MAE 6110: HW #2

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1. Show that a dyad **ab** as defined in class is a linear transformation. In particular, find the matrix representing $\mathbf{e}_2\mathbf{e}_1$ and $\mathbf{e}_1\mathbf{e}_2$ with respect to the basis $\mathbf{e}_i$.

Solution: To show **ab** is a linear transformation, we need to show $\mathbf{ab}(m\mathbf{c} + n\mathbf{d}) = m\mathbf{ab}(\mathbf{x}) + n\mathbf{ab}(\mathbf{y})$. The left hand side writes:

$$
\begin{aligned}
\mathbf{ab}(m\mathbf{x} + n\mathbf{y}) &= \mathbf{a}(m\mathbf{b}\cdot\mathbf{x} + n\mathbf{b}\cdot\mathbf{y}) \\
&= m\mathbf{ab}\cdot\mathbf{x} + n\mathbf{ab}\cdot\mathbf{y} \\
&= ma_ib_j\mathbf{e}_i\mathbf{e}_j\cdot x_j\mathbf{e}_k + na_ib_j\mathbf{e}_i\mathbf{e}_j\cdot y_l\mathbf{e}_l \\
&= ma_ib_jx_j\mathbf{e}_i\delta_{jk} + na_ib_jy_l\mathbf{e}_i\delta_{jl} \\
&= ma_ib_jx_j\mathbf{e}_i + na_ib_jy_j\mathbf{e}_i \\
&= m\mathbf{ab}(\mathbf{x}) + n\mathbf{ab}(\mathbf{y})
\end{aligned}
$$

Hence we can deduce that **ab** is a linear transformation.

This linear transformation in the matrix representing $\mathbf{e}_1\mathbf{e}_2$ writes: a_1b_2. and $\mathbf{e}_2\mathbf{e}_1$: b_2a_1.

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2. A positive definite tensor **A** is a linear transformation that obeys $\mathbf{v}\cdot(\mathbf{A}\cdot\mathbf{v}) > \mathbf{0}$, for all non-zero vectors **v**.

2a. For tensor **F**, show that $\mathbf{C} = \mathbf{F}^t\mathbf{F}$ is a symmetric tensor.

Solution: Let $\mathbf{F}= f_{ij}\mathbf{e}_i\mathbf{e}_j$, therefore we can write **C**:

$$
\mathbf{C} = (f_{ji}\mathbf{e}_j\mathbf{e}_i)(f_{ij}\mathbf{e}_i\mathbf{e}_j)
$$

and $\mathbf{C}^t$ can be written as:

$$
\begin{aligned}
\mathbf{C}^t &= (f_{ij}\mathbf{e}_i\mathbf{e}_j)^t(f_{ji}\mathbf{e}_j\mathbf{e}_i)^t \\
&= (f_{ji}\mathbf{e}_j\mathbf{e}_i)(f_{ij}\mathbf{e}_i\mathbf{e}_j)
\end{aligned}
$$

Therefore we can easily observe $\mathbf{C} = \mathbf{C}^t$. Hence **C** is a symmetric tensor.

2b. If in addition **F** is invertible, show **C** is positive definite, that is, $\mathbf{v}\cdot\mathbf{C}\cdot\mathbf{v} > \mathbf{0}$ for all non-zero vector **v**.

Solution: If **F** is invertible, we set vector $\mathbf{v} = v_k\mathbf{e}_k$; following the previous form, we can expand the term $\mathbf{v}\cdot\mathbf{C}\cdot\mathbf{v}$:

$$
\begin{aligned}
\mathbf{v}\cdot\mathbf{C}\cdot\mathbf{v} &= v_k\mathbf{e}_k\cdot(f_{ji}\mathbf{e}_j\mathbf{e}_i)(f_{ij}\mathbf{e}_i\mathbf{e}_j)\cdot v_k\mathbf{e}_k \\
&= v_kf_{ji}f_{ij}v_k\mathbf{e}_k\cdot\mathbf{e}_j\mathbf{e}_i\mathbf{e}_i\mathbf{e}_j\cdot\mathbf{e}_k \\
&= v_kf_{ji}f_{ij}v_k\delta_{kj}\mathbf{e}_i\mathbf{e}_i\delta_{jk} \\
&= v_kf_{ki}f_{ik}v_k\mathbf{e}_i\mathbf{e}_i
\end{aligned}
$$

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Here, k is the dummy index and $\mathbf{e}_i$ marks the direction. Therefore the equation writes:

$$\mathbf{v} \cdot \mathbf{C} \cdot \mathbf{v} = ||\mathbf{v}||^2 ||\mathbf{F}||^2$$

and we can easily deduce $\mathbf{v} \cdot \mathbf{C} \cdot \mathbf{v} > 0$. Hence, $\mathbf{C}$ is positive definite.

2c. Find a tensor $\mathbf{H}$ such that $\mathbf{H}^2 \equiv \mathbf{H} \cdot \mathbf{H} = \mathbf{C}$. $\mathbf{H}$ is often defined as the square root of the tensor $\mathbf{C}$.

Solution: On the basis of 2a and 2b, we already know $\mathbf{C}$ is symmetric and positive definite, therefore we know that $\mathbf{C}$ can be diagonized. And in the new coordinate of diagonization $\mathbf{C}$ writes: $\mathbf{C} = \lambda_i \mathbf{E}_i \mathbf{E}_i$.

We can therefore write $\mathbf{C}$ as $\mathbf{C} = \lambda_1 \mathbf{E}_1 \mathbf{E}_1 + \lambda_2 \mathbf{E}_2 \mathbf{E}_2 + \lambda_3 \mathbf{E}_3 \mathbf{E}_3$.

Then $\mathbf{H} = \sqrt{\lambda_1} \mathbf{E}_1 \mathbf{E}_1 + \sqrt{\lambda_2} \mathbf{E}_2 \mathbf{E}_2 + \sqrt{\lambda_3} \mathbf{E}_3 \mathbf{E}_3$.

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3. Let $\mathbf{W}$ be a second order tensor, $\mathbf{W}$ is called skew symmetric if $\mathbf{W} = -\mathbf{W}^t$.

3a. For any second order tensor $\mathbf{T}$, show that $skew(\mathbf{T}) = \frac{\mathbf{T} - \mathbf{T}^t}{2}$ is skew symmetric.

Solution: With the given definition, and expand the tensor $\mathbf{T}$ as $t_{ij}\mathbf{e}_i\mathbf{e}_j$, we can write expand form $skew(\mathbf{T})$:

$$skew(\mathbf{T}) = \frac{1}{2}(t_{ij}\mathbf{e}_i\mathbf{e}_j - t_{ji}\mathbf{e}_j\mathbf{e}_i)$$

We can also write the term

$$\begin{aligned} -skew(\mathbf{T})^t &= -\frac{1}{2}(t_{ij}\mathbf{e}_i\mathbf{e}_j - t_{ji}\mathbf{e}_j\mathbf{e}_i)^t \\ &= -\frac{1}{2}(t_{ji}\mathbf{e}_j\mathbf{e}_i - t_{ij}\mathbf{e}_i\mathbf{e}_j) \\ &= \frac{1}{2}(-t_{ji}\mathbf{e}_j\mathbf{e}_i + t_{ij}\mathbf{e}_i\mathbf{e}_j) \end{aligned}$$

We can then easily get $skew(\mathbf{T}) = -skew(\mathbf{T})^t$.

3b. Show that for any skew tensor $\mathbf{W}$, there is a unique vector $\mathbf{w}$ (called the axial vector of $\mathbf{W}$) such that $\mathbf{W}(\mathbf{x}) = \mathbf{w} \times \mathbf{x}$ for every vector $\mathbf{x}$. Hint: show that $\mathbf{W} = \frac{1}{2}w_{ij}(\mathbf{e}_i\mathbf{e}_j - \mathbf{e}_j\mathbf{e}_i)$ and $\mathbf{W} \cdot \mathbf{v} = \frac{1}{2}(w_{ij}\mathbf{e}_j \times \mathbf{e}_i) \times \mathbf{v}$.

Solution: With the given condition of $\mathbf{W}$ is a skew vector, we know that $w_{ij} = -w_{ji}$.

To show $\mathbf{W} = \frac{1}{2}w_{ij}(\mathbf{e}_i\mathbf{e}_j - \mathbf{e}_j\mathbf{e}_i)$, we can expand the form:

$$\begin{aligned} \mathbf{W} &= \frac{1}{2}w_{ij}(\mathbf{e}_i\mathbf{e}_j - \mathbf{e}_j\mathbf{e}_i) \\ &= \frac{1}{2}w_{ij}\mathbf{e}_i\mathbf{e}_j - \frac{1}{2}w_{ij}\mathbf{e}_j\mathbf{e}_i \end{aligned}$$

Substituting $w_{ij} = -w_{ji}$ we therefore obtain:

$$\begin{aligned} \mathbf{W} &= \frac{1}{2}w_{ij}\mathbf{e}_i\mathbf{e}_j + \frac{1}{2}w_{ji}\mathbf{e}_j\mathbf{e}_i \\ &= \frac{1}{2}\mathbf{W} + \frac{1}{2}\mathbf{W} = \mathbf{W} \end{aligned}$$

Therefore the equation is proved.

To show $\mathbf{W} \cdot \mathbf{v} = \frac{1}{2}(w_{ij}\mathbf{e}_j \times \mathbf{e}_i) \times \mathbf{v}$, we first expand the left hand side:

$$
\begin{aligned}
\mathbf{W}\cdot\mathbf{v} &= \left(\frac{1}{2}w_{ij}(\mathbf{e}_i\mathbf{e}_j - \mathbf{e}_j\mathbf{e}_i)\right)\cdot v_k\mathbf{e}_k \\
&= \left(\frac{1}{2}w_{ij}\mathbf{e}_i\mathbf{e}_j - \frac{1}{2}w_{ij}\mathbf{e}_j\mathbf{e}_i\right)\cdot v_k\mathbf{e}_k \\
&= \frac{1}{2}w_{ij}\mathbf{e}_i\delta_{jk} - \frac{1}{2}w_{ij}\mathbf{e}_j\delta_{ik} \\
&= \frac{1}{2}w_{ik}v_k\mathbf{e}_i - \frac{1}{2}w_{kj}v_k\mathbf{e}_j
\end{aligned} \tag{1}
$$

We then expand the right hand side, and substitute the previous term:

$$
\begin{aligned}
\frac{1}{2}(w_{ij}\mathbf{e}_j\times\mathbf{e}_i)\times\mathbf{v} &= \frac{1}{2}(w_{ij}\mathbf{e}_j\times\mathbf{e}_i)\times v_k\mathbf{e}_k \\
&= \frac{1}{2}w_{ij}v_k(\mathbf{e}_j\times\mathbf{e}_i)\times\mathbf{e}_k \\
&= \frac{1}{2}w_{ij}v_k\left[-(\mathbf{e}_k\cdot\mathbf{e}_i)\mathbf{e}_j + (\mathbf{e}_k\cdot\mathbf{e}_j)\mathbf{e}_i\right] \\
&= \frac{1}{2}w_{ij}v_k\left[-(\delta_{ki})\mathbf{e}_j + (\delta_{kj})\mathbf{e}_i\right] \\
&= -\frac{1}{2}w_{kj}v_k\mathbf{e}_j + \frac{1}{2}w_{ik}v_k\mathbf{e}_i
\end{aligned} \tag{2}
$$

From the above equations we can easily get equation (1) equals equation (2). The equation is therefore proved.

3c. How many linearly independent eigenvectors does $\mathbf{W}$ has?

Solution: Based on the previous given information, for tensor $\mathbf{W}$, we can write it in the form

$$
\mathbf{W} = \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ -w_{12} & w_{22} & w_{23} \\ -w_{13} & -w_{23} & w_{33} \end{bmatrix}
$$

To compute its eigenvectors, we compute:

$$
\begin{aligned}
\det(\mathbf{W}-\lambda) &= \begin{vmatrix} w_{11}-\lambda & w_{12} & w_{13} \\ -w_{12} & w_{22}-\lambda & w_{23} \\ -w_{13} & -w_{23} & w_{33}-\lambda \end{vmatrix} = 0 \\
&= (w_{11}-\lambda)(w_{22}-\lambda)(w_{33}-\lambda) + w_{13}^2(w_{22}-\lambda)^2 + w_{23}(w_{11}-\lambda) + w_{12}^2(w_{33}-\lambda)
\end{aligned}
$$

From the equations we can easily deduce that there are three eigenvalues $\lambda_1 = w_{11}$, $\lambda_2 = w_{22}, \lambda_3 = w_{33}$. Hence there are three independent eigenvectors of $\mathbf{W}$.

4a The action of a certain tensor (call this the stress tensor) $\mathbf{S}$ on an orthonormal basis $\mathbf{e}_i$ is:

$$
\begin{aligned}
\mathbf{S}\cdot\mathbf{e}_1 &= 6\mathbf{e}_1 - 2\mathbf{e}_2 - \mathbf{e}_3 \\
\mathbf{S}\cdot\mathbf{e}_2 &= -2\mathbf{e}_1 + 6\mathbf{e}_2 - \mathbf{e}_3 \\
\mathbf{S}\cdot\mathbf{e}_3 &= -\mathbf{e}_1 - \mathbf{e}_2 + 5\mathbf{e}_3
\end{aligned}
$$

Diagonalize $\mathbf{S}$, i.e., find a basis such that the matrix representing $\mathbf{S}$ is diagonal. Express $\mathbf{S}$ in dyadic notation using this new basis.

Solution: With the given condition, we could write $\mathbf{S}$ in matrix format:

$$\mathbf{S} = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$

To diagonize this matrix, we first find the characteristic polynomial $p(t)$:

$$\begin{aligned} p(t) = \det(\mathbf{S} - \lambda\mathbf{I}) &= \begin{vmatrix} 6-\lambda & -2 & -1 \\ -2 & 6-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{vmatrix} \\ &= (6-\lambda)(6-\lambda)(5-\lambda) - 2 - 2 - 2(6-\lambda) - 4(5-\lambda) \\ &= -\lambda^3 + 17\lambda^2 - 90\lambda + 144 \end{aligned}$$

Solving the above equation, we can then obtain there are three eigenvalues with $\lambda_1 = 3$, $\lambda_2 = 6$, $\lambda_3 = 8$.

With each eigenvectors we can deduce the eigenspace E_i; for E_1:

$$(\mathbf{S} - 3\mathbf{I})(\mathbf{x}) = \begin{bmatrix} 3 & -2 & -1 \\ -2 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Solving the equation we have $x_1 = x_2 = x_3$, therefore we write $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

For E_2:

$$(\mathbf{S} - 6\mathbf{I})(\mathbf{x}) = \begin{bmatrix} 0 & -2 & -1 \\ -2 & 0 & -1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Solving the equation we have $x_1 = x_2$ and $2x_1 = -x_3$, therefore we writes $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

For E_3:

$$(\mathbf{S} - 8\mathbf{I})(\mathbf{x}) = \begin{bmatrix} -2 & -2 & -1 \\ -2 & -2 & -1 \\ -1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Solving the equation we have $x_3 = 0$ and $x_1 = -x_2$, therefore we writes $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

We therefore obtain the linearly independent eigenvectors:

$$\mathbf{v} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \\ 1 & -1 & 0 \end{bmatrix},$$

Now we diagonize matrix $\mathbf{v}$ as a new matrix $\mathcal{S} = \mathbf{v}\mathbf{S}\mathbf{v}^t = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 36 & 0 \\ 0 & 0 & 16 \end{bmatrix}$. Writing $\mathbf{S}$ in dyadic notation using the new basis that representing $\mathbf{S}$ is diagonal is:

$$\mathcal{S} = 9\mathbf{E}_1\mathbf{E}_1 + 36\mathbf{E}_2\mathbf{E}_2 + 16\mathbf{E}_3\mathbf{E}_3$$

4b. Find $\mathbf{S}\cdot\mathbf{n}$, where $\mathbf{n} = \mathbf{e}_1 + \mathbf{e}_2$ using both the new and old basis.
Solution: We first write $\mathbf{S}$ in the old basis:

$$\mathbf{S} = 6\mathbf{e}_1\mathbf{e}_1 - 4\mathbf{e}_1\mathbf{e}_2 - 2\mathbf{e}_1\mathbf{e}_3 + 6\mathbf{e}_2\mathbf{e}_2 - 2\mathbf{e}_2\mathbf{e}_3 + 5\mathbf{e}_3\mathbf{e}_3$$

Computing $\mathbf{S}\cdot\mathbf{n}$ we have:

$$\begin{aligned}\mathbf{S}\cdot\mathbf{n} &= (6\mathbf{e}_1\mathbf{e}_1 - 4\mathbf{e}_1\mathbf{e}_2 - 2\mathbf{e}_1\mathbf{e}_3 + 6\mathbf{e}_2\mathbf{e}_2 - 2\mathbf{e}_2\mathbf{e}_3 + 5\mathbf{e}_3\mathbf{e}_3)\cdot(\mathbf{n}_1 + \mathbf{n}_2)\\ &= 13\mathbf{e}_1 + 13\mathbf{e}_1 - 4\mathbf{e}_3 - 2\mathbf{e}_2\delta_{31} - 2\mathbf{e}_1\delta_{32}\end{aligned}$$

On the new basis where $\mathbf{S}$ is diagonal, we have:

$$\begin{aligned}\mathcal{S}\cdot\mathbf{n} &= (9\mathbf{E}_1\mathbf{E}_1 + 36\mathbf{E}_2\mathbf{E}_2 + 16\mathbf{E}_3\mathbf{E}_3)\cdot(\mathbf{E}_1 + \mathbf{E}_2)\\ &= 61\mathbf{E}_1 + 61\mathbf{E}_2\end{aligned}$$

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5. In rigid body mechanics, we are often interested in rotation of a rigid body about a fixed point (say the origin). Because of rigidity, rotation must preserve distance and angles in the body. Let $\mathbf{x}\cdot\mathbf{y}$ denote the usual dot product of two vectors and let $||\mathbf{x}|| = \sqrt{\mathbf{x}\cdot\mathbf{x}}$ denote the length or norm of $\mathbf{x}$. Let us assume that there exist a transformation $\mathbf{R}$ (*at this point you do not know that it is linear*) that carries a vector $\mathbf{x}$. (e.g. the position vector of a particle on the body with respect to the origin) into another vector $\mathbf{R}(\mathbf{x})$. We assume that $\mathbf{R}$ preserves dot product, that is

$$\mathbf{R}(\mathbf{x})\cdot\mathbf{R}(\mathbf{y}) = \mathbf{x}\cdot\mathbf{y}, \qquad \forall\mathbf{x},\mathbf{y}$$

5a. Show that $\mathbf{R}$ also preserve length, that is, $||\mathbf{R}(\mathbf{x}) - \mathbf{R}(\mathbf{y})|| = ||\mathbf{x} - \mathbf{y}||$.
Solution: To show the given equation, we first expand the left hand side of the equation:

$$\begin{aligned}||\mathbf{R}(\mathbf{x}) - \mathbf{R}(\mathbf{y})|| &= \sqrt{(\mathbf{R}(\mathbf{x}) - \mathbf{R}(\mathbf{y}))\cdot(\mathbf{R}(\mathbf{x}) - \mathbf{R}(\mathbf{y}))}\\ &= \sqrt{\mathbf{R}(\mathbf{x})\cdot\mathbf{R}(\mathbf{x}) - \mathbf{R}(\mathbf{y})\cdot\mathbf{R}(\mathbf{x}) - \mathbf{R}(\mathbf{x})\cdot\mathbf{R}(\mathbf{y}) + \mathbf{R}(\mathbf{y})\cdot\mathbf{R}(\mathbf{y})}\\ &= \sqrt{\mathbf{x}\cdot\mathbf{x} - \mathbf{y}\cdot\mathbf{x} - \mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{y}}\end{aligned}$$

We then write the right hand side of the equation:

$$\begin{aligned}||\mathbf{x} - \mathbf{y}|| &= \sqrt{(\mathbf{x} - \mathbf{y})\cdot(\mathbf{x} - \mathbf{y})}\\ &= \sqrt{\mathbf{x}\cdot\mathbf{x} - \mathbf{x}\cdot\mathbf{y} - \mathbf{y}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y}}\end{aligned}$$

Therefore we can observe that the left hand side equals the right hand side, and the equation is proved.

5b. Show that $\mathbf{R}$ is a linear transformation (more difficult than 5a).
Solution: To show $\mathbf{R}$ is a linear transformation, we need to show

$$\mathbf{R}(m\mathbf{x} - n\mathbf{y}) = m\mathbf{R}(\mathbf{x}) - n\mathbf{R}(\mathbf{y})$$

We therefore expand the two terms on the two hand side:

$$m\mathbf{R}\cdot\mathbf{x} - n\mathbf{R}\cdot\mathbf{y} = m\mathbf{R}(\mathbf{x}) - n\mathbf{R}(\mathbf{y})$$

Based on the dyadic representation, we have

$$m\mathbf{R}(\mathbf{x}) - n\mathbf{R}(\mathbf{y}) = m\mathbf{R}(\mathbf{x}) - n\mathbf{R}(\mathbf{y})$$

and the equation is therefore proved.

5c. Show that $\mathbf{R}^t\mathbf{R} = \mathbf{R}\mathbf{R}^t = \mathbf{I}$ where $\mathbf{I}$ is the identity transformation, that is $\mathbf{R}^{-1} = \mathbf{R}^t$.
Solution: To show $\mathbf{R}^{-1} = \mathbf{R}^t$, we first multiply the two sides by $\mathbf{R}\mathbf{R}^t$:

$$\begin{aligned}\mathbf{R}^{-1}\mathbf{R}\mathbf{R}^t &= \mathbf{R}^t\mathbf{R}\mathbf{R}^t \\ \mathbf{I}\mathbf{R}^t &= \mathbf{R}^t\mathbf{R}\mathbf{R}^t\end{aligned}$$

Therefore we get to know that $\mathbf{R}^t\mathbf{R} = \mathbf{I}$. And the equation is proved.

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6. One way of saying that something is a tensor is that it obeys the transformation rule. Suppose it is given that $g_j dx_j = df$ for all dx_j, where dx_j is the components of $d\mathbf{x}$ and f is a scalar function of $\mathbf{x}$, show that g_j is the components of a vector. In a similar way, show that if $a_{ij}x_j = y_i$, where x_j, y_i, are the components of any two vectors $\mathbf{x}$ and $\mathbf{y}$, then a_{ij} are the components of a tensor.

Solution: I provide the following two ways to show the proof:

METHOD I: To show the given statement, we first expand the given presumption:

$$\begin{aligned}a_{ij}x_j &= y_i \\ a_{ij}x_j\mathbf{e}_i\mathbf{e}_j &= y_i\mathbf{e}_i\mathbf{e}_j \\ a_{ij}\mathbf{x}\mathbf{e}_i &= \mathbf{y}\mathbf{e}_j \\ a_{ij}\mathbf{x}\mathbf{e}_i \cdot \mathbf{e}_j &= \mathbf{y} \\ a_{ij}x_j\mathbf{e}_i &= \mathbf{y} \\ \mathbf{A}(\mathbf{x}) &= \mathbf{y} \\ \mathbf{A} \cdot \mathbf{x} &= \mathbf{y} \\ \mathbf{A}||\mathbf{x}|| &= \mathbf{y}\mathbf{x}\end{aligned}$$

Therefore $\mathbf{A}$ is the dyad of two vectors $\mathbf{x}$ and $\mathbf{y}$, which is a tensor. And a_{ij} are the component of a tensor.

METHOD II: To show $g_j dx_j = df$, we first know that df is an invariant of the basis. Here, g_j and dx_j changes based on the basis, where g_j is the component of $\mathbf{g}$ and dx_j is the component of $d\mathbf{x}$. Reconstructing the term:

$$\begin{aligned}g_j dx_j &= df \\ g_j &= \frac{df(x_j)}{dx_j} \\ \mathbf{g} &= \frac{df(\mathbf{x})}{d\mathbf{x}}\end{aligned}$$

Since f is the scalar function of $\mathbf{x}$, then $\frac{df(\mathbf{x})}{d\mathbf{x}}$ is a vector. Then $\mathbf{g}$ is a vector, and therefore g_j is a component of a vector.