MAE 6110: HW #12

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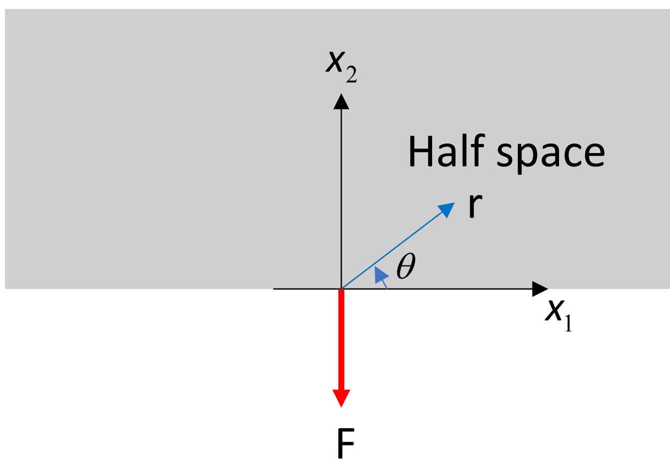
December 8, 2021

1. A line load, of magnitude F, is imposed on the surface of a linear elastic incompressible isotropic half space with shear modulus G. Assuming plane strain deformation, find the stress σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$, σ_{zz} and the associated strains ϵ_{rr}, $\epsilon_{\theta\theta}$, $\epsilon_{r\theta}$, ϵ_{zz} using a polar coordinate system. Also find the displacement fields. *Discuss in detail your displacement result.*



Hints:

1. Use Polar coordinate system (r, θ) (see Figure 1 above). Specifically, write all governing equations in Polar coordinates.

2. Dimensional analysis implies that the in-plane stress fields must have the form:

$$\sigma_{\alpha\beta} = \frac{F}{r} f_{\alpha\beta}(\theta) \tag{1}$$

where F is the applied line force, $F > 0$ implies tension. Here r, θ is a polar coordinate system with $0 \le \theta \le \pi$. The line force acts at the origin where $r = 0$.

3. The definition of the line force means that on any semi-circle with radius R with center at the origin, we must have

$$R \int_0^\pi \mathbf{T} \cdot \mathbf{e}_r d\theta - F\mathbf{e}_2 = 0 \tag{2}$$

4. Calculation of displacements can be a bit tedious since you have to integrate the strains.

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Solution: We first write out all the governing equations for this problem under polar coordinate. The first is the **equilibrium equation**:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0 \tag{3}$$

The **constitutive equation** for linear elasticity (*Hooke's law*):

$$\begin{aligned} \epsilon_{rr} &= \frac{1}{2G}((1-\nu)\sigma_{rr} - \nu\sigma_{\theta\theta}) \\ \epsilon_{\theta\theta} &= \frac{1}{2G}((1-\nu)\sigma_{\theta\theta} - \nu\sigma_{rr}) \end{aligned} \tag{4}$$

And the **kinematics**:

$$\epsilon_{rr} = u_{r,r}; \quad \epsilon_{\theta\theta} = u_{\theta,\theta}; \quad \epsilon_{r,\theta} = \frac{1}{2}(u_{r,\theta} + u_{\theta,r}) \tag{5}$$

The **boundary conditions** write:

$$\sigma_{rr}(\theta = \frac{\pi}{2}) = F \tag{6}$$

Taking the **hints** we can also write the BCs as:

$$r\int_0^{\pi} \mathbf{T}\cdot\mathbf{e}_r d\theta = F\mathbf{e}_2 \tag{7}$$

which can be written into:

$$\int_0^{\pi} \sigma_{rr}\mathbf{e}_r d\theta = \frac{F}{r}\mathbf{e}_2 \tag{8}$$

Recall the stress function, we can write out the three stresses:

$$\begin{aligned} \sigma_{rr} &= \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \\ \sigma_{\theta\theta} &= \frac{\partial^2 \phi}{\partial r^2} \\ \sigma_{r\theta} &= \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} + \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} \end{aligned} \tag{9}$$

Now we recall the general form of the stress function in **lecture notes**:

$$\begin{aligned} \phi = a_0 \log r + b_0 r^2 + c_0 r^2 \log r + d_0 r^2\theta + a_0'\theta + \\ \frac{a_1'}{2} r\theta\sin\theta + (b_1 r^3 + a_1' r^{-1} + b_1' r\log r)\cos\theta - \frac{c_1}{2} r\theta\cos\theta + (d_1 r^3 + c_1' r^{-1} + d_1' r\log r)\sin\theta \\ + \sum_{n=2}^{\infty}\left(a_n r^n + b_n r^{n+2} + a_n' r^{-n} + b_n' r^{-n+2}\right)\cos n\theta + \sum_{n=2}^{\infty}\left(c_n r^n + d_n r^{n+2} + c_n' r^{-n} + d_n' r^{-n+2}\right)\sin n\theta \end{aligned} \tag{10}$$

Due to the observed symmetric nature of this problem, we know that for both the stresses and the stress function should be symmetric in the range $\theta \subset [0, \pi]$. Therefore, only the $\sin\theta$ terms will be left and the $\cos\theta$ terms are eliminated automatically.

Now, taking the hint that the stress should take the form $\sigma_{\alpha\beta} = F f_{\alpha\beta}(\theta)/r$, substitute it in Equation (9) we have:

$$\frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = \frac{F f_{\alpha\beta}(\theta)}{r} \tag{11}$$

we further have

$$\frac{\partial\phi}{\partial r} + \frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2} = F f_{\alpha\beta} \tag{12}$$

Some analysis we can get the general form of the stress function ϕ:

$$\phi = r g(\theta) \tag{13}$$

Now we can apply the *del* operator in polar coordinate

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} \tag{14}$$

Applying this to the stress function to generate the biharmonic equation we have

$$\nabla^4\phi = 0 \Longrightarrow \frac{1}{r^3}\left(g(\theta) + 2\frac{d^2}{d\theta^2}g(\theta) + \frac{d^4}{d\theta^4}g(\theta)\right) = 0 \tag{15}$$

Solving this equation with either in `python` or MATLAB, the solution can be simplified to:

$$g(\theta) = A\cos\theta + B\theta\cos\theta + C\sin\theta + D\theta\sin\theta \tag{16}$$

We thence know the stress function:

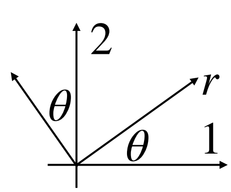
$$\phi(\theta, r) = rA\cos\theta + rB\theta\cos\theta + rC\sin\theta + rD\theta\sin\theta \tag{17}$$

Recall Equation (9), we can write out the general form of the stresses:

$$\begin{aligned} \sigma_{rr} &= -\frac{2B\sin\theta}{r} + \frac{2D\cos\theta}{r} \\ \sigma_{\theta\theta} &= 0 \\ \sigma_{r\theta} &= 0 \end{aligned} \tag{18}$$

For the stress in the longitudinal direction:

$$\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta}) \tag{19}$$



Now, given the hints, since we know $\mathbf{e}_r = \mathbf{e}_2\sin\theta$, we can substitute the BCs from the hints:

$$\int_0^{\pi} \sigma_{rr}\sin\theta d\theta = \frac{F}{r} \tag{20}$$

Since in our problem formulation we should expect σ_{rr} is symmetric in range $[0, \pi]$, then we should neglect the cos terms. The stress writes

$$\sigma_{rr} = -\frac{2B\sin\theta}{r} \tag{21}$$

Plug in the BCs we have

$$\begin{aligned} -\int_0^{\pi} \frac{2B}{r}\sin^2\theta d\theta &= \frac{F}{r} \\ -\frac{B\pi}{r} &= \frac{F}{r} \end{aligned} \tag{22}$$

We hence have

$$B = -\frac{F}{\pi} \tag{23}$$

Then we have

$$\sigma_{rr} = \frac{2F\sin\theta}{\pi r} \tag{24}$$

We can also compute the stress σ_{zz}:

$$\sigma_{zz} = \frac{\nu F\sin\theta}{\pi r} \tag{25}$$

Now, we recall the constitutive model of linear elasticity for the terms of strain:

$$\begin{aligned} \epsilon_{rr} &= \frac{1}{2G}((1-\nu)\sigma_{rr} - \nu\sigma_{\theta\theta}) \\ \epsilon_{\theta\theta} &= \frac{1}{2G}((1-\nu)\sigma_{\theta\theta} - \nu\sigma_{rr}) \end{aligned} \tag{26}$$

Since this problem already take the assumption of incompressibility we also know $\nu = 0.5$. Substitute it back into the constitutive model we have

$$\begin{aligned} \epsilon_{rr} &= \frac{1}{2G}\frac{F\sin\theta}{\pi r} \\ \epsilon_{\theta\theta} &= -\frac{1}{2G}\frac{F\sin\theta}{\pi r} \end{aligned} \tag{27}$$

Due to **kinematics**, we can hence compute the displacements with integration:

$$\begin{aligned} u_r &= \int \epsilon_{rr} dr = \frac{F\sin\theta}{2G\pi}\log(r) + C_1 \\ u_\theta &= \int \epsilon_{\theta\theta} d\theta = \frac{F\cos\theta}{2G\pi r} + C_2 \end{aligned} \tag{28}$$

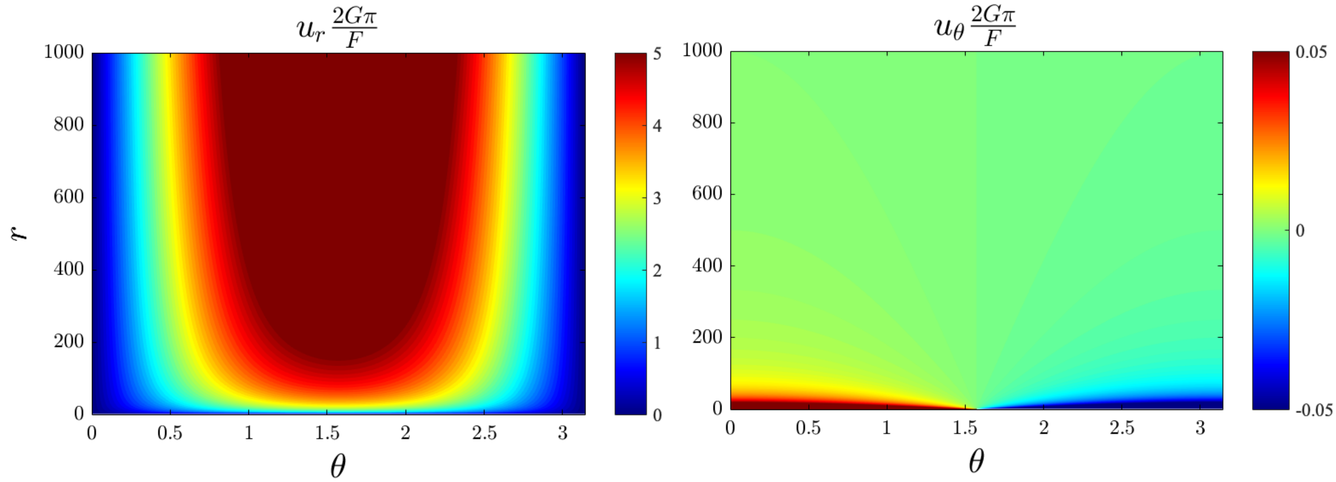
where C_1 and C_2 are constants to be determined.

Due to the fact that when $r \Longrightarrow \infty$, $u_r = u_\theta = 0$; we can deduce that $C_1 = C_2 = 0$.

Therefore we generate the final expression for the two displacements field, (I was struggled with the nonphysical expression of the displacement u_r, and referred to https://imechanica.org/node/319):

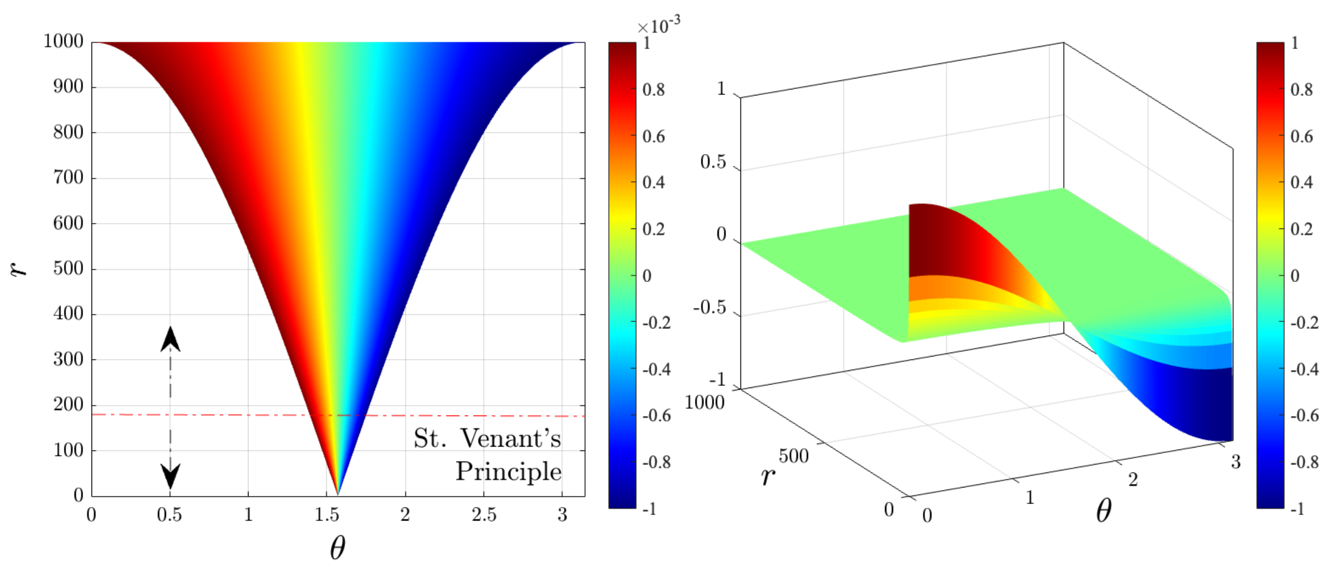
$$\begin{aligned} u_r &= \frac{F\sin\theta}{2G\pi}\log r \\ u_\theta &= \frac{F\cos\theta}{2G\pi r} \end{aligned} \tag{29}$$

To better illustrate our results, we plotted the following graph:



From the graphical representation we can deduce that for u_r the displacements increases as r keep increasing and the its value drops to zero as $r \longrightarrow 0$; from u_θ we observe that the value goes extremely huge as $r \longrightarrow 0$, same as u_r, and the value are symmetric whose directions are opposite to each other as r increases.

However, though the illustration for u_r is clear and easy to understand, the solution for u_θ is bit subtle and confusing. According to Saint Venant's principle, we know that the physical fields approaches to the loading can be somewhat imprecise, hence we set another detailed analysis plot for u_θ (normalized to the form $u_\theta \frac{2G\pi}{F}$), as shown in the following figure.



In the left sub figure we deliberately mark a red dotted line (≈ 200) to illustrate the area considered "close" to the loading area as governed by the St. Venant's principle, and the right sub figure is a 3D illustration for the displacement field. From the figure we can observe the displacement field behave like a "type-2" crack and as r approaches a large value the displacement value are reversed directions in the same magnitude, shaping like the intersection of two parabolic curves, with values approximating to zero as $\theta \longrightarrow 0$.

Also, from the link I just attached (https://imechanica.org/comment/930#comment-930), Dr. Huang raised a good point and provide some explanations which **make sense** to me: "*I agree with you that the unbounded displacement may be inevitable for the 2D problem. In 3D, the displacement in an* *infinite domain remains bounded under a concentrated force. Consequently, the effect of the load stays local, decaying from the point of action. In 2D, however, the effect of a concentrated force (line force for plane strain) extends to infinity, which seems unphysical to me. As I mentioned in a previous comment, this may be due to the fact that the line force itself extends to infinity by plane strain assumption."*

2. Solve the same problem in (1), except the half space is linear viscoelastic, with shear relaxation modulus $Y_1(t) = Y_\infty + (Y_0 - Y_\infty)e^{-t/t_R}$. *The initial condition is that the half space has no deformation and stress for $t < 0$ and the line load is imposed such that*

$$\begin{cases} 0, & t < 0 \\ \frac{F_0}{T}t, & T \geq t \geq 0 \\ F_0, & t \geq T \end{cases} \tag{30}$$

Express your answers in terms of Y_0, Y_∞ and t_R, T and F_0. You do not have to compute the displacement field for this problem, just the stresses and strains.

Solution: In this problem, we first recall the solutions of the stresses in linear elastic system:

$$\begin{aligned} \sigma_{rr} &= \frac{2F\sin\theta}{\pi r} \\ \sigma_{\theta\theta} &= \sigma_{r\theta} = 0 \\ \sigma_{zz} &= \frac{F\sin\theta}{\pi r} \end{aligned} \tag{31}$$

where the only difference with linear elasticity is the applied force is time-dependent:

$$\begin{cases} 0, & t < 0 \\ \frac{F_0}{T}t, & T \geq t \geq 0 \\ F_0, & t \geq T \end{cases} \tag{32}$$

With the given condition of $Y_1(t)$, we can use Laplace transform to evaluate $\tilde{Y}_1(s)$, then apply Fourier's transform to obtain $\tilde{C}_1(s)$, then use reverse Laplace transform to get $C_1(t)$. Note that in the Laplace domain the Creep and Relaxation obeys

$$\tilde{Y}_1\tilde{C}_1 = \frac{1}{s^2} \tag{33}$$

Using MATLAB to evaluate the mentioned process we get the following code:

```
syms Y_infty t t_R Y_0 Y(t) s
Y(t) = Y_infty + exp(-t/t_R) * (Y_0 - Y_infty)
Ys = laplace(Y(t))
Cs = 1/(Ys*s^2)
C(t) = ilaplace(Cs, s, t)
```

And we generate the form for $C_1(t)$:

$$C_1(t) = \frac{1}{Y_\infty} - \frac{e^{-\frac{Y_\infty t}{Y_0 t_R}}(Y_0 - Y_\infty)}{Y_0 Y_\infty} \tag{34}$$

It can also be observed that the stresses are independent of the modulus, hence they also work for linear viscoelastic problems. We can thence calculate strain based on the correspondence principle, writes:

$$\epsilon_{ij}(r,t) = \sigma_{ij}(0^+)C_1(t) + C_1 * \sigma'_{ij} \tag{35}$$

The two strains at short time $t = 0^+$:

$$\begin{aligned} \epsilon_{rr} = \sigma_{rr}(0^+)C_1(t) &= \frac{0^+}{T}\frac{2F_0\sin\theta}{\pi r}\left(\frac{1}{Y_\infty} - \frac{(Y_0 - Y_\infty)}{Y_0\,Y_\infty}\right) \\ \epsilon_{\theta\theta} = -\sigma_{rr}(0^+)C_1(t) &= -\frac{0^+}{T}\frac{2F_0\sin\theta}{\pi r}\left(\frac{1}{Y_\infty} - \frac{(Y_0 - Y_\infty)}{Y_0\,Y_\infty}\right) \end{aligned} \tag{36}$$

When $0 \leq t \leq T$, the strains take the form:

$$\begin{aligned} \epsilon_{rr} &= \sigma_{rr}(0^+)C_1(t) + \frac{2F_0\sin\theta}{\pi r T}\int_{0^+}^{T} C_1(t-\tau)\frac{dt}{d\tau}d\tau \\ &= \sigma_{rr}(0^+)C_1(t)\frac{2F_0\sin\theta}{\pi r T} + \int_{0^+}^{T} C_1(t-\tau)d\tau \\ \epsilon_{\theta\theta} &= \sigma_{rr}(0^+)C_1(t) - \frac{2F_0\sin\theta}{\pi r T}\int_{0^+}^{T} C_1(t-\tau)\frac{dt}{d\tau}d\tau \\ &= \sigma_{rr}(0^+)C_1(t) - \frac{2F_0\sin\theta}{\pi r T} + \int_{0^+}^{T} C_1(t-\tau)d\tau \end{aligned} \tag{37}$$

Therefore we have

$$\begin{aligned} \epsilon_{rr} =& -\frac{0^+}{T}\frac{2F_0\sin\theta}{\pi r}\left(\frac{1}{Y_\infty} - \frac{(Y_0 - Y_\infty)}{Y_0\,Y_\infty}\right) \\ &+ \frac{2F_0\sin\theta}{\pi r T}\left(\frac{T}{Y_\infty} + \frac{t_R\,\mathrm{e}^{-\frac{Y_\infty\,t}{Y_0\,t_R}}\left(\mathrm{e}^{\frac{T\,Y_\infty}{Y_0\,t_R}} - 1\right)}{Y_\infty} - \frac{Y_0\,t_R\,\mathrm{e}^{-\frac{Y_\infty\,t}{Y_0\,t_R}}\left(\mathrm{e}^{\frac{T\,Y_\infty}{Y_0\,t_R}} - 1\right)}{{Y_\infty}^2}\right) \\ \epsilon_{\theta\theta} =& -\frac{0^+}{T}\frac{2F_0\sin\theta}{\pi r}\left(\frac{1}{Y_\infty} - \frac{(Y_0 - Y_\infty)}{Y_0\,Y_\infty}\right) \\ &- \frac{2F_0\sin\theta}{\pi r T}\left(\frac{T}{Y_\infty} + \frac{t_R\,\mathrm{e}^{-\frac{Y_\infty\,t}{Y_0\,t_R}}\left(\mathrm{e}^{\frac{T\,Y_\infty}{Y_0\,t_R}} - 1\right)}{Y_\infty} - \frac{Y_0\,t_R\,\mathrm{e}^{-\frac{Y_\infty\,t}{Y_0\,t_R}}\left(\mathrm{e}^{\frac{T\,Y_\infty}{Y_0\,t_R}} - 1\right)}{{Y_\infty}^2}\right) \end{aligned} \tag{38}$$

When $t \geq T$, we have

$$\begin{aligned} \epsilon_{rr} &= \frac{2F_0\sin\theta}{\pi r}C_1 \\ &= \frac{2F_0\sin\theta}{\pi r}\left(\frac{1}{Y_\infty} - \frac{e^{-\frac{Y_\infty\,t}{Y_0\,t_R}}\,(Y_0 - Y_\infty)}{Y_0\,Y_\infty}\right) \\ \epsilon_{\theta\theta} &= -\frac{2F_0\sin\theta}{\pi r}C_1 \\ &= \frac{2F_0\sin\theta}{\pi r}\left(\frac{1}{Y_\infty} - \frac{e^{-\frac{Y_\infty\,t}{Y_0\,t_R}}\,(Y_0 - Y_\infty)}{Y_0\,Y_\infty}\right) \end{aligned} \tag{39}$$