CEE 6736: HW #1

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[65]:
```
# !sudo apt install cm-super dvipng texlive-latex-extra texlive-latex-recommended
```

Please show all work (i.e. include source code, and include thoughtful, analytical discussions):

(1) Please revisit our example from class, where the pond experienced salt water runoff from a bridge crossing over it. Please use Python to plot the salt concentration, $q(t)$, given in metric tons, as it fluctuates during the first 20 years that the bridge is in service. I am simply asking you to plot the solution to our math model that we obtained in class.

Solution:

Recall the governing equation for pond runoff discussed in class (in metric tons case):

$$\frac{dq}{dt} + \frac{1}{2}q = 10 + 5\sin(t)$$

Consider there is no runoff at the "very beginning": $q(0) = 0$. Recall the analytical solution given in the lecture:

$$q(t) = 20 - \frac{40}{17}\cos(2t) + \frac{10}{17}\sin(2t) - \frac{300}{17}e^{-\frac{t}{2}}$$

Plotting this:

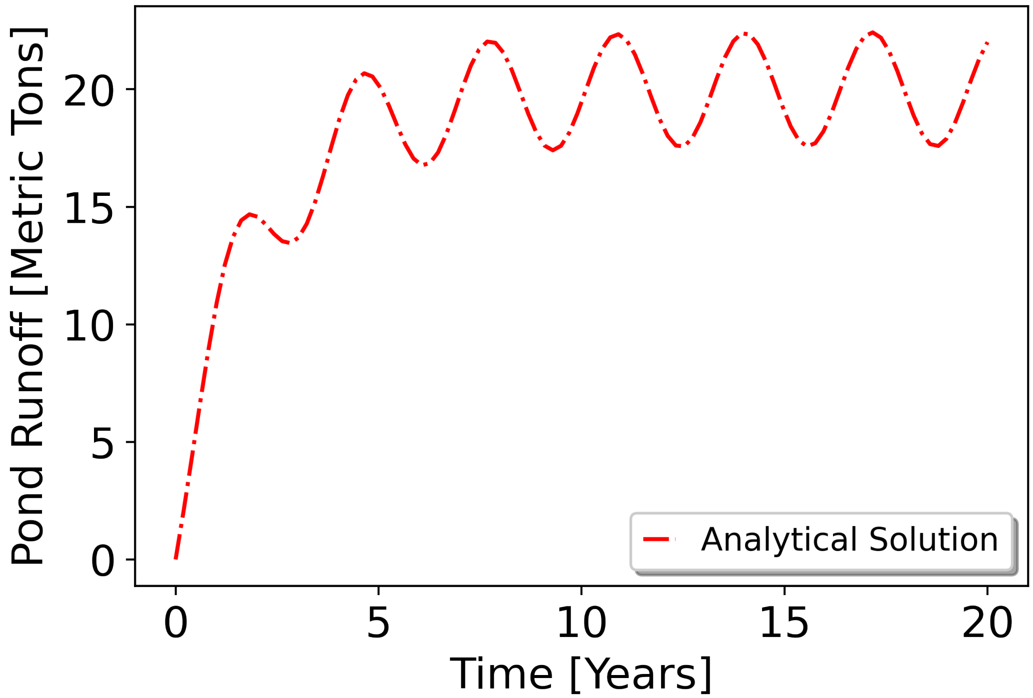
[5]:
```python
import numpy as np
import matplotlib.pyplot as plt
import matplotlib as mpl
from scipy.integrate import odeint

t = np.linspace(0,20,100)
q_analy = 20 - (40/17)*np.cos(2*t) + (10/17)*np.sin(2*t) - (300/17)*np.exp(-t/2)

plt.plot(t,q_analy,'r-.',label='Analytical Solution')
plt.xlabel("Time [Years]")
plt.ylabel("Pond Runoff [Metric Tons]")

# set plotting
plt.legend(shadow=True, handlelength=1, fontsize=12)
plt.rcParams['figure.dpi'] = 500
plt.show()
```

```
plt.figure(figsize=(5, 3))
mpl.rcParams.update({'font.size': 16})
```

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(2) Please write a small Python program that uses Euler's method to approximate the same response of our pond system during the first 20 years. Vary time step size, Δt, and compare the approximate solutions against the closed form solution for this problem (given in class, and used in (1)).

Solution:

Recall the Euler method discretization taught in class, the derivative is discretized in the form (lecture notes):

$$\frac{q_1 - q_0}{\Delta t} = 10 + 5\sin(2t) - 0.5q_0$$

where the current point q_1 can be computed as

$$q_1 = q_0 + \underbrace{(10 + \sin(2t) - 0.5q_0)}_{f(t,q_0)}(t - t_0)$$

This can be promoted to a more general case as

$$\underbrace{q_i}_{\text{current step}} = \underbrace{q_{i-1}}_{\text{previous step}} + \left(10 + \sin(\underbrace{t_{i-1}}_{\text{previous time}}) - 0.5\underbrace{q_{i-1}}_{\text{previous step}}\right)\Delta t$$

We hence write a loop to solve this:

[4]:

```
year = 20.0
q_t1 = np.zeros(20)
q_t2 = np.zeros(200)
q_t3 = np.zeros(2000)
Delta_t1 = 1
Delta_t2 = .1
Delta_t3 = .01
q0 = 0
i=0
for i in range(0,int(year/Delta_t1)):
  q_t1[i] = q_t1[i-1] + np.float64((10 + 5*np.sin(2*((i-1)/1)) - .5*q_t1[i-1]) * Delta_t1)
  i += 1
for i in range(0,int(year/Delta_t2)):
  q_t2[i] = q_t2[i-1] + np.float64((10 + 5*np.sin(2*((i-1)/10)) - .5*q_t2[i-1]) * Delta_t2)
  i += 1
for i in range(0,int(year/Delta_t3)):
  q_t3[i] = q_t3[i-1] + np.float64((10 + 5*np.sin(2*((i-1)/100)) - .5*q_t3[i-1]) * Delta_t3)
  i += 1

t_1 = np.linspace(0,20,20)
t_2 = np.linspace(0,20,200)
t_3 = np.linspace(0,20,2000)
plt.plot(t_1, q_t1, 'ro-.', label='Euler\'s Method: $\Delta t = 1$ [yr]')
plt.plot(t_2, q_t2, 'b>-.', label='Euler\'s Method: $\Delta t = 0.1$ [yr]')
plt.plot(t_3, q_t3, 'g.', label='Euler\'s Method: $\Delta t = 0.01$ [yr]')
plt.plot(t,q_analy, 'k-.', label='Analytical Solution')
plt.xlabel("Time [Years]")
plt.ylabel("Pond Runoff [Metric Tons]")

# set plotting
plt.legend(shadow=True, handlelength=1, fontsize=12)
plt.rcParams['figure.dpi'] = 500
plt.show()
plt.figure(figsize=(5, 3))
mpl.rcParams.update({'font.size': 16})
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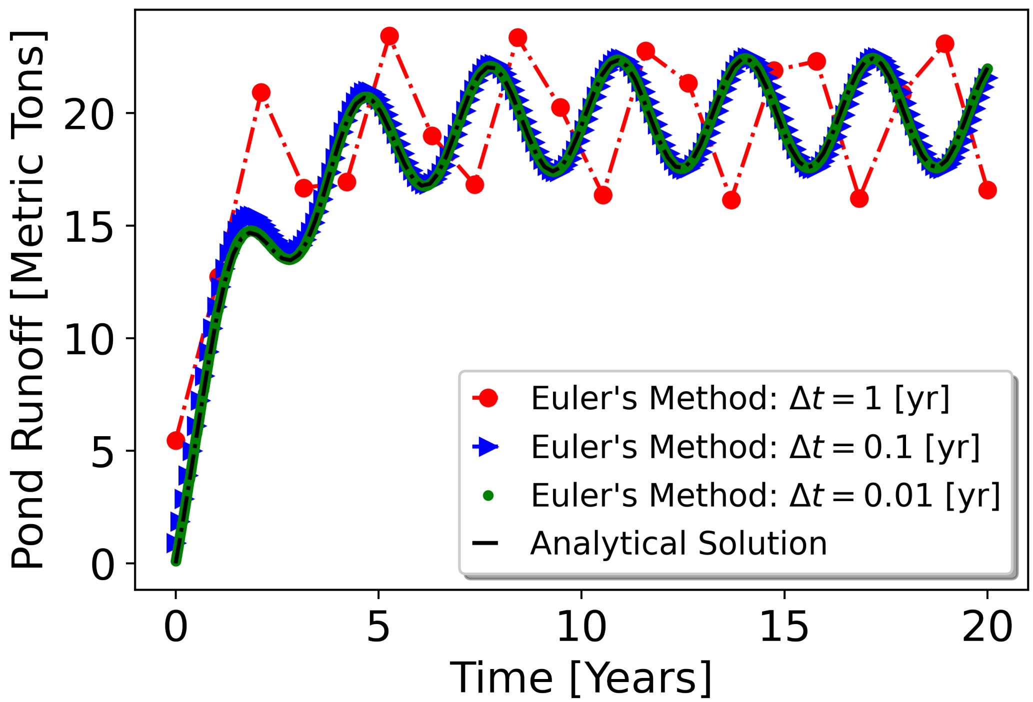


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(3) Please discuss your results from (1) and (2)

Solution:

As we are discretizing the solutions, when $\Delta t = 1$ [yr], the discretized solution is highly inaccurate — the solution points are observed to be "lagged behind" the standard analytical solution. Increasing discretization fidelity to $\Delta t = 0.1$ and 0.01 the approximated solutions are generally accurate — observed to be agreeing well with the analytical solution.

(4) Please analytically solve the following ODE using separation of variables: $\frac{dy}{dx} = \frac{4x-x^3}{4+y^3}$

Solution:

Rewrite the equation in the form:

$$(4+y^3)dy = (4x-x^3)dx$$

Integrate both sides:

$$\int (4+y^3)dy = \int (4x-x^3)dx$$

The general solution writes:

$$4y + \frac{1}{4}y^4 + C = 2x^2 - \frac{1}{4}x^4$$

Or written in the form:

$$2x^2 - \frac{1}{4}x^4 - 4y - \frac{1}{4}y^4 + C = 0$$

which is an implicit solution.

Now, let $x^2 = \mathcal{X}$, the equation writes:

$$-\frac{1}{4}\mathcal{X}^2 + 2\mathcal{X} = 4y + \frac{1}{4}y^4 + C$$

The explicit solution can be obtained via the quadratic equation:

$$\mathcal{X}_1 = 4 - 2\sqrt{4 + 4y + \frac{1}{4}y^4 + C}$$
$$\mathcal{X}_2 = 4 + 2\sqrt{4 + 4y + \frac{1}{4}y^4 + C}$$

The solutions of the overall equation hence write:

$$x_1 = \sqrt{4 - 2\sqrt{4 + 4y + \frac{1}{4}y^4 + C}}$$
$$x_2 = -\sqrt{4 - 2\sqrt{4 + 4y + \frac{1}{4}y^4 + C}}$$
$$x_3 = \sqrt{4 + 2\sqrt{4 + 4y + \frac{1}{4}y^4 + C}}$$
$$x_4 = -\sqrt{4 + 2\sqrt{4 + 4y + \frac{1}{4}y^4 + C}}$$