# MAE 6110: HW #3

Hanfeng Zhai*

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This problem is about the simple shear deformation in class notes.

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1a. Find $\mathbf{C}$ for simple shear and find the eigenvalues, you need to simplify, e.g. the largest eigenvalue of $\mathbf{C}$ is $\lambda_1 = (\sec\beta + \tan\beta)^2$, where $\tan\beta = \frac{1}{2}\tan\lambda$.

**Solution:** Based on the definition of simple shear:

$$\begin{cases} x_1 = X_1 + X_2 \tan\gamma \\ x_2 = X_2 \\ x_3 = X_3 \end{cases}$$

Therefore we can write the deformation gradient tensor:

$$\mathbf{F} = \begin{bmatrix} 1 & \tan\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can therefore write $\mathbf{C}$:

$$\mathbf{C} = \begin{bmatrix} 1 & \tan\gamma & 0 \\ \tan\gamma & \tan^2\gamma + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We compute the eigenvalues of $\mathbf{C}$ via $\det|\mathbf{C} - \lambda \cdot \mathbf{I}| = 0$, obtains that the three eigenvalues are

$$[\lambda_1, \lambda_2, \lambda_3] = \left[ \frac{\tan\gamma\sqrt{\tan^2\gamma + 4}}{2} + \frac{\tan^2\gamma}{2} + 1,\ \frac{\tan^2\gamma}{2} - \frac{\tan\gamma\sqrt{\tan^2\gamma + 4}}{2} + 1,\ 1 \right]$$

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Substituting the condition $\tan\beta = \frac{1}{2}\tan\gamma$ we get:

$$[\lambda_1, \lambda_2, \lambda_3] = \left[ (\tan\beta + \sec\beta)^2,\ (\tan\beta - \sec\beta)^2,\ 1 \right]$$

Therefore, the largest eigenvalue of $\mathbf{C}$ is $\lambda = (\tan\beta + \sec\beta)^2$.

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* [redacted] www.hanfengzhai.net
*Sibley School of Mechanical and Aerospace Engineering, Cornell University*

1b. Find the normalized eigenvectors for $\mathbf{C}$ – again, you need to simplify, all your answer should be expressed in term of the parameter $\beta$.

**Solution:** Computing the eigenvectors of $\mathbf{C}$, and write them up into a whole matrix we have:

$$\mathbf{V} = [\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3]$$
$$= \begin{bmatrix} \frac{1}{\tan\gamma}\left(\frac{\tan\gamma+\tan\gamma\sqrt{\tan^2\gamma+4}}{2} - \tan^2\gamma + 2\right) & \frac{1}{\tan\gamma}\left(\frac{\tan\gamma-\tan\gamma\sqrt{\tan^2\gamma+4}}{2} - \tan^2\gamma + 2\right) & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Substituting the given relations $\tan\beta = \frac{1}{2}\tan\gamma$ we further write the eigenvectors' matrix:

$$\mathbf{V} = \begin{bmatrix} \tan\beta + \sec\beta & \tan\beta - \sec\beta & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By normalizing this matrix we have the eigenvectors' matrix:

$$\mathcal{E} = \frac{\mathbf{V}}{||\mathbf{V}||} = \begin{bmatrix} \sqrt{\frac{1-\sin\beta}{2}} & -\sqrt{\frac{1+\sin\beta}{2}} & 0 \\ \sqrt{\frac{1+\sin\beta}{2}} & \sqrt{\frac{1-\sin\beta}{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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1c. Find the matrices representing $\mathbf{R}$ and $\mathbf{U}$ in the polar decomposition with respect to the original basis. Interpret $\mathbf{R}$ geometrically.

**Solution:** According to course note, the polar decomposition takes the form:

$$\mathbf{F} = \begin{bmatrix} 1 & \tan\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2\tan\beta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{RU}$$

We also know that $\mathbf{U}$ is the right stretch tensor, writes $\mathbf{U^2} = \mathbf{C}$.

From 1a we already compute the eigenvalues of $\mathbf{C}$, we can hence say that $\mathbf{C}$ can be rewritten into the form as follows in principal direction:

$$\mathbf{C} = \begin{bmatrix} (\tan\beta + \sec\beta)^2 & 0 & 0 \\ 0 & (\tan\beta - \sec\beta)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore we derive the tensor $\mathbf{U}$:

$$\mathbf{U} = \sqrt{\mathbf{C}} = \begin{bmatrix} \sec\beta + \tan\beta & 0 & 0 \\ 0 & \sec\beta - \tan\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that this $\mathbf{U}$ is in the prime basis, for calculation of $\mathbf{R}$, we need to transfer this $\mathbf{U}$ back to the original basis, called $\mathcal{U}$:

$$\begin{aligned} \mathcal{U} &= \mathcal{E}\mathbf{U}\mathcal{E}^T \\ &= \begin{bmatrix} \sqrt{\frac{1-\sin\beta}{2}} & \sqrt{\frac{1+\sin\beta}{2}} & 0 \\ -\sqrt{\frac{1+\sin\beta}{2}} & \sqrt{\frac{1-\sin\beta}{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sec\beta + \tan\beta 0 & 0 & 0 \\ 0 & \sec\beta - \tan\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{\frac{1-\sin\beta}{2}} & -\sqrt{\frac{1+\sin\beta}{2}} & 0 \\ \sqrt{\frac{1+\sin\beta}{2}} & \sqrt{\frac{1-\sin\beta}{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ \sin\beta & \frac{1+\sin^2\beta}{\cos\beta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Therefore we can compute $\mathbf{R}$ on the original basis:

$$\begin{aligned} \mathbf{R} = \mathbf{F}\mathcal{U}^{-1} &= \begin{bmatrix} 1 & 2\tan\beta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sin^2\beta+1}{\cos\beta} & -\sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

We can therefore geometrically interpret $\mathbf{R}$ as the rotational transformation with the angle $\beta$.

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1d. What are $\mathbf{R}$ and $\mathbf{U}$ for small shear deformations?

**Solution:** According to the lecture notes, for small shear deformations, one has to follow the *small strain theory*, which is:

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) \approx \epsilon$$

We can therefore compute $\mathbf{E}$:

$$\mathbf{E} \overset{\text{small shear strain}}{\approx} \epsilon = \begin{bmatrix} 0 & \tan\beta & 0 \\ \tan\beta & 2\tan^2\beta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Also, with small strain theory, we know that $\mathbf{F} \approx \mathbf{I} + \omega + \epsilon$. We therefore know $\omega$: $\omega = \mathbf{F} - \mathbf{I} - \epsilon$.

For small deformation, $\beta \approx 0 \rightarrow \sin\beta \approx \beta, \tan\beta \approx \beta, \cos\beta \approx 1$, then we have

$$\mathcal{U} = \begin{bmatrix} 1 & \beta & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \& \mathbf{R} = \begin{bmatrix} 1 & \beta & 0 \\ -\beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$