Problem Session #1

Finite Element Analysis

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Problem 1.1

- Recall strong form:

$$\underbrace{-\left(k(x)u'(x)\right)' + b(x)u'(x) + c(x)u(x)}_{\mathcal{L}(x,u)} = f(x)$$

- For all $x \in \Omega$ domain:

$$\mathcal{L}(u,x) - f = \emptyset$$

- An approximate solution:

$$\hat{\mathcal{L}}(\hat{u},x) - f \neq \emptyset$$

- Construct residual:

$$R_\Omega = \mathcal{L} - \hat{\mathcal{L}} \rightarrow 0$$

Weak Form

- Variational formulation by multiplying test function & integrating:

$$\int_\Omega R_\Omega v \, d\Omega = \emptyset$$

- Construct weak form with integration by part on the boundaries:

$$\int_\Omega R_\Omega v \, d\Omega + \int_\Gamma R_\Gamma v \, d\Gamma = \emptyset$$

- Residual over domain and boundaries
- R_Ω: linear combinations of basis functions (Galerkin method)

$$\sum_{m=1}^{M} A_m N_m$$

- $v(x)$: test function, integral must be well-behaved

Boundary Conditions

- Dirichlet B.C.s: $u(x = a) = g_0$
- Neumann B.C.s: $u'(x = b) = d_L$
- Robin B.C.s: $u'(x = c) + U(x = c) = \alpha$
- Trial space (for random function $w(x)$):

$$\mathcal{S} = \{w : \Omega \to \mathbb{R} \text{ smooth}\}$$

- Test space:

$$\mathcal{V} = \{w : \Omega \to \mathbb{R} \text{ smooth}\}$$

- Purpose of trial functions: *approximate the solution*

$$u(x) = a + bx + cx^2 + \dots$$

- Purpose of test functions: test how well the trial function satisfies governing equations (e.g., PDE)

Example (1.10)

Problem definition:

$$\text{Find } u : [a, b] \to \mathbb{R}, \text{ such that}$$

PDE:

$$u''' = f, \quad x \in (a, b)$$

w/ B.C.s:

$$u(a) = 1$$

$$u'(b) = 2$$

$$u''(a) = 3$$

Solution (Exact)

$$\int_a^x f(y)\,dy = \int_a^x u'''\,dy = u''(x) - u''(a) = u''(x) - 3$$

Example Cont.

$$\int_a^b \int_a^z f(y)\,dy\,dz = \int_a^b u''(z) - 3\,dz = u'(x) - u'(b) - 3(x-b)$$

$$\int_a^x \int_a^w \int_a^z f(y)\,dy\,dz\,dw = \int_a^x u''(w)\,dw - 3\int_a^x (w-b)\,dw$$

$$= u(x) - u(b) - \frac{3}{2}(x^2 - a^2) + 3b(x-a)$$

$\cdots$ *further integration will lead us to the exact solution*

Exact Solution and Variational Form

Exact solution writes:

$$u(x) = 1 + (x - a) + \frac{3}{2}(x^2 - a^2) + \int_a^x \int_b^w \int_a^z f(y)\,dy\,dz\,dw$$

Solving it with variational method:

(a) Form the residual, $R = u'' - f$

$$\int_a^b (u'' - f)v\,dx = 0, \qquad \text{for all } v \text{ smooth}$$

(b) Integration by parts:

$$u'(b)v(b) - u'(a)v(a) - \int_a^b u''v'\,dx + \int_a^b fv\,dx = 0$$

Exact Solution and Variational Form

(c) Substitute boundary conditions:

We know $u''(a) = 3$, hence $-3v(a) - \int_a^b u''v' + \int_a^b fv\,dx = 0$

(d) Formulate the weak form:

- Essential B.C.s: $u(a) = 1$ and $u'(b) = 2$
- Let:

$$\mathcal{S} = \{u : [a,b] \to \mathbb{R} \text{ smooth} \mid u(a) = 1, u'(b) = 2\}$$
$$\mathcal{V} = \{v : [a,b] \to \mathbb{R} \text{ smooth} \mid v(b) = 0\}$$

(e) Weak form of the problem:

Find $u \in \mathcal{S}$ such that $\forall v \in \mathcal{V}$: $\int_a^b u''v'\,dx = \int_a^b fv\,dx - 3v(a)$