

1/9/2024. lecture 1

- Introduction: applications, contexts, examples
- classification & solutions
- solution methods.

Introduction

PDEs: Systems that evolve in space & time

are often described via PDEs

Applications → { thermo.
mechanics of solid/liquid/gas
electromagnetics
chemical
dynamical
material science.
population

PDE: An equation that relates a multivariable function ψ & its partial derivatives in 2 or more independent variables.

$\psi(x, t)$ → general: tensor
dependent variable
ind. vars.
 $\psi(x, y, z, t)$ or $\psi(\vec{x}, t)$
or $\psi(\vec{r})$.

ψ } scalar: temp., pressure, density. ~ potentials.
vectors: velocity, \vec{E} , \vec{B} , force, ...
tensor: stress, strain, Reynolds stress.

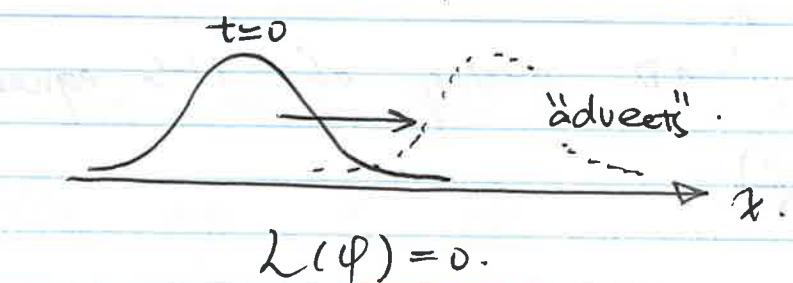
Examples

• Advections.

$$\frac{\partial \psi}{\partial t} + U \frac{\partial \psi}{\partial x} = 0 \quad (1D)$$

↑ const.

$$\frac{\partial \psi}{\partial t} + (\mathbf{U} \cdot \nabla) \psi = 0 \quad (N\text{-dim.})$$



*...? $L(\alpha\psi_1 + \beta\psi_2) = \alpha L(\psi_1) + \beta L(\psi_2)$

↑ const. ↓ flux for ψ

conservation form: $\frac{\partial \psi}{\partial t} + \frac{\partial (U\psi)}{\partial x} = 0$

general form for conservation law

$$\frac{\partial \psi}{\partial t} + \frac{\partial (F(\psi))}{\partial x} = 0 \quad F = \text{flux of } \psi$$

*...? flux.

*...? ∇ .

in N -dim.: $\frac{\partial \psi}{\partial t} + \nabla \cdot \vec{F} = 0$.
 $\hookrightarrow \vec{F}(\psi)$

if F only
depends ψ :

Remark: \vec{F} has to be one dimension
higher than ψ .

conservation law: $\frac{\partial \psi}{\partial t} + \frac{\partial F}{\partial \psi} \frac{\partial \psi}{\partial x} = 0$
 \hookrightarrow adv. vel.

* Nonlinear adv. eqn.: Burgers eqn.

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0. \quad \text{inviscid Burgers.}$$

\downarrow 1D analog of N-S equation.

$$\frac{\partial^2 u}{\partial x^2}.$$

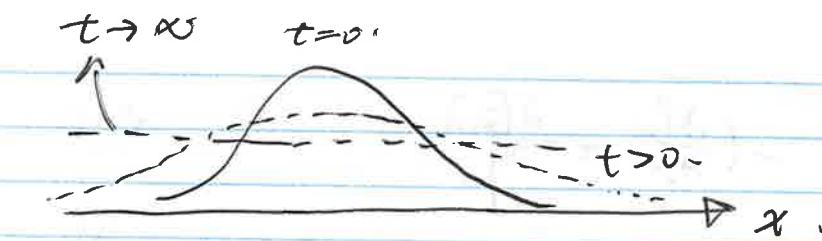
Characteristics Method.

T : temperature.

* Diffusion $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$. α : thermal diffusivity.

\leadsto heat, mass (concentration),
momentum, ...

$$= \frac{k}{\rho C_p} \cdot (\text{m}^2/\text{s}).$$



N -dim.: $\frac{\partial T}{\partial t} = \alpha \nabla^2 T$

$$\nabla^2(\cdot) = \frac{\partial^2(\cdot)}{\partial x_i \partial x_i} = \frac{\partial^2(\cdot)}{\partial x^2} + \frac{\partial^2(\cdot)}{\partial y^2} + \frac{\partial^2(\cdot)}{\partial z^2}$$

$$T_t = \alpha(T_{xx} + T_{yy} + T_{zz}).$$

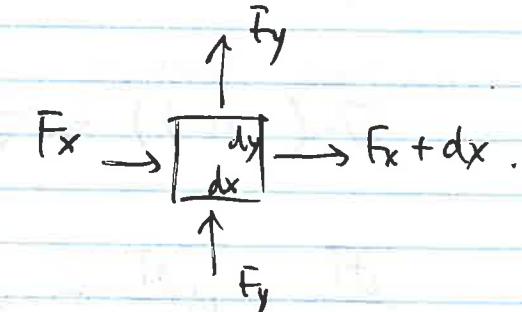
\Rightarrow how to use conservation law to derive the heat equation?

Note on PDE derivation

Control volume balance (Eulerian):

Heat eqn. in 2D:

$$(dx+dy) \rho C_p \frac{\partial T}{\partial t} = (F_x - F_{x+dx}) dy + (F_y - F_{y+dy}) dx.$$



$$\rho C_p \frac{\partial T}{\partial t} = - \left[\frac{F_{x+dx} - F_x}{dx} + \frac{F_{y+dy} - F_y}{dy} \right]$$

Take lim of $dx, dy \rightarrow 0$

$$\rho C_p \frac{\partial T}{\partial t} = -\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right).$$

$$\vec{F} = -k \nabla T \rightarrow \begin{cases} F_x = -k \frac{\partial T}{\partial x} \\ F_y = -k \cdot \frac{\partial T}{\partial y} \end{cases}$$

"Fourier's law"

$$\rho C_p \frac{\partial T}{\partial t} = k \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) T = k \nabla^2 T.$$

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_p} \nabla^2 T = \alpha \nabla^2 T.$$

$$\rho C_p \frac{\partial T}{\partial t} + \nabla \cdot \vec{F} = 0$$

$$^u \rho C_p \frac{\partial T}{\partial t} + \nabla \cdot (k \nabla T) = 0.$$

$$\nabla \cdot (\nabla \cdot \cdot) = \nabla^2 \cdot \cdot \rightarrow \text{Heat eqn.}$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = \alpha \cdot \frac{\partial^2 T}{\partial x^2} \quad \text{adv.-diff. eqn.}$$

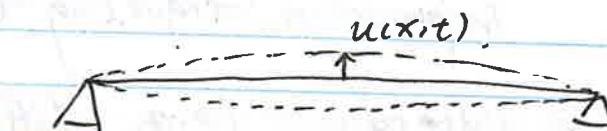
$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \xrightarrow{\text{steady state}} \nabla^2 T = 0.$$

$$\nabla^2 T = S \leftarrow \begin{matrix} \text{Poisson's eqn.} \\ \uparrow \text{source} \end{matrix} \quad \text{Laplace equation.}$$

Waves & Vibration

$$\frac{\partial^2 u}{\partial t^2} - C^2 \cdot \frac{\partial^2 u}{\partial x^2} = 0. \quad \text{D'Alembert's eqn.}$$

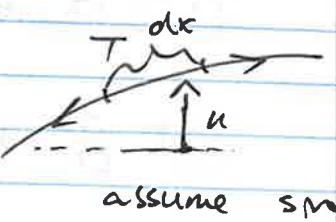
$$\frac{\partial^2 u}{\partial t^2} - C^2 \nabla^2 u = 0 \quad \text{wave eqn.}$$



String vibration.

Surface sound waves.

$$C^2 = \frac{T}{m} \leftarrow \begin{matrix} \text{tension} \\ \uparrow \text{mass per length.} \end{matrix}$$



assume sm

Classification of PDE

- Order or degree (of partial derivatives).

$\varphi(x, y)$: First-order:

$$a(\varphi, x, y) \frac{\partial \varphi}{\partial x} + b(\varphi, x, y) \frac{\partial \varphi}{\partial y} + c(\varphi, x, y) + d = 0$$

Second-order:

$$A(\varphi_x, \varphi_y, \varphi, x, y) \varphi_{xx} + B(\dots) \varphi_{xy} + C(\dots) \varphi_{yy} + \dots = 0$$

1st order
terms =

$$A^2 + B^2 + C^2 \neq 0.$$

$$B^2 - 4AC : \text{discriminant}$$

① $B^2 - 4AC > 0 \Rightarrow$ hyperbolic (e.g. wave eqn.)

wave-like solns, information + traveling characteristics

② $B^2 - 4AC = 0 \Rightarrow$ parabolic (e.g. diffusion)

③ $B^2 - 4AC < 0 \Rightarrow$ elliptic (e.g. Laplace, Poisson)

- Linear or nonlinear.

$$\rightsquigarrow L(\alpha\varphi_1 + \beta\varphi_2) = \alpha L(\varphi_1) + \beta L(\varphi_2).$$

↑ PDE .

- Homogeneous or inhomogeneous \rightsquigarrow trivial soln.
 $\hookrightarrow \varphi = 0$.

- I.C.s & B.C.s \rightarrow well-posedness.

(existence, uniqueness of solns)
 \hookleftarrow depends on domain
& independent variables.

Solution Methods

Methods

- characteristics.
- Separation of vars.
(eigenval. expns).
- Super-position
- Integral transformations.
- Similarity solutions.

Linear

Non-linear

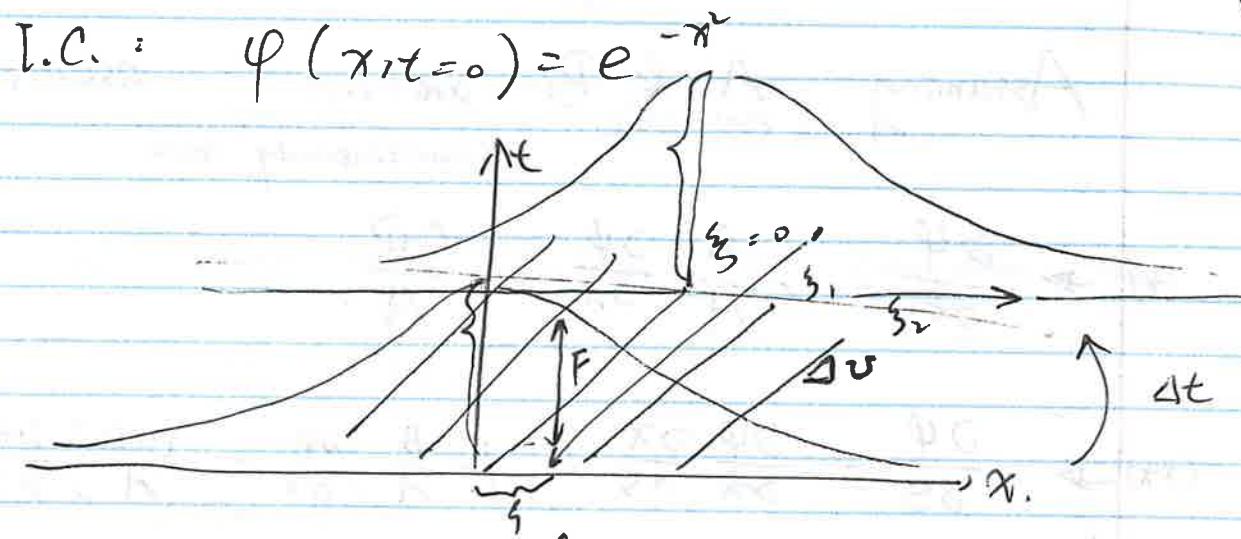


built solutions based on basis functions.



General theme:

convert "PDE" to a system of ODEs.



Goal: find solution for all s

Along charac.: $\frac{dx}{dt} = \frac{v}{1}$. & $\frac{d\varphi}{dt} = 0$.

$$dx = v dt, \Rightarrow x = vt + \xi. \text{ & } \varphi = F.$$

characteristics are labeled by ξ .

$$\Downarrow \\ F = F(\xi).$$

Goal: $\rightarrow \varphi(x, t)$?

$$\xi = x - vt$$

$$\varphi = F(\xi) = F(x - vt).$$

$$\text{I.C.: } \varphi(x, t=0) = F(x) = e^{-x^2}$$

$$\Downarrow \\ \varphi(x, t) = F(\xi) = F(x - vt) = e^{-(x-vt)^2}$$

Example 2

$$u(x, t).$$

$$\frac{\partial u}{\partial t} - \frac{x}{2} \cdot \frac{\partial u}{\partial x} = 0. \quad \& \quad u(x, t=0) = e^{-x^2}$$

$$\text{on characteristics: } \left. \frac{dx}{dt} \right|_{\xi} = \frac{-x}{2} = -\frac{x}{t} \Rightarrow \frac{dx}{x} = -\frac{1}{2} dt \quad \& \quad \left. \frac{du}{dt} \right|_{\xi} = 0.$$

$$\Rightarrow \frac{dx}{x} = -\frac{1}{2} dt, \rightarrow \ln x = -\frac{t}{2} + C.$$

$$x = e^C \cdot e^{-\frac{t}{2}} = \xi \cdot e^{-\frac{t}{2}} \rightarrow \xi = x e^{\frac{t}{2}}.$$

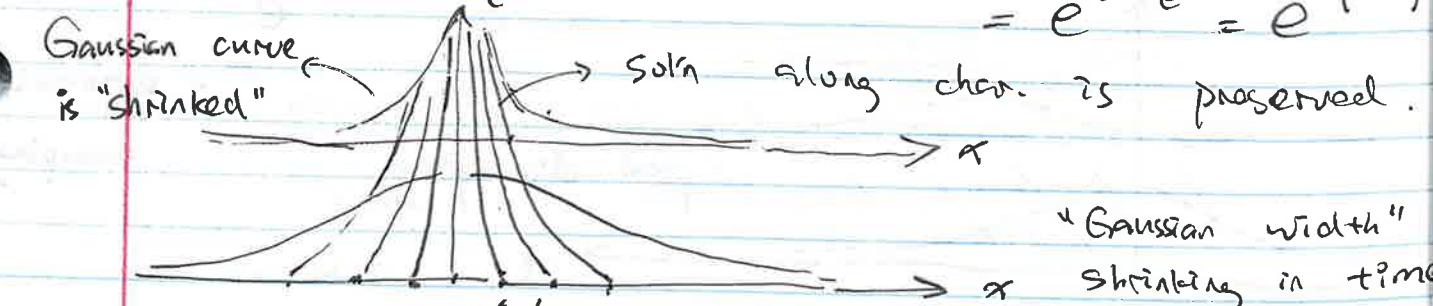
$$\left. \frac{du}{dt} \right|_{\xi} = 0 \Rightarrow u = F(\xi).$$

$$\text{I.C.: } u(x, t=0) = F(\xi) = F(x \cdot e^{\frac{t}{2}}).$$

$$= F(x) = e^{-x^2}.$$

$$u(x, t) = F(\xi) = F(x e^{\frac{t}{2}}) = e^{-(x e^{\frac{t}{2}})^2}$$

$$= e^{-x^2 e^t} = e^{-\frac{(x e^{\frac{t}{2}})^2}{e^{\frac{t}{2}}}}$$



"Gaussian width" shrinking in time

Practice problem: $\frac{\partial u}{\partial t} + t \frac{\partial u}{\partial x} = 0$.

$$u(x, t=0) = e^{-x^2}$$

Example 4 $u(x, t=0)$.

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = -u, \quad u(x, t=0) = e^{-x^2}. \quad \text{char.}$$

solvn not preserved along char.

linear & homogeneous.

along char.: $\frac{dx}{dt} \Big|_{\xi} = 1, \quad \frac{du}{dt} \Big|_{\xi} = -u.$

$$\frac{dx}{dt} \Big|_{\xi} = 1, \Rightarrow x = t + \xi. \quad \rightarrow \xi = x - t.$$

$$\frac{du}{dt} \Big|_{\xi} = -u \Rightarrow \frac{du}{u} = dt \Rightarrow u = u_0(\xi) e^{-t}.$$

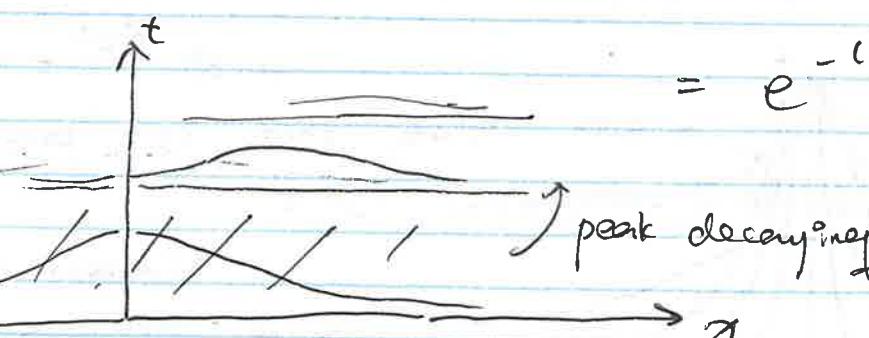
I.C.: $u(x, t=0) = u_0(\xi) e^{-t} = u_0(x) e^{-t}.$

$\stackrel{\parallel}{e^{-x^2}} \quad \stackrel{\rightarrow}{=} \quad \text{when } t=0.$

$$u(x, t) = u_0(\xi) e^{-t} = e^{-\xi^2} e^{-t}$$

$$= e^{-(x-t)^2} e^{-t}.$$

↑ exponential damping



Sample 5 Burger's eqn. (inviscid).

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$$

$$u(x, t=0) = e^{-x^2}$$

$$u(x, t) \rightarrow u(x(t), t).$$

on the characteristics:

$$\frac{dx}{dt} \Big|_{\xi} = u, \quad \frac{du}{dt} \Big|_{\xi} = 0.$$

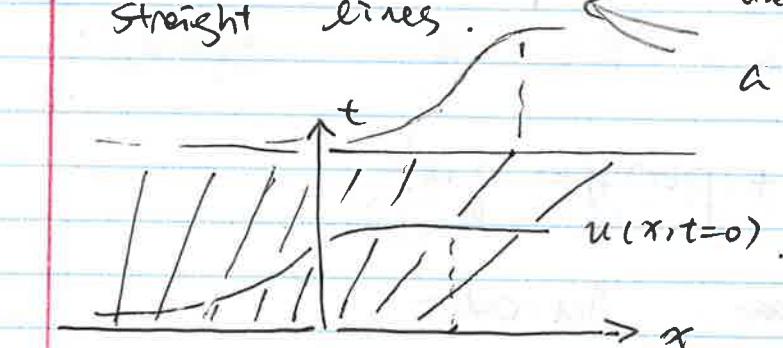
char. should be

straight lines.

along char., soln

doesn't change, u is
a slope of char.

*** IMPORT



slope of the char. does not change, ($u(\xi)$)

↓
straight lines

more rigorously, integrate:

$$x = ut + \xi \Rightarrow \xi = x - ut.$$

$$u = F(\xi).$$

therefore, $u = F(x - ut)$. ← implicit soln.

depends on I.C.

PROBLEM SESSION I

Review of ODEs.

Classification.

- Order: highest derivative.
- Linearity: are there y^2 , yy' , etc. terms?
- Homogeneity: is $y=0$ a solution or not.
- Coefficient: are coefficients y , y' , y'' , ... function of x or not?

1st-order ODE:

$$y'(x) + P(x)y = q(x).$$

→ linear, 1st-order.

Integrating factor: "reverse product rule".

$$\text{Example 1: } y' + 2y = 1.$$

Multiply by $\mu(x) = e^{2x} \rightarrow \mu(x) = 2e^{2x}$.

$$e^{2x}y' + 2e^{2x}y = e^{2x}$$

$$(e^{2x}y)' = e^{2x}.$$

$$e^{2x}y = \frac{1}{2}e^{2x} + C$$

$$y = \frac{1}{2} + C e^{-2x}.$$

In general, $\mu(x) = \exp(\int p(x) dx)$.

Separable equations.

$$N(y) \frac{dy}{dx} = M(x).$$

"Separate" the derivative.

$$\int N(y) dy = \int M(x) dx.$$

$$\text{Example: } y' - 6y^2x - x = 0.$$

$$\frac{dy}{dx} = (6y^2 + 1)x.$$

$$\frac{dy}{6y^2+1} = x dx.$$

$$\frac{\arctan(\sqrt{6}y)}{\sqrt{6}} = \frac{1}{2}x^2 + C$$

$$\arctan(\sqrt{6}y) = \sqrt{6}\left(\frac{1}{2}x^2 + C\right).$$

$$y = \frac{1}{\sqrt{6}} \tan\left(\sqrt{6}\left(\frac{1}{2}x^2 + C\right)\right)$$

1/11/2024

Lecture 2

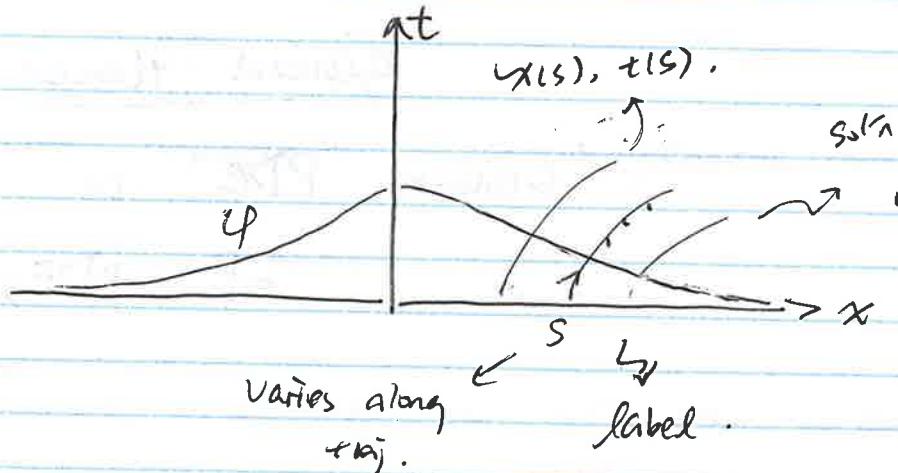
Characteristics (6 lectures).

"find coordinate transform to transform the PDE to ODEs".

First-order PDE

$$\begin{aligned} \psi(x, t) & \\ A(\psi, x, t) \frac{\partial \psi}{\partial t} + B(\psi, x, t) \frac{\partial \psi}{\partial x} + C(\psi, x, t) & \\ + D(x, t) = 0 & \end{aligned} \quad (*)$$

Think geometrically,



$\psi(x, t)$ on traj. $\rightarrow \psi(x(s), t(s))$.

$$\frac{\partial \psi}{\partial s} = \frac{\partial \psi}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial \psi}{\partial t} \cdot \frac{\partial t}{\partial s}. \quad (**)$$

factoring of characteristic curves along \rightarrow PDE \rightarrow ODEs.

sln along the traj.

\rightarrow will be ODE.

Assuming A & B are not simultaneously zero, assume $A \neq 0$,

$$(*) \rightarrow \frac{\partial \psi}{\partial t} = -\frac{B}{A} \cdot \frac{\partial \psi}{\partial x} - \frac{C+D}{A}.$$

$$(**) \rightarrow \frac{\partial \psi}{\partial s} = \frac{\partial \psi}{\partial x} \frac{\partial x}{\partial s} + \left(-\frac{B}{A} \cdot \frac{\partial \psi}{\partial x} - \frac{C+D}{A} \right) \frac{\partial t}{\partial s}.$$

along traj.

$$= \frac{\partial \psi}{\partial x} \left\{ \frac{dx}{ds} - \frac{B}{A} \cdot \frac{dt}{ds} \right\} - \frac{C+D}{A} \cdot \frac{dt}{ds}.$$

choose $s=t$:

$$\boxed{\frac{dx}{ds} = \frac{B}{A}(\psi, x, t) = \frac{dx}{dt}} \quad | \quad \text{1st-order ODE}$$

$$\boxed{\frac{d\psi}{dt} = -\frac{C(\psi, x, t) + D(x, t)}{A(\psi, x, t)}} \quad |$$

\rightarrow family of characteristic curves. $(x(s), t(s), \psi)$

Example 1

$$\begin{aligned} \frac{\partial \psi}{\partial t} + 2 \frac{\partial \psi}{\partial x} &= 0 \\ -\infty < x < \infty, \quad 0 < t < \infty & \end{aligned}$$

Exact equations.

$$M(x,y) + N(x,y) \frac{dy}{dx} = 0.$$

Trying to find some function $\psi(x,y)$ s.t.

$$\psi_x = M \quad \& \quad \psi_y = N.$$

Condition: $\psi_{xy} = \psi_{yx} \rightarrow My = Nx$

compute: $\psi = \int M dx$ or $\psi = \int N dy$

and compare $\psi_y = N$ or $\psi_x = M$.

Example $2xy - 9x^2 + \underbrace{(2y + x^2 + 1)}_N \frac{dy}{dx} = 0$

check if exact: $My = 2x = Nx$.

Then: $\psi_x = M \rightarrow \psi = \int M dx$.

$$\psi = \int 2xy - 9x^2 dx.$$

$$\psi = x^2y - 3x^3 + h(y).$$

how to find $h(y) \leftarrow \psi_y$

$$\psi_y = x^2 + h'(y) = 2y + x^2 + 1 = N.$$

$$\rightarrow h'(y) = 2y + 1 \rightarrow h(y) = y^2 + y + \text{const.}$$

$$\rightarrow y = x^2y - 3x^3 + y^2 + y + \text{const.} = \text{const.}$$

$$y^2 + (x^2 + 1)y - 3x^3 = C.$$

Second-order ODEs:

$$P(t)y'' + Q(t)y' + R(t)y = g(t).$$

$$ay'' + by' + cy = g(t)$$

Usually encounter and solve const. coeff. ODE
homog. $g(t) = 0$ vs. inhomog. $g(t) \neq 0$

homogeneous soln will be superposition of 2 s.

$$y_h(t) = C_1 y_1(t) + C_2 y_2(t)$$

how to get y_h ? Ansatz: $y = \text{exp}(rt)$

$$(ar^2 + br + c)\text{exp}(rt) = 0$$

$$\begin{cases} r = -\text{exp}(rt) \\ y'' = r^2 \text{exp}(rt) \end{cases}$$

Characteristic equation: $ar^2 + br + c = 0$

For those 2nd-order, linear, const-coeff ODEs,

You can start w/ characteristic eqns:

Quadratic equation: $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

- $b^2 - 4ac > 0 \rightarrow$ 2 real roots
- $b^2 - 4ac < 0 \rightarrow$ 2 complex conjugate roots
- $b^2 - 4ac = 0 \rightarrow$ repeated real roots.

For 2 real roots r_1 & r_2 :

$$y_h(t) = C_1 \exp(r_1 t) + C_2 \exp(r_2 t)$$

For 2 complex conjugate roots ($r_{1,2} = \alpha \pm \beta i$).

$$y_h(t) = \exp(\alpha t) [C_1 \cos(\beta t) + C_2 \sin(\beta t)]$$

$$y_1 = \exp((\alpha + \beta i)t) . \quad y_2 = \exp((\alpha - \beta i)t)$$

$$= \exp(\alpha t) [\cos \beta t + i \sin \beta t] \quad = \exp(\alpha t) [\cos \beta t - i \sin \beta t]$$

$$y_3 = C_1 y_1 + C_2 y_2 \rightarrow C_1 = C_2 = \frac{1}{2}$$

$$\hookrightarrow y_3 = \exp(\alpha t) \cos \beta t$$

$$y_4 = C_1 y_1 + C_2 y_2 \rightarrow C_1 = \frac{1}{2}i, \quad C_2 = \frac{1}{2}i$$

$$y_4 = \exp(\alpha t) \sin \beta t$$

For repeated real root (r):

$$y_h(t) = C_1 \exp(rt) + C_2 t \exp(rt)$$

$$\downarrow y_1 = \exp(rt)$$

$$\text{guess } y_2 = V(t) y_1$$

calculate y_2' & y_2'' & substitute.

Everything cancels except for $V'' = 0$.

$$\rightarrow \text{ODE for } V \rightarrow V = Ct + K = t$$

\uparrow
 \uparrow
const.

Always need $y_h(t)$.

If inhomogeneous $\rightarrow y^{(t)} = y_h(t) + y_p^{(t)}$.

2 methods: { Methods of undetermined coefficients.
Variation of parameters.

• Apply BCs/ICs after getting full soln.

→ Undetermined coefficients.

Example: $y'' - y = 3t^2 + t + 1$

try: $y_p(t) = at^2 + bt + c$

$$y'_p = 2at + b, \quad y''_p = 2a$$

Substitute: $2a - (at^2 + bt + c) = 3t^2 + 2t + 1$

$$at^2 + bt + c - 2a = -3t^2 - 2t - 1$$

$$\rightarrow a = -3, \quad b = -2, \quad c = -1$$

$$y_p = -3t^2 - 2t - 1$$

general sol'n: $y = y_p(t) + y_h(t)$.

to formulate (guess) y_p form:

n-deg. polynomial \leftrightarrow n-deg poly.

$$e^{at} \leftrightarrow e^{at}$$

$$\cos \beta t \leftrightarrow \cos \beta t + \sin \beta t$$

$$\sin \beta t \leftrightarrow \dots$$

→ Variation of Parameters.

More general method to find particular sol'n

$$y'' + q(t)y' + r(t)y = g(t)$$

* Requires homogeneous solution.

$$y_h(t) = C_1 y_1(t) + C_2 y_2(t)$$

Trying to find u_1 & u_2 s.t.

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

is a solution to the inhomogeneous sys.

$$y_p'(t) = u'_1 y_1 + u_1 y'_1 + u'_2 y_2 + u_2 y'_2$$

Now, assume $u_1 y'_1 + u_2 y'_2 = 0$

$$y_p''(t) = u''_1 y_1 + u'_1 y'_1 + u''_2 y_2 + u'_2 y'_2$$

Plug in & simplify:

$$u'_1 y'_1 + u'_2 y'_2 = g(t)$$

Solve for u'_1 , u'_2 and integrate

$$\begin{cases} u_1(t) = - \int \frac{y_2 g(t)}{W(y_1, y_2)} dt \\ u_2(t) = \int \frac{y_1 g(t)}{W(y_1, y_2)} dt \end{cases}$$

For $\int g(t) dt$,

$$u_1(t) = \int \frac{y_2 g(t)}{W(y_1, y_2)} dt$$

these to for

$y_p(t)$.

Wronskian $W(y_1, y_2) = y_1 y'_2 - y_2 y'_1 \neq 0$

then $y_p(t) = u_1 y_1 + u_2 y_2$.

Example $y'' - 4y' + 3y = e^{-t}$

$$\rightarrow y_1 = e^{3t}, \quad y_2 = e^t$$

$$W = y_1 y'_2 - y_2 y'_1 = -2e^{4t}$$

$$u_1 = \int \dots dt, \quad u_2 = \int \dots dt.$$

$$\rightarrow y_p = \frac{1}{8} e^{-t}$$

Wronskian.

$$W(y_1, y_2) = y_1 y'_2 - y_2 y'_1$$

↳ check whether y_1 & y_2 are a fundamental set of solutions.

↳ fundamental set if $W \neq 0$.

* Basically a check on linear independence.

If $y_2 = \text{const. } y_1 \rightarrow$ are not truly

2 Solutions

Lecture 3

1/16/2024

* Characteristics as coordinate transformations

* Nonlinear PDE (Burgers' equation)

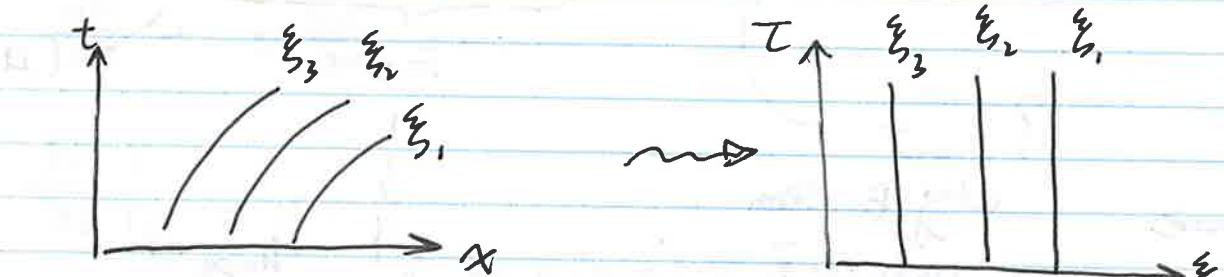
* Expansion & Compression waves & shocks.

1st order PDE: $\varphi(x, t)$. (*)

$$A(\varphi, x, t) \frac{\partial \varphi}{\partial t} + B(\varphi, x, t) \frac{\partial \varphi}{\partial x} + C(\varphi, x, t) = 0$$

$$\tilde{C}(\varphi, x, t) = C(\varphi, x, t) + D(x, t).$$

$$(x, t) \rightarrow (\xi(x, t), \tau(x, t))$$



$$\frac{\partial \varphi}{\partial t} = \frac{\partial \tau}{\partial t} \frac{\partial \varphi}{\partial \tau} + \frac{\partial \xi}{\partial t} \frac{\partial \varphi}{\partial \xi}$$

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \tau}{\partial x} \cdot \frac{\partial \varphi}{\partial \tau} + \frac{\partial \xi}{\partial x} \cdot \frac{\partial \varphi}{\partial \xi}$$

$$(*) : \left\{ A \frac{\partial \tau}{\partial t} + B \frac{\partial \tau}{\partial x} \right\} \frac{\partial \varphi}{\partial \tau} + \left\{ A \cdot \frac{\partial \xi}{\partial t} + B \frac{\partial \xi}{\partial x} \right\} \frac{\partial \varphi}{\partial \xi} = -C$$

We need the PDE \rightarrow ODE along characteristics
 ↓
 trajectory of solutions.
 along characteristics: $\xi(x, t) = \text{const.}$
 let $T = t \Rightarrow (\xi(x, t), T)$
 ↑
 const. along char.
 varies along char.
 (plays role of time).

$$(\star\star): A \frac{\partial \psi}{\partial T} + \left\{ A \frac{\partial \xi}{\partial t} + B \frac{\partial \xi}{\partial x} \right\} \cdot \frac{\partial \psi}{\partial \xi} = -\tilde{C}$$

= 0 \square

? ...
 why B term vanishes?
 "hope".

Along char., we want to be zero:

$$d\xi = 0 = \frac{\partial \xi}{\partial x} dx + \frac{\partial \xi}{\partial t} dt \Rightarrow \frac{\partial \xi}{\partial t} = -\frac{dx}{dt} \Big|_{\xi} \frac{\partial \xi}{\partial x}$$

Along char.:

$$\square: A \left(-\frac{dx}{dt} \Big|_{\xi} \frac{\partial \xi}{\partial x} \right) + B \left(\frac{\partial \xi}{\partial x} \right) = \frac{\partial \xi}{\partial x} \left(-A \frac{dx}{dt} \Big|_{\xi} + B \right)$$

$$\rightsquigarrow \frac{dx}{dt} \Big|_{\xi} = \frac{B}{A}. \quad \left. \begin{array}{l} \frac{\partial \psi}{\partial T} \Big|_{\xi} = -\frac{\tilde{C}}{A}. \\ \end{array} \right\} \text{System of ODEs.}$$

What we have derived

last week for 1st-order ODE

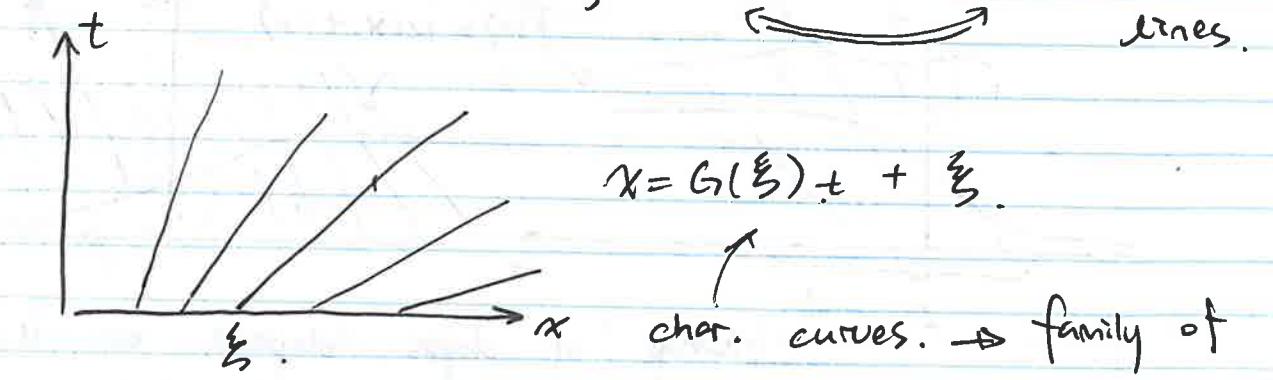
Burgers' Equation

1st order nonlinear PDE.

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0.$$

$$\left. \begin{array}{l} \frac{dx}{dt} \Big|_{\xi} = u. \\ \frac{du}{dt} \Big|_{\xi} = 0 \end{array} \right\} \rightarrow u(x, t) = G(\xi).$$

$\frac{dx}{dt} \Big|_{\xi} = \text{const.}$ char. are straight lines.



→ Impose I.C.: $u(x, t=0) = G(x - G(\xi)t) = G(x)$
 → e.g., $u(x, t=0) = F(x)$.

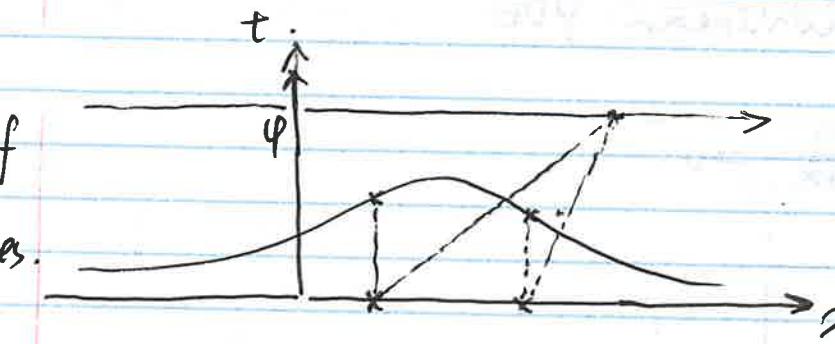
Solution: $u(x, t) = F(\xi)$.

$$u(x, t) = F(x - ut).$$

↙

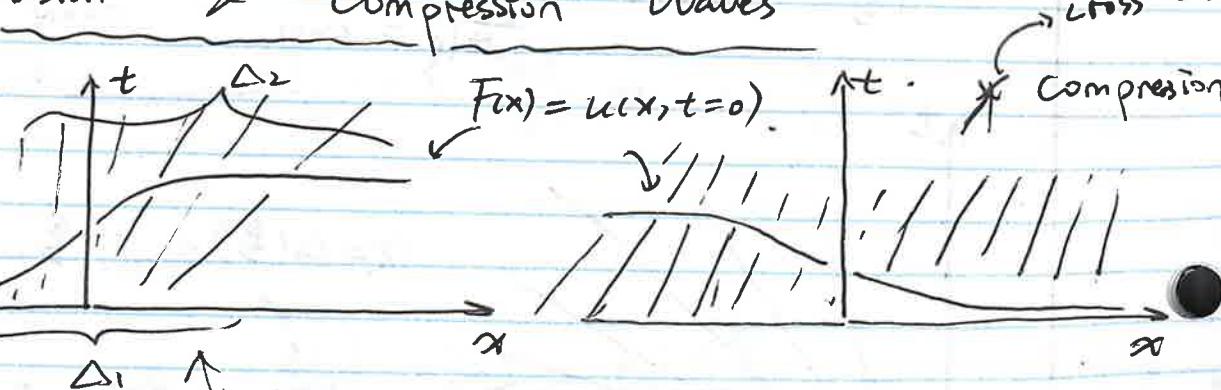
Implicit equation.

$$F(x) = e^{-x^2} \rightsquigarrow u = e^{(x-ut)^2}.$$



e.g.) Bisection method, Newton-Raphson, Secant method.

Expansion & Compression Waves



change of slopes depend on $u(0)$.

\rightarrow expansion waves

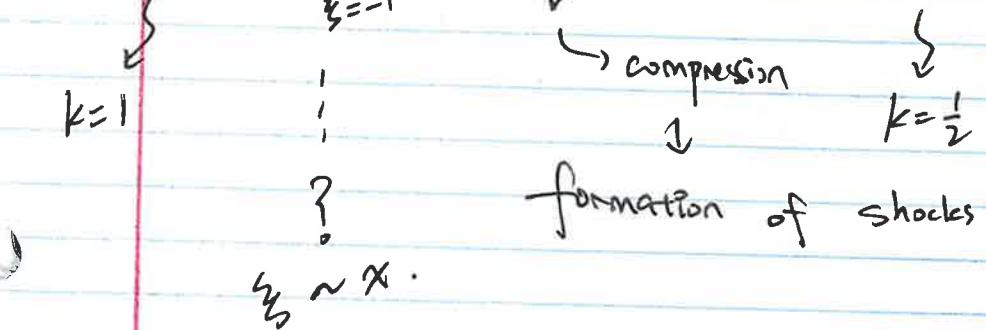
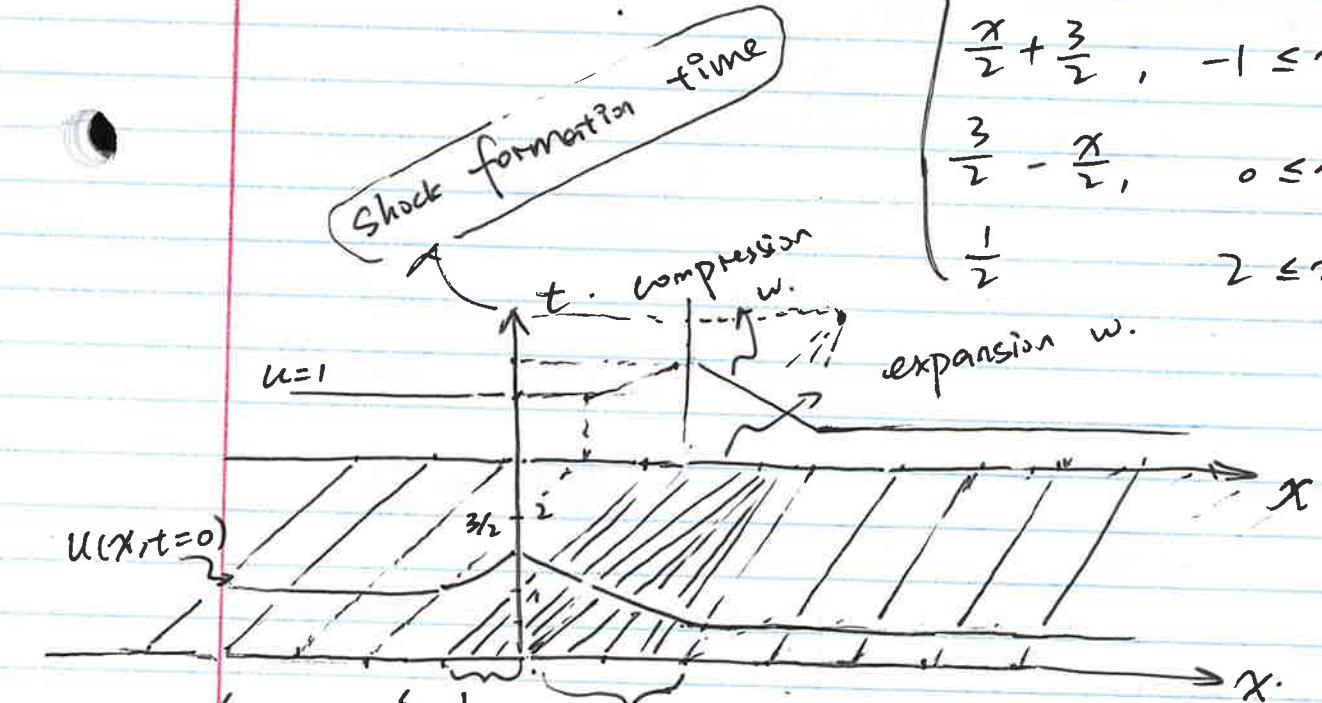
$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0.$$

→ If "characteristic crossing": formation of shocks.

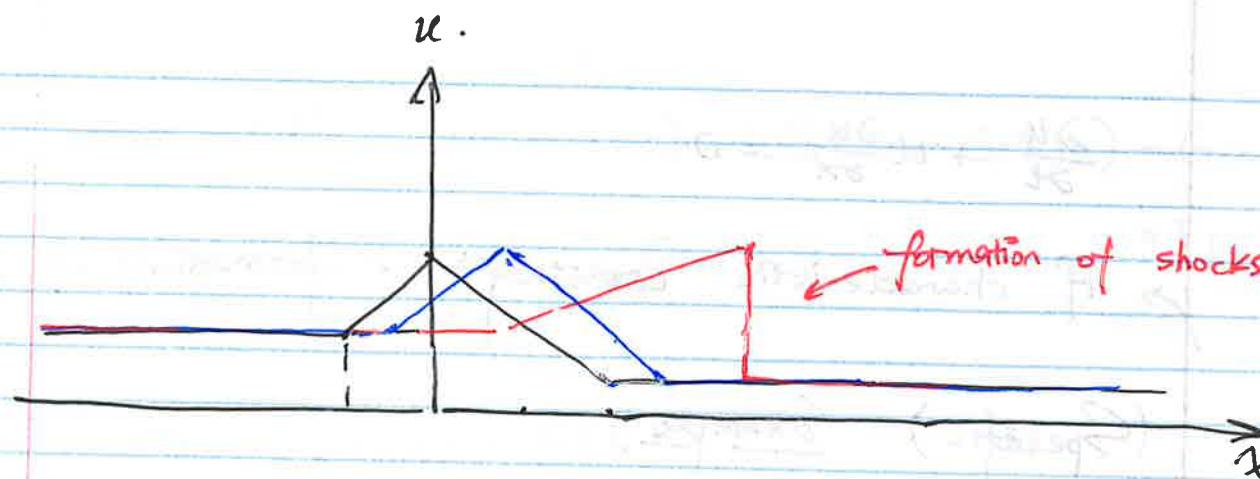
(Specific) Example.

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0.$$

$$u(x, t=0) = F(x) = \begin{cases} 1, & -\infty < x < -1, \\ \frac{x}{2} + \frac{3}{2}, & -1 \leq x < 0 \\ \frac{3}{2} - \frac{x}{2}, & 0 \leq x < 2 \\ \frac{1}{2}, & 2 \leq x < \infty \end{cases}$$



Remark: the definition of "slope" is reversed w.r.t. char. lines compared w/ normal context



lecture 4. 1/18/2024.

Recap for HW: equation of char.: $\frac{dx}{dt} \Big|_{\xi} = \frac{\beta}{A}$.

Pb. 2 ~ char. solution: $\frac{du}{dt} \Big|_{\xi} = -\frac{\tilde{\alpha}}{A}$.

unique analytical: $u(x, t)$

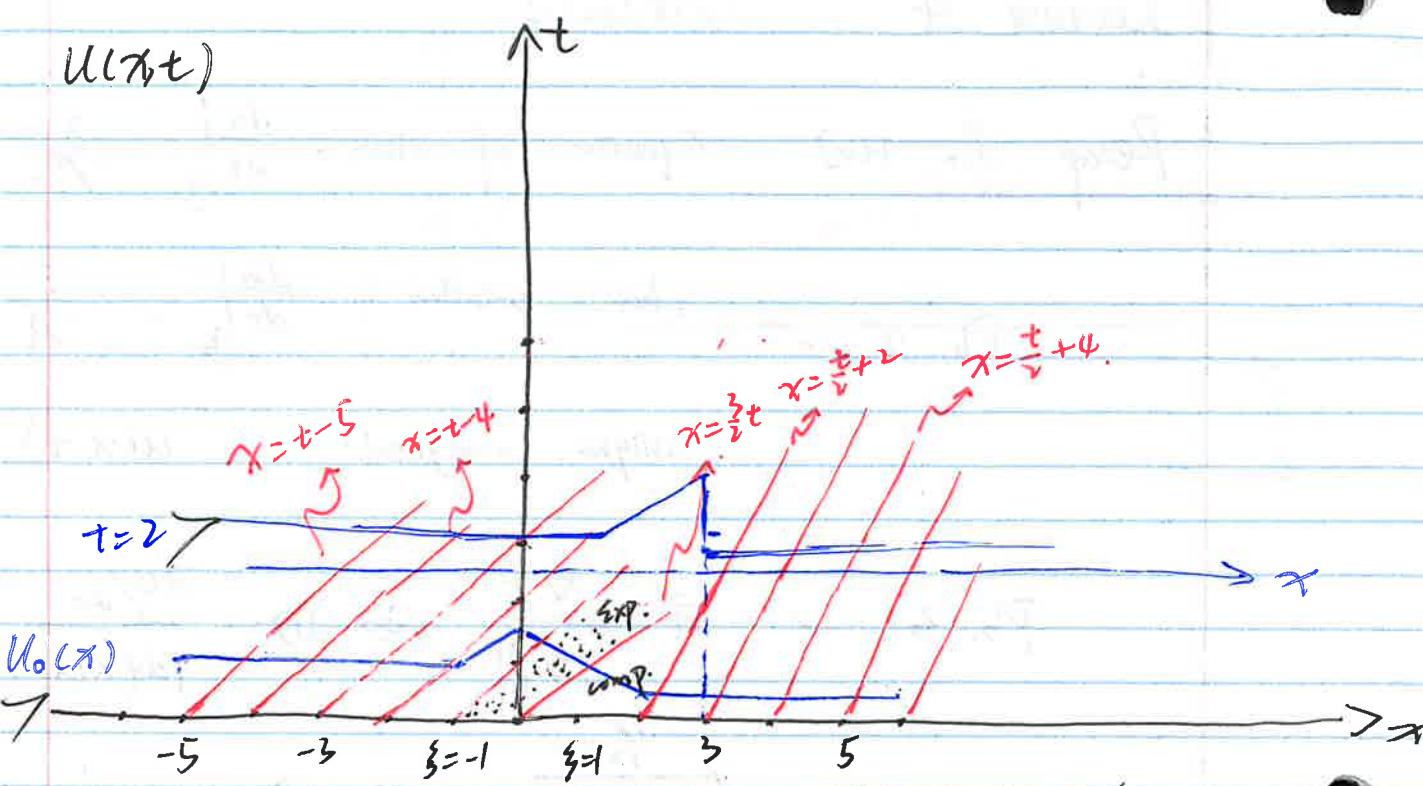
$$\text{Pb. 3 } \checkmark \quad \vec{n} = \frac{\nabla \varphi}{|\nabla \varphi|} \rightsquigarrow \text{1D: } \frac{d\varphi/dx}{|d\varphi/dx|}$$

$$\varphi_0: \begin{cases} \varphi_x = 0 & (t=0) \\ \downarrow \\ n=1 \end{cases}$$

Burgers' eqn.

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0.$$

$$u(x, 0) = u_0(x) = \begin{cases} 1 & x < -1 \\ \frac{3}{2} + \frac{x}{2} & -1 \leq x < 0 \\ \frac{3}{2} - \frac{x}{2} & 0 \leq x < 2 \\ \frac{1}{2} & x > 2 \end{cases}$$



$$\text{Equation for char. : } \frac{dx}{dt} \Big|_{\xi} = u. \quad x = u(\xi) + \xi.$$

char. soln curve: $u = F(\xi)$.

$$\begin{aligned} \text{I.C.S: } F(\xi(x, t=0)) &= F(x) = u_0(x) \\ u(x, t=0) &\stackrel{=} F(\xi) = u_0(\xi) \end{aligned}$$

$$\xi < -1 \quad \& \quad u_0(\xi) = 1 \Rightarrow x = t + \xi.$$

$$2 < \xi : u_0(\xi) = \frac{1}{2} \Rightarrow x = \frac{t}{2} + \xi$$

$$-1 \leq \xi < 0 : u_0(\xi) = \frac{3}{2} + \frac{\xi}{2}.$$

$$x = \xi + u_0(\xi)t = \xi + \left(\frac{3}{2} + \frac{\xi}{2}\right)t$$

$$= \xi \left(1 + \frac{t}{2}\right) + \frac{3}{2}t.$$

$$\Rightarrow \xi(x, t) = \frac{x - \frac{3}{2}t}{1 + \frac{t}{2}}$$

$$u(x, t) = u_0(\xi) = \frac{3}{2} + \frac{\xi}{2} = \frac{3}{2} + \frac{x/2}{1+t/2} - \frac{3t/2}{1+t/2}.$$

expansion zone: $t-1 \leq x < \frac{3}{2}$

$$\text{Recall: } u(x, t) = \frac{3}{2} + \frac{x/2}{1+t/2} - \frac{3t/2}{1+t/2}$$

$$0 \leq \xi < 2$$

compression zone

$$\frac{3}{2}t \leq x < \frac{t}{2} + 2$$

$$u = u_0(\xi) = \frac{3}{2} - \frac{\xi}{2}$$

$$x = u_0(\xi) + \xi t = \left(\frac{3}{2} - \frac{\xi}{2}\right)t + \xi$$

$$x = \xi \left(1 - \frac{t}{2}\right) + \frac{3}{2}t.$$

$$\Rightarrow \xi(x, t) = \frac{x - \frac{3}{2}t}{1 - \frac{t}{2}}$$

$$\text{@ } t=2 : x = \xi \left(1 - \frac{2}{2}\right) + \frac{3}{2}t$$

$$= \frac{3}{2}t.$$

$$u(x, t) = u_0(\xi) = \left(\frac{3}{2} - \frac{\xi}{2}\right)t + \xi.$$

$$= \left(\frac{3}{2} - \frac{x - \frac{3}{2}t}{2-t}\right)t + \frac{x - \frac{3}{2}t}{1 - \frac{t}{2}}$$

$$u(x,t) = \frac{3}{2} - \frac{x - \frac{3}{2}t}{2-t} = \frac{3}{2} - \frac{x}{2-t} + \frac{\frac{3}{2}t}{2-t}.$$

(a) $x=3$
 $t=2$ u is multi-valued.

Analytical solution:

$$u(x,t) = \begin{cases} 1 & x < -1+t \\ \frac{3}{2} - \frac{3t/4}{1+t/2} + \frac{x/2}{1+t/2} & -1+t \leq x < \frac{3}{2}t \\ \frac{3}{2} + \frac{3t/2}{2-t} - \frac{x}{2-t} & \frac{3}{2}t \leq x < \frac{t}{2}+2 \\ \frac{1}{2} & \frac{t}{2}+2 \leq x \end{cases}$$

Shock formation time.

* WHY??

Burgers' eqn. $u(x,t) = F(\underbrace{x - u(x,t)t}_{\xi})$

$$\frac{\partial u}{\partial x} \text{ blow up. } \textcircled{a} \text{ shock. } \rightarrow \frac{\partial u}{\partial x} = \frac{\partial F(\xi)}{\partial x}$$

$$= \frac{\partial F}{\partial \xi} \frac{\partial \xi}{\partial x} \\ = \frac{\partial F}{\partial \xi} \left(1 - \frac{\partial u(x,t)}{\partial x} t \right)$$

$u_0(\xi)$

$$\frac{\partial u}{\partial x} = \frac{\partial F}{\partial \xi} - \frac{\partial F}{\partial \xi} t \cdot \frac{\partial u}{\partial x}.$$

$$= \frac{\partial u}{\partial x} \left(1 + \frac{\partial F}{\partial \xi} t \right). = \frac{\partial F}{\partial \xi}.$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\frac{\partial F}{\partial \xi}}{1 + \frac{\partial F}{\partial \xi} t}.$$

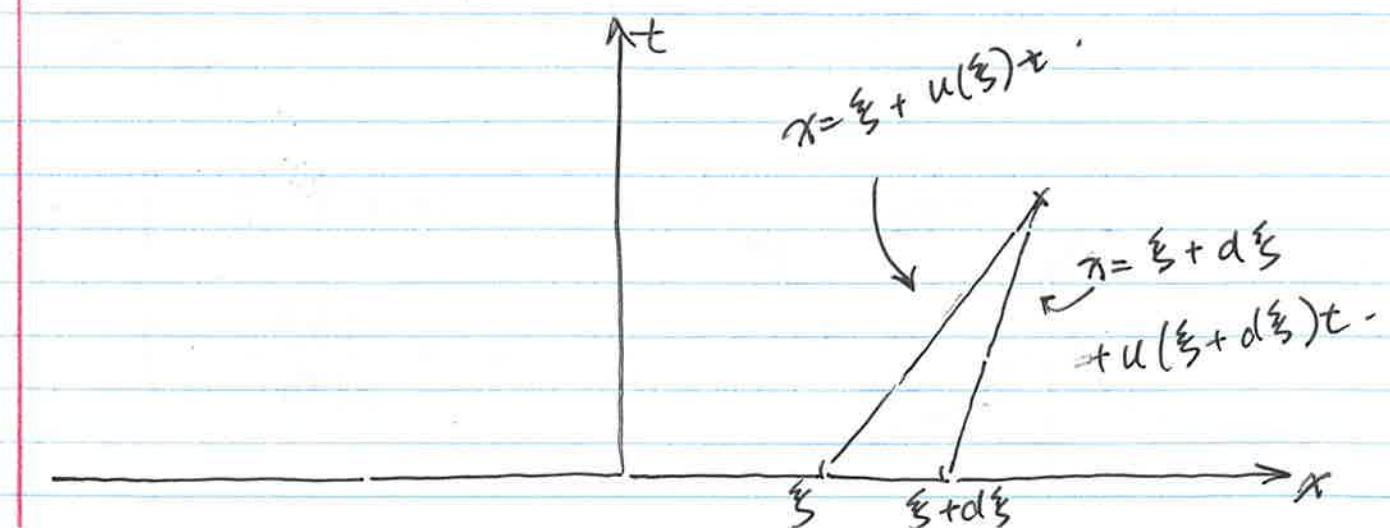
F is a smooth function.

$$\frac{\partial u}{\partial x} \rightarrow \infty \text{ iff } 1 + \frac{\partial F}{\partial \xi} t = 0.$$

$$\Rightarrow t = \frac{-1}{\frac{\partial F}{\partial \xi}(\xi)}.$$

$$t_{\text{shock}} = \min_{\xi} \left(\frac{-1}{\frac{\partial F}{\partial \xi}(\xi)} \right)$$

Geometric interpretation.



$$0 = \xi + d\xi + u_0(\xi + d\xi)t - \xi - u_0(\xi)t.$$

$$d\xi = [u_0(\xi) - u_0(\xi + d\xi)] t_{\text{shock}}.$$

assume: u_0 is smooth, \rightarrow Taylor's expansion.

$$d\xi = - \frac{\partial u_0}{\partial \xi} \times d\xi t.$$

$$(u_0(\xi) + \frac{\partial u_0}{\partial \xi} d\xi)$$

$$+ O(d\xi^2).$$

more generally: $x = u_0(\xi)t + \xi$.

$$\boxed{\left. \frac{dx}{d\xi} \right|_t = 0}$$

$$\frac{\partial u_0(\xi)}{\partial \xi} t + 1 = 0 \Rightarrow t_{\text{shock}} = \frac{-1}{\frac{\partial u_0}{\partial \xi}}.$$

Problem Session 2

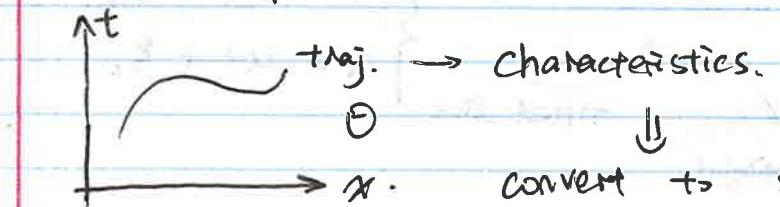
1/19/2024.

$$\frac{\partial \phi}{\partial t} + u(\phi, x, t) \frac{\partial \phi}{\partial x} = S(\phi, x, t).$$

domain: $x \in (-\infty, +\infty)$, $t \in [0, \infty)$.

I.C., $\phi(x, t=0) = f(x)$.

Method of Characteristics (MoC).

 traj. \rightarrow characteristics.

convert to sys. of ODEs.

$$x = x(\theta)$$

$$\frac{d\theta}{dt} = \frac{dt}{d\theta} \cdot \frac{\partial}{\partial t} + \frac{dx}{d\theta} \cdot \frac{\partial}{\partial x}, \quad t = t(\theta).$$

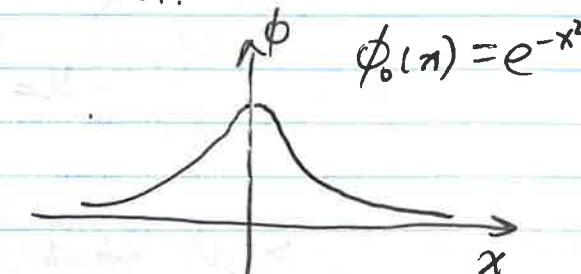
$$\theta \equiv t. \quad \left. \frac{d}{dt} \right|_{\text{traj.}} = 1 \cdot \frac{\partial}{\partial t} + \left. \frac{dx}{dt} \right|_{\text{traj.}} \cdot \frac{\partial}{\partial x}.$$

$$\left. \frac{dx}{dt} \right|_{\text{traj.}} = u(\phi, x, t). \rightarrow \text{ODE for eqn. of char. lines.}$$

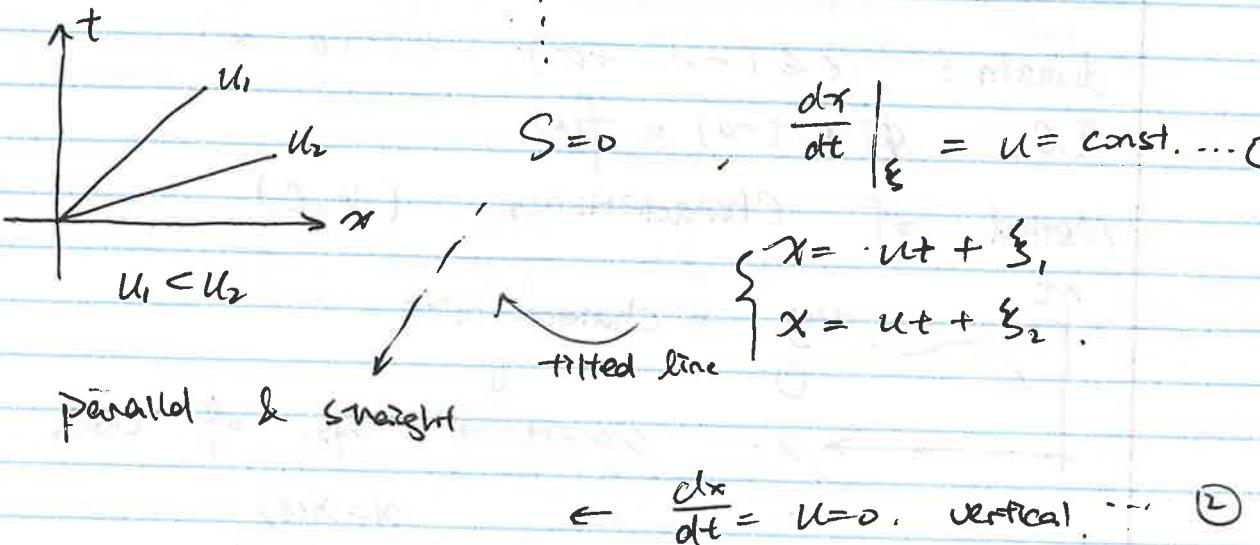
$$\left. \frac{d\phi}{dt} \right|_{\text{traj.}} = S(\phi, x, t). \rightarrow \text{ODE for char. solns. curves.}$$

- if $S(\phi, x, t) = 0$. $\rightarrow \left. \phi \right|_{\text{traj.}} = \text{const.}$

- if $\left. \frac{d\phi}{dt} \right|_S = \phi$...



$\frac{dx}{dt} \Big|_{\xi} = u(\phi, x, t)$. \Rightarrow propagation speed of information.



not parallel & straight $\leftarrow \frac{dx}{dt} = u(\phi)$. $S \neq 0 \dots \textcircled{3}$

$$\text{Example: } 2xt \frac{\partial \phi}{\partial x} + (1+t^2) \left[\frac{\partial \phi}{\partial t} + \phi \right] = 0.$$

I.C., $\phi(x, t=0) = \tanh(x)$

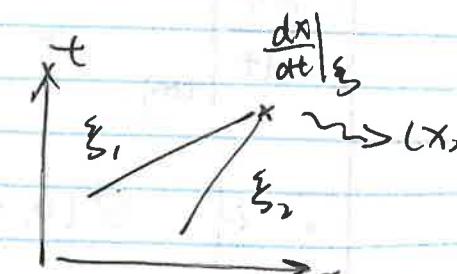
1st order in x & t , homog., \rightarrow $\phi=0$ satisfies.
linear, PDE.

$$\frac{\partial \phi}{\partial t} + \frac{2xt}{(1+t^2)} \cdot \frac{\partial \phi}{\partial x} = -\phi.$$

$$u = f(x, t).$$

↳ uniquely defined.

\Rightarrow No shock for a linear PDE.



$$\frac{dx}{dt} \Big|_{\xi} = u(., x, t) = \frac{2xt}{1+t^2}.$$

$$x = \xi(1+t^2) \rightarrow \xi = \frac{x}{1+t^2}$$

$$\frac{d\phi}{dt} \Big|_{\xi} = -\phi \rightarrow \phi = \phi(x, 0) \exp(-t).$$

\downarrow

$$\phi(\xi) = \tanh(\xi) \exp(-t)$$

\downarrow

$$\phi(x, t) = \tanh\left(\frac{x}{1+t^2}\right) \exp(-t).$$

$$2. \frac{\partial u}{\partial x} + u^2 \frac{\partial u}{\partial x} = \beta u. \quad (\text{when } \beta > 0).$$

(a).

$$\frac{du}{dt} = u^2, \Rightarrow u = [F(\xi)]^2 \exp(2\beta t).$$

$$\frac{du}{dt} = \beta u \Rightarrow u = F(\xi) \exp(\beta t).$$

$$\int_{\xi}^x dx = \int_{\xi}^t [F(\xi)]^2 \exp(2\beta t) dt.$$

$$\Rightarrow x = \frac{[F(\xi)]^2}{2\beta} \exp(2\beta t) - \frac{[F(\xi)]^2}{2\beta} + \xi.$$

shock time $= \frac{dx}{d\xi} \Big|_{+} = 0$ check for crossing times/shock f.

$$I.C. u = \exp(-2x^2).$$

$$u(\xi, 0) = F(\xi) = \exp(-2\xi^2).$$

$$\frac{dx}{d\xi} = \frac{(4\xi) \exp(-2\xi^2)}{2\beta} \left[1 - \exp(2\beta t_c) \right] + 1 = 0.$$

$$\Rightarrow 1 - \exp(2\beta t_c) = \frac{-2\beta \exp(2\xi^2)}{4\xi}.$$

$$t_c = \frac{1}{2\beta} \ln \left[1 + \frac{\beta \exp(2\xi^2)}{2\xi} \right].$$

↑

"Smallest t_c is the shock formation time".

To find t_{shock} , minimize t_c as a function of ξ .

$$\frac{dt_c}{d\xi} = 0$$

$$\hookrightarrow t_{shock} = \frac{1}{2\beta} \ln(1 + \beta\sqrt{e}).$$

for $\beta=0$ case, $u=F(\xi)$.

$$\frac{dx}{dt} = [F(\xi)]^2$$

$$\Rightarrow x = [F(\xi)]^2 t + \xi.$$

$$x = [\exp(-2\xi^2)]^3 t + \xi.$$

$$\text{if } \beta \rightarrow 0, t_{shock} = \frac{\beta\sqrt{e}}{2\beta} = \frac{\sqrt{e}}{2} \text{ for } \beta=0.$$

HW: 3b.

$$\begin{cases} x' = x + Ut \\ t' = t \end{cases}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} + \frac{\partial}{\partial x'} \cdot \frac{dx'}{\partial t}.$$

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x} + \frac{\partial}{\partial t'} \cdot \frac{\partial t'}{\partial x'} \neq 0$$

$$\hookrightarrow \frac{\partial \phi}{\partial t'} + \frac{\partial \phi}{\partial x'} U + U \frac{\partial \phi}{\partial x'} \leftarrow U \text{ not eliminated.}$$

if chose $\begin{cases} x' = x - Ut \\ t' = t \end{cases}$

$$\frac{\partial \phi}{\partial t'} = \frac{\partial \phi}{\partial x'} U + U \frac{\partial \phi}{\partial x'} \underbrace{\quad}_{\text{cancels.}}$$

$$\frac{\partial \phi}{\partial t'} = \text{RHS.}$$

$$= \gamma - \frac{d}{dx'} \left[\epsilon \frac{d\phi}{dx'} - \phi(1-\phi) \right]$$

$\phi_{eq}(x') : f(x')$ \uparrow replace $x' = x + Ut$

$$\left(\frac{\phi}{1-\phi}\right)' = \frac{(1-\phi)\phi' - \phi(1-\phi)}{(1-\phi)^2} = \frac{1}{(1-\phi)^2}$$

$$\frac{d\phi}{d(1-\phi)} = \frac{1}{\varepsilon} dx'$$

Using the coordinate transformation

$$\frac{\partial \phi}{\partial t} + U \frac{\partial \phi}{\partial x} = \gamma \frac{\partial}{\partial x} \left[\varepsilon \frac{\partial \phi}{\partial x} - \phi(1-\phi) \right]$$

ODE w.r.t. x'

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial t'} - \frac{\partial \phi}{\partial x'} \cdot U$$

choose $x' = x - Ut$

$$\frac{\partial \phi}{\partial x'} = \frac{\partial \phi}{\partial x}$$

$$\frac{\partial \phi}{\partial t'} - \frac{\partial \phi}{\partial x'} \cdot U + U \frac{\partial \phi}{\partial x'} = \gamma \frac{\partial}{\partial x'} \left[\varepsilon \frac{\partial \phi}{\partial x'} - \phi(1-\phi) \right]$$

Setting LHS = 0 → Solve for RHS = 0.

$$\gamma \frac{\partial}{\partial x'} \left[\varepsilon \frac{\partial \phi}{\partial x'} - \phi(1-\phi) \right] = 0$$

$$\varepsilon \frac{\partial^2 \phi}{\partial x'^2} = \frac{\partial}{\partial x'} [\phi(1-\phi)].$$

Integrating on both sides.

$$\varepsilon \frac{d\phi}{dx'} - \phi(1-\phi) = C_1 \rightarrow \text{away from interface}$$

LHS = 0

Need to satisfy the I.C.s

further integration:

$$(*) \quad \varepsilon \frac{d\phi}{dx'} = \phi(1-\phi). \quad \int \frac{1}{\phi(1-\phi)} d\phi = \left(\frac{x'}{\varepsilon} \right) + C$$

$$\phi_{x'} = \frac{1}{\varepsilon} (\phi - \phi^2)$$

$$\phi = \frac{1}{\varepsilon} \int \phi(1-\phi) dx'$$

@ $x' = 0$ & random x' ↓ ϕ preserves properties.

$$\frac{\phi}{1-\phi} = D \exp\left(\frac{x'}{\varepsilon}\right).$$

$$\phi(x') = \frac{D \exp(x'/\varepsilon)}{1 + D \exp(x'/\varepsilon)}. = 0.5 + 0.5x$$

Partial fraction for integration.

expanding (*):

$$\ln \phi + \ln(1-\phi) = \left(\frac{x'}{\varepsilon} \right) + C$$

$$\frac{d\phi}{\phi(1-\phi)} = \frac{dx'}{\varepsilon}$$

$$\frac{(1-\phi+\phi)d\phi}{\phi(1-\phi)} = \frac{dx'}{\varepsilon}.$$

$$\ln[\phi(1-\phi)] = \left(\frac{x'}{\varepsilon} \right) + C$$

$$\phi(1-\phi) =$$

$$\int \left[\frac{1}{\phi} + \frac{1}{1-\phi} \right] d\phi = \frac{1}{\varepsilon} \int dx'$$

Substituting ICs:

$$\begin{aligned} C_2 \exp\left(\frac{x}{\varepsilon}\right) &= \left(\frac{1}{2} + \frac{1}{2}x\right) \left[1 + C_2 \exp\left(\frac{x}{\varepsilon}\right) \right] \\ &= \frac{1}{2} + \frac{1}{2}C_2 \exp\left(\frac{x}{\varepsilon}\right) + \frac{x}{2} + \frac{x}{2}C_2 \exp\left(\frac{x}{\varepsilon}\right) \\ &= \frac{1}{2} + \frac{x}{2} + C_2 \left[\frac{1}{2} \exp\left(\frac{x}{\varepsilon}\right) + \frac{x}{2} \exp\left(\frac{x}{\varepsilon}\right) \right] \\ C_2 \cdot \exp\left(\frac{x}{\varepsilon}\right) \left[1 - \frac{1}{2} - \frac{x}{2} \right] &= \frac{1}{2} + \frac{x}{2}. \end{aligned}$$

$$C_2 = \frac{1}{\exp(x/\varepsilon)} \cdot \frac{1+x}{1-x}.$$

→ Interface solution:

$$\begin{aligned} \phi(x-vt) &= \frac{\frac{1}{\exp(x/\varepsilon)} \frac{1+x}{1-x} \exp[(x-vt)/\varepsilon]}{1 + \frac{1}{\exp(x/\varepsilon)} \frac{1+x}{1-x} \exp[(x-vt)/\varepsilon]} \\ &\quad \cdot (1+x) \exp\left(\frac{x-vt}{\varepsilon}\right). \\ \phi(x-vt) &= \frac{(1+x) \exp\left(\frac{x-vt}{\varepsilon}\right)}{(1-x) \exp\left(\frac{x}{\varepsilon}\right) + (1+x) \exp\left(\frac{x-vt}{\varepsilon}\right)} \end{aligned}$$

For the "non-interfacial" part:

$$\frac{C_2 \exp(x/\varepsilon)}{1 + C_2 \exp(x/\varepsilon)} = 0 \text{ or } 1$$

if $\phi \equiv 1$:

$$C_2 \exp\left(\frac{x}{\varepsilon}\right) = 1 + C_2 \exp\left(\frac{x}{\varepsilon}\right)$$

Equation Derivation for 3(c).

$$0.5 + 0.5 \left\{ x - \gamma \left(1 - \frac{x-\gamma t+1}{2-2\gamma t} \right) \right\}$$

$$0.5 + 0.5 \left[x - \gamma + \gamma \cdot \frac{x-\gamma t+1}{1-\gamma t} \right]$$

$$0.5 + 0.5x - 0.5\gamma + 0.5\gamma \frac{(x-\gamma t+1)}{1-\gamma t}$$

$$0.5 + 0.5x + 0.5\gamma \left(\frac{x-\gamma t+1}{1-\gamma t} - \frac{2-2\gamma t}{1-\gamma t} \right)$$

$$0.5 + 0.5x + 0.5\gamma \left(\frac{x-\gamma t+1-2+2\gamma t}{1-\gamma t} \right).$$

$$0.5(1+x) + 0.5\gamma \left(\frac{\gamma t-1+x}{1-\gamma t} \right)$$

$$0.5(1+x) + 0.5\gamma \left(-1 + \frac{x}{1-\gamma t} \right).$$

$$0.5(1+x) = \frac{\gamma}{2} + \frac{\gamma}{2} \cdot \frac{x}{1-\gamma t}$$

Simplifying 3(d).

$$0.5 + 0.5 \left\{ x - \gamma t - \gamma t \frac{x - \gamma t - \nu t + 1}{1 - \gamma t} - \nu t \right\}.$$

$$0.5 + 0.5x - 0.5\gamma t - 0.5 \cdot \gamma t \cdot \frac{x - \gamma t - \nu t + 1}{1 - \gamma t} - \nu t.$$

$$\frac{x+1}{2} - 0.5\gamma t \left[\frac{1 - \gamma t + x - \gamma t - \nu t + 1}{1 - \gamma t} \right] - \nu t$$

$$\frac{x+1}{2} - 0.5\gamma t \cdot \left[\frac{2(1 - \gamma t) + x - \nu t}{1 - \gamma t} \right] - \nu t$$

$$\frac{x+1}{2} - \gamma t - \frac{0.5\gamma t(x - \nu t)}{1 - \gamma t} - \nu t$$

$$x = \gamma t - \gamma t \frac{x - \gamma t + 1}{1 - \gamma t} + \frac{x}{1 - \gamma t}$$

Lecture 5

1/23/2024.

$\varphi(x, t)$.

$$A \frac{\partial \varphi}{\partial t} + B \frac{\partial \varphi}{\partial x} + C = 0 \Rightarrow \frac{\partial \varphi}{\partial t} + \frac{B}{A} \cdot \frac{\partial \varphi}{\partial x} = -\frac{C}{A}.$$

$$\frac{\partial \varphi}{\partial t} \Big|_{\xi} = \frac{\partial \varphi}{\partial t} + \frac{\partial x}{\partial t} \Big|_{\xi} \frac{\partial \varphi}{\partial x} = -\frac{C}{A}.$$

$$\frac{\partial x}{\partial t} \Big|_{\xi} = \frac{B}{A} \quad \& \quad \frac{\partial \varphi}{\partial t} \Big|_{\xi} = -\frac{C}{A}.$$

on characteristics:

$$\frac{dt}{A} = \frac{dx}{B} = \frac{d\varphi}{-C}.$$

i.e., const.

$\varphi(x, y, z, t)$

$$A \frac{\partial \varphi}{\partial t} + B \frac{\partial \varphi}{\partial x} + C \frac{\partial \varphi}{\partial y} + D \frac{\partial \varphi}{\partial z} = -\tilde{E}$$

$$\frac{\partial \varphi}{\partial t} + \frac{B}{A} \cdot \frac{\partial \varphi}{\partial x} + \frac{C}{A} \cdot \frac{\partial \varphi}{\partial y} + \frac{D}{A} \cdot \frac{\partial \varphi}{\partial z} = -\frac{\tilde{E}}{A}.$$

$$\frac{\partial \varphi}{\partial t} \Big|_{\xi} \frac{\partial \varphi}{\partial t} + \frac{\partial x}{\partial t} \Big|_{\xi} \frac{\partial \varphi}{\partial x} + \frac{\partial y}{\partial t} \Big|_{\xi} \frac{\partial \varphi}{\partial y} + \frac{\partial z}{\partial t} \Big|_{\xi} \frac{\partial \varphi}{\partial z} = -\frac{\tilde{E}}{A}.$$

$$\frac{\partial x}{\partial t} \Big|_{\xi} = \frac{B}{A}, \quad \frac{\partial y}{\partial t} \Big|_{\xi} = \frac{C}{A}, \quad \frac{\partial z}{\partial t} \Big|_{\xi} = \frac{D}{A}.$$

$$\frac{\partial \varphi}{\partial t} \Big|_{\xi} = -\frac{\tilde{E}}{A}$$

Alternatively:

$$\frac{dt}{A} = \frac{dx}{B} = \frac{dy}{C} = \frac{dz}{D} = -\frac{d\varphi}{E}$$

Conservation laws.

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0. \quad \text{Primitive Form.}$$

↑ "speed-up" of characteristics.

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left(\frac{u^2}{2} \right) = 0 \quad \text{Conservative Form.}$$

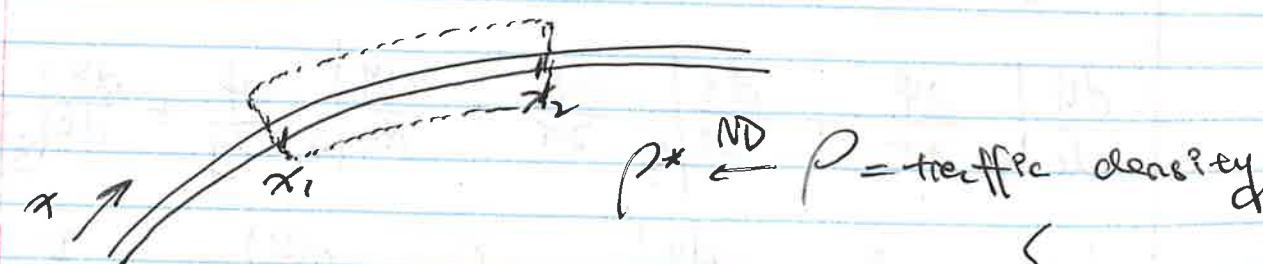
↑

flux term.

General conservation law. (1D).

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} F(u, x, t) = 0.$$

Example Traffic flow.



$$F(p, x, t) = \text{flux of cars: } \frac{\# \text{cars}}{\text{length}}$$

↳ F^*

% Conservation within the control volume.

$$\int_{x_1}^{x_2} p^*(x, t_2) dx - \int_{x_1}^{x_2} p^*(x, t_1) dx \\ = \int_{t_1}^{t_2} [F^*(x_2, t) - F^*(x_1, t)] dt.$$

Conservation law in integral form.

$x_2 \rightarrow x_1$] \rightarrow $x_2 = x_1 + dx$.
 $t_2 \rightarrow t_1$. . . \rightarrow $t_2 = t_1 + dt$. @ $\lim dx \& dt \rightarrow 0$

*** always holds!!!

$$\frac{\partial p^*}{\partial t} + \frac{\partial}{\partial x} F^* = 0. \quad \text{differential conservation law of cars.}$$

✓

Remark: F^* has to be differentiable, so this form may not hold universally!

For cars: $F^*(p^*) = p^* C^*(p^*)$.

traffic density $(\frac{\text{Cars}}{\text{m}})$ ↗
traffic speed $(\frac{\text{m}}{\text{s}})$.

assumed.

$$\text{Conservative form: } \frac{\partial p^*}{\partial t} + \frac{\partial}{\partial x} \left(C^*(p^*) p^* \right) = 0.$$

$$\text{Primitive form: } \frac{\partial p^*}{\partial t} + \left(\frac{\partial C^*}{\partial p^*} p^* + C^* \right) \frac{\partial p^*}{\partial x} = 0.$$

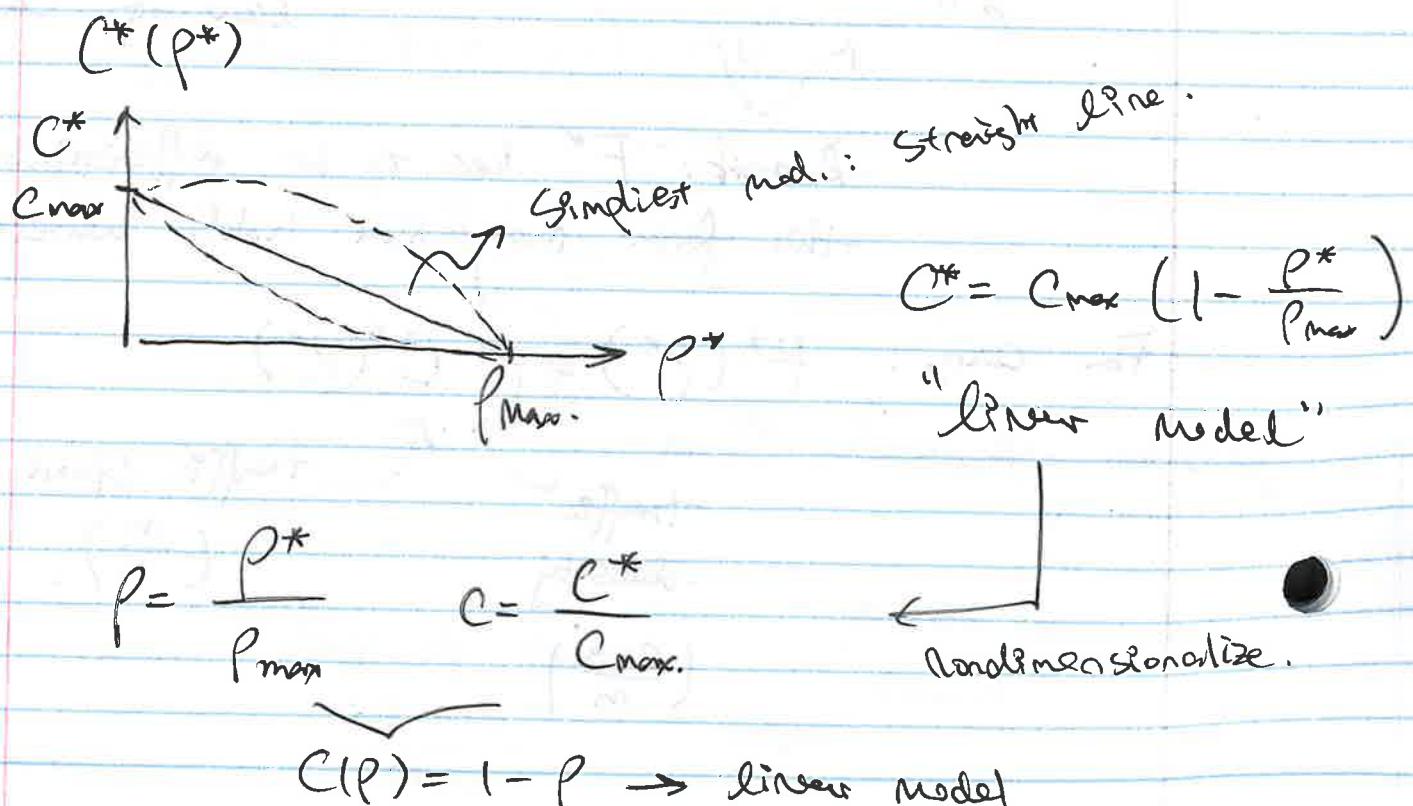
$\underbrace{\frac{\partial F^*}{\partial p^*}}$

i.e., Speed of characteristics

$$\frac{dx}{dt} \Big|_{\xi} = \frac{dc^*}{dp^*} p^* + C^* \leftrightarrow \text{func.}(p^*)$$

$$\frac{dp^*}{dt} \Big|_{\xi} = 0$$

Characteristics are a family of straight lines.



in conservative form:

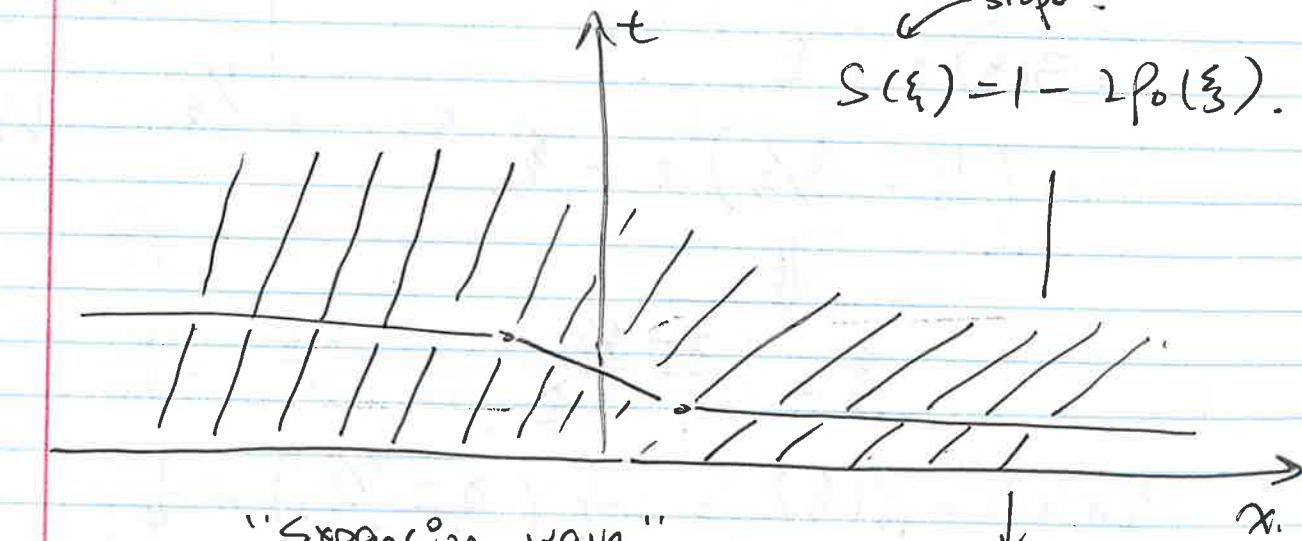
$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x} (p C(p)) = 0.$$

$$\frac{\partial p}{\partial t} + \left(C + p \frac{dc}{dp} \right) \frac{\partial p}{\partial x} = 0.$$

$$\frac{\partial p}{\partial t} + (1 - 2p) \frac{\partial p}{\partial x} = 0.$$

$$\frac{dx}{dt} \Big|_{\xi} = 1 - 2p. \quad \frac{dp}{dt} \Big|_{\xi} = 0$$

slope.
 $S(\xi) = 1 - 2p_0(\xi)$.



$$x = S(\xi)t + \xi.$$

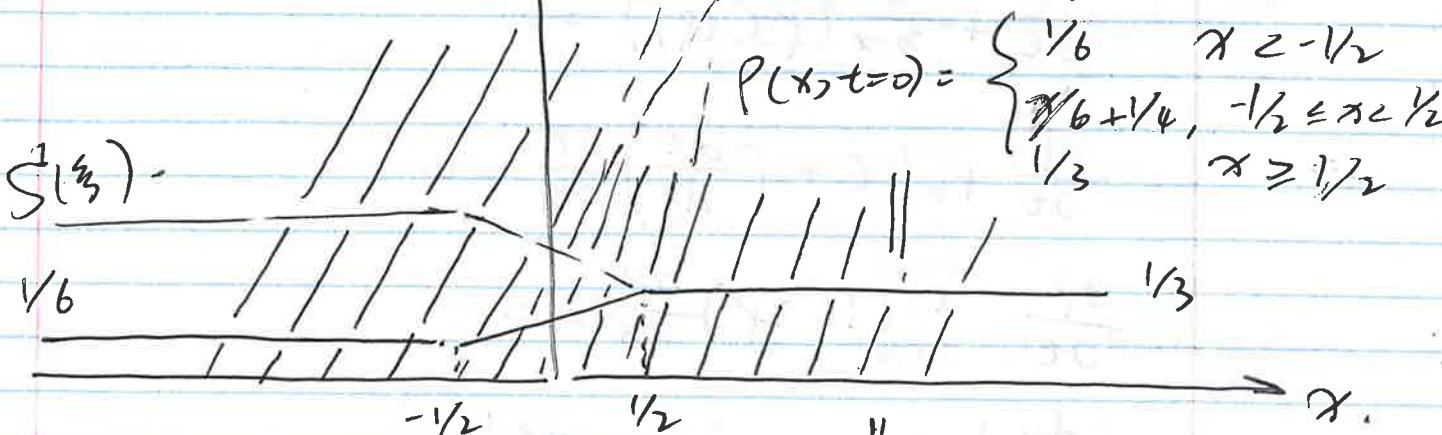
$$p(x, t) = p_0(\xi).$$

?!! - defined later.

"discontinuous form"

Compression wave.

$$S(\xi) = 1 - 2\rho(\xi).$$



$$\frac{dx}{dt} = 1 - 2\rho(\xi).$$

$$x = S(\xi)t + \xi.$$

$$x = \left(\frac{1}{2} - \frac{\xi}{3}\right)t + \xi.$$

$$\xi = \frac{x - t/2}{1 - t/3}.$$

$$\rho(x,t) = \rho_0(\xi) = \frac{1}{6} \left(\frac{x - t/2}{1 - t/3} \right) + \frac{1}{4}.$$

For all char. @ $\frac{t=3}{\downarrow}$, $\rightarrow \frac{\partial \rho}{\partial \xi} \rightarrow \infty$

Shock formation time.

$$\left. \frac{dx}{d\xi} \right|_t = 0$$

\Rightarrow look for a weak soln: Smooth soln + jumps.

We are seeking: shock speed. (shock location).

Soln in diff. parts

Jump across shock.

Concentration on one shock: $X_s(t)$.

Coordinate change: $Z = x - X_s(t)$.

Riding the shock.

$$\rho(x,t) \rightarrow \rho(Z, T) \quad T=t.$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho c) = 0 \Rightarrow Z \& T. \\ \frac{\partial \cdot}{\partial t} = \frac{\partial Z}{\partial T} \cdot \frac{\partial \cdot}{\partial Z} + \frac{\partial T}{\partial t} \cdot \frac{\partial \cdot}{\partial T}.$$

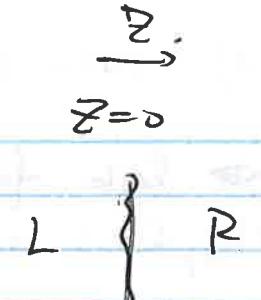
$$\frac{\partial \cdot}{\partial x} = \frac{\partial Z}{\partial x} \cdot \frac{\partial \cdot}{\partial Z} + \frac{\partial T}{\partial x} \cdot \frac{\partial \cdot}{\partial T}. \quad \frac{\partial Z}{\partial x} = 1. \\ \frac{\partial Z}{\partial t} = -\dot{X}_s$$

original

$$\text{cons. PDE} \Rightarrow \frac{\partial \rho}{\partial Z} + \frac{\partial}{\partial Z} [\rho c - \rho \dot{X}_s] = 0$$

New flux in moving frame of reference.

Flux across the shock in frame of reference moving with the shock is continuous.



$$F_L \rightarrow F_R.$$

$$F_L = F_R. \quad \leftarrow \text{frame of ref. of shock.}$$

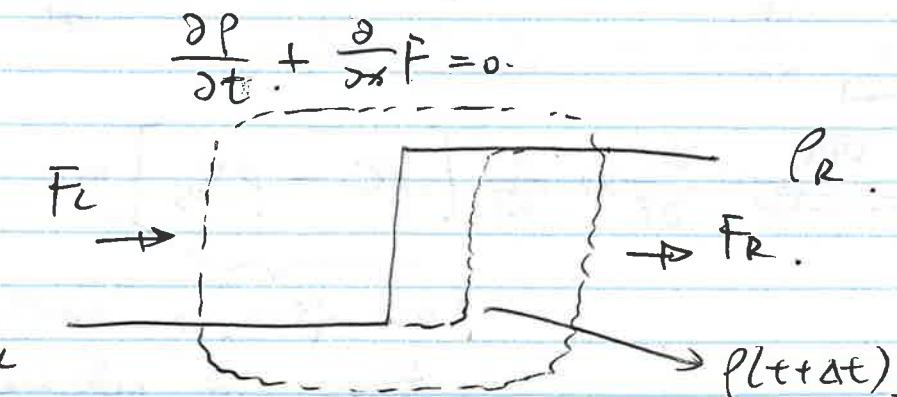
$$P_L C_L(P_L) - P_L \dot{x}_S = P_R C_R(P_R) - P_R \dot{x}_S$$

$$\dot{x}_S = \frac{P_R C_R(P_R) - P_L C_L(P_L)}{P_R - P_L}.$$

Hugoniot cond.

traffic problem

$$\text{in general: } \dot{x}_S = \frac{F_R - F_L}{P_R - P_L} \quad \begin{array}{l} \text{fluxes} \\ \text{state variables} \end{array}$$



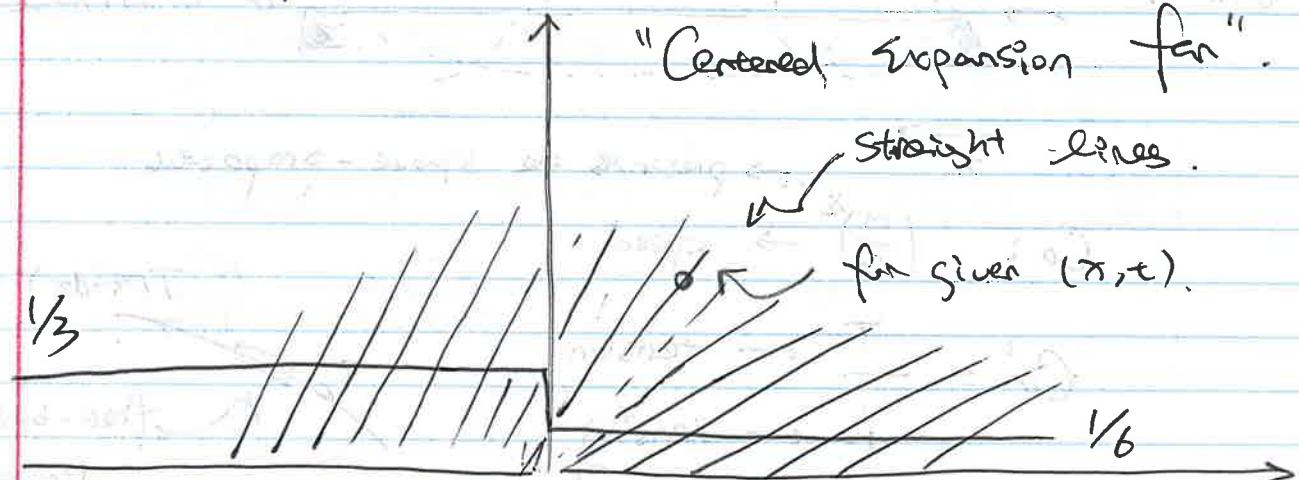
$$\int_{C\#}^{} P(t+\Delta t) dx - \int_{C\#}^{} P(t) dx = \dot{x}_S \Delta t (P_L - P_R)$$

$$(F_L - F_R) \Delta t$$

$$\dot{x}_S = \frac{P_R C_R - P_L C_L}{P_R - P_L} \xrightarrow{\substack{\rightarrow 1/3 \\ \downarrow \\ 1/3}} = \frac{1}{2}.$$

Integration of fluxes w.r.t. time.

traffic problem continued, with discontinuous I.C.:



$$\frac{dx}{dt} \Big|_{\xi_S} = 1 - 2\rho.$$

$$\frac{dp}{dt} \Big|_{\xi_S} = 0.$$

$$\text{slope: } \frac{dx}{dt} \Big|_{\xi_S} = \frac{x}{t}.$$

because char. passes (0,0) = 1 - 2ρ.

$$p(x, t) = \frac{1}{2} (1 - \frac{x}{t}).$$

lecture 6

1/25/2024.

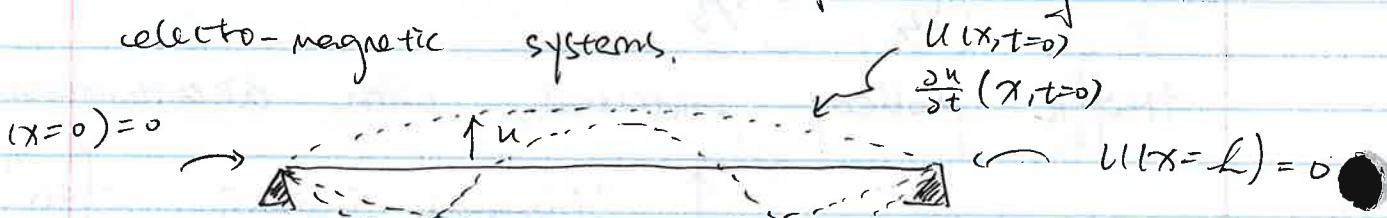
#Chapter 3 (Lele). \rightarrow 2nd-order PDE \rightarrow 1st orders.

Second-order PDE:

wave-eqn. $u(x, t)$.

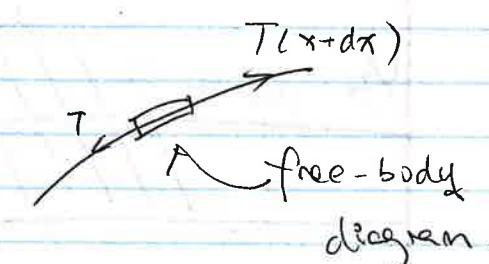
$$\frac{\partial^2 u}{\partial t^2} - C_0^2 \frac{\partial^2 u}{\partial x^2} = 0$$

Sound waves. vibration of a string / membrane.
electro-magnetic systems.



$$C_0: \left(\frac{m}{s}\right)^{\frac{1}{2}} \rightarrow \text{speed}$$

$$C_0^2 = \frac{T}{\rho} \quad \begin{matrix} \leftarrow \text{tension} \\ \leftarrow \text{density} \end{matrix}$$



D'Alembert's S.Rn. (holds for infinite domain).

$$-\infty < x < \infty$$

$$0 \leq t < \infty$$

$$\text{I.C.} = u(x, t=0) = f(x)$$

$$\frac{\partial u}{\partial t}(x, t=0) = g(x)$$

$$u(x, t) = \frac{1}{2} [f(x - c_0 t) + f(x + c_0 t)] + \frac{1}{2c_0} \int_{x-c_0 t}^{x+c_0 t} g(x') dx'$$

in multi-dimensions, the PDE writes: (wave eqn.).

$$\frac{\partial^2 u}{\partial t^2} - C_0^2 \nabla^2 u = 0$$

linear PDE: \rightarrow transform methods.

eigenfunction expansions.

Today: Method of characteristics to convert to a system of 1st-order PDEs

Approach:

- Rewrite as a system of coupled 1st order PDEs
 - Decouple the system
 - Solve each decoupled ODE/PDEs.
 - Construct solution of original PDEs.
- $$\rightarrow \frac{\partial u}{\partial t} - C_0^2 \frac{\partial u}{\partial x} = 0 \quad (*)$$

$$u_1 = \frac{\partial u}{\partial x}, \quad u_2 = \frac{\partial u}{\partial t}$$

$$\vec{U} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial t} \end{Bmatrix}$$

$$(*) : \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial t} \right) - C_0^2 \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = 0$$

$$\boxed{\frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial x} = 0}$$

$$\frac{\partial u_2}{\partial t} - C_0^2 \frac{\partial u_1}{\partial x} = 0$$

$$\frac{\partial u_1}{\partial t} = \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial t} \right) = \frac{\partial u_2}{\partial x}$$

Dir 20.

\vec{B} .

$$\vec{U} \cdot \frac{\partial \vec{U}}{\partial t} + \begin{bmatrix} 0 & -1 \\ -c_0^2 & 0 \end{bmatrix} \frac{\partial \vec{U}}{\partial x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{c} \uparrow \\ \frac{\partial}{\partial t} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{array} \quad \begin{array}{c} \uparrow \\ \frac{\partial}{\partial x} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{array}$$

decompose the matrix

$$\frac{\partial \vec{U}}{\partial t} + \vec{B} \frac{\partial \vec{U}}{\partial x} = 0. \leftarrow \text{coupled system of 1st-order PDEs.}$$

We see a combination of u_1 & u_2 that decouples the system.

eigenvalues & eigenvectors of \vec{B} :

$$\vec{B} \vec{x} = \lambda \vec{x}.$$

$$\vec{B} = \begin{bmatrix} 0 & -1 \\ -c_0^2 & 0 \end{bmatrix}.$$

$$\det(\vec{B} - \lambda I) = 0 \rightarrow \begin{vmatrix} -\lambda & -1 \\ -c_0^2 & -\lambda \end{vmatrix} = 0.$$

$$\lambda^2 - c_0^2 = 0 \Rightarrow \lambda^{(1)} = c_0 \text{ & } \lambda^{(2)} = -c_0.$$

$$\lambda^{(1)} = c_0, \lambda^{(2)} = -c_0.$$

$$\vec{x}^{(1)} = \begin{bmatrix} 1 \\ -c_0 \end{bmatrix}, \quad \vec{x}^{(2)} = \begin{bmatrix} 1 \\ c_0 \end{bmatrix}$$

$$Q = [\vec{x}^{(1)} \ \vec{x}^{(2)}] = \begin{bmatrix} 1 & 1 \\ -c_0 & c_0 \end{bmatrix}$$

$$\vec{B} = Q \Lambda Q^{-1}.$$

$$\Lambda = \begin{bmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{bmatrix} \rightarrow Q^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2c_0} \\ \frac{1}{2} & \frac{1}{2c_0} \end{bmatrix}.$$

$$\vec{V} = Q^{-1} \vec{U} \Rightarrow \vec{U} = Q \vec{V}$$

$$\vec{V} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}u_1 - \frac{1}{2c_0}u_2 \\ \frac{1}{2}u_1 + \frac{1}{2c_0}u_2 \end{bmatrix} \Rightarrow v_1 = \frac{1}{2} \left[\frac{\partial u}{\partial x} - \frac{1}{2c_0} \frac{\partial u}{\partial t} \right] \\ v_2 = \frac{1}{2} \left[\frac{\partial u}{\partial x} + \frac{1}{2c_0} \frac{\partial u}{\partial t} \right]$$

Recall:

$$\frac{\partial \vec{U}}{\partial t} + \vec{B} \frac{\partial \vec{U}}{\partial x} = 0.$$

$$\frac{\partial \vec{U}}{\partial t} + Q \Lambda Q^{-1} \frac{\partial \vec{U}}{\partial x} = 0.$$

$$Q^{-1} \cdot \frac{\partial \vec{V}}{\partial t} + \Lambda \frac{\partial \vec{V}}{\partial x} = 0.$$

$$\frac{\partial}{\partial t} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{bmatrix} \begin{bmatrix} \frac{\partial v_1}{\partial x} \\ \frac{\partial v_2}{\partial x} \end{bmatrix} = 0.$$

Decoupled: $\begin{cases} \frac{\partial v_1}{\partial t} + c_0 \cdot \frac{\partial v_1}{\partial x} = 0 \\ \frac{\partial v_2}{\partial t} - c_0 \cdot \frac{\partial v_2}{\partial x} = 0 \end{cases}$ characteristic method.

System.

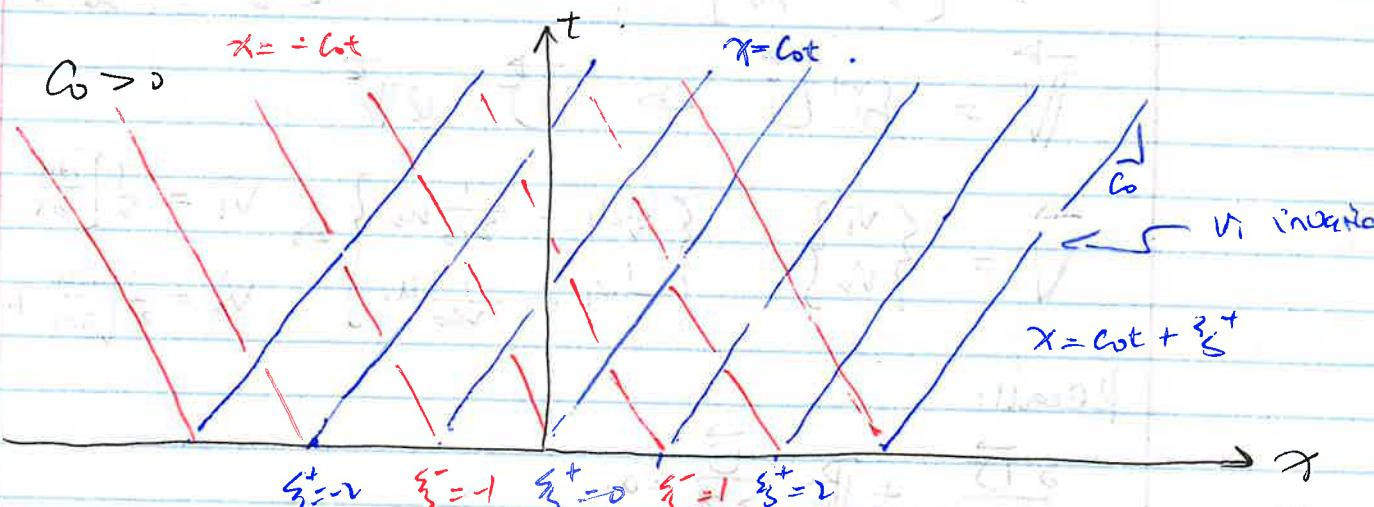
$$\frac{\partial v_1}{\partial t} + c_0 \cdot \frac{\partial v_1}{\partial x} = 0$$

$$v_1: \text{on char.: } \frac{\partial v_1}{\partial t} \Big|_{\xi^+} = c_0 \text{ & } \frac{\partial v_1}{\partial t} \Big|_{\xi^+} = 0$$

$$x = C_0 t + \xi^+ \Rightarrow v_1(x, t) = F^+(\xi^+) \quad \text{just nomination!!}$$

$$V_2: \text{on char. : } \frac{dx}{dt} \Big|_{\xi^-} = -c_0 \quad \& \quad \frac{dV_2}{dt} \Big|_{\xi^-} = 0.$$

$$x = -c_0 t + \xi^- \Rightarrow V_2(x, t) = F^-(\xi^-)$$



L.C. for V_i & V_r :

$$V_i = \frac{1}{2} \left[\frac{\partial u}{\partial x} - \frac{1}{c_0} \frac{\partial u}{\partial t} \right] \rightarrow V_i(x, t=0) = \frac{f'(x)}{2}$$

$$V_r = \frac{1}{2} \left[\frac{\partial u}{\partial x} + \frac{1}{c_0} \frac{\partial u}{\partial t} \right] \rightarrow V_r(x, t=0) = \frac{f'(x)}{2}.$$

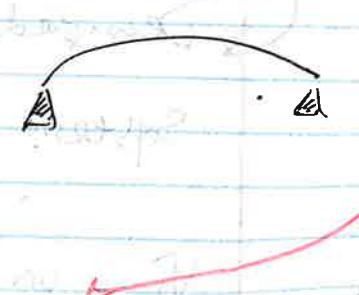
Sol'n for $u(x, t=0) = f(x)$.

$$\frac{\partial u}{\partial t}(x, t=0) = 0$$

$$V_i(x, t) = F^+(\xi^+) - F^+(x - c_0 t).$$

$$V_i(x, t=0) = F^+(x) = \frac{1}{2} f'(x)$$

$$V_i(x, t) = F^+(x - c_0 t) = \frac{1}{2} f'(x - c_0 t)$$



$$V_2(x, t) = F^-(\xi^-) = F^-(x + c_0 t),$$

$$V_2(x, t=0) = F^-(x) = \frac{1}{2} f'(x),$$

$$V_2(x, t) = F^-(x + c_0 t) = \frac{1}{2} f'(x + c_0 t)$$

V_i & V_r

$$\vec{U} = \alpha \vec{F}$$

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -c_0 & c_0 \end{bmatrix} \begin{bmatrix} V_i \\ V_r \end{bmatrix} \rightarrow \begin{aligned} U_1 &= V_i + V_r \\ U_2 &= c_0(V_r - V_i) \end{aligned}$$

$$U_1 = \frac{1}{2} [f'(x - c_0 t) + f'(x + c_0 t)]$$

$$\underbrace{\frac{\partial u}{\partial x}}_{\text{L.C.}} \quad \underbrace{\frac{\partial}{\partial x} f'(x - c_0 t)}_{\text{L.C.}}$$

$$\int dx \rightarrow U = \frac{1}{2} [f(x - c_0 t) + f(x + c_0 t)] + \chi(0).$$

$$U(x, t=0) = f(x) + \chi(0) = f(x).$$

$$\chi(0) = 0.$$

$$\frac{\partial^2 u}{\partial t^2} \quad \frac{\partial u}{\partial t}(x, t=0) = (-c_0 f' + c_0 f')/2 = 0.$$

$$\chi(0) \parallel x$$

$$\frac{\partial^2 u}{\partial t^2} - c_0^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$$U = \frac{1}{2} [f(x + c_0 t) + f(x - c_0 t)]$$

$$\chi(0) = \chi'(0) = 0 \rightarrow \chi \text{ is a line}$$

$$\leftarrow \chi'' \leftarrow \chi'' - c_0^2 \cdot 0 = 0$$

Generalization for 2nd-order PDEs in $x \& y$:

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = D \quad (\text{first order term.})$$

$$u_1 = \frac{\partial u}{\partial x} \quad \& \quad u_2 = \frac{\partial u}{\partial y}$$

$$\frac{\partial}{\partial x} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \begin{bmatrix} B/A & C/A \\ -1 & 0 \end{bmatrix} \frac{\partial}{\partial y} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} D/A \\ 0 \end{Bmatrix}$$

$$\frac{\partial \vec{U}}{\partial x} + B \frac{\partial \vec{U}}{\partial y} = \vec{0}$$

* eigenvalues

if B can be diagonalized \rightarrow

$$\frac{\partial \vec{V}}{\partial x} + \lambda \frac{\partial \vec{V}}{\partial y} = \vec{0} \quad \text{decoupled.}$$

$$B = \begin{bmatrix} B/A & C/A \\ -1 & 0 \end{bmatrix} \quad \det(B) = 0 \quad B \rightarrow I$$

eigenvalues.

$B^2 - 4AC > 0 \rightarrow$ two distinct real eigenvalues.
hyperbolic 2nd order PDE.

$B^2 - 4AC < 0 \Rightarrow$ two complex eigenvalues.
 \rightarrow elliptic system

$$\sqrt{\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2}} \Rightarrow \text{Heat eqn.}$$

$B^2 - 4AC = 0 \rightarrow$ repeated eigenvalues.

\rightarrow parabolic 2nd-order PDE.

$$\frac{\partial^2 u}{\partial x^2} - C^2 \frac{\partial^2 u}{\partial y^2} = 0 \quad A = C^2 > 0. \quad (\text{Example})$$

Problem Session 3

$$Q1. \quad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0. \quad u(x, t=0) = \begin{cases} 0, & x \leq 0 \\ x/\varepsilon, & 0 < x \leq \varepsilon \\ 1, & x > \varepsilon. \end{cases}$$

Find Solution. when $\varepsilon \rightarrow 0$.

$$\frac{dx}{dt} \Big|_{\xi} = u, \quad \frac{du}{dt} \Big|_{\xi} = 0 \rightarrow u = F(\xi).$$

$$\rightarrow u(\xi, t) = f(\xi)$$

$$u(x, t) = \begin{cases} 0 & x \leq 0 \\ x/t + \varepsilon & 0 < \frac{x}{t + \varepsilon} \leq \varepsilon \\ 1 & x - t > \varepsilon. \end{cases}$$

$$u(x, t=0) = F(\xi(x, t=0))$$

$$u(x, t=0) = f(x).$$

$$x = \frac{\xi}{\varepsilon}t + \xi$$

$$= F(\xi)t + \xi.$$

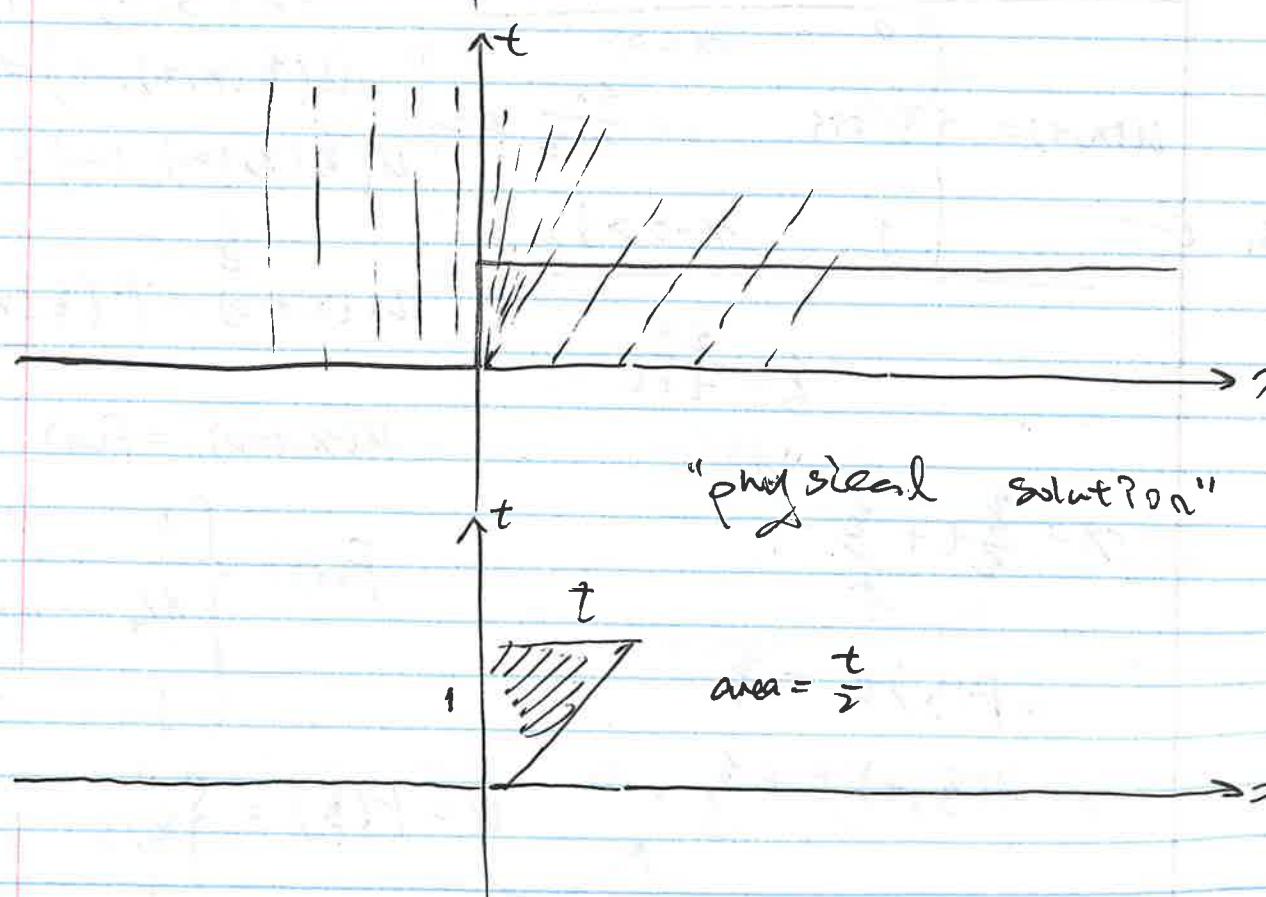
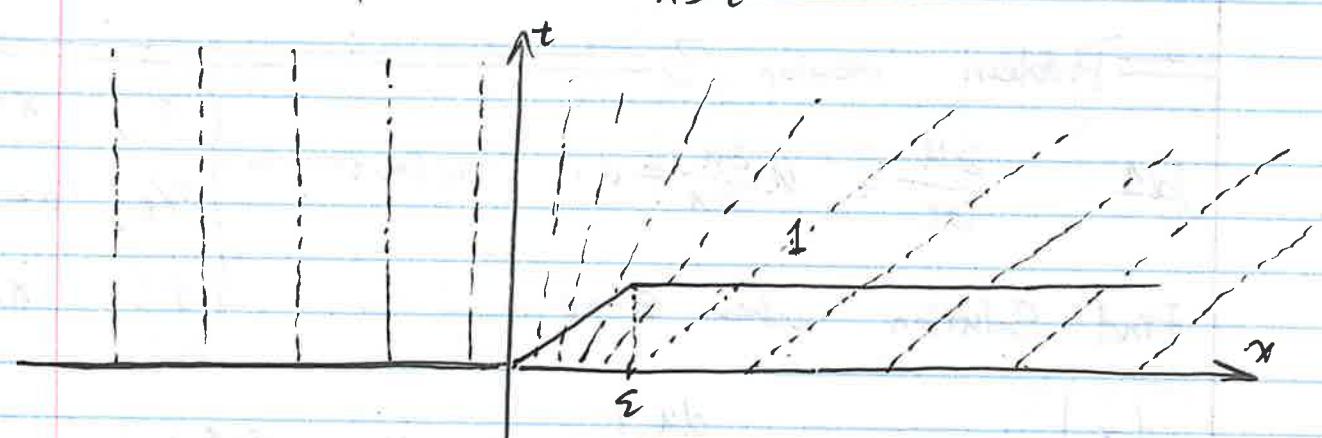
$$x = u(\xi, t) + \xi.$$

$$F(x) = \begin{cases} 0 & x \leq 0 \\ x/\varepsilon & 0 < x \leq \varepsilon \\ 1 & x > \varepsilon \end{cases}$$

$$u = F(\xi) = \begin{cases} 0 & x \leq 0 \\ \xi/\varepsilon & 0 < x \leq \varepsilon \\ 1 & x > \varepsilon \end{cases}$$

Now, letting $\varepsilon \rightarrow 0$.

$$u(x,t) = \begin{cases} 0, & x \leq 0 \\ x/t, & 0 < x \leq t \\ 1, & x > t \end{cases}$$



Multi-Lane Traffic Transport Problem

$$\frac{\partial AP}{\partial t} + \frac{\partial ACP}{\partial x} = 0 \quad \text{"flow"}$$

ρ = density of cars per unit length, $\frac{\# \text{ cars}}{\text{length}}$

$$A = \begin{cases} 1, & x \leq 0 \\ 2, & x > 0 \end{cases}$$

$$C = \text{Speed of traffic} = 1 - \frac{1}{4}\rho.$$

$$\rho(x, t=0) = \begin{cases} 2, & x \leq -3 \\ 1, & x > -3 \end{cases} \quad \rho(x, t) \text{ at } t > 0$$

$$M = AP.$$

$$\hookrightarrow m(x, t).$$

$$\frac{\partial M}{\partial t} + \frac{\partial m \cdot C(m/A)}{\partial x} = 0.$$

"switch the form for I.C.s"

$$m(x, t=0) = \begin{cases} 2, & x \leq -3 \\ 1, & -3 < x \leq 0 \\ 2, & x > 0 \end{cases}$$

$$\frac{\partial M}{\partial t} + \left[\frac{\partial C(m/A)}{\partial x} \right] \cdot m + \frac{\partial m}{\partial x} \cdot C(m/A) = 0.$$

$$\frac{\partial M}{\partial t} + \frac{\partial C(m/A)}{\partial (m/A)} \cdot \frac{\partial (m/A)}{\partial x} \cdot m + C(m/A) \frac{\partial}{\partial x} = 0.$$

$$\frac{\partial m}{\partial t} + C(m/A) \cdot \frac{m}{A} \cdot \frac{\partial m}{\partial x} + C(m/A) \frac{\partial m}{\partial x} = 0.$$

$$\frac{\partial m}{\partial t} + \left[1 - \frac{m}{4A} - \frac{m}{4A} \right] \frac{\partial m}{\partial x} = 0.$$

$$\frac{\partial m}{\partial t} + \left[1 - \frac{m}{2A} \right] \cdot \frac{\partial m}{\partial x} = 0 \quad \dots \text{(1st-order)}.$$

$$\Rightarrow \frac{dx}{dt} \Big|_{S_3} = 1 - \frac{m}{2A}(\pi(S_3, t)).$$

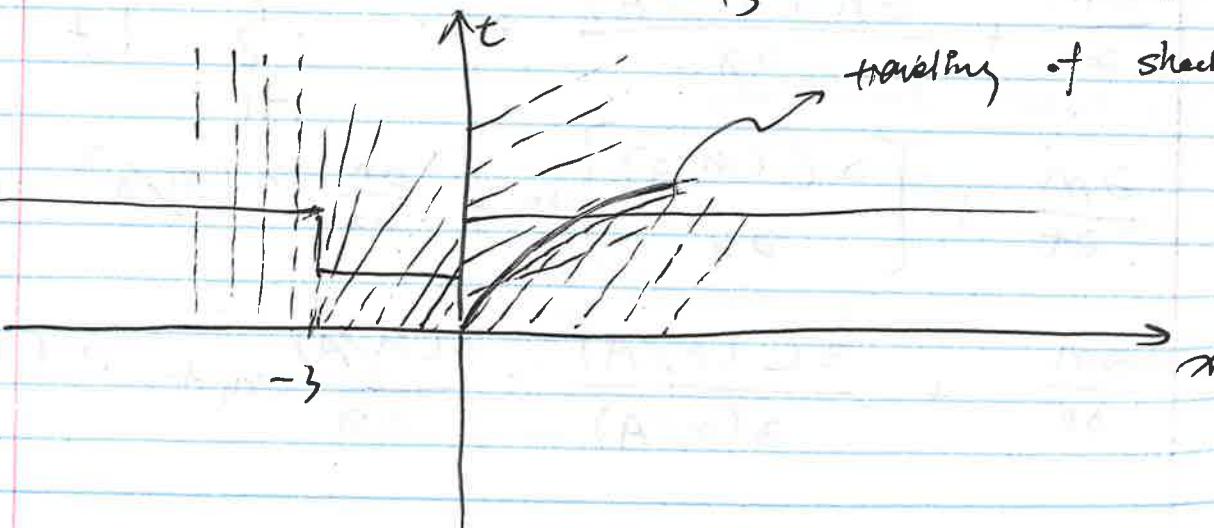
$$\frac{dm}{dt} \Big|_{S_3} = 0$$

$$\hookrightarrow m = M(\{S\})$$

Consider 2 cases:

$$x < 0.$$

$$\frac{dx}{dt} \Big|_{S_3} = 1 - \frac{m}{2} \quad . \quad \frac{dx}{dt} \Big|_{S_3} = 1 - \frac{m}{4x_2}.$$



Recall I.C.S:

$$m(x, t=0) = \begin{cases} 2 & x \leq -3 \\ 1 & -3 < x \leq 0 \\ 2 & x > 0 \end{cases}$$

Basic formula for shock speed.

$$= \frac{F_L - F_R}{m_L - m_R}.$$

$$f = mc \left(\frac{m}{A} \right).$$

$$F_L = \left(1 - \frac{m}{4A} \right) m, \quad F_R = \left(1 - \frac{m}{4A} \right) m.$$

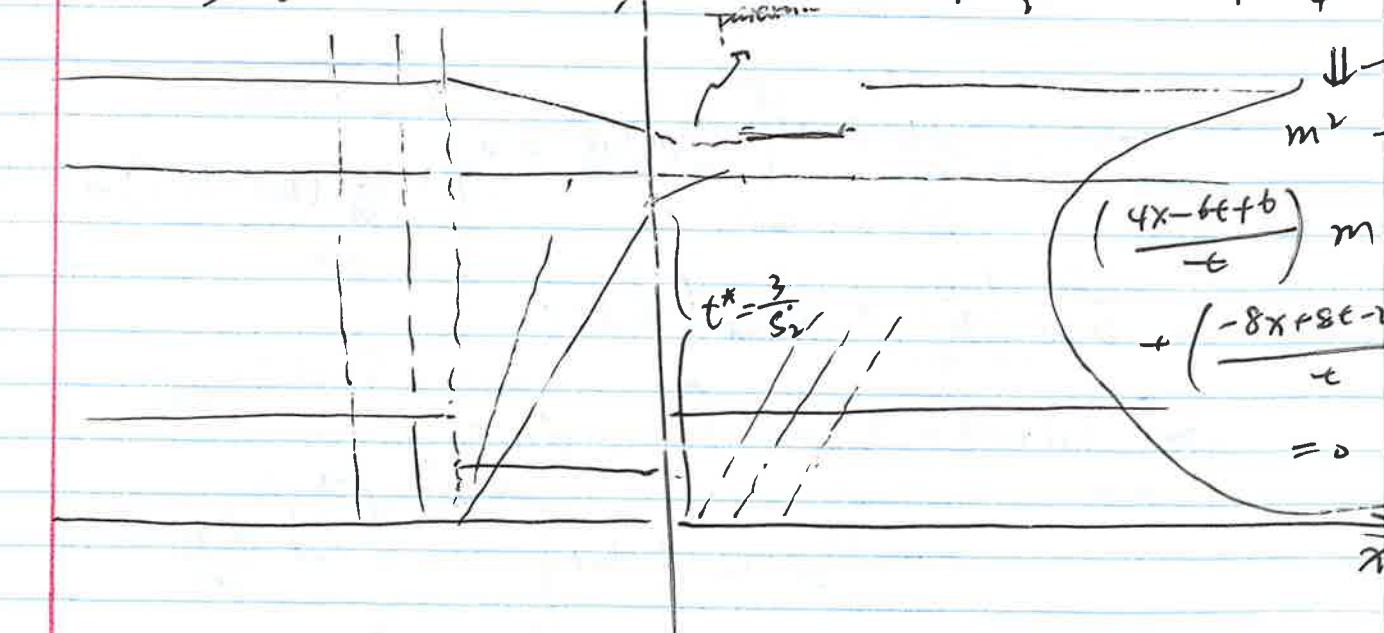
$$\downarrow \quad \downarrow$$

$$\left(1 - \frac{1}{4x_2} \right) = \frac{7}{8}$$

$$\left(1 - \frac{2}{4x_2} \right) x_2 = \frac{6}{4}$$

$$= \frac{7/8 - 6/4}{1-2} \quad \begin{matrix} \text{expansion} \\ \text{fan} \end{matrix} \quad \frac{3}{S_2} = t - \frac{x}{S_1}$$

$$= 5/8. \quad \begin{matrix} + \\ \text{parabola} \end{matrix} \quad \frac{3}{1-\frac{m}{4}} = t - \frac{x}{1-\frac{m}{4}}$$



Homework Problem. Derivation

$$3. \frac{dx}{dt} \Big|_{\xi} = \frac{ux}{t+1} \quad \frac{du}{dt} \Big|_{\xi} = \frac{-ut}{t+1}$$

Convert to matrix-vector format.

Recall lecture 5: on characteristics:

$$\frac{dt}{t+1} = \frac{dx}{ux} = \frac{du}{-ut}.$$

Integration on three sides:

$$\ln(t+1) + \xi = \int \frac{1}{ux} dx = \int \frac{1}{-ut} dt$$

$$\text{From } \int \frac{1}{ux} dx = - \int \frac{1}{ut} dt$$

$$\Rightarrow \int \frac{1}{ux} dx + \int \frac{1}{ut} dt = 0$$

$$\Rightarrow \int \frac{1}{u} d \ln x + \int \frac{1}{u} d \ln t = 0 \quad \left(-\frac{1}{u} (\ln t - \ln \xi) + \int \ln t \cdot \frac{-u}{ut} dt \right)$$

$$\frac{\partial}{\partial \ln t} \left(\frac{1}{u} \right) + \frac{\partial}{\partial \ln x} \left(\frac{1}{u} \right) = 0$$

$$\Rightarrow \ln(t+1) + \xi = \int -\frac{1}{u} d \ln t \quad \left(-\frac{1}{u} \ln t \Big|_{\xi}^{t+1} + \int \ln t \frac{d}{dt} \frac{1}{u} dt \right)$$

$$u_1 = \frac{\partial u}{\partial x}$$

$$u_2 = \frac{\partial u}{\partial t}$$

Converting to systems of ODEs:

$$\int \frac{1}{u} d \ln x + \int \frac{1}{u} d \ln t = 0$$

$$\begin{cases} u_1 = ux \\ u_2 = ut \end{cases} \rightarrow \begin{bmatrix} -\frac{t}{t+1} & \frac{u}{t+1} \\ -\frac{u}{t+1} & \frac{t}{t} \end{bmatrix} \Rightarrow \frac{-ut}{(t+1)^2} + \left(\lambda - \frac{u}{t}\right)(\frac{u}{t+1})$$

$$\frac{d}{dt} \begin{bmatrix} ux \\ ut \end{bmatrix} = \begin{bmatrix} -\frac{t}{t+1} & \frac{u}{t+1} \\ -\frac{u}{t+1} & \frac{t}{t} \end{bmatrix} \begin{bmatrix} ux \\ ut \end{bmatrix}$$

$$(ux)' = ux + u'x = \frac{-ut}{t+1} \cdot x + u \cdot \frac{ux}{t+1} = \frac{ux}{t+1}(-t+u).$$

$$(ut)' = ut + u' t = \frac{-ut}{t+1} \cdot t + u = ut \left(-\frac{t}{t+1} + \frac{1}{t} \right)$$

$$u = \exp(-t) \cdot (t+1) \exp(F(\xi))$$

$$= \frac{(t+1) \exp(F(\xi))}{\exp(t)} \rightarrow u(x) = F(x).$$

$$\exp(F(\xi)) = F(x). \quad \rightarrow u = \frac{(t+1)F(x)}{\exp(t)}$$

$$F(\xi) = \ln(F(x))$$

lns

Lecture 7. 1/30/2024.

"last session on Methods of Characteristics".

Wave Equation.

$$u(x, t).$$

$$\rightarrow \frac{\partial^2 u}{\partial t^2} - C_0^2 \frac{\partial^2 u}{\partial x^2} = 0.$$

$$-\infty < x < \infty$$

$$0 \leq t < \infty$$

$$\text{I.C.s: } u(x, t=0) = f(x).$$

$$\frac{\partial u}{\partial t}(x, t=0) = g(x).$$

Solution: D'Alembert sol'n

$$\frac{1}{2} \left[f(x - c_0 t) + f(x + c_0 t) \right] + \frac{1}{2c_0} \int_{x - c_0 t}^{x + c_0 t} g(s) ds$$

right-going wave

left-going wave

$$\text{Recall: } \frac{\partial^2 u}{\partial t^2} - C_0^2 \cdot \frac{\partial^2 u}{\partial x^2} = 0.$$

$$\left(\frac{\partial}{\partial t} - C_0 \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} + C_0 \frac{\partial}{\partial x} \right) u = 0$$

Variable: Reverses of advection

... "double-advection".

$$\xi^+ = x - c_0 t, \text{ const.}$$

$$\xi^- = x + c_0 t, \text{ const.}$$

$$(x, t) \rightarrow (\xi^+, \xi^-)$$

⇒ wave equation in ξ^+ & ξ^- :

$$\frac{\partial}{\partial \xi^+} \frac{\partial}{\partial \xi^-} u = 0$$

$$u_{\xi^+ \xi^-} = u_{\xi^- \xi^+} = 0.$$

$f(x, y)$, for which $f_{xy} = 0 \rightarrow f = G_1(x) + H_1(y)$

$$u = G^+(\xi^+) + G^-(\xi^-). \quad \leftarrow u(x, t).$$

$$= G^+(x - c_0 t) + G^-(x + c_0 t).$$

$$\text{I.C.: } u(x, t=0) = f(x).$$

$$\frac{\partial u}{\partial t}(x, t=0) = 0$$

$$\begin{cases} G^+(x) + G^-(x) = f(x), \\ -C_0 G^+(x) + C_0 G^-(x) = 0. \end{cases}$$

$$\begin{cases} G^+(x) + G^-(x) = f(x), \\ G^+(x) - G^-(x) = C. \end{cases}$$

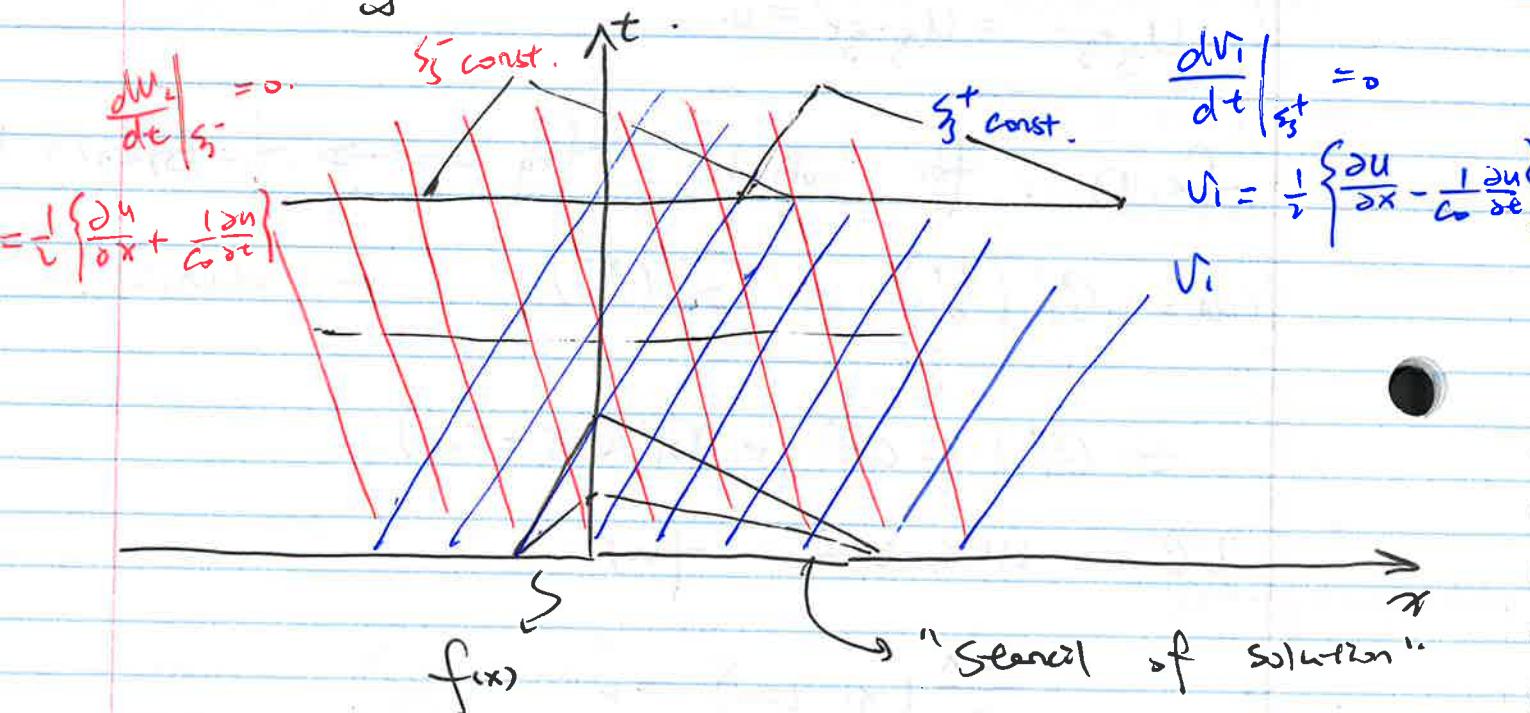
$$\rightarrow G^+ = \frac{1}{2} f + \frac{C}{2}$$

$$G^- = \frac{1}{2} f - \frac{C}{2}$$

$$u(x, t) = \frac{1}{c} [f(x - ct) + f(x + ct)]$$

Interpreting the sum.

$$\begin{cases} f(x) \neq 0, \\ g(x) = 0. \end{cases}$$

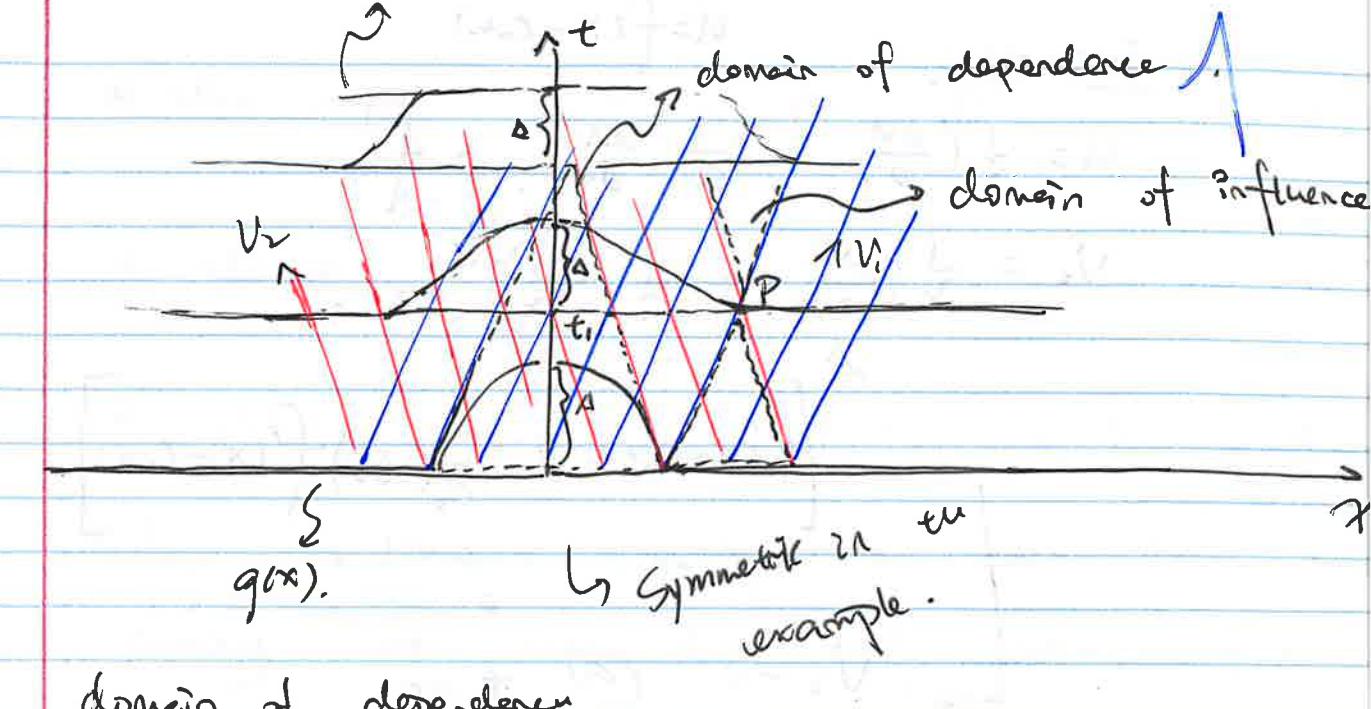


↓
the initial wave shape is preserved
during wave-propagation

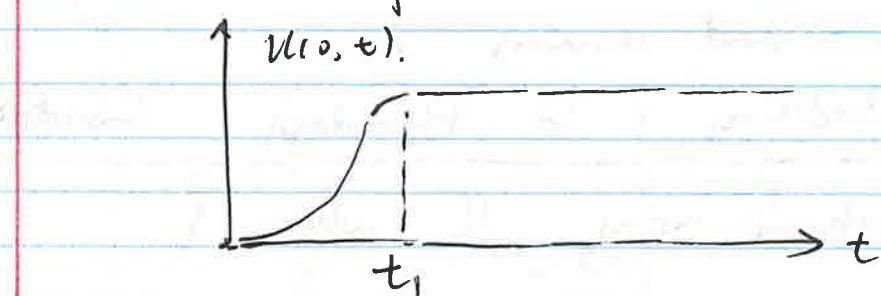
$$\begin{cases} f(x) \neq 0, \\ g(x) \neq 0. \end{cases}$$

constant-value line.

Q: unique?



domain of dependence.



Ex 3: One way waves?

(Wave actuators, wave absorbers, boundary conditions in numerical simulations ...)

$u(x, t) = f(x - ct)$ is a sol'n.

$$\frac{\partial u}{\partial t} := -c \cdot f'(x - ct).$$

$$\text{T.C.: } u(x, 0) = f(x).$$

$$\frac{\partial u}{\partial x}(x, 0) = -c \cdot f'(x)$$

Example

$$u = f(x - c_0 t)$$

$$V_1 = \frac{1}{2} \left(\frac{\partial u}{\partial x} - \frac{1}{c_0} \cdot \frac{\partial u}{\partial t} \right) = f'$$

$$V_2 = \frac{1}{2} \left(\frac{\partial u}{\partial x} + \frac{1}{c_0} \cdot \frac{\partial u}{\partial t} \right) = 0$$

$$\begin{aligned} & \left[f'(x - c_0 t) + \frac{1}{c_0} (-c_0) f'(x - c_0 t) \right] \\ & \Downarrow \\ & V_2 = 0 \quad @ \quad t = 0 \end{aligned}$$

and remains 0.

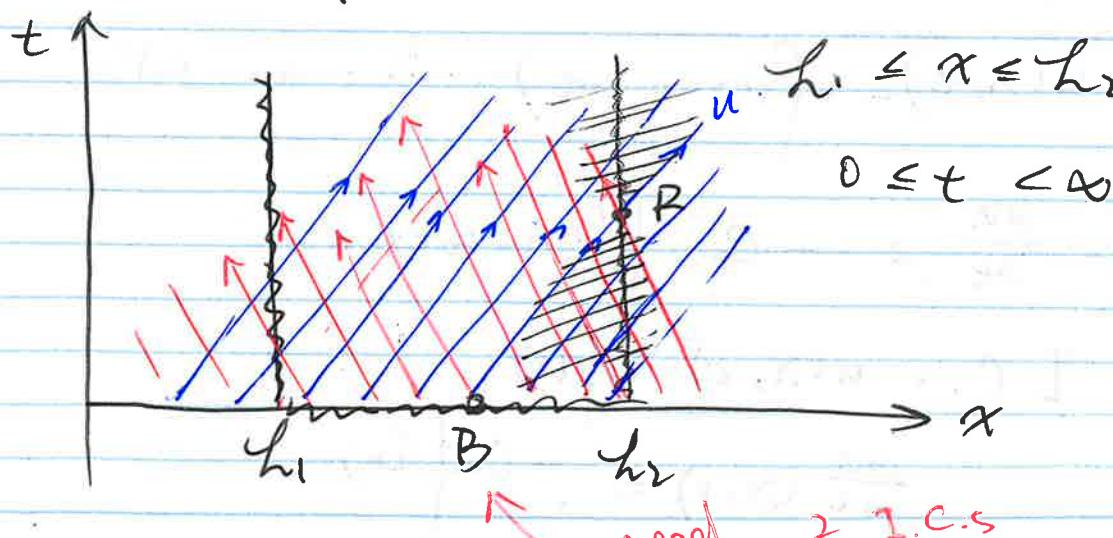
Initial Conditions & Boundary Conditions

where, how many & what?

↳ well-posed PDE problem.

Finite domain.

(Domain of interest).

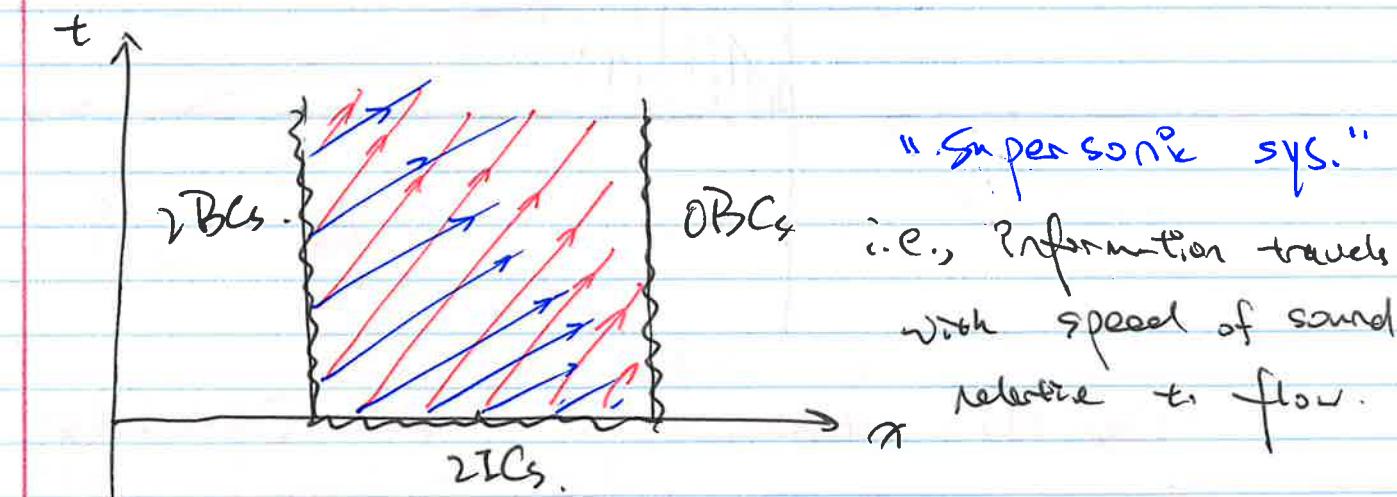


* How many I.C.s for \mathcal{B} ? 2 I.C.s

* How many B.C.s for \mathcal{R} ? 1. B.C.s

* How many B.C.s for \mathcal{L} ? 1.B.C.s

Edge conditions specify the char. that are coming into the domain of interest. (from outside)



Sol'n to wave eqn. in Finite Domain

D'Alembert $\rightarrow -\infty < x < \infty$

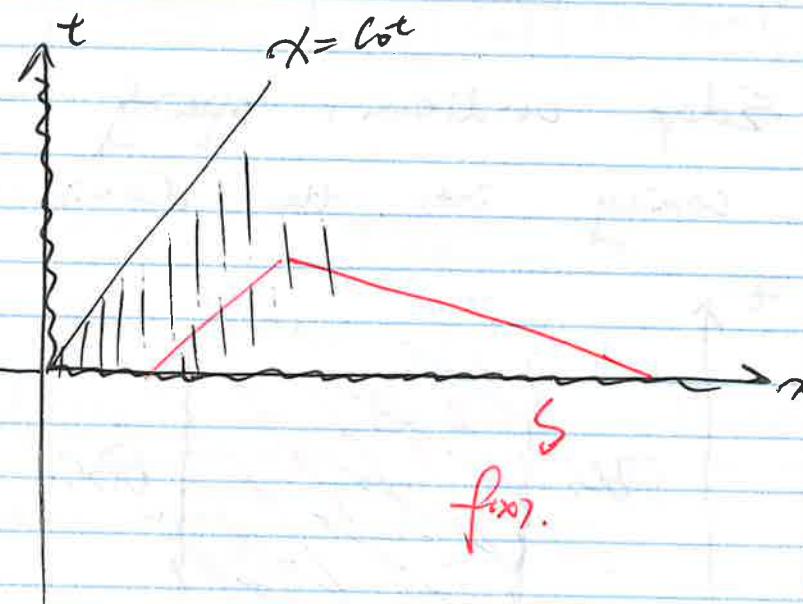
Semi-infinite domain

- $0 \leq x < \infty$
- $0 \leq t < \infty$

$$\text{Recall: } \frac{\partial^2 u}{\partial t^2} - c_0^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$$\text{T.C.S: } u(x, t=0) = f(x)$$

$$\frac{\partial u}{\partial t}(x, t=0) = 0$$



$$\text{Case (1): } u(x=0, t) = 0 \quad \leftarrow \text{Dirichlet B.C.s}$$

"Method of Images".

Embed into a problem with $-\infty < x < \infty$

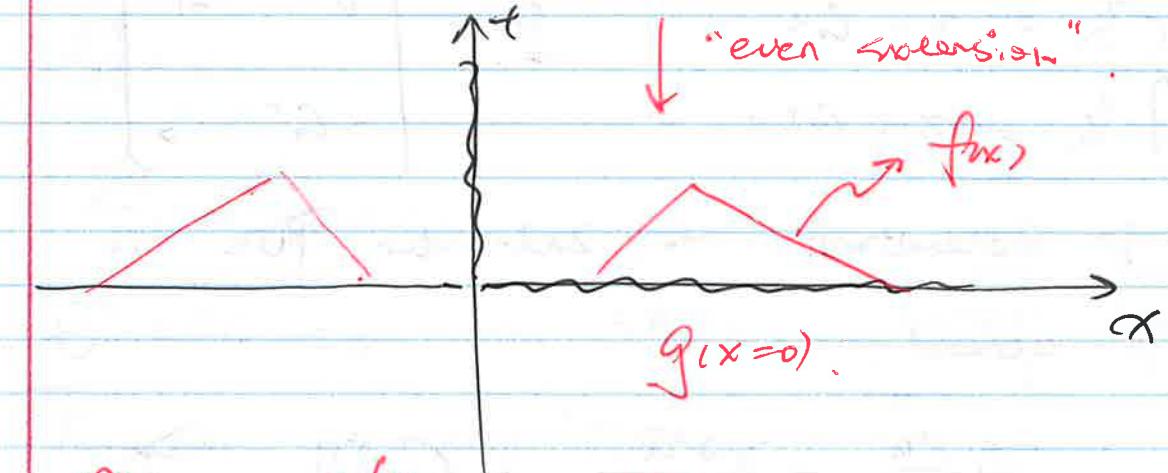
$$\tilde{f}(x, t=0) = \begin{cases} f(x), & 0 \leq x < \infty \\ -f(-x), & -\infty < x < 0 \end{cases}$$

odd extension f_m from f

$$\text{In other words, } u(x, t) = \frac{1}{2} [\tilde{f}(x - c_0 t) + \tilde{f}(x + c_0 t)]$$

- for $x > c_0 t : \frac{1}{2} [f(x+c_0 t) - f(x-c_0 t)]$
- for $x < c_0 t : \frac{1}{2} [f(x+c_0 t) + f(x-c_0 t)]$.

$$\text{Case (2). } \frac{\partial u}{\partial x}(x=0, t) = 0. \quad \leftarrow \text{Neumann B.C.s}$$



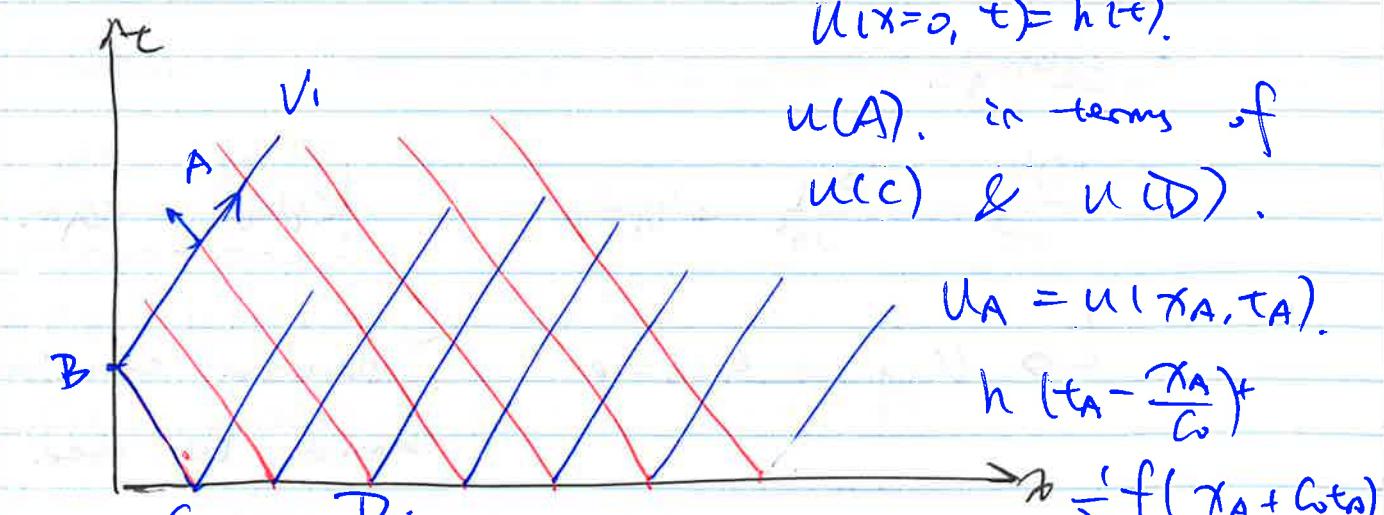
$$f(x) = \begin{cases} f(x), & 0 \leq x < \infty \\ f(-x), & -\infty < x < 0 \end{cases}$$

Time-dependent B.Cs

i.e., time-dep.

$$u(x=0, t) = h(t)$$

$u(A)$, in terms of $u(C)$ & $u(D)$.



$$h(t_A - \frac{x_A}{c_0})$$

$$f(x_A + c_0 t_A)$$

→ think back to B.C.s & I.C.s $- \frac{1}{2} f(c_0 t_A - x_A)$

lecture 8 2/1/2024

$$\frac{\partial^2 u}{\partial t^2} - C_0 \frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial x}{\partial t} = \pm C_0 \quad \det |B - \lambda I| = 0.$$

$$\begin{cases} \xi_+ = x - C_0 t \\ \xi_- = x + C_0 t \end{cases} \quad B = \begin{bmatrix} 0 & -1 \\ -C_0 & 0 \end{bmatrix}$$

→ Generalization to 2nd-order PDE ...

In general:

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial xy} + C \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial y}{\partial x} = \frac{B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A}$$

$$u_{xy} = 0$$

Example

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u = 0.$$

KdV eqn.

$$u_{xy} + 4u = 0 \quad \text{D'Alembert solution}$$

cannot be used anymore.

Example heat equation.

$$\frac{\partial u}{\partial t} - K \frac{\partial^2 u}{\partial x^2} = 0.$$

$$B^2 - 4AC \stackrel{?}{=} 0 \rightarrow \text{parabolic}$$

$$\frac{\partial x}{\partial t} = 0.$$

$$\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$B^2 - 4AC < 0 \rightarrow$ elliptic. complex characteristics.

Formulate a new PDE:

$$\frac{\partial^2 u}{\partial x^2} - \chi \frac{\partial^2 u}{\partial y^2} = 0. \quad \chi > 0: \text{hyperbolic eqn.}$$

$$\checkmark \quad \chi < 0: \text{elliptic eqn.}$$

"Euler-Triommi Eqn":

constant sol'n using superposition. (linearized system).

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0.$$

$$\text{B.C.s: } u(0,t) = u(1,t) = 0, \quad t > 0.$$

$$\text{I.C.s: } u(x,0) = f(x), \quad 0 < x < 1.$$

$$f(x) \in C^2[0,1].$$

Start with an Ansatz:

$$u(x,t) = \Phi(t) \Psi(x).$$

Assumption.

Goal: find a non-trivial sol'n.

If $u \neq 0$,

$$\Phi'(t) \cdot \Psi(x) \neq 0.$$

$$\Phi'(t) \Psi(x) = \Phi(t) \Psi''(x) \quad \dots \text{plug in original eqn.}$$

$$\Rightarrow \frac{\Phi'(t)}{\Phi(t)} = \frac{\Psi''(x)}{\Psi(x)}$$

$$= -\lambda \quad \begin{array}{l} \text{as long as the function} \\ \text{is not a zero-func.} \\ \text{we good :).} \\ \text{↑ defined as} \\ \sim \text{"separation const."} \end{array}$$

$$\Phi'(t) + \lambda \Phi(t) = 0 \quad \dots \text{solved via 1.}$$

Q: what's going on w. our B.C.s?

A: $\Psi(0) = \Psi(1) = 0$.

Sol: Harmonic Oscillator.

General sol'n: $\Psi(x) = C_1 \sin(\sqrt{\lambda}x) + C_2 \cos(\sqrt{\lambda}x)$.

Plugging in the B.C.s:

$$\rightarrow 0 + C_2 = 0$$

Second B.C.: $\Psi(1) = 0 \rightarrow C_1 \sin(\sqrt{\lambda}) = 0$.

General Sol'n:

$$\Psi_k(x) = C_{1k} \sin(k\pi x)$$

$$\sqrt{\lambda} = \pi, 2\pi, \dots$$

→ What if $\lambda < 0$? (Aside).

$$\Psi(x) = C_1 e^{kt} + C_2 e^{-kt}$$

$$\lambda = k\pi,$$

$$\forall k = 1, 2, \dots \infty$$

prove it: this will only be working if $\lambda = 0$.

→ Second ODE

$$\Phi'(t) + \lambda \Phi(t) = 0.$$

$$\frac{d\Phi'(t)}{\Phi(t)} = dt.$$

$$\Phi(t) = C_{k\phi} e^{-\lambda t}.$$

$$\Rightarrow \Phi(t) = e^{-k^2\pi^2 t}.$$

$$\Rightarrow u_k = C_{k\phi} e^{-k^2\pi^2 t} C_{1k} \sin(k\pi t).$$

$$u_k = C_k e^{-k^2\pi^2 t} \sin(k\pi x).$$

$$\text{General sol'n: } u(x, t) = \sum_{k=1}^{\infty} A_k u_k(x, t)$$

↳ Q: is it going to converge to
a finite number?
 ∇

$$A_k \rightarrow 0.$$

"sufficiently rapidly"

$$f(x) = \sum_{k=1}^{\infty} A_k \sin(k\pi x).$$

\uparrow function can be
represented as this

"series"

$$\int_0^1 \sin(m\pi x) f(x) dx = \int_0^1 \sum_{k=1}^{\infty} A_k \sin(k\pi x) \sin(m\pi x) dx$$

$$= A_m \int_0^1 \sin^2(m\pi x) dx$$

$$= \frac{A_m}{2}$$

$$\text{Since, } \int_0^1 \sin(k\pi x) \sin(m\pi x) dx = 0, \quad \forall k \neq m.$$

... why? because $A_k \rightarrow$ "sufficiently rapidly"
 k^{th} coefficient.

$$A_k = 2 \int_0^1 f(x) \sin(k\pi x) dx.$$

the solution:

$$u(x) = \sum_{k=1}^{\infty} A_k u_k(x, t)$$

... we use these "sin" functions due to,
"Orthogonality".

———— Problem Session 4 ————

2/1/2024 → Coupled system of 1st-order PDEs.
Methods of Merges.

$$\left\{ \begin{array}{l} \rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0 \\ \frac{\partial p}{\partial t} + \rho c_o \frac{\partial u}{\partial x} = 0 \end{array} \right.$$

L.C.S: $u(x, t=0) = f(x)$ → pressure disturbances.
 $p(x, t=0) = g(x)$

Step 1.

Matrix form

$$\begin{bmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial p}{\partial t} \end{bmatrix} + \begin{bmatrix} 0 & 1/\rho_o \\ \rho_o c_o^2 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial p}{\partial x} \end{bmatrix} = 0$$

Defn:

$$\vec{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} u \\ p \end{bmatrix}$$

$$\frac{\partial \vec{\phi}}{\partial t} + A \cdot \frac{\partial \vec{\phi}}{\partial x} = 0$$

$$2. A = Q \Lambda Q^{-1}$$

up to:

$$Q = [\vec{q}_1, \vec{q}_2], \quad \Lambda = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix},$$

$$\det(A - \lambda I) = 0.$$

$$\det \begin{bmatrix} -\lambda & 1/p_0 \\ p_0 c_0^2 & -\lambda \end{bmatrix} = 0 \rightarrow \begin{cases} \lambda_1 = c_0 \\ \lambda_2 = -c_0 \end{cases}$$

Find eigenvalues: $(A - \lambda_1 I) \vec{q}_1 = 0$.

$$\begin{bmatrix} -c_0 & 1/p_0 \\ p_0 c_0^2 & -c_0 \end{bmatrix} \begin{bmatrix} q_1^{(1)} \\ q_1^{(2)} \end{bmatrix} = 0.$$

$$-c_0 q_1^{(1)} + \frac{1}{p_0} q_1^{(2)} = 0.$$

$$p_0 c_0^2 q_1^{(1)} - c_0 q_1^{(2)} = 0.$$

$$\rightarrow q_1^{(2)} = p_0 c_0 q_1^{(1)}.$$

$$\vec{q}_1 = \begin{bmatrix} 1 \\ p_0 c_0 \end{bmatrix},$$

$$(A - \lambda_2 I) \vec{q}_2 = 0. \quad \rightarrow \vec{q}_2 = \begin{bmatrix} 1 \\ -p_0 c_0 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} c_0 & 0 \\ 0 & -c_0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1 \\ p_0 c_0 & -p_0 c_0 \end{bmatrix}.$$

$$Q^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1/p_0 c_0 \\ 1 & -1/p_0 c_0 \end{bmatrix}$$

$$\frac{\partial \vec{\phi}}{\partial t} + A \cdot \frac{\partial \vec{\phi}}{\partial x} = 0.$$

$$\frac{\partial \vec{\phi}}{\partial t} + Q \Lambda Q^{-1} \frac{\partial \vec{\phi}}{\partial x} = 0.$$

$$Q^{-1} \frac{\partial \vec{\phi}}{\partial t} + Q^{-1} Q \Lambda Q^{-1} \frac{\partial \vec{\phi}}{\partial x} = 0.$$

$$\frac{\partial}{\partial x} Q^{-1} \vec{\phi} + \Lambda \frac{\partial}{\partial x} Q^{-1} \vec{\phi} = 0.$$

$$\vec{\psi} = Q^{-1} \vec{\phi} \rightarrow \frac{\partial \vec{\psi}}{\partial t} + \Lambda \frac{\partial \vec{\psi}}{\partial x} = 0.$$

$$\frac{\partial}{\partial t} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} + \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = 0$$

$$\left\{ \begin{array}{l} \frac{\partial \psi_1}{\partial t} + c_0 \frac{\partial \psi_1}{\partial x} = 0 \\ \frac{\partial \psi_2}{\partial t} - c_0 \frac{\partial \psi_2}{\partial x} = 0 \end{array} \right.$$

$$\left\{ \begin{array}{l} \frac{\partial \psi_1}{\partial t} + c_0 \frac{\partial \psi_1}{\partial x} = 0 \\ \frac{\partial \psi_2}{\partial t} - c_0 \frac{\partial \psi_2}{\partial x} = 0 \end{array} \right.$$

Step 3.

Methods of Characteristics

... Decoupling Step.

$$\text{Eq. 1. } x = c_0 t + \xi_1, \quad \psi_i(\xi_1, t) = \psi_i(\xi_1, 0).$$

$$\text{Eq. 2. } x = -c_0 t + \xi_2.$$

$$\psi_2(\xi_2, t) = \psi_2(\xi_2, 0).$$

4. Given ICs: $\begin{cases} u(x, t=0) = f(x) \\ p(x, t=0) = g(x). \end{cases}$

Transform ICs. $x = \xi_1 = \xi_2 = \xi$.

$$u(\xi, 0) = f(\xi),$$

$$p(\xi, 0) = g(\xi).$$

$$\vec{\psi} = Q^{-1} \vec{\phi} \Rightarrow \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & \frac{1}{p_0 c_0} \\ 1 & -\frac{1}{p_0 c_0} \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix}$$

$$\psi_1(\xi_1, 0) = \frac{1}{2} \underbrace{u(\xi_1, 0)}_{f(\xi_1)} + \frac{1}{2p_0 c_0} \underbrace{p(\xi_1, 0)}_{g(\xi_1)} \rightarrow \psi_1(\xi_1, t)$$

$$\psi_2(\xi_2, 0) = \frac{1}{2} \underbrace{u(\xi_2, 0)}_{f(\xi_2)} + \frac{1}{2p_0 c_0} \underbrace{p(\xi_2, 0)}_{g(\xi_2)} \rightarrow \psi_2(\xi_2, t)$$

Final-form solution.

$$\xi_1 = x - c_0 t, \quad \xi_2 = x + c_0 t.$$

$$\psi_1(x, t) = \frac{1}{2} f(x - c_0 t) + \frac{1}{2p_0 c_0} g(x - c_0 t).$$

$$\psi_2(x, t) = \frac{1}{2} f(x + c_0 t) - \frac{1}{2p_0 c_0} g(x + c_0 t).$$

→ Final Step: $\vec{\phi} = Q \vec{\psi}$

$$\begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ p_0 c_0 & -p_0 c_0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}.$$

$$u(x, t) = \frac{1}{2} f(x - c_0 t) + \frac{1}{2p_0 c_0} g(x - c_0 t).$$

$$+ \frac{1}{2} f(x + c_0 t) - \frac{1}{2p_0 c_0} g(x + c_0 t).$$

$$p(x, t) = \frac{p_0 c_0}{2} f(x - c_0 t) + \frac{1}{2} g(x - c_0 t) - \frac{p_0 c_0}{2} f(x + c_0 t) + \frac{1}{2} g(x + c_0 t)$$

Only right-going waves.

$$\psi_2(\xi_2) = \frac{1}{2} f(\xi_2) - \frac{1}{2p_0 c_0} g(\xi_2) = 0.$$

$$\rightarrow g(\xi_2) = p_0 c_0 f(\xi_2).$$

$$\rightarrow g(x) = p_0 c_0 f(x)$$

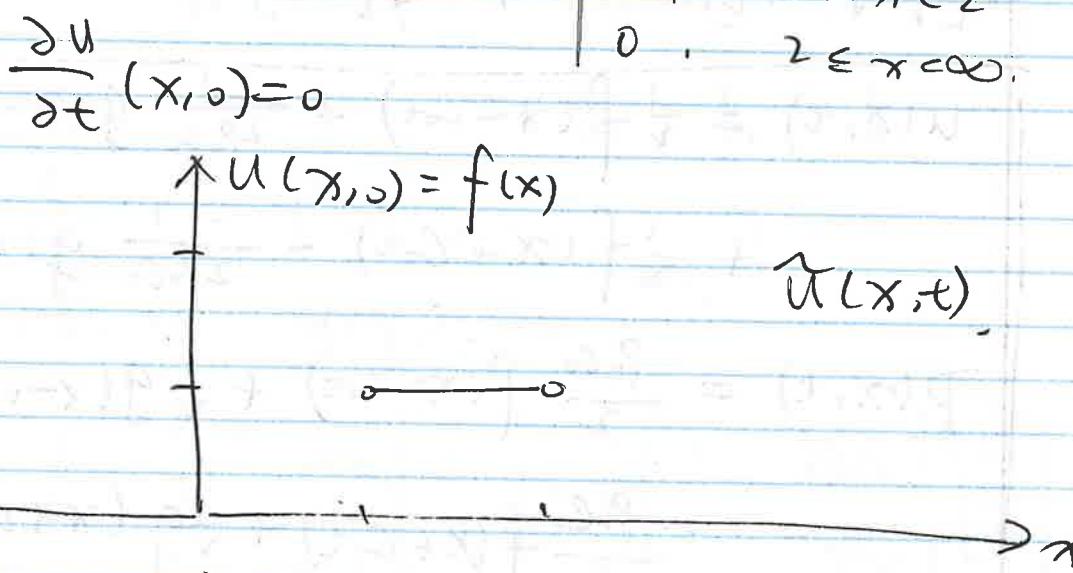
try to satisfy the BCs.

Methods of Images.

Wave equation on semi-infinite domain.

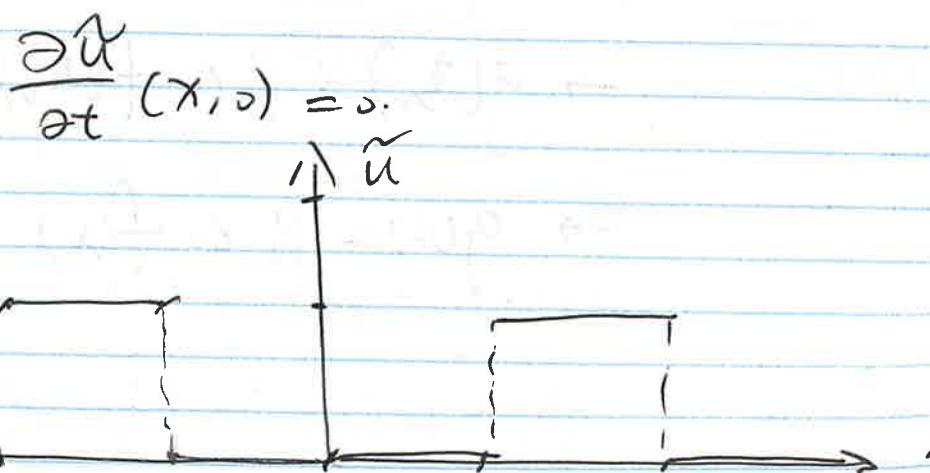
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x < \infty, \quad t \geq 0.$$

$$BC: \frac{\partial u}{\partial x}(0, t) = f(x) = \begin{cases} 0, & 0 \leq x < 1, \\ 1, & 1 \leq x < 2 \\ 0, & 2 \leq x < \infty. \end{cases}$$



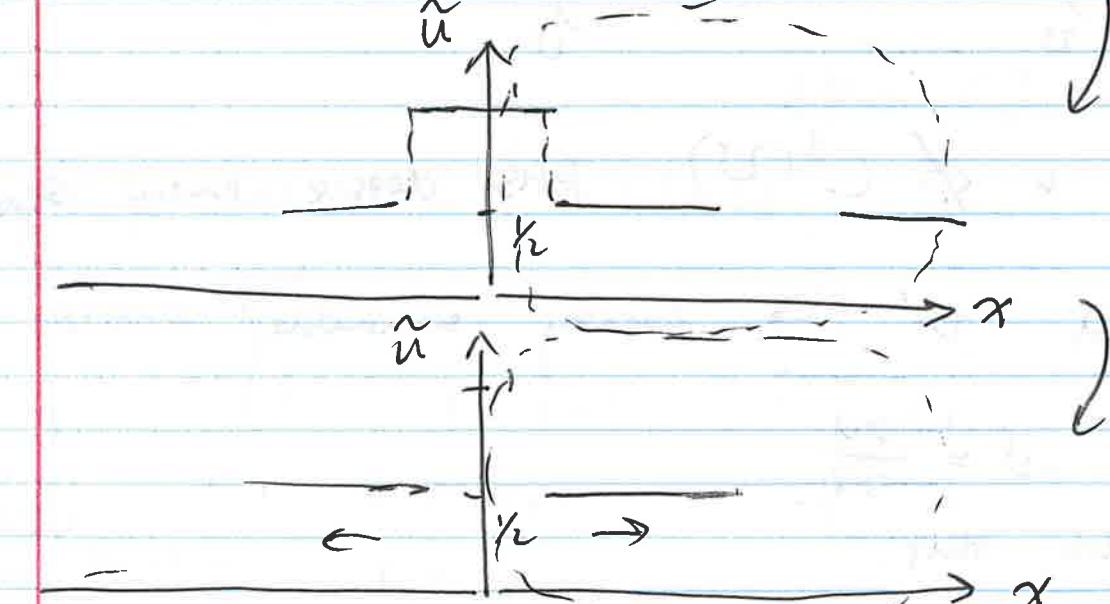
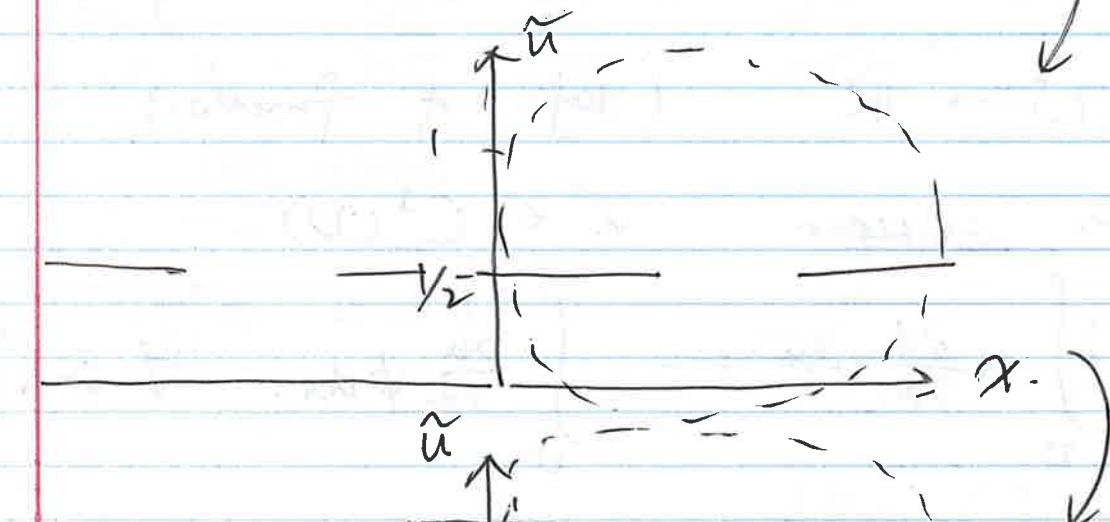
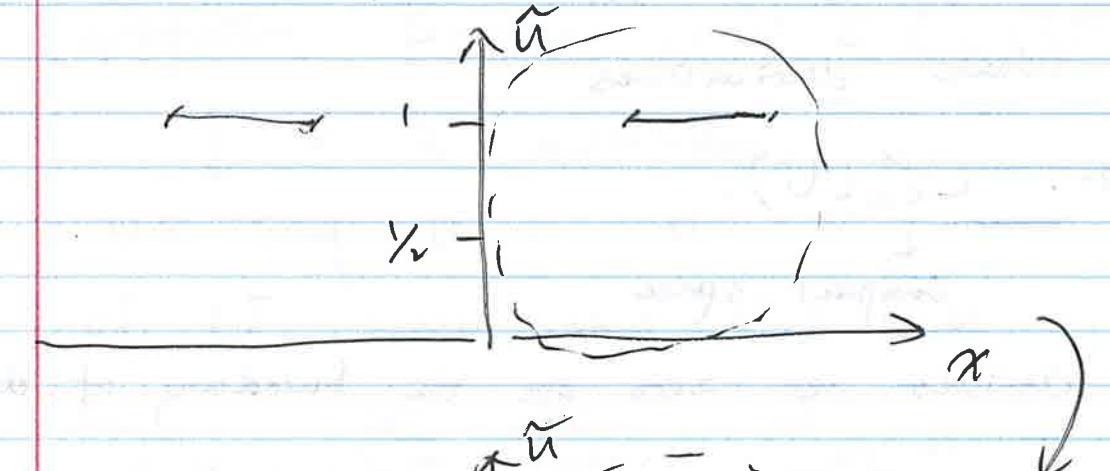
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t \geq 0.$$

$$I.C.s: \tilde{u}(x, 0) = f(x) = ?$$



$$f(x) = \begin{cases} f(x), & 0 \leq x < \infty \\ f(-x), & -\infty < x \leq 0. \end{cases}$$

$$\hat{u}(x, t) = \frac{1}{2} \tilde{f}(x - ct) + \frac{1}{2} \tilde{f}(x + ct).$$



became 9 2/6/2024.

Review:

- { Separation of variables.
- ODEs + B.C.s.
- General sol'n \rightarrow coeff.
- Series sol'n.

\rightarrow Weak derivatives.

Let. $C_c^\infty(U)$.

compact space.

Variables \rightarrow zero at the boundary of domain.

$f: U \rightarrow \mathbb{R}$. (defn of function).

Now, consider $u \in C^1(U)$.

$$\int_U u \frac{\partial \phi}{\partial x_i} dx = - \int_U \frac{\partial u}{\partial x_i} \phi dx. \quad \forall i=1, 2, \dots, n.$$

If $u \notin C^1(U)$ RHS doesn't make sense

find ' v ' \rightarrow locally summable

$$v = \frac{\partial u}{\partial x_i}$$

such that.

$$\int_U u \frac{\partial \phi}{\partial x_i} dx = - \int_U v \phi dx$$

↑ this form: weak derivatives

Definition: Suppose $u, v \in L^1_{loc}(U)$.

↓ v is multi-index

We say v is the α^{th} weak derivatives of U such that $v = D^\alpha u$.

$$D^\alpha u = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \dots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u$$

$|\alpha|$ times.

L1 Defn:

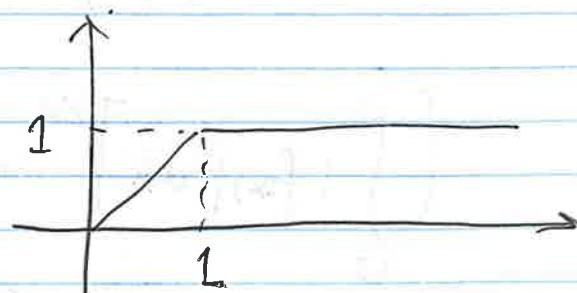
$$\int_U |f| dx < \infty$$

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v \phi dx$$

$v \in C_c^\infty(U)$.

e.g. let $n=1$, $U=(0, 2)$.

$$u(x) = \begin{cases} x & 0 < x \leq 1, \\ 1 & 1 \leq x \leq 2. \end{cases}$$



$$\text{Define: } v(x) = \begin{cases} 1 & 0 \leq x \leq 1, \\ 0 & 1 \leq x \leq 2. \end{cases}$$

$$\int_0^2 u\phi' dx = \int_0^1 \bar{u}\phi' dx + \int_1^2 \bar{u}' dx$$

$$= - \int_0^1 \phi dx + \phi(1) - \phi(0) + \overset{\circ}{\phi}(k) - \phi(1)$$

$$= - \int_0^2 u\phi dx$$

$W^{k,p}(U) \rightarrow$ Sobolev spaces.

$$u: U \rightarrow \mathbb{R}$$

s.t., for each α , with $|\alpha| \leq k$.

$D^\alpha u$ exists, in the weak sense & belongs to

$$L^p(U) \left(\int_U |f(x)|^p dx \right)^{1/p}.$$

A special case. If $p=2$. then

$$W^{k,2}(U) = H^k(U), \quad (k=0, 1, \dots)$$

$$\left[\int_U |f(x)|^2 dx \right]^{1/2} < \infty$$

Hilbert

the "norm" of the space

Defn if $u \in H^k(U)$.

$$\|u\|_{H^k} = \sum_{|\alpha| \leq k} \left(\int_U |D^\alpha u|^p dx \right)^{1/p}.$$

if $k=0$. it gives

$$\left(\int_U |f(x)|^2 dx \right)^{1/2} < \infty$$

"norm is bounded".

$$\text{e.g. } \|u\|_H = \left(\int_U |u|^p dx \right)^{1/2} + \left(\int_U \left| \frac{\partial u}{\partial x} \right|^p dx \right)^{1/2}$$

$$+ \left(\int_U \left| \frac{\partial^2 u}{\partial x^2} \right|^p dx \right)^{1/2} < \infty$$

* Properties of Hilbert spaces.

Defn For every $f, g \in H$. \subset Hilbert sp.

we can define a scalar

"scalar" not $\rightarrow (f, g)$.
"inner product"

1). $(f, g) \geq 0$, for all $f, g \in H$.

2) $(f, f) = 0$, iff $f = 0$.

can be complex.

$$3). (\bar{f}, g) = \bar{g}(f, g)$$

$$4). (f, g) = (g, f).$$

$$5). (f+g, h) = (f, h) + (g, h).$$

example $L^2(\Omega)$ with $\Omega \subset \mathbb{R}^n$.

↓ define.

$$(f, g) = \int_{\Omega} f(x) \overline{g(x)} dx$$

for real-valued func.

$$\int_{\Omega} f(x) g(x) dx$$

example $L^2_w(\Omega)$

$$(f, g)_w = \int_{\Omega} f(x) g(x) w(x) dx$$

↑ weighting func.

$\forall w(x) > 0$

Theorem: For H being Hilbert, if we set

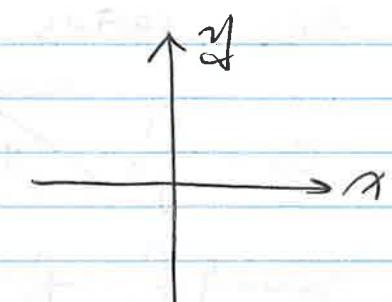
$$\|f\| = \sqrt{(f, f)}$$
 then $\|\cdot\|$ is a norm.

Orthogonality

example: \mathbb{R}^2

$$(x, y) = \|x\| \|y\| \cos \theta$$

$$\text{and } x \perp y, (x, y) = 0 \quad (\text{IFF})$$



Defn. If H is Hilbert, & $f, g \in H$, then f & g are orthogonal iff $(f, g) = 0$.

e.g.: $f_n(x) = \sin(nx)$.

then $\{f_n\}_{n=1}^{\infty}$ is orthogonal in $L^2(0, \pi)$.

Infinite Orthogonal sequences

- equipped with projection operators. P_E .
- if $\{f_n\}_{n=1}^{\infty}$ is a countable orthogonal set. in H . & $f_n \neq 0$ for all "n".
- we expect, simply $n \rightarrow \infty$.

$$H = \text{span}(\{f_n\}_{n=1}^{\infty}).$$

$$\text{then } P_E g = \sum_{n=1}^{\infty} \frac{(g, f_n)}{(f_n, f_n)} f_n.$$

if this series converges.

Let $e_n = f_n / \|f_n\|$, $C_n = (g, e_n)$.

$$T_n = \text{Span}\{f_1, f_2, \dots, f_n\}$$

S.t. $\{e_n\}_{n=1}^{\infty}$ is an orthogonal set.

$$P_E g = \sum_{n=1}^N C_n e_n$$

Bessel's inequality.

"just stare it"

$$\left| \sum_{n=1}^{\infty} |C_n|^2 \right| = \left| \sum_{n=1}^{\infty} |(g, e_n)|^2 \right| \leq \|g\|^2$$

Consequently, $\lim_{n \rightarrow \infty} C_n = 0$

Theorem Riesz - Fischer

$\sum_{n=1}^{\infty} C_n e_n$ is convergent in H .

$$\rightarrow P_E g = g$$

homework 10 2/8/2024

Recap: 1) weak derivatives.

2) Sobolev - Hilbert

3) general properties of Hilbert space.

Summarizing: Thm: Let $\{e_n\}_{n=1}^{\infty}$ $\frac{f_n}{\|f_n\|}$.

be an orthonormal set in H . form the basis.

→ a). $\{e_n\}_{n=1}^{\infty}$ form a basis in H .

→ b). $g = \underbrace{\sum_{n=1}^{\infty} (g, e_n) e_n}_{\text{generalized Fourier series}} \quad \forall g \in H$.

→ c). $\|g\|^2 = \sum_{n=1}^{\infty} |(g, e_n)|^2 \rightarrow \text{Bessel's equality}$.

→ d). $\{e_n\}_{n=1}^{\infty}$ is complete in H .

* Bounded Linear Operators.

→ e.g., derivatives, integrals, ...

$B(x, y) = \{T: X \rightarrow Y \text{ is linear } \|T\|_{x,y} < \infty\}$

where $\|T\|_{x,y} = \sup_{f \neq 0} \frac{\|Tf\|_Y}{\|f\|_X}$

\rightarrow on $[a, b]$

$$\text{let } Lu = a_2(x)u'' + a_1(x)u' + a_0(x)u.$$

a general second-order differential operator.

$$a_j(x) \in C([a, b]) \leftarrow \text{compact support.}$$

$$\& a_2(x) \neq 0.$$

Boundary condition.

$$B_1 u = C_1 u(a) + C_2 u'(a).$$

$$B_2 u = C_3 u(b) + C_4 u'(b).$$

$$\text{also. } |C_1| + |C_2| \neq 0, \quad |C_3| + |C_4| \neq 0$$

One can then write a general problem of the form, $Lu = f(x), \quad a < x < b$.

$$B_1 u = 0$$

$$B_2 u = 0$$

Intuition of the conjugate function $\bar{\psi}$.

$$\lambda_1 = a_1 + i b_1, \quad \lambda_2 = a_2 - i b_2$$

Adjoint problems

$$(Lu, \psi) = \int_a^b (a_2 \phi'' + a_1 \phi' + a_0 \phi) \bar{\psi} dx$$

$$\xrightarrow{\text{IBP}} = \int_a^b \phi ((a_2 \bar{\psi})'' - (a_1 \bar{\psi})' + a_0 \bar{\psi}) dx$$

then we can say,

$$(Lu, \psi) = (\phi, L^* \psi).$$

If we expand the forms,

$$L^* \psi = a_2 \psi'' + (2a_2' - a_1) \psi' + (a_2'' - a_1' + a_0) \psi.$$

$$\text{if } a_1 = a_2', \quad L^* \psi = L \psi.$$

$$\Rightarrow (Lu, \psi) = (\phi, L\psi).$$

"Self-adjoint."

e.g.

$$\int \frac{d^2 u}{dx^2} v dx$$

operator for 1D
heat eqn. $\frac{d^2 u}{dx^2}$

$$h =$$

$$\int u \frac{d^2 v}{dx^2} dx$$

Integrated by parts.

"no boundary terms"

$$(Lu, v) = (u, Lv).$$

$$(Lu, \psi) - (\phi, L^* \psi) = J(\phi, \psi) \Big|_a^b.$$

Some algebra, we can show:

$$J(\phi, \psi) = a_2 (\phi' \bar{\psi} - \phi \bar{\psi}') + (a_1 - a_2') \phi \bar{\psi}$$

if $J(\phi, \psi)|_a^b = 0$, then T is self-adjoint.

\Rightarrow here $L\phi = a_2(x)\phi'' + a_1(x)\phi' + a_0(x)\phi$.

in addition to (4), $a_2(x) < 0$.

A special case

$$P(x) = \exp \left(\int_a^x \frac{a_1(s)}{a_2(s)} ds \right).$$

\uparrow

a function of x .

$P(x) > 0$,
 $w(x) > 0$,
 $\in C([a, b])$.

$$w(x) = -\frac{P(x)}{a_2(x)}. \leftarrow \text{"weight function"}$$

$$q(x) = a_0(x) \cdot w(x)$$

then $L\phi = \lambda\phi \rightarrow$ is an eigenvalue problem.

$$-(P\phi)' + q_0\phi = \lambda w(x)\phi$$

\uparrow
functions. \uparrow

This ODE is called the
Sturm-Liouville Equation.

"the distinction of λ s are to be discussed"

Let $L_1\phi = (P\phi)' + q\phi$.

$$L\phi = \frac{L_1\phi}{w(x)}$$

Notice, for a test func. ψ .

$$(L_1\phi, \psi) = (\phi, L_1\psi)$$

$$\Rightarrow \int_a^b [(P\phi)' \psi + q\phi \psi] dx$$

$$\Rightarrow - \int_a^b \psi' (P\phi') dx + \int_a^b q\phi \psi dx$$

$$\Rightarrow - \int_a^b (\psi' P)' \phi dx + \int_a^b q\phi \psi dx$$

$$\Rightarrow \int [(\psi' P)' + q\psi] \phi dx$$

$$\Rightarrow (\phi, L\psi)$$

$$L\phi = \frac{h\phi}{w(x)},$$

→ claim is that

$$L\phi = \Delta\phi.$$

if & only if

$$L\phi = \Delta\phi.$$

In order to show that $L\phi$ is self-adjoint,

$$(\phi, \psi)_w = \int_a^b \phi(x) \overline{\psi(x)} w(x) dx$$

$$\|\phi\|_{hw}^2 = \left(\int_a^b |\phi(x)|^2 w(x) dx \right)^{1/2}$$

$$= \sqrt{(\phi, \phi)}$$

Problem Session #5

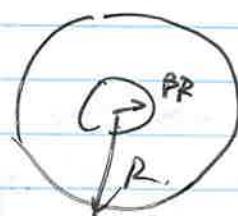
21/9/2024.

Unsteady Heat Conduction in an annular heat pipe.

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T = \alpha \Delta T.$$

↑
Laplacian operator.

inner radius βR . $0 < \beta < 1$.



outer radius R .

a). PDE in polar coordinates.

$$\frac{\partial T}{\partial t} = \alpha \left[\frac{1}{R} \cdot \frac{\partial}{\partial R} \left(R \frac{\partial T}{\partial R} \right) + \frac{1}{R^2} \cdot \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right]$$

axisymmetric, infinitely long

b).

$$\frac{\partial T}{\partial t} = \alpha \cdot \frac{1}{R} \cdot \frac{\partial}{\partial R} \left(R \frac{\partial T}{\partial R} \right).$$

"2nd-order in R , 1st-order in t !"

→ we need 1 T.C.s. & 2 B.C.s.

$$c). - \frac{\partial T}{\partial R} = \lambda q \quad \text{at } R = \beta R \quad q \neq 0 \rightarrow \text{dimensionless constant}$$

$$\frac{\partial T}{\partial R} = -h(T - T_{\infty}) \text{ at } R = R.$$

$$T = T_{\infty} \text{ at } t=0.$$

$R, t,$ → ind. var

$T.$ → dep. var

$h, T_{\infty}, R, q, \beta, \mathcal{R}, \alpha$ → Parameters

→ So we need to non-dimensionalize the system

$$\mathbb{H} = \frac{T - T_{\infty}}{qR}, \quad r^* = \frac{R}{\mathcal{R}}, \quad N = hR, \quad \tau = \frac{t}{t_c}$$

Dimensionless variables.

Dimensionless PDE.

$$\frac{1}{t_c} \frac{\partial \Theta}{\partial \tau} = \frac{\alpha}{\mathcal{R}} \frac{1}{r^*} \cdot \frac{\partial}{\partial r^*} \left(r^* \cdot \frac{\partial \mathbb{H}}{\partial r^*} \right)$$

$$\frac{1}{t_c} \sim \frac{\alpha}{R^2} \rightarrow t_c = \frac{R^2}{\alpha}$$

$$\boxed{T = \frac{\alpha \tau}{R^2}} \quad \text{"leaving the const = 1"}$$

the final dimensionless PDE looks like:

$$\frac{\partial \mathbb{H}}{\partial \tau} = \frac{1}{r^*} \frac{\partial}{\partial r^*} \left(r^* \frac{\partial \mathbb{H}}{\partial r^*} \right).$$

$$-\frac{\partial \mathbb{H}}{\partial r^*} = q_0, \quad r^* = \beta.$$

$$\frac{\partial \mathbb{H}}{\partial r^*} = -N(\mathbb{H}), \quad r^* = 1.$$

$$\mathbb{H} = 0, \quad \tau = 0$$

... # Sturm-Liouville problems require the B.C.s to be homogenized.

$$\rightarrow \text{decompose: } \mathbb{H} = \tilde{\mathbb{H}} + \mathbb{H}_{ss} \rightarrow \frac{\partial \mathbb{H}_{ss}}{\partial \tau} = 0.$$

$$\begin{aligned} & \uparrow \quad \uparrow \text{particular soln} \\ & \text{General soln} \quad ; \\ & \quad \quad \quad \text{analogy in ODE} \end{aligned}$$

$$0 = \frac{1}{r^*} \frac{d}{dr^*} \left(r^* \cdot \frac{d\mathbb{H}_{ss}}{dr^*} \right) \Rightarrow \mathbb{H}_{ss} = k_1 \ln(r^*) + k_2$$

$$\text{GOAL: } \frac{\partial \tilde{\mathbb{H}}}{\partial \tau} = \frac{1}{r^*} \cdot \frac{\partial}{\partial r^*} \left(r^* \cdot \frac{\partial \tilde{\mathbb{H}}}{\partial r^*} \right).$$

$$\frac{\partial \tilde{\mathbb{H}}}{\partial r^*} = 0, \quad \frac{\partial \tilde{\mathbb{H}}}{\partial r^*} = -N(\tilde{\mathbb{H}}), \quad \tilde{\mathbb{H}} = -\mathbb{H}_{ss}$$

$$\frac{d\langle H \rangle_{ss}}{dr^*} = -q, \quad \text{at } r^* = \beta.$$

$$\frac{d\langle H \rangle_{ss}}{dr^*} = -N\langle H \rangle_{ss}, \quad \text{at } r^* = 1.$$

After solving the constants:

$$\rightarrow \langle H \rangle_{ss} = \beta q \left[\frac{1}{N} - \ln(N) \right]$$

S.O.F.:

$$\tilde{\langle H \rangle}(r^*, \tau) = T(\tau) \cdot \tilde{X}(r^*).$$

$$\frac{T'}{T} = \frac{1}{\tilde{X}'} \cdot \frac{1}{r^*} \cdot \frac{\partial}{\partial r^*} \left(r^* \frac{\partial \tilde{X}}{\partial r^*} \right) = -\gamma^2$$

separation constant
definition

↓
the. Beuels equation

→ get the B.C.s terms:

$$\tilde{X}'(r^* = \beta) = 0.$$

$$\tilde{X}'(r^* = 1) = -N\tilde{X}. \quad \rightarrow \text{both are homogeneous}$$

B.C.s

is this eigenfunction universal?

$$X_m = A_m J_0(\gamma_m r^*) + B_m Y_0(\gamma_m r^*)$$

↑ the eigenfunctions.

→ eigenvalue condition

$$\text{at } r^* = \beta, \quad X_n' = 0.$$

$$\rightarrow A_m \gamma_m J_0'(\gamma_m \beta) + B_m \gamma_m Y_0'(\gamma_m \beta) = 0$$

$$B_m = -A_m \cdot \frac{J_0'(\gamma_m \beta)}{Y_0'(\gamma_m \beta)}.$$

$$X_m(r^*) = A_m \left[J_0(\gamma_m r^*) - \frac{J_0'(\gamma_m \beta)}{Y_0'(\gamma_m \beta)} \cdot Y_0(r^*) \right]$$

$$\frac{T'}{T} = -\gamma^2 \cdot \Rightarrow T(\tau) = \exp(-\gamma^2 \tau)$$

$$\tilde{\langle H \rangle}(r^*, \tau) = \sum_{n=1}^{\infty} A_n \exp(-\gamma_n^2 \tau) X_n(r^*)$$

→ 2nd boundary conditions:

$$X' = -N X, \quad \text{at } r^* = 1.$$

$$A_m \gamma_m J_0'(\gamma_m) + B_m \gamma_m Y_0'(\gamma_m) = -N [A_m J_0(\gamma_m) + B_m Y_0(\gamma_m)]$$

i.e., the eigenvalue condition

2

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -x.$$

$$\text{I.C. } u(x, 0) = f(x).$$

$$\left. \frac{dx}{dt} \right|_{\xi} = u.$$

$$\left. \frac{du}{dt} \right|_{\xi} = -x$$

$$\frac{d}{dt} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \rightarrow \vec{v} = \begin{bmatrix} x \\ u \end{bmatrix}.$$

Next step(s): find eigenvalues, eigenvectors, ...

$$\frac{d\vec{v}}{dt} = A\vec{v} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{v} \quad \lambda^2 + 1 = 0 \quad \lambda = \pm i$$

$$\lambda = i, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

$$\lambda = -i, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}. \quad \vec{v} = \begin{bmatrix} x \\ u \end{bmatrix}$$

$$C_1 \vec{v}_1 \exp(i\lambda t) + C_2 \vec{v}_2 \exp(-i\lambda t)$$

$$\begin{bmatrix} x \\ u \end{bmatrix} = C_1 \exp(it) \begin{bmatrix} 1 \\ i \end{bmatrix} + C_2 \exp(-it) \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Set up systems of ODEs

Subs. the I.C.s. $\rightarrow \begin{cases} u(\xi, 0) = f(\xi) \\ x(t=0) = \xi \end{cases}$

$$C_1 = \frac{\xi - if(\xi)}{2}, \quad C_2 = \frac{\xi + if(\xi)}{2}$$

$$\exp(it) = \cos(t) + i \sin(t).$$

$$\vec{v} = \begin{bmatrix} x \\ u \end{bmatrix} = (k_1 \cos(t) + k_2 \sin(t))$$

$$k_1 = \xi, \quad k_2 = -f(\xi)$$

Once we see the purely imaginary roots, the solution takes the form: $k_1 \cos(t) + k_2 \sin(t)$

$$x = \xi \cos(t) - f(\xi) \sin(t)$$

$$u = -k_1 \sin(t) + k_2 \cos(t)$$

$$= -\xi \sin(t) - f(\xi) \cos(t)$$

Shocks & expansion fan.

$$3. \quad \frac{\partial u}{\partial t} + u^2 \cdot \frac{\partial u}{\partial x} = 0.$$

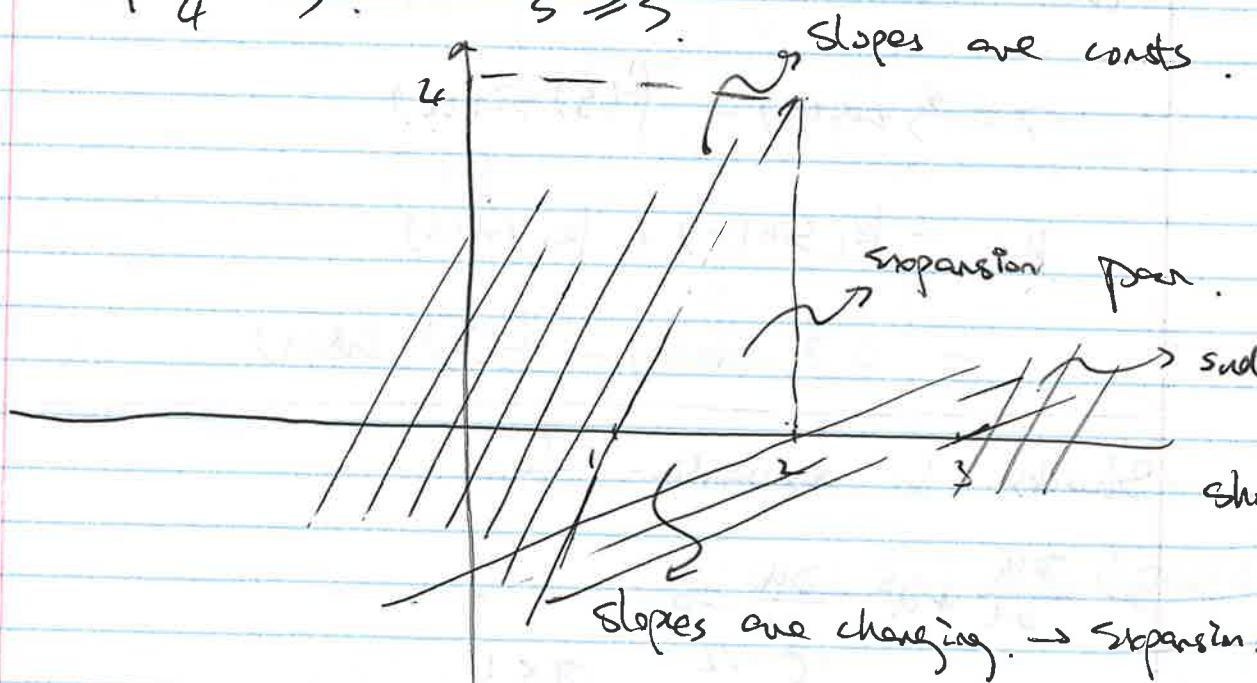
$$\rightarrow u(x, 0) = \begin{cases} \sqrt{x}, & x < 1 \\ 1, & 1 \leq x \leq 2 \\ 2, & 2 < x < 3 \\ 3, & x > 3 \end{cases}$$

$$\frac{du}{dt} \Big|_{\xi} = 0 \rightarrow u = F(\xi) = \begin{cases} 1/2, & \xi < 1 \\ \xi/2, & 1 \leq \xi \leq 2 \\ 1, & 2 < \xi < 3 \\ 1/2, & \xi \geq 3 \end{cases}$$

$$\frac{dx}{dt} = u^2 = (F(\xi))^2.$$

$$x = [F(\xi)]^2 t + \xi.$$

$$\begin{cases} \frac{1}{4}t + \xi, & \xi < 1 \\ (\frac{\xi}{2})^2 t + \xi, & 1 \leq \xi \leq 2 \\ t + \xi, & 2 < \xi < 3 \\ \frac{t}{4} + \xi, & \xi \geq 3 \end{cases}$$



Shock location & time: $(x_{\text{shock}}, t_{\text{shock}}) = (3, 0)$

Equation of shock \rightarrow find shock speed
convert PDE to conservative form.



$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left(\frac{u^3}{3} \right) = 0$$

$$F = \frac{u^3}{3}$$

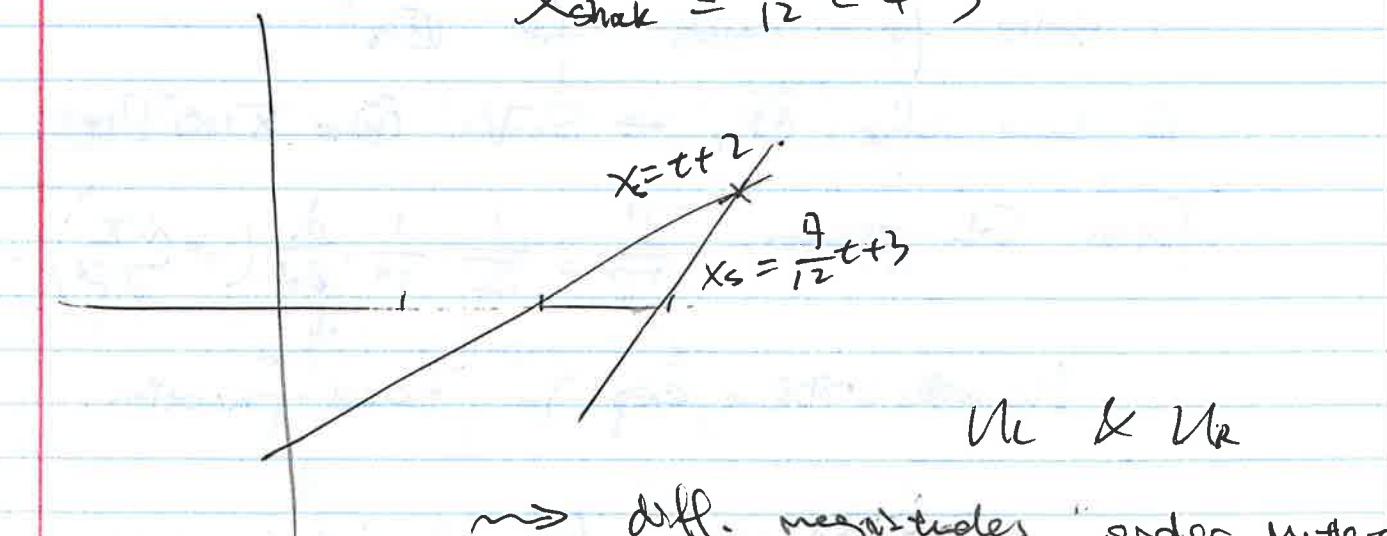
$$x_{\text{shock}} = \frac{u_r^3 - u_l^3}{3(u_r - u_l)}$$

$$u_l(\xi=3^-) = 1.$$

$$u_r(\xi=3^+) = 1/2$$

$$x_{\text{shock}} = \frac{7}{12} t + 3$$

$$x_{\text{shock}} = \frac{7}{12} t + 3$$



\Rightarrow diff. magnitudes, order matters

If $u_L > u_R \rightarrow$ expansion fan.

$u_L < u_R \rightarrow$ shock.

#Problem Session 5

Problem 1 (d).

$$\Theta = \tilde{\Theta} + \Theta_{ss}$$

↓ ↓

Unsteady. Steady-state

for steady-state: $\frac{d\Theta_{ss}}{dt} = 0$

Recall the dimensionless PDE:

$$\frac{1}{t^*} \frac{\partial \Theta_{ss}}{\partial t^*} = \frac{\alpha}{R^2} \frac{1}{r^*} \frac{\partial}{\partial r^*} \left(r^* \cdot \frac{\partial \Theta_{ss}}{\partial r^*} \right).$$

!!

$\frac{d\Theta_{ss}}{dr^*} = 0 \rightarrow$ the steady state form is
inhomogeneous \rightarrow Form SL problem.

Second-order ODE for Θ_{ss} : $k_1 I_n(r^*) + k_2$.

\rightarrow solve for consts. for Θ_{ss} .

To solve for $\tilde{\Theta}$, \rightarrow SoI: $\tilde{\Theta} = \bar{X}(r^*) T(t)$.

Form SL problem: $\frac{T'}{T} = \frac{1}{\bar{X}} \cdot \frac{1}{r^*} \cdot \frac{d}{dr^*} \left(r^* \frac{d\bar{X}}{dr^*} \right) = -\gamma^2$

1st order ODE \rightarrow exp(.) Bessel function.

Recall general form in Leib's notes:

$$R(r) = C_m J_m(\gamma r) + G_m Y_m(\gamma r),$$

Practice Problems: \rightarrow Shock & expansion fan.

$$\text{PDE: } \frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} = 0 \Rightarrow -\infty < x < \infty, 0 \leq t < \infty$$

$$\text{I.C.s: } u(x, t=0) = f(x) = \begin{cases} 1/2, & x < 1 \\ x/2, & 1 \leq x \leq 2 \\ 1, & 2 < x < 3 \\ 1/2, & x > 3. \end{cases}$$

\rightarrow the PDE in flux form: $\frac{\partial u}{\partial t} + \frac{\partial (\frac{1}{3}u^3)}{\partial x} = 0$
(conservative)

(a). \rightarrow the characteristic line: $\frac{\partial x}{\partial t} \Big|_{\xi} = u^2$.

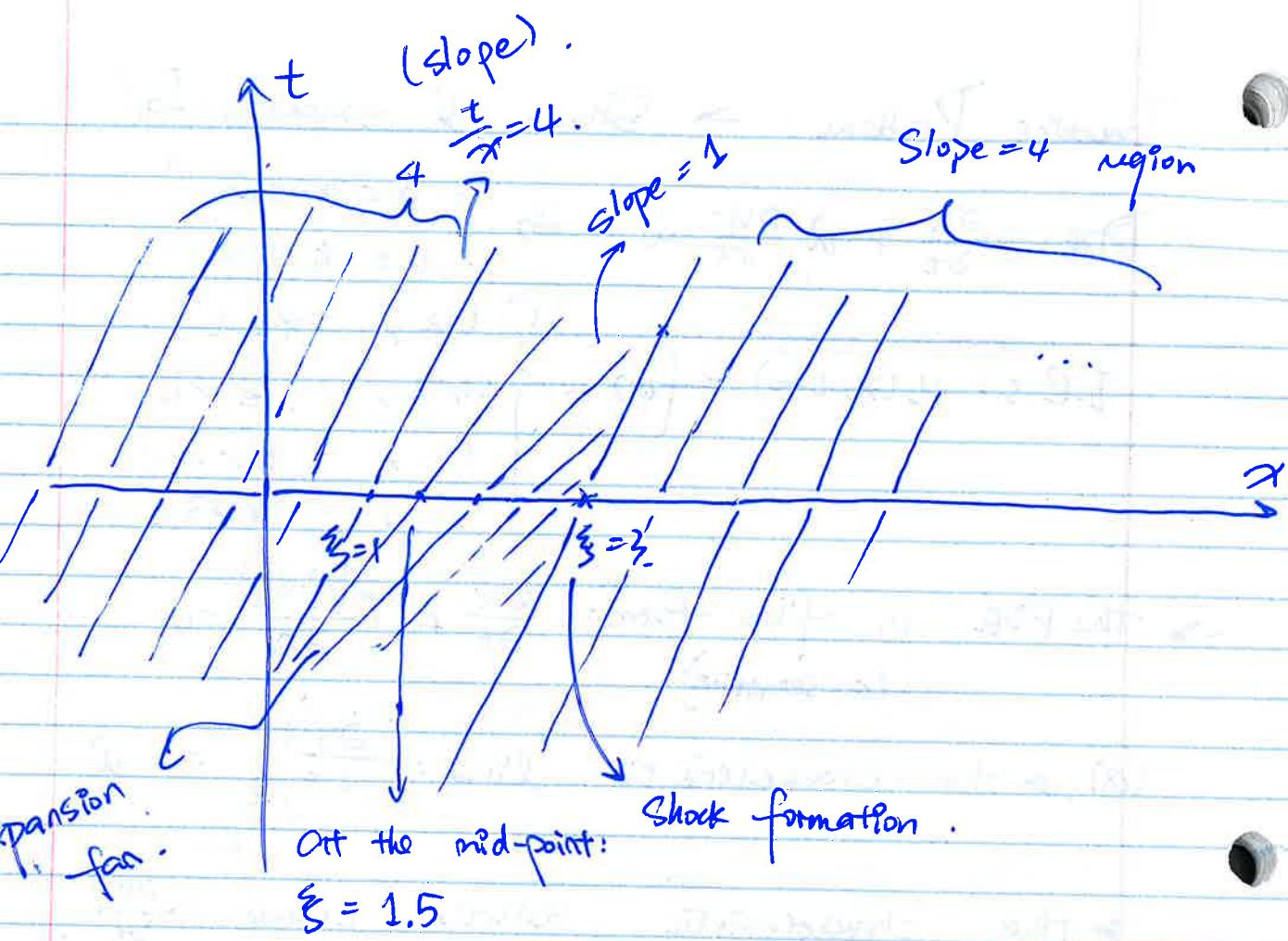
\rightarrow the characteristic solution curve: $\frac{\partial u}{\partial t} \Big|_{\xi} = 0$

$\rightarrow x = u^2 t + \xi \rightarrow \xi = x - u^2(\xi) t$.
 $u = F(\xi)$.

Applying the ICs: $F(\xi) = \begin{cases} 1/2, & \xi < 1 \\ \xi/2, & 1 \leq \xi < 2 \\ 1, & 2 < \xi < 3 \\ 1/2, & \xi > 3. \end{cases}$

$$\rightarrow t = \frac{x - \xi}{[F(\xi)]^2} = \begin{cases} 4(x - \xi), & \xi < 1/2 \\ \frac{4x}{\xi^2} - \frac{4}{\xi}, & 1 \leq \xi < 2 \\ x - \xi, & 2 < \xi < 3 \\ 4(x - \xi), & \xi > 3. \end{cases} \dots (*)$$

Based on (*), we can sketch the characteristics,



(c) Determine time & location for $t_{\text{shock/fan}}$.

We may begin with deriving the formula for the position of the expansion fan.

$$\text{Recall: } t = \frac{4x}{\xi^2} - \frac{4}{\xi}. \rightarrow \xi \in [1, 2].$$

$$x = \frac{\xi^2}{4}t + \xi.$$

$$\frac{\xi^2}{4}t + \xi = \frac{7}{12}t + 3 \quad \text{can solve for position.}$$

$$\left(\frac{\xi^2}{4} - \frac{7}{12}\right)t = 3 - \xi.$$

$$t = \frac{3 - \xi}{\frac{\xi^2}{4} - \frac{7}{12}} = \frac{36 - 12\xi}{3\xi^2 - 7}.$$

$$\text{Recall: } \xi = x - \frac{\xi^2}{4}t.$$

$$\frac{t}{4}\xi^2 + \xi - x = 0.$$

$$\xi = \frac{-1 \pm \sqrt{1+t}}{\frac{t}{2}}.$$

$$= \frac{1}{t}(-2 \pm 2\sqrt{1+t}).$$

$$\xi^2 = \frac{1}{t^2} [4 \pm 8\sqrt{1+t} + 4(1+t)] = \frac{1}{t^2} [8 + 4t \pm 8\sqrt{1+t}]$$

b) Determine the shock position.

Recall the equation for calculating the shock speed:

$$\dot{x}_s = \frac{F_L - F_R}{U_L - U_R} = \frac{\frac{1}{3}U_L - \frac{1}{3}U_R}{U_L - U_R}$$

Since shock is being formed at $\xi = 3$.

$$\dot{x}_s = \frac{\frac{1}{3} - \frac{1}{3} \cdot \frac{1}{8}}{\frac{1}{2}} = \frac{7}{24} \cdot 2 = \frac{7}{12}.$$

$$x_s = \frac{7}{12}t + 3.$$

From the sketch, one can tell that the expansion wave hits the shock at $\xi = 2$.

$$t+2 = \frac{7}{12}t + 3$$

$$\frac{5}{12}t = 1$$

$$t = \frac{12}{5} \rightarrow x = \frac{12}{5} + 2 = \frac{22}{5}$$

$$\xi = t + 2$$

$$x = t + 2$$

(d). determine the solution of the expansion fan.

Recall the characteristic for the expansion fan:

$$t = \frac{4x}{\xi^2} - \frac{4}{\xi} \rightarrow \xi^2 t + 4\xi = 4x \quad \hookrightarrow \xi = \frac{-4 \pm \sqrt{16 + 16xt}}{2t}$$

$$\rightarrow \text{solution: } u = F(\xi) = \frac{\xi}{2}$$

$$\hookrightarrow u = \frac{-1 \pm \sqrt{1+xt}}{t}$$

Since ξ is defined in the region $[1, 2]$

$$\rightarrow \xi > 0 \rightarrow u = \frac{-1 + \sqrt{1+xt}}{t}$$

e). determine the expression for the shock speed after it hits the fan.

After the expansion fan hits the shock, u_L is the expression we derived, $u_L = \frac{1}{2}$.

$$\dot{\xi}_s = \frac{F_L - F_R}{u_L - u_R} = \frac{\frac{1}{3} \left(\frac{-1 + \sqrt{1+xt}}{t} \right)^3 - \frac{1}{3} \left(\frac{1}{2} \right)^3}{\sim}$$

4. when forming the matrix-vector ODE:

1). original equation

2). consistency condition.

$$\vec{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

$$\frac{\partial \phi_1}{\partial t} + 2U \frac{\partial \phi_1}{\partial x} + (U^2 - c^2) \frac{\partial \phi_2}{\partial x} = 0. \quad \left\{ \begin{array}{l} \phi_1 = \frac{\partial P}{\partial t} \\ \phi_2 = \frac{\partial P}{\partial x} \end{array} \right.$$

$$\frac{\partial \phi_2}{\partial t} - \frac{\partial \phi_1}{\partial x} = 0$$

3) separate the " ∂_t " and " ∂_x ".

4) → focus on the ∂_x part.

$$\frac{\partial}{\partial x} \begin{bmatrix} 2U & U^2 - c^2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \rightarrow \text{original system.}$$

5) the two rows of the matrix stand for the two equations

Practice Midterm 2019-2020.

$$\frac{\partial u}{\partial t} + \left(1 - \frac{u}{2} + u^2 \right) \frac{\partial u}{\partial x} = 0 \quad \left\{ \begin{array}{l} -\infty < x < \infty \\ 0 \leq t < \infty \end{array} \right.$$

$$u(x, 0) = F(x) = \begin{cases} 2, & -\infty < x < 0 \\ 2-x, & 0 \leq x < 1 \\ 1, & 1 \leq x < \infty \end{cases}$$

To determine the characteristics, \Rightarrow equation of characteristic
and the characteristic solution curves:

$$\frac{\partial x}{\partial t} \Big|_{\xi} = 1 - \frac{u}{2} + u^2 \rightarrow \left[1 - \frac{u}{2} + u^2 \right] t + \xi = x.$$

$$\frac{\partial u}{\partial t} \Big|_{\xi} = 0 \rightarrow u = F(\xi). \quad \dots (*)$$

Recall the initial condition:

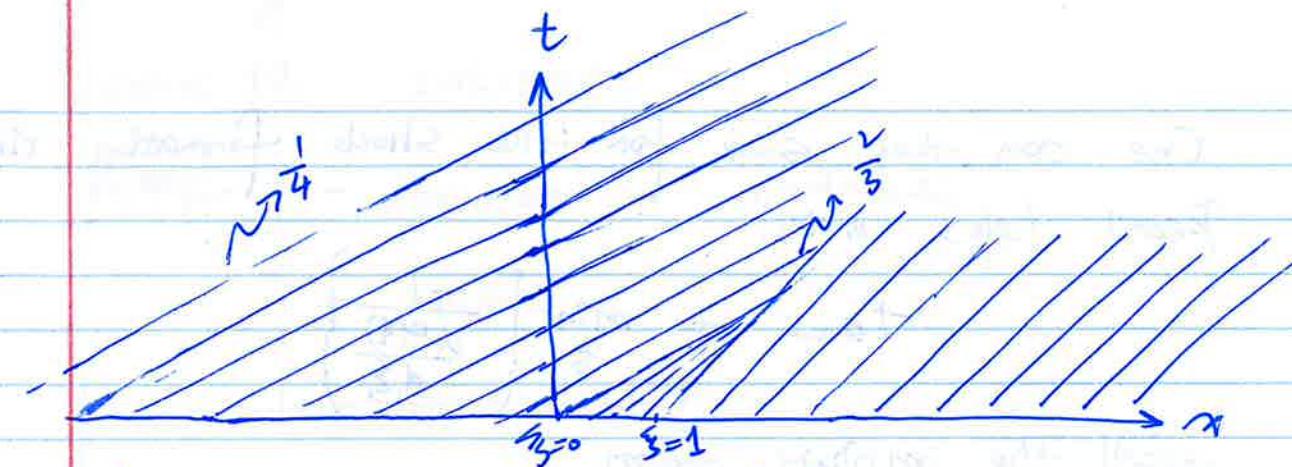
$$F(\xi) = \begin{cases} 2, & -\infty < \xi < 0, \\ 2-\xi, & 0 \leq \xi < 1, \\ 1, & 1 \leq \xi < \infty \end{cases}$$

modifying eqn. (*).

$$t = \frac{x-\xi}{1 - \frac{u}{2} + u^2} = \frac{x-\xi}{1 - \frac{F(\xi)}{2} + [F(\xi)]^2}$$

$$t = \begin{cases} \frac{x-\xi}{4}, & -\infty < \xi < 0, \\ \frac{x-\xi}{\xi^2 - \frac{3}{2}\xi + 4}, & 0 \leq \xi < 1 \\ \frac{x-\xi}{3/2}, & 1 \leq \xi < \infty \end{cases}$$

We can then sketch the characteristics



2. the solution contains a shock.

3. to find the explicit solution, recall the equation of the characteristics.

$$\xi = x - \left[1 - \frac{u}{2} + u^2 \right] t \cdot = x - t + \frac{ut}{2} - u^2 t.$$

$$u = F(\xi) = F(x - t + \frac{ut}{2} - u^2 t).$$

Substitute into the shock region:

$$\xi=0: \xi = x - t + t - 4t = x - 4t$$

$$\xi=1: \xi = x - t + \frac{t}{2} - t = x - \frac{3}{2}t. \quad \text{transition region!}$$

$$2-u = x - t + \frac{ut}{2} - u^2 t$$

$$+tu^2 - (1+\frac{t}{2})u + 2 - x + t = 0$$

$$u = \frac{(1+\frac{t}{2}) \pm \sqrt{(1+\frac{t}{2})^2 - 4t(2-x+t)}}{2t}$$

combine $u = 2 - \xi$

One can then solve for the shock formation time.

Recall Lele's notes,

$$t_{\text{shock}} = \min_{\xi} \left\{ \frac{-1}{\frac{dF(\xi)}{d\xi}} \right\}$$

recall the implicit form,

$$u = F(x - t + \frac{ut}{2} - u^2)$$

$$\frac{\partial u}{\partial x} = \left[1 - t(2u - \frac{1}{2}) \frac{\partial u}{\partial x} \right] F'(\xi).$$

$$\frac{\partial u}{\partial x} = F'(\xi) - t(2u - \frac{1}{2}) \frac{\partial u}{\partial x} \cdot F'(\xi).$$

$$F'(\xi) = \frac{\frac{\partial u}{\partial x}}{1 - t(2u - \frac{1}{2}) \frac{\partial u}{\partial x}}$$

$$t = \min \left\{ \frac{\frac{t(2u - \frac{1}{2}) \frac{\partial u}{\partial x} - 1}{\frac{\partial u}{\partial x}}}{1 - t(2u - \frac{1}{2}) \frac{\partial u}{\partial x}} \right\}.$$

$$t(2u - \frac{1}{2}) - \frac{1}{\frac{\partial u}{\partial x}}$$

Problem 2.

$$u(x, 0) = G(x) = \begin{cases} 1, & -\infty < x < 0. \\ 2, & 0 \leq x < \infty \end{cases}$$

$$F(\xi) = \begin{cases} 1, & -\infty < \xi < 0. \\ 2, & 0 \leq \xi < \infty \end{cases}$$

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↑

Lecture 12. 2/15/2024.

Recap

- Operators \rightarrow differential.
- Adjoints.
- Sturm-Liouville

$$\text{Setting } L_0 \phi = a_2(x) \phi'' + a_1(x) \phi' + a_0(x) \phi, \quad \forall x \in [a, b]$$

$$a_j \in C([a, b]) \quad \& \quad a_2 < 0.$$

$$P(x) = \exp \left(\int_a^x \frac{a_1(s)}{a_2(s)} ds \right).$$

$$W(x) = \frac{-P''}{a_2(x)}$$

$$Q(x) = a_0(x) W(x)$$

then $L_0 \phi = \lambda \phi$ is equivalent to

$$-(P\phi)' + Q\phi = \lambda W\phi.$$

$L_0 \phi$ is self-adjoint

$$(\psi, L_0 \phi) = (\phi, L_0 \psi)$$

$$(f, g) = \int_a^b f \tilde{g} dx$$

$$\text{let } h\phi = \frac{L\phi}{w(x)}$$

$$\Rightarrow h\phi = -\frac{(P\phi)'}{w} + \frac{q\phi}{w}$$

$h\phi = \gamma\phi \leftarrow \text{some eigenvalue problem.}$

...this is not self-adjoint

Recall: A new weighted space.

$$L^2_w(a,b) = \left\{ \phi : \|\phi\|_w = \left(\int_a^b |\phi(x)|^2 w(x) dx \right)^{1/2} < \infty \right\}$$

"bounded"

$$(\phi, \psi)_w = \int_a^b \phi(x) \bar{\psi}(x) w(x) dx$$

$$(h\phi, \psi)_w = (\phi, h\psi)_w$$

in our setting, boundary condns.

$$B_1\phi = C_1\phi(a) + C_2\phi'(a) = 0$$

$$B_2\phi = C_3\phi(b) + C_4\phi'(b) = 0$$

$$|C_1| + |C_2| \neq 0 \quad \& \quad |C_3| + |C_4| \neq 0$$

For such B.C.s. \rightarrow char. along B.C.s self-adjoint

for self-adjoint.



$$J(\phi, \psi) \Big|_a^b = p(x)(\phi' \bar{\psi} - \phi \bar{\psi}') \Big|_a^b =$$

if ψ also has the same B.C.s as ϕ .

$$\text{then } J(\phi, \psi) \Big|_a^b = 0$$

$T = \{h, B_1, B_2, L\}$ is self-adjoint.

Theorem:

- $h\phi = \gamma\phi$ is equivalent to $h\phi = \gamma\phi$.
- $T = \{h, B_1, B_2\}$ is self-adjoint.
- T has a countable sequence of real distinct eigenvalues. $\{\lambda_n\}_{n=1}^{\infty}$ with $\lambda_n \rightarrow \infty$.
 i) the corresponding eigenfunctions $\{\phi_n\}$ may be chosen to form a basis of $L^2_w(a,b)$
- These eigenfunctions are orthogonal in the weighted space, $L^2_w(a,b)$, s.t.

$$\int_a^b \frac{\phi_n \bar{\phi}_m w(x) dx}{\int_a^b \phi_n \bar{\phi}_n w(x) dx} = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases} \rightarrow \delta_{nm}$$

↑ normalization const.

f). Any $f \in L^2(a,b)$ can be written as

$$f = \sum_{n=1}^{\infty} C_n \phi_n.$$

$$\& C_n = (f, \phi_n)_W.$$

* Normalization of Orthogonal Eigenfunctions.

$\{\phi_n\}_{n=1}^{\infty}$ are not orthonormal.

they can be normalized using the inner

$$\text{product: e.g., } \gamma_n = \frac{\phi_n}{(\phi_n, \phi_n)_W}$$

\Rightarrow handling Periodic B.C.s.

$$\phi(a) = \phi(b) \& \phi'(a) = \phi'(b)$$

$$\equiv \phi(a) - \phi(b) = 0 \& \phi'(a) - \phi'(b) = 0$$

$\underbrace{\qquad\qquad\qquad}_{\downarrow}$
not separable !!!

\rightarrow admits $\lambda=0$ and $\phi=1$

Therefore, $J_W(\phi, \eta) = 0$ still holds

$$\{L, P_1, P_2\} \text{ is still self-adjoint}$$

\downarrow

$PBC_1 \quad PBC_2$

* Additionally, eigenvalues are not simple.

i.e., for each eigenvalue, there are two eigenfunctions.

Consider the heat equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$\text{B.C.s: } u(0,t) = u(1,t) = 0, \quad \forall t > 0.$$

$$\text{I.C.s: } u(x,0) = f(x), \quad 0 < x < 1$$

the solution u_k writes.

$$u_k = C_k e^{-k^2 \pi^2 t} \sin(k\pi x)$$

the general sol'n:

$$u = \sum_{k=1}^{\infty} A_k u_k(x,t). \quad \leftarrow \text{generalized Fc series}$$

1). as long as $f \in L^2(0,1)$.

2). $\{\sin(k\pi x)\}_{k=1}^{\infty}$ are orthogonal

3). $u(x, 0) = f(x)$.

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

just the Fourier series

$$A_n = \frac{(f, \phi_n)}{(\phi_n, \phi_n)}$$

projections along each basis

$$\phi_n = \sin(n\pi x)$$

i.e., normalization.

$$A_n = \frac{\int_0^1 f \sin(n\pi x) dx}{\int_0^1 \sin(n\pi x) \sin(n\pi x) dx} \quad \text{weight = 1.}$$

Corresponding ODE:

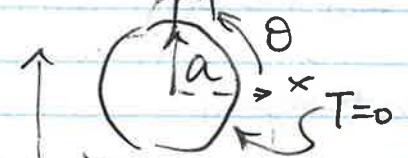
$$X'' + \gamma X = 0. \quad \leftarrow \text{harmonic oscillator}$$

$$X(0) = 0$$

i.e., Sturm-Liouville

$$X(1) = 0 \quad \text{problem.}$$

Consider: a heated cylindrical rod



$$\frac{\partial T}{\partial t} = \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial r^2} \right) \quad \text{non-dimensional.}$$

$$x = a \cos \theta, \quad y = a \sin \theta$$

Using chain rule,

$$\frac{\partial T}{\partial t} = \frac{1}{r} \cdot \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right). \quad \leftarrow \text{modified PDE.}$$

$$\text{I.C.s } T(r, 0) = f(r).$$

$$T(a, t) = 0$$

$$T(0, t) = \text{finite.}$$

Using the Sturm ansatz, we have

$$T(r, t) = R(r), T(t)$$

Plugging in the solution Ansatz:

$$\frac{1}{R(r)} \frac{1}{r} \cdot \frac{d}{dr} \left(r \frac{dR(r)}{dr} \right) = \frac{1}{T(t)} \frac{dT(t)}{dt} = -\gamma^2$$

$\rightarrow R(r)$ equation.

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dR}{dr} \right) = -\gamma^2 R(r).$$

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \cdot \frac{dR}{dr} = -\gamma^2 R(r)$$

$$\text{B.C.s: } T(a, t) = R(a)T(t) = 0$$

$$\Rightarrow R(a) = 0$$

$$\text{Multiplying by } r^2: r^2 R''(r) + r R'(r) + r^2 \gamma^2 R(r) = 0$$

↑
separation
const.

→ General Bessel Eqn.:

$$x^2 \phi'' + x \phi' + (x^2 - \nu^2) \phi = 0$$

of order 2).

$$\phi(x) = C_1 J_\nu(x) + C_2 Y_\nu(x)$$

\uparrow
Bessel's func
of 1st kind
 \uparrow
Bessel's func
of 2nd kind.

- Problem Session 6 ——————

Sturm-Liouville Problem

$$\frac{d}{dx} \left[P(x) y' \right] + Q(x) y = \lambda R(x) y.$$

$$\frac{d}{dx} \left[\frac{y'}{R(x)} \right] = \lambda y.$$

new operator $\left(\frac{d}{dx} \frac{1}{R(x)} \right)' = \lambda$.

$$\frac{d}{dx} \left(\frac{y'}{R(x)} \right) = \lambda y.$$

P, Q, R are real-valued, continuous func

$$P(x) > 0, R(x) > 0, \quad x \in [a, b]$$

→ finite domain

B.C.s. $C_1 y(a) + C_2 y'(a) = 0. \quad (B.C.1)$

$$C_3 y(b) + C_4 y'(b) = 0. \quad (B.C.2)$$

(1). All eigenvalues are real, distinct, & can be ordered.

$$\lambda_1 < \lambda_2 < \dots < \infty.$$

(2) the eigenfunctions corresponding to distinct eigenvalues are orthogonal to each other.

$$\lambda_m \rightarrow \phi_m, \Rightarrow L_1(\phi_m) = \lambda_m \phi_m.$$

$$\lambda_n \rightarrow \phi_n, \quad \langle \phi_m, \phi_n \rangle = \int_a^b P(x) \phi_m(x) \phi_n(x) dx = N_m \delta_{mn}.$$

$$N_m = \langle \phi_m, \phi_m \rangle.$$

(3). Any function f_m in L_r^2 and satisfy B.C.s:

$$f_m(x) = \sum_{m=1}^{\infty} C_m \phi_m(x).$$

where $C_m = \frac{\langle f, \phi_m \rangle}{\langle \phi_m, \phi_m \rangle}$. $f_m(x) \in L_r^2$ lies in L_r^2 space

$$\|f\|_r^2 = \left(\int_a^b |f(x)|^r dx \right)^{1/r}$$

$$\{f : \left(\int_a^b |f(x)|^r dx \right)^{1/r} < \infty\}$$

Singular St. problem.

P(a) and/or M(a) \Rightarrow \rightarrow a singular point

P(b) and/or M(b) \Rightarrow \rightarrow b singular point.

Let's say a is a singular pt.

1). B.C.s on \rightarrow .

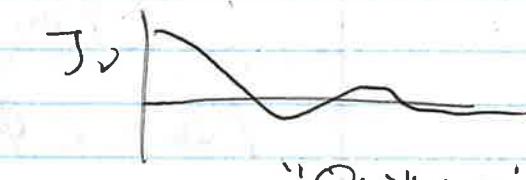
2). regularity requirement on a .

$$\text{i.e., } |y(a)| < \infty.$$

Example on Singular Sturm-Liouville Problem.

Bessel's function

$$t^2 y'' + t y' + (t^2 - v^2) y = 0$$



2 Ind. solns:

1) $J_v \Rightarrow$ Bessel functions of first kind.
of order v .

2) $Y_v \Rightarrow$ Bessel functions of second kind
of order v .

$$y = C_1 J_v(t) + C_2 Y_v(t)$$

Modified Bessel Equation

$$t^2 y'' + t y' - (t^2 + v^2) y = 0.$$

① $I_v(x) = i^{-v} J_v(ix)$. \rightarrow exponentially increasing
② well-behaved @ 0.

② $K_v = \frac{\pi}{2} \left\{ \frac{I_{-v}(x) - I_v(x)}{\sin(\pi v)} \right\}$. \rightarrow exponentially decaying.
 \rightarrow singular at $t=0$.

$$y = C_1 I_v(t) + C_2 K_v(t).$$

\rightarrow Not Oscillatory in nature.

Real Sturm-Liouville Problem.

$$-(p y')' + q(x) y = \lambda m(x) y \quad (**)$$

$$t^2 y'' + t y' + (t^2 - v^2) y = \lambda \quad \text{Let } t = \sqrt{x}.$$

$$\Rightarrow x^2 y'' + x \frac{dy}{dx} + (\lambda x^2 - v^2) y = 0$$

$$\frac{d^2 y}{dx^2} \Rightarrow x \frac{d^2 y}{dx^2} + \frac{dy}{dx} + (\lambda x + \frac{v^2}{x}) y = 0$$

$$\Rightarrow d(x \cdot \frac{dy}{dx}) - \frac{v^2}{x} y = -\lambda x y.$$

$$\Rightarrow -d\left(\pi \frac{dy}{dx}\right) + \frac{v^2}{x} y = \lambda x y. (*)$$

Compare eqn (**) & (*):

$$p(x) = x, \quad q(x) = \frac{v^2}{x}, \quad m(x) = \lambda$$

domain $\Rightarrow x \in [0, 1]$.

impose B.C.s on the domain.

$$\textcircled{1}. \quad y(1) = 0$$

\textcircled{2}. y is well-behaved at $x=0$.

In general, the solution writes:

$$y(x) = C_1 J_0(\sqrt{\lambda}x) + C_2 Y_0(\sqrt{\lambda}x)$$

$$D=0.$$

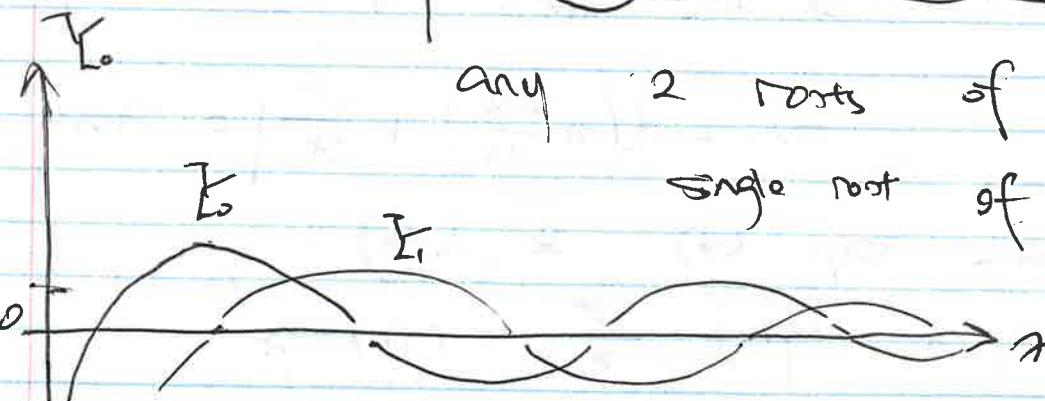
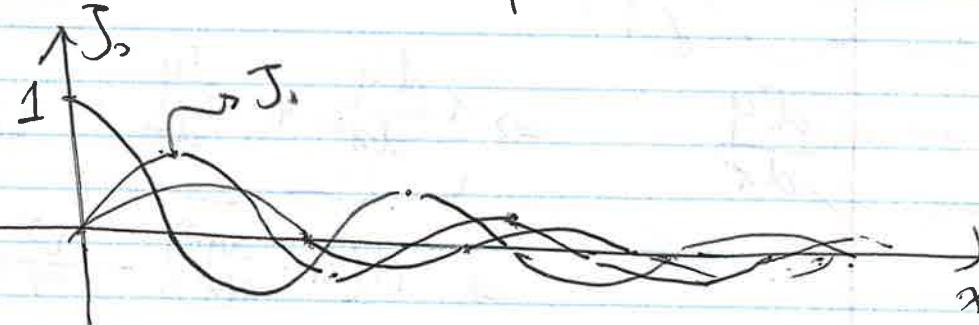
$$\Rightarrow y(x) = C_1 J_0(\sqrt{\lambda}x) + C_2 Y_0(\sqrt{\lambda}x)$$

$$y(0) = C_1 J_0(0) + C_2 Y_0(0) \Rightarrow C_2 = 0$$

$$y(1) = C_1 J_0(\sqrt{\lambda}) + C_2 Y_0(\sqrt{\lambda}) = 0$$

$$y(1) = C_1 J_0(\sqrt{\lambda}) = 0$$

In general,



$$J_0(\sqrt{\lambda}) = 0$$

$$\sqrt{\lambda_n} = j_{0,n}$$

$$\lambda_n = j_{0,n}^2$$

$$\phi_n = J_0(j_{0,n} x)$$

$$\int_0^1 x J_0(j_{0,n} x) J_0(j_{0,m} x) dx = 0$$

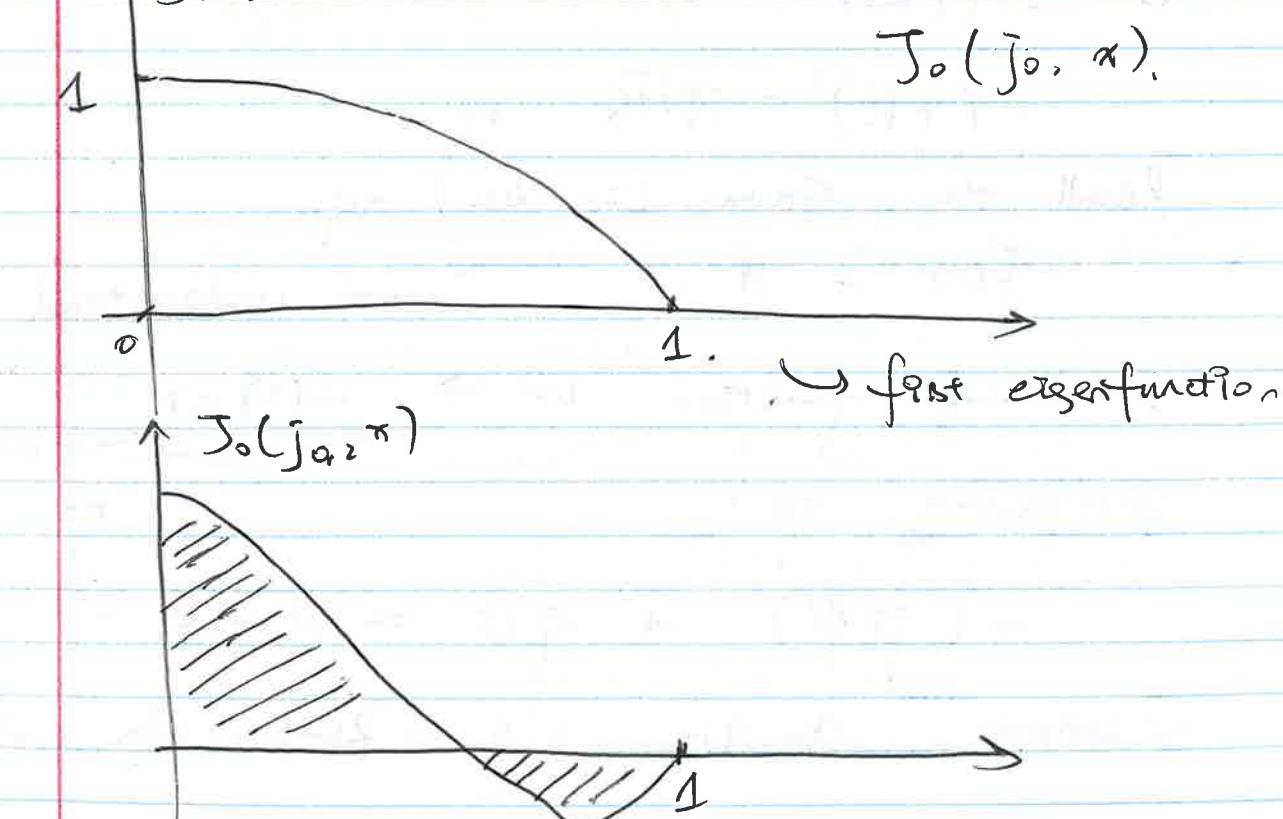
if $n \neq m$.

"orthogonality property"

$$\text{Final soln: } y = \sum_{n=1}^{\infty} C_n \phi_n = \sum_{n=1}^{\infty} C_n J_0(j_{0,n} x)$$

$$J_0(j_{0,1} x)$$

$$J_0(j_{0,1} x)$$



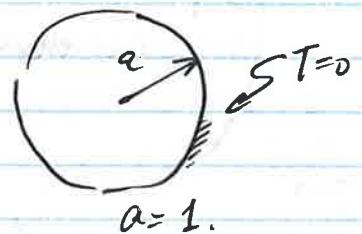
Solution is the linear combination of the eigenfunctions (orthogonal).

Lecture 13. 2/20/2024.

Recap: Formulate S.L. properties.

↳ 1D heat equation.

↳ 2D heat equation.



$$\text{Solving } \frac{\partial T}{\partial t} = \frac{1}{r} \cdot \frac{\partial}{\partial r} \left(r \cdot \frac{\partial T}{\partial r} \right).$$

↑ "non-dimensionalized"

$$\text{For space. } rR'' + rR' + \gamma^2 r^2 R \quad (\star)$$

$$-rR'' - R' - \gamma^2 r^2 R = 0 \quad \downarrow \text{"dividing by } r\text{"}$$

$$\Rightarrow - (rR')' - \gamma^2 r^2 R = 0.$$

$$- (rR')' = \gamma^2 r^2 R.$$

Recall the Sturm-Liouville eqn.
 $\psi(r) = r$.

Weighting function $w \rightarrow w(r) = r$ \downarrow
 \rightarrow understand singular behavior
 \rightarrow orthogonality

Comparing to

$$-(P\phi)' + q\phi = \gamma w\phi$$

Substitute $\chi = \gamma r$, & $R(r) = \Phi(\gamma r)$.

$$\text{then } \frac{dR}{dr} = \gamma \frac{d\Phi}{dr}$$

$$\frac{d^2R}{dr^2} = \gamma^2 \frac{d^2\Phi}{dr^2}.$$

$$\text{then. } r^2 \gamma^2 \Phi'' + r \gamma \Phi' + \gamma^2 r^2 \Phi = 0.$$

$$x^2 \Phi'' + x \Phi' + \gamma^2 \Phi = 0.$$

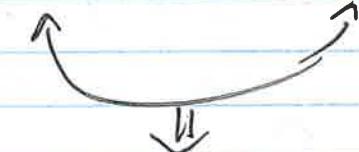
Bessel equation of 0^{th} order.

General form:

$$x^2 \Phi'' + x \Phi' + (\gamma^2 - v^2) \Phi = 0$$

the soln to this eqn:

$$\Phi(x) = C_1 J_v(\gamma x) + C_2 Y_v(\gamma x)$$



Bessel function of the "2nd" order.

Note: for our problem, $x = \gamma r$, $v = 0$.

Soln: $\Rightarrow C_1 J_0(\gamma r) + C_2 Y_0(\gamma r)$.

Recall the cond.: $T(r=0, t) = \text{finite}$.

$Y_0 \rightarrow -\infty$ as $r \rightarrow 0$.

Note: we can handle only one inhomogeneity.

therefore, the soln writes:

$$R(r) = C_1 J_0(\lambda r).$$

$$R(1) = C_1 J_0(\lambda) = 0. \quad \leftarrow \text{B.C.}$$

either $C_1 = 0$ or $J_0(\lambda) = 0.$

$$\Rightarrow J_0(\lambda) = 0.$$

$\lambda = Z^n \rightsquigarrow$ zeroes of the Bessel functions.

The eigenfunctions.

$$\{J_0(Z_n r)\}_{n=1}^{\infty}$$

* Orthogonality

$$\int_{S^2} J_0(Z_m r) J_0(Z_n r) r d\Omega = N_n \delta_{nm}.$$

$$\rightsquigarrow N_n = \int_{S^2} J_0(Z_n r) J_0(Z_n r) r d\Omega$$

General soln to this problem:

$$\text{Now, } T(r, t) = \sum_{n=1}^{\infty} C_n J_0(Z_n r) e^{-Z_n^2 t}$$

$$T(r, t=0) = \sum_{n=1}^{\infty} C_n J_0(Z_n r) = f(x)$$

$$C_n = \frac{(f(x), J_0(Z_n r))_W}{(J_0(Z_n r), J_0(Z_n r))_W} + \text{w.r.t}$$

Schrödinger's Eqn.

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi, \quad \hbar = \frac{h}{2\pi}.$$

\hookrightarrow Hamiltonian operator
(total energy).

\Rightarrow in classical mechanics,

$$H = \frac{P^2}{2m} + V,$$

$$P = mv. \quad \hookrightarrow \text{potential. ener.}$$

$$\hat{H} = \frac{\hat{P}^2}{2m} + \hat{V} \quad \checkmark \text{"a formulation"}$$

$$\hat{V} = \frac{\hbar^2}{i} \nabla \Rightarrow \hat{P}^2 = -\frac{\hbar^2}{2m} \nabla \cdot (\nabla).$$

Ψ is wave function.

$$\int_{S^2} |\Psi|^2 d\Omega = 1 \quad \text{"normalized".}$$

$\Psi(x, t)$. sum separate in to products of functions (Goal).

Recall the Ansatz: $\Psi(x, t) = X(x) T(t)$.

Subs. the Ansatz back to Schrödinger eqn.

$$i\hbar \frac{1}{\Psi(x)} \cdot \frac{d\Psi(x)}{dt} = -\frac{\hbar^2}{2m} \cdot \frac{1}{\Psi(x)} \nabla^2 \Psi + V.$$

↳ only when $E = \text{const.}$

Recall $i\hbar \frac{df}{dt} = Ef$.

$$\frac{df}{dt} = -i\frac{E}{\hbar} f \quad \left. \begin{array}{l} \\ \end{array} \right\} E$$

$$f(t) = \exp\left(-\left(\frac{iE}{\hbar}\right)t\right).$$

$$-\frac{\hbar^2}{2m} \cdot \nabla^2 \Psi + V\Psi(x) = E\Psi(x) \quad \text{space.}$$

↳ time-independent Schrödinger eqn.

Sols are called "Stationary states".

$$\Psi(x, t) = \sum_{n=1}^{\infty} C_n \psi_n(x) \cdot e^{-\frac{iEt}{\hbar}}.$$

$$\hat{H}\Psi(x) = E\Psi(x)$$

Transform ∇^2 to spherical coordinates.

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \cdot \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \left(\frac{\partial^2}{\partial \phi^2} \right)$$

polar coord. expansion.

$$\frac{\hbar^2}{2m} \nabla^2 = \frac{\hbar^2}{2m} \left[\begin{array}{c} \uparrow \\ \sim \\ \downarrow \end{array} \right] \Psi + V(r, \theta, \phi)\Psi = E\Psi.$$

$$\Psi(r, \theta, \phi) = R(r)Y(\theta, \phi).$$

$$-\frac{\hbar^2}{2m} \left[\frac{Y}{r^2} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{R}{r^2 \sin\theta} \cdot \frac{\partial}{\partial \theta} \left(\sin\theta \cdot \frac{\partial Y}{\partial \theta} \right) + \frac{R}{r^2 \sin^2\theta} \frac{\partial^2 Y}{\partial \phi^2} \right]$$

$$+ V R Y = E R Y$$

... Subs. the Schrödinger formulation.

dividing by RY .

$$-\frac{\hbar^2}{2m} \left\{ \frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + (V - E) \right\}$$

$$-\frac{\hbar^2}{2m} \left\{ \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \cdot \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \cdot \frac{\partial^2 Y}{\partial \phi^2} \right\} = 0.$$

* Y_{nlm} is the spherical harmonics.

Lecture #14 2/20/2024

- Heat eqn. in polar coordinates.

- Schrödinger's eqn.

↳ SoH, time-independent S.E.

→ Stationary states soln.

→ Spatial S.E.

↳ Radial eqn.

↳ Angular eqn.

$$\text{Ansatz: } \Psi(r, \theta, \phi) = R(r) Y(\theta, \phi).$$

$$-\frac{\hbar^2}{2m} \left[\frac{Y}{r^2} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{R}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \cdot \frac{\partial Y}{\partial \theta} \right) + \frac{R}{r^2 \sin^2 \theta} \frac{\partial^2 Y}{\partial \phi^2} \right]$$

$$+ V(r) R Y = E R Y$$

$$-\frac{\hbar^2}{2m} \left\{ \frac{1}{R} \cdot \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) \right\} + (\mathcal{H}(r) - E) r^2$$

Eigenvalues

Vibrational

modes

$$= \frac{\hbar^2}{2m} \frac{1}{Y} \left\{ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \cdot \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \phi^2} \right\} \Rightarrow$$

Positive const. ← $\ell(\ell+1)$

2 eqns.

... eigenvalues are the

"so-called" ignatum sts.

$$-\frac{\hbar^2}{2m} \left\{ \frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) \right\} + (\mathcal{H}(r) - E) r^2 = \ell(\ell+1)$$

$$\& \frac{1}{Y} \left\{ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \cdot \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \phi^2} \right\} = -\ell(\ell+1)$$

$$\sin \theta \frac{\partial}{\partial \theta} \left(\sin \theta \cdot \frac{\partial Y}{\partial \theta} \right) + \frac{\partial^2 Y}{\partial \phi^2} = -\ell(\ell+1) \sin^2 \theta \}$$

$$\text{Ansatz } Y(\theta, \phi) = \Theta(\theta) \Phi(\phi)$$

$$\Rightarrow \left\{ \frac{1}{\Theta} \left[\sin \theta \frac{d}{d\theta} \left(\sin \theta \cdot \frac{d\Theta}{d\theta} \right) \right] + \ell(\ell+1) \sin^2 \theta \right\} + \frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = 0$$

$$\frac{1}{\Theta} \left[\sin \theta \frac{d}{d\theta} \left(\sin \theta \cdot \frac{d\Theta}{d\theta} \right) \right] + \ell(\ell+1) \sin^2 \theta = m^2$$

$$\& \frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = -m^2$$

Another separation

$$\Rightarrow \Phi(\phi) = e^{im\phi}$$

constant

Since the Ansatz $e^{im\phi}$ is periodic.

$$\Phi(\phi + 2\pi) = \Phi(\phi).$$

$$\exp[i m(\phi + 2\pi)] = \exp(im\phi)$$

$$\therefore \exp(2\pi im) = 1.$$

$\forall m$ being an integer, $m=0, \pm 1, \pm 2, \dots$

$$\sin \theta \frac{d}{d\theta} \left(\sin \theta \frac{d\Psi}{d\theta} \right) + [l(l+1) \sin^2 \theta - m^2] \Psi = 0$$

\rightsquigarrow A form of associated Legendre eqn.

$$\text{Let } x = \cos \theta, \quad \frac{d}{d\theta} = \frac{dx}{d\theta} \cdot \frac{d}{dx} = -\sin \theta \frac{d}{dx}.$$

Writing in Legendre eqn.:

$$\frac{d}{d\theta} \left(\sin \theta \frac{d\Psi}{d\theta} \right) = -\sin \theta \frac{d}{dx} \left[-\sin^2 \theta \cdot \frac{d\Psi}{dx} \right]$$

$$\Rightarrow \sin \theta \cdot \frac{d}{dx} ((1 - \cos^2 \theta) \frac{d\Psi}{dx}).$$

$$\Rightarrow \sin \theta \frac{d}{dx} \left[(1 - x^2) \frac{d\Psi}{dx} \right]$$

Eventually we have:

$$\frac{d}{dx} \left((1 - x^2) \frac{d\Psi}{dx} \right) + \left(l(l+1) - \frac{m^2}{1-x^2} \right) \Psi = 0$$

the solution:

$$P_l^m(x) \equiv (1-x^2)^{m/2} \left(\frac{d}{dx} \right)^{m/2} P_l(x)$$

$$P_l(x) = \frac{1}{2^l l!} (x^2 - 1)^l$$

... Rodrigues formula.

$$\Psi(\theta) = AP_l^m(x) = AP_l^m(\cos \theta)$$

Associated Legendre polynomials are orthogonal.

l must be non-negative integers.

$|m| > l$, then $P_l^m = 0$.

\rightarrow for any given " l ", there are $(2l+1)$ possible values of m .

Second solution blows up \rightarrow "singular".

at $\theta = 0$ $\times \theta = \pi$ \uparrow factors of $P_l^m(x)$, $l_m > l$ kind

$$\int |Y_l|^2 r^2 \sin \theta dr d\theta d\phi = 1.$$

dS^2 ... in polar coordinates.

$$\int_0^\infty (Rr)^2 dr = 1 \quad \& \quad \int_0^{2\pi} \int_0^\pi |Y_l|^2 \sin \theta d\theta d\phi = 1$$

By formulating Sturm-Liouville,

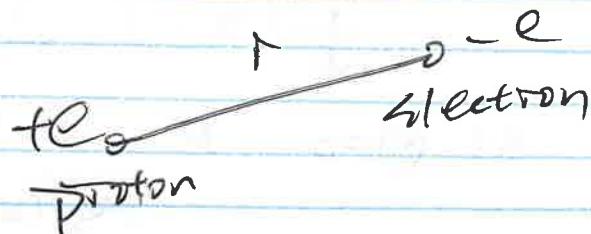
$$Y_l^m(\theta, \phi) = N_{lm}^{-1} e^{im\phi} P_l^m(\cos \theta).$$

$$N_{lm}^{-1} = \sqrt{\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}} \Rightarrow \sum = \begin{cases} (-1)^m & m \geq 0 \\ 1 & m < 0 \end{cases}$$

$$\int_0^{2\pi} \int_0^\pi [Y_l^m(\theta, \phi)]^* [Y_{l'}^{m'}(\theta, \phi)] \underbrace{\sin \theta d\theta d\phi}_{dS^2} = \delta_{ll'} \delta_{mm'}$$

Y_l^m are called spherical harmonics.

Hydrogen atom.



Using Coulomb's law, $\mathcal{F}(r) = \frac{-e^2}{4\pi \epsilon_0 r^2}$.

$$\frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) - \frac{2mr^2}{\hbar^2} [\mathcal{F}(r) - E] R = l(l+1)R$$

$$\text{Let } u = rR(r), \quad R = u/r.$$

$$\frac{dR}{dr} = \frac{[r \frac{du}{dr} - u]}{r^2} \Rightarrow r^2 \frac{dR}{dr} = [r \frac{du}{dr} - u]$$

$$\frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) = r \frac{d^2u}{dr^2} + \frac{du}{dr} - \frac{du}{dr}$$

$$\Rightarrow r \frac{d^2u}{dr^2} - \frac{2mr}{\hbar^2} [\mathcal{F}(r) - E] u = l(l+1)R.$$

Subs. u into the original PDE.

$$\Rightarrow -r \frac{d^2u}{dr^2} + \frac{2mr}{\hbar^2} [\mathcal{F}(r) - E] u = -l(l+1)R'$$

$$- \frac{\hbar^2}{2m} \frac{d^2u}{dr^2} + \mathcal{F}(r) + l(l+1) \frac{R}{r} \frac{\hbar^2}{2mr} = Eu.$$

Eventually, we land in the following eqn:

$$\left[-\frac{\hbar^2}{2m} \frac{d^2u}{dr^2} + \mathcal{F}(r) + \frac{\hbar^2}{2m} \cdot \frac{l(l+1)}{r^2} \right] u = Eu$$

\hookrightarrow mass of the particle.

Problem Session #7.

Legendre Eqn.

$$(1-x^2)y'' - 2xy' + \lambda^2 y = 0. \quad (\text{general form.})$$

$$\frac{d}{dx} \left[(1-x^2) \frac{dy}{dx} \right] = -\lambda^2 y. \quad (\text{standard form.})$$

$$P(x) = 1-x^2 \quad r(x) = 1. \quad q(x) = 0.$$

$x=\pm 1$ are the singular points of this ODE

... no more a regular Sturm-Liouville problem.

$$\lim_{x \rightarrow \pm 1} \left[\psi_n' \psi_m - \psi_n \psi_m' \right] \rightarrow 0$$

$$\lambda^2 = m(m+1), \quad \text{where } m=0, 1, 2, \dots$$

$P_m(x) \rightarrow \text{Legendre Polynomials}$

$$m=0, \quad P_0(x) = 1.$$

$$m=1, \quad P_1(x) = x.$$

$$m=2, \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$-1 \leq x \leq 1$$

$$\}$$

$$\lambda \leq 1$$

Inner Product.

$$\int_{-1}^1 P_n(x) P_m(x) dx = \frac{2}{2m+1} \delta_{nm}$$

$$\text{let } x = \cos \theta, \quad dx = -\sin \theta d\theta$$

$$\psi(x) \Leftrightarrow \psi(\theta)$$

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \cdot \frac{d\psi}{d\theta} \right) = -\lambda^2 \psi$$

$$\psi(\theta) = \sin \theta.$$

$$M(\theta) = \sin \theta. \quad \dots \text{weighting function}$$

$$q(\theta) = 0$$

$$-1 \leq x \leq 1, \quad \theta \in [0, \pi].$$

$$\lambda^2 = m(m+1), \quad m=0, 1, 2, \dots$$

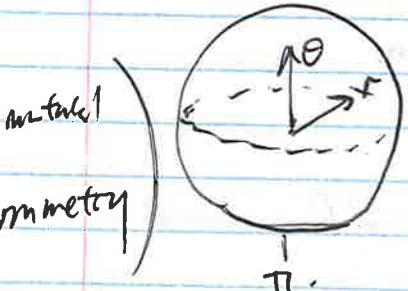
Weight function

$$\psi_m = P_m(x) = P_m(\cos \theta).$$

"finite soln at $x = \pm 1$ "

$$\int_0^\pi \sin \theta \cdot P_n(\cos \theta) P_m(\cos \theta) d\theta = \frac{2}{2n+1} \delta_{nm}$$

Example cooling of a sphere.



$$\text{I.C.} \rightarrow T(r, \theta, t=0) = \Delta T f(r, \theta)$$

$$\text{B.C.} \rightarrow T(r=R, \theta, t) = 0$$

$T(t) \rightarrow$ finite. @ $r=0, \theta=0 \text{ or } \pi$

$$\frac{\partial T}{\partial t} = \alpha \cdot \nabla_{(r, \theta, \phi)}^2 T = \alpha \left[\frac{1}{r^2} \cdot \frac{\partial}{\partial r} \cdot r^2 \left(\frac{\partial T}{\partial r} \right) + \right.$$

$$\left. \frac{1}{r^2 \sin \theta} \cdot \frac{\partial}{\partial \theta} \left(\sin \theta \cdot \frac{\partial T}{\partial \theta} \right) \right]$$

unsteady heat conduction (cooling).

Non-dimensionalization:

$$H = \frac{T}{\Delta T}, \quad \xi = \frac{r}{R}, \quad \tau = \frac{\alpha t}{R^2}$$

Dimensionless Problem:

$$\frac{\partial H}{\partial \tau} = \frac{1}{\xi^2} \frac{\partial}{\partial \xi^2} \left(\xi^2 \cdot \frac{\partial H}{\partial \xi} \right) + \frac{1}{\xi^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \cdot \frac{\partial H}{\partial \theta} \right)$$

$$\text{I.C.: } H(\xi, \theta, \tau=0) = f(\xi, \theta)$$

$$\text{B.C.s: } H(\xi=1) = 0, \quad H(\tau) \rightarrow \text{finite, } @ \begin{cases} \xi=0 \\ \theta=0 \text{ or } \pi \end{cases}$$

$$H = T(\tau) + H(\xi, \theta)$$

$$\frac{\partial H}{\partial \tau} = \nabla_{(\xi, \theta)}^2 H$$

$$HT' = T \nabla_{(\xi, \theta)}^2 H$$

$$\rightarrow \frac{T'}{T} = \frac{1}{H} \cdot \nabla_{(\xi, \theta)}^2 H = -\gamma^2$$

time \downarrow Sturm-Liouville Problem (eventually).
first order, recall Ansatz.

$$\frac{T'}{T} = -\gamma^2$$

further expand the Laplacian portion:

$$\frac{1}{\xi^2} \frac{\partial}{\partial \xi} \left(\xi^2 \cdot \frac{\partial H}{\partial \xi} \right) + \frac{1}{\xi^2 \sin \theta} \cdot \frac{\partial}{\partial \theta} \left(\sin \theta \cdot \frac{\partial H}{\partial \theta} \right) = -\gamma^2 H$$

etc

homogeneous, can separate H .

$$\frac{\partial}{\partial \xi} \left(\xi^2 \cdot \frac{\partial H}{\partial \xi} \right) + \gamma^2 H \xi^2 = \frac{-1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \cdot \frac{\partial H}{\partial \theta} \right)$$

$$\text{S.O.E: } H(\xi, \theta) = R(\xi) \Psi(\theta) \quad \text{SL-problem.}$$

$$-\frac{1}{R} \left[\frac{d}{d\xi} \xi^2 \frac{dR}{d\xi} + \gamma^2 R \xi^2 \right] = \frac{1}{\Psi} \frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Psi}{d\theta} \right) = -\beta^2$$

etc
@ $\begin{cases} \theta=0 \\ \theta=\pi \end{cases}$

$$\begin{cases} R(1) = 0 \\ R(0) \rightarrow \text{finite} \end{cases}$$

... eigenfunctions in θ -space:

$$Y(\theta) = P_m(\cos\theta), \quad \beta^2 = m(m+1), \quad m=0, 1, 2, \dots$$

ξ -space:

$$\frac{1}{\xi^2} \frac{d}{d\xi} \xi^2 \left(\frac{dR}{d\xi} \right) - \frac{m(m+1)}{\xi^2} R = -\lambda^2 R.$$

λ^2 is the eigenvalue

B.C.s in ξ -space:

$$R(\xi=1) = 0, \quad R(\xi=0) \rightarrow \text{finite}.$$

... Spherical Bessel's equation of n^{th} order.

↳ Solution: spherical Bessel's function

$$R = A J_0(\lambda \xi) + B Y_0(\lambda \xi) \xrightarrow{\text{finite}}$$

$$R_m(\xi) = j_m(\lambda_{nm}\xi)$$

Applying the B.C.s

$$R(\xi=1) = 0 \rightarrow j_m(\lambda_{nm}\xi) \xrightarrow{\text{eigenvalue conditions}}$$

$$H(\xi, \theta) = R(\xi) \Psi(\theta) \dots \text{spherical harmonics}$$

(H)(\xi, \theta, t).

$$(H) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{nm} \exp[-(\lambda_{nm})^2 t] j_m(\lambda_{nm}\xi)$$

↑ in Bessel's functions $P_m(\cos\theta)$.

How to find the Fourier coefficients?

Applying T.C.s

$$(H)(\xi, \theta, t=0) = f(\xi, \theta) \Rightarrow f(r, \theta)$$

$$= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} A_{nm} \exp[-(\lambda_{nm})^2 t] j_m(\lambda_{nm}\xi)$$

double summation,
double inner-product.

Integrate w.r.t. θ ,

$$\int_{\theta=0}^{\pi} f(\xi, \theta) P_S(\cos\theta) d\theta = \sum_{m=1}^{\infty} A_{sm} j_s(\lambda_{sm}) \frac{2}{(2s+1)}$$

Integrate w.r.t. ξ . (weighting function ξ^2)

$$\int_0^1 \xi^2 j_s(\lambda_{sm}) j_s(\lambda_{sp}) d\xi = N_p \delta_{mp} \quad A_{sp} N_p \frac{2}{2s+1}$$

↳ ξ goes from 0 to 1

$$\Rightarrow \int_0^1 \xi^2 j_s(\lambda_{sp}\xi) \int_{\theta=0}^{\pi} \sin\theta f(\xi, \theta) P_S(\cos\theta) d\theta d\xi$$

Lecture #15 2/27/2024

"last lecture on $S_0\text{H}_1$."

Review. HW#5 (Pb. 3).

$$\Delta u = 4, \quad (x, y) \in D.$$

$$\begin{cases} u(x, y) = 1, & x^2 + y^2 = 1, \quad y > 0. \\ u(x, y) = 0, & x^2 + y^2 = 1, \quad y < 0. \end{cases}$$

$$v = u - (x^2 + y^2).$$

subs. the "ansatz". getting rid of
the inhomogeneity.

$$\nabla^2 v = 0$$

$$v(x, y) = 1 - 1 = 0, \quad y > 0, \quad x^2 + y^2 = 1.$$

$$v(x, y) = 0 - 1 = -1, \quad y < 0, \quad x^2 + y^2 = 1.$$

"Needs to understand the B.C.s."

$$v(r, \theta) = R(r) H(\theta).$$

$$R(1) H(0) = 0. \quad \forall 0 \leq \theta < \pi.$$

$$R(1) H(\pi) = -1 \quad \forall \pi < \theta \leq 2\pi.$$

$$H(0) = 0, \quad \forall 0 \leq \theta \leq \pi.$$

known P.D.E.

$$H(\theta) = -\frac{1}{R(1)}, \quad \forall \pi < \theta \leq 2\pi.$$

↳ piecewise constant $H(\theta)$.

→ So your soln does not dep. on H .

So we need to look for $\neq 0$ solns.

$\neq 0$. → $H \sim$ piecewise const.

$$\neq 0: H(\theta) = K_1 \sin(\sqrt{\lambda} \theta) + K_2 \cos(\sqrt{\lambda} \theta).$$

↳ trivial soln. *** IMPORTANT

~~Recall~~ → Soln to spherical harmonics (S.E.).

→ Radial eqn.

recall:

$$-\frac{\hbar^2}{2m^2} \left\{ \frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + [Tr(r) - \Sigma] \right\}.$$

$$-\frac{\hbar^2}{2m^2} \frac{1}{Y} \left\{ \frac{1}{\sin\theta} \cdot \frac{\partial}{\partial\theta} \left(\sin\theta \cdot \frac{\partial Y}{\partial\theta} \right) \right\} = 0$$

$$+ \frac{1}{\sin^2\theta} \cdot \frac{\partial^2 Y}{\partial\phi^2} \right\} = 0$$

Multiplying by $\left(-\frac{2mr^2}{\hbar^2}\right)$

$$\frac{1}{R} \frac{d}{dr} \left[r^2 \frac{dR}{dr} \right] - \frac{2mr^2}{\hbar^2} [f(r) - E] R$$

Recall:

H-atom:

$$f(r) = \frac{-e^2}{4\pi\epsilon_0} \frac{1}{r}$$

$$\text{Let } u = rR(r), \quad R = \frac{u}{r}$$

$$\frac{dR}{dr} = \left[r \frac{dR}{dr} - u \right] / r^2$$

$$\frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) = \frac{d}{dr} \left(r \frac{du}{dr} - u \right)$$

$$= r \frac{d^2u}{dr^2} + \frac{du}{dr} - \frac{du}{dr}$$

$$= r \cdot \frac{d^2u}{dr^2}$$

$$\Rightarrow r \frac{d^2u}{dr^2} - \frac{2mr}{\hbar^2} [f(r) - E] u = l(l+1)R.$$

$$r \frac{d^2u}{dr^2} - \frac{2mr}{\hbar^2} f(r) + \frac{2mr}{\hbar^2} Eu = l(l+1)R.$$

$$-r \frac{d^2u}{dr^2} + \frac{2mr}{\hbar^2} f(r) + l(l+1)R = \frac{2mr}{\hbar^2} Eu.$$

multiplying by $\frac{\hbar^2}{2mr}$

$$-\frac{\hbar^2}{2m} \frac{d^2u}{dr^2} + f(r) + l(l+1) \frac{Rr}{r} \frac{\hbar^2}{2mr} = Eu.$$

→ find forms of eigenvalue problem.

this implies:

$$-\frac{\hbar^2}{2m} \cdot \frac{d^2u}{dr^2} + \left[f(r) + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} \right] u$$

Eigenvalue prob. ← → = Eu.

hope to subs. the Coulomb p.t. into $f(r)$.

$$\text{let } K = \frac{\sqrt{-2me}}{\hbar} \rightarrow E < 0. \text{ (Energy st.)}$$

$$f(r) = \frac{-e^2}{4\pi\epsilon_0} \cdot \frac{1}{r}$$

bound st. of atoms

for $E < 0, K \in \mathbb{R}$.

Divide the both sides by E ,

and use the substitution of K :

$$\frac{1}{K^2} \frac{d^2u}{dr^2} = \left[1 - \frac{me^2}{2\pi\epsilon_0\hbar^2 K} \frac{1}{(Kr)} \right]$$

Let $p = Kr$, &

$$+ \frac{l(l+1)}{(Kr)^2} u.$$

$$p_0 = \frac{me^2}{2\pi\epsilon_0\hbar^2 K}$$

$$\frac{d^2u}{dp^2} = \left[1 - \frac{p_0}{p} + \frac{l(l+1)}{p^2} \right] u.$$

as $p = kr \rightarrow \infty$

$$\frac{d^2u}{dp^2} = u.$$

the soln to this eqn:

$$u(p) = A e^{-p} + B p e^p.$$

$e^p \rightarrow \infty$ as $p \rightarrow \infty$, so we don't want this term.

$$u(p) \sim A e^{-p}; \quad \xrightarrow{\text{large } p}$$

IMPORTANT: always follow the soln ansa.

if $p \rightarrow 0$, $\frac{l(l+1)}{p^2}$ dominates

Methods of dominance balances.

$$\frac{d^2u}{dp^2} = \frac{l(l+1)}{p^2} u$$

$$p^l \rightarrow u(p) = C p^{l+1} + D p^{-l}$$

as $p \rightarrow 0$, $p^{-l} \rightarrow \infty \therefore D=0$.

$$\therefore u(p) = C p^{l+1} \quad (l>0)$$

$$\text{but } v(p) = p^{l+1} e^{-p} u$$

Ideally, we want to find $\frac{du}{dx}$ & $\frac{d^2u}{dx^2}$.

$$p \frac{d^2v}{dp^2} + 2(l+1-p) \frac{dv}{dp} + [p_0 - 2(l+1)] v = 0$$

NEW EQUATION. in terms of v

If you chose: $\begin{cases} v = 2l+1 \\ x = 2p \\ \gamma = \frac{j}{j_{\max}} = n-l-1. \end{cases}$

$$x \phi'' + (v+1-\gamma) \phi' + \gamma \phi = 0 \quad \phi = u$$

↳ special eqn: Associated

the solns to this eqn, Laguerre eqn.

is called Associated Laguerre polynomials.

$$L_{q-p}^P(x) = (-1)^P \underbrace{\left(\frac{d}{dx} \right)^P}_{\hookrightarrow} L_q(x)$$

"they are orthogonal"

↳ Laguerre

polynomials.

$$L_q(x) = e^x \left(\frac{d}{dx} \right)^q (e^{-x} x^q)$$

$$a_{j+1} = \left\{ \frac{2(j+l+1) - p_0}{(j+1)(j+2l+2)} \right\} a_j$$

$$\downarrow a_{j_{\max}} = 0$$

$$\rightarrow 2(j_{\max} + l + 1) - p_0 = 0$$

$$\text{Let } n = j_{\max} + l + 1.$$

$$2n - p_0 = 0 \rightarrow p_0 = 2n$$

↓

Principal Quantum Number

$$E_n = - \left[\frac{m}{2\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \right] \frac{1}{n^2}, \quad \forall n = 1, 2, \dots$$

$$E = \frac{\hbar^2 k^2}{2m} = \frac{-me^4}{8\pi^2 \epsilon_0 \hbar^2 p_0^2} \quad \xrightarrow{\text{Bohr formula}}$$

We can then express the soln:

$$\psi_{nml}(r, \theta, \phi) = R_{nl}(r) \cdot Y_l^m(\theta, \phi)$$

$$R_{nl}(r) = \frac{1}{r} r^{l+1} e^{-pr} V(p)$$

$$e^{V(p)} = \sum_{n-l-1}^{2l+1} (2p)$$

Legendre polynomials

We can then write the soln for hydrogen atoms: Normalization

$$\psi_{nml} = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}} e^{-r/na} \left(\frac{2r}{na}\right)^l \frac{Y_l^m}{n-l-1} \left(\frac{2r}{na}\right) Y_l^m(\theta, \phi).$$

IMPORTANT RESULTS

where $a = \frac{4\pi \epsilon_0 k^2}{me^2}$, denoted as "Bohr radius"

$$\int \psi_{n'm'l'}^* \psi_{nml} \underbrace{r^2 \sin\theta dr d\theta d\phi}_{d\Omega} = \delta_{nn'} \delta_{ll'} \delta_{mm'}$$

... Orthogonality

Inhomogeneities

fill now. → linear PDE for Sch.

→ Separable Sch ansatz.

↳ derivatives are w.r.t. only one ind. var.

→ All but "1" condition have to be homogeneous.

→ Simple Inhomogeneities

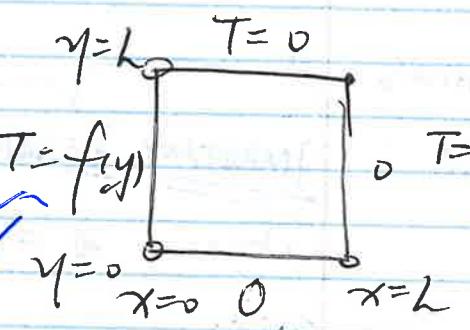
e.g. Poisson's eqn.

$$\frac{\partial}{\partial x} \left(\frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial T}{\partial y} \right) = 0$$

One can solve with

S. of

different



what happens if ...

we are solving
linear eqns. ...

we may constnot.

$$T(x, y) = T_1(x, y) + T_2(x, y).$$

↳ superposition

$$\begin{cases} T_1(x, 0) = f(x), \\ T_1(x, L) = 0, \\ T_1(0, y) = 0, \\ T_1(L, y) = 0. \end{cases}$$

$$\begin{cases} T_2(x, 0) = 0, \\ T_2(x, L) = 0, \\ T_2(0, y) = g(y), \\ T_2(L, y) = 0. \end{cases}$$

↳ S. of.
Superimpose two fields!

Lecture #16 2/19/2024.

Recall

Schrödinger eqn.

→ Associated Legendre eqn.

→ radial, polar, azimuthal.

→ eigenfunc., quantum number were eigenvals.

→ Inhomogeneous B.C. (more than 1).

Inhomogeneities

$$T_{xx} + T_{yy} = 0$$

$$T(x, 0) = 0 ; T(x, L) = 0$$

$$T(0, y) = f(y); T(L, y) = 0.$$

S. of:

$$T(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{L}\right) \sinh\left(\frac{n\pi(L-x)}{L}\right)$$

Recall the separated ODEs:

$$x'' - \lambda^2 x = 0 \rightarrow SL \text{ (nonsq.)}$$

$$y'' + \lambda^2 y = 0.$$

→ Coefficients (using homogeneity).

$$T(0, y) = \sum_{n=1}^{\infty} K_n \sin\left(\frac{n\pi y}{L}\right) \sinh(n\pi x) = f(y)$$

$$k_n = \frac{(f(y), \sin(\frac{n\pi y}{L}))}{(\sin(\frac{n\pi y}{L}), \sin(\frac{n\pi y}{L}))} \rightarrow (f(y), \phi_n)$$

(ϕ_n, ψ_n)

if $T(x, 0) = f(x)$, $T(x, L) = 0$,

$T(0, y) = g(y)$. $T(L, y) = 0$.

By linearity:

$$T(x, y) = T_1(x, y) + T_2(x, y).$$

T_2 will satisfy

$$\begin{cases} T(x, 0) = f(x), \\ T(x, L) = 0, \\ T(0, y) = 0, \\ T(L, y) = 0 \end{cases}$$

T_1 will satisfy

$$T(x, 0) = 0, \quad T(x, L) = 0, \quad T(0, y) = g(y), \quad T(L, y) = 0$$

Inhomogeneities

- two strategies

1). Use superposition to reduce the problem

→ multiple standard S.O.I. problems.

separately
superposition

2). If there is a bulk of inhomogeneity.

... Can one guess a substitution s.t.

PDE becomes homogeneous.

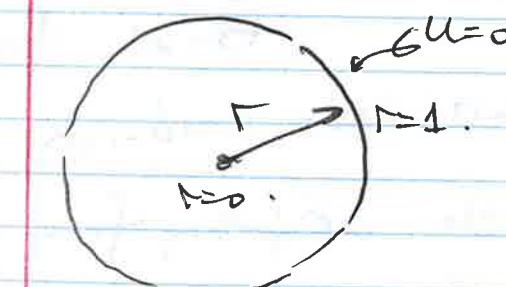
... New variable: $v = u + u_p$.

General inhomogeneities.

$$\text{Recall } \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) = \frac{\partial u}{\partial t} - f(r, t).$$

i.e., 2D heat conduction in polar coordinates.

(infinitely long cylinder).



$$\text{B.C.s: } \begin{cases} u(r=0, t) = \text{finite}, \\ u(r=R, t=0) = 0 \end{cases}$$

→ there is an inhomogeneity in the eqn. Need eigenfunction exp.:

Necessarily, assuming: $f(r, t) = \sum_{n=1}^{\infty} C_n \phi_n$

$$\text{S.O.I.: } r^2 \ddot{\phi}_n + r \dot{\phi}'_n + n^2 \phi_n = 0$$

$$T' = -r^2 T$$

$$\rightarrow T(t) = a e^{-\frac{\lambda^2}{4}t}$$

Contains all the temporal terms.

$$R(r) = C_n J_0(Z_n r)$$

$$u(r, t) = \sum_{n=1}^{\infty} A_n J_0(Z_n r) \cdot e^{-\frac{\lambda_n^2}{4}t}$$

why
dry
dry
at B.C.

$$f(r, t) = \sum_{n=1}^{\infty} C_n(t) J_0(Z_n r) \text{ homogeneous PDE.}$$

$$C_n(t) = \frac{1}{N_m} \int_0^1 f(r, t) J_0(Z_n r) r dr$$

$$N_m = \frac{J_1(Z_m)}{2}$$

General sol'n writes:

$$u = u^h + u^p$$

\downarrow Sol'n to inhomogeneous PDE with zero B.C.s.

Sol'n to homogeneous PDE

$$u^p(r, t) = \sum_{n=1}^{\infty} A_n(t) J_0(Z_n r)$$

Substitute in the PDE:

Bessel func.,

- inner prod. is weighted,
the weighting func is r .

$$\sum_{n=1}^{\infty} A_n(t) [J_0(Z_n r)]'' = \sum_{n=1}^{\infty} \frac{dA_n}{dt} J_0(Z_n r)$$

Special derivatives

$$-\sum_{n=1}^{\infty} C_n(t) J_0(Z_n r)$$

Notation on the 1st

Bessel form (w.r.t. deriv. of r) $f(r, t)$.

$$[J_0(r)]'' \rightarrow \frac{1}{r} \cdot \frac{\partial}{\partial r} \left[r \frac{\partial}{\partial r} (J_0(Z_n r)) \right]$$

$$\text{LHS: } \sum_{n=1}^{\infty} A_n(t) [Z_n^2 J_0(Z_n r)]$$

$$\text{LHS} = \sum_{n=1}^{\infty} \frac{dA_n(t)}{dt} J_0(Z_n r) \quad (\text{properties of the Bessel functions})$$

$$- \sum_{n=1}^{\infty} C_n(t) J_0(Z_n r)$$

$$- A_n(t) Z_n^2 = - C_n(t) + \frac{dA_n(t)}{dt}$$

$$\frac{dA_n(t)}{dt} = C_n(t) - A_n(t) Z_n^2$$

Solving 1st-order ODE w/ inhomog.

coeff. from int. by.

$$\hat{A}_n(t) = \int_0^t e^{2\pi i t'} C_n(t') dt'.$$

↑ dummy var.
Solv in terms
of func. of t .

$$C_n(t) = \frac{(f(x, t), \phi_n)_W}{(\phi_n, \phi_n)}$$

... End of S.T

Fourier Transform

$\mathcal{F}(f)$.

↳ transform PDE into ODEs.



Def. : $\hat{f}(\underline{k})$



Inverse Fourier map

$$\hat{f}(\underline{k}) = \frac{1}{(2\pi)^{N/2}} \int_{-\infty}^{\infty} f(\underline{x}) e^{i\underline{k}\cdot\underline{x}} d\underline{x}.$$

wave vector ←

dimension, i.e., \mathbb{R}^N

$$f(\underline{x}) = \frac{1}{(2\pi)^{N/2}} \int_{-\infty}^{\infty} \hat{f}(\underline{k}) e^{-i\underline{k}\cdot\underline{x}} d\underline{k}.$$

Careless see.

General form.

$$\mathcal{F}[f_{\underline{x}}] = \hat{f}(\underline{k}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\underline{x}) e^{i\underline{k}\cdot\underline{x}} d\underline{x}$$

.. Fourier transform

$$f(x) = \frac{1}{x} \int_{-\infty}^{+\infty} \hat{f}(k) e^{-ak\cdot x} dk$$

$$a = \pm 1, \quad x = \sqrt{2\pi}.$$

$f(x)$ & $\hat{f}(k)$ are integrable:

$$\mathcal{L} = \int_{\mathbb{R}^n} |f| dx < \infty$$

"bounded".

Derivatives.

→ Boundary terms

$$\mathcal{F}(f') = -ik\mathcal{F}(f). \quad \begin{cases} \text{go away} \\ \text{or fine decay} \end{cases} \quad \text{at infinity.}$$

(BP →
check K)

$$\mathcal{F}(f'') = -k^2 \mathcal{F}(f).$$

Convolution theorem.

$$\mathcal{F}(f) = \hat{f}(k) \cdot \left(\mathcal{F}^{-1}(\hat{f}\hat{g}) \right).$$

$$\mathcal{F}(g) = \hat{g}(k) \cdot \left(\mathcal{F}^{-1}(\hat{f}\hat{g}) \right).$$

↑ hoping
to find.

1) case:

$$\begin{aligned} \mathcal{F}^{-1}\{\hat{f}\hat{g}\} &= H(x) \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\xi) g(x-\xi) d\xi, \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(\xi) f(x-\xi) d\xi. \end{aligned}$$

Fundamental soln

$$u_t - u_{xx} = \delta(x, t), \quad -\infty < x < \infty$$

$$\Rightarrow u_t - u_{xx} = \delta(x-\xi) \delta(t-t')$$

$$\delta(x-a) = \begin{cases} \infty & x=a \\ 0 & x \neq a \end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(x-a) = 1, \quad \int_{-\infty}^{+\infty} \delta(x-a) f(x) dx = f(a)$$

Fundamental soln: $G(x-\xi, t-t')$.

$$u(x, t) = \int_{-\infty}^{\infty} \int_{t=0}^t G(x-\xi', t-t') \delta(\xi' t') dt' d\xi'$$

$$\begin{aligned} \frac{u_t - u_{xx}}{t} &= \delta(x-\xi) \delta(t-t') \\ \text{B.C.s} \quad u \rightarrow 0 & \text{ as } |x| \rightarrow \infty \quad \left. \begin{array}{l} -\infty < x < \infty \\ 0 < t < \infty \\ |\xi| < \infty \end{array} \right\} \\ \hat{u}_t &\sim \text{"Fourier transform of } u \text{"} \\ &+ k^2 \hat{u} \\ \dots \text{ turning PDE in ODE and F.T. only in space.} \\ &= \frac{\delta(t-t')}{\sqrt{2\pi}} \int \delta(x-\xi) e^{ikx} dx e^{ik\xi}. \end{aligned}$$

Recall the convolution thm.
Solve for ξ and $t' = 0$

$$\hat{u}_t + k^2 \hat{u} = \frac{\delta(t)}{\sqrt{2\pi}}$$

Using Green's func. to solve the ODE.

$$\hat{u}(k, t) = \frac{1}{\sqrt{2\pi}} e^{-k^2 t}.$$

$$u(x, t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-k^2 t} e^{-ikx} dk$$

$$m = \zeta_3 + i \frac{x}{2t^{1/2}} \quad \{ e^{-mx} \text{ is analytic}$$

$G(x, t) = \sqrt{\pi} e^{x^2/2t} \sum_{n=0}^{\infty} \sin(n\pi x)$ can
be calculated

Problem Session 8

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2} + Q(x, t), \quad 0 \leq x < \infty, \quad t > 0$$

$$u(0, t) = u(l, t) = 0$$

source term

$$u(x, 0) = G(x)$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

$$u = X(x) T(t).$$

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

$$X(x) = A \cos(\lambda x) + B \sin(\lambda x)$$

$$X_n(x) = B_n \sin(n\pi x), \quad n=0, 1, 2, \dots$$

series expansion

$$u(x, t) = \sum_{n=0}^{\infty} B_n \sin(n\pi x) T(t).$$

incorporate the const.
into $T(t)$

$$= \sum_{n=0}^{\infty} \sin(n\pi x) T(t)$$

... Eigenfunction expansion

$$\sum_{n=0}^{\infty} \sin(n\pi x) T = \sum_{n=0}^{\infty} - (n\pi)^2 \sin(n\pi x) T + Q$$

$$\sum_{n=0}^{\infty} [T' + (n\pi)^2 T] \sin(n\pi x) = Q$$

$$\left[\sum_{n=0}^{\infty} [T' + (n\pi)^2] \sin(n\pi x), \sin(n\pi x) \right] = [Q, \sin(n\pi x)]$$

$$[T' + (n\pi)^2] [\sin(n\pi x), \sin(n\pi x)] = [Q, \sin(n\pi x)]$$

$$T' + (n\pi)^2 T = \frac{(Q, \sin(n\pi x))}{(\sin(n\pi x), \sin(n\pi x))} \approx \frac{Q}{n}$$

$\begin{cases} \sin((n+1)\pi x) \\ \vdots \\ \sin(n\pi x) \\ \vdots \\ \sin(1\pi x) \end{cases}$

$$M(t) = e^{\int_0^t (n\pi)^2 dt} = e^{\int_0^t (m^2) \cdot t} = e^{(m^2)t}.$$

$$\frac{d}{dt} (e^{(m^2)t} T) = 2e^{(m^2)t} q_n.$$

$$e^{(m^2)t} T - T(0) = \int_0^\pi 2e^{(m^2)t} q_n(t') dt'.$$

modify a little bit:

$$T(t) = e^{-(m^2)t} \left[\int_0^t 2e^{(m^2)t'} q_n(t') dt' + T(0) \right].$$

$$\rightarrow T(t) = e^{-(m^2)t} \left[\int_0^1 2e^{(m^2)t'} q_n(t') dt' + 2q_n \right].$$

$$u(x, t) = \sum_{n=0}^{\infty} \sin(n\pi x) T(t)$$

$$= \sum_{n=0}^{\infty} \sin(n\pi x) e^{-(m^2)t} \left[\int_0^1 2e^{(m^2)t'} q_n(t') dt' + 2q_n \right]$$

Fourier transform.

$$\frac{\partial \hat{u}}{\partial x} = ik \hat{f}$$

$$\frac{\partial^2 \hat{u}}{\partial x^2} = \overset{\longleftarrow}{\frac{\partial}{\partial x}} \cdot \overset{\longleftarrow}{\left(\frac{\partial u}{\partial x} \right)} = ik \frac{\partial u}{\partial x} \\ = ik(i k) \hat{u} = -k^2 \hat{u}(k, t)$$

$$\frac{\partial^2 \hat{u}}{\partial t^2} = \frac{\partial^2 \hat{u}}{\partial t'^2}.$$

$$\frac{\partial^2 \hat{u}(k, t)}{\partial t^2} + c^2 k^2 \hat{u}(k, t) = 0$$

$$\hat{u}(k, 0) = -(k)$$

$$\frac{\partial \hat{u}(k, 0)}{\partial t} = 0$$

$$\frac{\partial^2 \hat{u}(k, t)}{\partial t^2} = -ikc \hat{A} e^{-ikt} + ikc \hat{B} e^{ikt}.$$

$$\rightarrow \hat{A} + \hat{B} = \hat{f} \rightarrow \hat{B} = \hat{f} - \hat{A} \rightarrow 2\hat{A} = \hat{f}$$

$$\hat{A} = \frac{1}{2}\hat{f} = \hat{B}$$

$$\rightarrow -ikc \hat{A} + ikc \hat{B} = 0 \rightarrow \hat{B} = \hat{A}$$

only 2 vars. govern the PDE

→ we can find a similarity soln.

$$G = \hat{f}(k) \cdot \left(\frac{e^{ikct} + e^{-ikct}}{2} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{f}(k)}{2} e^{-ikx} dk$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{f}(k)}{2} e^{ikct} e^{-ikx} dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{f}(k)}{2} e^{-ik(x+ct)} dk$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{-ik(x-ct)} dk$$

$$\hat{f}(x) = \text{IFT.}(\hat{f}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{-ikx} dk$$

$$u(x,t) = \frac{f(x+ct)}{2} + \frac{f(x-ct)}{2}$$

$$(Q1). \rho C_p \cdot \frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2}$$

$$(Q2). \text{Prove } \int_0^\infty e^{-s^2} ds = \frac{\sqrt{\pi}}{2}.$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy = \int_0^\infty \int_0^\infty e^{-r^2} r dr d\theta$$

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^2 = \pi$$

$$(Q3). q_b = -K \frac{\partial T}{\partial x}$$

$$q_b(x=0) \sim t^{-1/2}$$

	\tilde{T}	x	t	β	α
deg.	1	0	0	1	0
length	0	1	0	-1	2
time	0	0	1	-1	-1
rank	$m=3$	$n=5$			$\# \text{PI} \text{ SPS} = 2$

$$\Pi_1 = \text{fcn}(\Pi_2)$$



apply similarly
at 2nd-order in space
(\rightarrow 1st order)

b). $n = \frac{1}{2}$
 $m = \frac{3}{2}$

$$A =$$

$$B =$$

$\partial \tilde{e}$ for $F(\eta)$.

$$\tilde{T} = T - T_0 = Bt^m F(\eta)$$

$\sim \partial e$ for $F(\eta)$.

Bcs for $F(\eta)$

$$\tilde{T} + T_0 = T$$

$$T = Bt^m F(\eta) + T_0. \quad \# \text{match the order of } t$$

$$\frac{\partial (Bt^m F(\eta) + T_0)}{\partial t} = \alpha \frac{\partial^2 [Bt^m F(\eta) + T_0]}{\partial x^2}$$

$$\frac{\partial [Bt^m F(\eta)]}{\partial t} = \alpha \frac{\partial^2 [Bt^m F(\eta)]}{\partial x^2}$$

$$\frac{\partial}{\partial t} \left[Bt^m F\left(\frac{x}{At^n}\right) \right] = \alpha \cdot \frac{\partial^2}{\partial x^2} \left[Bt^m F\left(\frac{x}{At^n}\right) \right]$$

$$Bt^{m-1} \cdot F\left(\frac{x}{At^n}\right) + Bt^m \cdot F\left(\frac{x}{At^n}\right) \cdot \frac{x}{A} \cdot t^{-n-1} \cdot (-n).$$

choose A, B
so more diff for
 A, B, C is simple

LHS:

$$F(\eta) \cdot Bt^{m-1} + F(\eta) \cdot \frac{x}{A} \cdot B \cdot t^{m-n-1} (-n).$$

RHS:

$$\frac{\partial}{\partial x} \left[\alpha B \cdot t^m \cdot F(\eta) \cdot \frac{1}{At^n} \right]$$

$$\alpha \cdot B \cdot t^m \cdot F(\eta) \cdot \frac{1}{(At^n)^2}$$

$$\frac{\alpha B t^m F(\eta)}{A^2} \cdot t^{m-2n}.$$

$$m-1 = m-2n = m-n-1.$$

$$m=1. \rightarrow n=\frac{1}{2}$$

$$F(\eta) \cdot Bt^{m-1} + F(\eta) \cdot \frac{x}{A} \cdot B \cdot t^{m-\frac{3}{2}} (-\frac{1}{2})$$

$$= \frac{\alpha B F(\eta)}{A^2} \cdot t^{m-1}$$

$f(\eta) \sim$ superposition

ODE for $F(\eta)$

$$2F'' + \eta F' - 3F = 0. \quad F(0) = 1. \quad F(\infty) = 0$$

$$A = \sqrt{\alpha^2}, \quad B = -\beta \sqrt{\alpha^2}$$

... these ODEs are all valid options

depending on A & B .

$$F'' + \eta F' - 3F = 0.$$

$$A = \sqrt{2\alpha^2}, \quad B = -\beta A = -\beta \sqrt{2\alpha^2}$$

(P2). Domain should be $0 \leq x < \infty$

$$\frac{\partial C}{\partial x} \Rightarrow \frac{\partial}{\partial x} \left(D_0 C \cdot \frac{\partial C}{\partial x} \right).$$

$$\frac{\partial C}{\partial x} \propto -t \cdot D_0$$

$$n = 1/2$$

$$m = 0$$

$$\frac{\partial [B t^m F(\eta)]}{\partial t} = \frac{\partial}{\partial x} \left\{ D_0 C \cdot \frac{\partial^2 [B t^m F(\eta)]}{\partial x^2} \right\}$$

$$B t^{m-1} F(\eta) + B t^m F'(\eta) \cdot (-n) \cdot t^{-n-1} \cdot \frac{x}{A}$$

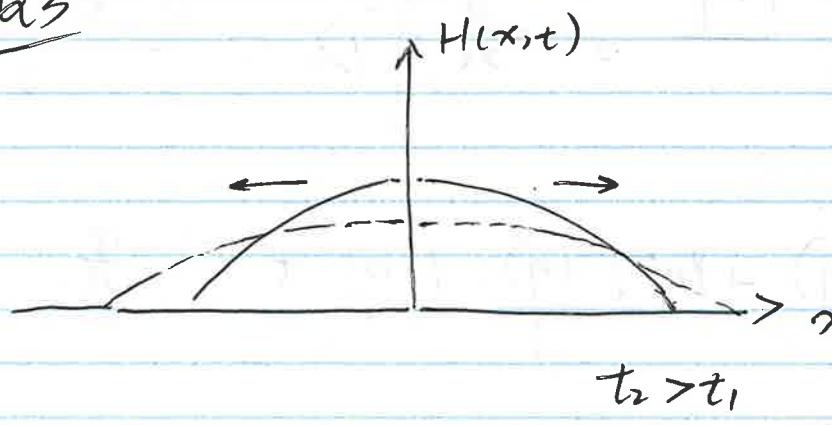
$$= \frac{\partial}{\partial x} \left\{ D_0 C \cdot B t^m \right\}$$

$$b). \quad 0 = F'' F + (F')^2 + 2\eta F'$$

$$F(0) = 1, \quad F(\infty) = 0$$

HW #8 -

Q3



$$\frac{\partial H}{\partial t} = -\frac{r}{3\mu} \cdot \frac{\partial}{\partial x} \left[H^3 \cdot \frac{\partial^3 H}{\partial x^3} \right]$$

	H	x	t	α
depth	1	0	0	0
length	0	1	0	1
time	0	0	1	-1

$\rightarrow -\frac{r}{3\mu}$

$$n=4 \\ m=3 \quad > \# \pi \text{ group} = 1.$$

}

there exists a similarity

soln.

$$\frac{\partial H}{\partial t} = Bt^{m-1} \cdot [m\bar{F} - m\eta F']$$

$$H(x,t) = Bt^m \bar{F}(\eta) \cdot \eta = \frac{x}{At^n}.$$

$$\frac{\partial \eta}{\partial t} = (-n)t^{-n-1} \cdot \frac{x}{A}.$$

$$\frac{\partial \eta}{\partial \pi} = \frac{1}{At^n}.$$

$$\frac{\partial^3 H}{\partial x^3} = \frac{\partial^2}{\partial x^2} \cdot \left(Bt^m \bar{F}'(\eta) \cdot \frac{1}{At^n} \right)$$

$$= (Bt^m)^3 \bar{F}''(\eta) \cdot \frac{1}{(At^n)^3}$$

$$= \left(\frac{B}{A}\right)^3 t^{3m-3n} \cdot \bar{F}''(\eta).$$

$$\frac{\partial}{\partial x} [H^3 \cdot H''] = (H^3)' H'' + H^3 \cdot H'''$$

$$= 3H^2 \cdot H' \cdot \frac{1}{At^n} H''' + H^3 \cdot \left[\left(\frac{B}{A}\right)^4 t^{4m-4n} \right]$$

$$H' = \frac{Bt^m}{At^n} \cdot F'(\eta) \quad F'''(\eta)$$

$$R.H.S. = -\frac{r}{3\mu} \cdot \frac{\partial}{\partial x} \left(H^3 \cdot \frac{\partial^3 H}{\partial x^3} \right).$$

$$= -\frac{r}{3\mu} \left[3H^2 \frac{\partial H}{\partial x} \cdot \frac{\partial^3 H}{\partial x^3} + H^3 \frac{\partial^4 H}{\partial x^4} \right]$$

$$ds = At^m d\eta$$

$$\int_0^{s(t)} Bt^m F(\eta) \cdot At^m d\eta = M.$$

$$\rightarrow ABt^{m-n} \int_{\eta=0}^{\frac{b(t)}{At^m}} F(\eta) d\eta = M.$$

$$\frac{\partial H}{\partial x} = \frac{B}{A} \cdot t^{m-n} \cdot F'(\eta).$$

$$\frac{\partial^2 H}{\partial x^2} = \frac{B}{A^2} \cdot t^{m-2n} \cdot F''(\eta).$$

$$\frac{\partial^3 H}{\partial x^3} = \frac{B}{A^3} \cdot t^{m-3n} \cdot F'''(\eta).$$

$$\frac{\partial^4 H}{\partial x^4} = \frac{B}{A^4} \cdot t^{m-4n} \cdot F''''(\eta).$$

Plugging in the derivatives to RHS.

$$RHS: = -\frac{r}{3\mu} \left[3H^2 \cdot \frac{B}{A} \cdot t^{m-n} \cdot F'(\eta) \cdot \frac{B}{A^3} t^{m-3n} F'''(\eta) + H^3 \cdot \frac{B}{A^4} \cdot t^{m-4n} \cdot F''''(\eta) \right]$$

$$\begin{aligned} LHS: \frac{\partial H}{\partial t} &= Bm t^{m-1} F + Bt^m \cdot F' \frac{\partial F}{\partial t} \\ &= Bt^{m-1} [mf - m\eta F'] \end{aligned}$$

Condition: t exponential coefficients are the same ... ?

$$RHS = -\frac{r}{3\mu} \left[3H^2 \cdot \frac{B^2}{A^4} \cdot t^{2m-4n} F'(\eta) \cdot F''(\eta) + H^3 \cdot \frac{B}{A^4} \cdot t^{m-4n} \cdot F''''(\eta) \right]$$

$$\begin{aligned} RHS &= -\frac{r}{3\mu} \left[3 \cdot F^2 F' F''' + F^3 F'''' \right] \frac{B^4}{A^4} t^{4m-4n} \\ 4m-4n &= m-1 \quad \checkmark \\ 4m &= 3n+1 \end{aligned}$$

$$m+n=0.$$

$$m = \frac{1}{7}, \quad n = -\frac{1}{7}.$$

$$\text{Q4} \quad \frac{\partial}{\partial x} \left(\sqrt{T} \cdot \frac{\partial T}{\partial x} \right) = \gamma T^2 \frac{\partial T}{\partial t}.$$

$$\textcircled{2} \quad \frac{\partial T}{\partial x}(0, t) = 0, \quad T(\infty, t) = 0$$

$$\textcircled{1} \quad T(x, 0) = 0, \quad \textcircled{3} \quad \gamma \int_0^\infty T^3 dx = \beta.$$

	\tilde{T}	x	t	T	β
deg	1	0	0	-3/2	3/2
length	0	1	0	-2	-1
time	0	0	1	1	1

Semi-infinite slab

$$x=0$$

Dimensional analysis.

$$\frac{[E]^3/2}{[L]^2} = \gamma \frac{[E]^3}{T}$$

$$[\gamma][E]^3[L] = [\beta].$$

$$\text{rank} = 3, \quad (m=3)$$

of dimensionless groups $n=5$

π groups: $n-m=2 < 3$.

$$\pi_1, \pi_2 = f^n(\pi_i) \quad \begin{array}{l} \text{num. ind} \\ \text{var. + num.} \\ \text{dep. var.} \end{array}$$

$$\text{let. } \pi_1 = x^a t^b \gamma^c \beta^d = \eta. \quad \left. \begin{array}{l} \\ \\ \end{array} \right\}^{2+!} \\ \pi_2 = T^a t^b \gamma^c \beta^d = \Theta. \quad \left. \begin{array}{l} \\ \\ \end{array} \right\}$$

$$\eta = \frac{x}{At^m}, \quad T = Bt^m F(\eta). \quad \begin{array}{l} \text{dimensional} \\ \text{analysis} \end{array}$$

$$\begin{aligned} a-2c-d &= 0 \\ b+c+d &= 0 \end{aligned} \quad \begin{array}{l} \xrightarrow{?} (d-c)=0 \\ \xrightarrow{?} \end{array}$$

lecture 20

3/15/2014

Review of the Course.

→ Start:

PDEs → Classifications.
→ Properties.

↪ Order: $A_1 \frac{\partial^2 \phi}{\partial x^2} + B_1 \frac{\partial^2 \phi}{\partial xy} + C_1 \frac{\partial^2 \phi}{\partial y^2}$

Independent. $+ a \frac{\partial \phi}{\partial x} + b \frac{\partial \phi}{\partial y} + c(\phi)$

var. x, y

$$+ d(x, y) = 0$$

$$\Delta = B_1^2 - 4AC_1$$

{ $\Delta = 0$, parabolic. ← eigenval. non-distinct.
 $\Delta > 0$, hyperbolic
 $\Delta < 0$, elliptic. ← eigenval. complex.

Characteristics

Solv - Eigenfunction expansions

Integral Transforms

Similarity

• Linear

• Nonlinear

Definition on Linearity.

$$h\phi = 0.$$

$$h(C_1\phi_1 + C_2\phi_2) = C_1h(\phi_1) + C_2h(\phi_2)$$

Linear



Non-linear



char.

Solv.



Integral.



Similarity.



→ requires the sgn. to be hyperbolic.

• for hyperbolic. $\frac{dx}{dt} = \pm c_0 \rightsquigarrow$ wave sgn.

↑ to be finite.

• for parabolic. $\frac{dx}{dt} = 0$

$t \rightarrow \text{const.}$

• for elliptic. $\frac{dx}{dt} \rightarrow \text{complex.}$

... characteristics exists, just we

can't really use it to solve sufficiently.

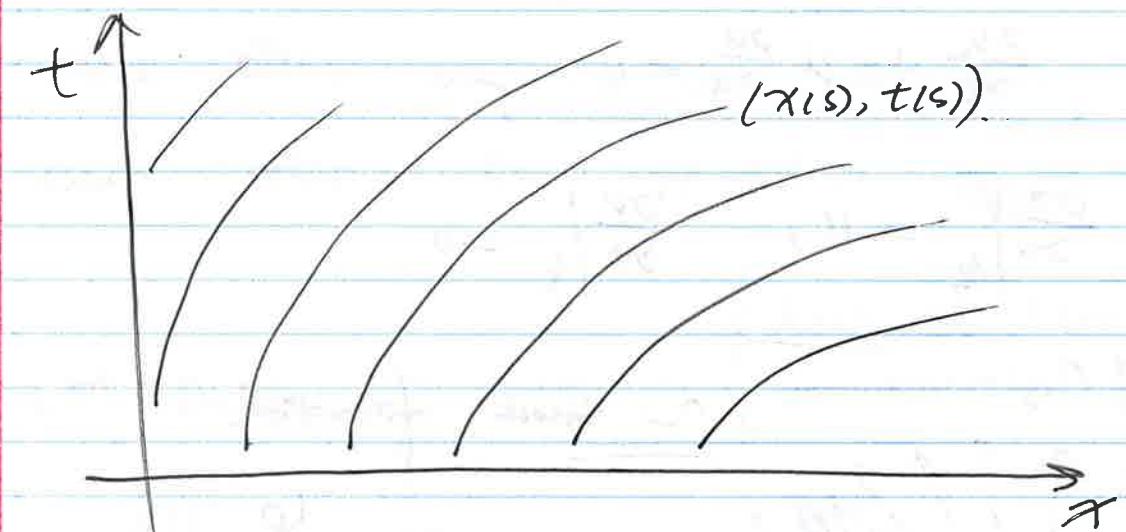
Characteristics

$$A \frac{\partial \phi}{\partial t} + B \frac{\partial \phi}{\partial x} + C(\phi) + D = 0$$

$$\phi(x, t) \xrightarrow{\text{Parametrized}} \phi(x(s), t(s)).$$

... coordinate transformation.

$$\frac{\partial \phi}{\partial s} = \frac{\partial \phi}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial \phi}{\partial t} \cdot \frac{\partial t}{\partial s}.$$



Name the PDE as:

$$\frac{\partial \phi}{\partial t} + \frac{B}{A} \frac{\partial \phi}{\partial x} + \frac{C(\phi)}{A} + \frac{D}{A} = 0$$

... assuming $A \neq 0$.

$$\frac{dx}{ds} - \frac{B}{A} \cdot \frac{dt}{ds} = 0 \rightarrow \frac{d\phi}{ds} = - \frac{(C+D)}{A} \cdot \frac{dt}{ds}$$

choose $t=S$.

$$\frac{dt}{ds} = 1 \Rightarrow \frac{dx}{ds} = \frac{B}{A}, \frac{d\phi}{ds} = -\frac{(C+D)}{A}.$$

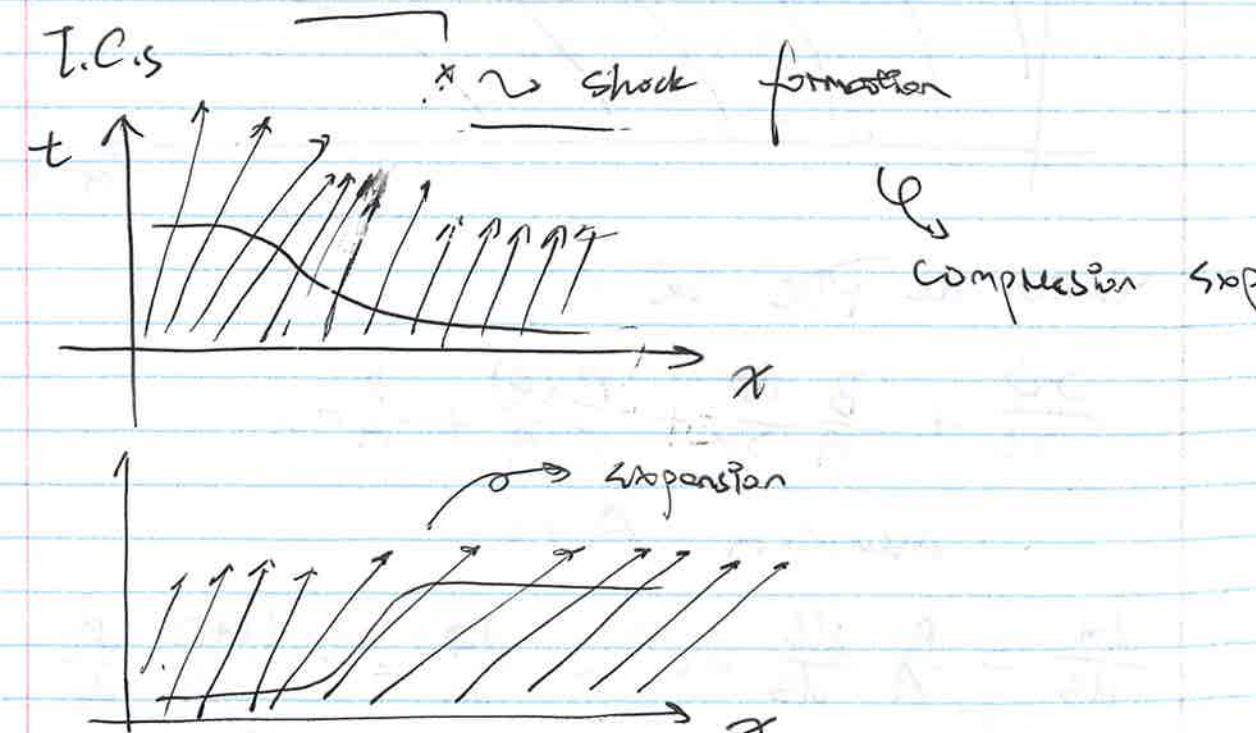
It can also be written as:

$$\frac{dt}{A} = \frac{dx}{B} = -\frac{d\phi}{(C+D)}$$

Burgers' Eqn.

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$$

$$\left. \frac{\partial x}{\partial t} \right|_{\text{L}} = u, \quad \left. \frac{\partial u}{\partial t} \right|_{\text{L}} = 0$$



Generalize to conservation law.

$$\frac{\partial f}{\partial t} + \frac{\partial F}{\partial x} = 0$$

If a shock forms, I can use

Rankine-Hugoniot to find shock speed:

$$U_S = \frac{F_R - F_L}{P_R - P_L}$$

these ODEs may not be independent of each other in M.C.

2nd Order eqn.

$$\frac{\partial^2 U}{\partial t^2} + \beta \frac{\partial^2 U}{\partial x^2} = 0$$

$$\beta = \begin{bmatrix} 0 & -1 \\ C & 0 \end{bmatrix}, \quad U = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial t} \end{bmatrix}$$

for wave eqn

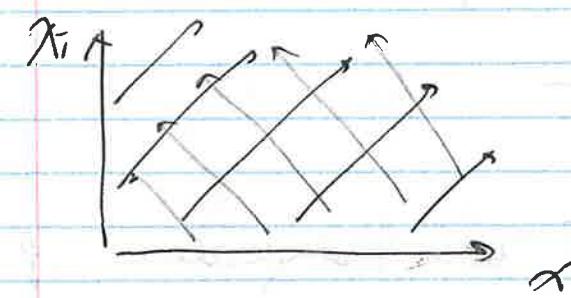
positive eigenvalues
normalize β

Check the HW
question for ref.

$$\{f(x - ct) + f(x + ct)\}$$

$$\frac{\partial u}{\partial t} + c_0 \frac{\partial u}{\partial x} = 0 \quad \frac{\partial u}{\partial t} - c_0 \frac{\partial u}{\partial x} = 0$$

two sets of solutions for waves



If $\underline{\beta}$ has complex eigenvalues, then what?

... Solvs. \rightarrow All types
↓

has to be linear.

dependent variables: u .

independent variables: x, y .

Ansatz: $u = X(x) Y(y)$.

Subs. back to PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \leftarrow \text{Laplace}$$

$$X''Y + Y''X = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = \lambda \rightarrow \text{const.}$$

$$\begin{cases} X'' - \lambda X = 0 \\ Y'' + \lambda Y = 0 \end{cases}$$

All B.C.s except 1 to be homogeneous.

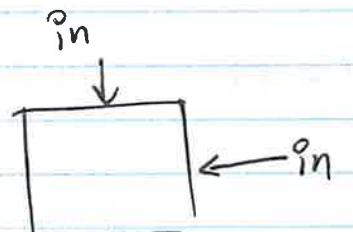
Solutions to ODEs: eigenfunctions.

$$f = \sum C_n \phi_n$$

this "1"

is the
"inhomogeneous".

$$u = \sum f(x) \phi(y)$$



Similarity :-

→ Buckingham-Pi to find non-dim. group.

→ $\eta = f(t, x)$.
 ↑ inherent scaling.

→ Write the solution in terms of η .

↳ appropriate scaling.

$$\eta = \frac{x}{At^n}, \quad \tilde{T} = Bt^m F(\eta).$$

PDE \rightarrow ODE (η)
↳ function.

Subs. scaling. in PDE.

Get an ODE in $F(\eta)$.

use B.C.s to determine n & m .

$$\eta = \frac{x}{2\sqrt{At}}$$