MAE 6110: HW #8

Hanfeng Zhai*

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1. The problem of a spherical cavity in a neo-Hookean solid under internal pressure is a famous problem in the theory of nonlinear elasticity. In this HW, I will lead you through the solution.

A spherical cavity in an incompressible neo-Hookean solid under a constant internal pressure P. The key assumptions are:

- Spherical symmetry: the deformation is spherically symmetric.
- The pressure on the surface of deformed sphere remains the same.
- The initial radius of the sphere is and its center is placed at the origin O.

1a. Due to symmetry, in spherical coordinates, all the displacements are zero except the radial displacement, that is

$$\mathbf{X} = R\mathbf{E}_R, \ \mathbf{x} = r(R)\mathbf{e}_r$$

where R is the radial distance from the origin O. Here $(\mathbf{E}_R, \mathbf{E}_\Phi, \mathbf{E}_\Theta)$ are the orthonormal basis vectors in the spherical coordinates (R, Θ, Φ) in the reference configuration. Draw a figure to illustrate (1) to convince yourself. We denote the basis vectors in the current configuration by $(\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_\theta)$. The spherical coordinates in the current configuration is (r, θ, ϕ). First, show

$$\mathbf{F} = \frac{dr}{dR}\mathbf{e}_r\mathbf{E}_R + \frac{r}{R}\mathbf{e}_\theta\mathbf{E}_\Theta + \frac{r}{R}\mathbf{e}_\phi\mathbf{E}_\Phi$$

(remember the basis vectors are no longer independent of position!)

Solution: based on the previous lecture notes, we already know that

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \tag{1}$$

Here, $\mathbf{x} = \mathbf{x}(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ and $\mathbf{X} = \mathbf{X}(\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_\Phi)$.

Equation (1) can be further expanded to

$$\frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial x_1}{\partial X_1}\mathbf{e}_1\mathbf{E}_1 + \frac{\partial x_2}{\partial X_2}\mathbf{e}_2\mathbf{E}_2 + \frac{\partial x_3}{\partial X_3}\mathbf{e}_3\mathbf{E}_3 \tag{2}$$

* [redacted]ww.hanfengzhai.net
Sibley School of Mechanical and Aerospace Engineering, Cornell University

According to the relation between Cartesian and sphereical coordinates, we know that

$$\begin{aligned} x_1 &= r\sin\theta\cos\phi \\ x_2 &= r\sin\theta\sin\phi \\ x_3 &= r\cos\theta \end{aligned} \tag{3}$$

and

$$\begin{aligned} X_1 &= R\sin\Theta\cos\Phi \\ X_2 &= R\sin\Theta\sin\Phi \\ X_3 &= R\cos\Theta \end{aligned} \tag{4}$$

Plugging in Equations (3) *&* (4) into Equation (2) we have

$$\begin{aligned} \mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} &= \frac{\partial(r\sin\theta\cos\phi)}{\partial(R\sin\Theta\cos\Phi)}\mathbf{e}_r\mathbf{E}_R + \frac{\partial(r\sin\theta\sin\phi)}{\partial(R\sin\Theta\sin\Phi)}\mathbf{e}_\theta\mathbf{E}_\Theta + \frac{\partial(r\cos\theta)}{\partial(R\cos\Theta)}\mathbf{e}_\phi\mathbf{E}_\Phi \\ &= \left(\frac{\partial r}{\partial R}\frac{\partial(\sin\theta\cos\phi)}{\partial(\sin\Theta\cos\Phi)}\right)\mathbf{e}_r\mathbf{E}_R + \left(\frac{\partial r}{\partial R}\frac{\partial(\sin\theta\sin\phi)}{\partial(\sin\Theta\sin\Phi)}\right)\mathbf{e}_\theta\mathbf{E}_\Theta + \left(\frac{\partial r}{\partial R}\frac{\partial(\cos\theta)}{\partial(\cos\Theta)}\right)\mathbf{e}_\phi\mathbf{E}_\Phi \\ &= \frac{\partial r}{\partial R}\frac{\partial(\sin\theta\cos\phi)}{\partial(\sin\Theta\cos\Phi)}\mathbf{e}_r\mathbf{E}_R + \frac{r}{R}\left(\frac{\partial(\sin\theta\sin\phi)}{\partial(\sin\Theta\sin\Phi)}\right)\mathbf{e}_\theta\mathbf{E}_\Theta + \frac{r}{R}\left(\frac{\partial(\cos\theta)}{\partial(\cos\Theta)}\right)\mathbf{e}_\phi\mathbf{E}_\Phi \end{aligned} \tag{5}$$

Here, as given in the instructions, applying the symmetry condition in spherical coordinates, we know θ and ϕ remains the same between the current configuration and reference configuration, therefore we have

$$\frac{\partial(\sin\theta\cos\phi)}{\partial(\sin\Theta\cos\Phi)} = 1, \quad \frac{\partial(\sin\theta\sin\phi)}{\partial(\sin\Theta\sin\Phi)} = 1, \quad \frac{\partial(\cos\theta)}{\partial(\cos\Theta)} = 1 \tag{6}$$

Substitute Equation (6) back to Equation (5) we have:

$$\mathbf{F} = \frac{dr}{dR}\mathbf{e}_r\mathbf{E}_R + \frac{r}{R}\mathbf{e}_\theta\mathbf{E}_\Theta + \frac{r}{R}\mathbf{e}_\phi\mathbf{E}_\Phi \tag{7}$$

The equation is therefore proved.

1b. Show that, for an incompressible neo-Hookean solid, the true stress tensor is diagonal and

$$\sigma_{rr} = -p + \mu\lambda^{-4}, \quad \sigma_{\theta\theta} = \sigma_{\phi\phi} \equiv \sigma = -p + \mu\lambda^2$$

where $\mu = 2c_1$ is the small strain shear modulus and $\lambda = r/R$. Give a physical interpretation of λ.

Solution: First, recall the equation for strain energy for incompressible neo-Hookean solids:

$$\phi = c_1(I_1 - 3) \tag{8}$$

where $I_1 = \mathrm{tr}\mathbf{C}$, $\longrightarrow \mathbf{C} = \mathbf{F}^T\mathbf{F}$.

Recall the form of the Cauchy stress in lecture notes {Equation (26) in <u>CONSTITUTIVE MODEL</u>}:

$$\sigma = -p\mathbf{I} + 2\left[\left(\frac{\partial\Phi}{\partial I_1} + I_1\frac{\partial\Phi}{\partial I_2}\right)\mathbf{b} - \frac{\partial\Phi}{\partial I_2}\mathbf{b}^2\right] \tag{9}$$

which in this problem turns into

$$\sigma = -p\mathbf{I} + 2\left(\frac{\partial \Phi}{\partial I_1}\right)\mathbf{b} \tag{10}$$

where $\mathbf{b} = \mathbf{F}^T\mathbf{F}$. Further expanding Equation (10) we have

$$[\sigma] = -p\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} + 2\begin{bmatrix}c_1 & 0 & 0\\ 0 & c_1 & 0\\ 0 & 0 & c_1\end{bmatrix}\begin{bmatrix}\left(\frac{dr}{dR}\right)^2 & 0 & 0\\ 0 & \left(\frac{r}{R}\right)^2 & 0\\ 0 & 0 & \left(\frac{r}{R}\right)^2\end{bmatrix} \tag{11}$$

Further expanding the term we have

$$[\sigma] = \begin{bmatrix}-p + 2c_1\left(\frac{dr}{dR}\right)^2 & 0 & 0\\ 0 & -p + 2c_1\left(\frac{r}{R}\right)^2 & 0\\ 0 & 0 & -p + 2c_1\left(\frac{r}{R}\right)^2\end{bmatrix} \tag{12}$$

Due to $\lambda = \frac{r}{R}$, and due to incompressibility: $J = \det \mathbf{F} = (\frac{dr}{dR})^2(\frac{r}{R})^4 = 1$, we know that $(\frac{dr}{dR}) = (\frac{R}{r})^2$. Therefore Equation (12) can be rewritten into

$$[\sigma] = \begin{bmatrix}-p + 2c_1\lambda^{-4} & 0 & 0\\ 0 & -p + 2c_1\lambda^2 & 0\\ 0 & 0 & -p + 2c_1\lambda^2\end{bmatrix} \tag{13}$$

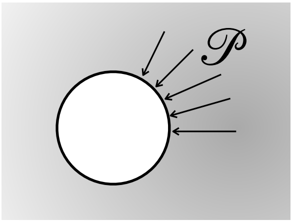
The condition is therefore proved.

1c. Show that the equilibrium equation in current configuration and in spherical coordinates is

$$r\frac{d\sigma_{rr}}{dr} + 2[\sigma_{rr} - \sigma] = 0$$

where σ is the true stress. If the radius of the deformed sphere is b, what is the boundary condition, i.e.,

$$\sigma_{rr}(r = -b) = ??$$



Solution: To show the given form, we apply the *del* operator to the Cauchy stress tensor in spherical coordinates, writing out the equilibrium equation:

$$\begin{aligned}\nabla \cdot \sigma = &\left[\frac{\partial \sigma_{rr}}{\partial r} + 2\frac{\sigma_{rr}}{r} + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\cot\theta}{r}\sigma_{\theta r} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi r}}{\partial \phi} - \frac{1}{r}(\sigma_{\theta\theta} + \sigma_{\phi\phi})\right]\mathbf{e}_r\\ &+ \left[\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\cot\theta}{r}\sigma_{\theta\theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\theta}}{\partial \phi} + \frac{\sigma_{\theta r}}{r} - \frac{\cot\theta}{r}\sigma_{\phi\phi}\right]\mathbf{e}_\theta\\ &+ \left[\frac{\partial \sigma_{r\phi}}{\partial r} + 2\frac{\sigma_{r\phi}}{r} + \frac{1}{r}\frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{\sigma_{\phi r}}{\partial r} + \frac{\cot\theta}{r}(\sigma_{\theta\phi} + \sigma_{\phi\theta})\right]\mathbf{e}_\phi\end{aligned} \tag{14}$$

Due to the symmetric nature of the problem, we know that $\nabla_\phi, \nabla_\theta = 0$. And the shear terms $\sigma_{\alpha\beta} = 0\ (\alpha \neq \beta)$. Therefore the equation can be further expanded to

$$\begin{aligned}
0 &= \left[\frac{\partial \sigma_{rr}}{\partial r} + 2\frac{\sigma_{rr}}{r} + \frac{\cot\theta}{r}\sigma_{\theta r} - \frac{1}{r}(\sigma_{\theta\theta} + \sigma_{\phi\phi})\right]\mathbf{e}_r \\
&+ \left[\frac{2\sigma_{r\theta}}{r} + \frac{\cot\theta}{r}\sigma_{\theta\theta} + \frac{1}{r\sin\theta} + \frac{\sigma_{\theta r}}{r} - \frac{\cot\theta}{r}\sigma_{\phi\phi}\right]\mathbf{e}_\theta + \left[2\frac{\sigma_{r\phi}}{r} + \frac{1}{r}\frac{\partial\sigma_{\theta\phi}}{\partial\theta} + \frac{\sigma_{\phi r}}{\partial r} + \frac{\cot\theta}{r}(\sigma_{\theta\phi} + \sigma_{\phi\theta})\right]\mathbf{e}_\phi \\
&= \frac{\partial\sigma_{rr}}{\partial r} + \left[2\frac{\sigma_{rr}}{r} - \frac{\sigma_{\theta\theta}}{r} - \frac{\sigma_{\phi\phi}}{r}\right]
\end{aligned} \tag{15}$$

The equation is therefore proved.

For the boundary conditions, we assume a constant pressure (internal) $\mathcal{P}$ acting on the hollow sphere, the BCs thence writes:

$$\sigma_{rr}(r = -b) = -\mathcal{P}$$

1d. Change the independent variable from r to λ and show that the equilibrium equation (4) can be written as

$$\frac{d\sigma_{rr}}{d\lambda} = -2\mu\lambda^{-5}(1 + \lambda^3)$$

Solution: To derive the given equation, according to chain rule, we know:

$$\frac{d\sigma_{rr}}{dr} = \frac{d\sigma_{rr}}{d\lambda}\frac{d\lambda}{dr} \tag{16}$$

Here, r is the independent variable, therefore

$$\frac{d}{dr} = \frac{d}{d\lambda}\frac{d\lambda}{dr} = \frac{d}{d\lambda}\frac{d\left(\frac{r}{R(r)}\right)}{dr} = \frac{d}{d\lambda}\frac{R - r\frac{dR}{dr}}{R^2} \tag{17}$$

Hence, we can deduce

$$\frac{d\lambda}{dr} = \frac{1}{R} - \frac{r}{R^2}\frac{dR}{dr} \tag{18}$$

Now, we substitute Equation (18) back into the Equilibrium equation in spherical coordinate, we have

$$\begin{aligned}
r\frac{d\sigma_{rr}}{d\lambda}\frac{d\lambda}{dr} + 2(\sigma_{rr} - \sigma) &= 0 \\
r\frac{d\sigma_{rr}}{d\lambda}\left(\frac{1}{R} - \frac{r}{R^2}\frac{dR}{dr}\right) + 2(-p + \mu\lambda^{-4} - \sigma) &= 0 \\
r\frac{d\sigma_{rr}}{d\lambda}\left(\frac{Rdr - rdR}{R^2dr}\right) &= 2(p - \mu\lambda^{-4} + \sigma)
\end{aligned} \tag{19}$$

Further,

$$\begin{aligned}\frac{d\sigma_{rr}}{d\lambda} &= \frac{2}{r}\frac{R^2dr}{Rdr - rdR}(p - \mu\lambda^{-4} + \sigma) \\ &= \frac{2R^2dr(p - \mu\lambda^{-4} + \sigma)}{r(Rdr - rdR)} \\ &= \frac{2(p - \mu\lambda^{-4} + \sigma)}{\frac{r}{R^2dr}(Rdr - rdR)} \\ &= \frac{2(p - \mu\lambda^{-4} - p + \mu\lambda^2)}{\lambda - \lambda^2(\frac{dR}{dr})}\end{aligned} \tag{20}$$

$$\begin{aligned}&= \frac{2(-\mu\lambda^{-4} + \mu\lambda^2)}{\lambda\left[1 - \lambda(\frac{dR}{dr})\right]} \\ &= \frac{-2\mu(\lambda^{-5} - \lambda)}{1 - \lambda(\frac{dR}{dr})} \\ &= \frac{-2\mu(\lambda^{-5} - \lambda)}{1 - \lambda^3} \\ &= \frac{-2\mu(\lambda^{-5} - \lambda)}{\lambda^5(\lambda^{-5} - \lambda^{-2})} \\ &= \frac{-2\mu(\lambda^{-5} - \lambda^{-2} + \lambda^{-2} - \lambda)}{\lambda^5(\lambda^{-5} - \lambda^{-2})} \\ &= \frac{-2\mu(\lambda^{-5} - \lambda^{-2})}{\lambda^5(\lambda^{-5} - \lambda^{-2})} + \frac{-2\mu(\lambda^{-2} - \lambda)}{\lambda^5(\lambda^{-5} - \lambda^{-2})} \\ &= \frac{-2\mu}{\lambda^5} + \frac{-2\mu(\lambda^{-2} - \lambda)}{\lambda^2(\lambda^{-2} - \lambda)} \\ &= \frac{-2\mu}{\lambda^5} - \frac{2\mu}{\lambda^2}\end{aligned} \tag{21}$$

The equation is therefore proved.

1e. Solve (6) with the appropriate boundary condition and show that the solution is:

$$\sigma_{rr}(\lambda) = 2\mu\left(\frac{-5 + \lambda^{-4}}{4} + \lambda^{-1}\right)$$

Solution: Now, we substitute the condition $\{\text{when } \lambda = 1 : \sigma_{rr} = 0\}$ in the integration of Equation (21), and compute the constant:

$$\begin{aligned}\sigma_{rr} &= -2\mu\int \lambda^{-5}(1 + \lambda^3)d\lambda \\ &= -2\mu\int [\lambda^{-5} + \lambda^{-2}]d\lambda \\ &= \frac{1}{2}\mu\lambda^{-4} + 2\mu\lambda^{-1} + \mu Const. \\ &\xrightarrow{\lambda=1:\sigma_{rr}=0} Const. = -\frac{5}{2} \\ \Longrightarrow \frac{\sigma_{rr}}{\mu} &= \frac{1}{2}\lambda^{-4} + 2\lambda^{-1} - \frac{5}{2}\end{aligned} \tag{22}$$

The equation is therefore proved.

1f. Denote the stretch ratio λ at the surface of the deformed sphere by λ_0 and use your solution in 1e to find out the dependence of λ_0 on the applied pressure P. Plot this dependence. Explain in your own words what this solution mean.

Solution: The pressure P dependence to the stretch λ_0 graph is shown as below. The code for plotting it is also attached as below. As can be deduced from the figure, when the pressure approach ≈ 2, the deformed stretch λ_0 approaches infinity. This solution means when the pressure on the sphere approaches 2, the body approximates the compressed limit.

```
close;clc;clear

for l0 = 0.5:0.01:5
    R = 5; % set initial radius
    l = l0; % stretch
    P = -(0.5 .* l.^(-4) + 2.*(1./l) - 2.5); %calculated from normalized stress
    scatter(P,l0,30,'black');hold on
end
```