MAE 7750: HW #1

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February 6, 2023

1. Obtain an expression for $\partial \mathbf{A}^{-1}/\partial \mathbf{A}$, where $\mathbf{A}$ is a second order tensor. (Hint: use indicial notation and start from identity $A^{-1}_{ik}A_{kj} = \delta_{ij}$).

We start from the term $A^{-1}_{ik}A_{kj}$, by applying partial derivative on this term:

$$\frac{\partial\left(A^{-1}_{ik}A_{kj}\right)}{\partial A_{pq}} = \frac{\partial A^{-1}_{ik}}{\partial A_{pq}}A_{kj} + A^{-1}_{ik}\frac{\partial A_{kj}}{\partial A_{pq}}$$

Due to $A^{-1}_{ik}A_{kj} = \delta_{ij}$, we have $\partial \delta_{ij}/\partial A_{pq} = 0$. We then have

$$\frac{\partial A^{-1}_{ik}}{\partial A_{pq}}A_{kj} + A^{-1}_{ik}\frac{\partial A_{kj}}{\partial A_{pq}} = 0$$
$$\frac{\partial A^{-1}_{ik}}{\partial A_{pq}}A_{kj} = -A^{-1}_{ik}\frac{\partial A_{kj}}{\partial A_{pq}}$$

We thus obtain

$$\begin{aligned}\frac{\partial A^{-1}_{ik}}{\partial A_{pq}} &= -A^{-1}_{ik}\frac{\partial A_{kj}}{\partial A_{pq}}A^{-1}_{kj} \\ &= -A^{-1}_{ik}\delta_{kp}\delta_{jq}A^{-1}_{kj} \\ &= -A^{-1}_{ip}A^{-1}_{kq}\end{aligned}$$

Hence, we obtain

$$\frac{\partial \mathbf{A}^{-1}}{\partial \mathbf{A}} = -\mathbf{A}^{-1}\mathbf{A}^{-1}$$

2. Obtain the expression $\partial \det\mathbf{A}/\partial \mathbf{A} = \det(\mathbf{A})\mathbf{A}^{-\mathsf{T}}$ using direct notation.

According to the definition of derivative, consider $\partial \det\mathbf{A}/\partial \mathbf{A}$ multiplies another tensor $\mathbf{B}$[1]:

[1]Ref.: https://en.wikipedia.org/wiki/Tensor_derivative_(continuum_mechanics)

$$
\begin{aligned}
\frac{\partial \det \mathbf{A}}{\partial \mathbf{A}} : \mathbf{B} &= \frac{d}{d\alpha} \det(\mathbf{A} + \alpha \mathbf{B}) \Big|_{\alpha=0} \\
&= \frac{d}{d\alpha} \det \left[\alpha \mathbf{A} \left(\frac{1}{\alpha} \mathbf{I} + \mathbf{A}^{-1} \cdot \mathbf{B} \right) \right] \Big|_{\alpha=0} \\
&= \frac{d}{d\alpha} \left[\alpha^3 \det(\mathbf{A}) \det \left(\frac{1}{\alpha} \mathbf{I} + \mathbf{A}^{-1} \cdot \mathbf{B} \right) \right] \Big|_{\alpha=0} .
\end{aligned}
\tag{1}
$$

writing the determinant in the form of invariants:

$$
\det(\lambda \mathbf{I} + \mathbf{A}) = \lambda^3 + I_1(\mathbf{A}) \lambda^2 + I_2(\mathbf{A}) \lambda + I_3(\mathbf{A}) \tag{2}
$$

Substitute equation (2) back into equation (1) we have:

$$
\begin{aligned}
\frac{\partial \det \mathbf{A}}{\partial \mathbf{A}} : \mathbf{B} = \frac{d}{d\alpha} & \left[\alpha^3 \det(\mathbf{A}) \left(\frac{1}{\alpha^3} + I_1 \left(\mathbf{A}^{-1} \cdot \mathbf{B} \right) \frac{1}{\alpha^2} + I_2 \left(\mathbf{A}^{-1} \cdot \mathbf{B} \right) \frac{1}{\alpha} + I_3 \left(\mathbf{A}^{-1} \cdot \mathbf{B} \right) \right) \right] \Big|_{\alpha=0} \\
&= \det(\mathbf{A}) \frac{d}{d\alpha} \left[1 + I_1 \left(\mathbf{A}^{-1} \cdot \mathbf{B} \right) \alpha + I_2 \left(\mathbf{A}^{-1} \cdot \mathbf{B} \right) \alpha^2 + I_3 \left(\mathbf{A}^{-1} \cdot \mathbf{B} \right) \alpha^3 \right] \Big|_{\alpha=0} \\
&= \det(\mathbf{A}) \left[I_1(\mathbf{A}^{-1} \cdot \mathbf{B}) + 2 I_2 \left(\mathbf{A}^{-1} \cdot \mathbf{B} \right) \alpha + 3 I_3 \left(\mathbf{A}^{-1} \cdot \mathbf{B} \right) \alpha^2 \right] \Big|_{\alpha=0} \\
&= \det(\mathbf{A}) I_1 \left(\mathbf{A}^{-1} \cdot \mathbf{B} \right) \\
&= \det(\mathbf{A}) \mathrm{tr} \left(\mathbf{A}^{-1} \cdot \mathbf{B} \right) \\
&= \det(\mathbf{A}) \left[\mathbf{A}^{-1} \right]^{\mathsf{T}} : \mathbf{B}
\end{aligned}
$$

3. Consider the dyad $\mathbf{D} = \mathbf{a} \otimes \mathbf{a}$.

- Write out the components of $\mathbf{D}$ in matrix form.

  Let $\mathbf{a} = [a_1, a_2, \cdots, a_n]^{\mathsf{T}}$, the components of the dyad $\mathbf{D}$ writes:

$$
\mathbf{D} = \begin{bmatrix} a_1 a_1 & a_1 a_2 & \cdots & a_1 a_n \\ a_2 a_1 & a_2 a_2 & \cdots & a_2 a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n a_1 & a_n a_2 & \cdots & a_n a_n \end{bmatrix}
$$

- Compute the three principal invariants of $\mathbf{D}$ simplifying as much as possible.

  The first principal invariant I_1 is the trace of $\mathbf{D}$:

$$
I_1 = \sum_{i=1}^{n} \mathbf{D}_{ii} = \sum_{i=1}^{n} a_i^2
$$

  The second principal invariant I_2 is the determinant of the $\mathbf{D}$:

$$I_2 = \det(\mathbf{D}) = \det(\mathbf{a} \otimes \mathbf{a}) = (\det(\mathbf{a}))^2 = (\prod_{i=1}^{n} a_i)^2$$

The third principal invariant I_3 is the product of the non-zero eigenvalues of $\mathbf{D}$, which is the product of all the components of $\mathbf{a}$:

$$I_3 = \prod_{i=1}^{n} a_i$$

- Compute the eigenvalues of $\mathbf{D}$.

  The eigenvalue equation writes:

$$\sum_{j=1}^{n} a_j^2 v_j = \lambda v_i, \text{ for } i = 1, 2, \cdots, n$$

  Since $\mathbf{D}$ is a scalar multiple of $\mathbf{a} \otimes \mathbf{a}$, all the eigenvalues are equal to $\mathbf{a} \cdot \mathbf{a} = \sum_{i=1}^{n} a_i^2$. Hence, the eigenvalues of $\mathbf{D} = \mathbf{a} \otimes \mathbf{a}$ are

$$\lambda = \sum_{i=1}^{n} a_i^2$$

  with corresponding eigenvectors $\mathbf{v} = \mathbf{a}$

4. Let tensor $\mathbf{A}$ be given by $\mathbf{A} = \alpha(\mathbf{I} - \mathbf{e}_1 \otimes \mathbf{e}_1) + \beta(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$. where α, β are non-zero scalars and $\mathbf{e}_1, \mathbf{e}_2$ orthonormal vectors.

- Show that the eigenvalues of $\mathbf{A}$ are

$$\lambda_1 = \alpha$$
$$\lambda_{2,3} = \alpha/2 \pm (\alpha^2/4 + \beta^2)^{1/2}$$

  Since $\mathbf{A} = \alpha(\mathbf{I} - \mathbf{e}_1 \otimes \mathbf{e}_1) + \beta(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$, then

  Given a second-order tensor $\mathbf{A}$, the eigenvalues λ_i are found by solving the characteristic equation $\det(\mathbf{A} - \lambda_i \mathbf{I}) = 0$, where $\mathbf{I}$ is the identity tensor.

  $\mathbf{A}$ can be expanded as

$$[\mathbf{A}] = \alpha \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) + \beta \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & \beta & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}$$

Solving the characteristic equation we have

$$\det \begin{bmatrix} -\lambda & \beta & 0 \\ \beta & \alpha-\lambda & 0 \\ 0 & 0 & \alpha-\lambda \end{bmatrix} = 0 \ \rightarrow \ \left(-\lambda(\alpha-\lambda) - \beta^2\right)(\alpha-\lambda) = 0$$

We can hence obtain the three eigenvalues:

$$\begin{aligned} \lambda_1 &= \alpha \\ \lambda_{2,3} &= \alpha/2 \pm (\alpha^2/4 + \beta^2)^{1/2} \end{aligned}$$

- Compute the associated eigenvectors.

  Solving the three linear equations by substituting the eigenvalues we have:

$$\begin{cases} \left(\begin{bmatrix} 0 & \beta & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} - \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} \right) \mathbf{v}_1 = 0 \\ \left(\begin{bmatrix} 0 & \beta & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} - \begin{bmatrix} \frac{\alpha}{2} + (\frac{\alpha^2}{4} + \beta^2)^{1/2} & 0 & 0 \\ 0 & \frac{\alpha}{2} + (\frac{\alpha^2}{4} + \beta^2)^{1/2} & 0 \\ 0 & 0 & \frac{\alpha}{2} + (\frac{\alpha^2}{4} + \beta^2)^{1/2} \end{bmatrix} \right) \mathbf{v}_2 = 0 \\ \left(\begin{bmatrix} 0 & \beta & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} - \begin{bmatrix} \frac{\alpha}{2} - (\frac{\alpha^2}{4} + \beta^2)^{1/2} & 0 & 0 \\ 0 & \frac{\alpha}{2} - (\frac{\alpha^2}{4} + \beta^2)^{1/2} & 0 \\ 0 & 0 & \frac{\alpha}{2} - (\frac{\alpha^2}{4} + \beta^2)^{1/2} \end{bmatrix} \right) \mathbf{v}_3 = 0 \end{cases}$$

  Solving these equations we then obtain the three eigenvectors:

```python
from sympy import *
from sympy.solvers.solveset import linsolve

alpha, beta = symbols('alpha beta')

A = Matrix([[0, beta, 0], [beta, alpha, 0], [0, 0, alpha]])
B1 = Matrix([[alpha, 0, 0], [0, alpha, 0], [0, 0, alpha]])
B2 = Matrix([[alpha/2 + sqrt(alpha**2/4 + beta**2), 0, 0], [0, alpha/2 + sqrt(alpha**2/4 + beta**2), 0], [0, 0, alpha/2 + sqrt(alpha**2/4 + beta**2)]])
B3 = Matrix([[alpha/2 - sqrt(alpha**2/4 + beta**2), 0, 0], [0, alpha/2 - sqrt(alpha**2/4 + beta**2), 0], [0, 0, alpha/2 - sqrt(alpha**2/4 + beta**2)]])

M1 = A - B1; M2 = A - B2; M3 = A - B3;
v1 = Matrix([Symbol('v1(1)'), Symbol('v1(2)'), Symbol('v1(3)')])
v2 = Matrix([Symbol('v2(1)'), Symbol('v2(2)'), Symbol('v2(3)')])
v3 = Matrix([Symbol('v3(1)'), Symbol('v3(2)'), Symbol('v3(3)')])

sol1 = linsolve((M1*v1), (v1[0], v1[1], v1[2]))
sol2 = linsolve((M2*v2), (v2[0], v2[1], v2[2]))
sol3 = linsolve((M3*v3), (v3[0], v3[1], v3[2]))
```

Written in symbolic forms as:

$$\begin{cases} \mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ \mathbf{v}_2 = \begin{bmatrix} \frac{2\beta}{\alpha+\sqrt{\alpha^2+4\beta^2}} \\ 1 \\ 0 \end{bmatrix} \\ \mathbf{v}_3 = \begin{bmatrix} \frac{2\beta}{\alpha-\sqrt{\alpha^2+4\beta^2}} \\ 1 \\ 0 \end{bmatrix} \end{cases}$$

- Under which conditions on α, β is $\mathbf{A}$ positive definite.

  We can compute $\det \mathbf{A} = -\alpha\beta^2$.

  To satisfy the positive-definite conditions $-\alpha\beta^2 > 0$, one should obtain $\alpha < 0$ & $\beta \neq 0$.

5. Let ϕ and $\mathbf{u}$ be smooth scalar and vector fields defined on the surface $\mathcal{S}$ and curve $\mathcal{C}$, and let $\mathbf{n}$ the unit outward normal on $\mathcal{S}$. Show that

$$\oint_{\mathcal{C}} \phi d\mathbf{x} = \int_{\mathcal{S}} \mathbf{n} \times \mathrm{grad}\phi ds$$

$$\oint_{\mathcal{C}} \mathbf{u} \times d\mathbf{x} = \int_{\mathcal{S}} \left[(\mathrm{div}\mathbf{u})\mathbf{n} - (\mathrm{grad}^{\mathsf{T}}\mathbf{u})\mathbf{n} \right] ds$$

For the first equation, recall Stoke's theorem, considering the definition of a circulation per unit area[2], we have:

$$\sum \phi d\mathbf{x} = (\nabla \times \phi) ds$$

we can then write:

$$\begin{aligned} \oint_{\mathcal{C}} \phi d\mathbf{x} &= \oint_{\mathcal{C}} d\mathbf{x} \times \mathrm{grad}\phi ds \\ &= \int_{\mathcal{S}} \mathbf{n} \times \mathrm{grad}\phi ds \end{aligned}$$

[2]Ref.: https://www.lehman.edu/faculty/anchordoqui/VC-4.pdf

For the second equation, we can start with the LHS:

$$\begin{aligned}\oint_{\mathcal{C}} \mathbf{u} \times d\mathbf{x} &= \oint_{\mathcal{C}} d(\mathbf{x} \times \mathbf{u}) \\ &= \int_{\mathcal{S}} d\mathbf{x} \times \mathbf{u} ds \\ &= \int_{\mathcal{S}} \left[(\text{div}\mathbf{u})\mathbf{n} - (\text{grad}^{\mathsf{T}}\mathbf{u})\mathbf{n}\right] ds\end{aligned}$$

6. Apply the operator ∇ to product of smooth tensor fields $\mathbf{A}, \mathbf{B}$ to establish the identity:

$$\text{div}(\mathbf{AB}) = \text{grad}\mathbf{A} : \mathbf{B} + \mathbf{A}\text{div}\mathbf{B}$$

Starting with the LHS:

$$\begin{aligned}\text{div}(\mathbf{AB}) &= \nabla \cdot (\mathbf{AB}) \\ &= \sum_{i=1}^{n} \frac{\partial(A_j B_j)}{\partial x_i} \\ &= \sum_{i=1}^{n} \frac{\partial A_j}{\partial x_i} B_j + \sum_{i=1}^{n} A_j \frac{\partial B_j}{\partial x_i} \\ &= \text{grad}\mathbf{A} : \mathbf{B} + \mathbf{A}\text{div}\mathbf{B}\end{aligned}$$

7. A certain motion of a continuum body in the material description is given in the form

$$x_1 = e^t X_1 - e^{-t} X_2, x_2 = e^t X_1 + e^{-t} X_2, x_3 = X_3$$

for $t > 0$. Find the velocity and acceleration components in terms of the material and spatial coordinates and time.

The velocity in the material description:

$$\mathbf{V}(\mathbf{X}, t) = \frac{\partial \mathbf{x}}{\partial t} = \begin{cases} \frac{\partial x_1}{\partial t} = e^t X_1 + e^{-t} X_2 \\ \frac{\partial x_2}{\partial t} = e^t X_1 - e^{-t} X_2 \\ \frac{\partial x_3}{\partial t} = 0 \end{cases}$$

which can also be written in matrix form

$$\mathbf{V} = \begin{bmatrix} e^t X_1 + e^{-t} X_2 \\ e^t X_1 - e^{-t} X_2 \\ 0 \end{bmatrix}$$

Acceleration in the material description writes:

$$\mathbf{A}(\mathbf{X},t) = \frac{\partial \mathbf{V}(\mathbf{X},t)}{\partial t} = \begin{cases} \frac{\partial V_1}{\partial t} = e^t X_1 + e^{-t} X_2 \\ \frac{\partial V_2}{\partial t} = e^t X_1 - e^{-t} X_2 \\ \frac{\partial V_3}{\partial t} = 0 \end{cases}$$

which can also be written in matrix form

$$\mathbf{A} = \begin{bmatrix} e^t X_1 - e^{-t} X_2 \\ e^t X_1 + e^{-t} X_2 \\ 0 \end{bmatrix}$$

Rearranging the given condition we can write:

$$\mathbf{X} = \begin{cases} X_1 = \frac{1}{2e^t}(x_1 + x_2) \\ X_2 = \frac{1}{2e^{-t}}(x_2 - x_1) \\ X_3 = x_3 \end{cases}$$

we can hence write the velocity and acceleration in the spatial description:

$$\mathbf{v}(\mathbf{x},t) = \begin{bmatrix} -\frac{1}{2e^t}(x_1 + x_2) \\ \frac{1}{2e^{-t}}(x_2 - x_1) \\ 0 \end{bmatrix}, \quad \mathbf{a}(\mathbf{x},t) = \begin{bmatrix} \frac{1}{2e^t}(x_1 + x_2) \\ \frac{1}{2e^{-t}}(x_2 - x_1) \\ 0 \end{bmatrix}$$

8. In a deformation of a three-dimensional problem, the displacement components of $\mathbf{u}$ are found to be

$$u_1 = x_1 - \frac{1}{4}x_2, u_2 = x_1 + 2x_2, u_3 = -3x_3$$

- Compute the matrix representations of the deformation gradient and its inverse and show the deformation is isochoric.

  Considering $\mathbf{u} = \mathbf{x} - \mathbf{X}$, we have

$$\mathbf{I} - \mathbf{F}^{-1} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}$$

  We can thence calculate $\mathbf{F}^{-1}$:

$$\mathbf{F}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -\frac{1}{4} & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

By calculating the inverse we get $\mathbf{F}$:

$$\mathbf{F} = \begin{bmatrix} -4 & -1 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

We can then calculate the determinant: $\det \mathbf{F} = 4\frac{1}{4} = 1$. So we can say that the deformation is isochoric.

- Determine the components of the material and spatial strain tensors $\mathbf{C}$, $\mathbf{E}$ and $\mathbf{b}$, $\mathbf{e}$.

  We can calculate material strain tensor from $\mathbf{F}$:

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^{\mathsf{T}}\mathbf{F} - \mathbf{I}\right) = \begin{bmatrix} 15.5 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -0.46875 \end{bmatrix}$$

  The spatial strain tensor:

$$\mathbf{e} = \frac{1}{2}\left(\mathbf{I} - \mathbf{F}^{\mathsf{T}}\mathbf{F}^{-1}\right) = \begin{bmatrix} 2.5 & 2.5 & 0 \\ 0 & 0.625 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

  The right Cauchy–Green deformation tensor

$$\mathbf{C} = \mathbf{F}^{\mathsf{T}}\mathbf{F} = \begin{bmatrix} 17 & -16 & 0 \\ -16 & 16 & 0 \\ 0 & 0 & 0.0625 \end{bmatrix}$$

  The left Cauchy–Green deformation tensor

$$\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathsf{T}} = \begin{bmatrix} 32 & 4 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 0.0625 \end{bmatrix}$$

  The results are calculated from the following codes:

```
F = np.linalg.inv(np.array([[0, 1/4, 0], [-1, -1, 0], [0, 0, 4]]))
E = .5*(np.dot(np.transpose(F), F) - np.eye(3));print(E)
e = .5*(np.eye(3) - np.dot(np.transpose(F),np.linalg.inv(F)));print(e)
b = np.dot(F,np.transpose(F));print(b)
C = np.dot(np.transpose(F),F); print(C)
```

9. Since the strain energy function, Ψ of an isotropic hyperelastic material is an invariant, we may regard it a function of the principal stretches $\lambda_a, a = 1, 2, 3$ and thus write

$$\Psi = \Psi(\mathbf{C}) = \Psi(\lambda_1, \lambda_2, \lambda_3)$$

- Stating the necessary conditions derive the three principal Cauchy stress components (note: σ_a), along with the principal 1st and 2nd Piola-Kirchhoff stresses.

  To derive the three principal Cauchy stress components, one can first write out the first Piola-Kirchhoff stress tensor from the strain energy function Ψ by[3]:

$$\mathbf{P} = \frac{\partial \Psi}{\partial \mathbf{F}}$$

  One can then get the Cauchy stress[4]:

$$\sigma = J^{-1}\mathbf{P}\mathbf{F}^{\mathsf{T}}$$

  where $J = \det \mathbf{F}$. By computing the eigenvalues of the σ one can hence obtain the three principal Cauchy stress components.

  Based on σ, one can also compute the second Piola–Kirchhoff stress:

$$\mathbf{S} = J\mathbf{F}^{-1}\sigma\mathbf{F}^{-\mathsf{T}}$$

  From the above equations, we know that one should know $\mathbf{F}$ and its relationship to Ψ to obtain the corresponding components.

- Formulate the strain energy function Ψ of an incompressible isotropic hyperelastic material in terms of principal stretches and obtain the three principal Cauchy stress components, along with the principal 1st and 2nd Piola-Kirchhoff stresses.

  Recall the strain energy function of an incompressible isotropic hyperelastic material, we may assume a simple strain energy function

$$\begin{aligned}\Psi(I_1, I_2, I_3) &= \frac{\mu}{2}(I_1 - 3) + \frac{\lambda}{2}(I_2 - 3)^2 \\ &= \frac{\mu}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)\end{aligned}$$

  According to the chain rule, we have

$$\frac{\partial \Psi}{\partial \mathbf{C}} = \sum_i^3 \frac{\partial \Psi}{\partial \lambda_i^2}\frac{\partial \lambda_i^2}{\partial \mathbf{C}}$$

  Since $\frac{\partial \lambda_i^2}{\partial \mathbf{C}} = \mathbf{N}_i \otimes \mathbf{N}_i$, we have

$$\frac{\partial \Psi}{\partial \mathbf{C}} = \sum_i^3 \frac{\partial \Psi}{\partial \lambda_i^2}\mathbf{N}_i \otimes \mathbf{N}_i$$

[3] Ref.: https://www.cs.toronto.edu/jacobson/seminar/sifakis-course-notes-2012.pdf

[4] Ref.: https://pkel015.connect.amazon.auckland.ac.nz/SolidMechanicsBooks/Part_III/Chapter_3_Stress_Mass_Momentum/Stress_Balance_Principles_05_Stress_Measures_NonLinear.pdf

We can then compute the second Piola-Kirchhoff stress

$$
\begin{aligned}
\mathbf{S} &= \sum_{i}^{3} \frac{1}{\lambda_i} \frac{\partial \Psi}{\partial \lambda_i} \mathbf{N}_i \otimes \mathbf{N}_i \\
&= \frac{\mu}{2} \sum_{i}^{3} \frac{1}{\lambda_i} \frac{\partial \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right)}{\partial \lambda_i} \mathbf{N}_i \otimes \mathbf{N}_i \\
&= \mu \left(\mathbf{N}_1 \otimes \mathbf{N}_1 + \mathbf{N}_2 \otimes \mathbf{N}_2 + \mathbf{N}_3 \otimes \mathbf{N}_3\right)
\end{aligned}
$$

from which we can also compute the first Piola-Kirchhoff stress

$$
\begin{aligned}
\mathbf{P} &= \mathbf{F}\mathbf{S} \\
&= \sum_{i}^{3} \frac{\partial \Psi}{\partial \lambda_i} \mathbf{N}_i \otimes \mathbf{N}_i \\
&= \mu \left(\lambda_1 \mathbf{N}_1 \otimes \mathbf{N}_1 + \lambda_2 \mathbf{N}_2 \otimes \mathbf{N}_2 + \lambda_3 \mathbf{N}_3 \otimes \mathbf{N}_3\right)
\end{aligned}
$$

We can then compute the Cauchy stress

$$
\begin{aligned}
\sigma &= J^{-1} \mathbf{F} \mathbf{P}^{\mathsf{T}} \\
&= \frac{1}{\lambda_1 \lambda_2 \lambda_3} \sum_{i}^{3} \frac{\partial \Psi}{\partial \lambda_i} \lambda_i \mathbf{N}_i \otimes \mathbf{N}_i \\
&= \frac{\mu}{\lambda_1 \lambda_2 \lambda_3} \left(\lambda_1^2 \mathbf{N}_1 \otimes \mathbf{N}_1 + \lambda_2^2 \mathbf{N}_2 \otimes \mathbf{N}_2 + \lambda_3^2 \mathbf{N}_3 \otimes \mathbf{N}_3\right)
\end{aligned}
$$

For the general case of incompressible hyperelastic materials[5], we have

$$
\begin{aligned}
\sigma &= -p\mathbf{I} + 2\mathbf{F} \cdot \frac{\partial \Psi}{\partial \mathbf{C}} \cdot \mathbf{F}^{\mathsf{T}} \\
\mathbf{P} &= -p\mathbf{F}^{\mathsf{T}} + 2\mathbf{F} \cdot \frac{\partial \Psi}{\partial \mathbf{C}} \\
\mathbf{S} &= -\mathbf{F}^{-1} p \mathbf{F}^{\mathsf{T}} + 2\frac{\partial \Psi}{\partial \mathbf{C}}
\end{aligned}
$$

10. Consider an incompressible hyperelastic membrane under biaxial deformation with the following kinematic assumptions:

$$
x_1 = \lambda_1 X_1, x_2 = \lambda_2 X_2, x_3 = \frac{1}{\lambda_1 \lambda_2} X_3
$$

In particular, the two principal stretches λ_1 and λ_2 are given. According to the membrane theory, assume a plane stress state (out-of-plane stress is zero) and specify the Cauchy stresses (note: use results from problem 9) in the plane of the membrane by applying Ogden's strain energy function.

[5]Ref.: https://en.wikipedia.org/wiki/Hyperelastic_material

Applying Ogden's strain energy function[6], we have

$$\Psi = \sum_i^3 \frac{2\mu_i}{\alpha_i^2}\left(\left(\frac{\lambda_1}{J^{1/3}}\right)^{\alpha_i} + \left(\frac{\lambda_2}{J^{1/3}}\right)^{\alpha_i} + \left(\frac{\lambda_3}{J^{1/3}}\right)^{\alpha_i}\right) + \frac{K_1}{2}(J-1)^2$$

According to the plane stress assumption, we have $\sigma_3 = 0$.

From the given deformation gradient tensor $\mathbf{F}$, we know that $J = \det \mathbf{F} = 1$.

From the solution in Prob. 9, we can write out the Cauchy stress:

$$\begin{aligned}
\frac{\partial \Psi}{\partial \mathbf{C}} &= \sum_i^3 \frac{\partial \Psi}{\partial \lambda_i^2} \mathbf{N}_i \otimes \mathbf{N}_i \\
&= \sum_i^3 \frac{2\mu_i}{\alpha_i}\left(\frac{1}{J^{1/3}}\right)^{\alpha_i} \left(\lambda_1^{\alpha_i-2} + \lambda_2^{\alpha_i-2} + \lambda_3^{\alpha_i-2}\right) \\
&= \sum_i^3 \frac{2\mu_i}{\alpha_i}\left(\lambda_1^{\alpha_i-2} + \lambda_2^{\alpha_i-2} + \left(\frac{1}{\lambda_1\lambda_2}\right)^{\alpha_i-2}\right) \\
\rightarrow 2\mathbf{F}\frac{\partial \Psi}{\partial \mathbf{C}}\mathbf{F}^{\mathsf{T}} &= \sum_i^3 \frac{4\mu_i}{\alpha_i}\left(\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \left(\frac{1}{\lambda_1\lambda_2}\right)^{\alpha_i}\right)
\end{aligned}$$

We can hence write out the matrix form of the Cauchy stress:

$$\sigma = \begin{bmatrix} \sum_i^3 \frac{4\mu_i}{\alpha_i}\lambda_1 - p & 0 & 0 \\ 0 & \sum_i^3 \frac{4\mu_i}{\alpha_i}\lambda_2 - p & 0 \\ 0 & 0 & \sum_i^3 \frac{4\mu_i}{\alpha_i}\frac{1}{\lambda_1\lambda_2} - p \end{bmatrix}$$

Since we know that $\sigma_3 = 0$, we have $\sum_i^3 \frac{4\mu_i}{\alpha_i}\frac{1}{\lambda_1\lambda_2} = p$. We can then write out the form of Cauchy stress:

$$\sigma = \begin{bmatrix} \sum_i^3 \frac{4\mu_i}{\alpha_i}\left(\lambda_1 - \frac{1}{\lambda_1\lambda_2}\right) & 0 & 0 \\ 0 & \sum_i^3 \frac{4\mu_i}{\alpha_i}\left(\lambda_2 - \frac{1}{\lambda_1\lambda_2}\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

[6] Ref.: http://solidmechanics.org/text/Chapter3_5/Chapter3_5.htm