MAE 5350: HW #1

Hanfeng Zhai*

Multidisciplinary Design Optimization

September 20, 2021

Part (a)

Q1. Motivation

Summarize in 5-10 sentences why you decided to take this class. What do you expect to learn? How does this knowledge fit in with your career or research plans?

- Summarize in 5-10 sentences why you decided to take this class. This class attracts me mainly because its name: *Multidisciplinary Design Optimization*. For *Multidisciplinary*: I was involved in many projects are in the intersections of many disciplines: thermal engineering, structural engineering, chemical engineering, bioengineering, etc., and I'm a huge fan of multiphysics system simulation. Usually we solve each modules separately and couple them from specific variables that occurred in both the governing equations. So I'm curious whether the same strategy applies for the *Multidisciplinary* here. Second, I'm excited about the term *Optimization*: data-driven, quasi-static, & physics-informed machine learning methods are extremely powerful and popular terms in my field, and started to be adopted by both academia and the industry. All of these methods are based on different optimization methods to "train" a learning structure. Therefore, the know more details about the optimization methods is the main attraction for me.

- What do you expect to learn? Basic optimization methods that can be applied to many engineering & scientific fields. Currently, MDO already proved its effectiveness in spacecraft engineering, underwater vehicles, aerofoil design, wind turbine system etc. and became very successful in the industry. I believe it can also be applied to micro&nano system, circuits, batteries, lab-on-chip, and many "small" systems!

- How does this knowledge fit in with your career or research plans? I hope to apply what I've learnt in MDO course to my current research of multiscale mechanics, and many potential field of interests

* [redacted] www.hanfengzhai.net
Sibley School of Mechanical and Aerospace Engineering, Cornell University

including microfluidics, lab-on-chip system, batteries, etc. Also, I believe I got more chances to be involved in many industrial projects to view them from the perspective of MDO.

Q2. System decomposition

In Sections 1.1 and 1.2 of *Papalambros and Wilde*, the authors discuss hierarchical levels in system definition and hierarchical system decomposition. To answer the questions below, consider one of the following engineering systems: a wind turbine power system, an Unmanned Aerial Vehicle (UAV), or an underwater vehicle: Woods Hall Oceanographic Institute's Alvin.

1) Describe the system boundary that you would choose in setting up a model for your system. What are the inputs and outputs that cross this system boundary and characterize your system?

- System Chosen: Wind turbine power system.
- System Boundary: The constraints in this system includes [1]Basic rule of physics (i.e. Navier-Stokes equation for fluid dynamics, thermodynamics, equilibrium of forces applied to structures), [2]Boundaries of local area for setting up the turbines, [3]budget, [4]the maximum energy that area can support, [5]the height limit based on the turbine structure, etc.
- Inputs: The inputs of the system includes [1]the number of turbines, [2]the geometrical parameters of wind turbine (i.e., volume / size, height, occupation area, etc.), [3]The shape of the blade (i.e., angles, area shapes, etc.), [4]the materials for turbine, [5]the average distance between turbines.
- Outputs: The outputs of the system includes [1]the power generation per turbine, [2]stress acting on the blade, [3]the deformation of the turbine body, [4]costs per turbine, [5]vibration caused by the turbine working.

2) Propose a component decomposition for your system (use a similar level of detail to that shown in *Papalambros and Wilde* Fig. 1.10).

See Figure 1.

3) Propose an aspect decomposition for your system (use a similar level of detail to that shown in *Papalambros and Wilde* Fig. 1.12).

See Figure 2.

Q3. Simple Unconstrained Optimization/Math Review

When solving the following problem you may use computational tools, but please apply the methods taught in class before doing any computation and show your derivations.

The Rosenbrock function is often used as a simple test problem for optimization algorithms:

$$f(\mathbf{x}) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$

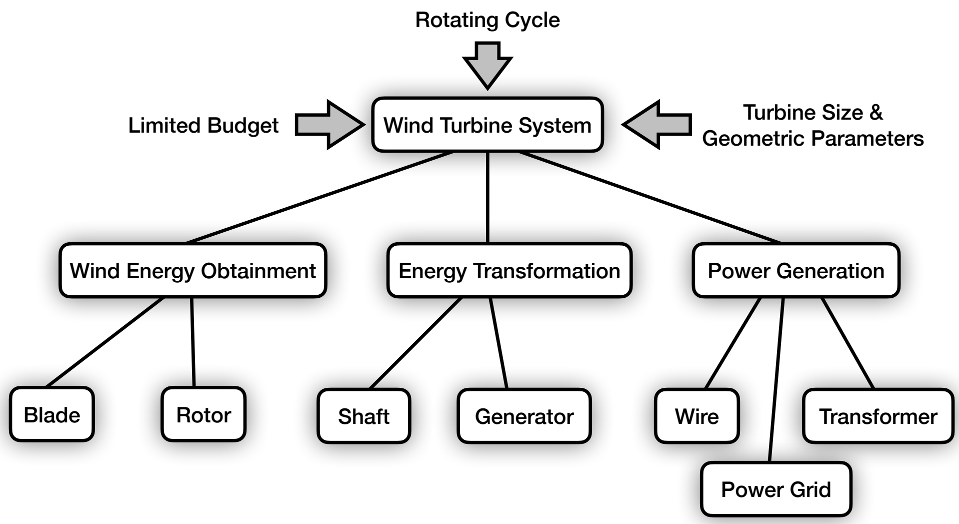

Figure 1: Component decomposition for the wind turbine system (*solution for (2)*).

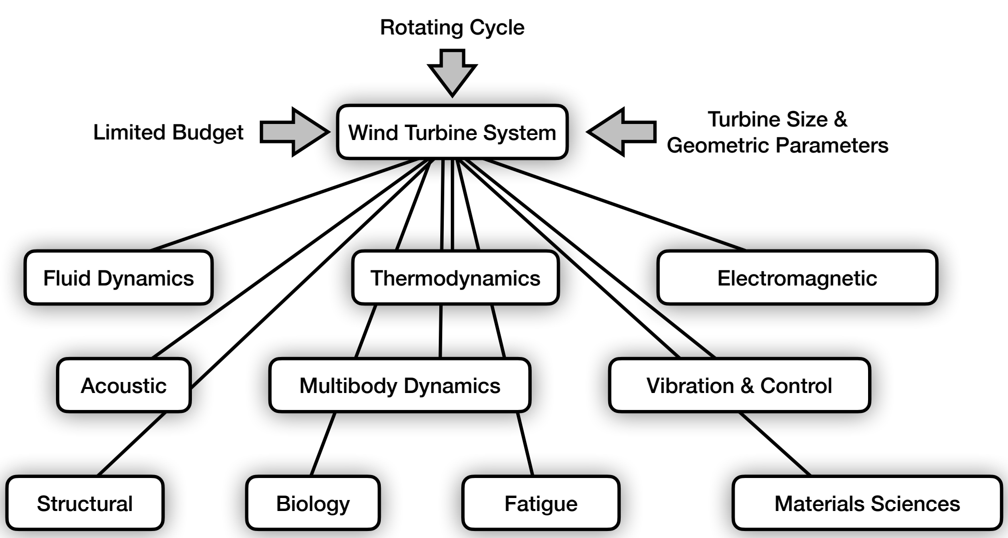

Figure 2: Aspect decomposition for the wind turbine system (*solution for (3)*).

a) Compute the gradient (vector of first derivatives) and Hessian (matrix of second derivatives) of $f(\mathbf{x})$.

Gradient:

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -400x_1x_2 + 400x_1^3 + 2x_1 - 2 \\ 200x_2 - 200x_1^2 \end{bmatrix}$$

Hessian:

$$\mathbf{H}f(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} -400x_2 + 1200x_1^2 + 2 & -400x_1 \\ -400x_1 & 200 \end{bmatrix}$$

b) Show that $\mathbf{x}^* = (1,1)$ is the only local minimizer of this function, and that the Hessian matrix at that point is positive definite.

If we expand the form, we get $f(\mathbf{x}) = 100x_2^2 - 200x_1^2x_2 + 100x_1^4 + 1 - 2x_1 + x_1^2$ If we let $f_{x_1} = 0$ and $f_{x_2} = 0$, we get

$$\begin{cases} -400x_1x_2 + 400x_1^3 + 2x_1 - 2 = 0 \\ 200x_2 - 200x_1^2 = 0 \end{cases}$$

Solving the equation we obtain $x_1 = 1$ and $x_2 = 1$ (because from second row we have $x_2 - x_1^2 = 0$ and from the first row implies both $x_2 - x_1^2 = 0$ and $x_1 - 1 = 0$), thus we can say that $\mathbf{x}^*$ only critical (stationary) point.

Here we obtain the critical point $(1,1)$. Substitute this point into the Hessian $\mathbf{H}f(1,1)$ we get $\begin{bmatrix} 802 & -400 \\ -400 & 200 \end{bmatrix}$. Now we can observe that $D_1 = f_{xx} = 802 > 0$ and $D_2 = \det \mathbf{H} = 400 > 0$, therefore the point $(1,1)$ is a local minimum.

c) Make a contour plot of the objective value of the Rosenbrock function versus the design variables x_1 and x_2 and verify the local minimum graphically.

See Figure 3.

d) Show that the function

$$f(\mathbf{x}) = 8x_1 + 12x_2 + x_1^2 - 2x_2^2$$

has only one stationary point, and that it is neither a maximum or minimum, but a saddle point.

We first need to compute the gradient of the function

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 8 + 2x_1 \\ 12 - 4x_2 \end{bmatrix}$$

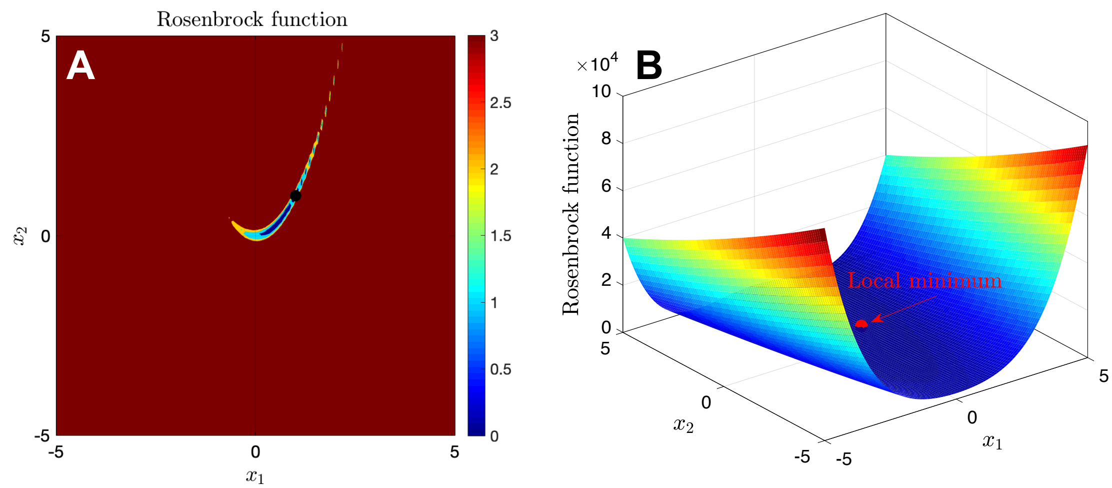

Figure 3: Rosenbrock function contour plot with regards to the coordinates in 2D and 3D. The local minimum point was plotted black in the left sub figure and plotted red in the right sub figure.

Computing the Hessian of the matrix we obtain

$$\mathbf{H}f(\mathbf{x}) = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}$$

Let the gradient equals zero we obtain $x_1 = -4$ and $x_2 = 3$, from $\mathbf{H}$ we know that $x_1 = -4$ is the local maximum and $x_2 = 3$ is the local minimum. Computing the determinant of matrix we have $\det \mathbf{H} = -8 < 0$, therefore we say $(-4,\ 3)$ is a saddle point.

To verify this point, we substitute these two points and plot them we obtain figure 4.

From figure 4 we observe that the critical point is the lowest in the x_1 direction and highest for x_2 direction, points verified.

Q4. ***Papalambros and Wilde***

When solving the following problem you may use computational tools, but please apply the methods taught in class before doing any computation and show your derivations.

a) Sketch the function $f(\mathbf{x}) = (x_2 - x_1)^4 + 8x_1x_2 - x_1 + x_2 + 3$ in the interval $-2 < x_i < 2$. Solve for the optimal point $(\mathbf{x}^*)$ and add the point to your plot. Does the result match your intuition?

Let the gradient of the function equals zero we obtain

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 8x_2 + 4(x_1 - x_2)^3 - 1 \\ 8x_1 - 4(x_1 - x_2)^3 + 1 \end{bmatrix} = 0 \tag{1}$$

We can then derive the numerical solution with MATLAB `vpasolve` module, generating the following code:

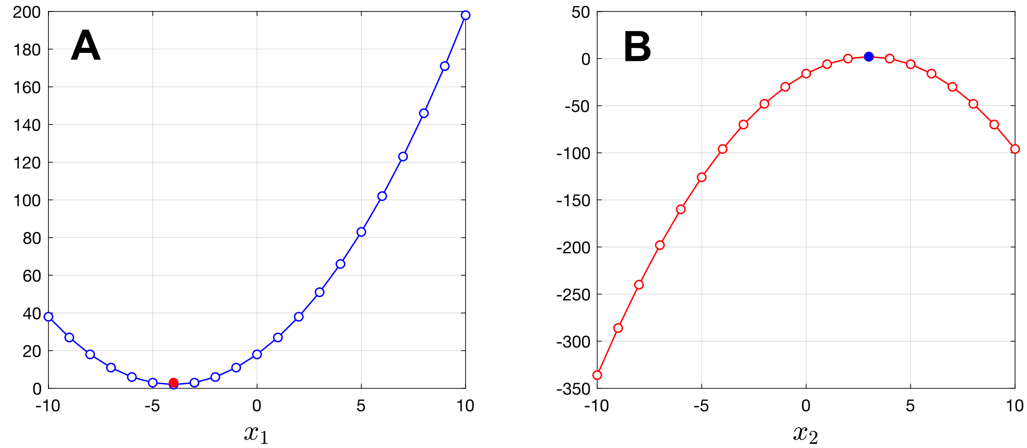

Figure 4: The plot of $f(\mathbf{x})$ on fixed $x_2 = 3$ and $x_1 = -4$.

```
>> syms x1 x2
>> f = @(x1,x2) (x2-x1)^4+8*x1*x2-x1+x2+3;
>> eq1 = diff(f,x1)
    eq1 = 8*x2 + 4*(x1 - x2)^3 - 1
>> eq2 = diff(f,x2)
    eq2 = 8*x1 - 4*(x1 - x2)^3 + 1
% according to equation (1), we know that eq1 - eq2 = 0, of which we derive that x1 = -x2, then substitute into Eq. (1)
>> eqn = 8*(-x1) + 4*(x1 + x1)^3 - 1
>> fpasolve(eqn, x1)
ans =

-0.13479721820272227913146897567463
 0.55357993584438379685387442521215
-0.41878271764166151772240544953752
% then we can get the three points' coordinate based on x1 = -x2
```

The function and the three points is sketched as in figure 5. Based on the subfigure **B** and **C** we can see that the optimal point is $(0.55357993584438379685387442521215, -0.55357993584438379685387442521215)$.

b) Sketch $f(x)$ in the same interval $-2 < x_i < 2$ but additionally sketch the constraint $g(\mathbf{x}) = x_1^4 - 2x_2x_1^2 + x_2^2 + x_1^2 - 2x_1 \leq 0$. Solve for the the optimal point ($\mathbf{x}^*$) given this new constraint and add the point to your plot. Is it different from the result you got in part a)? Why? Hint: is $g(x)$ an active or inactive constraint? How can we use this information to help solve the new optimization problem?

As we can observe from figure 6 (especially from subfigure **C**), the inequality constraint $g(\mathbf{x}) < 0$ doesn't include any area where $f(\mathbf{x})$ locate. Therefore, the critical point remains the same, and the constraint is an inactive constraint.

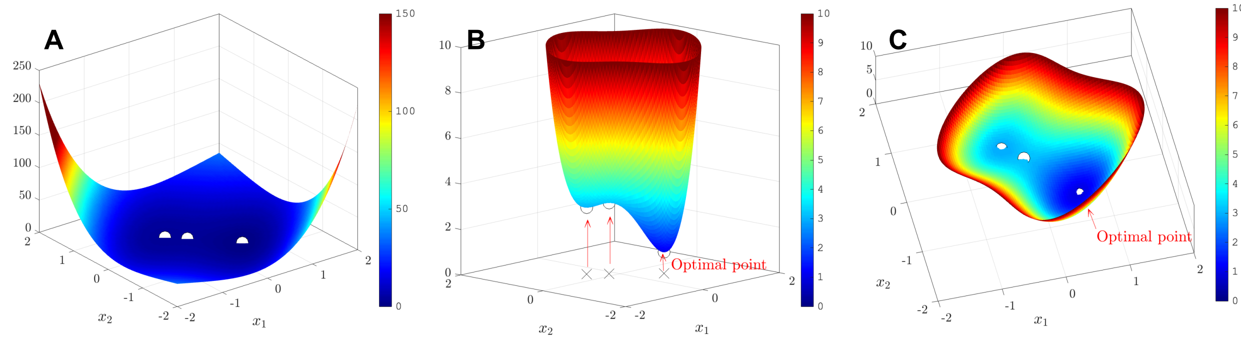

Figure 5: The sketch of function $f(\mathbf{x})$, the three solution points for $\nabla f = 0$, with highlighting the optimal point, in different numerical ranges. **A**. the sketch in the range of $[0, 150]$. **B**. the sketch in the range pf $[0, 10]$. **C**. a different angular view for subfigure **B**.

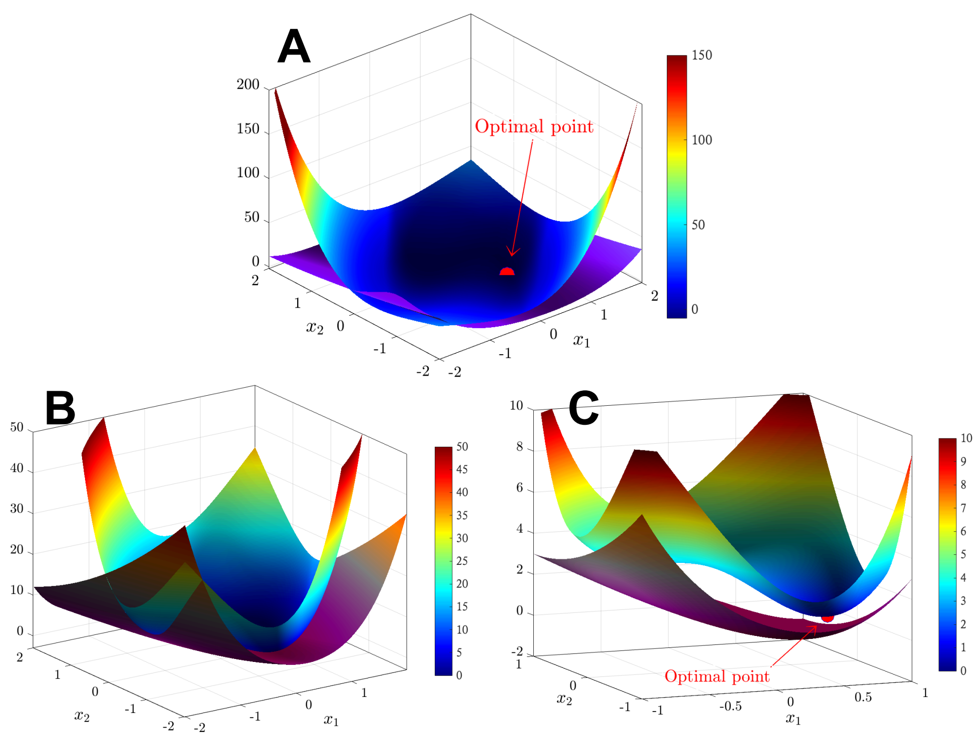

Figure 6: The sketch of the objective function $f(\mathbf{x})$ and inequality constraint $g(\mathbf{x})$ from different angles and different numerical ranges. **A**. the sketch in the range of $[0, 150]$ as can be compared with Figure 5, with the optimal point. **B**. the sketch in $[0, 50]$. **C**. the sketch in $[0, 10]$, highlighting the optimal point.