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Mechanics of biomaterials. Below is an SEM image of dentin, consisting of dentin tubule, peritubular (PT), and intertubular (IT) dentin. You may ask any questions for clarification. Note that (d) is the bonus question. List and justify all your assumptions.

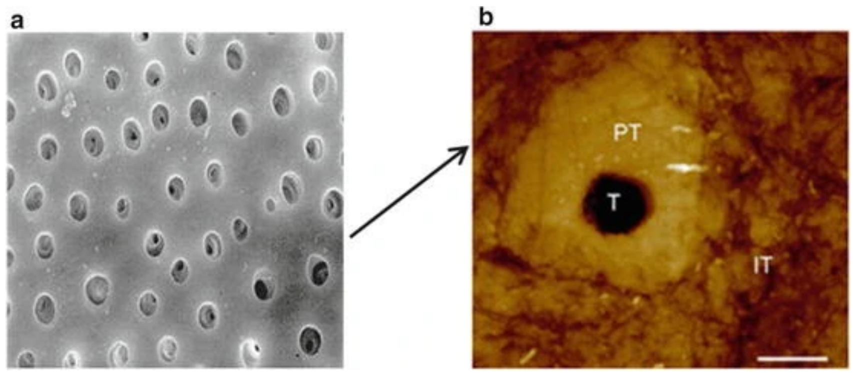

Figure 1: SEM image of dentin from Ref. [1]

(a) Assume PTD and ITD have the same elastic moduli. Simplify a model with just one tubule. During the chewing process, (the material within) the tooth mostly undergoes shear. Derive the stress concentration factor under such a condition.

(b) Now view ITD and PTD as separate objects. PTD has an initial radius of r_P, and r_I for ITD, where $r_P > r_I$. The tubule has a radius of r_T. After inserting PTD into ITD, they both have radii of r_D. Solve the stress fields.

(c) After inserting the PTD, assuming the same loading condition with (a). Solve the stress fields. (Hint: you may assume there are no shape change for the hole on ITD)

(d) In reality, PTD is much stiffer than ITD (approximately 2 to 3 times Young's modulus). Assume they both have yield stress k. Solve for the stress fields after only one of the materials yields. Discuss the effects of plasticity (Comment on related problems for fracture).

References

[1] Mahdi Shahmoradi, Luiz E. Bertassoni, Hunida M. Elfallah, and Michael Swain. *Fundamental Structure and Properties of Enamel, Dentin and Cementum*, page 511–547. Springer Berlin Heidelberg, 2014.

Table 8.1. The Michell solution — stress components

ϕ	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
r^2	2	0	2
$r^2\ln(r)$	$2\ln(r)+1$	0	$2\ln(r)+3$
$\ln(r)$	$1/r^2$	0	$-1/r^2$
θ	0	$1/r^2$	0
$r^3\cos\theta$	$2r\cos\theta$	$2r\sin\theta$	$6r\cos\theta$
$r\theta\sin\theta$	$2\cos\theta/r$	0	0
$r\ln(r)\cos\theta$	$\cos\theta/r$	$\sin\theta/r$	$\cos\theta/r$
$\cos\theta/r$	$-2\cos\theta/r^3$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$
$r^3\sin\theta$	$2r\sin\theta$	$-2r\cos\theta$	$6r\sin\theta$
$r\theta\cos\theta$	$-2\sin\theta/r$	0	0
$r\ln(r)\sin\theta$	$\sin\theta/r$	$-\cos\theta/r$	$\sin\theta/r$
$\sin\theta/r$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$	$2\sin\theta/r^3$
$r^{n+2}\cos n\theta$	$-(n+1)(n-2)r^n\cos n\theta$	$n(n+1)r^n\sin n\theta$	$(n+1)(n+2)r^n\cos n\theta$
$r^{-n+2}\cos n\theta$	$-(n+2)(n-1)r^{-n}\cos n\theta$	$-n(n-1)r^{-n}\sin n\theta$	$(n-1)(n-2)r^{-n}\cos n\theta$
$r^n\cos n\theta$	$-n(n-1)r^{n-2}\cos n\theta$	$n(n-1)r^{n-2}\sin n\theta$	$n(n-1)r^{n-2}\cos n\theta$
$r^{-n}\cos n\theta$	$-n(n+1)r^{-n-2}\cos n\theta$	$-n(n+1)r^{-n-2}\sin n\theta$	$n(n+1)r^{-n-2}\cos n\theta$
$r^{n+2}\sin n\theta$	$-(n+1)(n-2)r^n\sin n\theta$	$-n(n+1)r^n\cos n\theta$	$(n+1)(n+2)r^n\sin n\theta$
$r^{-n+2}\sin n\theta$	$-(n+2)(n-1)r^{-n}\sin n\theta$	$n(n-1)r^{-n}\cos n\theta$	$(n-1)(n-2)r^{-n}\sin n\theta$
$r^n\sin n\theta$	$-n(n-1)r^{n-2}\sin n\theta$	$-n(n-1)r^{n-2}\cos n\theta$	$n(n-1)r^{n-2}\sin n\theta$
$r^{-n}\sin n\theta$	$-n(n+1)r^{-n-2}\sin n\theta$	$n(n+1)r^{-n-2}\cos n\theta$	$n(n+1)r^{-n-2}\sin n\theta$

8.4.1 Hole in a tensile field

To illustrate the use of Table 8.1, we consider the case where the the body of Figure 8.2 is subjected to uniform tension at infinity instead of shear, so that the boundary conditions become

$$\sigma_{rr} = 0 \;\; ; \;\; r=a \tag{8.60}$$

$$\sigma_{r\theta} = 0 \;\; ; \;\; r=a \tag{8.61}$$

$$\sigma_{xy}, \sigma_{yy} \to 0 \;\; ; \;\; r\to\infty \tag{8.62}$$

$$\sigma_{xx} \to S \;\; ; \;\; r\to\infty \; . \tag{8.63}$$

The unperturbed problem in this case can clearly be described by the stress function

$$\phi = \frac{Sy^2}{2} = \frac{Sr^2\sin^2\theta}{2} = \frac{Sr^2}{4} - \frac{Sr^2\cos(2\theta)}{4} \; . \tag{8.64}$$

Table 9.1. The Michell solution — displacement components

ϕ	$2\mu u_r$	$2\mu u_\theta$
r^2	$(\kappa-1)r$	0
$r^2\ln(r)$	$(\kappa-1)r\ln(r)-r$	$(\kappa+1)r\theta$
$\ln(r)$	$-1/r$	0
θ	0	$-1/r$
$r^3\cos\theta$	$(\kappa-2)r^2\cos\theta$	$(\kappa+2)r^2\sin\theta$
$r\theta\sin\theta$	$\frac{1}{2}\{(\kappa-1)\theta\sin\theta-\cos\theta$ $+(\kappa+1)\ln(r)\cos\theta\}$	$\frac{1}{2}\{(\kappa-1)\theta\cos\theta-\sin\theta$ $-(\kappa+1)\ln(r)\sin\theta\}$
$r\ln(r)\cos\theta$	$\frac{1}{2}\{(\kappa+1)\theta\sin\theta-\cos\theta$ $+(\kappa-1)\ln(r)\cos\theta\}$	$\frac{1}{2}\{(\kappa+1)\theta\cos\theta-\sin\theta$ $-(\kappa-1)\ln(r)\sin\theta\}$
$\cos\theta/r$	$\cos\theta/r^2$	$\sin\theta/r^2$
$r^3\sin\theta$	$(\kappa-2)r^2\sin\theta$	$-(\kappa+2)r^2\cos\theta$
$r\theta\cos\theta$	$\frac{1}{2}\{(\kappa-1)\theta\cos\theta+\sin\theta$ $-(\kappa+1)\ln(r)\sin\theta\}$	$\frac{1}{2}\{-(\kappa-1)\theta\sin\theta-\cos\theta$ $-(\kappa+1)\ln(r)\cos\theta\}$
$r\ln(r)\sin\theta$	$\frac{1}{2}\{-(\kappa+1)\theta\cos\theta-\sin\theta$ $+(\kappa-1)\ln(r)\sin\theta\}$	$\frac{1}{2}\{(\kappa+1)\theta\sin\theta+\cos\theta$ $+(\kappa-1)\ln(r)\cos\theta\}$
$\sin\theta/r$	$\sin\theta/r^2$	$-\cos\theta/r^2$
$r^{n+2}\cos n\theta$	$(\kappa-n-1)r^{n+1}\cos n\theta$	$(\kappa+n+1)r^{n+1}\sin n\theta$
$r^{-n+2}\cos n\theta$	$(\kappa+n-1)r^{-n+1}\cos n\theta$	$-(\kappa-n+1)r^{-n+1}\sin n\theta$
$r^n\cos n\theta$	$-nr^{n-1}\cos n\theta$	$nr^{n-1}\sin n\theta$
$r^{-n}\cos n\theta$	$nr^{-n-1}\cos n\theta$	$nr^{-n-1}\sin n\theta$
$r^{n+2}\sin n\theta$	$(\kappa-n-1)r^{n+1}\sin n\theta$	$-(\kappa+n+1)r^{n+1}\cos n\theta$
$r^{-n+2}\sin n\theta$	$(\kappa+n-1)r^{-n+1}\sin n\theta$	$(\kappa-n+1)r^{-n+1}\cos n\theta$
$r^n\sin n\theta$	$-nr^{n-1}\sin n\theta$	$-nr^{n-1}\cos n\theta$
$r^{-n}\sin n\theta$	$nr^{-n-1}\sin n\theta$	$-nr^{-n-1}\cos n\theta$

condition that the body be in equilibrium. This requires that

$$\int_0^{2\pi}(F_1(\theta)\cos\theta-F_3(\theta)\sin\theta)\,ad\theta - \int_0^{2\pi}(F_2(\theta)\cos\theta-F_4(\theta)\sin\theta)bd\theta=0 \tag{9.36}$$

$$\int_0^{2\pi}(F_1(\theta)\sin\theta+F_3(\theta)\cos\theta)ad\theta - \int_0^{2\pi}(F_2(\theta)\sin\theta+F_4(\theta)\cos\theta)bd\theta=0 \tag{9.37}$$