Notes on Constitutive Model for Surface Contact*

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1 Normal contact

Impose contact conditions: (1) Formulate **non-penetration condition** as geometrical constraint. (2) Develop an elastic or elasto-plastic **constitutive laws**.

- Non-penetration condition: $g_N \leq 0$

Two bodies: $\mathcal{B}_1$ & $\mathcal{B}_2$; Contact takes place when g_N is equal to zero. In this case, the associated normal component p_N^1 of the stress vector:

$$\mathbf{t^1} = \sigma^{\mathbf{1}} \overline{\mathbf{n}}^{\mathbf{1}} = p_N^1 \overline{\mathbf{n}}^{\mathbf{1}} + t_T^{1\beta} \overline{\mathbf{a}}_\beta^{\mathbf{1}}$$

Stress vector $\rightarrow$ obeys $\mathbf{t^1} = \mathbf{t^2}$ in contact point $\overline{\mathbf{x}}^{\mathbf{1}}$. $p_N = p_N^1 = p_N^2 < 0$ (No adhesive stresses will not be allowed in the contact interface throughout our considerations.) Frictionless contact: $t_T^{1\beta} = 0$.

- Contact condition: $g_N = 0$ & $p_N < 0$.
- Gap between the bodies: $g_N > 0$ & $p_N = 0$.

Leads to HERTZ–SIGNORINI–MOREAU conditions for frictionless contact:

$$g_N \geq 0, \quad p_N \leq 0, \quad p_N g_N = 0$$

.(*We here ignore how to formulate nominal stress to simplify*)

- CAUCHY's stress: $\mathbf{t} = \sigma \mathbf{n}$.
- The normal and tangential components follow: $\mathbf{t} = \mathbb{P}\mathbf{t} + (\mathbb{I} - \mathbb{P})\mathbf{t}$

*Personal note on [Wriggers, 2006]

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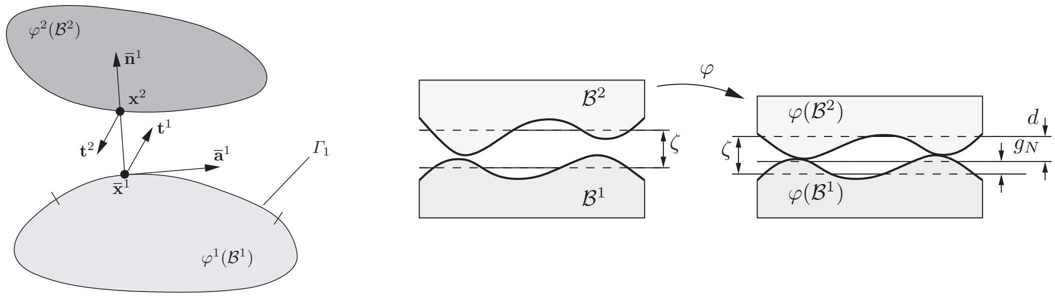

Figure 1: The schematic view for two bodies in contact. The left is the general schematic for two-body contact. The right is the normal contact process.

- Fourth order unit tensor $\mathbf{t} = \mathbb{I}\mathbf{t}$.
- Projection tensor: $\mathbb{P} = \bar{\mathbf{n}}^1 \otimes \bar{\mathbf{n}}^1$, with properties: $\mathbb{P}^2 = \mathbb{P}$ and $\mathbb{P}(\mathbb{I} - \mathbb{P}) = 0$.

1.1 *Constitutive model*

Derivation of constitutive contact equations of two rough surfaces:

- Find a mathematical description of the surface geometry by statistical
- A model which describes the mechanical behavior of one summit of the rough surface under loading

Contact law for normal contact:

$$p_N(\gamma) = \frac{1}{h^2} \int_\gamma^\infty \int_0^\infty N_i(\gamma) P(\zeta_\sigma, \kappa_\sigma) d\kappa d\zeta$$

Where

- $\zeta_\sigma = z/\sigma_z$: normalized asperity height. (height z is measured on a regular grid with spacing h.)
- Curvature k_s yields: $\kappa_\sigma = k_s/\sigma_k$ (rms-curvature σ_k).
- $N_i(\gamma)$: the normal contact force.
- $\gamma = g_N/\sigma_z$: the gap function normalized by the rms-height.
- $P(\zeta_\sigma, \kappa_\sigma)$: the probability distribution of a joint.

$\Longrightarrow$ In most applications, it is sufficient to formulate the constitutive relation for the **apparent contact pressure**:

$$p_N = f(d)$$

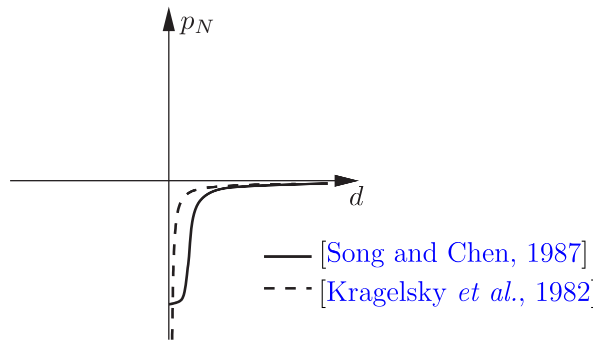

Figure 2: A graphical comparison of the theory of [Song and Chen, 1987] and [Kragelsky *et al.*, 1982].

- f: nonlinear functions
- d: current mean plane distance
- p_N: contact pressure

Mean plane distance d is related to the geometrical approach g_N:

$$g_N = \zeta - d$$

- ζ is the initial mean plane distance in the contact area Γ_c,

1.2 *Some empirical models*

- [Song and Chen, 1987]: constitutive relation for the approach of both surfaces yields an exponential law of the form

  $$p_N = c_3 e^{-c_4 d^3}$$

  A more detailed description:

  $$p_N = \frac{c_1(1617646.152\frac{\sigma}{m})^{c_2}}{5.589^{1+0.0711c_2}} \exp\left[-\frac{1+0.0711c_2}{(1.363\sigma)^2}d^2\right]$$

  $\rightarrow$ σ: statistical parameters; m: surface profile.

- [Kragelsky *et al.*, 1982]: constitutive equation for the real contact pressure:

  $$p_N = c_N d^n = c_N \left(\zeta - g_N\right)^{-1/n}$$

  $\rightarrow$ c_N and n are constitutive parameters that have to be determined by experiments.

2 Tangential contact

(*Same situation as for normal contact when stick in the contact interface is considered.*)

Tangential sliding between bodies: (1) derive a constitutive equation for friction which can be stated in the form of an **evolution equation**.

The interfacial behavior related to frictional response: **Tribology**

$$\text{Tribology} \to \begin{cases} \textbf{Adhesion} \\ \text{Friction} \\ \text{Wear} \\ \text{Lubrication} \\ \text{Thermal contact} \\ \text{Electric contact} \end{cases}$$

There are two ways for solid shear: *stick & slip*.

For stick condition write: $\dot{\mathbf{g}}_T = 0 \Longleftrightarrow \mathbf{g}_T = 0$.

For sliding, adopt the COULOMB law, writes:

$$\mathbf{t}_T = -\mu |p_N| \frac{\dot{\mathbf{g}}_T}{||\dot{\mathbf{g}}_T||}, \quad \text{if } ||\mathbf{t}_T|| > \mu |p_N|$$

μ: sliding friction coefficient (constant in classical COULOMB law) $\to \mu = \mu(\dot{\mathbf{g}}_T, p_N, \theta)$.

A heuristic friction law, incorporates the relative sliding velocity $\dot{\mathbf{g}}_T$:

$$\mu(\dot{\mathbf{g}}_T) = \mu_D + (\mu_S - \mu_D) e^{-c||\dot{\mathbf{g}}_T||}$$

which depends upon three constitutive parameters μ_S, μ_D, & c.

Elasto-plastic analogy for friction

- regularize COULOMB's law
- match experimental observations

Split the slip into three parts: tangential

References

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[Kragelsky *et al.*, 1982] I.V. Kragelsky, M.N. Dobychin, V.S. Kombalov. (1982) Friction and Wear Calculation Methods. Pergamon Press, Oxford.