Article

On stability of Forced van der Pol Equation by control

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Abstract: Nonlinear dynamics by nature are extremely hard to control, which were usually linearized as implemented in feedback control design. van der Pol oscillatory circuits exhibit such behavior, possess wide engineering applications background. Here, a deterministic algorithm control approach was introduced and investigated on controlling the nonlinear behavior of van der Pol dynamics. Five different scenarios were implemented for controlling the dynamics as for comparison and study the behavior of the deterministic algorithm: the circular trajectory of radius equals zero with no control, the circular trajectory of radius equals five with no control, the circular trajectory of radius equals five with feed-back control, the circular trajectory of radius equals five with feed-forward control, the circular trajectory of radius equals five with both feed-back and feed-forward control. It is observed that the feed-forward control on the targeted radius successfully converges the control algorithm to the desired trajectory. With further emphasis on the controller's behavior, it is observed that as the desired trajectory's radius changes, the stability of the controller may vary. Further investigation on the different radius of desired trajectory one deduce the controller approach to a stable state while the radius is approaching 2, and the CPU computation time increases as radius increases. It is also observed that even for the considered "stable" state the errors still exhibit a sinusoidal fluctuation. The proposed study on nonlinear controllers could proffer insights on microelectronics, system design, and many potential fields.

Keywords: Nonlinear system; Control; Linearization; van der Pol dynamics

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1. Introduction

Controlling nonlinear dynamics and chaos has always been an important goal for scientists and engineers since achieving such can tackle numerous issues in engineering and systems design, i.e., circuits and microelectronics [1], biology and organic systems [2], earth sciences and natural systems [3], etc. Controlling such a complex system usually requires scrutinized systems and model formulation. By 1927, regarding the complex inherent dynamics of oscillating circuits, Balthazar van der Pol sought circuits that oscillated at a fixed frequency for use in signal transmission and receipt [4,5]. The van der Pol equation exhibits strong chaotic dynamics behavior, and the forced van der Pol equation also exhibits chaotic behavior, i.e., global bifurcation [6]. Considering the wide applications background and all these delicate behavior that are extremely hard to control in engineering, proposing methods for controlling such dynamics are hence important and meaningful. In control science, linearizing nonlinear systems for either feedback or feed-forward controls are of common ways to tackle the issue of chaotic system [7]. As of a common practice in control theory, linearization has indeed been applied to many nonlinear chaotic systems such as the Lorenz system [8], the forced Duffing oscillator [9], Poincaré map [10], etc. Considering very little literature truly addressed the issue of controlling van der Pol oscillator, in 2017, Cooper *et al.* study the nonlinear behavior of deterministic algorithm imposing controlled trajectory to forced van der Pol equation [5].

Following the work by Cooper *et al.*, the aim of this paper is to investigate how different controlling schemes may variate the behavior of the van der Pol dynamics. The first goal of the paper is to study how different control methods may change the

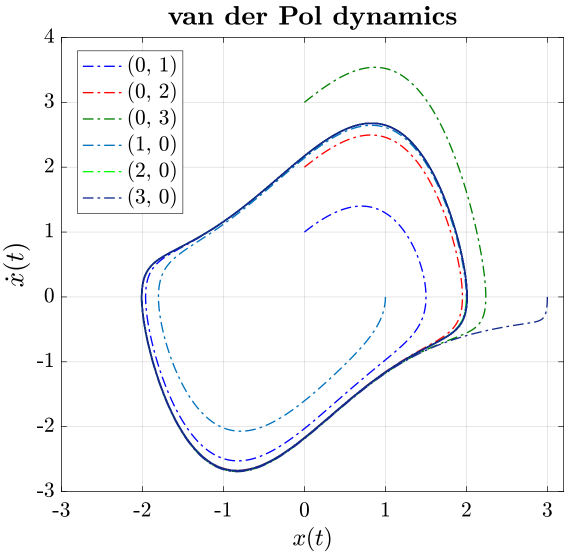

Figure 1. The inherent dynamics of van der Pol equation. The phase was initiated from six different initial points: $(i, 0)$, $(0, i)$, $i = 1, 2, 3$, to illustrate the inherent dynamics.

controlled behavior of the van der Pol dynamics. What's more, the stability of the imposed control scheme is also to be discussed since such hasn't been touched much by the field. Different desired trajectories may variate the stability of the controlled scheme, and the variations behavior are also of interest in this paper. To answer these questions, a controlled framework was designed and implemented for comparison with feed-forward, feedback controls of the dynamics. The paper will address how each method changes the controlled scheme of the dynamics.

The paper is arranged as follows: In Section 2, the problem for nonlinear van der Pol dynamics control is formulated and the related methods are elicited. In Section 3 the materials and methodology for both nonlinear feed-forward control and linearized feedback control are introduced and explained, in Sections 3 and 3.2, respectively. This is followed by a brief description of the stability analysis of the van der Pol dynamics. Section 4 presented the results on controlled dynamics comparing the five different methods: the circular trajectory of radius equals zero with no control, the circular trajectory of radius equals five with no control, the circular trajectory of radius equals five with feedback control, the circular trajectory of radius equals five with feed-forward control, the circular trajectory of radius equals five with both feed-back and feed-forward control, in Sections 4.1.1, 4.1.2, 4.1.3, 4.1.4, 4.1.5, respectively. The stability of the controlled schemes is also discussed in Section 4.2. The paper is eventually concluded and summarized in Section 5.

2. Problem Formulation

Van der Pol articulated that the oscillatory behavior fit the class of nonlinear equations that are now referred to the van der Pol equation. The equation exhibits an oscillatory behavior, but the amplitude is not constant, it instead represents an invariant set called a "limit cycle" [5]. System trajectories converge to this invariant that is set from any initial conditions, as can be referred in Figure 1.

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0 \tag{1}$$

Seeking to produce a fixed-amplitude oscillation, forcing functions are added to the nonlinear equation resulting in Equation (2). Typical control design procedures would begin with a linear time-invariant (LTI) feedback controller based on a linearized version

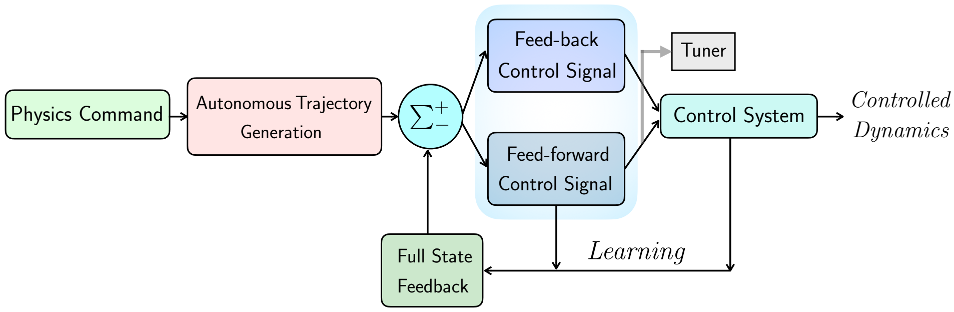

Figure 2. Schematic diagrams for both deterministic algorithm control. Note that the tuner can switch and "tune" the choice of feed-forward and feed-back controls.

of the system equation. The paper aim to optimize the feedback control gains via the Ricatti equation (linear quadratic regulator, LQR):

$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = F(t) \tag{2}$$

Here, the purpose of this paper is to examine and investigate how different control methods can be employed for forced van der Pol equation as for controlling chaos and nonlinearities. As referred in Figure 1, the van der Pol equation will converge to a specific trajectory given different initial points and difficult to variate the route. The robustness of the nonlinearity may also variate the controlled scheme, which may influence the stability of the control.

3. Methodology

In this paper, as introduced in Section 2, both feedback and feed-forward control will be implemented in our control scheme as can be tuned on/off in the framework. The framework is enabled through SIMULINK in MATLAB®, and the main framework is illustrated as in Figure 2. The tuner can turn on/off either feedback or feed-forward controllers as to design different control processes for our implementation and comparison. The "learning" of the system is enabled through the full state feedback process as reducing errors and passing on commands to the controllers. Note that in the framework parameters were picked as $\mu = 1$ and $t = 100[s]$.

3.1. Nonlinear Feedforward Control

The basic premise of idealized nonlinear feedforward control design [11] is to define the inherent dynamics as the idealized feedforward control [5], as shown in Equation (3); where the control designer is required to specify the desired trajectory to be followed. In this study, the desired trajectory is a circular route. The controller is idealized only with respect to the assumed parameter estimates as the μ in Equation (3) [5]

$$F(t) = u \equiv u_{ff} = \frac{d^2x_d}{dt^2} - \mu(1-x_d^2)\frac{dx_d}{dt} \tag{3}$$

where $F(t)$ is the forced term as in Equation (2):

$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = F(t) \equiv \frac{d^2x_d}{dt^2} - \mu(1-x_d^2)\frac{dx_d}{dt} \tag{4}$$

3.2. Linearized Feedback Control Design

In control theory and sciences, the normal first procedure for control design involves linearizing the nonlinear dynamic equation, and based upon designing the control signal. For the van der Pol dynamics, Equation (1) can be linearized into Equation (5), expressed in state-variable formulation from which state space trajectories are displayed on phase portraits [5]:

$$\begin{aligned} [A] = [1\ -1;\ 1\ 0];\ [B] = [-1;\ 0];\ [C] = [0;\ 1];\ [D] = [0]; \\ [Q] = [1\ 0;\ 0\ 1];\ [R] = [1];\ [S] = [2.6818\ 0.4142;\ 0.4142\ 3.3784] \\ K_p = 2.6818;\ K_d = 0.4142 \end{aligned} \tag{5}$$

where the states from $[A]$ to $[S]$ are the expressions used in the linear-quadratic optimization leading to a feedback controller with proportional and derivative gains for K_p and K_d. The closed loop dynamics are established by Equation (2) where the van der Pol forcing function $F(t)$ is a proportional-derivative (PD) controller by Sands [12] who's gains are in Equation (5) [5]:

$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = F(t) \equiv -K_d(\dot{x}_d - \dot{x}) - K_p(x_d - x) \tag{6}$$

which we add the forced term $F(t)$ back to the original van der Pol equation and obtain:

$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = F(t) \equiv \frac{d^2x_d}{dt^2} - \mu(1-x_d^2)\frac{dx_d}{dt} + x_d - K_d(\dot{x}_d - \dot{x}) - K_p(x_d - x) \tag{7}$$

3.3. Stability Analysis

As per Cooper *et al.* [5] discussed, directly implementing the feedforward control signal make converge the desired trajectory pretty successfully. But during the numerical experiments it is found that as the trajectory radius varies, the controlled scheme sometimes does not converge well. Hence, an estimation of effect of radius to the stability of the controlled signal was carried out by investigating four different desired radius: $r = 1$ to 4, implemented with idealized nonlinear feedforward control. The trajectories are to be analyzed considering both trajectory history, CPU computation time and errors of the controlled trajectories.

4. Results and discussion

4.1. Control analysis with different methods

The overall results of trajectories were accumulated into one graph for concise visualization as in Figure 3. It can be observed in Figure 3 that for both uncontrolled signal with $r = 5$ and $r = 0$ the trajectories converged to a same route: the inherent dynamics of van der Pol equation. The algorithm of pure feedback control does not successfully impose the desired trajectory, howbeit a pure feed-forward control successfully enforce the circuits to the desired trajectory: a circle with radius equals to five. Such results agree with Cooper *et al.*'s findings [5]. The combined feedback - feed-forward control scheme also seems not to successfully enforce the desired trajectory, as in the light blue dots in Figure 3.

Now, to characterize how different control signals and frameworks alternate the trajectories in details, a full position, rate, acceleration trajectories $(x(t),\ \dot{x}(t),\ \ddot{x}(t))$ estimation in comparison with the desired control phase are analyzed as follows.

4.1.1. Zero desired trajectory with no control

As can be observed in Figure 4, with the desired control of circular trajectory $r = 0$ as shown in the red dotted lines, the systematic response (blue solid lines) doesn't seem to change much along the time, which is explained as no control signals were implemented. As indicated by Cooper *et al.*, this simulation can be considered as a baseline for further controlled schemes. The first and second order derivatives of $x(t)$ increases the oscillation along with time.

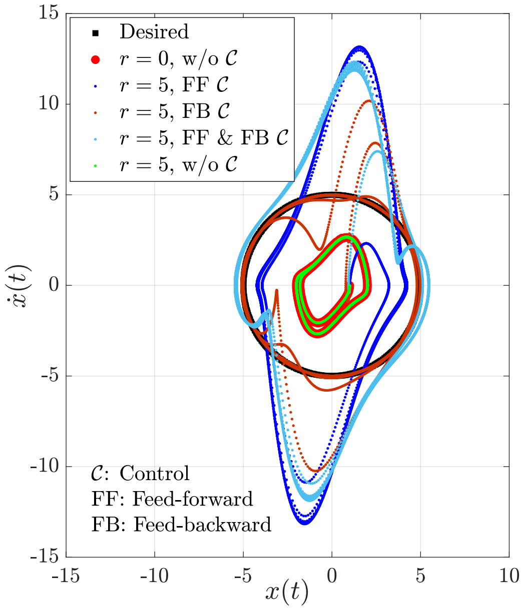

Figure 3. The phase portrait of the van der Pol equations by five different control methods. Note that the black solid squares are the desired trajectory; the light red dots are the trajectory given no control with circular radius $r = 0$; the dark blue dots are the trajectory with feed-back control with circular radius $r = 5$; the dark red dots are the trajectory with feed-forward control with circular radius $r = 5$; the light blue dots are the trajectory combined feed-back and feed-forward control with circular radius $r = 5$; the green dots are the trajectory given no control with circular radius $r = 5$. In the figure, $\mathcal{C}$ stands for the term control and FB & FF are abbreviations for feed-back and feed-forward.

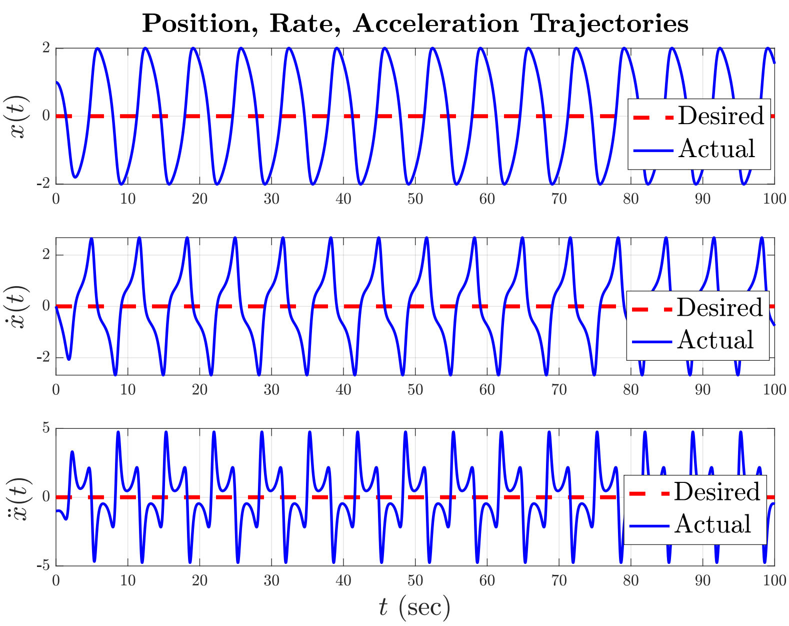

Figure 4. The trajectory comparison of $x(t)$, $\dot{x}(t)$, $\ddot{x}(t)$, for desired trajectory: $r = 0$ with no control activated.

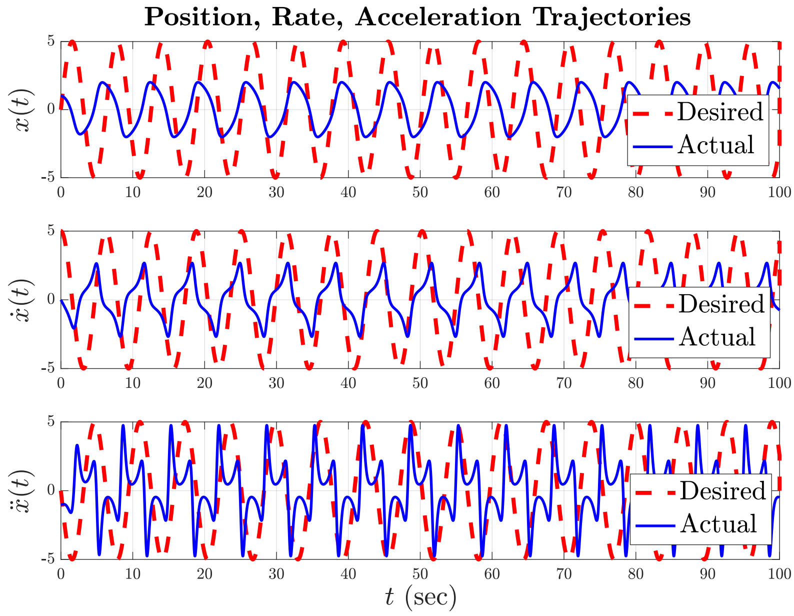

Figure 5. The trajectory comparison of $x(t)$, $\dot{x}(t)$, $\ddot{x}(t)$, for desired trajectory: $r = 5$ with no control activated.

4.1.2. Circular desired trajectory with no control

In Figure 5 it can be observed that the actual trajectories as in the blue solid lines are same as in Figure 4, $r = 0$, corresponding to Figure 3. As the imposed circular trajectory of $r = 5$ (red dotted line) the systematic does not displayed the any response signals, which can be accounted to the no control blocks were implemented in the framework.

4.1.3. Circular desired trajectory with feedback control

By adding up a feedback control signal, one can observe that the data of $x(t)$ are fitted with good approximation as on the top sub figure in Figure 6. However, for the terms of $\dot{x}(t)$ and $\ddot{x}(t)$ the errors grows evidently, which are accounted for the increasing erroneous trajectory in Figure 3.

4.1.4. Circular desired trajectory with feed-forward control

If only impose a feed-forward control, the systematic response, as shown in Figure 7, will converge to the desired trajectory following the executed signals. The results in Figure 7 agrees with what's been observed in Figure 3. Also, the rate and accelerations reveals how the signals been implemented for control: fluctuations happened at the initial stage later converged to the desired trajectory at approximately 10s.

4.1.5. Circular desired trajectory with both controls

If both feedback and feed-forward controls were implemented to the system, Figure 8 indicates that the position and the rate were fitted pretty well, yet the accelerations display higher error.

In fine, the implementation and comparison of the five different control schemes imply using directly a feed-forward nonlinear adaptive control will best impose the desired signal (trajectory), same as reported by Cooper *et al.* [5]; Yet linearized feedback control do have fair dynamics implementation for position x, yet the errors increases evidently for rate (velocity) controls.

Also, Cooper *et al.* [5] reported how the standard deviation and the mean errors variate compared with three of the methods we implemented: Unforced, indicating given a desired trajectory with no control; Forced by linear feedback control and forced

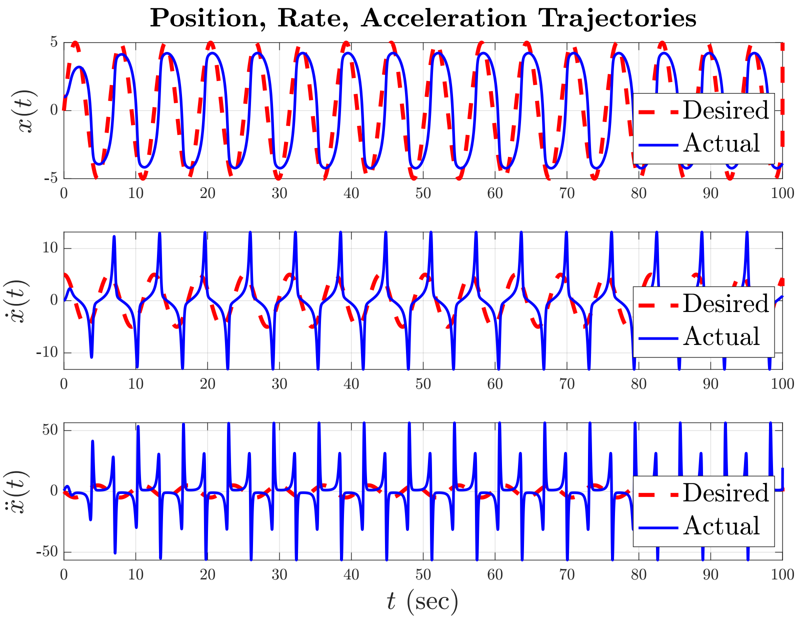

Figure 6. The trajectory comparison of $x(t)$, $\dot{x}(t)$, $\ddot{x}(t)$, for desired trajectory: $r = 5$ with feed-back control.

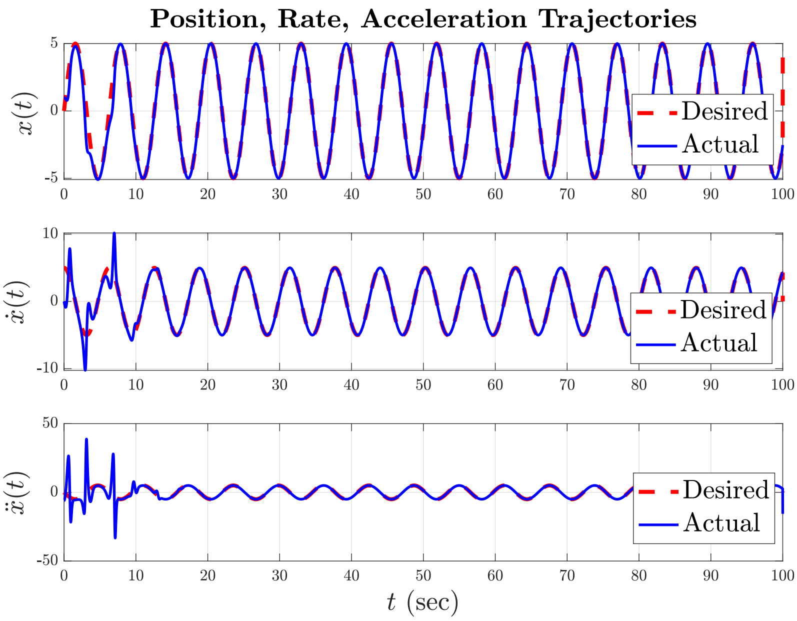

Figure 7. The trajectory comparison of $x(t)$, $\dot{x}(t)$, $\ddot{x}(t)$, for desired trajectory: $r = 5$ with feed-forward control.

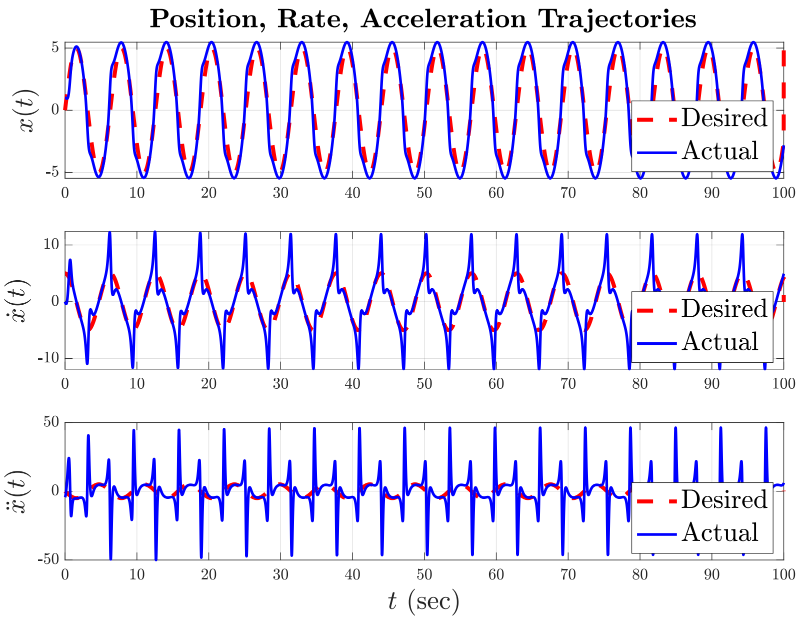

Figure 8. The trajectory comparison of $x(t)$, $\dot{x}(t)$, $\ddot{x}(t)$, for desired trajectory: $r = 5$ with both feed-back and feed-forward control.

Initial Position	Unforced	Forced by LQR Linear Feedback K_p = 1.1913 K_d = 0.9050	Forced only by Idealized Nonlinear Feedforward
(0,0)	μ = 2.858 σ = 3.357	μ = 0.28163 σ= 1.41	μ = 0.083008 σ = 0.30951
(5,0)	μ = 2.2539 σ = 4.4174	μ = 0.15107 σ = 2.0607	μ = −0.5283 σ = 1.2645
(1,0)	μ = 2.8491 σ = 3.4741	μ = 0.28427 σ = 1.4116	μ = 0.092108 σ = 0.33885
(0,1)	μ = 2.8756 σ = 3.1283	μ = 0.2784 σ = 1.3977	μ = 0.066252 σ = 0.24343
(6,0)	μ = 1.7929 σ = 4.62	μ = 0.0041396 σ = 2.9392	μ = −1.0166 σ = 2.0182
(0,6)	μ = 2.7794 σ = 3.7424	μ = 0.26159 σ = 1.3979	μ = 0.01697 σ = 0.067415

Table 1: Comparison of forcing method (non-optimal observers). LQR. μ and standard deviation σ for comparison. Note that the table is reproduced from Cooper *et al.* [5]

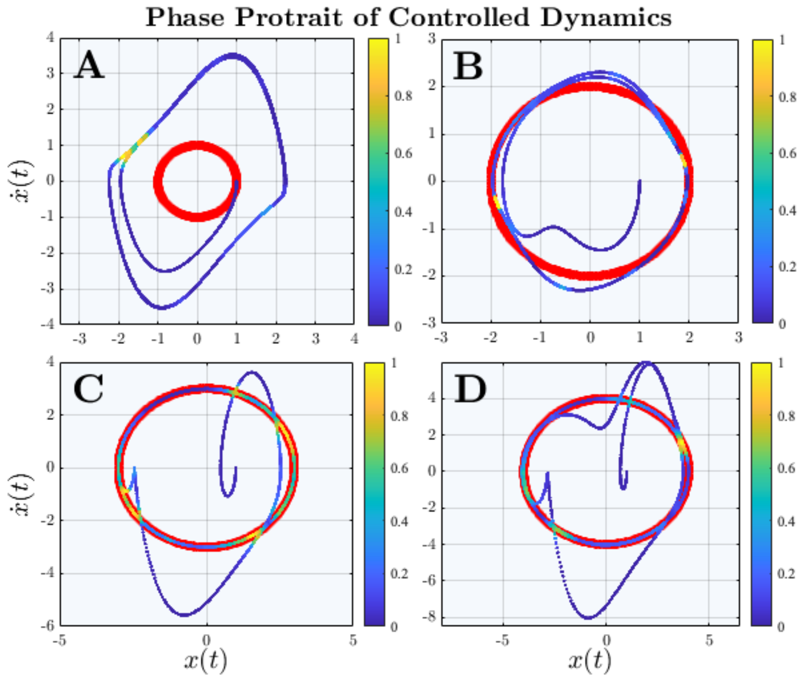

Figure 9. The stability of the van der Pol dynamics regarding four different desired trajectory with radius: $r = 1$ to 4 from subfigures **A** to **D**.

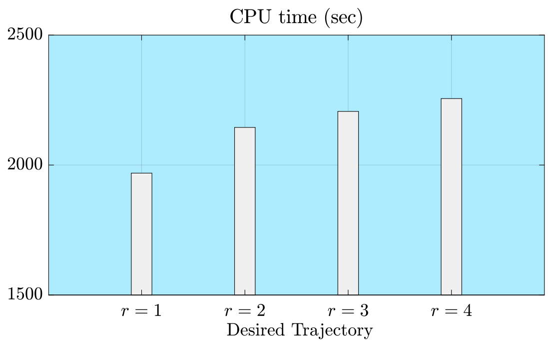

Figure 10. The CPU computation time for the four different desired trajectories.

by nonlinear feed-forward control for the van der Pol dynamics in Table 1. The table quantitatively showed that the forced feed-forward control does display smaller errors.

4.2. *Stability of feed-forward control*

As the desired trajectory changes, the stability of the controlled scheme may also change. Figure 9 shows how changing of radius may change the convergence of controlled signals: when radius are at 3 to 4, the controlled signal converge to the desired trajectories (subfigures **C** and **D** in Figure 9), similar with what has been observed when $r = 5$, as in Figure 3. However, when the radius is approximately 1 to 2, it has been observed that the signal did not converge to the desired trajectories so well as in subfigures **A** and **B** in Figure 9. Also, based on the color bar one can deduce the signal tend to be strong at points neighboring the trajectory of inherent unforced van der Pol dynamics. Further, we proposed an explanation for subfigure **A** as the trajectory does not follow the implemented signal: the desired trajectory's radius was way smaller than the original shape of the van der Pol dynamics, which makes the controller exhibit the unforced state. As it is only roughly discussed for the relation between the desired trajectory and controlled state, further investigation including detailed quantitative analysis on how radius variate final state is to be carried out.

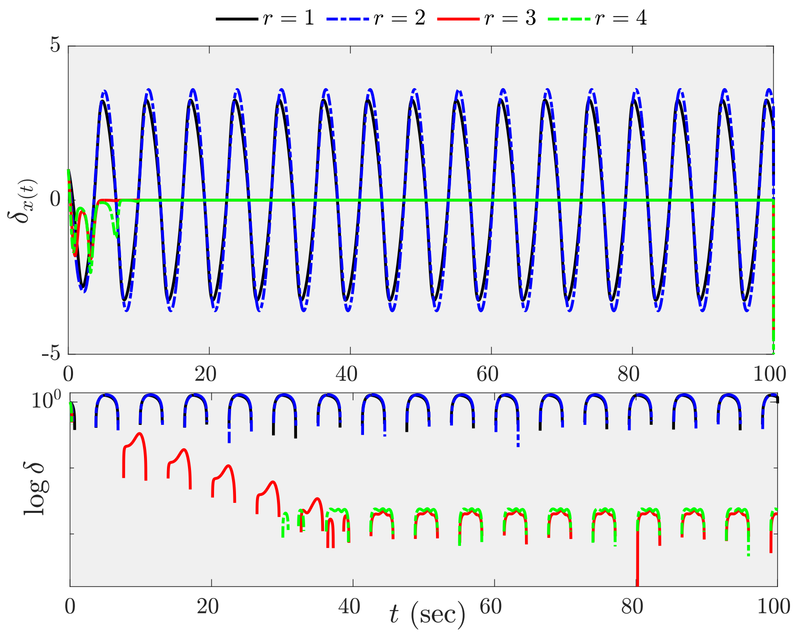

Figure 11. The errors of the four different desired trajectories with plotted in both linear (top) and log scale (down).

Besides, estimation on the different trajectory's effect on computation time are also carried out, as shown in Figure 10. It can be observed that with larger value of radius of desired trajectory, the CPU time increases at a mild rate.

Targeting Figure 9, it remains an open problem on how the errors variate between desired and actual trajectory. From Figure 11 it can be evidently observed that as time increasing, the $r = 3$ and $r = 4$ trajectories converge to the desired trajectory yet the $r = 1$ and $r = 2$ trajectories displays strong periodic fluctuation errors. Plotting the errors in log scale as in the bottom subfigure one can discern that even for $r = 3$ and 4 the errors still displays a periodic fluctuating form, which is of interest for further investigations.

5. Conclusion and outlook

The robust nonlinearity of the van der Pol oscillator has been a strenuous task to control, yet possesses wide applications background from microelectronics to biological systems. Here, taking Cooper *et al.*'s approach [5], we investigate the behavior of forced van der Pol equation with implemented deterministic control algorithm of feedback, feed-forward, and feedback feed-forward combined control approach in comparison with the benchmark problems of two different radii with no control. Simulation results indicate the idealized nonlinear feed-forward control signal displays the best control accuracy, same as proposed by Cooper *et al.* [5]. Further investigation it can be deduced that for linearized feedback control and the combined approach the signal fitting for position $x(t)$ still displays good accuracy, yet the rate and positions display evident errors. What's more, it is found that the changing of the radius can variate the stability of the controlled signal. Targeting this issue, an investigation on the different desired trajectory of radius $r = 1, 2, 3, 4$ was carried out and it has been found that for $r = 3$ and 4 and controlled state still converge to the desired signals, yet for $r = 1$ and 2, a strong trajectory fluctuation occurred. The CPU computation time indicates that as the radius of the desired trajectory increases the computation time increases. The log scaled error plot showed that even for $r = 1$ and 2, which seems to converge to the desired trajectory, there are still periodic fluctuating errors that occur along with time.

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References

1. Chen, G. and Ueta, T., 2002. Chaos in Circuits and Systems, World Scientific Publishing Co., Singapore, ISBN 981-02-4933-00.
2. Gierer A., and Meinhardt H. 1972. A theory of biological pattern. Kybernetik. 12:30–39.
3. Lorenz, E. 1963. Deterministic Non periodic flow, Journal of Atmospheric Sciences
4. van der Pol, B.; van der Mark, J. Frequency de-multiplication. Nature 1927, 120, 363–364.
5. Cooper, M.; Heidlauf, P.; Sands, T. Controlling Chaos—Forced van der Pol Equation. Mathematics 2017, 5, 70.
6. John Guckenheimer, Kathleen Hoffman, and Warren Weckesser. The Forced van der Pol Equation I: The Slow Flow and Its Bifurcations. SIAM J. Appl. Dyn. Syst., 2(1), 1–35.
7. Matthew M. Peet. Systems Analysis and Control. URL: http://control.asu.edu/Classes/MMAE443/443Lecture03.pdf.
8. Alvarez-Gallegos, J., Nonlinear Regulation of a Lorenz System by Feedback Linearization Technique, J. Dynam. Control, 1994, no. 4, pp. 277–298.
9. Chen, L.Q. and Liu, Y.Z., A Modified Exact Linearization Control for Chaotic Oscillators, Nonlin. Dynam., 1999, vol. 20, pp. 309–317.
10. Fradkov, A., Guzenko, P., and Pavlov, A., Adaptive Control of Recurrent Trajectories Based on Linearization of Poincar´e Map, Int. J. Bifurcat. Chaos, 2000, vol. 10, pp. 621–637.
11. Slotine, J. Applied Nonlinear Control; Prantice-Hall: Englewood Cliffs, NJ, USA, 1991; Chapter 9.
12. Sands, T. Physics-Based Control Methods. In Advances in Spacecraft Systems and Orbit Determination; Rushi, G., Ed.; In-Tech Publishers: Rijeka, Croatia, 2012; pp. 29–54.