MAE 6110: HW #5

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1. Let D_t be any fixed region in the current configuration Ω_t at time t.

In class, we derive the symmetric of the true stress using a local argument. In this HW problem, you will derive the symmetry of the true stress tensor based on angular momentum balance (AMB):

The statement of AMB is (rate of change of angular momentum) = applied moment due to contact and body forces.

$$\frac{D}{Dt}\int_t \rho\mathbf{x}\times\mathbf{v}dv_t = \int_{D_t}\rho\mathbf{x}\times\mathbf{b}dv_t + \int_{\partial D_t}\mathbf{x}\times(\sigma\cdot\mathbf{n})ds_t$$

1a. 1^{st} establish:

$$\frac{D}{Dt}\int_{D_t}\rho\mathbf{x}\times\mathbf{v}dv_t = \int_{D_t}\mathbf{x}\times\mathbf{a}(\mathbf{x},t)\rho dv_t$$

Solution: We first expand the LHS:

$$\begin{aligned}\frac{D}{Dt}\int_{D_t}\rho\mathbf{x}\times\mathbf{v}dv_t &= \int_{D_t}\rho\left[\frac{D\mathbf{x}}{Dt}|_{\mathbf{x}}\times\mathbf{v} + \frac{D\mathbf{v}}{Dt}|_{\mathbf{x}}\times\mathbf{x}\right]dv_t \\ &= \int_{D_t}\rho\left[\mathbf{v}\times\mathbf{v}+\mathbf{a}\times\mathbf{x}\right]dv_t \\ &= \int_{D_t}\rho\left[\mathbf{a}(\mathbf{x},t)\times\mathbf{x}\right]dv_t\end{aligned}$$

1b. Use your result in 1a and divergence theorem to show that AMB is equivalent to

$$\int_{D_t}\mathbf{x}\times\left[\rho\mathbf{a}(\mathbf{x},t)-\rho\mathbf{b}-\frac{\partial\sigma_{ks}}{\partial x_s}\mathbf{e}_k\right]dv_t - \int_{D_t}\epsilon_{ijk}\mathbf{e}_i\sigma_{kj}dv_t = 0 \tag{1}$$

Solution: Based on Equation 1, we can expand the AMB in terms of

$$\begin{gathered}\int_{D_t}\mathbf{x}\times\mathbf{a}(\mathbf{x},t)\rho dv_t - \int_{D_t}\rho\mathbf{x}\times\mathbf{b}dv_t - \int_{\partial D_t}\mathbf{x}\times(\sigma\cdot\mathbf{n})ds_t = 0 \\ \int_{D_t}\mathbf{x}\times\left[\mathbf{a}(\mathbf{x},t)\rho-\rho\mathbf{b}\right]dv_t - \int_{\partial D_t}\mathbf{x}\times(\sigma\cdot\mathbf{n})ds_t = 0\end{gathered} \tag{2}$$

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Based on Green's theorem, we can expand the second term as:

$$\begin{aligned}
\int_{\partial D_t} \mathbf{x} \times (\sigma \cdot \mathbf{n}) ds_t &= \int_{D_t} [\mathbf{x} \times (\nabla \cdot \sigma) + \nabla \cdot (\mathbf{x} \times \sigma \cdot \mathbf{n})] \, dv_t \\
&= \int_{D_t} \left[\mathbf{x} \times \frac{\partial \sigma}{\partial \mathbf{x}} + \mathbf{e}_i \times \sigma_{kj} \mathbf{e}_k \mathbf{e}_j \cdot \mathbf{e}_s \right] dv_t \\
&= \int_{D_t} \left[\mathbf{x} \times \frac{\partial \sigma}{\partial \mathbf{x}} + \sigma_{kj} \mathbf{e}_k \epsilon_{ijs} \right] dv_t
\end{aligned} \tag{3}$$

Substituting Equation 3 into Equation 2, we have

$$\begin{aligned}
\int_{D_t} \mathbf{x} \times [\mathbf{a}(\mathbf{x}, t)\rho - \rho \mathbf{b}] \, dv_t - \int_{D_t} \left[\mathbf{x} \times \frac{\partial \sigma}{\partial \mathbf{x}} + \sigma_{kj} \mathbf{e}_k \epsilon_{ijs} \right] dv_t = 0 \\
\rightarrow \int_{D_t} \mathbf{x} \times \left[\rho \mathbf{a}(\mathbf{x}, t) - \rho \mathbf{b} - \frac{\partial \sigma_{ks}}{\partial x_s} \mathbf{e}_k \right] dv_t - \int_{D_t} \epsilon_{ijk} \mathbf{e}_i \sigma_{kj} dv_t = 0
\end{aligned}$$

1c. Use 1c to conclude that $\sigma_{kj} = \sigma_{jk}$.

Solution: Based on the given condition:

$$\int_{D_t} \mathbf{x} \times \left[\rho \mathbf{a}(\mathbf{x}, t) - \rho \mathbf{b} - \frac{\partial \sigma_{ks}}{\partial x_s} \mathbf{e}_k \right] dv_t = \int_{D_t} \epsilon_{ijk} \mathbf{e}_i \sigma_{kj} dv_t \tag{4}$$

Now we recall the balance law:

$$\nabla_{\mathbf{x}} \cdot \sigma = -\rho \mathbf{b} + \rho \mathbf{a} \tag{5}$$

Substituting Equation 5 into Equation 4 we have:

$$\begin{aligned}
\int_{D_t} \mathbf{x} \times [\nabla_{\mathbf{x}} \cdot \sigma - \nabla_{\mathbf{x}} \cdot \sigma] \, dv_t &= \int_{D_t} \epsilon_{ijk} \mathbf{e}_i \sigma_{kj} dv_t \\
0 &= \int_{D_t} \int_{D_t} \epsilon_{ijk} \mathbf{e}_i \sigma_{kj} dv_t \\
\rightarrow \epsilon_{ijk} \mathbf{e}_i \sigma_{kj} &= 0
\end{aligned} \tag{6}$$

Now we expand the permutation symbol of RHS:

$$\epsilon_{i23} \mathbf{e}_i \sigma_{23} + \epsilon_{i32} \mathbf{e}_i \sigma_{32} + \epsilon_{i12} \mathbf{e}_i \sigma_{12} + \epsilon_{i21} \mathbf{e}_i \sigma_{21} + \epsilon_{i13} \mathbf{e}_i \sigma_{13} + \epsilon_{i31} \mathbf{e}_i \sigma_{31} = 0, \;\; i = 1, 2, 3$$

Therefore we can conclude:

$$\begin{aligned}
\sigma_{12} = \sigma_{21}, \;\; \sigma_{23} = \sigma_{32}, \;\; \sigma_{13} = \sigma_{32} \\
\rightarrow \sigma_{kj} = \sigma_{jk}
\end{aligned}$$

2. The components of the true stress tensor $\sigma = \sigma_{ij} \mathbf{e}_i \mathbf{e}_j$ at a point $\mathbf{p}$ are given in appropriate units by

$$[\sigma_{ij}] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix}$$

The calculations below are respect to the basis $\mathbf{e}_j$.

2a. Find the traction on the plane at $\mathbf{p}$ normal to the x_1 axis

Solution:

We define the traction at $\mathbf{p}$ as $\mathbf{t}$:

$$\mathbf{t} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

2b. Find the traction at $\mathbf{p}$ on the plane whose normal has direction ratios $1 : -1 : 2$.

Solution:

Here, we first normalize the direction vector as $\mathbf{n}'$:

$$\mathbf{n}' = \frac{1}{\sqrt{1^2 + 1^2 + 2^2}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

Then we can calculate the the traction $\mathbf{t}$ on $\mathbf{p}$:

$$\mathbf{t} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{5\sqrt{6}}{6} \\ \frac{5\sqrt{2}\sqrt{3}}{3} \\ -\frac{\sqrt{6}}{6} \end{bmatrix}$$

2c. Find the principal stress components and direction of principal axis of stress through $\mathbf{p}$.

Solution: To get the principal stress, we calculate the determinant $\det|\sigma - \lambda| = 0$, expanded as:

$$\det|\sigma - \lambda| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 6 \\ 3 & 6 & 1-\lambda \end{vmatrix} = 0$$

And generated three solutions: $\lambda_1 = 10, \lambda_2 = 0, \lambda_3 = -4$. Then the three principal stresses are: $\mathbf{T}_1 = 10, \mathbf{T}_2 = 0, \mathbf{T}_3 = -4$. And we have the eigenvector:

$$\mathbf{Q} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{bmatrix} \frac{3\sqrt{70}}{70} & \frac{2\sqrt{5}}{5} & \frac{\sqrt{14}}{14} \\ \frac{3\sqrt{2}\sqrt{35}}{35} & -\frac{\sqrt{5}}{5} & \frac{\sqrt{2}\sqrt{7}}{7} \\ \frac{\sqrt{5}\sqrt{14}}{14} & 0 & -\frac{3\sqrt{14}}{14} \end{bmatrix}$$

Therefore we have the principal directions:

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

3. Assume all principal stresses are different, and denote the maximum and minimum principal stress at $\mathbf{p}$ by σ_I & σ_{III}. Show that as the orientation of the surface through $\mathbf{p}$ varies, σ_I is the greatest and σ_{III} is the least normal component of traction on the surface.

Solution: For a stress matrix σ, we have

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Where in the principal direction, the stress matrix takes the form

$$\sigma = \begin{bmatrix} \sigma_I & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & \sigma_{III} \end{bmatrix}$$

Due to $\sigma = \sigma_{ij}\mathbf{e}_i\mathbf{e}_j$, we can therefore write the form of stress in principal direction:

$$\sigma = \sigma_I \mathbf{e}_1^2 + \sigma_{II}\mathbf{e}_2^2 + \sigma_{III}\mathbf{e}_3^2 \tag{7}$$

We also know there is a constraint for the system:

$$\mathbf{e}_1^2 + \mathbf{e}_2^2 + \mathbf{e}_3^2 = 1 \tag{8}$$

We now subject the constraints (8) to the equation 7 and form a Lagrangian with multiplier λ to solve for the extrema:

$$\begin{aligned} \sigma_I \mathbf{e}_1 - \lambda_1 \mathbf{e}_1 &= 0 \\ \sigma_{II} \mathbf{e}_2 - \lambda_2 \mathbf{e}_2 &= 0 \\ \sigma_{III} \mathbf{e}_3 - \lambda_3 \mathbf{e}_3 &= 0 \\ \mathbf{e}_1^2 + \mathbf{e}_2^2 + \mathbf{e}_3^2 &= 1 \end{aligned}$$

We therefore obtain the solutions:

$$\mathbf{e}_1 = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}, \sigma = \sigma_I, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ \pm 1 \\ 0 \end{bmatrix}, \sigma = \sigma_{II}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ \pm 1 \end{bmatrix}, \sigma = \sigma_{III}. \tag{9}$$

Since σ_I and σ_{III} are the maximum and minimum principal stresses. The maximum and minimum traction in Equation 9 are therefore obtained as σ_I and σ_{III}.