MAE 6110: HW #6

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1. What is the relationship between the 1st Piola stress tensor and the true stress tensor when the deformation is small? Justified your answer.

Solution: Recall the lecture note we first write out the 1st Piola stress & Cauchy stress:

$$\mathbf{P} = J\sigma\mathbf{F}^{-T} \text{ or } \sigma = \frac{1}{J}\mathbf{P}\mathbf{F}^{T} \tag{1}$$

Now we adopt the second method to keep the derivation; Expand the gradient deformation tensor $\mathbf{F}$:

$$\mathbf{F} = \mathbf{I} + \epsilon + \omega \text{ or } \mathbf{F}^{\mathrm{T}} = \mathbf{I} + \epsilon - \omega$$

When the deformation is small, the higher order terms ϵ, ω can be neglected, hence the gradient deformation tensor becomes $\mathbf{F} = \mathbf{I}$. Substituting it into Equation 1, we have

$$\sigma = \frac{1}{\det \mathbf{F}}\mathbf{P}\mathbf{I} \Longrightarrow \sigma = \frac{1}{\mathbf{I}}\mathbf{P} \Longrightarrow \sigma = \mathbf{P}$$

Hence we can deduce that when the deformation is small the 1st Piola stress tensor equals Cauchy stress.

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2. Use the simple tension test to illustrate your result in 1.

Solution: We first formulate the simple tension test, assuming a bar is tensiled at x_3 axis, the gradient deformation tensor writes:

$$\mathbf{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{2}$$

Now we assume the material is isotropic and hyper-elastic, the 1st Piola stress writes:

$$\mathbf{P} = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}}$$

and the Cauchy stress writes

$$\sigma = \frac{1}{J}\frac{\partial W(\mathbf{F})}{\partial \mathbf{F}}\mathbf{F}^{T}$$

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Since the strain density fucntion W obeys:

$$\frac{\partial W}{\partial \mathbf{F}} = 2\mathbf{F}\frac{\partial \hat{W}}{\partial \mathbf{C}}$$

and $\hat{W}$ obeys (lecture notes):

$$\frac{\partial \hat{W}}{\partial \mathbf{C}} = \frac{\partial \Phi}{\partial I_1}\mathbf{I} + \frac{\partial \Phi}{\partial I_2}(I_1\mathbf{I} - \mathbf{C}) + \frac{\partial \Phi}{\partial I_3}\mathbf{C}^{-1}$$

Hence the first Piola stress and Cauchy stress can be further expanded to

$$\mathbf{P} = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} = 2\left[\left(\frac{\partial \Phi}{\partial I_1} + I_1\frac{\partial \Phi}{\partial I_2}\right)\mathbf{F} - \frac{\partial \Phi}{\partial I_2}\mathbf{FC} + I_3\frac{\partial \Phi}{\partial I_3}\mathbf{F}^{-1}\right]$$

$$\sigma = \frac{1}{J}\frac{\partial W(\mathbf{F})}{\partial \mathbf{F}}\mathbf{F}^T = \frac{2}{\det \mathbf{F}}\left[\left(\frac{\partial \Phi}{\partial I_1} + I_1\frac{\partial \Phi}{\partial I_2}\right)\mathbf{b} - \frac{\partial \Phi}{\partial I_2}\mathbf{b}^2 + I_3\frac{\partial \Phi}{\partial I_3}\mathbf{I}\right]$$

Now we assume a neo-Hookean constitutive model where $\Phi = c_1(I_1 - 3)$ and subject it into 1st Piola and Cauchy stresses:

$$\mathbf{P} = 2c_1\mathbf{F} \ \& \ \sigma = \frac{1}{\det \mathbf{F}}(c_1\mathbf{b})$$

We therefore subject the simple tensile test (Equation 2) into the two stresses terms, and recall our previous derivation, we hence have $\mathbf{F} = \mathbf{I}$, $\det \mathbf{F} = 1$, $\mathbf{b} = \mathbf{FF}^T = \mathbf{I}$.

Hence $\mathbf{P} \approx c_1$ and $\sigma \approx c_1$, and hence $\mathbf{P} \approx \sigma$.

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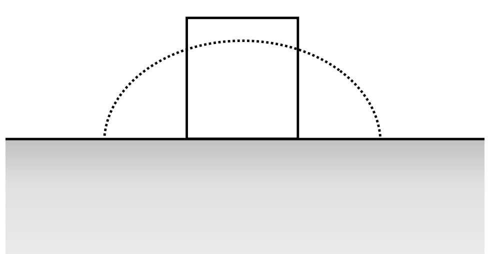
3. A soft incompressible elastic solid has form of a circular cylinder with radius a_0 and height h_0. Both top and bottom end of the cylinder is initially flat. The cylinder is then put into contact with a flat rigid substrate and is in static equilibrium. The interface between the flat rigid substrate and this end of the cylinder is lubricated (by water or oil) so it is free to slide (frictionless). The undeformed cylinder has mass density ρ_0. You may assume gravity is constant and equal to g.

3a. Without calculation, sketch the deform configuration. Explain your reasoning.

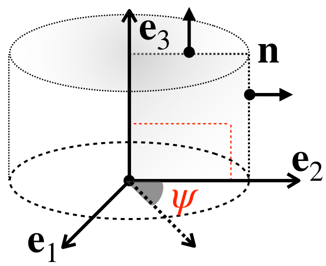
Solution:



The original shape (solid line) deformed into the circular shape (dashed line). First, gravity is a body force acting on all part of the materials body, hence the whole body turned into a parabolic shape that's been "dragged down" by gravity. Second, since the bottom is lubricated by oil, so the bottom shapes changed to a larger area. Third, the shape is smooth and uniform because the gravity was acted on the whole body slowly so there is no evident local shape deformation.

3b. What are the boundary conditions in reference configuration? You are required to express these boundary conditions mathematically using notations developed in class.

Solution: In the reference configuration, there are no displacements on the vertical axis; also there are no traction on the outer surface; also there are no friction at the bottom between the surface between the materials and the substrate since the bottom is lubricated.



In the reference configuration, the lateral and top side of the cylinder should be considered separately, as shown in the beyond schematic.

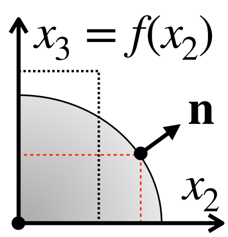
On the top side, the stresses obey traction free condition: when $X_3 = h_0 : P_{ij}n_i = 0 \quad (\mathbf{P} \cdot \mathbf{n} = 0)$, where $\mathbf{n} = [0,\ 0,\ 1]$.

On the lateral side, we first assume there exists a angle ψ between $\mathbf{e}_1$ and $\mathbf{e}_2$ axis as illustrated in the schematic, then the normal vector can be written as $\mathbf{n} = \mathbf{e}_2 \cos\psi + \mathbf{e}_1 \sin\psi$. Therefore the traction free condition can be expressed on the lateral side: when $X_1 + X_2 = a_0^2 : P_{ij}n_i = 0 \quad (\mathbf{P} \cdot \mathbf{n} = 0)$, where $\mathbf{n} = [\sin\psi,\ \cos\psi,\ 0]$.

On the bottom side, the BCs agrees with friction free condition: when $X_3 = 0 : u_3 = 0; P_{13} = P_{23} = 0$.

3c. What are the boundary conditions (BCs) in current or deformation configuration? As in (2b), you are asked to these BCs mathematically. This means that you have to quantity the shape of the deformed configuration using symbols. Hint: the deformed configuration is axis-symmetric, so it can be described by a surface of revolution.

Solution: To describe the whole process and express the BCs, we first adopt the surface of evolution to describe the shape of outer surface. The whole body can be considered as a 2D shape in x_2-x_3 plane that revolve around the x_3 axis. The shape in x_2-x_3 plane can be expressed as $x_2 = f(x_3)$. Now we need to find the normal vector on the surface



From the previous sub-questions, we know that the cylinder deformed into a uniform and smooth shape that can be described by surface revolution, which agrees with axis-symmetric, as illustrated in the above figure.

To compute the normal vector, we need tot compute the tangent of the shape in two directions. We first need to Find a tangent vector to your curve by differentiating the parametric function. Since x_2

here is the variable, we took the partial derivative on x_2:

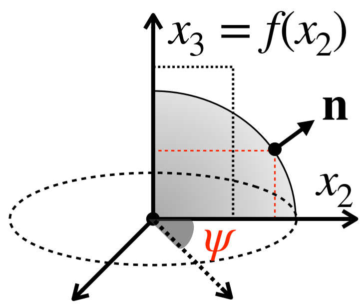
$$\frac{d}{dx_2}\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \frac{d}{dx_2}\begin{bmatrix} x_2 \\ f(x_2) \end{bmatrix} = \begin{bmatrix} 1 \\ \dot{f}(x_2) \end{bmatrix}$$

Then, Rotate this vector by 90° degrees by swapping the coordinates and making one negative:

$$\text{Tanget vector} \rightarrow \text{Normal vector}: \begin{bmatrix} 1 \\ \dot{f}(x_2) \end{bmatrix} \rightarrow \begin{bmatrix} -\dot{f}(x_2) \\ 1 \end{bmatrix}$$

The magnitude of the vector is $\sqrt{1+\dot{f}^2(x_2)}$. And the unit normal vector writes:

$$\mathbf{n} = \frac{1}{\sqrt{1+\dot{f}^2(x_2)}}\begin{bmatrix} -\dot{f}(x_2) \\ 1 \end{bmatrix}$$



If the shape in this x_2-x_3 is revolved around the x_3 axis, and we assume the angle of the revolution is ψ (as shown in the above figure). then the normal vector takes the form:

$$\mathbf{n} = \frac{1}{\sqrt{1+\dot{f}^2(x_2)}}\begin{bmatrix} -\dot{f}(x_2)\sin\psi \\ -\dot{f}(x_2)\cos\psi \\ 1 \end{bmatrix}$$

Now we can write out how does the shape transformed from the original configuration $\mathbf{X}$ to the current configuration $\mathbf{x}$. To write it out, we need to consider from the points on the lateral and top side of the cylinder differently.

We assume the deformations on x_2 and x_3 axes are u_2 and u_3. On the lateral side:

$$\begin{aligned} x_1 &= 0 \\ x_2 &= a_0 + u_2(X_1 = 0, X_2 = a_0) \\ x_3 &= X_3 + u_3(X_1 = 0, X_2 = a_0) \end{aligned}$$

On the top side:

$$\begin{aligned} x_1 &= 0 \\ x_2 &= X_2 + u_2(X_1 = 0, X_3 = h_0) \\ x_3 &= h_0 + u_3(X_1 = 0, X_3 = h_0) \end{aligned}$$

With these given conditions we know what is the form of normal vectors $\mathbf{n}$ and $\mathbf{m}$ in the deformed shape. Now we can write out the BCs in the current configuration:

$$\text{Traction free on the outer surface} : \sigma \cdot \mathbf{m}' = 0 \ \& \ \sigma \cdot \mathbf{n}' = 0$$
$$\text{No friction \&vertical displacement on the bottom surface } X_3 = 0 : u_3 = 0$$
$$\sigma_{13} = \sigma_{23} = 0$$

3e. Is your formulation in 1b valid when the cylinder is glued to the substrate? If not, what boundary condition(s) do you need to impose? Which boundary condition gives the larger stress (a physical reasoning is fine)?

Solution: When the boundary is fixed, the shape of the deformed body becomes different, i.e. there will be a curve between the fixed bottom and the deformed upper body. All the displacements become zero at the bottom since it's fixed and there are also stress occurs. The traction free BCs also agrees. The BCs are therefore mathematically expressed as:

$$\text{When } X_3 = 0 : u_1 = 0, u_2 = 0, u_3 = 0$$
$$\text{On the outer surface} : \sigma_{ij} n_i = 0 \quad (\sigma \cdot \mathbf{n} = 0)$$

3f. Express the stress (σ) and the small strain tensor (ϵ) in cylindrical coordinates (R, θ, Z). For the strain tensor, you may use the symmetry condition that $u_\theta = 0$.

Solution: We first write out the displacements vector:

$$\mathbf{u} = u_R \mathbf{e}_R + u_\theta \mathbf{e}_\theta + u_Z \mathbf{e}_Z \xrightarrow{u_\theta = 0} \mathbf{u} = u_R \mathbf{e}_R + u_Z \mathbf{e}_Z$$

The stress tensor writes:

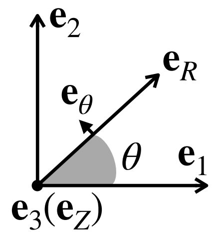
$$\sigma = \sigma_{ij} \mathbf{e}_i \mathbf{e}_j = \sigma_{RR} \mathbf{e}_R \mathbf{e}_R + \sigma_{R\theta} \mathbf{e}_R \mathbf{e}_\theta + \sigma_{\theta\theta} \mathbf{e}_\theta \mathbf{e}_\theta + \sigma_{RZ} \mathbf{e}_R \mathbf{e}_Z + \sigma_{ZZ} \mathbf{e}_Z \mathbf{e}_Z + \sigma_{Z\theta} \mathbf{e}_Z \mathbf{e}_\theta$$

Since axis-symmetry nature of the material body, we know that the terms $\sigma_{R\theta} = 0$ and $\sigma_{Z\theta} = 0$ Considering the transformation between the two coordinates:

$$\sigma_{RR} = \mathbf{e}_R \cdot \sigma \cdot \mathbf{e}_R$$
$$\sigma_{R\theta} = \mathbf{e}_R \cdot \sigma \cdot \mathbf{e}_\theta$$
$$\sigma_{\theta\theta} = \mathbf{e}_\theta \cdot \sigma \cdot \mathbf{e}_\theta$$
$$\sigma_{RZ} = \mathbf{e}_R \cdot \sigma \cdot \mathbf{e}_Z$$
$$\sigma_{ZZ} = \mathbf{e}_Z \cdot \sigma \cdot \mathbf{e}_Z$$
$$\sigma_{Z\theta} = \mathbf{e}_Z \cdot \sigma \cdot \mathbf{e}_\theta$$

The beyond equations can be further written into:

$$\begin{aligned} \sigma_{RR} &= \mathbf{e}_R \cdot \sigma_{ij} \mathbf{e}_i \mathbf{e}_j \cdot \mathbf{e}_R \\ \sigma_{\theta\theta} &= \mathbf{e}_\theta \cdot \sigma_{ij} \mathbf{e}_i \mathbf{e}_j \cdot \mathbf{e}_\theta \\ \sigma_{RZ} &= \mathbf{e}_R \cdot \sigma_{ij} \mathbf{e}_i \mathbf{e}_j \cdot \mathbf{e}_Z \\ \sigma_{ZZ} &= \mathbf{e}_Z \cdot \sigma_{ij} \mathbf{e}_i \mathbf{e}_j \cdot \mathbf{e}_Z \end{aligned} \tag{3}$$



From the beyond figure we can deduce that

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{e}_R \cos\theta \\ \mathbf{e}_2 &= \mathbf{e}_R \sin\theta \\ \mathbf{e}_1 &= \mathbf{e}_\theta \sin\theta \\ \mathbf{e}_2 &= \mathbf{e}_R \cos\theta \end{aligned} \tag{4}$$

we can hence further write

$$\begin{aligned} \mathbf{e}_1 \cdot \mathbf{e}_R &= \cos\theta \\ \mathbf{e}_2 \cdot \mathbf{e}_R &= \sin\theta \\ \mathbf{e}_1 \cdot \mathbf{e}_\theta &= \sin\theta \\ \mathbf{e}_2 \cdot \mathbf{e}_\theta &= \cos\theta \end{aligned}$$

We therefore substitute Equation 4 into Equation 3, the stresses terms can be expanded to (Z and 3 are the same axis, using different symbols)

$$\begin{aligned} \sigma_{RR} &= \mathbf{e}_R \cdot \sigma_{ij}\mathbf{e}_i\mathbf{e}_j \cdot \mathbf{e}_R \\ &= \mathbf{e}_R \cdot \mathbf{e}_1\sigma_{11}\mathbf{e}_1 \cdot \mathbf{e}_R + \mathbf{e}_R \cdot \mathbf{e}_1\sigma_{12}\mathbf{e}_2 \cdot \mathbf{e}_R + \mathbf{e}_R \cdot \mathbf{e}_1\sigma_{21}\mathbf{e}_2 \cdot \mathbf{e}_R + \mathbf{e}_R \cdot \mathbf{e}_2\sigma_{22}\mathbf{e}_2 \cdot \mathbf{e}_R \\ \sigma_{\theta\theta} &= \mathbf{e}_\theta \cdot \sigma_{ij}\mathbf{e}_i\mathbf{e}_j \cdot \mathbf{e}_\theta \\ &= \mathbf{e}_\theta \cdot \mathbf{e}_1\sigma_{11}\mathbf{e}_1 \cdot \mathbf{e}_\theta + \mathbf{e}_\theta \cdot \mathbf{e}_1\sigma_{12}\mathbf{e}_2 \cdot \mathbf{e}_\theta + \mathbf{e}_\theta \cdot \mathbf{e}_1\sigma_{21}\mathbf{e}_2 \cdot \mathbf{e}_\theta + \mathbf{e}_R \cdot \mathbf{e}_2\sigma_{22}\mathbf{e}_2 \cdot \mathbf{e}_\theta \\ \sigma_{RZ} &= \mathbf{e}_R \cdot \sigma_{ij}\mathbf{e}_i\mathbf{e}_j \cdot \mathbf{e}_Z \\ &= \mathbf{e}_R \cdot \mathbf{e}_1\sigma_{13}\mathbf{e}_3 \cdot \mathbf{e}_Z + \mathbf{e}_R \cdot \mathbf{e}_2\sigma_{23}\mathbf{e}_3 \cdot \mathbf{e}_Z \\ \sigma_{ZZ} &= \mathbf{e}_Z \cdot \sigma_{ij}\mathbf{e}_i\mathbf{e}_j \cdot \mathbf{e}_Z \\ &= \mathbf{e}_Z \cdot \mathbf{e}_3\sigma_{33}\mathbf{e}_3 \cdot \mathbf{e}_Z \end{aligned} \tag{5}$$

Equation 5 can thence be expanded to

$$\begin{aligned} \sigma_{RR} &= \sigma_{11}\cos^2\theta + 2\sigma_{12}\cos\theta\sin\theta + \sigma_{22}\sin^2\theta \\ \sigma_{\theta\theta} &= \sigma_{11}\sin^2\theta + 2\sigma_{12}\cos\theta\sin\theta + \sigma_{22}\cos^2\theta \\ \sigma_{RZ} &= \sigma_{13}\cos\theta + \sigma_{23}\sin\theta \\ \sigma_{ZZ} &= \sigma_{33} \end{aligned}$$