MAE 6110: HW #11

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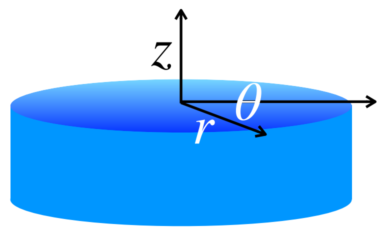
November 25, 2021

1. In a torsion rheology test, the specimen is typically a *circular cylinder* of radius R sandwiched between two rigid plates. At time $t = 0$, the machine imparts a sinusoidal twist $\gamma(t) = \gamma_0 e^{i\omega t}$ to the top plate (the bottom plate is fixed) at a frequency ω.



1a. Assuming that the specimen is linearly viscoelastic, isotropic and homogeneous and the shear relaxation function is given by $G(t)$, formulate the problem by writing down all the initial and boundary conditions and the governing equations.

Solution: We first write out the **equation of equilibrium** for linear viscoelastic body:

$$\sigma_{ij,j} + f_i = 0 \tag{1}$$

Also with the **kinematics**:

$$\epsilon_{ij} = \frac{u_{i,j} + u_{j,i}}{2} \tag{2}$$

The **constitutive model** for linear viscoelasticity writes of the shear stresses and strains write:

$$s_{ij}(t) = 2G\gamma_{ij} \tag{3}$$

where G is the shear modulus, where we can elicit the short time relaxation modulus Y_1 taking the form $Y_1 = 2G$, the equation further writes:

$$s_{ij}(t) = \gamma_{ij}(0^+)Y_1(t) + Y_1^* \gamma_{ij}' \tag{4}$$

where we also have

$$Y_1^* \gamma_{ij}' = \int_0^t Y_1(t-\tau)\frac{\partial \gamma_{ij}}{\partial \tau} d\tau \tag{5}$$

The constitutive law therefore can be rewritten:

$$s_{ij} = 2\gamma_{ij}(0^+)G_1(t) + 2\int_{0^+}^t G(t-\tau)\frac{d\gamma_{ij}}{d\tau} d\tau \tag{6}$$

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For the **initial conditions**, we assume the body is undisturbed at time $t < 0$:

$$u_i = 0,\ \gamma_{ij} = 0,\ \sigma_{ij} = 0, \quad t < 0 \tag{7}$$

and natural for $t \geq 0$ we have

$$\gamma(t) = \gamma_0 e^{iwt} \tag{8}$$

For the **boundary conditions**, on the rounding sides of the cylinder, the traction free BCs stands:

$$\sigma_{ij} n_j = 0, \quad r = R \tag{9}$$

On the downside, the displacements are prescribed:

$$u_i = 0, \quad z = 0 \tag{10}$$

1b. After an initial transient, the stress and strain field in the sample reaches a steady state (they are independent of time). Find the steady state relation between the torque M measured by the load cell and the angle of twist per unit length γ at steady state oscillation. You are encourage to use equations derive in class.

Solution: When $t \to \infty$, according to lecture note, we know that the modulus converge to

$$\hat{Y}_1(\omega) \xrightarrow{t\to\infty} Y_1(\infty) + i\omega \int_0^\infty \left[Y_1(\eta) - Y_1(\infty)\right] e^{i\omega\eta} d\eta \tag{11}$$

Here, $Y_1(t)$ is the modulus of the materials that is dependent on the model we used. If the standard solid model is employed, we have

$$Y_1(t) = G_\infty + (G_0 - G_\infty) e^{-t/t_\infty} \tag{12}$$

In this case taking Equation (11) we have

$$\hat{Y}_1(\omega) = G_\infty + \frac{i\omega(G_0 - G_\infty)}{t_0^{-2} + \omega^2} \tag{13}$$

Therefore we can compute the stress when time approximate infinity:

$$\sigma(t \to \infty) = \gamma_0 e^{i\omega t} \hat{Y}_1(\omega) \tag{14}$$

Therefore we can compute M:

$$M = \frac{J}{R}\sigma = \frac{\pi R^3 \sigma}{2} \tag{15}$$

Where we need to plug in σ and compute the moment M.

1c. Assuming that $G(t) = G_\infty + \frac{G_0 - G_\infty}{(1+t/t_R)^n}$ with $n = 1$ and $G_0/G_\infty = 10$, find the storage and loss modulus, as well as the loss tangent. Plot these quantities versus normalized frequency. You do not need to know the actual values of G_0, G_∞, t_R provided that you normalized the physical quantities appropriately.

Solution: Since we already have for 1b:

$$\hat{Y}_1(\omega) = Y_1(\infty) + i\omega \int_0^\infty \left[Y_1(\eta) - Y_1(\infty)\right] e^{i\omega\eta} d\eta \tag{16}$$

And based on the instructions,

$$G(t) = G_\infty + \frac{G_0 - G_\infty}{(1 + t/t_R)^n} \tag{17}$$

And we know that $t \to \infty$, $G(\infty) \approx G_\infty$; And substitute $Y_1 = 2G_1$ further Equation (16) can be rewritten into:

$$\hat{Y}_1 = \hat{G}(\omega) = G_\infty + i\omega(G_0 - G_\infty)\int_0^\infty \frac{1}{(1+\eta/t_R)^n} e^{i\omega\eta} d\eta \tag{18}$$

By normalizing the equation with G_∞ the equation further writes

$$\frac{\hat{Y}_1}{G_\infty} = 1 + i\omega(\frac{G_0}{G_\infty} - 1)\int_0^\infty \frac{1}{(1+\eta/t_R)^n} e^{i\omega\eta} d\eta \tag{19}$$

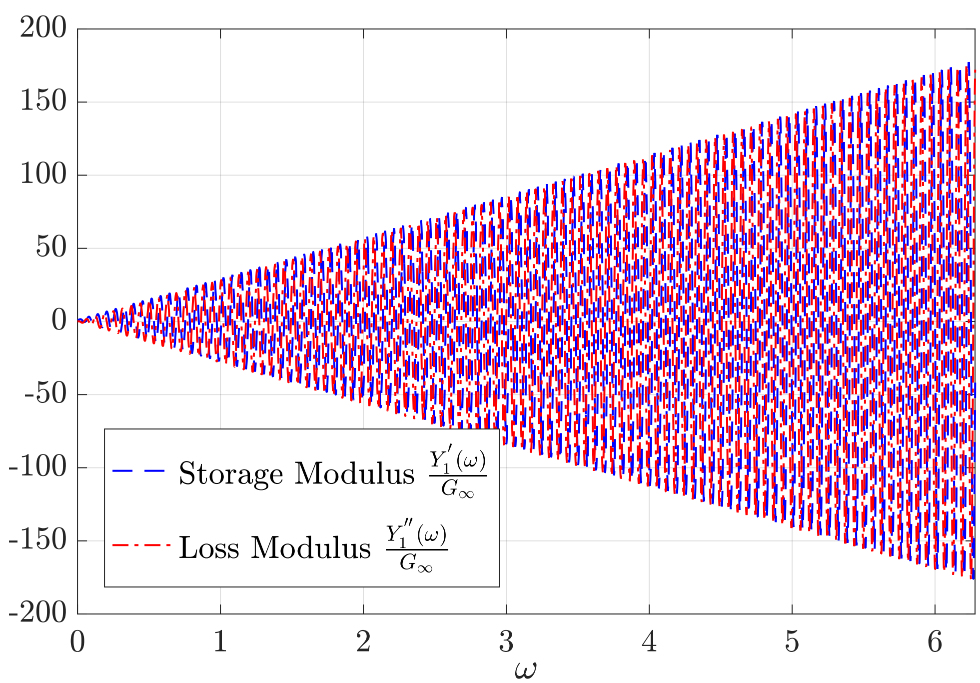
The storage and loss modulus hence writes

$$\frac{Y_1'(\omega)}{G_\infty} = 1 + \omega(\frac{G_0}{G_\infty} - 1)\int_0^\infty \frac{t_R}{t_R + \eta} e^{i\omega\eta} d\eta \tag{20}$$

and

$$\frac{Y_1''(\omega)}{G_\infty} = \omega(\frac{G_0}{G_\infty} - 1)\int_0^\infty \frac{1}{t_R + \eta} e^{i\omega\eta} d\eta \tag{21}$$

Plotting the dependencies we have



2a. Derive the equation

$$\dot{\gamma} = \frac{\dot{\tau} - G_1\dot{\gamma}}{G_2} + \frac{\tau - G_1\gamma}{\eta}$$

in class notes for a standard solid with Cartoon given below, and find the shear creep and relaxation functions. Note this cartoon is different from what is given in class notes.

Solution: Based on the given model, we know

$$\gamma_1 = \frac{\tau_1}{G_1} \quad \& \quad \dot{\gamma}_2 = \frac{\dot{\tau}_2}{G_2} + \frac{\tau_2}{\eta} \tag{22}$$

In this model, it is obvious that

$$\dot{\gamma} = \frac{\dot{\tau} - \dot{\tau}_1}{G_1} + \frac{\tau - \tau_1}{\eta} \tag{23}$$

which can be further written into

$$\dot{\gamma} = \frac{\dot{\tau} - \dot{\gamma} G_1}{G_2} + \frac{\tau - \gamma G_1}{\eta} \tag{24}$$

The equation is therefore proved.

To find the creep and relaxation function, we rearrange Equation (24):

$$\tau + \frac{\eta}{G_2}\dot{\tau} = G\eta + \eta\frac{G_1 + G_2}{G_2}\dot{\eta} \tag{25}$$

We need to solve for $C(t \to 0^+)$ and $C(t \to \infty)$. When $t \to 0^+$, we know that $\dot{\gamma}$ and $\dot{\tau}$ dominates, Equation (25) further can be reduced to:

$$\frac{\eta}{G_2}\dot{\tau} = \eta\frac{G_1 + G_2}{G_2}\dot{\gamma} \tag{26}$$

In this case the creep function C_0 can be written into

$$C_0 = \frac{1}{G_1 + G_2} \tag{27}$$

While when $t \to \infty$, we know that $\dot{\tau}$ & $\dot{\gamma} \approx 0$, therefore Equation (25) can be represented into a linear model:

$$\tau = G_1\gamma \to \gamma = \frac{\tau}{G_1} \tag{28}$$

In this way we have $C_\infty = \frac{1}{G_1}$. We therefore have the creep function:

$$C(t) = \frac{1}{G_1} - \frac{G_2}{G_1(G_1 + G_2)}e^{-\frac{G_2 t}{\eta}} \tag{29}$$

In a similar way we can also compute the relaxation function, when $t \to \infty$, the gradients $\dot{\tau}$ and $\dot{\gamma} \approx 0$ and we hence have

$$\tau = G_1\gamma \ \to \ Y_0 = G_1 \tag{30}$$

And in short time when $\dot{\gamma}$ and $\dot{\tau}$ dominates we have

$$\tau = \dot{\gamma}(G_1 + G_2) \to Y_\infty = G_1 + G_2 \tag{31}$$

We hence have the relaxation function

$$Y(t) = Y_\infty + (Y_0 - Y_\infty)e^{-\frac{t}{t_0}} \tag{32}$$

2b. An initially stress free linear viscoelastic incompressible bar is loaded uni-axially at a constant strain rate of $\dot{\epsilon}_L > 0$ from time equal to 0 to t_{max}, then unloads at a different rate $-\dot{\epsilon}_{UL}$ ($\dot{\epsilon}_{UL} > 0$) until the bar is stress free. Assuming that material is a standard solid and the shear relaxation function is given by what you find in (2a), find the time t_f where the stress in the bar just reaches zero.

Solution: From the instructions, we know when $t = t_f$ we have $\sigma = Y(t) = 0$, which can also be written as $Y(t = t_f) = 0$; Hence

$$0 = Y_\infty + (Y_0 - Y_\infty)e^{-t_f/t_0} \tag{33}$$

Further deriving the equation we have

$$e^{-t_f/t_0} = \frac{-Y_\infty}{Y_0 - Y_\infty} \tag{34}$$

we hence have

$$\log\left(\frac{-Y_\infty}{Y_0 - Y_\infty}\right) = -\frac{t_f}{t_0} \tag{35}$$

therefore

$$t_f = -t_0 \log\left(-\frac{Y_\infty}{Y_0 - Y_\infty}\right) \tag{36}$$

where Y_0 and Y_∞ can be acquired from 2a.

2c. The amount of energy per unit volume dissipated at this time.

Solution: Recall lecture note, to calculate the energy dissipation, we do a integration on the stress to calculate the total energy

$$W = \int_{cycle} \sigma d\epsilon \tag{37}$$

which can be separated into energy losses and energy storage as $W = W_{storage} + W_{loss}$.

The energy loss can hence be written as

$$W_{loss} = \omega\epsilon_0^2 \hat{Y}_1^{''}(\omega) \int_{cycle} \sin^2(\omega t)dt \tag{38}$$

Here, the loss modulus $\hat{Y}_1^{''}$ can be computed from relaxation function as

$$Y_1^{''}(\omega) = \omega \int_0^\infty [Y_1(\eta) - Y_1(\infty)]\cos(\omega\eta)d\eta \tag{39}$$

And Y_1 can be acquired from 2a.

2d. What is the strain in the bar at t_f? Find the strain in the bar for t greater than t_f. Does the bar returns to its original state?

Solution: First, for linear viscoelastic problem the strains are dependent on the stress history. When $t = t_f$, the strain writes:

$$\epsilon(t) = \sigma(0^+)C(t) + \int_{0^+}^{t_f} C(t-\tau)\sigma^{'}(\tau)d\tau \tag{40}$$

When $t > t_f$ there are no more stresses in the bar $\rightarrow \sigma^{'}(\tau) = 0$. Hence,

$$\epsilon(t) = \sigma(0^+)C(t) \tag{41}$$

Hence, after the moment $t = t_f$, the strain is the creep function; when $t \rightarrow \infty$ the material will return back to the initial stage, and within this period the energy dissipated causing its loss.

3. At time = 0, a sudden internal pressure p is imposed on a spherical hole in an infinite linear viscoelastic elastic solid. The pressure is held constant for all time $t > 0$. Assuming that the solid is isotropic and incompressible, find the stress and strain field as a function of time and position. Find the if the pressure is a continuous function of time, e.g. $p = p_0 \sin \omega t$, $t > 0$?

Solution: For this problem, we recall HW 8, and apply a spherical coordinate to model the system. We write out the governing equations of this problem.

We first consider a linear elastic system and consider the **constitutive model**:

$$\sigma_{ij} = 2G\epsilon_{ij} + P\delta_{ij} \tag{42}$$

where P is the hydrostatic pressure.

We also need to consider the **equation of equilibrium** under spherical coordinate, taking the form

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{2}{r}(\sigma_{rr} - \sigma) = 0 \tag{43}$$

where $\sigma = \sigma_{\theta\theta} = \sigma_{\phi\phi}$.

And also we have **kinematics**:

$$\epsilon_{ij} = \frac{u_{i,j} + u_{j,i}}{2} \tag{44}$$

Now we assume the radius of the spherical hole is R_0, the **boundary condition** is:

$$\begin{aligned} \sigma_{rr}(r = R_0) = -p, \quad t \geq 0 \\ \sigma_{rr}(r \to \infty) = 0 \end{aligned} \tag{45}$$

Here, p is the assumed pressure acting on the inner sphere.

In the linear viscoelastic problem, we can assume there exist incompressibility, written:

$$\epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{\phi\phi} = 0 \tag{46}$$

Since $\sigma_{\theta\theta} = \sigma_{\phi\phi} = \sigma$, we can hence conclude $\epsilon_{\theta\theta} = \epsilon_{\phi\phi} = \epsilon$. Plugging in the incompressibility condition, we have:

$$2\epsilon = -\epsilon_{rr} \tag{47}$$

From Equations (47) and (42), we have

$$2\sigma = 3P - \sigma_{rr} \tag{48}$$

therefore we deduce

$$\sigma = \frac{3P - \sigma_{rr}}{2} \tag{49}$$

Now, plug in Equation (49) into Equation (43), we have

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{3}{r}(\sigma_{rr} - P) = 0 \tag{50}$$

We can solve this equation with `Mathematica`, with results

$$\sigma_{rr} = \frac{C}{r^3} + P \tag{51}$$

Substitute the boundary conditions and we can solve for C:

$$C = b^3(-p - P) \tag{52}$$

We can hence solve for the stress in radial direction for linear elasticity:

$$\sigma_{rr} = P + \frac{b^3(-p - P)}{r^3} \tag{53}$$

Substituting this into the constitutive model we therefore have the strain

$$\epsilon_{rr} = \frac{b^3(-p - P)}{2Gr^3} \tag{54}$$

Now, recall the correspondence principle, linear viscoelasticity can be mapped into linear elasticity by applying a Laplace transform:

$$\begin{aligned}\tilde{\sigma}_{ij,j} &= 0\\ \tilde{\epsilon}_{ij} &= \frac{\tilde{u}_{i,j} + \tilde{u}_{j,i}}{2}\\ \tilde{e}_{ij} &= s\tilde{C}_1\tilde{s}_{ij}\\ \tilde{\epsilon}_{kk} &= s\tilde{C}_2\tilde{\sigma}_{kk}\end{aligned} \tag{55}$$

Since the solution in Equation (53) is already independent of modulus, we can expect after a reverse Laplace transform the solution for both stresses and strains are also independent of modulus.