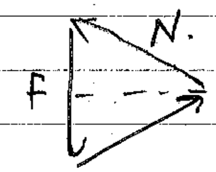
(*) & $1/(1-x) \approx 1+x$

Snap-through.

geometry: $(h-w)^2 + l^2 = L^2$ & $h^2 + l^2 = L_0^2$.







from this we can write length change:

$$f = L - L_0 = l\left[\sqrt{1+\left(\frac{h-w}{l}\right)^2} - \sqrt{1+\left(\frac{h}{l}\right)^2}\right]$$

final length initial

Equilibrium: $N\sin(\alpha-\varphi) = -\frac{F}{2}$.

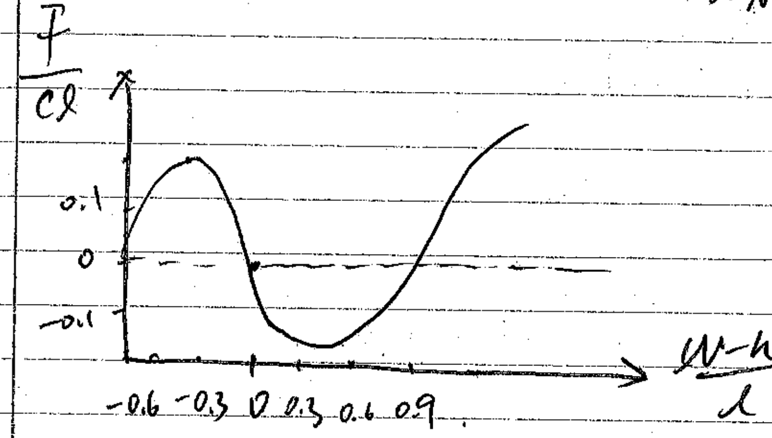
$$\sin(\alpha-\varphi) = \frac{(h-w)}{L}$$

$$\hookrightarrow N\frac{h-w}{L} = -\frac{F}{2}$$

constitutive law: $f_s = cf$. → final expression

$$c(h-w)\frac{L-L_0}{L} = -\frac{F}{2} \rightarrow$$

$$\frac{w-h}{l}\left[1 - \frac{L_0}{l\sqrt{1+\left(\frac{w-h}{l}\right)^2}}\right] - \frac{F}{2cl} = 0$$



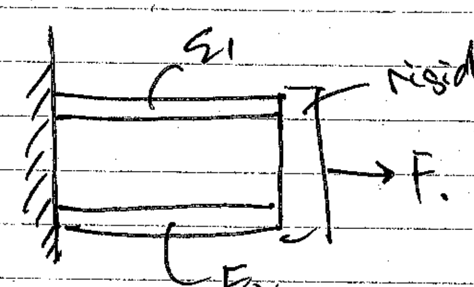
(*) F: applied force

Material nonlinearities

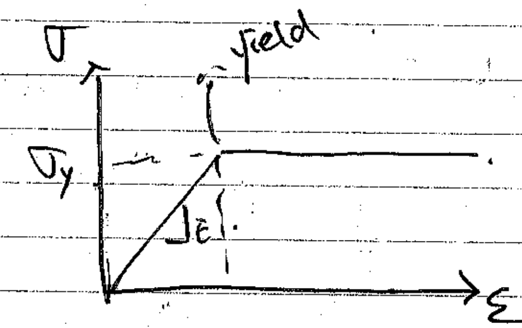
elastic moduli: $E_1 = 2E_2 = 2E$.

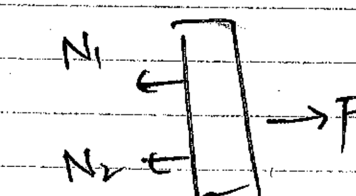
their yield stresses:

$$\sigma_{y1} = 3\sigma_{y2} = 3\sigma_y$$



same cross-sectional area A.





$N_1 + N_2 = F$

$\sigma_1 + \sigma_2 = \frac{F}{A}$.

$u_1 = u_2 = u$, $\varepsilon = \frac{u}{l}$.

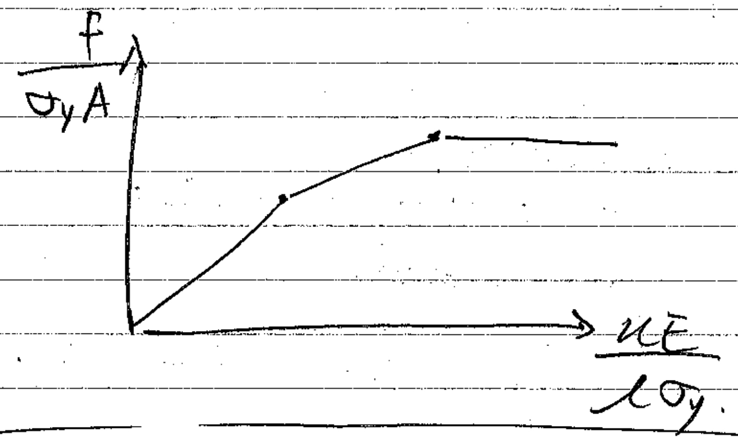
correct thru Hooke's law.

$$\sigma_i = E_i \varepsilon = E_i \frac{u}{l}.$$

$$u = \frac{Fl}{(E_1+E_2)A}.$$

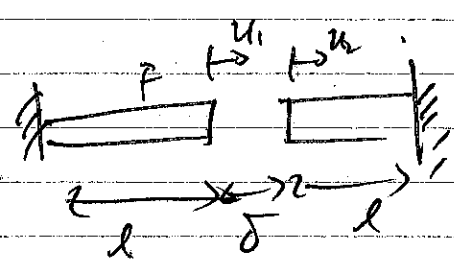
$\sigma_{y_2} < \sigma_{y_1}$.

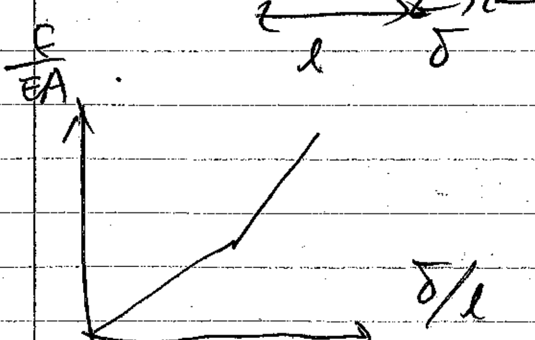
Bar 2 yields $\longrightarrow$ $F = 2A\sigma_y$



$\longrightarrow$ (*)

BC induced nonlinearity.





HW : notes.

$$\rightarrow \frac{\partial \underline{\underline{A}}^{-1}}{\partial \underline{\underline{A}}} \qquad \text{hint: } A^{-1}_{ik} A_{kj} = \delta_{ij}$$

$$\frac{\partial (A^{-1}_{ik} A_{kj})}{\partial A_{pq}} = - A^{-1}_{ik} \frac{\partial A_{k}}{\partial A_{pq}} A^{-1}_{ik} + A^{-1}_{ik} \frac{\partial A_{ik}}{\partial A_{pq}} A_{kj}$$

$$\frac{\partial \underline{\underline{A}}^{-1} \underline{\underline{A}}}{\partial} \qquad \frac{\partial A^{-1}_{ik}}{\partial A_{kj}} \qquad \frac{\partial A_{ik}}{\partial A_{kj}} = \delta_{ij}$$

$$\frac{\partial A^{-1}_{ij}}{\partial A_{pq}} = - A^{-1}_{ip} A^{-1}_{jq}. \qquad \frac{\partial (ab)}{\partial c} = \frac{\partial a}{\partial c} \cdot b + \frac{\partial b}{\partial c} \cdot a$$

$$\frac{\partial \det \underline{\underline{A}}}{\partial \underline{\underline{A}}} = \det(\underline{\underline{A}}) \cdot \underline{\underline{A}}^{-T}.$$

$$\frac{\partial \det (A_{ij})}{\partial A_{kl}} =$$

Prob 1. $\dfrac{\partial (A^{-1}_{ik} A_{kj})}{\partial A_{pq}}$

$$= \frac{\partial A^{-1}_{ik}}{\partial A_{pq}} \cdot A_{kj} + \frac{\partial A_{kj}}{\partial A_{pq}} \cdot A^{-1}_{ik}$$

$$\frac{\partial \delta_{ij}}{\partial A_{pq}} = \partial A_{kj} \quad = \frac{\partial \delta_{ij}}{\partial A_{pq}}$$

$$\hookrightarrow \frac{\partial A^{-1}_{ik}}{\partial A_{pq}} A_{kj} = -\frac{\partial A_{kj}}{\partial A_{pq}} A^{-1}_{ik}$$

$$A_{\ell k} A^{-1}_{kj} = \delta_{\ell j} \qquad \frac{\partial A^{-1}_{il}}{\partial A_{pq}} = - \cdots A^{-1}_{kj}$$

$$\frac{\partial A^{-1}_{ij}}{\partial A_{pq}} A_{jk} + A^{-1}_{ij} \frac{\partial A_{jk}}{\partial A_{pq}} = \frac{\partial A^{-1}_{il}}{\partial A_{pq}} A_{\ell k} + A^{-1}_{il} \frac{\partial A_{\ell k}}{\partial A_{pq}}$$

$$\hookrightarrow \frac{\partial A^{-1}_{ij}}{\partial A_{pq}} = -A^{-1}_{il} \frac{\partial A_{lk}}{\partial A_{pq}} A^{-1}_{kj}$$

$$= -A^{-1}_{il} \delta_{lp} \delta_{kq} A^{-1}_{kj}$$

$$= -A^{-1}_{ip} A^{-1}_{qj}$$

Prob. 2

$$\frac{\partial \det(\underline{A})}{\partial \underline{A}} = \frac{\partial \det(\underline{A})}{\partial A_{ij}} = \frac{\partial \|A_{ij}\|}{\partial A_{ij}}$$

$$= \sum_i \operatorname{sgn} \prod a$$

direct notation:

$$\det \underline{A} = \left\| \begin{bmatrix} a_{11} & \\ & \cdots \end{bmatrix} \right\|$$

∂A_1

Invariant:

$$\begin{cases} \dfrac{\partial I_1}{\partial \underline{A}} = \underline{I} \\ \dfrac{\partial I_2}{\partial \underline{A}} = I_1 \underline{I} - \underline{A}^T \\ \dfrac{\partial I_3}{\partial \underline{A}} = (\underline{A}^T)^2 - I_1 \underline{A}^T + I_2 \underline{I} = I_3 \underline{A}^{-T} \end{cases}$$

$$\det \underline{A} = \|A_{ij}\|$$

$$\sum_i \sum_l \frac{\partial A}{\partial A_{kl}} \cdot \frac{\partial A_{kl}}{\partial A_{ij}}$$

$$\frac{\partial \det A}{\partial A_{ij}} = \sum_k^n \sum_l^n C_{kl} \delta_{ik} \delta_{jl}$$

$$\frac{\partial \det A}{\partial A_{ij}} = \sum_k^n \sum_l^n \qquad \det(\underline{A}) \; \shortparallel$$

$$f(\underline{A}) = \det(\underline{A}).$$

$\cancel{A^3 +}$

$$A_{11} A_{22} - A_{12} A_{21}$$

$$\det(\underline{A}) = \frac{1}{6} \varepsilon_{ijk} \varepsilon_{abc} A_{ia} A_{jb} A_{kc}$$

$$\frac{\partial \frac{1}{6}\varepsilon_{ijk}\varepsilon_{abc} A_{ia} \cancel{A}_{jb} A_{kc}}{\partial A_{ij}}$$

$$\frac{\partial \det A_{ij}}{\partial A_{ij}} = (A_{ij}^T)^2 - I_1 A_{ij}^T + I_2 \underline{I}$$

$$= (A_{ij}^T)^2 - \cancel{a}\ \mathrm{tr}\, A_{ij} A_{ji} + \frac{1}{2}\left[(\mathrm{tr}\, A_{ij})^2 - \mathrm{tr}\, A_{ij}^2 \right]$$

$I_3 = \det A_{ij}$

$$= A_{ji}^2 - \mathrm{tr}\, A_{ij} A_{ji} + \frac{1}{2}[\sim]$$

$I_1 = \mathrm{tr}\, A_{ij}$

$I_2 = \frac{1}{2}\left[(\mathrm{tr}\, \underline{A})^2 - \mathrm{tr}\, (\underline{A}^2) \right]$

$$= \det A_{ij} A_{ji}^{-1}$$

$$= A_{ji}^2 - \mathrm{tr}\, A_{ij} A_{ji} + \frac{1}{2}\left[(\mathrm{tr}\, A_{ij})^2 - \mathrm{tr}\, A_{ij}^2 \right]$$

$$\frac{\partial (A_{11}A_{22} - A_{12}A_{21})}{\partial} = A_{22} \quad A_{12}$$

Prob. 3 $\quad \underline{\underline{D}} = \underline{a} \otimes \underline{b}$

$$= \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & \vdots \\ & \cdots & \end{bmatrix}$$

$$\det(\underline{A})\, \mathrm{Tr}(\underline{A}^{-1} \partial \underline{A}) \quad \uparrow \quad \overline{\partial \underline{A}}$$

$$\frac{\partial \det(A)}{\partial A} = \lim_{h \to 0} \frac{\det(A+h) - \det A}{\cancel{A}\, h}$$

$$= \det A \lim_{h\to 0} \frac{\det\left[A^{-1}(A+h)\right] - 1}{\cancel{A}}$$

$= \det$

$\det(\partial A)^{1/2} - \det A.$

Prob. 4. $\underline{\underline{A}} = \alpha(\underline{\underline{I}} - \underline{e}_1 \otimes \underline{e}_1) + \beta(\underline{e}_1 \otimes \underline{e}_2 + \underline{e}_2 \otimes \underline{e}_1)$

Compute . eigen values .

- eigenvectors

We can first write out the matrix form

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - [$

Prob 5. - Vector form

$$\oint_C \nabla \times \underline{F} \; d\underline{r} = \iint_S \text{curl } \underline{F} \; d\underline{S}$$

$$\iint_S \text{curl } \underline{F} \, d\underline{S} = \iiint_V \text{div}(\text{curl } \underline{F}) \, dV$$

$$\because \text{curl}(\text{curl } \underline{F}) = \nabla(\nabla \cdot \underline{F}) - \nabla^2 \underline{F}$$

$$\iiint_V \text{div}(\text{curl } \underline{F}) \, dV = \iiint_V [\nabla(\nabla \cdot \underline{F}) - \nabla^2 \underline{F}] \, dV$$

$$\iiint_V [\nabla(\nabla \cdot \underline{F}) - \nabla^2 \underline{F}] \, dV = \iint_S (\nabla \cdot \underline{F}) \, \underline{n} \, dS - \iint_S \underline{F} \cdot \underline{n} \, dS$$

$$\oint_C \nabla \times \underline{F} \cdot d\underline{r} = \iint_S (\nabla \cdot \underline{F}) \, \underline{n} \, dS - \oint_C \underline{F} \cdot d\underline{r}$$

Prob. 5

$$\oint_C \phi \, dx = \int_S \underline{n} \times \text{grad}\,\phi \, ds$$

$$\oint_C \underline{u} \times d\underline{x} = \int_S [(\text{div } \underline{u})\underline{n} - (\text{grad}^T \underline{u})\underline{n}] \, ds$$

$$\oint_C \phi \, dx = \int_S \underline{n} \times \text{grad}\,\phi \, ds$$

$$\oint_C \underline{u} \times d\underline{x}.$$

$$\int_C \underline{A} \cdot d\underline{r} = \int_S \underline{A} \times d\underline{S}$$

$$\oint_C \phi \, dx = \int_S \underline{n} \times \text{g}$$

$d(\underline{x} \cdot \text{grad}\,\phi)$

$d\underline{x} \cdot \text{grad}\,\phi$

scalar $\times \underline{x}$

Proof for 1:

$$\int_C \phi \, d\underline{x} = \int_C d(\underline{x} \cdot \text{grad}\,\phi)$$

$$= \int_C d\underline{x} \times \text{grad}\,\phi \, dS$$

$$= \int_S \underline{n} \times \text{grad}\,\phi \, dS$$

Proof for 2:

$$\boxed{\int_C \underline{u} \times d\underline{x} = \int_C d(\underline{X} \times \underline{u})}$$

$$= \int_S d\,\underline{X} \times \underline{u}\, dS.$$

$$\boxed{= \int_S \left[\operatorname{div} \underline{u}\, \underline{n} - \operatorname{grad}^T \underline{u}\, \underline{n} \right] ds}$$

Gauss's Theorem

$$\int_S \phi \cdot d\underline{S} = \int_V \operatorname{div} \phi\, dV$$

$$\oint_c \phi\, d\underline{x} = \int_S \underline{n} \times \operatorname{grad} \phi\, dS$$

Consider efflux dE: $dE = \phi \cdot d\underline{S}$

$$\operatorname{div} \phi\, dV = dE = \phi \cdot dS$$

Stokes' Theorem:

$$\oint_c \underline{a} \cdot d\underline{l} = \int_S \operatorname{curl} \underline{a} \cdot d\underline{S}.$$

$$\sum_{\text{de loop}} \underline{a} \cdot d\underline{l} = (\nabla \times \underline{a}) \cdot d\underline{S}$$

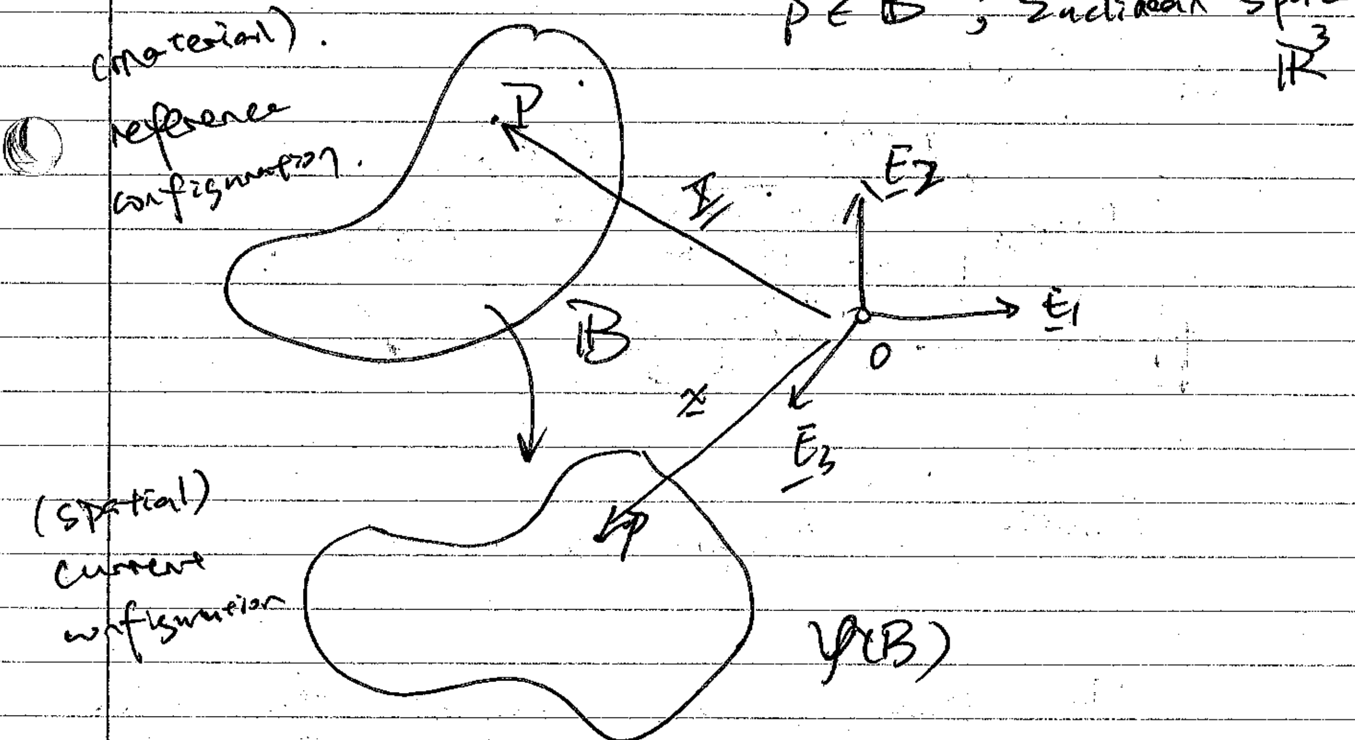
Lecture 2. Continuum Mechanics - important

- kinematics. (motion)
- Strain measures.
- vector & tensor transformation

} this.

- Balance equations.
- Stress measures. } .....
- Constitutive relations.

$p \in \mathcal{B}$; Euclidean space $\mathbb{R}^3$

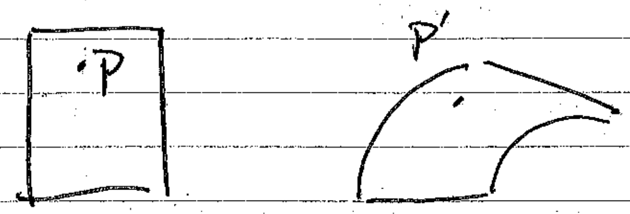


Configuration $\varphi: \mathcal{B} \to \mathbb{R}^3$.

x, X - positions in $\mathbb{R}^3$ w/ origin O

$x = \varphi(X, t)$

$X =$

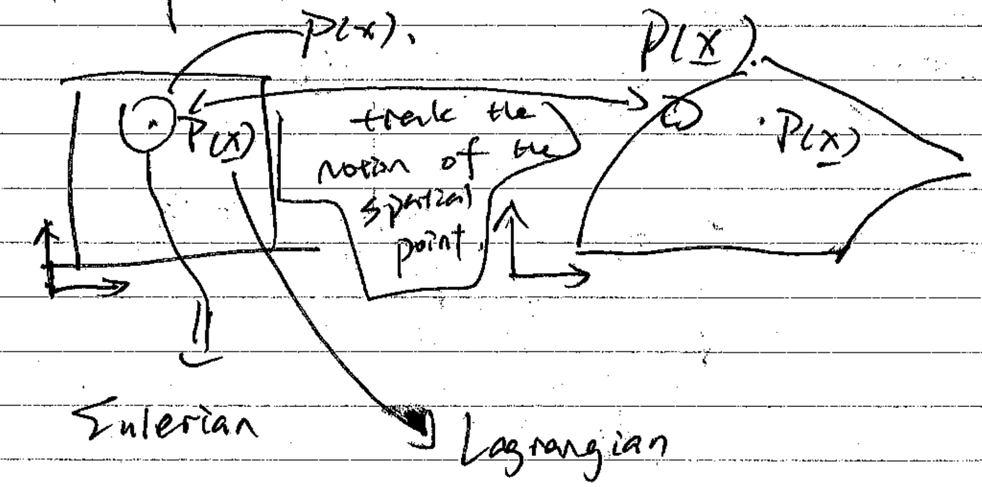


$\underset{\sim}{X} = X_A \underset{\sim}{e}_A$

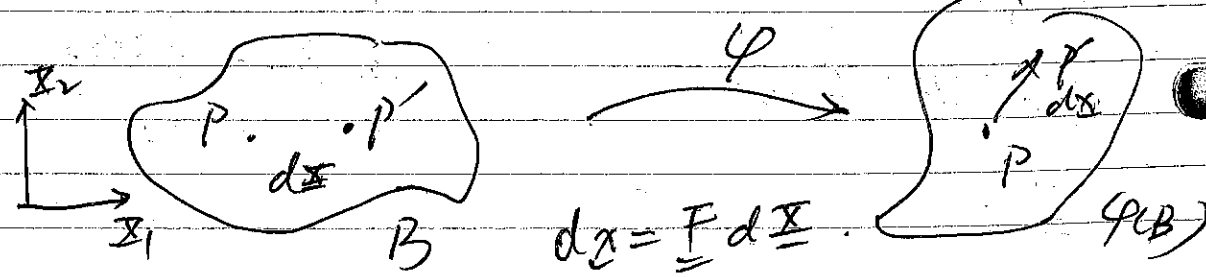
in linear FEM – Lagrangian

but does not make a diff.

in fluid



– How line elements mapped from reference to current config.



$d\underset{\sim}{x} = \underset{\sim}{F}\, d\underset{\sim}{X}$.

$\underset{\sim}{F} = \dfrac{\partial x}{\partial \underset{\sim}{X}}$. $\rightarrow$ $\dfrac{\partial x_i}{\partial X_A} = x_{i,A}$.

$\underset{\sim}{F} = \text{Grad}\, \varphi(\underset{\sim}{X}, t)$ $\quad \underset{\sim}{x} = \varphi(\underset{\sim}{X})$

* tensor vs. matrix

tensor ↓ preserves basis properties

matrix ↘ numerical form.

$\underset{\sim}{A} = \underset{\sim}{a} \otimes \underset{\sim}{b}$.

$\underset{\sim}{b} = b_i \underset{\sim}{e}_i$

$\underset{\sim}{a} = a_i \underset{\sim}{e}_i$

$[A] = [\ \cdots\ ]$

$dx_1 = F_{11}\, dX_1 + F_{12}\, dX_2 + F_{13}\, dX_3$.

$$[F_{iA}] = \begin{bmatrix} \frac{\partial x_1}{\partial X_1}, & \frac{\partial x_1}{\partial X_2}, & \\ & \cdots & \ddots \\ \frac{\partial x_3}{\partial X_1} & & \end{bmatrix}$$

(contined)

Kinematics – deformation gradients.

$\underline{F}$ has to be invertable.

$$d\underline{X} = \underline{F}^{-1} d\underline{x}.$$

$$\underline{F}^{-1} = (F^{-1})_{iA}\, \underline{E}_A \otimes \underline{e}_i$$

$\hookrightarrow \dfrac{\partial X_A}{\partial x_i}$

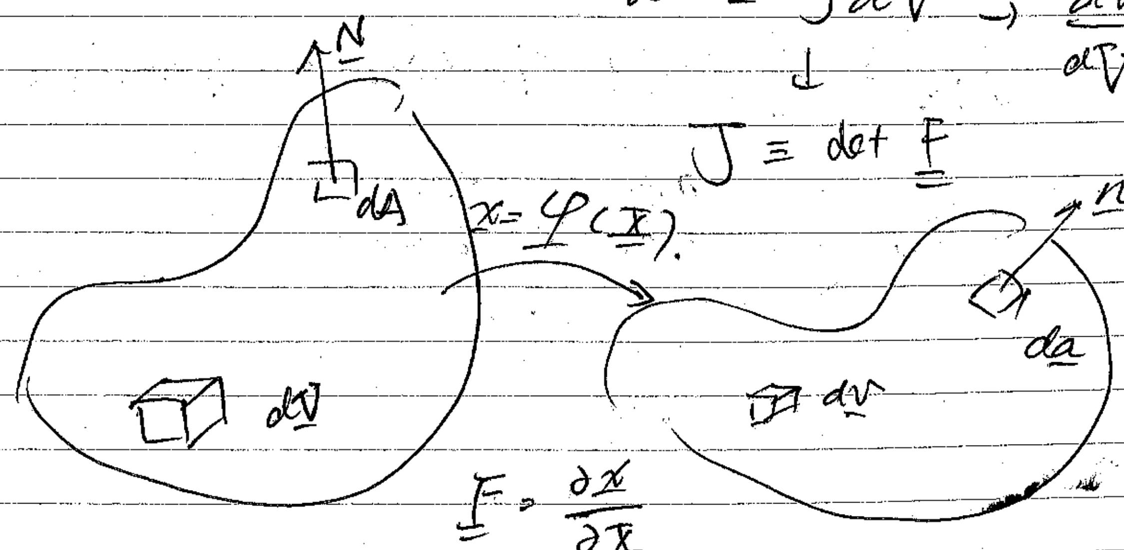
★ Nanson's formula. Capture transformation of surface element in ref. config. $d\underline{A}$.

$$d\underline{a} = \underline{n}\, da = J \underline{F}^{-T} \underline{N}\, dA = J \underline{F}^{-T} d\underline{A}$$

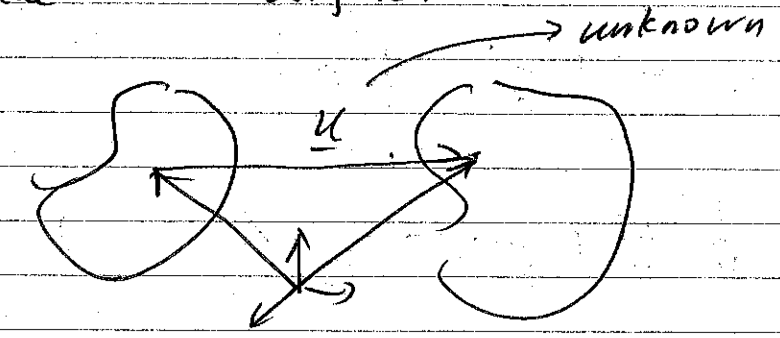
$$dv = J dV \rightarrow \frac{dv}{dV}$$

$\downarrow$

$J \equiv \det \underline{F}$



– Define the displacement



$$\underline{u}(\underline{X}, t) = \underline{\varphi}(\underline{X}, t) - \underline{X} = x - \underline{X}.$$

$$\underline{F} = \text{Grad}[\underline{X} + \underline{u}(\underline{X}, t)]$$

$$= \underline{\underline{I}} + \text{Grad}\,\underline{u} = \underline{\underline{I}} + \underline{\underline{H}}$$

$\downarrow$

displacement gradient

★ $\text{Grad} = \nabla_{\underline{X}}$.

– material gradient operation

$\text{grad} = \nabla_{\underline{x}}$

– spatial gradient operation.

$$\underline{\underline{I}} = \delta_{ij} E_i \otimes E_j.$$

– Strain measures.

Green-Lagrange strain tensor $\uparrow$ Right Cauchy-Green strain tensor

$$\underline{\underline{E}} \equiv \tfrac{1}{2}(\underline{F}^T \underline{F} - \underline{\underline{I}}) = \tfrac{1}{2}(\underline{\underline{C}} - \underline{\underline{I}})$$

high-order terms get rid of.

$$\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{H}}^T + \underline{\underline{H}} + \underline{\underline{H}}^T\underline{\underline{H}})$$

$$\underline{\underline{\varepsilon}} = \frac{1}{2}(\underline{\underline{H}} + \underline{\underline{H}}^T) = \frac{1}{2}(u \quad)$$

Define rotation $\underline{\underline{R}}$ ($\underline{\underline{R}}^{-1} = \underline{\underline{R}}^T$) & corresponding symmetric stretch tensors $\underline{\underline{U}}$ & $\underline{\underline{V}}$: $\underline{\underline{F}} = \underline{\underline{R}}\underline{\underline{U}} = \underline{\underline{V}}\underline{\underline{R}}$.

$$\begin{cases} F_{iB}\, \underline{e}_i \otimes \underline{E}_B = (R_{iA}\, \underline{e}_i \otimes \underline{E}_A)(U_{CB}\, \underline{E}_C \otimes \underline{E}_B) \\ F_{iB}\, \underline{e}_i \otimes \underline{E}_B = (V_{ik}\, \underline{e}_i \otimes \underline{e}_k)(R_{mB}\, \underline{e}_m \otimes \underline{E}_B) \end{cases}$$

generalized strain measure:

$$\underline{\underline{E}}^\alpha = \frac{1}{\alpha}(\underline{\underline{U}}^\alpha - \underline{\underline{I}}) \ \& \ \underline{\underline{e}} = \frac{1}{\alpha}(\underline{\underline{V}}^\alpha - \underline{\underline{I}}).$$

$$\Downarrow$$

$$\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{U}}^2 - \underline{\underline{I}})$$

$$\underline{\underline{U}} = \sum_i^3 \lambda_i \underline{N}_i \otimes \underline{N}_i \ ; \ \underline{\underline{V}} = \sum_i^3 \lambda_i \underline{n}_i \otimes \underline{n}_i$$

$$\underline{n}_i = \underline{\underline{R}}\, \underline{N}_i$$

– Strain measures.

Almansi Strain tensor.

$$\underline{\underline{e}} \equiv \underline{\underline{e}}^{(-2)} = \frac{1}{2}(\underline{\underline{I}} - \underline{\underline{V}}^{-2}) = \frac{1}{2}(\underline{\underline{I}} - \underline{\underline{b}}^{-1})$$
$$= \frac{1}{2}(\underline{\underline{I}} - \underline{\underline{F}}^{T}\underline{\underline{F}}^{-1}).$$

$$\underline{\underline{b}} \equiv \underline{\underline{F}}\underline{\underline{F}}^T = \underline{\underline{V}}\underline{\underline{R}}\underline{\underline{R}}^T\underline{\underline{V}}^T = \underline{\underline{V}}^2.$$

* $\begin{cases} \text{pull back} \\ \text{push forward} \end{cases}$

pull back

$$G(\underline{X}) = g(\underline{x}).$$

$$\text{Grad}\, G = \underline{\underline{F}}^T \text{grad}\, g \iff \frac{\partial G}{\partial X_A} = \frac{\partial g}{\partial x_i}\frac{\partial x_i}{\partial X_A}.$$

$$\text{grad}\, g = \underline{\underline{F}}^{-T}\, \text{Grad}\, G.$$

$$\frac{\partial g}{\partial x_i} = \frac{\partial g}{\partial X_A}\frac{\partial X_A}{\partial x_i}.$$

pull back + push forward

Vector & tensor transformation.

HW continued.

Prob. 9. $\psi \to \sigma_1, \sigma_2, \sigma_3$; P_1, P_2, P_3; S_1, S_2, S_3.

Strain Energy function $\Psi \to \lambda_1, \lambda_2, \lambda_3$.

$$\frac{\partial \Psi}{\partial \lambda_i} = \sigma_i \lambda_i, \quad i = 1, 2, 3.$$

$$P_i = \frac{\partial \Psi}{\partial \lambda_i} \qquad \sigma_i = \lambda_i P_i.$$

$$\underline{\underline{S}} = 2\rho_0 \frac{\partial \psi(\underline{\underline{C}})}{\partial \underline{\underline{C}}}$$

$$\underline{\underline{S}} = \underline{\underline{F}}^{-1} \underline{\underline{P}} = J \underline{\underline{F}}^{-1} \underline{\underline{\sigma}}\, \underline{\underline{F}}^{-T}.$$

Ψ - incompressible isotropic hyperelastic.

i.t.o. principal stretches.

=

3 Cauchy stress.

$$\Psi = \frac{1}{2}k(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + \frac{k}{\beta}(\lambda_1\lambda_2\lambda_3)^\beta$$

$$\begin{cases} \sigma_1 = k\lambda_1(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + (\lambda_1\lambda_2\lambda_3)^\beta k\beta\lambda_1 \\ \sigma_2 = k\lambda_2(\quad \sim \quad) + \quad \sim \quad \lambda_2 \\ \sigma_3 = k\lambda_3(\quad \sim \quad) + \quad \sim \quad \lambda_3 \end{cases}$$

$$\psi = (I_1, I_2, I_3) = \frac{\mu}{2}(I_1 - 3) + \frac{\lambda}{2}(I_1 - 3)^2$$

$$\psi = \frac{\mu}{2}(I_1 - 3).$$

$$= \frac{\mu}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3).$$

$$\Downarrow$$

$\sigma_1, \sigma_2, \sigma_3 = \ldots$

$$\underline{\underline{P}} = -p\underline{\underline{F}}^{-T} + 2\left[\left(\frac{\partial\Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2}\right)\underline{\underline{F}} - \frac{\partial \Phi}{\partial I_2}\underline{\underline{F}}\underline{\underline{C}}\right]$$

Prob. 10

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \frac{1}{\lambda_1\lambda_2} X_3$$

☆ Membrane theory:

Ogden's theory: $\sigma = \mu(\lambda^n - 1)$

$$W = \sum_{i}^{m} \frac{a_i}{n_i}\left[\left(\lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}\right)^{\frac{n_i}{2}} - 2\right]$$

Strain energy (Wiki) $W(\lambda_1, \lambda_2) = \sum_{i}^{N} \frac{\mu_i}{\alpha_i}(\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_1^{-\alpha_i}\lambda_2^{-\alpha_i} - 3)$

We know λ_1 & λ_2 & $\det \underline{\underline{F}} = 1$

HW note.

Prob 7: reverse to get $\underline{X}(x)$. φ^{-1} $\quad \frac{\partial u}{\partial x_1} + \underline{\underline{I}}$

$$\begin{cases} x_1 = e^t X_1 - e^{-t} X_2 \\ x_2 = e^t X_1 + e^{-t} X_2 \\ x_3 = X_3 \end{cases}$$

$\underline{u} = \underline{x} - \underline{X}$

$$\underline{\underline{F}} = \frac{\partial \underline{x}}{\partial \underline{X}} = \frac{\partial x(\underline{u} + \underline{X})}{\partial X_1}.$$

$$\begin{cases} x_1 + x_2 = 2e^t X_1 \rightarrow X_1 = \frac{1}{2e^t}(x_1 + x_2). \\ x_2 - x_1 = 2e^{-t} X_2 \rightarrow X_2 = \frac{1}{2e^{-t}}(x_2 - x_1). \\ X_3 = x_3. \end{cases}$$

$$\begin{bmatrix} e^t X_1 - e^{-t} X_2 \\ e^t X_1 - e^{-t} X_2 \\ 0 \end{bmatrix}$$

→ in the material description:

$$\underline{A}(\underline{X}, t) = \frac{\partial \underline{V}(\underline{X}, t)}{\partial t} = \begin{cases} \frac{\partial V_x}{\partial t} = e^t X_1 - e^{-t} X_2. \\ \frac{\partial V_y}{\partial t} = e^t X_1 - e^{-t} X_2 \\ \frac{\partial V_z}{\partial t} = 0 \end{cases}$$

$$+\frac{\cancel{2e^{t}}}{2e^{-t} \cdot \cancel{2e^{-t}}}$$

$$\underline{V}(\underline{X}, t) = \frac{\partial \underline{x}(\underline{X}, t)}{\partial t} = \begin{cases} \frac{\partial x_1}{\partial t} = e^t X_1 + e^{-t} X_2. \\ \frac{\partial x_2}{\partial t} = e^t X_1 - e^{-t} X_2 \\ \frac{\partial x_3}{\partial t} = 0. \end{cases}$$

$$\frac{-2\cancel{e^t}}{4e^{2t}\cancel{e^t}} \quad -\frac{1}{2e^t} = \begin{bmatrix} e^t X_1 + e^{-t} X_2 \\ e^t X_1 - e^{-t} X_2 \\ 0 \end{bmatrix}$$

Prob. 8:

$$d\underline{u} = (\underline{\underline{F}} - \underline{\underline{I}})\, d\underline{X}.$$

$$\frac{d\underline{u}}{d\underline{X}} = \underline{\underline{F}} - \underline{\underline{I}}$$

$$\frac{d\underline{u}}{d\underline{X}} + \underline{\underline{I}} = \underline{\underline{F}}.$$

$$\underline{x} - \underline{X} = \underline{u}.$$

$$\frac{\partial \underline{x}}{\partial \underline{x}} - \frac{\partial \underline{X}}{\partial \underline{x}} = \frac{\partial \underline{u}}{\partial \underline{x}}$$

$$\underline{\underline{I}} - \underline{\underline{F}}^{-1} = \frac{\partial \underline{u}}{\partial \underline{x}}$$

$$\underline{\underline{I}} - \frac{\partial \underline{u}}{\partial \underline{x}} = \underline{\underline{F}}^{-1}$$

$$\underline{\underline{F}}^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \begin{bmatrix} 1 & -\frac{1}{4} & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\underline{\underline{F}}^{-1} = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\underline{\underline{F}} = \begin{bmatrix} 0 & -\frac{1}{16} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$\underline{\underline{F}} = \frac{1}{\det(\underline{\underline{F}}^{-1})} \operatorname{adj}(\underline{\underline{F}}^{-1}).$$

$$\operatorname{adj}(\underline{\underline{F}}^{-1}) = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Prob. 9 & 10.

$$\frac{\partial \psi}{\partial \underline{\underline{C}}} = \sum_i^3 \frac{\partial \psi}{\partial \lambda_i^2} \frac{\partial \lambda_i^2}{\partial \underline{\underline{C}}}.$$

$$\frac{\partial \lambda_i^2}{\partial \underline{\underline{C}}} = \underline{n_i}\, \underline{n_i}$$

$$\hookrightarrow \frac{\partial \psi}{\partial \underline{\underline{C}}} = \sum_i^3 \frac{\partial \psi}{\partial \lambda_i^2} \cdot \underline{n_i}\, \underline{n_i}$$

$$\underline{\underline{S}} = 2\frac{\partial \psi}{\partial \underline{\underline{C}}} = \sum_i^3 \frac{1}{\lambda_i} \frac{\partial \psi}{\partial \lambda_i} \underline{n_i}\, \underline{n_i}$$

$$= \sum_i^3 \frac{\partial \frac{\mu}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)}{\partial \lambda_i^2} \, n_i n_i$$

$$= \cancel{\mu}(\mu \lambda_1 \underset{n_1 n_1}{} + \mu \lambda_2 \underset{n_2 n_2}{} + \mu \lambda_3 \underset{n_3 n_3}{})$$

$$\mu(\underline{N_1 N_1} + \underline{N_2 N_2} + \underline{N_3 N_3}).$$

$$\underline{\underline{P}} = \underline{\underline{F}}\, \underline{\underline{S}}$$

$$\hookrightarrow \cancel{\underline{\underline{F}}\, \underline{N_i} = \lambda_i \underline{n_i}}$$

$$\underline{\underline{F}}_i \underline{N_i} = \lambda_i \underline{n_i}$$

$$\underline{\underline{P}} = \underline{\underline{F}} \sum_i^3 \frac{\partial \psi}{\partial \lambda_i^2} \underline{N_i} \otimes \underline{N_i} = \sum_i \frac{\partial \psi}{\partial \lambda_i} \underline{N_i} \otimes \underline{N_i}$$

$$\mu(\lambda_1 \underline{N_1 N_1} + \lambda_2 \underline{N_2 N_2} + \lambda_3 \underline{N_3 N_3}).$$

$$\underline{\underline{\sigma}} = J^{-1} \underline{\underline{F}} \cdot \underline{\underline{P}}^T = J^{-1} \sum_i^3 \frac{\partial \psi}{\partial \lambda_i} \lambda_i \underline{N_i} \otimes \underline{N_i} \qquad \frac{1}{\lambda_1 \lambda_2 \lambda_3}$$

Prob 10.

Ogden model: Plane stress: $\sigma_3 = 0$

$$\psi = \sum_i^3 \frac{2\mu_i}{\alpha_i^2}\left(\bar\lambda_1^{\alpha_i} + \bar\lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3\right) + \frac{K_1}{2}(J-1)^2$$

↳ $\left(\frac{\lambda_i}{J^{1/3}}\right)^{\alpha_i}$

$$\frac{\partial\psi}{\partial\lambda_i} = \sum_i^3 \frac{2\mu_i}{}$$

↳ Const. $\boxed{\alpha_i \quad \lambda_i^{\alpha_i - 1}}$

$$\frac{2\mu_i}{\alpha_i^2}\left(\frac{1}{J^{1/3}}\right)^{\alpha_i} \alpha_i \lambda_i^{\alpha_i - 1}$$

$$= \frac{2\mu_i}{\alpha_i}\left(\frac{1}{J^{1/3}}\right)^{\alpha_i} \lambda_i^{\alpha_i - 1}$$

$$\underline{F} = \frac{\partial x}{\partial X} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \frac{1}{\lambda_1\lambda_2} \end{bmatrix}$$

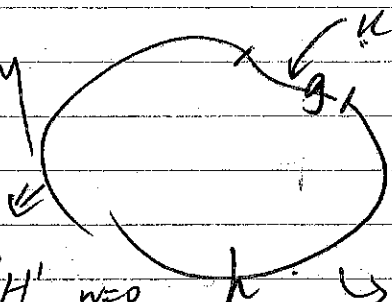
$J = 1$.

weak form

Integrability

BCs.

Week 3 - lecture 1.



$\Rightarrow u = \{u \mid u \in H^1;\ u = g \text{ on } \Gamma_g\}$

$\int_\Omega (\nabla w)^2\, dV < \infty$

doesn't blow up

integrability

$w = \{w \mid w \in H^1,\ w = 0 \text{ on } \Gamma_g\}$.

h ↳ $\underline{q} \cdot \underline{n}$.

* cannot prescribe both force & displacement

"Spring"-type BCs _Robin

essential vs. natural

Dirichlet vs. Neumann.

$$\underbrace{u = g \qquad -q_i n_i = h}$$

1st: multiply both sides by test function

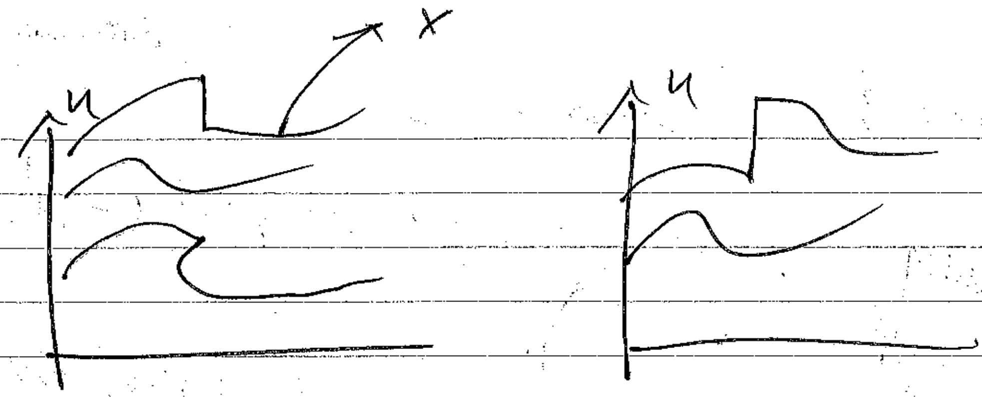
$q_{i,i} = f$

↳ $w \operatorname{div} \underline{q} = wf$

$\Gamma = \Gamma_g \cup \Gamma_h$

↳ $\int_\Omega w \operatorname{div} \underline{q}\, dV = \int_\Omega w f\, dV$

↳ $\int_{\partial\Omega} w \underline{q} \cdot \underline{n}\, dS - \int_\Omega \underline{q} \cdot \operatorname{grad} w\, dV = \int_\Omega w f\, dV$



$\therefore$ we find u has ~~to~~ be ~~smooth~~ continuous

$C^0 \mapsto$ zeroth derivative is continuous

$C^1 \mapsto$ first derivative is continuous

$C^n \mapsto n^{th}$ derivative is continuous.

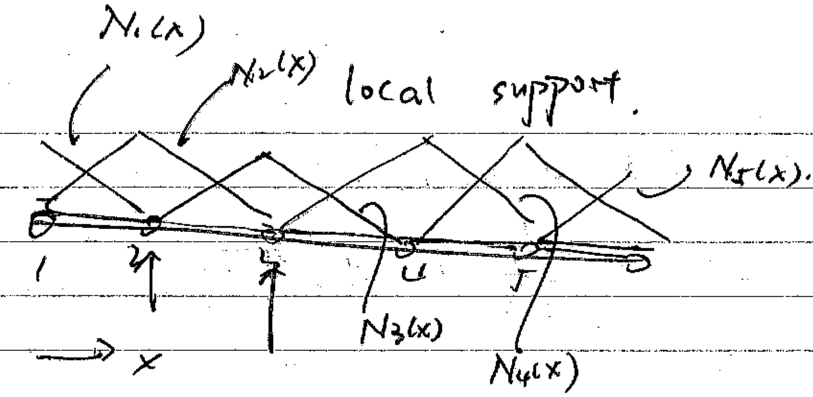
1) ☆ strong form: point-wise
weak form: on an average sense.

Order

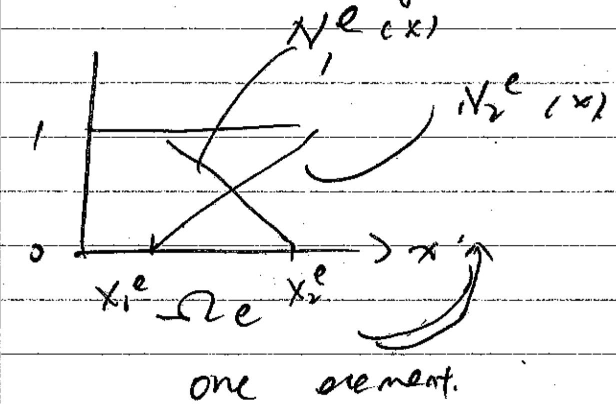
2) Strong form: $q_{i,i} = k_{ij} u_{,ij}$ - 2^{nd} order
weak form: - 1^{st} order deriv.
↳ symmetry between w & u.

— Shape function example:

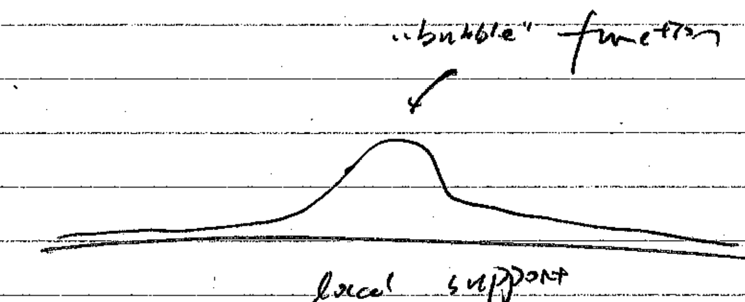
1D domain:

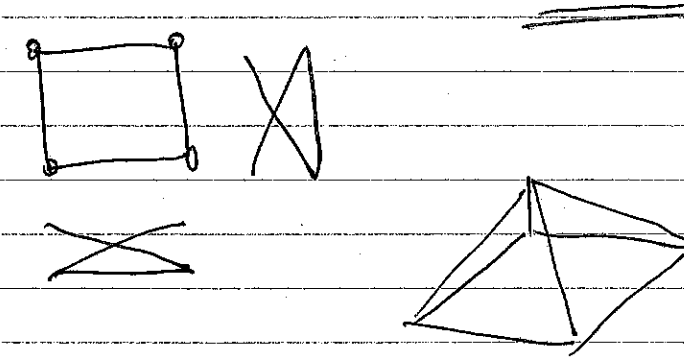


all the shape functions look the same.



— linear shape functions.





$\Rightarrow (W^h, h)_p = \sum_i C_A (N_A, h)$,

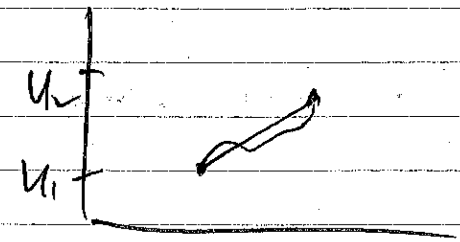
many integrals.

$= \int_\Gamma N_1 c_1 \, ds + \int_\Gamma N_2 c_2 \, ds + \dots\dots$

$\sum_{A \in \eta - \eta_g} C_A \{ \sim\sim \} = 0 \rightarrow$ we can set

$u \simeq u_1 N_1 + u_2 N_2$.

$$\frac{\partial u}{\partial x} \simeq u_1 \frac{\partial N_1}{\partial x} + u_2 \frac{\partial N_2}{\partial x}$$

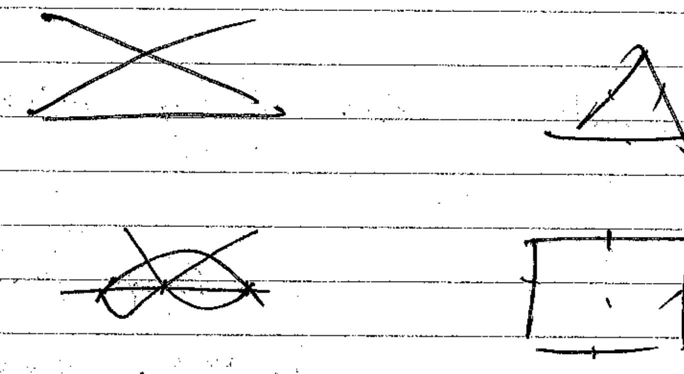


#FEM

Week 3 - lecture 2.

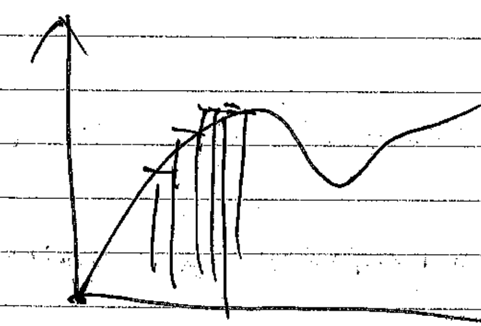
Galerkin formulation.

$$\left[\quad \right] \begin{Bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{Bmatrix} = \{ \quad \} \rightarrow \text{solve for } d.$$



quadratic

quadrilateral



Gauss quadrature

sample → integration method

⇒ Same shape function

Bubnov - Galerkin → same s.f.

Petrov-Galerkin → different s.f.

↳ supg.

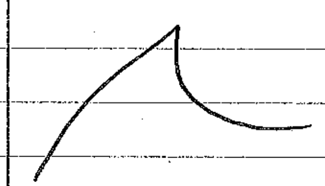
how to solve PDE → tune the shape function

Galerkin method vs. Riley-Ritz method

⇓

shape function

has to be global

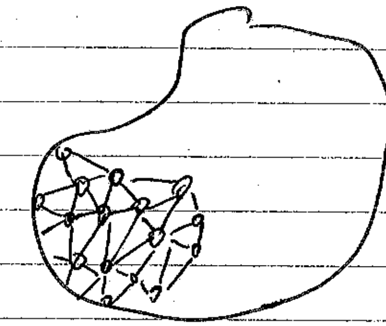


C_0 : derivative have a jump.

① what order derivatives in weak form

② what order continuities is required.

Week 4. lecture 1.



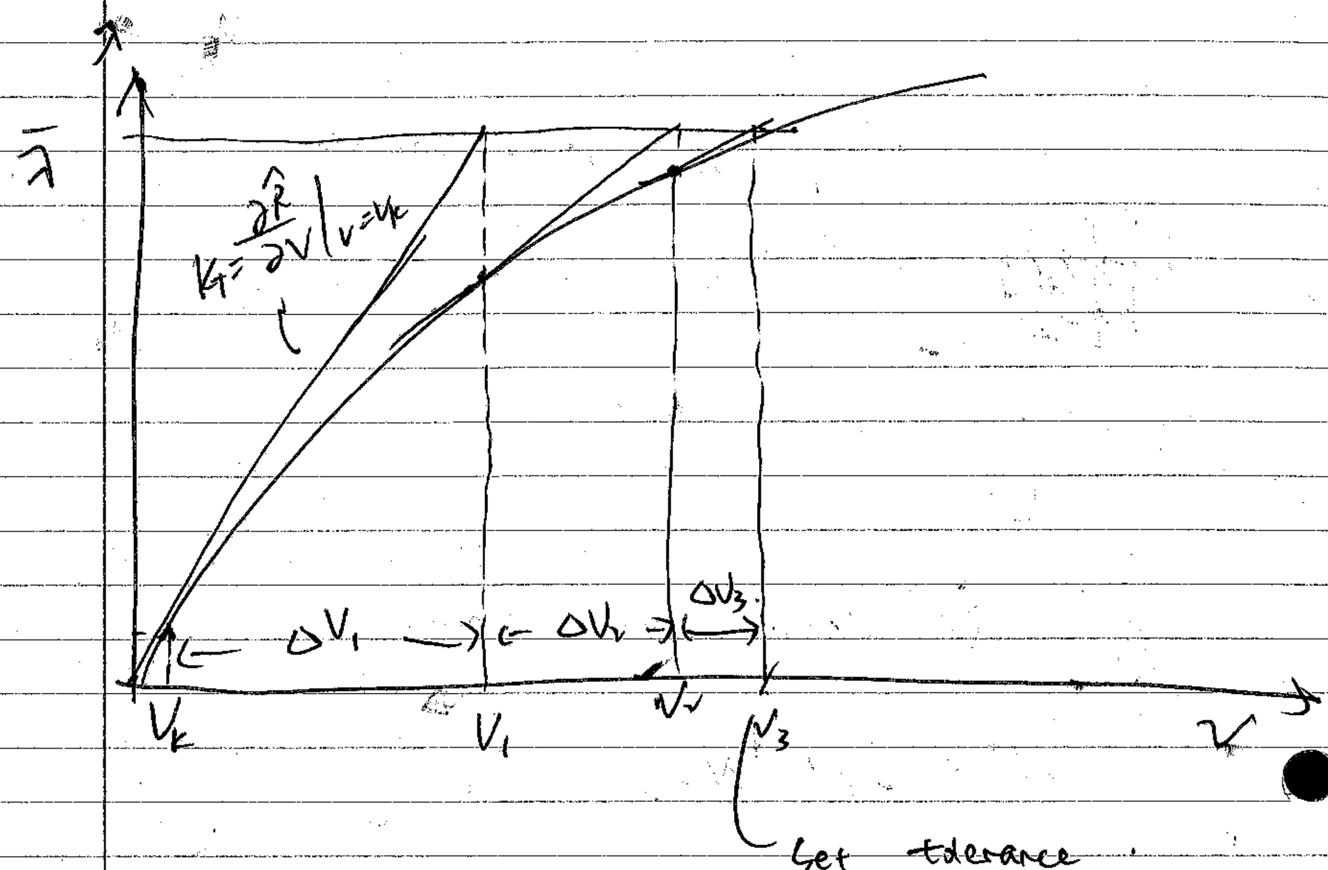
u, u^h → ele. size

$\underline{L}^e$

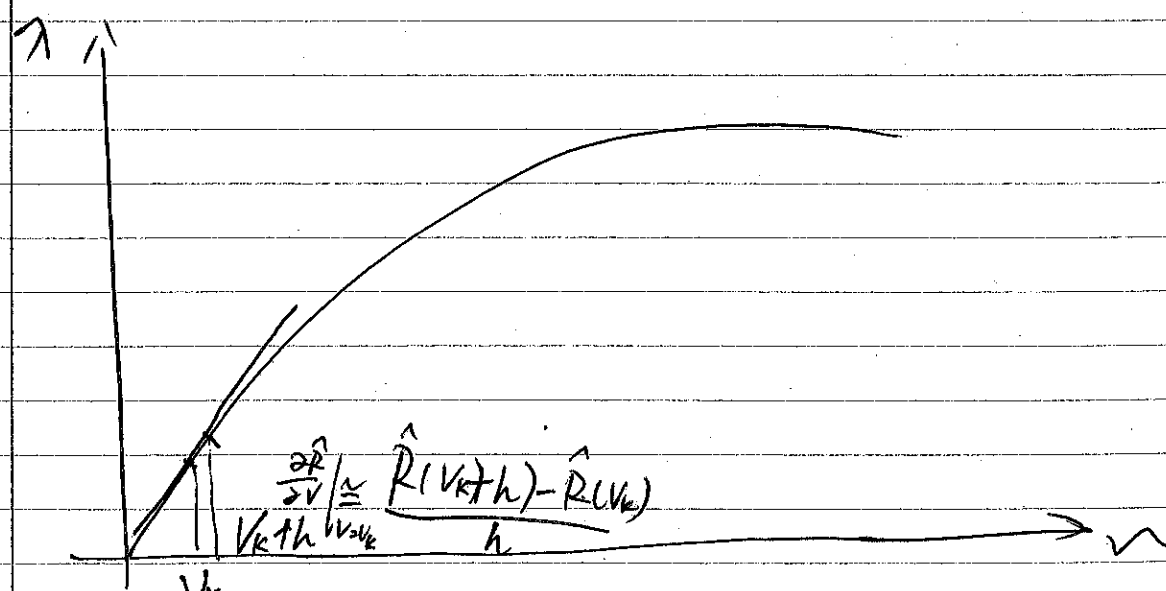
e.g. 1D linear shape func.

$\underline{N}^e = [N_1 \;\; N_2]$ $\quad \underline{L}^e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Week 5 . Lecture 1

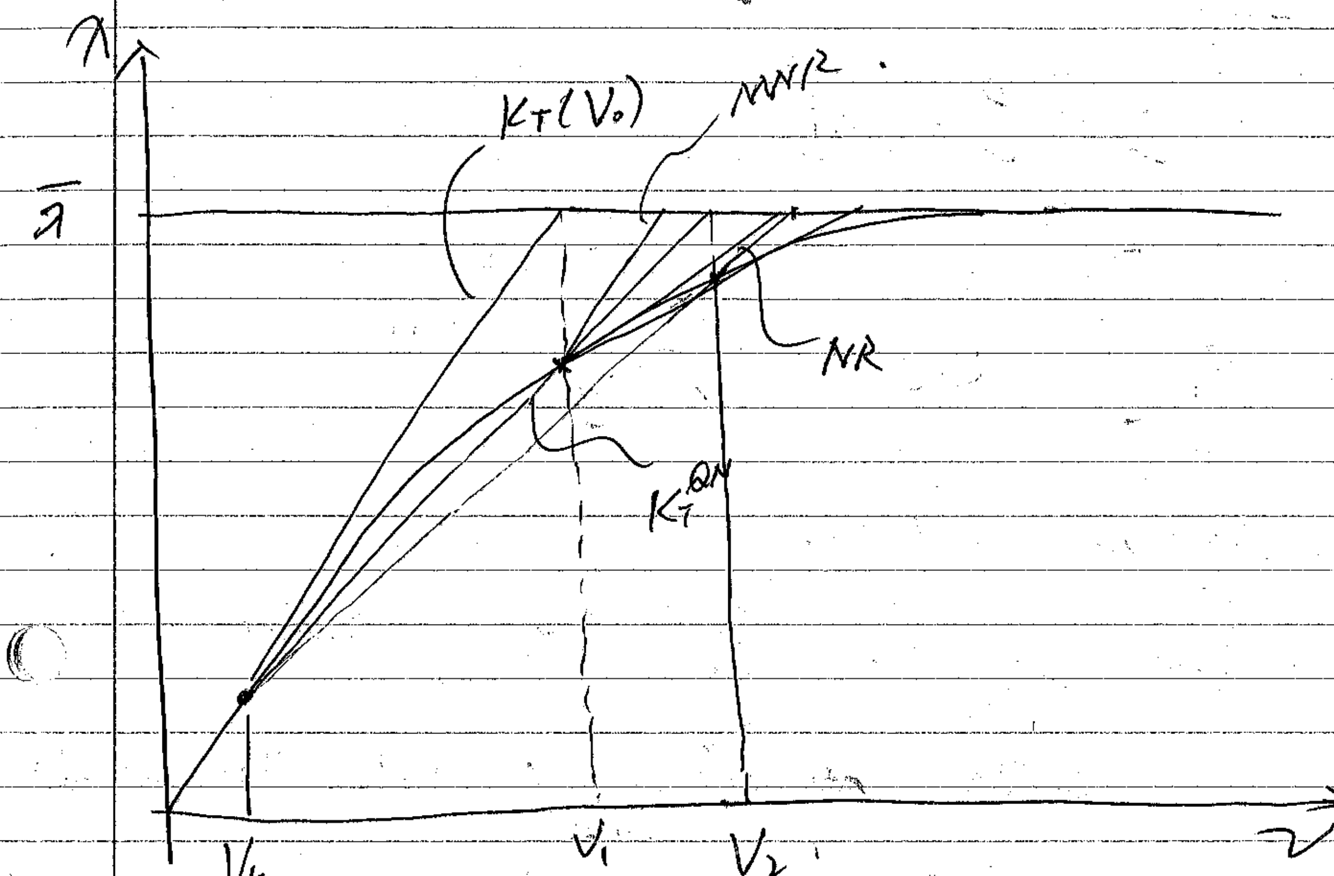


"Quadratic Convergence".



Lecture 2

Comparison between 3 method



HW2 :

Newton – Raphson

$\underline{G} = \underline{R} - \lambda \underline{P}$

$$\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} - \lambda \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \to 0.$$

$V_0 = V_K$ (ICs).

For λ_i in $\vec{\lambda}$ [0, ... 10] initialize

while $\varepsilon \geq$ 1e-4. and iter ≤ 15.

$\underline{G} + D\underline{G}^{K_T} \Delta \underline{V}$

$$\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} + \begin{bmatrix} D_1G_1 & D_1G_2 \\ D_2G_1 & D_2G_2 \end{bmatrix} \begin{bmatrix} \Delta V_1 \\ \Delta V_2 \end{bmatrix}$$

equiv.

$\underline{K}_T \Delta \underline{V} = -\underline{G}$ ... (1)

$\underline{V} = \underline{V} + \Delta \underline{V}$ ... (2)

$\|\underline{G}\| = \sqrt{G[0]^{\wedge}2 + G[1]^{\wedge}2}$ (3)

For j = [0, 1, 2, ..., 40].

$\Delta\lambda$ = 0.25

$\lambda = \Delta\lambda * j$

$\underline{G}_0 = \underline{K}_{T0} \, \Delta \underline{V}_0$.

~~For~~

While $\varepsilon >$ 1e-4 or iter ≤ 15.

$\underline{G} = \underline{R} - \lambda \underline{P}$

$V_{init.}$

$K_T = [\quad]$

$V = V + \Delta V$.

$\|G\| = \sqrt{\ldots}$

$$G = -\begin{bmatrix} 0.6V_1\Delta V_1 + 6\Delta V_1 - 2xV_2\Delta V_2 \\ -\Delta V_1 + \Delta V_2 \end{bmatrix}$$

$$K_T = \begin{bmatrix} 0.6V_1 + 6 & -2xV_2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \Delta V_1 \\ \Delta V_2 \end{bmatrix}$$

Newton-Ramphson implementation.

Initialization

for λ

while

$\underline{V}_0 = \underline{V}_k = \begin{bmatrix} \cdot \\ \cdot \end{bmatrix}$ & $i \leq 15$.

compute G

$\underline{G} = \underline{R} - \lambda \underline{P} \implies \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} - \lambda \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$

compute K_T.

$\underline{\underline{K_T}} = DG = \begin{bmatrix} D_1 G_1 & D_2 G_1 \\ D_1 G_2 & D_2 G_2 \end{bmatrix}$

compute ΔV_{i+1}.

$\Delta V_{i+1} = \frac{\underline{G}}{\underline{\underline{K_T}}} - \Phi \cdot \underline{\underline{K_T}}^{-1} \underline{G}$

compute new $\underline{V}$

$\underline{V}_{i+1} = \underline{V}_i + \Delta V_{i+1}$

$i = i+1$.

Compute convergence

$\varepsilon = \mathrm{sqrt}(\underline{G}[0]^2 + \underline{G}[1]^2)$

☆ How to solve I.C.s?

$\begin{bmatrix} 0.2 V_1^3 - x V_2^2 + 6 V_1 \\ V_2 - V_1 \end{bmatrix} = 0.$

$0.2 V_1^3 - x V_2^2 + 6 V_1$

HW Line Search Method

$\Downarrow$

Select α_i.

$g(\alpha_i) = \Delta \underline{V}_{i+1}^T \underbrace{\underline{G}(\underline{V}_i + \alpha_i \Delta \underline{V}_{i+1}, \bar{\lambda})}= 0.$

$\underline{G} = \begin{bmatrix} 0.2 V_1^3 - x V_2^2 + 6 V_1 \\ V_2 - V_1 \end{bmatrix} - \lambda_i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

ΔV_{i+1}^T

$g(\alpha_i) = \begin{bmatrix} \Delta V_{i+1}[0] \\ \Delta V_{i+1}[1] \end{bmatrix} \left\{ \begin{bmatrix} 0.2 (V_1 + \alpha_i \Delta V_1)^3 - x (V_2 + \alpha_i \Delta V_2)^2 + 6 (V_1 + \Delta V_1) \\ V_2 + \Delta V_2 - V_1 - \Delta V_1 \end{bmatrix} - \lambda_i \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

to solve for ~~α_{i+1}~~ ~~next~~ α:

$\alpha_i^{k+1} = \alpha_i^k - g(\alpha_i^k) \left[\frac{\alpha_i^k - \alpha_i^{k-1}}{g(\alpha_i^k) - g(\alpha_i^{k-1})} \right]$

BFGS Method

\# Initialization

$V_0 = V_k$

\# compute G

... #(same)

\# compute K_T

... (# same). $\Rightarrow K_T(0)$.

For loop ...

- compute G
- Compute H^{BN}

▶ $w_i = v_i - v_{i-1}$

$[\ ] = [\ ] - [\ ]$

▶ $g_i = G_i - G_{i-1}$

$[\,\vdots\,] = [\,\vdots\,] - [\,\vdots\,]$

▶ $a = \dfrac{1}{\text{np.transpose}(g_i) \cdot w_i} \cdot w_i$

▶ $b_i = -\left\{ g_i - \left[\dfrac{-w_i^T g_i}{w_i^T G_{i-1}} \right]^{\frac{1}{2}} G_{i-1} \right\}$

BFGS

def: compute_V(H, G, V) (ΔV: $\underbrace{H, G}_{\Delta V}$)

- $\Delta V = -HG$
- $V = V + \Delta V$

$\rightarrow V_{i-1}, \quad V_i$

def: compute G (R, λ, P, V)

$G_{prev} \leftarrow G = R - \lambda P$.

$G \leftarrow G\_new = R\_new - \lambda P$.

$\rightarrow G_{i-1}, \quad G_i$

$G_i, G_{i-1} \Longleftarrow$

$V_i, V_{i-1} \Longleftarrow$

$W_i = V_i - V_{i-1}$

$g_i = G_i - G_{i-1}$

$a_i = \dfrac{1}{g_i^T w_i} W_i$

$b_i = -\left\{ g_i - \left[\dfrac{-w_i^T g_i}{w_i^T G_{i-1}} \right]^{\frac{1}{2}} G_{i-1} \right\}$

$\rightarrow H_i = H_i(a_i, b_i, H_{i-1})$

For BFGS.

we have to solve exactly for

v & v_prev and G & G_prev

before the iter v loop:

V_{new} exact solution:

def compute $G(\lambda, v)$.

$\rightarrow G$.

def compute $H(V_i, V_{i-1}, G_i, G_{i-1}, \lambda)$.

$\rightarrow H$

def solve $v(\lambda)$.

"solve v analytically"!

$\rightarrow v$.

for $\lambda = [0, 0.25, 0.5, \dots\dots 10]$

$V_{i-1}, V_i, G_{i-1}, G_i$.

while not converge:

H = compute H

$\Delta v = H \cdot G$.

$\hookrightarrow V_{i+1} = V_i + \Delta V_{i+1}$; $V_i = V_{i-1}$.

Solve for next v:

$$\begin{bmatrix} 0.2V_1^3 - xV_2^2 + 6V_1 \\ \\ V_2 - V_1 \end{bmatrix} - 0.25 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

$$0.2V_1^3 - xV_2^2 + 6V_1 - 0.25 = 0$$

Week 6: Lecture 1.

Weak form & variational principles.

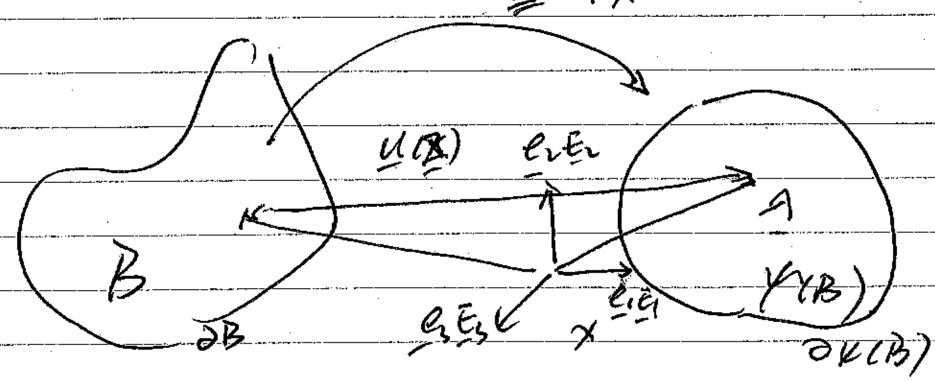
(Lecture 11).

- finite deformations
  - ↳ geometric nonlinearities
  - ↳ material nonlinearities (cons. law)

Strong form → Weak form

{ material description
  spatial desp.

{ material ~
  spat. ~



Strong form: hyperelasticity

↳ reference config.

- Kinematics:

$$\underline{x} = \varphi(\underline{X}) = \underline{u} + \underline{X}$$

$$\underline{\underline{F}} = \mathrm{Grad}\,\varphi(\underline{X}, t) = \underline{\underline{I}} + \mathrm{Grad}\,\underline{u}$$

- Constitutive law.

$$\underline{\underline{P}} = \frac{\partial W(\underline{\underline{F}})}{\partial \underline{\underline{F}}} \quad \boxed{\in B.}$$

fixed.

strain energy function.

$$W = W(I_1, I_2, I_3)$$

principle invariants of $\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}}$.

- Equilibrium.

body forces per unit ref. vol.

$$\mathrm{Div}\,\underline{\underline{P}} + \underline{b} = \rho_0 \dot{\underline{V}} \quad \boxed{\in B}$$

fixed.

material divergence.

"2nd order"

- BCs:

$\partial B_u \cup \partial B_\sigma = \partial B$ $\quad \underline{u} = \bar{\underline{u}}$ on ∂B_u essential

$\partial B_u \cap \partial B_\sigma = \emptyset$ $\quad \underline{\underline{P}}\,\underline{N} = \bar{\underline{t}}$ on ∂B_σ natural

E.L. strain tensor

- kinematics

$$\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{F}}^T\underline{\underline{F}} - \underline{\underline{I}}) \qquad \underline{\underline{F}} = \underline{\underline{I}} + \text{Grad}\,\underline{u}$$

- const. mod.

$$\underline{\underline{S}} = \frac{\partial W}{\partial \underline{\underline{E}}}$$

- equilibrium

$$\text{Div}\,(\underline{\underline{F}}\,\underline{\underline{S}}) + \bar{\underline{b}} = \rho_0 \bar{\underline{v}}$$

referential density

- BCs.

current config.

$\underline{b}$ — per unit current volume.

ρ — current density.

$\underline{\underline{\sigma}} = \frac{1}{J}\underline{\underline{\tau}}$ ← kirchhoff stress tensor.

Lecture 13.

$\underline{\eta} \cdot \underline{f} \rightarrow$ scalar

$\eta_i f_i$

→ single contraction.

$$\underline{\underline{A}}\,\underline{\underline{B}} = A_{IK} B_{KJ} \underline{E}_I \otimes \underline{E}_J = M_{IJ}$$

$$\underline{\underline{A}} \cdot \underline{\underline{B}} = A_{IJ} B_{IJ} = \alpha$$

→ trace of matrix

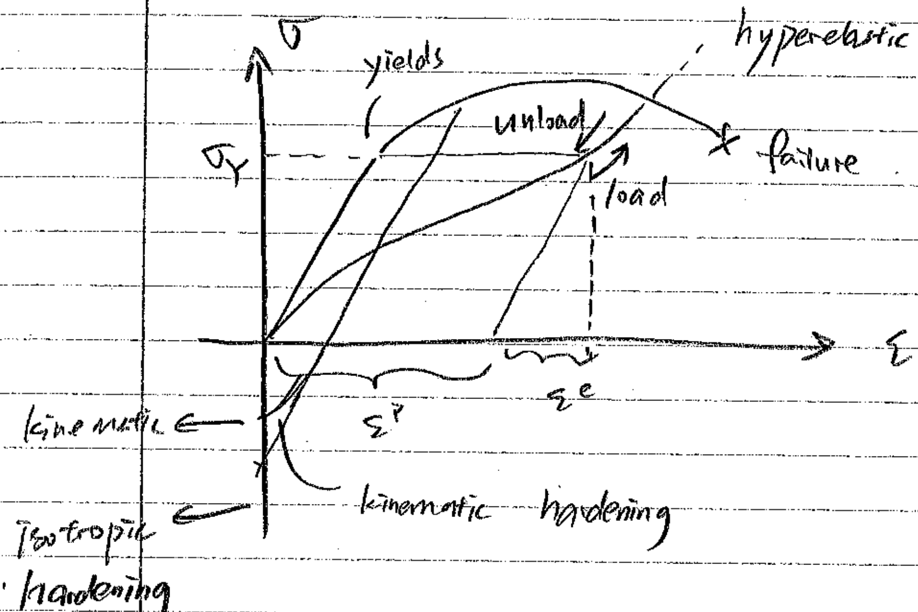
$$\underline{\underline{\underline{A}}}\,\underline{\underline{B}} = \underline{M}$$

$$\underline{\underline{\underline{A}}} \cdot \underline{\underline{\underline{B}}} = \alpha$$

Conclude: Course Notes for FEM

Additional Notes on Plasticity

plasticity vs. hyperelasticity.



hyperelasticity has a one-on-one mapping between the stress and strains

FEA solves

Hyperelastic: $\Rightarrow$ $\underline{\underline{S}}$ ... $\underline{\underline{\varepsilon}}$]

...ocal form.

$G(\bar{\varphi}, \eta) + \Delta($ ...

the same ...

$\underline{\underline{\alpha}}$ $\hat{\alpha}$

$\downarrow$ $\downarrow$

" q ... $|\underline{\underline{\alpha}}|^2$) 2nd order Scalar

plasticity - easies ...

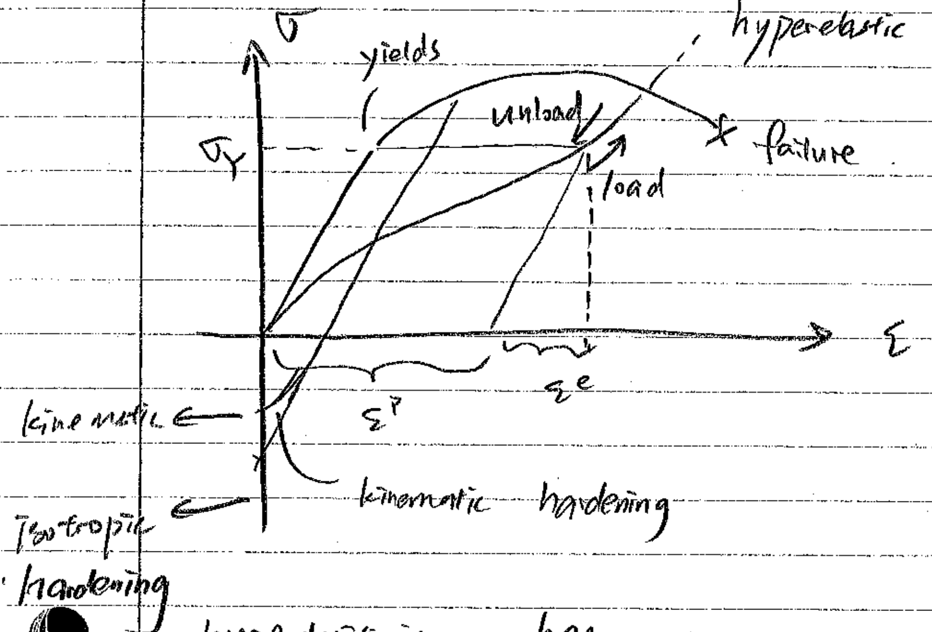
irr...

$q_{ij} = -\frac{2}{3} H \alpha_{ij}$ { isotropic / kinematic) hardening

dissipate energy , $\hat{q} = -\hat{H} \hat{\alpha}$

Additional Notes on Plasticity

plasticity vs. hyperelasticity.



hyperelasticity has a one-on-one mapping between the stress and strains

$$\text{Hyperelastic:} \rightarrow \underline{\underline{S}} = 2\frac{\partial W(\underline{\underline{C}})}{\partial \underline{\underline{C}}} = \varphi(\underline{\underline{C}})$$

FEA solves.

$$u = \bar{u} + \Delta u$$

$$C$$

$$G(\bar{\varphi}, \eta) + \Delta G \cdot \Delta u \simeq 0$$

Calculate stresses in this term.

plasticity - easiest version

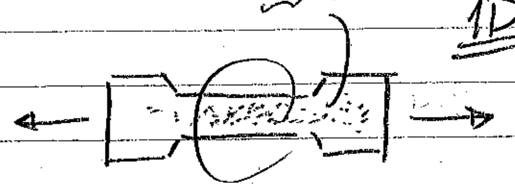
irreversibility.

dissipate energy to induce plastic deformation.

thinking an experiment

assume macroscopic poles

1D



homogeneous Stress state

$\Downarrow$

why we like it

assume small deformations: or kinematic "Empirical assumption from obs."

$$\underset{=}{\varepsilon} = \underset{=}{\varepsilon}^e + \underset{=}{\varepsilon}^p$$

plastic

elastic

(Additive strain decomposition)

divatoric

$$\underset{=}{e} = \underset{=}{\varepsilon} - \frac{1}{3}\,\mathrm{tr}\,\underset{=}{\varepsilon}\,\underset{=}{I}$$

$$\underset{=}{s} = \underset{=}{\sigma} - \frac{1}{3}\,\mathrm{tr}\,\underset{=}{\sigma}\,\underset{=}{I}$$

Conserv. angular momentum

The same way we define a Strain Energy

$$\underset{=}{\sigma} = \frac{\partial \psi(\underset{=}{\varepsilon}^e, \underset{=}{\alpha})}{\partial \underset{=}{\varepsilon}^e}$$

Stress & strain → external

$$\underset{=}{q} = -\frac{\partial \psi(\underset{=}{\varepsilon}^e, \underset{=}{\alpha})}{\partial \underset{=}{\alpha}}$$

microstructural $\Downarrow$ Internal - not obs.

$\underset{=}{q}$ - internal vars - we cannot directly observe

$\uparrow$

Some dissipation mechanism, has to be mathematically described by some thermodynamic conjugate variables - hardening.

$\underset{=}{q}$ is the thermodynamic conjugate of $\underset{=}{\alpha}$

Strain energy → elastic tensor, → hardening variables

$$\psi(\underset{=}{\varepsilon}^e, \underset{=}{\alpha}) = W_e(\underset{=}{\varepsilon}^e) + W_r(\underset{=}{\alpha})$$

→ From linear elasticity, for small deformation.

$$W_e = \frac{1}{2}\underset{=}{\varepsilon}^e \cdot \mathbb{C}^e[\underset{=}{\varepsilon}^e]$$

Hooke's law.

$\Delta \underset{=}{E} \cdot \mathbb{C}[\delta \underset{=}{E}]$ Variational form.

→ For hardening potential

kinematic

$\underset{=}{\alpha}$ → 2nd order, $\hat{\alpha}$ → Scalar

$$W_r = \frac{1}{2}\hat{H}\hat{\alpha}^2 + \frac{1}{3}H|\underset{=}{\alpha}|^2$$

Isotropic hardening

$$\underset{=}{q} = -\frac{2}{3}H\underset{=}{\alpha}, \quad q_{ij} = -\frac{2}{3}H\alpha_{ij}$$

and $\hat{q} = -\hat{H}\hat{\alpha}$

{ isotropic, kinematic } hardening

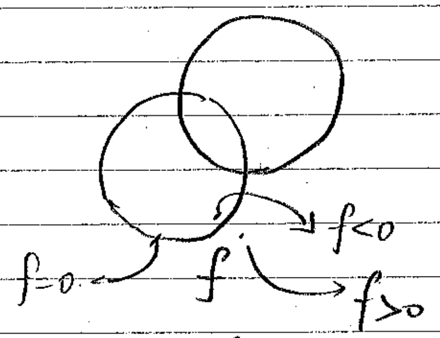
$$\begin{cases} \underline{\underline{q}} = -\dfrac{\partial W_r}{\partial \underline{\underline{\alpha}}} \\ \hat{q} = -\dfrac{\partial W_r}{\partial \hat{q}} \end{cases}$$

→ simplified version, not the only form

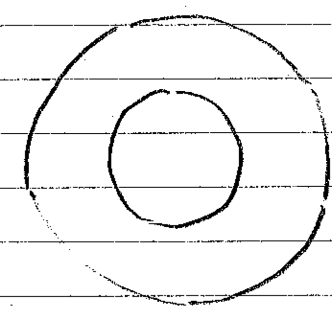
the yield condition: $f(\underline{\underline{\sigma}}, \underline{\underline{q}}, \hat{q}) \leq 0$

$\underline{\underline{q}} \downarrow$ kinematic, $\hat{q} \downarrow$ isotropic

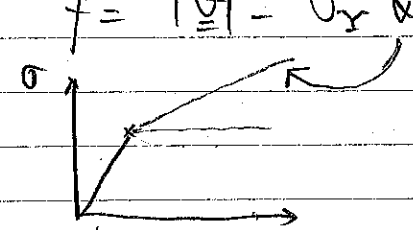
hardening



yield function



in 1D: $f = |\underline{\underline{\sigma}}| - \sigma_Y \hat{\alpha}$, modulate the yield stress



$\underline{\underline{s}} = \text{dev}\, \underline{\underline{\sigma}}$, $(\underline{\underline{s}} - \underline{\underline{q}}) \cdot (\underline{\underline{s}} - \underline{\underline{q}})$

$I_1^{\underline{\underline{s}}-\underline{\underline{q}}} = tr(\underline{\underline{s}} - \underline{\underline{q}})$ pressure independent

linear isotropic & kinematic hardening

$$f(\underline{\underline{s}}, \underline{\underline{q}}, \hat{q}) = \|\underline{\underline{s}} - \underline{\underline{q}}\| - \sqrt{\tfrac{2}{3}}(Y_0 - \hat{q}) \leq 0$$

Von Mises plasticity.

$$f(\underline{\underline{s}}, \underline{\underline{q}}, \hat{q}) = \sqrt{(\underline{\underline{s}} - \underline{\underline{q}}) \cdot (\underline{\underline{s}} - \underline{\underline{q}})} - k(\hat{q}) \leq 0$$

↓ if no internal variables ↓ perfect plasticity

Elasto-plastic material laws — flow rules

associate $\boxed{\text{flow rule}}$: → plastic flow = induce plasticity

$$\dot{\underline{\underline{e}}}^p = \lambda \frac{\partial f}{\partial \underline{\underline{s}}}$$

$$\begin{cases} \underline{\underline{\sigma}}(\underline{\underline{\varepsilon}}^e, \underline{\underline{\alpha}}, \hat{\alpha}) \\ \underline{\underline{q}}(\end{cases}$$

"associate plasticity" — define the plastic strain rate

$$\underbrace{\dot{\underline{\underline{\alpha}}} = \lambda \frac{\partial f}{\partial \underline{\underline{q}}}}_{\text{kinematic}}, \quad \underbrace{\dot{\hat{\alpha}} = \lambda \frac{\partial f}{\partial \hat{q}}}_{\text{isotropic}} \quad \text{hardening}$$

time is "quasi-time" — quasi-static exp.

flow direction:

$$\underline{\underline{n}} = \frac{\partial f}{\partial \underline{\underline{s}}} = \frac{\underline{\underline{s}} - \underline{\underline{q}}}{\|\underline{\underline{s}} - \underline{\underline{q}}\|}$$

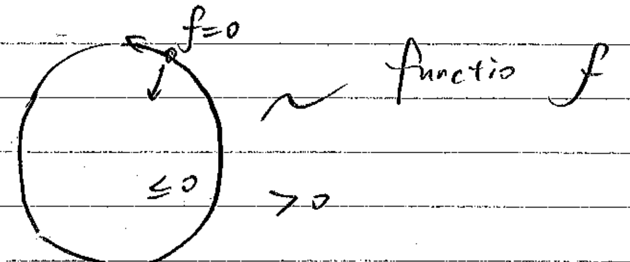
$$\frac{\partial f}{\partial \underline{\underline{q}}} = -\underline{\underline{n}}.$$

the evolution equations:

$$\dot{\underline{\underline{e}}}^p = \lambda \underline{\underline{n}}, \quad \dot{\underline{\underline{\alpha}}} = -\lambda \underline{\underline{n}}, \quad \dot{\hat{\alpha}} = \lambda\sqrt{\tfrac{2}{3}}$$

the latter yields plastic strain incre.

$$\dot{\hat{\alpha}} = \sqrt{\tfrac{2}{3}}\, \|\dot{\underline{\underline{e}}}^p\|, \quad \|\dot{\underline{\underline{e}}}^p\| = \lambda$$



$$f = f(\underline{\underline{s}}, \underline{\underline{q}}, \hat{q})$$

when stress be on $f=0$, distinguish 3 cases:

- elastic unloading: $\lambda = 0$, $\dot{f} < 0$
- neutral loading $\lambda = 0$ → specifically for 3D case $\dot{f} = 0$
- plastic flow. $\lambda > 0$, $\dot{f} = 0$

KKT condition summarizes the 3 cases.

$$\lambda \geq 0, \quad f \leq 0, \quad \lambda f = 0.$$

→ for any dissipative problem.

& consistency condition:

$$\lambda \dot{f} = 0, \quad \text{if } f = 0$$

$$\begin{cases} \dot{f} < 0, & \lambda = 0 \\ \dot{f} = 0, & \lambda = 0 \\ \dot{f} = 0, & \lambda > 0 \end{cases}$$

if $\lambda = 0$,

derive a incremental form of small deformation plasticity; consistency for isotropic / kinematic hardening.

$$\dot{f} = \frac{\partial f}{\partial \underline{\underline{s}}} \cdot \dot{\underline{\underline{s}}} + \frac{\partial f}{\partial \underline{\underline{q}}} \cdot \dot{\underline{\underline{q}}} + \frac{\partial f}{\partial \hat{q}} \dot{\hat{q}} = 0$$

$$\dot{f} = \frac{\partial f}{\partial \underline{\underline{s}}} \cdot \mathbb{C}^e [\dot{\underline{\underline{e}}} - \dot{\underline{\underline{e}}}^p] + \underbrace{\frac{\partial f}{\partial \underline{\underline{q}}}}_{-\underline{\underline{n}}} \cdot \dot{\underline{\underline{q}}} + \frac{\partial f}{\partial \hat{q}} \cdot \dot{\hat{q}}, \quad \dot{\hat{q}} = \hat{H}\dot{\hat{\alpha}}$$

$$= \underline{\underline{n}} \cdot \mathbb{C}^e[\dot{\underline{\underline{e}}}] - \lambda\left(\underline{\underline{n}} \cdot \mathbb{C}^e[\underline{\underline{n}}] + \tfrac{2}{3} H\, \underline{\underline{n}} \cdot \underline{\underline{n}} + \tfrac{2}{3}\hat{H}\right) = 0.$$

the incremental rule: $\underline{\underline{\sigma}} = \underbrace{\operatorname{dev} \underline{\underline{\sigma}}}_{\underline{\underline{s}}} + p \underline{\underline{I}}$

$$\dot{\underline{\underline{\sigma}}} = \mathbb{C}^{ep}[\dot{\underline{\underline{\varepsilon}}}]$$

We can solve for plastic multiplier:

$$\lambda = A^{-1} \underline{n} \cdot \mathbb{C}^e [\dot{e}]$$

linear elasticity "standard" ~

Q: ① when to use specific R?

② R_u vs. R ...?

↳ $\bar{v}$ vs. v ...?