COURSE NOTES

Foundations of Solid Mechanics

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2021

Week 1: Mon. 8/29/2021

Vectors & tensors. → Cartesian.

in Physics, we are familiar with

$$\begin{cases} \nabla \times \vec{E} = 0 \\ \nabla \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0} \end{cases} \Rightarrow \text{Electro-statics.}$$

which is, independent of coordinate system

in a RH coordinate,

$$\|\vec{e}_i\| = 1.$$

$$\vec{V} = \sum_{i=1}^{3} V_i \vec{e}_i$$

index subscript ← $\vec{e}_i \cdot \vec{e}_j \equiv \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$

⇓

called Kronecker delta.

(in orthonormal basis).

Here, V_i is component of $\vec{V}$

with a basis $\{\vec{e}_i\}$.

$$= \sum_{j=1}^{3} V_j' \vec{e}_j'$$

transformation of basis.

$\vec{x} \rightarrow$ cooridinate dependent.

<u>Summation convection</u>

$\downarrow$

(indicial notation).

Sum of $v_i e_i = v_i e_i = v_k e_k$

repeated indices.

term: subscripts. $\underbrace{a_{ij} b_k}_{\text{summing over } j} C_{jm} \rightarrow$ free indices.

contraction, if $i = m$.

$a_{ip} b_k C_{pi} \rightarrow$ double summation.

*** A dummy index cannot repeat more than 2 !!!

$\delta_{ij}\delta_{jk} = d_{ik} = \delta_{ik}$.

e.g. $\delta_{ij}\delta_{jk} = \delta_{11}\delta_{1k} + \underset{0}{\underset{\parallel}{\delta_{12}\delta_{2k}}} + \underset{0}{\underset{\parallel}{\delta_{13}\delta_{3k}}}$

$= \delta_{1k}$.

$\delta_{2k}\ (i=2)$, $\delta_{3k}\ (i=3)$.

$\vec{v}\cdot\vec{w} = v_i\vec{e}_i \cdot w_j\vec{e}_j = v_i w_j \underbrace{(\vec{e}_i\cdot\vec{e}_j)}_{\delta_{ij}}$

$= v_i w_i = v_j w_j$.

usual dot product.

Cross product.

$\vec{v}\times\vec{w} = v_i\vec{e}_i \times w_j\vec{e}_j = v_i w_j\, \vec{e}_i\times\vec{e}_j$ (a).

$\vec{e}_i\times\vec{e}_j = \underbrace{[(\vec{e}_i\times\vec{e}_j)\cdot\vec{e}_k]}_{= \epsilon_{ijk}\ \text{(Permutation Symbol)}}\vec{e}_k$

free index on two sides of Eqn. must be equal !!!

e.g.

$$(\vec{e}_1 \times \vec{e}_2) \cdot \vec{e}_3 = 1.$$

$$\epsilon_{ijk} = 1, \quad (1,2,3), (3,1,2), (2,3,1).$$
$$= -1, \quad (2,1,3), (1,3,2), (3,2,1).$$
$$= 0, \quad \text{otherwise}.$$

Review ~~ang~~ undergrad linear algebra

$\det(\sim)$.

eq. (a) writes. $v_i w_j \epsilon_{ijk} \vec{e}_k$

$$\Rightarrow \epsilon_{ijk} v_i w_j \vec{e}_k$$
$$= \epsilon_{kij} v_i w_j \vec{e}_k$$

Q. $\vec{a} \times (\vec{b} \times \vec{c}) = ?$

$$= a_k \vec{e}_k \times (b_i c_j \, \vec{e}_i \times \vec{e}_j)$$
$$= a_k \vec{e}_k \times (b_i c_j \, \epsilon_{ijm} \vec{e}_m).$$
$$= a_k b_i c_j (\delta_{iw}\delta_{jk} - \delta_{ik}\delta_{jw}) \vec{e}_w$$

$$= (a_i \vec{e}_i) \times [\epsilon_{pjk} b_j c_k \vec{e}_p].$$
$$= (\delta_{sj}\delta_{ik} - \delta_{sk}\delta_{ij}) a_i b_j c_k \vec{e}_s$$
$$= (b_s a_k c_k - b_i a_i c_s) \vec{e}_s$$
$$= [b_s(\vec{a}\cdot\vec{c}) - (\vec{a}\cdot\vec{b}) c_s] \vec{e}_s$$
$$= (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

- How vectors transform? (on basis).

$$\vec{v} = v_i \vec{e}_i = v_j' \vec{e}_j'$$
$$v_j' = (\vec{v}\cdot\vec{e}_j') = (v_i \vec{e}_i \cdot \vec{e}_j').$$
$$= v_i (\vec{e}_i \cdot \vec{e}_j').$$
$$P_{ji} \equiv \vec{e}_j' \cdot \vec{e}_i.$$

projection of one basis on another basis.

$$\vec{v}_j' = P_{ji} \cdot \vec{v}_i$$

$$\vec{v}' = \begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\vec{P} = [P_{ji}]$$
$$\vec{v}' = \vec{P}\vec{v}$$
$$\vec{v} = \vec{P}^{-1}\vec{v}'$$

☆☆☆ $\hookrightarrow \vec{P}^{-1} = \vec{P}^t$

off-course supplementary:

Week 1:, Wed. 9/1/2021.

Second order Tensor $\underline{\underline{A}}$.

A 2nd order tensor is a linear transformation from $\mathbb{E}^3$ to $\mathbb{E}^3$

LT = a special kind of mapping $\mathbb{E}^3 \to \mathbb{E}^3$

properties:

$\underline{\underline{A}}(\underline{u})$ to some vector

$\underline{\underline{A}} : (\underline{u}) \longrightarrow \underline{\underline{A}}(\underline{u})$.

$\to \underline{\underline{A}}(a\underline{u}) = a\,\underline{\underline{A}}(\underline{u})$.

real No.

$\to \underline{\underline{A}}(\underline{u} + \underline{w}) = \underline{\underline{A}}(\underline{u}) + \underline{\underline{A}}(\underline{w})$.

$$\Updownarrow$$

$$\underline{\underline{A}}(a\underline{u} + b\underline{w}) = a\,\underline{\underline{A}}(\underline{u}) + b\,\underline{\underline{A}}(\underline{w}).$$

$\forall\, \underline{u}, \underline{w}$ w/ $\mathbb{E}^3$ and a, b.

$\underline{\underline{A}}(\underline{0}) = \underline{0}$.

Example rigid body rotation about a fixed point.

$\begin{cases} \text{Defination gradient tensor.} \\ \text{Stress tensor} \end{cases}$

$$\underline{x} = x_i \vec{e}_i$$

$$\underline{\underline{A}}(\underline{x}) = \underline{\underline{A}}(x_j \vec{e}_j).$$

$$= \cancel{A_j}\, x_j \underline{\underline{A}}(\underline{e}_j)$$

this tells us that a linear transformation is completely determined by its action on the basis vectors.

$$\underline{\underline{A}}(\underline{e}_j) = a_{ij} \underline{e}_i$$

$$a_{ij} e_i (e_j \cdot \underline{x})$$

$$\underline{\underline{A}}(x_j e_j) = \cancel{a_{ij}}\, x_j \underline{\underline{A}}(e_j) = a_{ij} x_j \underline{e}_i$$

$$\underline{\underline{A}}(\underline{e}_1) \cdot \underline{e}_1 = a_{11}$$

$$\underline{\underline{A}}(\underline{e}_1) \cdot \underline{e}_2 = a_{21}$$

$$\underline{\underline{A}}(\underline{e}_1) \cdot \underline{e}_3 = a_{31}$$

$$A = \begin{bmatrix} \cancel{A}a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\underline{\underline{A}}(\underline{x}) = a_{ij} \underline{e}_i (\underline{e}_j \cdot \underline{x})$$

$$= \underbrace{a_{ij} \underline{e}_i \underline{e}_j}_{\underline{\underline{A}}} \cdot \underline{x}$$

Define $\underline{a}\underline{b}$ as the linear transformation.

$$(\underline{a}\underline{b})(\underline{x}) = \underline{a}(\underline{b} \cdot \underline{x}). \quad ??$$

$\downarrow$

check: this is a LT.

$$= \underline{\underline{A}} \cdot \underline{x} \quad \text{(we can skip the dot).} \; ??$$

$\underline{a}\underline{b} \rightarrow$ dyad

*** Any linear transformation can be written as sum of dyad

$$\underline{a}\underline{b} \Leftrightarrow \underline{a} \otimes \underline{b} \rightarrow \text{linear transformation.}$$

A simple representation: $\underline{\underline{A}} = a_{ij} \underline{e}_i \underline{e}_j$

$$\underline{a}\underline{b} \neq \underline{b}\underline{a}.$$

$\underline{e}_i \to \underline{e}'_r.$

$$\underline{\underline{A}} = a_{ij}\, \underline{e}_i\, \underline{e}_j = a'_{rs}\, \underline{e}'_r\, \underline{e}'_s$$

$$= a_{ij} \underbrace{(\underline{e}_i \cdot \underline{e}'_r)\, \underline{e}'_r}_{\underline{e}_i} \underbrace{(\underline{e}_j \cdot \underline{e}'_s)\, \underline{e}'_s}_{\underline{e}_j}$$

$$= a_{ij} \underbrace{(\underline{e}_i \cdot \underline{e}'_r)}_{P_{ri}} \underbrace{(\underline{e}_j \cdot \underline{e}'_s)}_{P_{sj}}$$

$$= a'_{rs}\, \underline{e}'_r\, \underline{e}'_s$$

$$a'_{rs} = a_{ij} P_{rs} P_{ij}.$$

$$[PA]_{rj} [P^T]_{js}.$$

$$A' = PAP^T \qquad P^T A' P = A.$$

$$\Downarrow$$

$$[a'_{rs}].$$

$$\underline{\underline{A}}(\underline{x}) = \underline{y}.$$

$$\det \underline{\underline{A}} \equiv \det A.$$

$$\det A = \det (P^T A' P) = \det P^T \det A' \det P$$

$$= \det \underbrace{(P^T P)}_{I} \det A'$$

$$= \det A'$$

*** det is invariant

$$(a\underline{\underline{A}} + b\underline{\underline{B}})(\underline{x}) \equiv a\underline{\underline{A}}(\underline{x}) + b\underline{\underline{B}}(\underline{x})$$

composition $\underline{\underline{A}}^{-1}$ of mapping.

$$(\underline{\underline{A}} \circ \underline{\underline{B}})(\underline{x}) \equiv \underline{\underline{A}}(\underline{\underline{B}}(\underline{x}))$$

$$= \underline{\underline{A}} \cdot (\underline{\underline{B}} \cdot \underline{x}).$$

$$\underline{\underline{B}} \cdot \underline{x} = b_{ij}\, \underline{e}_i \underline{e}_j \cdot \underline{x}.$$

$$\underline{\underline{A}} \cdot (\underline{\underline{B}} \cdot \underline{x}) = (a_{rs}\, \underline{e}_r \underline{e}_s) \cdot b_{ij}\, \underline{e}_i \underline{e}_j \cdot \underline{x}.$$

$$= a_{rs}\, \underline{e}_r\, b_{ij} \underbrace{(\underline{e}_s \cdot \underline{e}_i)}_{\delta_{si}} \underline{e}_j \cdot \underline{x}.$$

$$= a_{rs} b_{sj}\, \underline{e}_r\, \underline{e}_j \cdot \underline{x}.$$

$$[\underline{\underline{C}}] = C \qquad C = AB.$$

$$\underline{\underline{A}}^{-1} \circ \underline{\underline{A}} = \underline{\underline{I}}.$$

$$\underline{\underline{I}} = \delta_{ij}\, \underline{e}_i \underline{e}_j = \underline{e}_i \underline{e}_i$$

$\downarrow$

identity tensor.

$\underline{\underline{A}}^T$: transpose of $\underline{\underline{A}}$.

$$\underline{v} \cdot \underline{\underline{A}}^T \cdot \underline{u} = \underline{u} \cdot \underline{\underline{A}} \cdot \underline{v}$$

for all $\underline{u}$ & $\underline{v}$ in $\mathbb{E}^3$

$\underline{u} = \underline{e}_j$

$\underline{v} = \underline{e}_i$

$$\underline{e}_j \cdot \underline{\underline{A}} \cdot \underline{e}_i$$

$$= \underline{e}_j \cdot a_{rs}\, \underbrace{\underline{e}_r \underline{e}_s \cdot \underline{e}_i}_{\delta_{si}}$$

$$= a_{rs}\, \delta_{jr} \delta_{si} = a_{ji}.$$

$$\underline{e}_i \cdot \underline{\underline{A}}^T \cdot \underline{e}_j = \underbrace{a_{ij}^T = a_{ji}}.$$

$\Downarrow$

true for 1 com,

true for all ~

$$(\underline{\underline{A}} + \underline{\underline{B}})^T = \underline{\underline{A}}^T + \underline{\underline{B}}^T$$

$$(\underline{\underline{A}}^T)^T = \underline{\underline{A}} \qquad (\underline{\underline{A}}^{-1})^T = (\underline{\underline{A}}^T)^{-1}$$

$$(\underline{\underline{A}} \circ \underline{\underline{B}})^T = \underline{\underline{B}}^T \circ \underline{\underline{A}}^T$$

$\underline{\underline{A}}^T = \underline{\underline{A}}$ symmetric

$\underline{\underline{A}}^T = -\underline{\underline{A}}$ asymmetric $\rightarrow$ mechanics of solids.

*** Eigenvalue of asymmetric tensor

$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) \rightarrow$ independent of basis

$$= \det(A' - \lambda I)$$

$$= \det[\underbrace{PAP^T}_{A'} - \lambda I].$$

$$= \det[P(A - \lambda I)P^T]$$

$$= \underbrace{\det[PP^T]}_{1} \det(A - \lambda I)$$

$$\equiv \det(\underline{\underline{A}} - \lambda \underline{\underline{I}}).$$

$PP^T = I$

$$\det(A - \lambda I)$$

$$= P_3(\lambda) = (-\lambda)^3 + I_1 \lambda^2 - I_2 \lambda + \det A.$$

$$I_1 = \frac{1}{2}[(\mathrm{tr}(A))^2 - \mathrm{tr}(A^2)] \qquad I_2 = a_{ii} = \mathrm{tr}(A).$$

$\hookrightarrow a_{11} + a_{22} + a_{33}.$

I_1, I_2, $\det A$

are scalar movement of the Tensor $\underline{\underline{A}}$

$$\underline{\underline{A}} = \lambda_1 \underline{E}_1 \underline{E}_1 + \lambda_2 \underline{E}_2 \underline{E}_2 + \lambda_3 \underline{E}_3 \underline{E}_3$$

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix}$$

$\{E_i\}$

$\uparrow$

eigenvectors of $\underline{\underline{A}}$

$$\det A = \lambda_1 \lambda_2 \lambda_3$$

$$I_2 = \lambda_1 + \lambda_2 + \lambda_3 .$$

$$I_1 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 .$$

☆ how to diagonize 3×3 matrix.

(Labor day - Monday).

Week 2: Wed.

(Review)

$$\underline{\underline{A}}(\underline{v}) = \underline{\underline{A}} \cdot \underline{v}$$

$$\underline{\underline{A}} = a_{ij} \underline{e}_i \underline{e}_j$$

$$\underline{\underline{A}}(\underline{v}) = a_{ij} v_j \underline{e}_i$$

$$= a_{ij} \underline{e}_i \underbrace{\underline{e}_j \cdot \underline{v}}_{v_j}$$

$$(\underline{\underline{A}} \circ \underline{\underline{B}})(\underline{v}) = \underline{\underline{A}}(\underline{\underline{B}}(\underline{v}))$$

$$= (a_{ij} \underline{e}_i \underline{e}_j)(b_{kl} v_l \underline{e}_k).$$

$$= a_{ij} \underline{e}_i b_{kl} v_l \delta_{jk}$$

$$= a_{ij} b_{jl} v_l \underline{e}_i$$

in other words,

$$\underline{\underline{A}} \circ \underline{\underline{B}} = a_{ij} b_{jl} \underline{e}_i \underline{e}_l$$

$$= \underline{\underline{A}} \otimes \underline{\underline{B}} =$$

$$\underline{\underline{A}} = a_{ij} \underline{e}_i \underline{e}_j , \quad \underline{\underline{B}} = b_{kl} \underline{e}_k \underline{e}_l$$

$$\underline{\underline{A}} \cdot \underline{\underline{B}} = a_{ij} \underline{e}_i \underline{e}_j \cdot b_{kl} \underline{e}_k \underline{e}_l$$

$$\underline{\underline{A}}\underline{\underline{B}} = \underline{\underline{A}} \circ \underline{\underline{B}}$$

$\underline{\underline{A}}\underline{\underline{B}}$

$\underline{a}\underline{b} : \underline{c}\,\underline{d} \equiv (\underline{a}\cdot\underline{c})(\underline{b}\cdot\underline{d})$

$\underline{a}\underline{b} \cdot\cdot \underline{c}\,\underline{d} \equiv (\underline{a}\cdot\underline{d})(\underline{c}\cdot\underline{b})$

$\underline{\underline{A}} = a_{ij}\,\underline{e}_i\,\underline{e}_j$

$\underline{\underline{B}} = b_{kl}\,\underline{e}_k\,\underline{e}_l$

$\underline{\underline{A}} : \underline{\underline{B}} = (a_{ij}\,\underline{e}_i\,\underline{e}_j) : (b_{kl}\,\underline{e}_k\,\underline{e}_l)$.

$\underline{\underline{A}} : \underline{\underline{B}} = a_{ij} b_{kl} \underbrace{(\underline{e}_i\cdot\underline{e}_k)}_{\delta_{ik}} \underbrace{(\underline{e}_j\cdot\underline{e}_l)}_{\delta_{jl}}$.

$= a_{kj} b_{kj} \rightarrow$ scalar

↳ can be extended to 2 angular relations of linear transformations.

e.g. for vectors

$\underline{a}\cdot\underline{b}$

$tr(\underline{\underline{A}})$

$= a_{ij}\underbrace{(\underline{e}_i\cdot\underline{e}_j)}_{\delta_{ij}} = a_{ii}$.

$tr(\underline{\underline{A}}+\underline{\underline{B}}) = tr\,\underline{\underline{A}} + tr\,\underline{\underline{B}}$.

$tr(a\underline{\underline{A}}) = a\,tr(\underline{\underline{A}})$.

$tr(\underline{\underline{A}}^T) = tr(\underline{\underline{A}})$.

$tr(\underline{\underline{A}}\circ\underline{\underline{B}}) = tr(\underline{\underline{B}}\circ\underline{\underline{A}})$

Tensor field

Scalar field $f(\underline{x})$

$d\underline{x}_i = d\underline{x}\cdot\underline{e}_i$

$df = \frac{\partial f}{\partial x_i} dx_i$.

$= \frac{\partial f}{\partial x_i}(\underline{e}_i\cdot d\underline{x})$.

$\nabla f \equiv \frac{\partial f}{\partial x_i}\underline{e}_i$

$= \nabla f \cdot d\underline{x}$

$\nabla \underset{\sim}{u} = ?$

$\lim\left(\frac{\underline{u}(\underline{x}+t\underline{v}) - \underline{u}(\underline{x})}{t}\right) \Longleftarrow (\nabla \underline{u})\cdot\underline{v} \equiv \lim \frac{\underline{u}(\underline{x}+t\underline{v}) - \underline{u}(\underline{x})}{t}$

$= \frac{\left[\underline{u}(\underline{x}) + \frac{\partial u}{\partial x_i} + v_i\right] - u_i(\underline{x})}{t}$.

$$= \frac{\partial \underline{u}}{\partial x_k} v_k$$

$$= \frac{\partial (u_i \underline{e}_i)}{\partial x_k} v_k = \frac{\partial u_i}{\partial x_k} \underline{e}_i (\underline{e}_k \cdot \underline{v})$$

$$= \underbrace{\left(\frac{\partial u_i}{\partial x_k} \underline{e}_i \underline{e}_k \right)}_{\nabla \underline{u}} \cdot \underline{v}$$

$\nabla \underline{u} = \frac{\partial u_i}{\partial x_k} \underline{e}_i \underline{e}_k \longrightarrow$ bump up by 1 order (w/ gradients).

$$\nabla \underline{\underline{P}} = \frac{\partial \underline{\underline{P}}}{\partial x_k} \underline{e}_k$$

$$= \frac{\partial P_{ij}}{\partial x_k} \underline{e}_i \underline{e}_j \underline{e}_k$$

$$tr(\nabla \underline{u}) = \frac{\partial u_i}{\partial x_k} (\underline{e}_i \cdot \underline{e}_k)$$

$$= \frac{\partial u_i}{\partial x_i}$$

$$= \nabla \cdot \underline{u}.$$

$$\nabla \cdot \underline{\underline{P}} \equiv \frac{\partial P_{ij}}{\partial x_k} \underline{e}_i \underbrace{(\underline{e}_j \cdot \underline{e}_k)}_{\delta_{jk}} = \frac{\partial P_{ij}}{\partial x_j} \underline{e}_i$$

$$\iiint_V f_{,i} \, dV = \iint_S f \, n_i \, dS.$$

$\hookrightarrow$ Green's theorem.

$$f_{,i} \equiv \frac{\partial f}{\partial x_i}$$

$$\left.\begin{aligned} \int_V u_{j,i} \, dV &= \int_S u_j n_i \, dS \\ \int_V T_{kl,i} \, dV &= \int_S T_{kl} n_i \, dS. \end{aligned}\right\}.$$

Mechanics.

- kinematics. ✓ (geometry).
- Balance laws ✓ (forces, moments, ...)
- Contitutive laws ← "Research".

connect. &.

Week 3: Mon.

kinematics.

Material point is labeled by its coordinate $\underline{X}_i$ or $\underset{\sim}{X}$.

(Ref. config.).

Ω_0 is the configuration of the Body at $t=0$.

Normally we choose Ω_0 to be the undeformed State of the body.

$\underline{x} - \underline{X} = \underline{u}(\underline{X}, t)$.

$t>0$. Body deforms at occupies Ω_t.

$$\underset{\sim}{X} \rightarrow \underset{\sim}{x}$$

mapping.

function "kai".

this is a mapping: $\underset{\sim}{x} = \underset{\sim}{\chi}(\underset{\sim}{X}, t)$.

$= \underline{x}(\underline{X}, t)$.

$\underline{u} = \underline{x} - \underline{X}$ → displacement vector.

Always assume mapping $\underline{\chi}$ is one-one points

↓

given one point

↓

another point associated w/ it.

*** Some simple examples: (motion).

$\underline{x} = \underline{X} + \underline{c}(t)$. Rigid body translation.

X interesting cuz no defamation.

*** e.g. 2.

rectangular cross section.

let Ω_0 be a bar, (straight bar)

We define $\underline{x} = \underline{X} + \sum_{k=1}^{3} (\lambda_k - 1) X_k \underline{e}_k$.

(remember)

$\underline{u} = \underline{x} - \underline{X}$

$\underline{u} = \sum_{k=1}^{3} (\lambda_k - 1) X_k \underline{e}_k$

↓

real positive numbers.

$\lambda_k = 1$: no displacements → body remain initial state.

$\lambda_k \neq 1$: stretch & compress in $\underline{e}_1, \underline{e}_2, \underline{e}_3$ directions

→ stretch ratios

$\lambda_k = \frac{L}{L_0}$.

☆ you can always impose a displacement field on a body.

Rigid body rotation

$\underline{x} = \underline{\underline{R}}(\underline{X})$.

$\underline{e}_k \longrightarrow \underbrace{\underline{\underline{R}}(\underline{e}_k)}_{\underline{n}_k}$

$\boxed{\underline{\underline{R}} = \underline{n}_k \underline{e}_k}$ ⟷ rotation.

linear trans. → completely det. by action on its basis

does not increase any defination

$$d\underline{x} = \frac{\partial \underline{x}}{\partial \underline{X}_j} d\underline{X}_j \quad \text{(def of grad.)}$$

$$= \underbrace{\nabla_{\underline{X}} \underline{x}}_{\underline{\underline{F}}} \cdot d\underline{X}.$$

$\underline{\underline{F}} = \nabla_{\underline{X}}$ → deformation gradient tensor.

↳ contains all the information on local deformation.

$\underline{\underline{F}}(\underline{X}, t)$

$$\underbrace{\nabla_{\underline{X}} \underline{x}}_{\underline{\underline{F}}} = \frac{\partial (x_i \underline{e}_i)}{\partial \underline{X}_j} \underline{e}_j$$

$$\underline{\underline{F}} = \frac{\partial x_i}{\partial \underline{X}_j} \underline{e}_i \underline{e}_j$$

$$[\underline{\underline{F}}] = \begin{bmatrix} \frac{\partial x_1}{\partial \underline{X}_1} & \frac{\partial x_1}{\partial \underline{X}_2} & \frac{\partial x_1}{\partial \underline{X}_3} \\ \vdots & \vdots & \vdots \end{bmatrix}.$$

↓ respect to the basis $\underline{e}_i, \underline{e}_j$

$$d\underline{x} = \underline{\underline{F}} \cdot d\underline{X}.$$

<u>change of length (fiber)</u>

$$d\underline{x} \cdot d\underline{x} - d\underline{X} \cdot d\underline{X}$$

$$= (\underline{\underline{F}} \cdot d\underline{X}) \cdot (\underline{\underline{F}} \cdot d\underline{X}) - d\underline{X} \cdot d\underline{X}$$

$$= d\underline{X} \cdot \underbrace{(\underline{\underline{F}}^T \cdot \underline{\underline{F}})}_{\underline{\underline{C}}} \cdot d\underline{X} - d\underline{X} \cdot d\underline{X}$$

$\underline{\underline{C}}$ is the Cauchy - Green Tensor

$$= d\underline{X} \cdot \underbrace{(\underline{\underline{C}} - \underline{\underline{I}})}_{\underline{\underline{E}}} \cdot d\underline{X}.$$

$$\underline{\underline{I}} \cdot d\underline{X} = d\underline{X}.$$

$$\frac{\|d\underline{x}\|^2}{\|d\underline{X}\|^2} = \left(\frac{\|d\underline{x}\|}{\|d\underline{X}\|}\right)^2$$ Lagrangian Strain Tensor

$$\frac{d\underline{x} \cdot d\underline{x}}{\|d\underline{X}\|^2} - 1$$

$$= \frac{d\underline{X}}{\|d\underline{X}\|} \cdot \underline{\underline{C}} \cdot \frac{d\underline{X}}{\|d\underline{X}\|} - 1$$

$$= \underbrace{\underline{N}}_{\text{unit vector}} \cdot \underline{\underline{C}} \cdot \underline{N} - 1$$ stretch ratio

$\underline{X}, t$.

Solid: $\underline{X}$ → reference configuration.

Fluid: $\underline{x}$ ↳ spatial → current coordinates.

don't use summation with λ !!!

$$\underline{\underline{Ex}}\ 1: \quad \underline{u} = \sum_{k=1}^{3} (\lambda_k - 1)\, x_k\, \underline{e}_k \qquad (3k).$$

$$[\underline{\underline{F}}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \qquad \underline{\underline{F}} = \lambda_1 \underline{e}_1\underline{e}_1 + \lambda_2 \underline{e}_2\underline{e}_2 + \lambda_3 \underline{e}_3\underline{e}_3$$

$$\underline{\underline{F}} = \lambda_1 \underline{e}_1\underline{e}_1 + \lambda_2 \underline{e}_2\underline{e}_2 + \lambda_3 \underline{e}_3\underline{e}_3.$$

$$\lambda_1 = \lambda_2 = \lambda_3 > 1.$$

uniform expansion: $[\underline{\underline{F}}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$\lambda_1 = \lambda_2 = \lambda_3 < 1$

uniform compression.

$\det \underline{\underline{F}} = J$

invariant.

$= \lambda_1 \lambda_2 \lambda_3$

Rubber is almost incompressible, so $J \simeq 1$.

In general, $\det \underline{\underline{F}} = \dfrac{dV}{dV_0}$ the deform of the volume over the reference (original) volume.

always true.

$= \dfrac{V}{V_0}$ ← new

reference volume.

in a tension bar:

$$\underline{\underline{\varepsilon}} = \varepsilon_{ij}\, \underline{e}_i \underline{e}_j$$

$$\varepsilon_{ij} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\varepsilon_{ij}} + \frac{1}{2}\left(\frac{\partial u_k}{\partial x_i} \cdot \frac{\partial u_k}{\partial x_j}\right).$$

small strain tensor ($1\% \sim 2\%$).

effect of large defor. quadratic term.

$(10^{-2} \sim 4 \cdot 10^{-2})$

$10^{-2} \cdot 10^{-2} = 10^{-4}.$

Week 3.

Sep. 15th (Wed.)

Review. $\underline{\underline{F}}$ Deformation Gradient Tensor

completely characterize the local deformation at a point $\underline{X}$.

$\underline{\underline{F}}(\underline{X}, t)$, $\underline{X}$, t, independent variables.

Material description.

$\underline{\underline{C}} = \underline{\underline{F}}^T \underline{\underline{F}}$ $\underline{\underline{F}}^T \underline{\underline{F}}$ Right Caughty - Green Tensor

$\underline{\underline{E}} = \underline{\underline{F}}^T \underline{\underline{F}} - \underline{\underline{I}}$.

$d\underline{x} \cdot d\underline{x} = d\underline{X} \cdot \underline{\underline{C}} \cdot d\underline{X}$

$\underline{N}$ is a unit vector.

$$\frac{\|d\underline{x}\|^2}{\|d\underline{X}\|^2} = \overbrace{\frac{d\underline{X}}{\|d\underline{X}\|}}^{\underline{n}} \cdot \underline{\underline{C}} \cdot \overbrace{\frac{d\underline{X}}{\|d\underline{X}\|}}^{\underline{n}}$$

$\frac{\|d\underline{x}\|}{\|d\underline{X}\|} \equiv \lambda$. → Ratio of length $\underline{x}$ of material line element in the current configuration.

$$\text{Stretch Ratio} = \frac{\text{length of mat. line ele. in the current configuration.}}{\text{length of mat. line ele. in the Ref. configuration.}}$$

$\lambda_n^2 = \underline{N} \cdot \underline{\underline{C}} \cdot \underline{N}$.

$\underline{N} \cdot \underline{\underline{E}} \cdot \underline{N} = \lambda_n^2 - 1$. measure the deformation.

$\underline{\underline{E}}$ → Lagrangian strain tensor.

$$E_{ij} = \underbrace{\frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right]}_{\varepsilon_{ij}} + \frac{1}{2}\left[\frac{\partial u_k}{\partial X_i} \cdot \frac{\partial u_k}{\partial X_j}\right].$$

Remember:

$\underline{x} = \underline{X} + \underline{u}(\underline{X}, t)$.

Both $\underline{\underline{C}}$ & $\underline{\underline{E}}$ are symmetric Tensors.

$\underline{\underline{C}}^T = \underline{\underline{C}}$. Recall $\underline{\underline{F}}$ is revertible.

$\underline{\underline{F}}^T \underline{\underline{F}} = \underline{\underline{C}}$.

$\underline{\underline{F}}$ → revertible → $\underline{\underline{C}}$ is positive definite.

$d\underline{X} \cdot \underline{\underline{C}} \cdot d\underline{X} \geq 0$. only when $d\underline{X} = 0$.

identical (exactly) → $\|\underline{\underline{F}} \cdot d\underline{X}\|^2 > 0$. $d\underline{X} \neq 0$.

$\underline{\underline{C}}$ symmetric implies that $\underline{\underline{C}}$ has Real eigen values

$\lambda_1^2 > \lambda_2^2 > \lambda_3^2 \leftarrow \lambda_1^2, \lambda_2^2, \lambda_3^2$.

$\underline{\underline{C}}$ positive definite implies $\lambda_i^2 > 0$.

$i = 1, 2, 3$.

$\underline{\underline{C}}$ Can be diagonalized

that is, $\underline{\underline{C}}$ can be written as

$$\underline{\underline{C}} = \lambda_1^2 \underline{n}_1 \underline{n}_1 + \lambda_2^2 \underline{n}_2 \underline{n}_2 + \lambda_3^2 \underline{n}_3 \underline{n}_3$$

$\underline{n}$ are orthornormal eigen vectors of $\underline{\underline{C}}$

that is $\underline{n}_i \cdot \underline{n}_j = \delta_{ij}$

THIS $\lambda_1, \lambda_3, \lambda_2$ are called principal stretches.

$\underline{n}_i$'s are the principal direction.

Polar Decomposition Theorem

$$\boxed{\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}}} = \underline{\underline{V}}\,\underline{\underline{R}}.$$

$\underline{\underline{R}}$ is a rigid body rotation tensor, $\underline{\underline{R}}^T = \underline{\underline{R}}^{-1}$

$$\underline{\underline{R}}^T \underline{\underline{R}} = \underline{\underline{R}}\,\underline{\underline{R}}^T = \underline{\underline{I}}.$$

$\underline{\underline{U}}$ is symmetric, positive definite.

and $\underline{\underline{U}}^2 = \underline{\underline{C}}$ & $\underline{\underline{U}} = \sqrt{\underline{\underline{C}}}$.

$$\underline{\underline{U}} = \lambda_1 \underline{N}_1 \underline{N}_1 + \lambda_2 \underline{N}_2 \underline{N}_2 + \lambda_3 \underline{N}_3 \underline{N}_3$$

<u>check</u> $\underline{\underline{U}} \cdot \underline{\underline{U}} = \underline{\underline{U}}^2 = \underline{\underline{C}}$.

$\underline{\underline{F}}$ can be decompose into two simple tensor,

where first tensor, $\underline{\underline{U}}$ → stretch tensor

↓

it stretch the material point in principal directions

the it rotate with $\underline{\underline{R}}$.

$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}} \cdot \underline{\underline{X}}$. → this a local theorem

$\underline{\underline{U}}$ comes first, and $\underline{\underline{R}}$ comes second.

$\underline{n}_i = \underline{\underline{R}}(\underline{N}_i)$

☆ Only need to prove

$\underline{\underline{R}} \equiv \underline{\underline{F}}\,\underline{\underline{U}}^{-1}$ is a rotation.

$\underline{\underline{R}}^T\underline{\underline{R}} = \underline{\underline{I}}$.

$\underline{\underline{R}}^T\underline{\underline{R}} = (\underline{\underline{F}}\,\underline{\underline{U}}^{-1})^T(\underline{\underline{F}}\,\underline{\underline{U}}^{-1})$.

$\underline{\underline{C}} = \underline{\underline{U}}^2$

$= (\underline{\underline{U}}^{-T}\underline{\underline{F}}^T)(\underline{\underline{F}}\,\underline{\underline{U}}^{-1}) = \underline{\underline{U}}^{-1}\underline{\underline{F}}^T\underline{\underline{F}}\,\underline{\underline{U}}^{-1}$

symmetric $\downarrow$ $\underline{\underline{U}}^{-1}$

$= \underbrace{\underline{\underline{U}}^{-1}\underline{\underline{U}}}_{\underline{\underline{I}}}\;\underbrace{\underline{\underline{U}}\,\underline{\underline{U}}^{-1}}_{\underline{\underline{I}}}$

$= \underline{\underline{I}}$

Therefore we prove: $\underline{\underline{R}}\,\underline{\underline{R}}^T = \underline{\underline{R}}^T\underline{\underline{R}} = \underline{\underline{I}}$.

$\underline{n}_i = \underline{\underline{R}}(\underline{N}_i)$.

$\underline{\underline{R}} = \underline{n}_i\underline{N}_i = \underline{n}_1\underline{N}_1 + \underline{n}_2\underline{N}_2 + \underline{n}_3\underline{N}_3$.

*** Decomposition is unique.

if we define: $\underline{\underline{V}} = \lambda_1\underline{n}_1\underline{n}_1 + \lambda_2\underline{n}_2\underline{n}_2 + \lambda_3\underline{n}_3\underline{n}_3$.

$\underline{\underline{V}}\,\underline{\underline{R}} = (\lambda_1\underline{n}_1\underline{n}_1 + \lambda_2\underline{n}_2\underline{n}_2 + \lambda_3\underline{n}_3\underline{n}_3)\cdot(\underline{n}_1\underline{N}_1 + \underline{n}_2\underline{N}_2 + \underline{n}_3\underline{N}_3)$.

$= \lambda_1\cdot\underline{n}_1\underline{N}_1 + \lambda_2\underline{n}_2\underline{N}_2 + \lambda_3\underline{n}_3\underline{N}_3$

$\Downarrow$ then we can check:

$\underline{\underline{R}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{R}}$.

[IN CASE] in mechanics theorem paper,

Someone writes:

$\underline{\underline{F}} = \cancel{F_{iA}\, e}$

$F_{iA}\, \underline{e}_i\, \underline{E}_A \rightarrow$ two point tensor

$\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$

$\underline{E}_A \cdot \underline{E}_B = \delta_{AB}$

in the ref. config. $\underline{e}_1, \underline{e}_2, \underline{e}_3$

current config.

$\underline{E}_1, \underline{E}_2, \underline{E}_3$

<u>Simple Shear Deformation.</u>

$$\begin{cases} x_1 = X_1 + X_2\tan\gamma \\ x_2 = X_2 \\ x_3 = X_3 \end{cases}$$

fixed number $(0, \pi/2)$.

$\underline{x} = x_i\underline{e}_i$

$$[\underline{\underline{F}}] = \begin{bmatrix} 1 & \tan\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{F}} = \underline{e}_1\underline{e}_1 + \tan\gamma\, \underline{e}_1\underline{e}_2 + \underline{e}_2\underline{e}_2 + \underline{e}_3\underline{e}_3.$$

$\det \underline{\underline{F}} = 1.$

$\hookrightarrow \det(\underline{\underline{R}}\,\underline{\underline{U}}) = \underbrace{\det \underline{\underline{R}}}_{=1} + \det \underline{\underline{U}} \Rightarrow \lambda_1\lambda_2\lambda_3.$

$\downarrow$

$\frac{V}{V_0}.$

$\underline{\underline{F}} =$

Office Hour:

$$\underline{\underline{F}} = (\delta_{ij} + u_{ij})\, \underline{e}_i\underline{e}_j$$

$$\underline{\underline{E}} = \frac{1}{2}\left[u_{i,j} + u_{j,i} + u_{k,i}\, u_{k,j}\right].$$

$\downarrow$

$,j = \frac{\partial}{\partial X_j}$

Week 4 (3). Mon.

Linear Theory. (small deformation).

↳ pertubation theory.

geometry change small.

(gradients, of displacements small, $\ll 1$).

$$\underset{\approx}{E} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \underbrace{\frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}}_{\text{higher order terms}}\right).$$

$\underline{u} = u_k \underline{e}_k$.

$$\approx \underbrace{\frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right]}_{\text{leading order terms}}.$$

$$\underline{\underline{C}} - \underline{\underline{I}} \equiv 2\underline{\underline{E}}.$$

$$\underline{\underline{F}} = \underline{\underline{I}} + \frac{\partial u_i}{\partial x_j} \underline{e}_i \underline{e}_j = \underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}}.$$

small rotation tensor

$$\frac{\partial u_i}{\partial X_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)}_{\text{sym}} + \underbrace{\frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right]}_{\text{Anti-sym}}$$

ε_{ij} ω_{ij}

$$C_{ij} = (\delta_{ij} + \varepsilon_{ij} + \omega_{ij})(\delta_{ji} + \varepsilon_{ji} + \omega_{ji})$$

$$= \delta_{ij}\delta_{ji} + \delta_{ij}\varepsilon_{ji} + \delta_{ij}\omega_{ji} + \varepsilon_{ij}\delta_{ji} + \varepsilon_{ij}\varepsilon_{ji} + \varepsilon_{ij}\omega_{ji} + \omega_{ij}\delta_{ji} + \omega_{ij}\varepsilon_{ji} + \omega_{ij}\omega_{ji}$$

$$\underline{\underline{C}} = (\underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}})^T(\underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}}).$$

$$= (\underline{\underline{I}} + \underline{\underline{\varepsilon}} - \underline{\underline{\omega}})(\underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}}).$$

$$= \underline{\underline{I}} + 2\underline{\underline{\varepsilon}} + \text{H.O.T.}$$

$$\underline{\underline{C}} = \underline{\underline{U}}^2 \quad (???)$$

$$\underline{\underline{C}} \approx \underline{\underline{I}} + 2\underline{\underline{\varepsilon}}.$$

$$\underline{\underline{U}} \approx \underline{\underline{I}} + \underline{\underline{\varepsilon}}. \qquad \underline{\underline{U}}^2 = \underline{\underline{I}} + 2\underline{\underline{\varepsilon}} + \underline{\underline{\varepsilon}}\,\underline{\underline{\varepsilon}}$$

$$\approx \underline{\underline{I}} + 2\underline{\underline{\varepsilon}}.$$

$$\underline{\underline{U}}^{-1} \approx \underline{\underline{I}} - \underline{\underline{\varepsilon}}.$$

$$\underline{\underline{F}} = \underline{\underline{R}}\,\underline{\underline{U}}$$

$$\underline{\underline{R}} = \underline{\underline{F}}\,\underline{\underline{U}}^{-1}. \qquad \underline{\underline{R}} \approx (\underline{\underline{I}} + \underline{\underline{\varepsilon}} + \underline{\underline{\omega}})(\underline{\underline{I}} - \underline{\underline{\varepsilon}}).$$

$$= \underline{\underline{I}} - \cancel{\underline{\underline{\varepsilon}}} + \underline{\underline{\omega}}.$$

$$\underline{V} = \frac{\partial \underline{X}}{\partial t}\Big|_{\underline{X}\ \text{fix}} = \frac{\partial \underline{u}}{\partial t}\Big|_{\underline{X}}.$$

↳ (at a fixed material point).

$$\underline{x} = \underline{X} + \underline{u}(\underline{X}, t).$$

$$\underline{A} = \frac{\partial^2 \underline{X}}{\partial t^2}\Big|_{\underline{X}} = \frac{\partial^2 \underline{u}}{\partial t^2}\Big|_{\underline{X}}.$$

$\underline{V}(\underline{X}, t)$.

mechanics: quantities in spatial descrip.

density

$\rho(\underline{X}, t)$.

the material derivative:

$$f(\underline{x}, t) = f(\underline{\chi}(\underline{X}, t), t).$$

$$\frac{Df}{Dt} \equiv \dot{f} \qquad \text{Fixed } \underline{X}.$$

$$= \frac{\partial f}{\partial x_i} \underbrace{\frac{\partial \chi_i}{\partial t}\bigg|_{\underline{X} = \underline{\chi}^{-1}(\underline{x}, t)}}_{V_i(\underline{X}, t)} + \frac{\partial f}{\partial t}\bigg|_{\underline{x}}.$$

velocity.

$$= \frac{\partial f}{\partial x_i} V_i(\underline{x}, t) + \frac{\partial f}{\partial t}\bigg|_{\underline{x}} = \nabla_{\underline{x}} f \cdot \underline{V} + \frac{\partial f}{\partial t}\bigg|_{\underline{x}}$$

$$\nabla_{\underline{x}} f = \frac{\partial f}{\partial x_j} \underline{e}_j.$$

$$\nabla_{\underline{X}} g = \frac{\partial g}{\partial X_j} \underline{e}_j.$$

$$\underline{V} = \underline{V}(\underline{X}, t) = \frac{\partial \underline{\chi}}{\partial t}\bigg|_{\underline{X}}.$$

$$\underline{v} = \underline{V}(\underline{\chi}^{-1}(\underline{x}, t), t) = \underline{\chi}^{-1}(\underline{x}, t).$$

in the spatial configuration.

$$\underline{a} = \underline{A}(\underline{\chi}^{-1}(\underline{x}, t), t).$$

$$\underline{a} = \underline{v} \cdot \nabla_{\underline{x}} \underline{V} + \frac{\partial \underline{V}}{\partial t}\bigg|_{\underline{x}}.$$

Concept of Stress.

if given displacement field, & ref. config.

↳ then we can calculate everything.

Assumption. Cauchy's hypothesis.

Important: orientation → X shape.

$$\underline{t} = \frac{\Delta f}{\Delta S} \qquad \Delta S \to 0$$

$$= \underline{t}(\underline{n}, \cancel{\underline{x}}, t).$$

$\underline{n} \rightarrow$ outward unit normal vector in the current configuration.

$\underline{t} \rightarrow$ traction vector (stress).

** If we know traction in 3D, then we know the stress at this point.

Cauchy's theorem.

Force on pyramid?

Body forces: $\rightarrow$ depends on volume of element.

ρ = mass per unit volume in current configuration.

Body forces per unit volume in

linear momentum Balance.

$$\underline{t}\,dA - \underline{t}_1 dA_1 - \underline{t}_2 dA_2 - \underline{t}_3 dA_3 + \rho\, \underline{b}\, dV$$

$$= m \cancel{\frac{dV}{dt}} \quad (\rho\, dV)\underline{a}.$$

dV is the volume of small pyramid

$dA \gg dV, \quad \frac{dV}{dA} \rightarrow 0.$

(in small pyramid).

$$\underline{t} = \underline{t}_1 \frac{dA_1}{dV} + \underline{t}_2 \frac{dA_2}{dV} + \underline{t}_3 \frac{dA_3}{dV}$$

$$\underline{t} = \cancel{\underline{t}_1 \underline{n}_1 + \underline{t}_2 \underline{n}_2 + \underline{t}_3 \underline{n}_3}$$

$$\underline{t}_1 \underline{e}_1 \cdot \underline{n} + \underline{t}_2 \underline{e}_2 \cdot \underline{n} + \underline{t}_3 \underline{e}_3 \cdot \underline{n}.$$

$$\underline{t}_1 = \sigma_1 \underline{e}_1$$

$$\underline{t}_k = \sigma_k \underline{e}_k, \quad k = 1, 2, 3.$$

$$\underline{t}_j = \sigma_{ij} \underline{e}_i$$

$$= \sigma_{ij} \underline{e}_i \underline{e}_j \cdot \underline{n}.$$

(other text book: $\underline{t} = \underline{\underline{\sigma}}^T \cdot \underline{n}$).

$\underline{\underline{\sigma}}$ = Cauchy or True stress tensor

$$\underline{t}_j = \sigma_{ji} \underline{n}_i. \qquad \underline{t} = \underline{\underline{\sigma}} \cdot \underline{n}.$$

$$\underline{t}_2 = \sigma_{12}\,\underline{e}_1 + \sigma_{22}\,\underline{e}_2 + \sigma_{32}\,\underline{e}_3.$$

Week 4, Wed.

$\underline{\underline{\sigma}}$ True stress tensor in current config.

Cauchy

$\underline{\underline{\sigma}} \cdot \underline{n} = \underline{t} \rightarrow$ traction vector.

$$[\underline{t}] = \frac{\text{Force}}{\text{Unit area}}$$

* Equilibrium equation – Deformed configuration (LMB)

↓

Linear momentum balance.

Key Results.

$$\underbrace{\nabla_x \cdot \underline{\underline{\sigma}}}_{\text{Divergence}} + \rho \underbrace{\underline{b}}_{\text{Body force}} = \rho \cdot \underbrace{\underline{a}}_{\text{acceleration}}$$

spatial.

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \rho a_i. \quad \dots (N1)$$

Current config.

⇓

in real-world applications, cuz you ~~didn~~ donke know current cofig.

in elastic prob., we use reference config.

In Ref config.

Eq. (N1) become:

$$\nabla_X \cdot \underline{\underline{P}} + \rho_0 \underline{B} = \rho_0 \underline{A}.$$

$\underline{A} = \underline{A}(\underline{X}, t)$

$$\frac{\partial P_{ij}}{\partial X_j} + \rho_0 B_i = \rho_0 A_i.$$

First Piola Tensor ⇒ Nominal stress tensor

$\boxed{\text{AMB}}$ Angular momentum balance.

$$\sigma_{ij} = \sigma_{ji}.$$

$$\hookrightarrow \underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T.$$

AMB.

$$\underline{\underline{P}}\,\underline{\underline{F}}^T = \underline{\underline{F}}\,\underline{\underline{P}}^T.$$

Derivation

METHOD I: Forces acting on Ω_t.

(Force balance) integral of $\int_{\Omega_t} \rho \underline{b}\, dV$

$$+ \int_{\Omega_t} \underline{\underline{\sigma}} \cdot \underline{n}\, dS \quad \text{(traction)} \qquad (F.1)$$

LMB

(Newton's law). $+ \frac{D}{Dt} \int_{\Omega_t} \rho \underline{v}\, dV$ ★★★ cannot take inside.

$\rho\, dV = \rho_0\, dV_0$ conservation of mass.

$$\frac{dV}{dV_0} = \frac{\rho_0}{\rho} = \det \underline{\underline{F}}.$$

Jacobian.

Eq. (F1) becomes:

$$\frac{D}{Dt} \int_{\Omega_t} \rho \underline{v}\, dV = \frac{D}{Dt} \int_{\Omega_0} \rho_0 \underline{V}\, dV_0$$

fixed

So can take $\frac{D}{Dt}$ inside

$$= \int_{\Omega_0} \rho_0 \underline{A}\, dV_0.$$

$$= \int_{\Omega_t} \rho \underline{a}\, dV$$

$$\underbrace{\int_{\partial\Omega_t} \underline{\underline{\sigma}} \cdot \underline{n}\, ds}_{\text{Divergence theorem}} + \int_{\Omega_t} (\rho \underline{b} - \rho \underline{a})\, dV.$$

$$\Downarrow$$

$$\int_{\Omega_t} \nabla_x \cdot \underline{\underline{\sigma}}\, dV.$$

$$\int_{\Omega_t} [\nabla_x \cdot \underline{\underline{\sigma}} + \rho \underline{b} - \rho \underline{a}]\, dV = 0$$

This is true for any Ω_t ⇒

$$\Rightarrow \boxed{\nabla_{\underline{x}} \cdot \underline{\underline{\sigma}} + \rho \underline{b} = \rho \underline{a}}$$

FINISHED !!!

σ_{21} ...

Net force in 1D,

$$\left[\frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + \frac{\partial \sigma_{21}}{\partial x_1}\right] dx_1 dx_2 dx_3$$

$$+ \rho \underline{b}\, dx_1 dx_2 dx_3 = \rho \underline{a}\, dx_1 dx_2 dx_3$$

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \rho a_i$$

Δ Δ $\Delta \rightarrow$ total force.

just times

$$dx_1 dx_2 (\sigma_{23})\, dx_3 = (\sigma_{32} \cdot dx_2)\, dx_1 dx_3$$

force

$$\sigma_{23} = \sigma_{32}$$

$$\sigma_{ij} = \sigma_{ji}$$

$$\int_{\Omega_0} \rho_0 \underline{B}\, dV_0 + \int_{\partial R_t} \underline{\underline{\sigma}} \cdot \underline{n}\, dS = \int_{\Omega_0} \rho_0 \cdot \underline{A}\, dV_0$$

$\underline{n}\, dS_t$

$\underline{n}\, dS_t$

$$\underline{t} = \underline{\underline{\sigma}} \cdot \underline{n} = \frac{d\underline{f}}{dS_t} \Rightarrow \underline{t}\, dS_t = d\underline{f}$$

$\underline{N}_0\, dS_0$

$$dV = \underline{n}\, dS_t \cdot d\underline{x} = \underline{n} \cdot dS_t \cdot \underline{\underline{F}} \cdot d\underline{X}.$$

$$dV_0 = \underline{N}\, dS_0 \cdot d\underline{X}.$$

$$\frac{dV}{dV_0} = \det \underline{\underline{F}} \equiv J.$$

$$\underline{n}\, dS_t \cdot \underline{\underline{F}} \cdot d\underline{X} = J \underline{N}\, dS_0 \cdot d\underline{X}.$$

$$d\underline{X} \cdot dS_t\, \underline{\underline{F}}^T \cdot \underline{n} = d\underline{X} \cdot J \underline{N}\, dS_0.$$

$d\underline{X}$ is solitary!

$$dS_t \cdot \underline{\underline{F}}^T \cdot \underline{n} = J \underline{N}\, dS_0.$$

$$\underline{\underline{F}}^T \cdot \underline{n}\, dS_0 = J \underline{N}\, dS_0$$

$$\boxed{\underline{n}\, dS_0 = J \underline{\underline{F}}^T \cdot \underline{N}\, dS_0}$$

Substitute $\int_{\partial\Omega_t} \underline{\underline{\sigma}} \cdot \underline{n}\, dS$

~~Nansen's~~ Formula

Nansen's.

$$\int_{\partial\Omega_0} \underbrace{J \underline{\underline{\sigma}}\, \underline{\underline{F}}^{-1}}_{\underline{\underline{P}}} \cdot \underline{N}\, dS_0.$$

By definition $\underline{\underline{P}} = J \underline{\underline{\sigma}}\, \underline{\underline{F}}^{-1}.$

Divergence

$$\int_{\partial\Omega_0} \nabla_{\underline{X}} \cdot \underline{\underline{P}}\, dV_0 \rightarrow \nabla_{\underline{X}} \cdot \underline{\underline{P}} + \rho_0 \underline{B} = \rho_0 \underline{A}.$$

$$\underline{t}_\alpha\, dS_0 = d\underline{f} = \underline{t}\, dS_t.$$

$$\underline{\underline{P}} \cdot \underline{n} = \underline{t}$$

not a symmetric tensor

$$\underline{\underline{P}} = J \underline{\underline{\sigma}}\, \underline{\underline{F}}^{-T}$$

not symmetric

symmetric.

AMB.

$$\frac{(\underline{\underline{P}}\,\underline{\underline{F}}^T)^T}{J} = \frac{\underline{\underline{P}}\,\underline{\underline{F}}^T}{J} \Rightarrow \boxed{\underline{\underline{F}}\,\underline{\underline{P}}^T = \underline{\underline{P}}\,\underline{\underline{F}}^T}$$

In fluid mech., use current config. as variables.

☆☆☆ Basic balance laws of continuum mechanics.

(Derived in current configuration)

Office hour Fri. 3:30pm.

HW #2.

$\underline{\underline{A}} = a_{ij}\, \underline{e}_i\, \underline{e}_j$.

$\{\underline{e}_i\}$ original basis.

$\{\underline{E}_j\}$. New.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \vdots & & \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Symmetry.

whole num.

λ_i : eigenvalues.

$\underline{\underline{A}}$ in new basis : $\underline{\underline{A}} = \lambda_1 \underline{E}_1 \underline{E}_1 + \lambda_2 \underline{E}_2 \underline{E}_2 + \lambda_3 \underline{E}_3 \underline{E}_3$

$$\rightarrow \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 3 \end{bmatrix}.$$

eigenvalues: $\lambda_1 = 8$, $\lambda_2 = 6$, $\lambda_3 = 3$.

Original one: $\underline{\underline{A}} = 6\underline{e}_1\underline{e}_1 - 2\underline{e}_1\underline{e}_2 - 1\underline{e}_1\underline{e}_3 \ldots\ldots$

$\hookrightarrow$ So, $\underline{\underline{A}} = 8\underline{E}_1\underline{E}_1 + 6\underline{E}_2\underline{E}_2 + 3\underline{E}_3\underline{E}_3$

what is $\underline{E}_1$?

Original tensor $\underline{\underline{A}} = a_{ij}\underline{e}_i\underline{e}_j$.

$= \lambda_1 \underline{E}_1\underline{E}_1 + \lambda_2 \underline{E}_2\underline{E}_2 + \lambda_3 \underline{E}_3\underline{E}_3$.

$\underline{E}_1 \cdot \underline{E}_1 = 1$, $\underline{E}_2 \cdot \underline{E}_1 = 0$, $\underline{E}_3 \cdot \underline{E}_1 = 0$

need to normalize to 1

$(a_{ij}\underline{e}_i\underline{e}_j) \cdot \underline{E}_1 = \lambda_1 \underline{E}_1$

$\underline{E}_1 \cdot \underline{e}_j = P_{1j}$.

$a_{ij}\, \underline{e}_i \underbrace{(\underline{e}_j \cdot \underline{E}_1)}_{P_{1j}} = \lambda_1 \underline{E}_1$

$\Downarrow$

$\underline{E}_1 = P_{1i}\underline{e}_i$

$a_{ij} P_{1j}\, \underline{e}_i = \lambda_1 \underline{E}_1 = \lambda_1 P_{1i}\, \underline{e}_i$

or $\left[a_{ij}P_{1j} - \lambda_1 P_{1i}\right] \underline{e}_j = \underline{0}$

$\Rightarrow a_{ij} P_{1j} - \lambda P_{1i} = 0 \longrightarrow i = 1, 2, 3$

$A = [a_{ij}]$. $\quad A\vec{P}_1 - \lambda \vec{P}_1 = 0$

eigenvector of $\underline{A}$ for $\underline{E}$ $\longleftarrow$ $\vec{P}_1 = \begin{bmatrix} P_{11} \\ P_{12} \\ P_{13} \end{bmatrix}$.

or $\quad (A - \lambda I)\vec{P}_1 = \vec{0}$

$\lambda_1 \rightarrow$ eigenvalue of $\underline{A}$

$\vec{P}_1$ is a eigen vector for A $[\lambda_1]$.

$\underline{E}_1 = P_{1i}\, \underline{e}_i$

$\underline{E}_1 \cdot \underline{E}_1 = 1$. $\quad P_{1i} P_{1i} = 1$

Same idea applies to E_2

$\underline{E}_2 = P_{2i}\, \underline{e}_i \longleftarrow \lambda_2$

E_3 .....

$\underline{\underline{A}} \cdot (\underline{e}_1 + \underline{e}_2) = \ldots$

☆ get the same no matter which basis...

$\underline{\underline{A}} = (\underline{e}_1 \underline{e}_2)$. MATRIX.

linear transformation

$\underline{\underline{A}} \cdot \underline{e}_1 =$ first column of $\underline{\underline{A}}$.

$(\underline{e}_1 \underline{e}_2) \cdot \underline{e}_1 = \underline{0}$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\underline{\underline{A}} \cdot \underline{e}_2 = (\underline{e}_1 \underline{e}_2) \cdot \underline{e}_2 = \underline{e}_1 (\underline{e}_2 \cdot \underline{e}_2) = \underline{e}_1$

$= 1\underline{e}_1 + 0\underline{e}_2 + 0\underline{e}_3$

$\underline{\underline{A}} \cdot \underline{e}_3 = \underline{0}$ $\quad \rightarrow$ ☆ respect to basis $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$.

$\underline{\underline{A}} = a_{ij}\, \underline{e}_i \underline{e}_j$ $\qquad [\underline{e}_1 \underline{e}_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$[A] = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} +$$

$$= a_{11} \underline{e}_1 \underline{e}_1 + \cdots$$

$$\underline{V} = V_1 \underline{e}_1 + V_2 \underline{e}_2 + V_3 \underline{e}_3.$$

$$\underline{E}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\underline{V} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

$$V_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + V_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \cdots$$

$$\underline{V} = \underline{e}_1$$

$$\underline{e}_i \underline{e}_j$$

$$\underline{\underline{A}} = a_{11} \underline{e}_1 \underline{e}_1 + a_{12} \underline{e}_1 \underline{e}_2 + a_{13} \underline{e}_1 \underline{e}_3 + \cdots$$

$$[\underline{A}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

break it down into simple linear trans.

$$\underline{V} = V_1 \underline{e}_1 + V_2 \underline{e}_2 + V_3 \underline{e}_3$$

☆ eigenvalues → invariants

eigenvectors → be cave of the basis !!!

$\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ → $\underline{E}_i = P_{ij} \underline{e}_j$

$\{\underline{E}_1, \underline{E}_2, \underline{E}_3\}$.

with respect to this basis.

$$\vec{P}_1 = \begin{bmatrix} P_{11} \\ P_{12} \\ P_{13} \end{bmatrix}.$$

associated with P_{ij}.

$\underline{\underline{W}}$ = Skew symmetric

$\underline{\underline{W}} \cdot \underline{v} = \underline{\omega} \times \underline{v}$

*** need to tell what $\underline{W}$ is

expand:

$\underline{\underline{W}} = w_{12}\, \underline{e}_1 \underline{e}_2 + w_{21}\, \underline{e}_2 \underline{e}_1 + \cdots$

$w_{11},\ w_{22},\ w_{33}, = 0.$

$\underline{\underline{W}} \cdot \underline{v} = \underline{P} \times \underline{v}.$

$w_{21} = -w_{12}$

$= w_{12}\, \underline{e}_1 \underline{e}_2 - w_{12}\, \underline{e}_2 \underline{e}_1 + \cdots$

$\underline{\underline{W}} \cdot \underline{v} \rightarrow \underline{v} = v_i \underline{e}_i$

$\underline{P} \times \underline{v} = \cdots v_1 \cdots v_2 \cdots v_3$

$P_1 = w_{32}$

$P_2 = w_{13}.$

$P_3 = w_{21}.$

$\Rightarrow \underline{P} = w_{32}\, \underline{e}_1 + w_{13}\, \underline{e}_2 + w_{21}\, \underline{e}_3.$

Sep 27. Week 3.

Personal Review – So far:

→ Cartesian Tensors.

- Review on Notation.

Summation Convention (Indical notation)

Permutation symbol.

Transformation Rule for vectors

▷ Second Order tensors.

▷ Transpose of tensor

Symmetric & Skew-symmetric tensor

Tensor transformation (basis, ...)

Operation of tensors: (products, ...)

Symmetric tensors: Diagonalization.

High order tensor

Trace of second order tensor

~~High order tensor~~

Tensor fields.

Kinematics

Sep 27, Week 5. Mon.

Review: Last lecture: Balance laws.

True stress tensor → current coordinate.

→ $\nabla_{\underline{x}} \cdot \underline{\underline{\sigma}} + \rho \underline{b} = \rho \underline{a}$ invariante form

(× config. influence).

$$\underbrace{\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \rho a_i.}_{3\text{ PDEs.}} \quad i = 1, 2, 3.$$

→ in the current coordinate

independent spatial variable are x_i.

AMB $\sigma_{ij} = \sigma_{ji}$, $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$

Balance law in reference configuration, independent variables are X_i.

$\underline{\underline{P}}$ (Nominal or 1st Piola stress tensor).

$$\underline{\underline{P}}(\underline{X}) = J \underline{\underline{\sigma}}(\underline{x}) \underline{\underline{F}}^{-T}. \longrightarrow J = \det \underline{\underline{F}}.$$

true stress.

$$\underline{\underline{\sigma}} = \frac{1}{J} \underline{\underline{P}} \underline{\underline{F}}^T \qquad \underline{\underline{\sigma}}(\underline{x} = \underline{\chi}(\underline{X}, t), t).$$

$$\nabla_{\underline{X}} \underline{\underline{P}} + \rho_0 \underline{B}_0 = \rho_0 \underline{A}$$

$$\frac{\partial P_{ij}}{\partial X_j} + \rho_0 B_{0i} = \rho_0 A_i.$$

$\underline{\underline{P}}$ is not always symmetric

So. an.. (AMB) $\underline{\underline{P}}\underline{\underline{F}}^T = \underline{\underline{F}}\underline{\underline{P}}^T$

Proof.

$\det \underline{\underline{F}} = J = \frac{dV}{dV_0}$, $\rho_0 dV_0 = \rho dV$

$$\frac{\rho_0}{\rho} = \frac{dV}{dV_0} = J = \det \underline{\underline{F}}$$

Material is called incompressible if $J = 1$, $\forall \underline{X}$

Review Part

$\underline{\underline{P}} \cdot \underline{N}\, dS_0 = \underline{\underline{\sigma}} \cdot \underline{n} \cdot dS_0$

Simple example. $\underline{\underline{\sigma}} = \sigma_{11} \underline{e}_1 \underline{e}_1$

$\sigma_{11} = \frac{F}{A}$

$P_0 = \frac{F}{A_0}$

Incompressible solid. $J = \lambda_1 \lambda_2 \lambda_3 = 1$.

$\lambda = \lambda_2 = \lambda_3$. $\lambda = \frac{1}{\sqrt{\lambda_1}}$.

Plane stress.

$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0, \quad \forall X_3.$

Sym $\mapsto$ $\sigma_{31} = \sigma_{32} = 0.$

Non-vanishing Stress state. $\left\{ \begin{array}{l} \text{hydrostatic pressure} \\ \text{(atmosphere)} \end{array} \right.$ i, $w \gg t$.

$\sigma_{11}, \sigma_{12} = \sigma_{21}, \sigma_{22}$ $\qquad \sigma_{\alpha\beta}. \quad \alpha, \beta = 1, 2.$

Assumption: $\sigma_{\alpha\beta}(X_1, X_2, t)$. independent of X_3

Equilibrium Eq: $\dfrac{\partial \sigma_{\alpha\beta}}{\partial X_\beta} + \rho b_\alpha = \cancel{\rho}\, 0$

Quasi-static problem $\quad \underline{A} = 0.$

↳ Reduce to a 2D problem

Pure shear:

$$\underline{\underline{\sigma}} = \sigma_{12}\, \underline{e}_1 \underline{e}_2 + \sigma_{21}\, \underline{e}_2 \underline{e}_1$$

$$= \tau (\underline{e}_1 \underline{e}_2 + \underline{e}_2 \underline{e}_1).$$

Eg.

$\underline{\sigma} \cdot \underline{n} = 0$ (no external traction).

(no disp.)

$u_3(X_1, X_2, X_3 = 0) = 0.$

(no shear)

$\tau_{21} = \tau_{12} = 0$, on $X_3 = 0$

$|X_1^2 + X_2^2| = a.$

→ Solve the shape of gel.

↳ Laplace-Young Equation.

Equilibrium.

$$\gamma_{LV} \cos\theta - \gamma_{LS} = \gamma_{VS}$$

$$\frac{\gamma}{G a} = \text{elastocapillary no.}$$

G: shear modulus. $\quad a$: typical length scale

water leak out → poroelasticity

EASY WAY TO DO THIS:

$\underline{P} \cdot \underline{N} = 0 \qquad \nabla_{\underline{X}} \underline{P} = -\rho_0 \underline{B}_0$

glued. → $u_3 = 0$ $\qquad = -\rho_0 g \underline{e}_3$

$\nabla \cdot \underline{\underline{\sigma}} = \rho g \underline{e}_3$

3 equations → 6 unknown. ($\sigma_{\alpha\beta}$)

3 unknowns → $\hat{\underline{u}}$.

9 unknowns.

Nonlinear Elasticity

Continuum Mechanics.

Sep 29, Wed, Week 5.

Constitutive law.

$$\underline{\underline{\sigma}} = \underline{\underline{\Psi}}(\underline{\underline{F}}(t'), -\infty < t' \le t).$$

$\underline{\underline{\sigma}}(t)$ ↕ Follow the whole deformation history.

how to obtain the function → Q.

▽ Hyperelasticity (Green's elasticity).

Elasticity $\underline{\underline{\sigma}}$ depends only on $\underline{\underline{F}}(t)$.

$$\underline{\underline{\sigma}} = \underline{\underline{\Psi}}(\underline{\underline{F}}(t)).$$

↳ response function.

Mathematically,

$$\underline{\underline{P}} = \frac{\partial w(\underline{\underline{F}})}{\partial \underline{\underline{F}}}$$

w → the strain energy density.

w ← energy per unit volume

⇑

* the definition of hyperelastic materials.

$$P_{ij} = \frac{\partial w}{\partial F_{ij}}.$$

$$dW = \frac{\partial W}{\partial F_{ij}} dF_{ij} = \frac{\partial W}{\partial \underline{\underline{F}}} : d\underline{\underline{F}}$$

$$= P_{ij}\, dF_{ij} = \underline{\underline{P}} : d\underline{\underline{F}} \qquad \text{work}$$

$$d\phi = -\underline{E} \cdot d\underline{x} \qquad \text{Potential}$$

$\underline{\underline{\sigma}}$

external work rate

$$= \int_{\partial\Omega_t} (\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{v}\, ds + \int_{\Omega_t} \rho \cdot \underline{b} \cdot \underline{v}\, dV_t$$

in the reference configuration:

(Piola stress

$$\int_{\partial\Omega_0} (\underline{\underline{P}} \cdot \underline{N}) \cdot \underline{V}\, dS_0 + \int_{\Omega_0} \rho_0\, b_0 \cdot V\, dV_0$$

$$\int_{\partial\Omega_0} (\underline{\underline{P}} \cdot \underline{N})\, dS_0 = \int_{\partial\Omega_0} P_{ij} N_j V_i\, dS_0$$

div. Theo.

$$= \int_{\partial\Omega_0} (P_{ij} V_i)_{,j}\, dV_0 \qquad ,j = \frac{\partial}{\partial X_j}$$

$$= \int_{\partial\Omega_0} \underbrace{P_{ij,j}}_{\text{LMB}} V_i\, dV_0 + \int_{\partial\Omega_0} P_{ij} V_{i,j}\, dV_0$$

$$P_{ij,j} = -\rho_0 B + \rho_0 A_i$$

$$\text{EWR} = \int_{\Omega_0} \rho_0 \underline{A} \cdot \underline{v}\, dV_0 + \int_{\Omega_0} P_{ij} V_{i,j}\, dV_0$$

$$\int_{\Omega_0} P_{ij} \dot{F}_{ij}\, dV_0$$

$$F_{ij} = \delta_{ij} + u_{i,j}$$

$$V_{i,j} = \frac{\partial}{\partial X_j}\left[\frac{\partial u_i}{\partial t}\bigg|_{X_i}\right]$$

$$= \frac{\partial F_{ij}}{\partial t}\bigg|_{\underline{X}}$$

$$\frac{DF_{ij}}{Dt} = \dot{F}_{ij}$$

move the body

kinetic energy. (KE)

$$= \frac{1}{2}\frac{D}{Dt}\left(\int_{\Omega_0} \rho_0 \, (\underline{V} \cdot \underline{V}) \, dV_0\right)$$

in the reference configuration

$$+ \int_{\Omega_0} \underline{\underline{P}} : \dot{\underline{\underline{F}}} \, dV_0$$

completely general

work rate

deform the body.

there could be dissipation in the process.

☆ hyperelastic

$$\frac{D}{Dt}\int_{\Omega_0} W(\underline{\underline{F}}) \, dV_0$$

$$\frac{\partial W}{\partial \underline{\underline{F}}} = \underline{\underline{P}}$$

assume $P_{ij} = \dfrac{\partial W}{\partial F_{ij}}$.

$$dW = \frac{\partial W}{\partial F_{ij}} \, dF_{ij}$$

$$\cancel{(dW)^{\cdot} = \frac{\partial W}{\partial F_{ij}} \dot{dF}_{ij}}$$

$$\frac{DW}{Dt} = \underbrace{\frac{\partial W}{\partial F_{ij}}}_{P_{ij}} \underbrace{\frac{DF_{ij}}{Dt}}_{\dot{F}_{ij}}$$

$t_1 \to t_2$; have to integrate the rate to time.

Total work from $t_1 \to t_2$

$$\int_{t_1}^{t_2} EWR \, dt = \int_{t_1}^{t_2} \frac{D}{Dt} \cdot \int_{\Omega_0} \rho \, (\underline{V})^2 \, dV_0 \, dt.$$

$$+ \int_{t_1}^{t_2} \frac{D}{Dt}\left(\int_{\Omega_0} W(\underline{\underline{F}}) \cdot dV_0\right) dt$$

t_1: $\underline{\underline{F}}_1$ $\underline{V}_1$

t_2: $\underline{\underline{F}}_2$ $\underline{V}_2$

$\underline{\underline{F}}_2 = \underline{\underline{F}}$, $\underline{V}_2 = \underline{V}$

$$\int_{\Omega_0} W(\underline{\underline{F}}_2) \, dV_0 - \int_{\Omega_0} W(\underline{\underline{F}}_1) \, dV_0 = 0$$

Motivation for Hyperelasticity

↳ No energy loss during loading

Objectivity

- $W(\underline{\underline{F}}) = W(\underline{\underline{Q}}\,\underline{\underline{F}})$

$\downarrow$

rigid body rotation.

$\underline{\underline{Q}}$: rotation.

$\uparrow$ (any)

$\underline{\underline{F}} = \cancel{\underline{\underline{Q}}}\,\underline{\underline{R}}\,\underline{\underline{U}}$

only depends on stretch tensor

$W(\underline{\underline{F}}) = W(\underline{\underline{R}}^T\underline{\underline{R}}\,\underline{\underline{U}}) = W(\underline{\underline{U}})$.

☆ only the stretching parts make a diff.

$= \hat{W}(\underline{\underline{C}})$. $\quad (\because \underline{\underline{U}}^2 = \underline{\underline{C}})$.

$\underline{\underline{P}} \Rightarrow P_{ij} = \dfrac{\partial W}{\partial F_{ij}} = \dfrac{\partial \hat{W}}{\partial C_{kl}} \dfrac{\partial C_{kl}}{\partial F_{ij}}$

$= 2\underline{\underline{F}} \dfrac{\partial \hat{W}}{\partial C_{ij}}$ (HW).

not isotropic.

$W(\underline{\underline{F}}) = W(\underline{\underline{F}}\,\underline{\underline{Q}})$.

~~only for isotropic~~

Isotropic : true for all Q

depends on the symmetry of the materials.

$$\underline{x} = \underline{X} + \underline{u}(\underline{X}, t)$$

$$\underline{X} = \chi^{-1}(\underline{x}, t).$$

$$\underline{u}(\chi^{-1}(\underline{x}, t)) = \hat{\underline{u}}(\underline{x}, t).$$

$$\underline{x} - \underline{X} = \underline{u} \qquad \frac{\partial u_i}{\partial X_j} = \frac{\partial u_i}{\partial x_k} \cdot \frac{\partial x_k}{\partial X_j}$$

$$\frac{\partial u_i}{\partial X_j} = \frac{\partial \hat{u}_i}{\partial x_k} \cdot \frac{\partial x_k}{\partial X_j}.$$

$$\underline{x} - \underline{X} = \underline{u}.$$

$$\underline{X} - \underline{x} = \underline{u}$$

Oct. 5, Mon, Week 6.

$$\det(\underline{\underline{I}} + d\underline{\underline{B}}) = 1 + \frac{1}{\mathbb{R}}(d\underline{\underline{B}}) + \underline{\underline{O}}(d\underline{\underline{B}})^2$$

$$d(\det \underline{\underline{A}}) = \det A \cdot (1 + \cancel{\det}\,\mathrm{tr}(\underline{\underline{A}}^{-1} d\underline{\underline{A}})) - \det \underline{\underline{A}}$$

$$= (\det A)\,\mathrm{tr}(\underline{\underline{A}}^{-1} d\underline{\underline{A}}).$$

$$= (\det A)(\underline{\underline{A}}^{-T} : d\underline{\underline{A}})$$

$$= \frac{\partial \det \underline{\underline{A}}}{\partial \underline{\underline{A}}} : d\underline{\underline{A}}.$$

↳ hold true for all $d\underline{\underline{A}}$.

$$\Rightarrow \frac{\partial \det \underline{\underline{A}}}{\partial \underline{\underline{A}}} = (\det \underline{\underline{A}})\underline{\underline{A}}^{-T}.$$

$$\frac{\partial (\det A_{ij})}{\partial A_{ij}} = (\det \underline{\underline{A}}) A^{-1}_{jk}.$$

$I_1 \underline{\underline{I}} - \underline{\underline{C}}$

↳ $$\underline{\underline{P}} = \frac{\partial W}{\partial \underline{\underline{F}}} = 2\underline{\underline{F}} \frac{\partial \hat{W}}{\partial \underline{\underline{C}}} = 2\underline{\underline{F}}\left[\frac{\partial \Phi}{\partial I_1}\frac{\partial I_1}{\partial \underline{\underline{C}}} + \frac{\partial \Phi}{\partial I_2}\frac{\partial I_2}{\partial \underline{\underline{C}}} + \frac{\partial \Phi}{\partial I_3}\frac{\partial I_3}{\partial \underline{\underline{C}}}\right]$$

$\underline{\underline{I}}$

1st Piola stress.

$$\underline{\underline{P}} = 2\left[\left(\frac{\partial \Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2}\right)\underline{\underline{F}} + \frac{\partial \Phi}{\partial I_2}\underline{\underline{F}}\,\underline{\underline{C}} + I_3 \frac{\partial \Phi}{\partial I_3}\underline{\underline{F}}^{-T}\right]$$

$$\underline{\underline{F}}\,\underline{\underline{C}}^{-T} = \underline{\underline{F}}\,(\underline{\underline{F}}^T\underline{\underline{F}})^{-1}$$
$$= \underline{\underline{F}}\,(\underline{\underline{F}}^{-1}\underline{\underline{F}}^{-T})$$
$$= \underline{\underline{F}}^{-T}$$

Recall

$$\underline{\underline{\sigma}} = \frac{1}{J}\underline{\underline{P}}\,\underline{\underline{F}}^T.$$

implies. $J = \det \underline{\underline{F}}$

$$\hookrightarrow \frac{1}{J}\left[\left(\frac{\partial \Phi}{\partial I_1} + I_1 \cdot \frac{\partial \Phi}{\partial I_2}\right)\underline{\underline{F}}\,\underline{\underline{F}}^T - \frac{\partial \Phi}{\partial I_2}(\underline{\underline{F}}\,\underline{\underline{F}}^T)^2 + I_3 \frac{\partial \Phi}{\partial I_3}\underline{\underline{I}}\right] = \underline{\underline{\sigma}}.$$

Tension Test.

$$\underline{\underline{F}} \to \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix};$$

$$\underline{\underline{F}} = \lambda\, \underline{e}_1\underline{e}_1 + \lambda_2 \underline{e}_2\underline{e}_2 + \lambda_3 \underline{e}_3\underline{e}_3$$

$I_1 \neq$

$I_2 \neq$

$$\underline{\underline{F}}^T\underline{\underline{F}} \Rightarrow \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_1^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}$$

$$tr(\underline{\underline{C}}) = \lambda_1^2 + \lambda_1^2 + \lambda_3^2.$$

$\lambda_1 = \lambda_2 = \lambda.$

$\Rightarrow \det \underline{\underline{C}} = \lambda^4\lambda_3^2.$

$$\underline{\underline{P}} = \underline{\underline{\sigma}}/\lambda.$$

$$P_{33} = \frac{\sigma_{33}}{\lambda}. \qquad \sigma_{ij} = 0,\ i,j \neq 3.$$

$\lambda \to I_1, I_2, I_3$

loading $\to \lambda \to$ curve.

fit: $\downarrow$ constitutive modeling

$\lambda = 1 - \varepsilon \to \frac{\delta L}{L_0}$

strain ener. dens. function.

$$\Phi = C_1 (I_1 - 3 - 2\log(J)) + C(\ln J)^2.$$

$\det \underline{F}$

(C_1. C are material constants.

common model

locally linear model.

$$\underline{\sigma} = \lambda \underline{\varepsilon}.$$

simple tese & simple shear

determine different parameters.

fight against entropy

(molecular interactions)

Temperature is ~~[illegible]~~ important

↳ entropy.

Incompressible.

$$\frac{dV}{dV_0} = \overbrace{\det \underline{F}}^{J} = \underline{I}.$$

Isotropic deformation.

Define new energy function.

$$W_{new} = W(\underline{F}) - P(J-1)$$

Lagrangian multiplier

(impose constraints)

$$\underline{P} = \frac{\partial W_{new}}{\partial \underline{F}} = \frac{\partial W}{\partial \underline{F}} - P \cdot \frac{\partial \det(\underline{F})}{\partial \underline{F}}.$$

$$= \frac{\partial W}{\partial \underline{F}} - P \underbrace{(\det \underline{F})}_{1} \underline{F}^{-T}.$$

$J = 1$

$$\underline{\sigma} = \underline{P}\,\underline{F}^T.$$

$$\underline{\sigma} = \frac{\partial W}{\partial \underline{F}} \underline{F}^T - P \underline{I}. \longrightarrow \text{model.}$$

$$I_3 = \det \underline{C} = \det(\underline{F}^T \underline{F}) = \det(\underline{F}^T)\det \underline{F} = 1.$$

Isotropic incompressible solid.

$$\overbrace{W = \Phi(I_1, I_2)}.$$

$$\underline{\underline{\sigma}} = \left[\left(\frac{\partial \Phi}{\partial I_1} + I_2 \frac{\partial \Phi}{\partial I_2}\right)\underline{\underline{b}} - \frac{\partial \Phi}{\partial I_2}\underline{\underline{b}}^2\right] - p\underline{\underline{I}}$$

Corrected Note: (Based on Wiley books).

Review.

Hyperelasticity $\rightarrow$ $\underline{\underline{P}} = \dfrac{\partial W(\underline{\underline{E}})}{\partial \underline{\underline{F}}}$

$\rightarrow$ gradient of strain energy density with respect to $\underline{\underline{F}}$.

$\rightarrow$ end exactly where u start.

Objectivity $\sim$ $W(\underline{\underline{F}}) = \hat{W}(\underline{\underline{C}}) \equiv \bar{W}(\underline{\underline{E}})$.

$\underline{\underline{C}} = \underline{\underline{F}}^T\underline{\underline{F}}$.

Isotropic Material $\sim$ $W(\underline{\underline{F}}) = W(\underline{\underline{F}}\,\underline{\underline{Q}})$.

$\forall$ orthogonal tensor $\underline{\underline{Q}}$

Define $\bar{\underline{\underline{F}}} = \underline{\underline{F}}\,\underline{\underline{Q}}$ $\sim$ if isotropic.

$W(\bar{\underline{\underline{F}}}) = W(\underline{\underline{F}})$ $\forall Q$.

Objectivity $\sim$ $W(\bar{\underline{\underline{F}}}) = \hat{W}(\bar{\underline{\underline{C}}}) = W(\bar{\underline{\underline{F}}}^T\bar{\underline{\underline{F}}})$.

$= \hat{W}(\underline{\underline{F}}\,\underline{\underline{Q}})^T(\underline{\underline{F}}\,\underline{\underline{Q}}) = \hat{W}(\underline{\underline{Q}}^T\underline{\underline{C}}\,\underline{\underline{Q}})$.

$\hat{W}$ is a scalar invariant of Tensor $\underline{\underline{C}}$.

$$\underbrace{\det[\underline{\underline{C}} - \lambda\underline{\underline{I}}]}_{\text{independent of } Q} = (-\lambda)^3 + I_1\lambda^2 - I_2\lambda + I_3.$$

$$\rightarrow \begin{cases} I_1 = \operatorname{tr}\underline{\underline{C}}. \\ I_2 = \frac{1}{2}\left[(\operatorname{tr}\underline{\underline{C}})^2 - \operatorname{tr}\underline{\underline{C}}^2\right]. \\ I_3 = \det\underline{\underline{C}}. \end{cases}$$

$\rightarrow$ For isotropic material.

$$\hat{W} = \Phi(I_1, I_2, I_3).$$

isotropic.

$$\rightarrow \underline{\underline{P}} = \frac{\partial W}{\partial \underline{\underline{F}}} = 2\underline{\underline{F}}\frac{\partial \hat{W}}{\partial \underline{\underline{C}}}.$$

$$= 2\underline{\underline{F}}\left[\frac{\partial \Phi}{\partial I_1}\underbrace{\left(\frac{\partial I_1}{\partial \underline{\underline{C}}}\right)}_{\underline{\underline{I}}} + \frac{\partial \Phi}{\partial I_2}\underbrace{\left(\frac{\partial I_2}{\partial \underline{\underline{C}}}\right)}_{I_1\underline{\underline{I}} - \underline{\underline{C}}} + \frac{\partial \Phi}{\partial I_3}\underbrace{\left(\frac{\partial I_3}{\partial \underline{\underline{C}}}\right)}_{I_3\underline{\underline{C}}^{-T} = I_3\underline{\underline{C}}^{-1}}\right]$$

$I_3 = \det\underline{\underline{C}}$.

<u>most general</u>: how to find $\dfrac{\partial \det\underline{\underline{A}}}{\partial \underline{\underline{A}}}$.

$$d(\det\underline{\underline{A}}) = \frac{\partial \det\underline{\underline{A}}}{\partial \underline{\underline{A}}} : d\underline{\underline{A}}$$

$$= \frac{\partial(\det\underline{\underline{A}})}{\partial A_{ij}} dA_{ij}.$$

$$d(\det\underline{\underline{A}}) = \det(\underline{\underline{A}} + d\underline{\underline{A}}) - \det(\underline{\underline{A}})$$

$$= \det(\underline{\underline{A}}(\underline{\underline{I}} + \underbrace{\underline{\underline{A}}^{-1}d\underline{\underline{A}}}_{d\underline{\underline{B}}})) - \det\underline{\underline{A}}$$

$$\underline{\underline{I}} + d\underline{\underline{B}} \rightarrow \begin{bmatrix} 1+dB_{11} & dB_{12} & \cdots \\ \cdots & 1+dB_{22} & \cdots \\ \cdots & \cdots & 1+dB_{33} \end{bmatrix}$$

$$\det(\underline{\underline{I}} + d\underline{\underline{B}}) = 1 + \operatorname{tr}(d\underline{\underline{B}}) + o(d\underline{\underline{B}})^2.$$

$$d(\det(d\underline{\underline{B}})) = \det \underline{\underline{A}}\,(1 + \operatorname{tr}(\underline{\underline{A}}^{-1} d\underline{\underline{A}})) - \det \underline{\underline{A}}$$

$$= (\det \underline{\underline{A}}) \operatorname{tr}(\underline{\underline{A}}^{-1} d\underline{\underline{A}}).$$

$$= (\det \underline{\underline{A}}).(\underline{\underline{A}}^{-T} : d\underline{\underline{A}}) = \frac{\partial \det \underline{\underline{A}}}{\partial \underline{\underline{A}}} : d\underline{\underline{A}}$$

$$\Rightarrow \frac{\partial \det \underline{\underline{A}}}{\partial \underline{\underline{A}}} = (\det \underline{\underline{A}})\, \underline{\underline{A}}^{-T}.$$

$$\cdots + \frac{\partial \Phi}{\partial I_3} \frac{\partial I_3}{\partial \underline{\underline{C}}} \rightarrow I_3 \underline{\underline{C}}^{-T} = I_3 \underline{\underline{C}}^{-1}$$

$$I_3 = \det \underline{\underline{C}}.$$

$$\underline{\underline{P}} = 2\left[\left(\frac{\partial \Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2}\right)\underline{\underline{F}} - \frac{\partial \Phi}{\partial I_2} \underline{\underline{F}}\,\underline{\underline{C}} + I_3 \frac{\partial \Phi}{\partial I_3} \underline{\underline{F}}^{-T}\right]$$

Recall: $\underline{\underline{\sigma}} = \frac{1}{J} \cdot \underline{\underline{P}}\, \underline{\underline{F}}^T \rightarrow$ True Stress.

Allison Carter

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Detective Todd.

— Mike Hughes → detective

Tension Test.

$$\underline{\underline{F}} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_1 & \\ & & \lambda_3 \end{bmatrix}.$$

$$\underline{\underline{F}} = \lambda_1 \underline{e}_1 \underline{e}_1 + \lambda_1 \underline{e}_2 \underline{e}_2 + \lambda_3 \underline{e}_3 \underline{e}_3.$$

$$I_1 = \operatorname{tr} \underline{\underline{C}} = \operatorname{tr}(\underline{\underline{F}}^T \underline{\underline{F}}) = 2\lambda_1^2 + \lambda_3^2$$

$$I_3 = \det \underline{\underline{C}} = \lambda_1^4 \lambda_3^2.$$

$$I_2 = \cancel{P_{33} = \frac{\sigma_{33}}{\lambda}} \qquad P_{33} = \frac{\sigma_{33}}{\lambda}.$$

common model.

$$\rightarrow \Phi = C_1(I_1 - 3 - 2\ln J) + C_2(\ln J)^2$$

$J = \det \underline{\underline{F}}$. ~ C_1, C_2 material constants.

Compressible isotropic materials.

Incompressible Solid.

$$\frac{dV}{dV_0} = \det \underline{\underline{F}} = 1. \quad \text{"Isotropic Deformation"}$$

$$W_n = W(\underline{\underline{F}}) - p(J-1).$$

Lagrangian multiplier

$$\underline{\underline{P}} = \frac{\partial W_n}{\partial \underline{\underline{F}}} = \frac{\partial W}{\partial \underline{\underline{F}}} - p(J-1) = \frac{\partial W}{\partial \underline{\underline{F}}} - p(\det \underline{\underline{F}})(\underline{\underline{F}}^{-1})$$

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$J=1 \sim \underline{\underline{\sigma}} = \underline{\underline{P}}\,\underline{\underline{F}}^T = \frac{\partial W}{\partial \underline{\underline{F}}}\underline{\underline{F}}^T - p\underline{\underline{I}}.$

$$d(\det(d\underline{\underline{\beta}})) = \det \underline{\underline{A}}\,(1 + tr(\underline{\underline{A}}^{-1} d\underline{\underline{A}})) - \det \underline{\underline{A}}$$

$$= (\det \underline{\underline{A}})\, tr(\underline{\underline{A}}^{-1} d\underline{\underline{A}})$$

$$= (\det \underline{\underline{A}})(\underline{\underline{A}}^{-T} : d\underline{\underline{A}})$$

$$= \frac{\partial \det \underline{\underline{A}}}{\partial \underline{\underline{A}}} : d\underline{\underline{A}}$$

$$\rightarrow \frac{\partial \det \underline{\underline{A}}}{\partial \underline{\underline{A}}} = (\det \underline{\underline{A}})\,\underline{\underline{A}}^{-T}$$

$$\cdots + \frac{\partial \Phi}{\partial I_3}\frac{\partial I_3}{\partial \underline{\underline{C}}} \rightarrow I_3 \underline{\underline{C}}^{-T} = I_3 \underline{\underline{C}}^{-1}$$

$$I_3 = \det \underline{\underline{C}}$$

$$\underline{\underline{P}} = 2\left[\left(\frac{\partial \Phi}{\partial I_2} + I_1 \frac{\partial \Phi}{\partial I_2}\right)\underline{\underline{F}} - \frac{\partial \Phi}{\partial I_2}\underline{\underline{F}}\,\underline{\underline{C}} + I_3 \frac{\partial \Phi}{\partial I_3}\underline{\underline{F}}^{-T}\right]$$

$$\underline{\underline{\sigma}} = \frac{1}{J}\underline{\underline{P}}\,\underline{\underline{F}}^T \rightarrow \text{true str.}$$

<u>Tension Test</u>

For isotropic.

$$\underline{\underline{\sigma}} = 2\left[\left(\frac{\partial \Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2}\right)b - \frac{\partial \Phi}{\partial I_2} b^2\right] - p\underline{\underline{I}}$$

Oct 7. Wed, Week 6.

Incompressible hyperelasticity.

- Kinematics — quantities deformation.

  $\underline{\underline{C}}$, $\underline{\underline{E}}$, $\underline{\underline{U}}$ strain measures.

- Balance laws. — stresses.

  $\underline{\underline{P}}$, $\underline{\underline{\sigma}}$, ... other stresses measures.

  e.g. 2nd Piola stress.

  Biot stress.

- Constitutive Model.

  Relationship stress — strain.

$$\underline{\underline{P}} = -p\underline{\underline{F}}^{-T} + 2\left[\left(\frac{\partial \Phi}{\partial I_1} + I_1 \frac{\partial \Phi}{\partial I_2}\right)\underline{\underline{F}} - \frac{\partial \Phi}{\partial I_2}\underline{\underline{F}}\,\underline{\underline{C}}\right]$$

Recall I_1, I_2 are invariants of $\underline{\underline{C}}$.

$$I_1 = tr\,\underline{\underline{C}}, \quad I_2 = \frac{1}{2}\left[(tr\,\underline{\underline{C}})^2 - tr(\underline{\underline{C}})^2\right] \quad (1)$$

Lagrange multiplier enforce $\det \underline{\underline{F}} = J = 1$.

E.g. Uniaxial Tension or Compression test.

Ref. config.

$L_0 \gg W_0$ and t_0, Tension test.

$\underline{\underline{P}} \cdot \underline{N}$ on all lateral surface is 0.

$\underline{N} = \underline{e}_2$ or $\underline{e}_3$.

Undeformed lateral surfaces. $\leftarrow$ $\boxed{P_{13} = P_{23} = P_{33} = 0,\ P_{12} = P_{22} = P_{32} = 0}$

$\nabla_{\underline{X}} \cdot \underline{\underline{P}} = \underline{0}$. $\leftrightarrow$ Balance law.

Simplest model: $\Phi = \frac{\mu}{2}(I_1 - 3)$

ideal rubber $\rightarrow$ Neo-Hookean solid.

$= \frac{\mu}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)$

Principal coordinate.

Principal stretches.

Eqn (1): $\rightarrow$

$\underline{\underline{P}} = -p\underline{\underline{F}}^{-T} + \mu \underline{\underline{F}}$. $\leftarrow$ material model. (constitutive).

$u_1 = (\lambda_1 - 1)X_1$.

$u_2 = u_3 = (\lambda_2 - 1)X_2$

$\hookrightarrow (\lambda_2 - 1)X_3$ $\quad \lambda_2 = \lambda_3$

$$\underline{\underline{F}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$\det \underline{\underline{F}} = 1$. $\quad \lambda_1 \lambda_2^2 = 1$. $\rightarrow$ $\lambda_2 = \frac{1}{\sqrt{\lambda_1}}$

incompressibility.

Subs. into constitutive model.

make sure you satisfy boundary conditions.

* Not satisfy balance law $\rightarrow$ nonequilibrium states.

$$[\underline{\underline{F}}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda_1}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\lambda_1}} \end{bmatrix}$$

$$[\underline{\underline{F}}^{-T}] = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_1} & 0 \\ 0 & 0 & \sqrt{\lambda_1} \end{bmatrix}$$

$P_{11} = -p/\lambda_1 + \mu\lambda_1$.

$P_{22} = -p\sqrt{\lambda_1} + \mu / \sqrt{\lambda_1}$.

$= P_{33}$.

$P_{12} = P_{21} = P_{23} = P_{32}$

$= P_{13} = P_{31} = 0$.

$\lambda_1 = \text{const.} \Rightarrow$ equilibrium equation automatically satisfied.

B.C. are automatically satisfied.

* Now, determine P.

$$\underline{\underline{BC}} = P_{22} = P_{33} = 0. \Rightarrow P\sqrt{\lambda_1} = \mu/\sqrt{\lambda_1}$$

$$\Rightarrow P = \frac{\mu}{\lambda_1}.$$

Substitute back to P_{11}.

$$P_{11} = -\frac{\mu}{\lambda_1^2} + \mu\lambda_1.$$

$$\underline{\underline{\sigma}} = \underline{\underline{P}}\,\underline{\underline{F}}^T.$$

$$\sigma_{11} = P_{11}\lambda_1$$

$$\sigma_{11} = -\frac{\mu}{\lambda_1} + \mu\lambda_1^2.$$

$\lambda_1 \approx 1+\varepsilon.$ $\quad \sigma_{11} = -\frac{\mu}{1+\varepsilon} + \mu(1+\varepsilon)^2$ tensile modulus

$$\cong -\mu(1-\varepsilon) + \mu(1+2\varepsilon) = 3\mu\varepsilon$$

$\varepsilon << 1$

$$\mu = E/3.$$

$$\frac{E}{2(1+\nu)} = \mu$$

↳ * $\nu = 1/2.$

Poisson ratio.

<u>Plane stress deformation</u>

(Surface : traction free).

B.C. : $\underline{\underline{P}} \cdot \underline{e}_3 = 0$, on surface

$$\begin{cases} u_1(X_1, X_2). \quad u_2(X_1, X_2). & t_0 << L_0 \\ u_3(X_1, X_2). & t_0 << w_0. \end{cases}$$

(Assumption).

$P_{13} = P_{23} = P_{33} \equiv 0$, in the region (everywhere).

↳ plane stress assumption.

$$\underline{\underline{F}} = \begin{bmatrix} 1+\frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & 0 \\ \frac{\partial u_2}{\partial X_1} & 1+\frac{\partial u_2}{\partial X_2} & 0 \\ 0 & 0 & \cancel{\frac{\partial u_3}{\partial X_3}} \end{bmatrix}$$

→ wrong

$$\begin{matrix} \downarrow & \downarrow & \\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_1} & \lambda_3 \to \lambda_3(X_1, X_2). \\ \| & \| & \boxed{1+\frac{\partial u_3}{\partial X_3}} \\ 0 & 0 & \end{matrix}$$

$$\underline{\underline{P}} = -p\,\underline{\underline{F}}^{-T} + \mu\,\underline{\underline{F}}.$$

$$[\underline{\underline{P}}] = -p \underbrace{\begin{bmatrix} 1+\frac{\partial u_2}{\partial X_3} & -\frac{\partial u_2}{\partial X_1} & 0 \\ -\frac{\partial u_1}{\partial X_2} & 1+\frac{\partial u_1}{\partial X_1} & 0 \\ 0 & 0 & \frac{1}{\lambda_3} \end{bmatrix}}_{[\underline{\underline{F}}^{-T}]}$$

$$+\mu \begin{bmatrix} 1+\frac{\partial u_1}{\partial X_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \qquad \Phi = \frac{\mu}{2}(I_1 - 3).$$

$$P_{33} = -\frac{p}{\lambda_3(X_1, X_2)} + \mu\lambda_3(X_1, X_2) = 0.$$

equilibrium

$$\boxed{\begin{aligned} &\frac{\partial P_{11}}{\partial X_1} + \frac{\partial P_{12}}{\partial X_2} + \cancel{\frac{\partial P_{13}}{\partial X_3}}^{\,0} = 0 \\ &\frac{\partial P_{21}}{\partial X_1} + \frac{\partial P_{22}}{\partial X_2} = 0. \end{aligned}}$$

$$P_{11} = \mu\lambda_3^3 \cdot \left(1 + \frac{\partial u_2}{\partial X_2}\right) + \mu\left(1 + \frac{\partial u_1}{\partial X_1}\right).$$

Oct, 13, 2021. Wed.

<u>REVIEW</u>: plane stress : incompressible neoHookean Solid

$\underline{\underline{F}}$ for plane stress.

$$[\underline{\underline{F}}] = \begin{bmatrix} \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} & 0 \\ \begin{pmatrix} 0 & 0 \end{pmatrix} & \lambda_3 \end{bmatrix} \qquad [\underline{\underline{F}}_{in}] \qquad , \alpha = \frac{\partial}{\partial X_\alpha}$$

$$x_\alpha = X_\alpha + u_\alpha(X_1, X_2) \qquad \alpha = 1.2.$$

Independent of X_3.

λ_3 is the out-of-plane stretch ratio, $\lambda_3(X_1, X_2)$.

$$\underline{\underline{F}}_{in} = x_{\alpha,\beta}\, \underline{e}_\alpha \underline{e}_\beta$$

<u>neo-Hookean</u>

$$\underline{\underline{P}} = -p\,\underline{\underline{F}}^{-T} + \mu\,\underline{\underline{F}}$$

<u>for ~~any~~ incompressibility</u>

$$J = \det \underline{\underline{F}} = 1 = (\det \underline{\underline{F}}_{in})\lambda_3 = 1 \qquad (x_{1,1}x_{2,2} - x_{1,2}x_{2,1})$$

$$\Rightarrow \det \underline{\underline{F}}_{in} = \frac{1}{\lambda_3}$$

$$[\underline{\underline{F}}^{-T}] = \begin{bmatrix} x_{2,2}\lambda_3 & -x_{2,1}\lambda_3 & 0 \\ -x_{1,2}\lambda_3 & x_{1,1}\lambda_3 & 0 \\ 0 & 0 & 1/\lambda_3 \end{bmatrix}$$

(P11) <u>Next step:</u>

$$\begin{cases} P_{11} = -p\, x_{2,2}\lambda_3 + \mu x_{1,1} \\ P_{22} = +p\lambda_3 x_{2,1} + \mu x_{1,2} \\ P_{21} = p\lambda_3 x_{1,2} + \mu x_{2,1} \\ P_{22} = -p\lambda_2 x_{1,1} + \mu x_{2,2} \end{cases}$$

$P_{13} = P_{23} = P_{31} = P_{32} = 0$, consistent with the plane stress assump.

$\underline{\underline{P_{33} = 0}}$. $= -p\frac{1}{\lambda_3} + \mu\lambda_3 = 0$.

$\hookrightarrow p = \mu\lambda_3^2$.

Substitute

use LMB: (ignore body forces.) & acceleration.

$P_{11,1} + P_{12,2} = 0$

$\Leftrightarrow (p\lambda_3^2 x_{2,2})_{,1} + \mu x_{1,11} + (\mu\lambda_3^3 x_{1,2})_{,12}$

$+ \mu x_{1,22} = 0$

$0 = P_{21,1} + P_{22,2} = (\mu\lambda_3^3 + x_{1,2})_{,21} + \mu x_{2,11}$

$- (\mu\lambda_3^3 x_{1,1})_{,12} + \mu x_{2,22} = 0.$

$$\begin{cases} \mu\nabla_X^2 x_1 + \mu\left((\lambda_3^3 x_{1,2})_{,2} - (\lambda_3^3 x_{2,2})_{,1}\right) = 0 \\ \Updownarrow \\ \mu\nabla_X^2 x_2 + \mu[\quad\quad] = 0 \end{cases}$$

$$\lambda_3 = \frac{1}{x_{1,1}x_{2,2} - x_{2,1}x_{1,2}}$$

coupled PDEs for unknowns x_1, x_2.

BCs. $X_2 = \pm L_0$,

$u_1 = 0$. $u_2 = \frac{\pm\Delta}{2}$

on lateral sides:

$X_1 = -C_0$ $P_{21} = P_{11} = 0$

$X_1 = W_0 - C_0$ $P_{21} = P_{11} = 0$

on crack faces. $-C_0 < X_1 < 0$, $X_2 = 0^{\pm}$

$P_{12} = P_{22} = 0$

Plane strain

Assumption:

$$\begin{cases} u_1 = u_1(X_1, X_2) \\ u_2 = u_2(X_1, X_2) \\ u_3 = 0 \end{cases} \xrightarrow{\text{imply}} \underline{\underline{F}} = \begin{bmatrix} x_{1,1} & x_{1,2} & 0 \\ x_{2,1} & x_{2,2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\det \underline{\underline{F}} = 1$. $\rightarrow$ to determine $\underline{\underline{P}}$

P_{33} is not zero.

σ_{33}

Linear Elasticity

kinematics.

$$\underline{\underline{E}} = \frac{u_{i,j} + u_{j,i}}{2}. \qquad ,i = \frac{\partial}{\partial X_i}.$$

(one simple strain measure, only).

small strain tensor.

$$\frac{\partial \sigma_{ij}}{\partial X_j} = -\rho_0 B_i \qquad \text{Equilibrium.}$$

All you need, is constitutive model.

large deformation $\xrightarrow{\text{linearize}}$ constitutive model.

$$\frac{\partial W(\underline{\underline{E}})}{\partial \underline{\underline{E}}} = \underline{\underline{0}}.$$

small for all in linear stage ($\hat{W} = \bar{W} = W$).

$$\boxed{\sigma_{ij} = \underbrace{k_{ijkl}}_{\text{independent of strain}} \varepsilon_{kl}} \qquad (\text{Expect})$$

3^4

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}$$

$\rightarrow$ quadratic function of strain tensor

(try) $W = \frac{1}{2} \cdot K_{ijkl}\, \varepsilon_{ij}\, \varepsilon_{kl}$

$$\sigma_{rs} \leftarrow \frac{\partial W}{\partial \varepsilon_{rs}} = \frac{1}{2} \cdot \underbrace{K_{ijkl}\, \delta_{ir} \delta_{js}\, \varepsilon_{kl}}_{K_{rskl}} + \underbrace{K_{ijkl}\, \varepsilon_{ij}}_{K_{ijrs}} \delta_{kr} \delta_{ls} \Big].$$

$$\sigma_{rs} = \frac{1}{2}\left[K_{rskl}\, \varepsilon_{kl} + K_{ijrs}\, \varepsilon_{ij} \right]$$

$$= \frac{1}{2}\left[K_{rskl}\, \varepsilon_{kl} + K_{klrs}\, \varepsilon_{kl} \right]$$

$$\sigma_{ij} = \frac{1}{2}\left[K_{ijkl}\, \varepsilon_{kl} + K_{klij}\, \varepsilon_{kl} \right].$$

$$\sigma_{ij} = \frac{1}{2}\underbrace{\left[K_{ijkl} + K_{klij} \right]}_{\hat{\underline{\underline{K}}}} \varepsilon_{kl}$$

$\hat{K}_{ijkl} = \hat{K}_{klij}$ symmetric in kl, ij

$\hookrightarrow$ 81 component.

Symmetry of $\sigma_{ij} \Rightarrow K_{ijkl} = K_{jikl}$

Symmetry of $\varepsilon_{kl} \Rightarrow K_{ijkl} = K_{ijlk}$.

$9 \times 9 \longrightarrow 6 \times 6$

$\downarrow$

36 independent components.

The existence of W.

$\downarrow$

$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}$

$\downarrow$

implies $K_{ijkl} = K_{klij}$.

$\Downarrow$

~~18~~ 21 ind. comp.

$\downarrow$

most common model for elastic.

Oct. 15, office hours.

C_1 C_2, $\propto$ shear modulus
Poisson's ratio.

letting $\lambda \to 1$

at very small $\to$ agrees with Hooke's law.

Plot the curve.

normalize the stress for shear modulus

other terms $\propto$ ratio of C_1, C_2

$\downarrow$

function only of the Poisson's ratio

reasonable choice $\nu = 0.45$

0.5 (~~int~~ incompressible).

a lot of curve with different Poisson's ratio.

* normalized shear modulus G.

$\downarrow$

Piola & Cauchy normalized by

you can normal the stress by G.

λ_1, λ_3

$$\cancel{\lambda_3^2 - 1 + \frac{C_2}{C_1} \ln(\lambda_1^2 \lambda_3) = 0}$$

$$\lambda^2 - 1 + \frac{C_2}{C_1} \ln(\lambda^3 \lambda^2) = 0$$

$f(\nu)$ incompressible $\frac{C_2}{C_1} \to$ huge

$\downarrow$ G.

$\lambda_3 \lambda^2 = 1$. $\leftarrow$ Lambert function

1st order expansion John Hutchinson

$-\lambda$

$0.45\lambda_3 + 0.45 = \lambda_1 - 1$

$1.45 - 0.45\lambda_3 = \lambda_1$

$1.45 - 0.45 = \lambda_1$

$\lambda_1 = 0.5$

Oct. 18th, 2021, Mon.

Review.

Linear elasticity.

$$W = \frac{1}{2} k_{ijkl} \cdot \varepsilon_{ij} \varepsilon_{kl}.$$

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}.$$

k_{ijkl} has 21 independent constants.

$$k_{ijkl} = k_{jikl} = k_{ijlk} = k_{klij}.$$

Anisotropic

$$\sigma_{ij} = k_{ijkl} \varepsilon_{kl}.$$

$$W = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} = \frac{1}{2} \underline{\underline{\sigma}} \cdot \underline{\underline{\varepsilon}}$$

Isotropy solids.

$$k_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

λ, μ are constants.

General form of isotropic 4th order Tensor

"Introduction to Cartesian Tensors"

Jim ~~Knowlis~~ Knowles.

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + \mu [\varepsilon_{ij} + \varepsilon_{ji}].$$

$$= 2\mu \varepsilon_{ij} + \lambda \varepsilon_{kk} \delta_{ij}.$$

Generalized Hooke's Law

μ, λ are called Lame constants.

$$\sigma_{kk} = 2\mu \varepsilon_{kk} + 3\lambda \varepsilon_{kk}.$$

$$\sigma_{kk} = (2\mu + 3\lambda) \varepsilon_{kk}.$$

$$2\mu \varepsilon_{ij} + \lambda \frac{\sigma_{kk}}{(2\mu + 3\lambda)} \delta_{ij} = \sigma_{ij}.$$

$$\varepsilon_{ij} = \frac{\sigma_{ij}}{2\mu} - \frac{\lambda \sigma_{kk}}{(2\mu + 3\lambda)} \cdot \frac{1}{2\mu} \cdot \delta_{ij}$$

$$\varepsilon_{ij} = \frac{(1+\mu)}{E} \sigma_{ij} - \frac{\nu \sigma_{kk}}{E} \delta_{ij}.$$

μ — ν — Poisson's ratio.

E — Young's Modulus

$$\frac{1}{2\mu} = \frac{(1+\nu)}{E} \Rightarrow \mu = \frac{E}{2(1+\nu)}$$

Shear Modulus. $\frac{\nu}{E} = \frac{\lambda}{(2\mu + 3\lambda) 2\mu}$.

Tension test

$\sigma_{11} = \sigma$. $\quad \sigma_{ij} = 0$. $(i,j \neq 1)$.

$$\varepsilon_{11} = \frac{\sigma_{11}}{E}.$$

↳ tension modulus.

$$\varepsilon_{22} = \varepsilon_{33} = -\frac{\mu}{E}\sigma_{11}.$$

$$-\frac{\varepsilon_{22}}{\varepsilon_{11}} = \mu,$$

Poisson's ratio ≥ 0.

There are negative Poisson's ratio material but anisotropic.

Apply a pure hydrostatic tension,

$$\varepsilon_{ij} \leftarrow \varepsilon_{kk} = -\frac{(1+\nu)}{E} p\,\delta_{ij} + \frac{3p\nu}{E}\delta_{ij}.$$

if $\sigma_{ij} = -p\delta_{ij}$.

$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = -\underbrace{\frac{(1+2\nu)}{E}}_{1/K} p.$$

Bulk Modulus $= -1/K\, p \rightarrow -p/K$.

$$K = \cancel{\frac{E}{1-2\nu}}\ \frac{E}{1-2\nu}.$$

$\nu \rightarrow \frac{1}{2}$, $K \rightarrow \infty$.

$\Downarrow$

$\mathrm{tr}\,\underline{\underline{\varepsilon}} = 0$. incompressible solid

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}.$$

$$\sigma_{ij} = 2\mu\cancel{\varepsilon_{ij}} \rightarrow \sigma_{ij} = 2\mu\varepsilon_{ij} - p\delta_{ij}.$$

★★★. office hour.

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

General form relation for linear elasticity

① $\varepsilon_{ij} = \dfrac{u_{i,j} + u_{j,i}}{2}$

② $\sigma_{ij,j} = -\rho_0 B_i$.

② $\sigma_{ij} = \sigma_{ji}$

③ $\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$

- Substitute ① into ③ to express strains in terms of displacements.
- Substitute stress into ② to obtain.

$$G\nabla^2 \underline{u} + (\lambda + G)\nabla(\nabla\cdot\underline{u}) = -\rho_0 \underline{B}$$

↳ Navier's equation (3PDEs)

Subject Navier's Eq. to BCs

Typical e.g.

mixed BCs.

Displacement prescribed.

$$\begin{cases} \sigma_{ij} n_i = T_n(\underline{X}) & \underline{X} \in S_T \\ S_u: \ u_i = f(\underline{X}), & \underline{X} \in S_U \end{cases}$$

$$\left[2\mu \left[\frac{u_{i,j} + u_{j,i}}{2} \right] + \lambda u_{k,k} \delta_{ij} \right] n_j = T_i(\underline{X}).$$

fixed boundary

(thin sheet).

In linear Elasticity.

Build up complex solutions from simple ones.

$$\sigma_{ij,j} = -\rho_0 B_i$$

Suppose we guess a solution for ⊛ that also satisfies the Boundary Conditions (BC). (traction BC).

$\underline{B} = g\, \underline{e}_3$

If we guess: σ^*_{ij}.

compute displacement.

$$\varepsilon^*_{ij} = \frac{(1+\nu)}{E} \sigma^*_{ij} - \frac{\nu}{E} \sigma^*_{kk} \delta_{ij}$$

$\downarrow$

Integrate strain $\underline{\varepsilon}^*$ to get displacement field

There are three unknown disp.

$u_1, u_2, u_3 \rightarrow$ (position).

I guess $\sigma_{ij}^* \rightarrow \varepsilon_{ij}^*$

$$\varepsilon_{ij}^* = \frac{u_{i,j} + u_{j,i}}{2}$$

6 equations here.

6 equations, & 3 unknowns.

the solutions may not exists, if e.

not unique.

Plane strain.

linear elas.

$u_3 = 0$, $\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = u_{1,1}$

$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = u_{2,2}$.

$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{u_{1,2} + u_{2,1}}{2}$

ε_{ij} (rest) $= 0$.

$u_3 = 0$, u_1, u_2 depends on x_1, x_2 only.

You can show that

$-2\varepsilon_{12,12} + \varepsilon_{11,22} + \varepsilon_{22,11} = 0$

$$\frac{\partial^2 (\ )}{\partial x_1 \partial x_2} = (\ )_{,12}$$

↳ compatibility equation for plane strain.

: puts a constraint on the strain.

$$\varepsilon_{ij} = \frac{\sigma_{ij}(1+\nu)}{E} - \frac{\nu \sigma_{kk} \delta_{ij}}{E}.$$

↳ you will find this:

$$\nabla^2(\sigma_{11} + \sigma_{22}) = \frac{1}{(1-\nu)} \nabla \cdot (\rho_0 \underline{b}).$$

compatibility equation for stress.

Wed., Oct. 20, 2021. Week 9 (?).

Linear Elasticity.

6 Eqns. $\varepsilon_{ij} = \dfrac{u_{i,j} + u_{j,i}}{2}$ — kinematics.

3 Eqns. $\sigma_{ij,j} = -\rho_0 B_i$ — Balance laws.

6 Eqns. $\varepsilon_{ij} = \dfrac{(1+\nu)\sigma_{ij}}{E} - \dfrac{\nu \sigma_{kk} \delta_{ij}}{E}$

Constitutive model.

15 Eqns.

unknowns: ε_{ij}, ~~σ_{ij}~~. u_i, σ_{ij}. 15 unknowns.

Navier Eqns. (Displacement formulation).

$$G \nabla^2 \underline{u} + (\lambda + G) \nabla (\nabla \cdot \underline{u}) = -\rho_0 \underline{B}.$$

3 Eqns, & 3 unknowns. u_1, u_2, u_3.

independent variables. X_i.
positions.

dependent is u_i.

Most useful when body is subject to BCs.

Antiplane shear deformation.

$u_\alpha = 0$. $\alpha = 1, 2$. No in-plane disp

$u_3 = u(X_1, X_2)$. independent of X_3.

$\Downarrow$

$\varepsilon_{\alpha\beta} = 0$, $\alpha = 1, 2$.

$\varepsilon_{33} = \dfrac{\partial u_3}{\partial X_3} = 0$.

Only non-vanishing strain are.

engineering strain $\left\langle \begin{array}{l} \varepsilon_{13} = \varepsilon_{31} = \frac{1}{2} \frac{\partial u}{\partial X_1} = \frac{1}{2} \gamma_1 \\ \varepsilon_{23} = \varepsilon_{32} = \frac{1}{2} \frac{\partial u}{\partial X_2} = \frac{1}{2} \gamma_2 \end{array} \right.$

Constitutive model.

$\sigma_{\alpha\beta} = 0$ in-plane stress

$\sigma_{33} = 0$

$$\left\{ \begin{array}{l} \sigma_{13} = \sigma_{31} = G\gamma_1 \\ \sigma_{23} = \sigma_{32} = G\gamma_2 \end{array} \right.$$

Equilibrium Eqns are identically satisfied in 1 & 2 directions ($B_1 = B_2 = 0$)

$$\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} = 0$$

$$\Updownarrow$$

no body force.

$$\frac{\partial \tau_1}{\partial x_1} + \frac{\partial \tau_2}{\partial x_2} = 0 \iff \nabla_x \cdot \vec{\tau} = 0. \quad (1)$$

$$\underline{\tau} = \tau_1 \underline{e}_1 + \tau_2 \underline{e}_2.$$

γ_1 & γ_2 must satisfy the fact that.

$$\gamma_1 = \frac{\partial w}{\partial x_1}, \quad \gamma_2 = \frac{\partial w}{\partial x_2} \Rightarrow \frac{\partial \gamma_1}{\partial x_2} = \frac{\partial \gamma_2}{\partial x_1}.$$

Stress compatibility

the eqn. int.

$$\Rightarrow \frac{\partial \tau_1}{\partial x_2} = \frac{\partial \tau_2}{\partial x_1}. \quad (2)$$

Introduce a stress function. ϕ.

$$\tau_1 = \frac{\partial \phi}{\partial x_2} \quad \tau_2 = -\frac{\partial \phi}{\partial x_1}. \quad (3)$$

Subs. (2) into (1), we see that

(2) ↓ (3)

(1) is satisfied automatically.

$$\frac{\partial \tau_1}{\partial x_1} = \frac{\partial^2 \phi}{\partial x_1 \partial x_2} \qquad \frac{\partial \tau_2}{\partial x_2} = \frac{\partial^2 \phi}{\partial x_2 \partial x_1}$$

Substitute (3) into (2).

$$\frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_1^2} = 0 \quad \text{or} \quad \nabla_x^2 \phi = 0.$$

Laplace Eqn in 2D.

Stress function approach

Disp. Formulation.

$$\tau_1 = G \frac{\partial w}{\partial x_1}.$$

$$\tau_2 = G \frac{\partial w}{\partial x_2}. \quad \Big\} \nabla_x^2 w = 0.$$

Substitute into (1).

Simpliest form of Navier's Equation.

$$\phi + i w = f(z).$$

$$\phi + i G w$$

☆ Anti-plane shear.

Reminder: <u>Plane strain</u>

$u_\alpha(X_1, X_2)$, $\alpha = 1, 2$.

$u_3 \equiv 0$.

$\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}$ (all others strain components $= 0$).

<u>Compatibility</u>

$$\boxed{\frac{\partial^2 \varepsilon_{11}}{\partial X_2^2} - \nu \frac{\partial^2 \varepsilon_{12}}{\partial X_1 \partial X_2} + \frac{\partial^2 \varepsilon_{22}}{\partial X_1^2} = 0}$$

Note

$\frac{\partial \sigma_{3i}}{\partial X_i} = 0$, is automatically satisfied.

$\sigma_{31} = \sigma_{32} = 0$; σ_{33} is independent of X_3.

$\varepsilon_{33} = 0 \Rightarrow \frac{\sigma_{33}}{E} - \frac{\nu(\sigma_{11} + \sigma_{22})}{E} = 0$

$$\boxed{\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})}.$$

Equilibrium equation is identically satisfied in 3 direction.

Therefore: $\frac{\partial \sigma_{11}}{\partial X_1} + \frac{\partial \sigma_{12}}{\partial X_2} = \rho_0 B_1$

$$\frac{\partial \sigma_{21}}{\partial X_1} + \frac{\partial \sigma_{22}}{\partial X_2} = \rho_0 B_2$$

Equilibrium Eqns (LMB). $\rightarrow$ (4a, b)

Assuming $\underline{B} = \underline{0}$.

Airy stress function, ϕ.

$\sigma_{11} = \frac{\partial^2 \phi}{\partial X_2^2}$ $\quad \sigma_{12} = -\frac{\partial^2 \phi}{\partial X_1 \partial X_2}$.

$\sigma_{22} = \frac{\partial^2 \Phi}{\partial X_1^2}$. (5)

Substitute (5) into (4a, b).

$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \frac{\nu(\sigma_{22} + \sigma_{33})}{E}$.

$\varepsilon_{22} = \frac{\sigma_{22}}{E} - \frac{\nu(\sigma_{11} + \sigma_{33})}{E}$

$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$.

$\varepsilon_{12} = \cancel{\frac{\sigma_{33}}{E} - \frac{\nu(\sigma_{11} + \sigma_{22})}{E}}$. $\frac{\sigma_{12}}{2G} \rightarrow \tau$. $\Downarrow$

Simplified plane strain constitutive model

$\varepsilon_{11} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_{11} - \nu\sigma_{22}\right]$.

$\gamma = \frac{\tau}{G}$ $\quad \varepsilon_{22} = \frac{(1+\nu)}{E}\left[(1-\nu)\sigma_{22} - \nu\sigma_{11}\right]$.

Non-zero Inplane stress fields are

σ_{11}, σ_{12}, σ_{22}.

Non-zero Out-of-plane stress.

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22}).$$

S_T : Traction Boundary Conditions.

no shear stress.

free boundary.

▶ Solution independent of material properties.

Stresses $\propto$ stress function.

$$\begin{cases} \sigma_{11} = \dfrac{\partial^2 \phi}{\partial x_1^2} \\ \sigma_{22} = \dfrac{\partial^2 \phi}{\partial x_2^2} \\ \sigma_{12} = -\dfrac{\partial^2 \phi}{\partial x_1 \partial x_2}. \end{cases}$$

<u>plane stress</u> (finite deformation).

$t \ll L$. and other in-plane dimensions.

$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$. → three non zero stresses

σ_{11} σ_{22}, σ_{12}.

$\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{32} = \varepsilon_{31} \cong 0.$

$\varepsilon_{\alpha\beta}$ is approx. independent of x_3.

$$\frac{\delta V}{V} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 0$$

$\downarrow$

incompressible.

Constitutive model

Plane stress $\quad \sigma_{13} = \sigma_{23} = \sigma_{33} = 0.$

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \frac{\nu(\sigma_{22} + \overset{0}{\overset{\shortparallel}{\sigma_{33}}})}{E} = \frac{\sigma_{11}}{E} - \frac{\nu\sigma_{22}}{E}$$

$$\varepsilon_{12} = \frac{\sigma_{12}}{2G}.$$

$$\varepsilon_{22} = \frac{\sigma_{22}}{E} - \frac{\nu(\sigma_{11} + \overset{0}{\overset{\shortparallel}{\sigma_{33}}})}{E} = \frac{\sigma_{22}}{E} - \frac{\nu\sigma_{11}}{E}$$

Plane strain

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \frac{\nu(\sigma_{22} + \overset{\cancel{0} \to \nu(\sigma_{11}+\sigma_{22})}{\overset{\shortparallel}{\sigma_{33}}})}{\cancel{\nu} E}.$$

$$= \frac{\sigma_{11}}{E} - \frac{\nu(\sigma_{22} + \nu(\sigma_{11} + \sigma_{22}))}{E}.$$

$$= \frac{(1-\nu^2)\sigma_{11}}{E} - \frac{\nu(1+\nu)\sigma_{22}}{E}$$

Compatibility & Equilibrium.

$$\nabla^4 \phi = 0. \quad \text{(No Body force).}$$

$\nu \longrightarrow \dfrac{\nu}{1-\nu}$

plane stress. $\qquad$ plane strain.

$G \longleftrightarrow G.$

$$\underline{\underline{\varepsilon}} = \frac{\nabla \underline{u} + (\nabla \underline{u})^T}{2}.$$

$$\varepsilon_{ij} = \frac{u_{i,j} + u_{j,i}}{2}$$

$$\underline{u} = u_r \underline{e}_r + u_\theta \underline{e}_\theta.$$

$$\begin{cases} \underline{e}_r = \cos\theta\, \underline{e}_1 + \sin\theta\, \underline{e}_2 \\ \underline{e}_\theta = -\sin\theta\, \underline{e}_1 + \cos\theta\, \underline{e}_2. \end{cases}$$

In cartesian coordinates

$$\nabla \underline{u} = \frac{\partial u_i}{\partial x_j} \underline{e}_i \underline{e}_j$$

$$\overleftarrow{\nabla} \underline{u} = \frac{\partial u_i}{\partial x_j} \underline{e}_i \underline{e}_j$$

$$\overrightarrow{\nabla} \underline{u} = \frac{\partial u_i}{\partial x_j} \underline{e}_j \underline{e}_i = (\overleftarrow{\nabla} \underline{u})^T$$

$$\nabla = \cancel{\underline{e}_\theta \frac{\partial}{\partial r} + \underline{e}_r \frac{\partial}{\partial \theta}}.$$

$$\underline{e}_r \frac{\partial}{\partial r} + \frac{\underline{e}_\theta}{r} \frac{\partial}{\partial \theta}.$$

$$\nabla \cdot \underline{u} = \left(\underline{e}_r \frac{\partial}{\partial r} + \frac{\partial \underline{e}_\theta}{r} \frac{\partial}{\partial \theta} \right) (u_r \underline{e}_r + u_\theta \underline{e}_\theta)$$

$$= \underline{e}_r \cdot \frac{\partial (u_r \underline{e}_r)}{\partial r} + \underline{e}_r \cdot \frac{\partial (u_\theta \underline{e}_\theta)}{\partial r}$$

$$= \underline{e}_r \cdot \frac{\partial u_r}{\partial r} \cdot \underline{e}_r + \underline{e}_r \left(\frac{\partial u_\theta}{\partial r} \underline{e}_\theta \right)$$

$$+ \frac{\underline{e}_\theta}{r} \left[\frac{\partial u_r}{\partial \theta} \underline{e}_r + u_r \underbrace{\frac{\partial \underline{e}_r}{\partial \theta}}_{\underline{e}_\theta} \right]$$

$$+ \frac{\underline{e}_\theta}{r} \frac{\partial u_\theta}{\partial \theta} \underline{e}_\theta + \frac{\underline{e}_\theta}{r} u_\theta (\underline{e}_r)$$

$$\frac{\partial \underline{e}_\theta}{\partial \theta} = -\cos\theta \, \underline{e}_1 - \sin\theta \, \underline{e}_2 = -\underline{e}_r$$

$$\left(\sim \quad -\frac{u_\theta}{r} \right) \underline{e}_\theta \underline{e}_\theta$$

$$\nabla \underline{u} = \frac{\partial u_r}{\partial r} \underline{e}_r \underline{e}_r + \frac{\partial u_\theta}{\partial r} \underline{e}_\theta \underline{e}_\theta + \frac{1}{r} \cdot \frac{\partial u_\theta}{\partial \theta} \cdot \underline{e}_\theta \underline{e}_r$$

$$+ \left[\frac{u_r}{r} \cdot \underbrace{\underline{e}_\theta \underline{e}_\theta}_{\varepsilon_{rr}} + \frac{1}{r} \cdot \frac{\partial u_\theta}{\partial \theta} \right] \underline{e}_\theta \underline{e}_\theta$$

$$\underline{\underline{\varepsilon}} = \frac{\partial u_r}{\partial r} \underline{e}_r \underline{e}_r + \overbrace{\left[\frac{u_\theta}{r} + \frac{1}{r} \cdot \frac{\partial u_\theta}{\partial \theta} \right]}^{\varepsilon_{\theta\theta}} \underline{e}_\theta \underline{e}_\theta$$

$$+ \frac{1}{2} \underbrace{\left[\frac{\partial u_\theta}{\partial r} - \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} - \frac{u_\theta}{r} \right]}_{\varepsilon_{\theta r}} \underline{e}_\theta \underline{e}_r + \frac{1}{2} [\quad] \underline{e}_r \underline{e}_\theta$$

$\sigma_{\alpha\beta}$ (Cartesian) $\{\underline{e}_1, \underline{e}_2\}$.

$\downarrow$

$\sigma_{rr}, \sigma_{r\theta}, \sigma_{\theta\theta}$ polar coordinates.

$\{\underline{e}_r, \underline{e}_\theta\}$.

Oct. 27, 2021. Wed

☆☆☆ $\left\{ \begin{array}{l} \text{Anti-plane shear} \\ \text{plane strain} \\ \text{plane stress} \end{array} \right\}$ stress functions.

$\Downarrow$

$\nabla^2 \phi = 0$ harmonic

$\nabla^4 \phi = 0$ biharmonic

$\underline{u} = u_r \underline{e}_r + u_\theta \underline{e}_\theta$

$\underline{e}_r = \cos\theta \, \underline{e}_1 + \sin\theta \, \underline{e}_2$

$\underline{e}_\theta = -\sin\theta \, \underline{e}_1 + \cos\theta \, \underline{e}_2$.

$\underline{\underline{\sigma}} = \sigma_{ij} \underline{e}_i \underline{e}_j = \sigma_{rr} \underline{e}_r \underline{e}_r + \cdots$

$\nabla \cdot \underline{\underline{\sigma}} \rightarrow$ Easy in Cartesian Coordinate

<u>Plane strain or plane stress</u>

$$\begin{cases} \dfrac{\partial \sigma_{rr}}{\partial r} + \dfrac{1}{r} \cdot \dfrac{\partial \sigma_{r\theta}}{\partial \theta} + \dfrac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0. \\[2ex] \dfrac{1}{r} \cdot \dfrac{\partial \sigma_{\theta\theta}}{\partial \theta} + \dfrac{\partial \sigma_{r\theta}}{\partial r} + \dfrac{2\sigma_{r\theta}}{\partial r} = 0 \end{cases}$$

DERIVE THIS ONE !!!!

Third eqn. automatically satisfied.

$$\sigma_{rr} = \frac{\phi_{,r}}{r} + \frac{\phi_{,\theta\theta}}{r}.$$

$$\sigma_{\theta\theta} = \phi_{,rr} \qquad \sigma_{r\theta} = -(\phi_{,\theta}/r)_{,r}$$

$$\nabla^4 \phi = 0 \qquad = \overbrace{\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial r^2}\right)}^{\nabla^2}$$

$$\underbrace{\left(\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial \theta^2}\right)}_{\nabla^2}\phi = 0$$

<u>Technique of solution</u>

① Fourier transform.

[Strip or half space problem]

② superposition. (simple idea X technique)

↳ traction free.

sum of two solutions.

⟹ ~~XXX~~ work for malicem elasticity

③ separation of variables.

↳ works for simple geometry.

<u>Complex variable method</u>

function theory.

(Antiplane shear).

$\nabla^2 \phi = 0$ and $\nabla^2 w = 0$.

$$\phi_{,x} = \sigma_{23}, \quad \phi_{,y} = \sigma_{32} \quad ①$$

$x = x_1$, $y = x_2$. (stress function).

$$\sigma_{13} = G\frac{\partial w}{\partial x} = G w_{,x}$$

$$\sigma_{23} = G\frac{\partial w}{\partial y} = G w_{,y} \quad ②$$

①, ②.

$$\Rightarrow \phi_{,x} = G w_{,y}$$

③.

$$\phi_{,y} = -G w_{,x}$$

Define a complex function.

$$f(z) \equiv \phi + iGw$$

ϕ: real part of f. $\quad iGw$: imaginary part of f

③ is a rotation between real part of f and its imaginary parts.

③ is called the Cauchy - Riemann Eqns.

$h(z) = u + iv$. CR.

$$\begin{bmatrix} u_{,x} = v_{,y} \\ u_{,y} = -v_{,x} \end{bmatrix}$$

Any fct with Real & Imaginary parts that satisfies the CR Eqns is called an analytic fct in $\underline{\underline{D}}$

↳ in a Domain D.

$$\begin{cases} \phi_{,xx} = G w_{,xy} \\ \phi_{,xy} = -G w_{,xy} \\ \nabla^2 \phi = 0 \end{cases}$$

$\cos x$ ← Replace x by z. $= \cos z$.

$$\cos x = \frac{e^x + e^{-x}}{2}$$

$$\cos z = \frac{e^z + e^{-z}}{2}$$

$$e^z = e^{x+iy} = e^x e^{iy}$$

$$= e^x [\cos y + i \sin y]$$

$$\frac{(e^x \cos y + i e^x \sin y) + e^{-x}(\cos y - i \sin y)}{2} = e^x \cos y + i e^x \sin y.$$

$$= \frac{(e^x + e^{-x}) \cos y + i (e^x - e^{-x}) \sin y}{2}$$

$$= \underbrace{\cosh x \cos y}_{u} + i \underbrace{\sinh x \sin y}_{v} = \cos z.$$

$$\begin{cases} u_{,x} = v_{,y} \\ u_{,y} = -v_{,x} \end{cases} \checkmark \rightarrow CR$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \rightarrow a_n = \frac{1}{n!} f^n(z_0)$$

analytic solution.

$$= \frac{\partial^n f}{\partial z^n}\Big|_{z_0}$$

$$f'(z) = \sum_{i=0}^{\infty} a_n n (z - z_0)^{n+1}$$

$$f'(z) = \phi_{,x} + i G_{\alpha} w_{,x}$$

$$= \frac{\phi_{,y}}{i} + \frac{i G w_{,y}}{i}$$

$$= -i\phi_{,y} + G w_{,y}$$

$$\boxed{\phi = \mathrm{Re}\left[\bar{z}\varphi(z) + \chi(z)\right]}$$

Analytic

$\bar{z} \equiv x - iy$.

⇐ take this methods.

displacements :

$$2G(u_1 + iu_2) = \kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}$$

$$\psi(z) = \chi'(z) = \frac{d\chi}{dz}.$$

$\kappa = 3 - 4\nu$. plane strain.

$= \frac{3-\nu}{1+\nu}$ stress.

$$\sigma_{11} + \sigma_{22} = 2\left[\varphi'(z) + \overline{\varphi'(z)}\right].$$

$$\sigma_{22} + i\sigma_{12} = \varphi'(z) + \overline{\varphi'(z)} + \bar{z}\,\varphi''(z) + \psi'(z).$$

$$\frac{dz^n}{dz} = n z^{n-1}$$

☆ Any rule with differentiations can be applied to complex variable theory.

(doesn't depend on the hole size).

$\frac{\sigma_{11}^*}{\sigma_\infty}$ = const. independent of σ_∞

(infinite). → $w \gg a$.

$\sigma_{\alpha\beta} = \sigma_\infty f(\frac{x}{a}, 0)$

$= \sigma_\infty f(\frac{x}{a}, \frac{y}{a})$

$\frac{\sigma_{11}^*}{\sigma_\infty} = f(0, 1)$

$\shortparallel$

3

BC : $\sigma_{11} = \sigma_\infty$, as $|x| \to \infty$

I & II. $\sigma_{21} = 0$, as $|x| \to \infty$

III & IV : $\sigma_{22} = 0$. as $|y| \to \infty$

$\sigma_{12} = 0$, as $|y| \to \infty$.

BC on hole : $\underline{\sigma} \cdot \underline{n} = 0$ on hole.

$\underline{n} = \underline{e}_r = \cos\theta\, \underline{e}_1 + \sin\theta\, \underline{e}_2$.

$\sigma_{\alpha\beta} \underline{n}_\beta = 0 \rightarrow r = a = \sqrt{x^2 + y^2}$.

$\sigma_{11}\cos\theta + \sigma_{12}\sin\theta = 0.$

$\sigma_{21}\cos\theta + \sigma_{22}\sin\theta = 0$

$|\theta| \le \pi$.

∵ linear problem

1: σ_{ij} independent of G, ν.
↳ $\sigma_{\alpha\beta}$

→ $\sigma_{\alpha\beta}$, proportional to σ_∞

Mon. ~~Oct.~~ Nov. 1st, Week 11

Theory of Torsion

Semi-inverse Method

① Guess a form for the disp. field.

$\Downarrow$

strain

$\Downarrow$

stress $\rightarrow$ satisfy equilibrium.

check the BCs are satisfied.

* Cylinder with a uniform cross-section.

$\underbrace{x_1(s),\ x_2(s)}_{\text{parameterize by}}$

$\underline{u}$ is the displacement field.

$u_1 = -\alpha x_2 x_3$ $\qquad$ $u_2 = \alpha x_1 x_3$ $\qquad$ $u_3 = w(x_1, x_2)$

α: a constant. $\qquad$ warping function.

To motivate this, look at a special case

Assume $w = 0$. Bar is circular.

$\underline{u} = u_1 \underline{e}_1 + u_2 \underline{e}_2 = u_r \underline{e}_r + u_\theta \underline{e}_\theta$

$\underline{e}_r = \underline{e}_1 \cos\theta + \underline{e}_2 \sin\theta$.

$\underline{e}_\theta = -\underline{e}_1 \sin\theta + \cos\theta\, \underline{e}_2$. ①

$u_r = \underline{u} \cdot \underline{e}_r \qquad u_\theta = \underline{u} \cdot \underline{e}_\theta$ ②

$u_1 = -\alpha \sin\theta\, X_3 \qquad u_2 = \alpha r \cos\theta\, X_3$

$u_r = 0, \qquad u_\theta = \alpha r X_3$. ⊛

on surface: $u_\theta = \alpha R X_3$.
$(r = R)$

$u_\theta = \alpha r L = r\theta_0$

$rL = r\theta_0$. $\qquad \alpha = \dfrac{\theta_0}{L}$ $\Big\rangle$ the unit of trace per unit length

Strain tensor in cylindrical coordinate

the strain due to $\begin{cases} \varepsilon_{rr} = \varepsilon_{\theta\theta} = \varepsilon_{r\theta} = \varepsilon_{r3} \\ = \varepsilon_{33} = 0 \end{cases}$

$\varepsilon_{\theta 3} = \frac{1}{2}\alpha r$

$L? = R\theta_0 \qquad ? = \dfrac{R\theta_0}{L} = R\alpha$

$\tau_{\theta 3} = \frac{G}{2}\alpha r$ $\leftarrow$ stress!!

$\hookrightarrow$ only non-trivial stress component in polar coordinate

- traction free BCs automatically satisfied on surface of cylinder

element

$\iint \tau_{\theta 3}\, r\, dA = M$

(Recall the warping function of z)

$\varepsilon_{11}=0,\ \varepsilon_{22}=0,\ \varepsilon_{12}=0,\ \varepsilon_{33}=0.$

$$\varepsilon_{13}=\frac{1}{2}\left[-\alpha x_2+\frac{\partial w}{\partial x_1}\right]$$

$$\varepsilon_{23}=\frac{1}{2}\left[\alpha x_1+\frac{\partial w}{\partial x_2}\right].$$

$$\Rightarrow \begin{cases} \tau_{\alpha\beta}=0, & \alpha=1,2. \\ \tau_{33}=0 \end{cases}$$

$$\tau_{13}=G\left[-\alpha x_2+w_{,1}\right] \qquad \tau_{23}=G\left[\alpha x_1+w_{,2}\right]$$

Enforce equilibrium,

Equilibrium in 1, 2 directions are satisfied automatically. → No Body Force.

in 3 direction. $\boxed{w \text{ is harmonic.}}$

→ equilibrium is satisfied.

$$\frac{\partial \tau_{13}}{\partial x_1}+\frac{\partial \tau_{23}}{\partial x_2}=0. \Rightarrow \text{Traction free BCs on the side of the Bar}$$

$$\underbrace{\frac{\partial^2 w}{\partial x_1^2}+\frac{\partial^2 w}{\partial x_2^2}}_{\nabla^2 w=0.}=0$$

in some sense, anti-plane shear.

$\tau_{ij}n_j=0$ (traction free).

$$\underline{t}=\frac{dx_1}{ds}\underline{e}_1+\frac{dx_2}{ds}\underline{e}_2$$

$$\underline{n}=\frac{dx_2}{ds}\underline{e}_1-\frac{dx_1}{ds}\underline{e}_2$$

$\tau_{ij}n_j=0$

$$\Downarrow$$

$$\boxed{\tau_{31}\underline{n}_1+\tau_{32}\underline{n}_2=0} \quad (\text{BCs}).$$

$$\boxed{\tau_{31}\cdot\frac{dx_2}{ds}-\tau_{32}\frac{dx_1}{ds}=0}$$

$$G\left[-\alpha x_2+w_{,1}\right]\frac{dx_2}{ds}-G\left[\alpha x_1+w_{,2}\right]\frac{dx_1}{ds}=0.$$

(we can cancel the G).

$$\underbrace{w_{,1}\frac{dx_2}{ds}-w_{,2}\frac{dx_1}{ds}}_{\text{grad: } \underbrace{\nabla w\cdot\underline{n}}_{\frac{dw}{dn}}}=\alpha\left[x_1\frac{dx_1}{ds}+x_2\frac{dx_2}{ds}\right].$$

$$=\frac{\alpha}{2}\left[\frac{d(x_1^2+x_2^2)}{ds}\right]$$

traction free BCs

$$\frac{dw}{dn}=\frac{\alpha}{2}\cdot\frac{d(x_1^2+x_2^2)}{ds} \quad > \text{BCs for Laplace}$$

$\nabla^2 w=0.$ ↖ Neumann BCs.

guarantee this

$$\oint_C \frac{dw}{dn}\,ds=0 \quad \left(\text{existence of solution}\right) \text{ condition to be satisfied}$$

$$\Downarrow$$

automatically satisfied. → $\frac{dw}{dn}$

$$M=\underbrace{\alpha\left[G\iint_A\left[x_\alpha x_\alpha+x_1w_{,2}-x_2w_{,1}\right]dx_1dx_2\right]}_{K:\ \text{torsional stiffness}}$$

$$\boxed{M = k\alpha}$$

w is harmonic

$\Downarrow$

w is the real / Im part of an analytical function.

$f(z) = w + i\phi \quad . \quad i = \sqrt{-1}$

$z = x_1 + i x_2$

w, ϕ are related by CR Eqns.

$$\begin{cases} w_{,1} = \phi_{,2} \\ w_{,2} = -\phi_{,1} \end{cases}$$

$\sigma_{13} = G[-\alpha x_2 + w_{,1}] = G[-\alpha x_2 + \phi_{,2}]$

$\sigma_{23} = G[\alpha x_1 + w_{,2}] = G[\alpha x_1 - \phi_{,1}]$

Note: w, ϕ, is harmonic, so ϕ also satisfy $\nabla^2\phi = 0$.

Check. Equilibrium Eqn is automatically satisfied.

$\sigma_{13} n_1 + \sigma_{23} n_2 = 0 \Rightarrow$ BCs.

$\sigma_{13}\dfrac{dx_2}{ds} - \sigma_{23}\dfrac{dx_1}{ds} = 0.$

Wed., Nov. 3rd, Week 12

<u>REVIEW</u>. Displacement. $\rightarrow \nabla^2 w = 0$

TRACTION FREE BCs: $\dfrac{dw}{dn} = x_2 n_1 - x_1 n_2$

$= x_2\dfrac{dx_2}{ds} + x_1\dfrac{dx_1}{ds} = \dfrac{1}{2}\dfrac{d(x_1^2+x_2^2)}{ds}.$

$\underline{t} = \dfrac{dx_1}{ds}\underline{e}_1 + \dfrac{dx_2}{ds}\underline{e}_2$

$\underline{n} = \dfrac{dx_2}{ds}\underline{e}_1 - \dfrac{dx_1}{ds}\underline{e}_2.$

$$\begin{cases} \sigma_{13} = \frac{G}{2}\alpha\left[-x_2 + w_{,1}\right] \\ \sigma_{23} = \frac{G}{2}\alpha\left[x_1 + w_{,2}\right] \end{cases}$$

$$\begin{cases} u_1 = -\alpha x_2 x_3 \\ u_2 = \alpha x_1 x_3 \\ u_3 = \alpha\, w(x_1, x_2) \end{cases}$$

A different approach.

$f(z) = w + i\phi$ ← conjugate harmonic function to w.

$w_{,1} = \phi_{,2}$ & $w_{,2} = -\phi_{,1}$ <u>CR</u>

$\nabla^2\phi = 0$

$\sigma_{13} = \frac{G}{2}\alpha\left[-x_2 + \phi_{,2}\right], \quad \sigma_{23} = \frac{G}{2}\alpha\left[x_1 - \phi_{,1}\right]$

<u>T.F. BCs</u> $\sigma_{31} n_1 + \sigma_{32} n_2 = 0 \Rightarrow \left[-x_2 + \phi_{,2}\right] n_1 + \left[x_1 - \phi_{,1}\right] n_2 = 0.$

$\underbrace{-x_2\dfrac{dx_2}{ds} + x_1\left(-\dfrac{dx_1}{ds}\right)}_{\dfrac{d(x_1^2+x_2^2)}{ds}}$

$-\phi_{,1} n_2 + \phi_{,2} n_1 = 0.$

$\underbrace{\phi_{,1} t_1 \quad \phi_{,2} t_2}_{\nabla\phi\cdot\underline{t} = \frac{d\phi}{ds}}$

$$\Rightarrow \frac{d\phi}{ds} = \frac{d(x_1^2+x_2^2)}{ds} \Rightarrow \phi = \frac{x_1^2+x_2^2}{2} + \text{const} \Rightarrow \text{Curve } C$$

Set const $=0$.

BC: $\boxed{\phi = \frac{x_1^2+x_2^2}{2} \text{ on } C \qquad \nabla^2\phi = 0}$

Example

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$

$f(z) = w + i\phi$.

$$f(z) = i(\underset{\text{const.}}{C^2})z^2 \qquad z = x+iy$$

$$z^2 = x^2 - y^2 + 2ixy$$

$$iC^2z^2 + ik^2 \rightarrow \text{some other const.}$$

C k k are real numbers.

hopefully satisfy the boundary conditions.

$$= \underbrace{iC^2(x^2-y^2) + ik^2}_{\text{imaginary part.}} \underbrace{-2C^2xy}_{\text{real part}}$$

BC on C: $C^2(x^2-y^2) - k^2 = \frac{x^2+y^2}{2}$

$$k^2 = \underbrace{x^2\left[\frac{1}{2} - C^2\right]}_{\frac{1}{a^2}} + \underbrace{y^2\left[\frac{1}{2}+C^2\right]}_{\frac{1}{b^2}}$$

$$C^2 = \frac{1}{2}\frac{a^2-b^2}{a^2+b^2} \qquad k^2 = \frac{a^2b^2}{a^2+b^2}$$

that means $\rightarrow \phi$

$$\sigma_{13} = G\alpha[-x_2 + \phi_{,2}]$$

$$\sigma_{23} = G\alpha[x_1 - \phi_{,1}]$$

$$\Downarrow$$

$$\begin{cases} \sigma_{13} = \dfrac{-2G\alpha a^2 y}{a^2+b^2} \\ \sigma_{23} = \dfrac{2G\alpha b^2 x}{a^2+b^2} \end{cases}$$

$w = -2C^2xy$ Also know waping function.

Note $c=0$, $a=b$, circle

$$M = \iint_y (\sigma_{32}x - \sigma_{12}y)\,dA$$

$$= \frac{G\pi\alpha b^3a^3}{a^2+b^2} = k\alpha.$$

$$\hookrightarrow k = G(\pi ab)\frac{a^2b^2}{(a^2+b^2)} \qquad A = \pi ab$$

how to calculate waping function / stresses in torsion.

Prandtl stress function approach:

$$\nabla^2\phi = 0$$

$$\phi = \frac{1}{2}(x^2+y^2), \text{ on } C.$$

Define a function: $\Phi = \phi - \frac{1}{2}(x^2+y^2)$

on the BC, $\Phi = 0$. on C

$\nabla^2\Phi = -2 \rightarrow$ Poisson's equation

$$\phi = \Phi + \frac{1}{2}(x^2+y^2) \qquad \cancel{\sigma_{13} = \frac{G}{2}\alpha}$$

$$\begin{cases} \sigma_{13} = G\alpha\,\Phi_{,2} \\ \sigma_{23} = -G\alpha\,\Phi_{,1} \end{cases}$$

Calculate the moment:

$$M = \alpha 2G \iint_A \Phi \, dA.$$

Constant Φ curve.

$\Phi = c$

the gradient of Φ, normal to surface.

$$\nabla\Phi = \Phi_{,1}\, \underline{e}_1 + \Phi_{,2}\, \underline{e}_2$$

$\underline{\tau} = G\alpha$

$$\left[\Phi_{,2}\, \underline{e}_1 - \Phi_{,1}\, \underline{e}_2\right] \Rightarrow \underline{\tau} \cdot \nabla\Phi = 0.$$

The constant line are the direction of shear vector

lines of shear stresses.

$$\nabla\Phi = \Phi_{,1}^2 + \Phi_{,2}^2 \propto \left[\, |\Phi|_{Ga} \,\right]$$

Prandtl <u>soap film analogy</u>

$T \to$ surface tension. ($\ell \approx$ const)

↳ $\frac{\text{Force}}{\text{length}}$.

film deformed.

Deflection.

$$\nabla^2 D = -\frac{P}{T}$$

pressure.

small membrane deflection formula.

define new const.

$$\frac{1}{2}d = \frac{DT}{P}.$$

$$\nabla^2\phi = -2$$

Similar !!

also $D = 0$ on C

$$D = \frac{dP}{2T}$$

$$\nabla^2 d = -2 \Rightarrow d = 0 \text{ on } C$$

$$G = \frac{\mu}{2} \qquad C_2 = \frac{\Lambda}{2} \qquad E, \nu$$

$$\sigma = \frac{1}{J}\left(C_1(\lambda_1^2 - 1) + C_2 \ln(J)\right) = 0$$

$$\lambda_i = 1 + \varepsilon_{ii} \qquad \frac{1}{\lambda_i} = 1 - \varepsilon_{ii}$$

$$\lambda_i^2 = 1 + 2\varepsilon_{ii} + \cancel{\varepsilon_{ii}^2}$$

$$J = \det F = \lambda_1^2 \lambda_3 = 1 + 2\varepsilon_{11} + \varepsilon_{33}$$

$$\ln(1 + \varepsilon_{ii}) \approx \varepsilon_{ii}$$

$C_1 >$

Solid Mechanics

Nov. 8, Mon, Week 12.

Linear viscoelasticity

ideal model:

Deformation $\not\leftrightarrow$ stress history.

real material:

Deformation $\Rightarrow$ (history).

- Cartoon Models

- uniaxial tension test.

- MAXWELL model

spring & dash pot in series

long run : simple fluid.
(suddenly apply a stress)

$\sigma_s = E\varepsilon_s$.

$\varepsilon = \frac{\sigma}{E}$.

$\dot{\varepsilon}_{vis} = \frac{\sigma}{\eta}$

Same since they are in series.

$\dot{\varepsilon}_{vis} = \frac{\sigma_{vis}}{\eta}$.

$$\boxed{\underbrace{\dot{\varepsilon}}_{\dot{\varepsilon}_s + \dot{\varepsilon}_{vis}} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}}$$

$\Rightarrow$ ODE in time.

Voigt model.

$\sigma = \sigma_s + \sigma_{vis}$.

- $\sigma = E\varepsilon + \eta\dot{\varepsilon} \Rightarrow$ Voigt model.

Solve a ODE to get this curve.

$t > 0, \Rightarrow \eta\dot{\varepsilon} + E\varepsilon = 1$

$\varepsilon = A e^{-\frac{E}{\eta}t} + \frac{1}{E}$

$= \frac{1}{E}\left[1 - e^{-\frac{E}{\eta}t}\right]$.

Short time : fluid ; & long time : solid

- Standard model

Voigt element.

Solid like behavior

$\frac{E_1 + E_2}{E_1 E_2} = \frac{1}{E_2} + \frac{1}{E_1}$

Solid behavior for both long & short time

$$\dot{\sigma} + \underbrace{\frac{E_1 + E_2}{\eta}\sigma = E_1\dot{\varepsilon} + \frac{E_2 E_1}{\eta}\varepsilon}_{\text{3 parameters to determine.}}.$$

Linear ODE

- Concept of Creep function.

Creep function $C(t)$ strain history due to a unit stress $\sigma = H(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0. \end{cases}$

$C(t) = \left(\frac{1}{E} + \frac{t}{\eta}\right)H(t)$. $\Leftrightarrow$ Maxwell

$C(t) = \frac{1}{E}\left(1 - e^{-\frac{Et}{\eta}}\right)H(t)$. $\Leftrightarrow$ Voigt.

$C(t) = \underbrace{\frac{E_1 + E_2}{E_1 E_2}}_{1/E_\infty} - \frac{1}{E_2}e^{-E_2 t/\eta}$

$\frac{E_2}{\eta}$ units : $\frac{1}{[t]}$

$t_c \Leftrightarrow$ Creep relaxation time

$C(t=0) = \frac{1}{E_1}$ short time modulus.

$C(t=\infty) = \frac{1}{E_2} + \frac{1}{E_1} = \frac{1}{E_\infty}$

long time modulus.

$\frac{E_\infty}{E_1} = 10^{-3}$.

Voigt element

$\varepsilon(t) = H(t)$.

→ unit strain & stress history.

Maxwell model.

→ fluid.

$Y(t) = E e^{-\frac{E}{\eta}t} H(t)$.

Für standard solid:

$E_\infty + (E_0 - E_\infty) e^{-\frac{t}{t_R}}$

$t_R = \cancel{\frac{E_1 + E_2}{\eta}}$, $E_0 = E_1$.

$\frac{\eta}{E_1 + E_2}$ (typo)

Boltzmann superposition principle

① Assume system is linear.

② Assume Causality.

③ Non-Aging.

Creep.

Given $C(t)$

Response of system due to a unit stress step function.

$\sigma(t) = \sigma(0^+) H(t) + \sigma(\Delta t) H(t-\Delta\tau) + \sigma'(\Delta\tau)\Delta\tau H(t+\Delta\tau)$.

$\varepsilon(t) = \sigma(0^+) C(t) + \sigma'(0)\Delta\tau C(t-\Delta\tau) + \sigma'(\Delta\tau)\Delta\tau C(t-2\Delta\tau) H(t-2\Delta\tau) + \cdots\cdots$

$= \sigma(0^+) C(t) + \int_{0^+}^{t} \sigma'(\tau) C(t-\tau) d\tau$.

Nov. 10, Wed, Week 12.

REVIEW: Boltzmann superposition –

Any stress history can be broken down into sum of step functions.

↓ Each step fct. $H(t-\tau)$

$C(t-\tau)$.

Real Linear viscoelastic model for uniaxial tension ⇒ $\varepsilon(t) = \sigma(0^+)C(t) + \int_{0^+}^{t} C(t-\tau)\frac{d\sigma}{d\tau}\,d\tau$

Do a simple example.

[Convolution product

$f(t)$, $g(t)$, define for $t \in (0, \infty)$.

$$f * g = \int_0^t f(t-\tau)\,g(\tau)\,d\tau = \int_0^t g(t-\tau)\,f(\tau)\,d\tau.$$

(1) $$\boxed{\varepsilon = \sigma(0^+)\,C(t) + C * \sigma'} \qquad \sigma' \equiv \frac{d\sigma}{dt}$$

The Laplace transform of a function f defined in zero to infinity.

$$\mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt.$$

$\Downarrow$

$\hat{f}(s)$ ⇒ is a function of s.

↳ s in the transform variable.

↳ s is in general a complex variable

Properties.

$$\mathcal{L}[f'(t)] = s\hat{f}(s) - f(0^+).$$

$$\mathcal{L}[f * g] = \hat{f}(s)\,\tilde{g}(s) \leftarrow \text{convolution Theorem.}$$

Laplace transform ①:

$$\tilde{\varepsilon}(s) = \sigma(0^+)\,\tilde{Y}(s) + \underbrace{\mathcal{L}[Y * \sigma']}_{\tilde{C}(s)\,\underbrace{\mathcal{L}[\sigma'(t)]}_{[s\tilde{\sigma}(s) - \sigma(0^+)]}}$$

$$\boxed{\tilde{\varepsilon}(s) = s\,\tilde{C}(s)\,\tilde{\sigma}(s)} \quad ②$$

clear & linear. → transform domain.

Messy $\Rightarrow \varepsilon(t) = \sigma(0^+)\,C(t) + \int_{0^+}^{t} C(t-\tau)\,\frac{d\sigma}{d\tau}\,d\tau.$

Do the same thing with Relaxation function $Y(t)$.

Apply a strain history. $\varepsilon(t)$ ← given.

$$\sigma(t) = \varepsilon(0^+)\,Y(t) + \int_{0^+}^{t} Y(t-\tau)\,\frac{d\varepsilon}{d\tau}\,d\tau.$$

$$\boxed{\tilde{\sigma}(s) = s\,\tilde{Y}(s)\,\tilde{\varepsilon}(s)} \quad ③$$

on the transform plane, — it's trivial.

Combine ② & ③

$$\hat{\sigma}(s) = s\,\hat{Y}(s)\,.\,s\,\hat{C}(s)\,\hat{\sigma}(s).$$

$$\Rightarrow s\,\tilde{Y}(s)\cdot\tilde{C}(s) = 1$$ ← related.

$$\tilde{Y}(s) = \frac{1}{s^2\,\tilde{C}(s)}.$$

$$Y(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2\,\tilde{C}(s)}\right].$$

$$\hookrightarrow \tilde{Y}(s).\,\hat{C}(s) = \frac{1}{s^2}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t.$$

$$\int_0^t Y(t-\tau)\,C(\tau).\,d\tau = t. \qquad t \geq 0.$$

(Bromwich Integral) $\qquad s = s_1 + i s_2$

$$\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \tilde{Y}(s).\,e^{st}\,ds = Y(t).$$

$t > 0.$

$\tilde{Y}(s)$ is Analytic on $\mathrm{Re}\,s > 0$

Isotropic linear viscoelastic solid.

$$\sigma_{ij} = 2G\,\varepsilon_{ij} + \lambda\,\varepsilon_{kk}\,\delta_{ij}.$$ MEMORIZE THIS!!! ☆☆

$$\varepsilon_{ij} = \frac{\sigma_{ij}}{2G} - \frac{\nu}{E}\,\sigma_{kk}\,\delta_{ij} \qquad E = \frac{G}{2(1+\nu)}.$$

deviatoric stress tensor.

$$S_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

$$e_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}.$$

Strain deviator tensor

volumetric part.

in literature, $\frac{1}{3}\sigma_{kk} = \sigma$; $\varepsilon_{kk} = e$

change of volume.

e_{ij}

shear Modulus. (relaxation modulus)

$$\begin{cases} S_{ij} = 2G\, e_{ij}. \\ \sigma_{kk} = 3k\, \varepsilon_{kk}. \end{cases}$$

(relaxation) Bulk Modulus.

exactly the SAME

$$\begin{cases} \sigma_{ij} = 2G\,\varepsilon_{ij} + \lambda\,\varepsilon_{kk}\delta_{ij} \\ \varepsilon_{ij} = \dfrac{\sigma_{ij}}{2G} - \dfrac{\nu}{E}\sigma_{kk}\delta_{ij}. \end{cases}$$

$$S_{ij} = e_{ij}(0^+)\,Y_1(t) + \int_{0^+}^{t} Y_1(t-\tau)\frac{\partial e_{ij}}{\partial \tau}\,d\tau$$

$Y_1(t) \longleftrightarrow 2G$. $\quad \sigma_{kk} = \varepsilon_{kk}(0^+)\,Y_2(t) + \int_{0^+}^{t} Y_2(t-\tau)\frac{\partial \varepsilon_{kk}}{\partial \tau}d\tau$

$Y_2 \longleftrightarrow 3k$

Consider $\mathcal{L}$

$$\boxed{\begin{aligned} \tilde{S}_{ij} &= s\,\tilde{Y}_1(s)\,\tilde{e}_{ij} \\ \tilde{\sigma}_{kk} &= s\,\tilde{Y}_2(s)\,\tilde{\varepsilon}_{kk} \end{aligned}}$$

linear Elasticity

$\sigma_{ij,j} = 0$. No body Force.

$\varepsilon_{ij} = \dfrac{u_{i,j} + u_{j,i}}{2}$, kinematics.

Constitutive model.

$$\begin{cases} S_{ij} = e_{ij}(0^+)\cdot Y_1(t) + Y_1 * \dot{e}_{ij} \\ \sigma_{kk} = \varepsilon_{kk}(0^+)\,Y_2(t) + Y_2 * \dot{\varepsilon}_{kk} \end{cases}$$

General Problem

$\partial\Omega_u$

given.

$\underset{\sim}{u}(\underset{\sim}{x}, t) = \underset{\sim}{f}(\underset{\sim}{x}, t)$.

$\partial\Omega_T$. $\sigma_{ij}n_j = T_i(x, t)$. involve time.

given.

linear Elasticity

$$\begin{cases} \tilde{\sigma}_{ij,j} = 0. \\ \tilde{\varepsilon}_{ij} = \dfrac{\tilde{u}_{i,j} + \tilde{u}_{j,i}}{2} \\ \tilde{S}_{ij} = \underbrace{s\,\tilde{Y}_1(s)}_{2G}\cdot\tilde{e}_{ij}(s) \end{cases}$$

$\Downarrow$

correspondence principle

$$\tilde{\sigma}_{kk} = \overbrace{s\,\tilde{Y}_2(s)}^{3k}\,\tilde{\varepsilon}_{kk}(s)$$

$$\text{BC}\begin{cases} \tilde{u}(x, s) = \tilde{f}(x, s). \\ \tilde{\sigma}_{ij}n_j = \tilde{T}_i(x, s) \end{cases}$$

Simple Example

BCs. $\begin{cases} \sigma_{rr}(r=a,\ t>0) = P_0 \\ \sigma_{rr}(r=b,\ t\geq 0) = 0 \end{cases}$

$$SY(s)\,C_1(s)\,s = 1$$

$$\tilde{e}_{ij}(s) = s\,C_1(s)\,\tilde{S}_{ij}$$

$$e_{ij} = C_1(t)\,\tilde{S}_{ij}(0^+) + \int_{0^+}^{t} C_1(t-\tau)\underbrace{\frac{\partial S_{ij}}{\partial \tau}}_{=0}\,d\tau$$

$u_i = \Delta\, g\left(\frac{r}{a}, 0\right)$

$u_3(x_1, x_2, x_3=0) = -\Delta$

$(x_1^2 + x_2^2) < 1$

$\sigma_{ij} = \frac{G\Delta}{a} f\left(\frac{r}{a}, 0\right)$

Office Hour.

$\psi \rightarrow \phi$

$\nabla^2\phi = 0 \qquad \phi = \frac{x^2+y^2}{2}$

formulate

$f \rightarrow$ BCs to simple.

$f\big|_{BCs} = 0$

$f \rightarrow$ harmonic

$\hookrightarrow$ separation of variables

Differential EQs.

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \Rightarrow \frac{\partial^2\phi}{\partial x^2} = -\frac{\partial^2\phi}{\partial y^2}.$$

$$f = \frac{\partial^2\phi}{\partial x^2} + 1.$$

Example:

PRANDTL: Stress function

Three common formulations for solving torsion:
w, ϕ, Φ_p.

Solution for Laplacian
↓
Harmonic Analysis.

November 13, 2021. Week 13. Mon.

REVIEW: Correspondence principle.

$$Y(t) = Y_\infty + (Y_0 - Y_\infty)\sum_{j=1}^{n} a_j e^{-t/t_j}$$

$Y_\infty = Y(t=\infty)$ long time Modulus.

$Y_0 = Y(t=0)$. Instantaneous Modulus.

$\sum_{j=1}^{n} a_j = 1$

t_j = Relaxation times.

Power law model.

$$Y(t) = Y_\infty + (Y_0 - Y_\infty)\frac{1}{1+(t/t_0)^\alpha}$$

Corresponds to

n of this

Rheology.

Shear strain: $\varepsilon = \varepsilon_0 e^{i\omega t}$.

* What is the response.

(long time / steady state res)

Shear stress $\sigma(t) = \underbrace{\varepsilon(0^+)}_{\varepsilon_0} Y_1(t) + \int_{0^+}^{t} (Y_1(t-\tau)) \frac{d\varepsilon}{d\tau} d\tau$

$\varepsilon_0 i\omega e^{i\omega\tau}$

$= \varepsilon_0 Y_1(t) + \frac{\varepsilon_0}{i\omega} \int_0^t Y_1(t-\tau) e^{i\omega\tau} d\tau$

$\varepsilon_0 Y_1(t) + i\omega\varepsilon_0 \int_{0^+}^{t} [Y_1(t-\tau) - Y_1(t)] e^{i\omega\tau} d\tau$

$+ \underbrace{i\omega\varepsilon_0 Y_1(t) \int_{0^+}^{t} e^{i\omega\tau} d\tau}$

$i\omega\varepsilon_0 Y_1(t) \cdot \frac{e^{i\omega\tau}}{i\omega} \Big|_{0^+}^{t}$

$= \varepsilon_0 Y_1(t) e^{i\omega t} - \varepsilon_0 Y_1(t)$

$= \varepsilon_0 Y_1(t) e^{i\omega t} + i\omega\varepsilon_0 \int_0^t [Y_1(t-\tau) - Y_1(t)] e^{i\omega\tau} d\tau$

$\int_0^t Y_1(t-\tau) e^{i\omega\tau} d\tau$. $\quad \eta = t-\tau$.

$\tau = t-\eta$.

$= \int_t^0 Y_1(\eta) \cdot e^{i\omega(t-\eta)} (-d\eta)$

$= \int_0^t Y_1(\eta) e^{i\omega(t-\eta)} d\eta = e^{i\omega t} \int_0^t Y_1(\eta) e^{-i\omega\eta} d\eta$

$\int_0^t Y_1(t) \cdot e^{i\omega\tau} d\tau$.

$= Y_1(t) \cdot e^{i\omega t} \int_0^t e^{i\omega\eta} d\eta$.

$\sigma(t) = \varepsilon_0 Y_1(t) \cdot e^{i\omega t} + i\omega e^{i\omega t} \int_0^t [Y_1(\eta) - Y_1(t)] e^{i\omega\eta} d\eta$.

$= e^{i\omega t} \left[\varepsilon_0 Y_1(t) + i\omega \int_0^t [Y_1(\eta) - Y_1(t)] e^{-i\omega\eta} d\eta \right]$

$\xrightarrow{t \to \infty}$ converge to $\underbrace{\varepsilon_0 Y_1(\infty) + i\omega \int_0^\infty [Y_1(\eta) - Y_1(\infty)] e^{-i\omega\eta} d\eta}_{\hat{Y}_1(\omega)}$.

Complex Modulus

$\sigma(t \to \infty) = \varepsilon_0 e^{i\omega t} \hat{Y}_1(\omega)$. $\longrightarrow \cos\omega t + i\sin\omega t$.

$\hat{Y}_1 = \underbrace{\hat{Y}_1'(\omega)}_{\text{Storage modulus}} + i \underbrace{\hat{Y}_1''(\omega)}_{\text{lost modulus}}$.

$\tan\phi = \frac{\hat{Y}_1'(\omega)}{\hat{Y}_1''(\omega)}$

loss tangent.

$$\hat{Y}_1'(\omega) = Y_1(\infty) + \omega \int_0^\infty [Y_1(\eta) - Y_1(\infty)] \sin(\omega\eta)\, d\eta.$$

$$\hat{Y}_1''(\omega) = \omega \int_0^\infty [Y_1(\eta) - Y_1(\infty)] \cos(\omega\eta)\, d\eta.$$

$\hat{Y}_1'(\omega) = \hat{Y}_1'(-\omega)$ → even fct. of ω

$\hat{Y}_1''(\omega) =$ odd fct. of ω.

Change of energy in a cycle.

$$W = \int_{\text{cycle}} \sigma\, d\varepsilon$$

We apply $\varepsilon = \varepsilon_0 \left[\frac{e^{i\omega t} + e^{-i\omega t}}{2} \right]$

$$= \varepsilon_0 \cos(\omega t).$$

$$\varepsilon_0 \frac{e^{i\omega t}}{2} \rightarrow \varepsilon_0 \hat{Y}_1(\omega) e^{i\omega t} = \sigma(t).$$

$$\varepsilon_0 \frac{e^{-i\omega t}}{2} \rightarrow \frac{\varepsilon_0 \hat{Y}_1(-\omega) e^{-i\omega t}}{2} = \sigma(t).$$

$$\hat{Y}_1(-\omega) = \overline{\hat{Y}_1(\omega)}$$

$$\sigma = \frac{\varepsilon_0}{2} \left[\hat{Y}_1(\omega) e^{i\omega t} + \overline{\hat{Y}_1(\omega)} e^{i\omega t} \right]$$

$$= \varepsilon_0 \left[\hat{Y}_1'(\omega) \cos\omega t - \hat{Y}_1''(\omega) \sin\omega t \right].$$

$$\because e^{i\omega t} \hat{Y}_1(\omega) = \left[\hat{Y}_1'(\omega) + i \hat{Y}_1''(\omega) \right] \left[\cos\omega t + i \sin\omega t \right].$$

$$\mathrm{Re}(\quad) = \hat{Y}_1'(\omega) \cos\omega t - \hat{Y}_1''(\omega) \sin\omega t$$

$$-d\varepsilon = \omega \varepsilon_0 \sin\omega t\, dt$$

$$-\omega \varepsilon_0^2 \int_{\text{cycle}} \left[\hat{Y}_1'(\omega) \cos\omega t - \hat{Y}_1''(\omega) \sin\omega t \right] \sin\omega t\, dt.$$

work = W in a cycle.

we already show: $\varepsilon_0 e^{i\omega t} \rightarrow \varepsilon_0 \hat{Y}_1(\omega) e^{i\omega t}$.

1st integral

$$\hat{Y}_1'(\omega) \int_{\text{cycle}} \cos\omega t\, d\sin\omega t.$$

$$\frac{d \cos^2 \omega t}{2\omega} = 0.$$

$$\omega \varepsilon_0^2 \hat{Y}_1''(\omega) \underbrace{\int_{\text{cycle}} \sin^2(\omega t)\, dt\, \omega}_{\text{Some number}}$$

if you plot the lost modulus –

Nov. 17, 2021. Wed., Week 13.

$\varepsilon_0 e^{i\omega t}$.

Long time stress.

$$\sigma = \varepsilon_0 \underbrace{\hat{Y}_1(\omega)}_{\text{complex modulus}} e^{i\omega t}$$

$$\hat{Y}_1(\omega) = \hat{Y}_1'(\omega) + \underbrace{i}_{\sqrt{-1}}\,\underbrace{\hat{Y}_1''(\omega)}_{\text{loss modulus} \to \text{odd function.}}$$

Storage modulus

Energy loss per cycle $= \pi \varepsilon_0^2 \left|\hat{Y}_1''(\omega)\right|$

$\to$ Even function.

<u>loss tangent.</u>

$$\hat{Y}_1 = |\hat{Y}_1| e^{i\delta}$$

$$\underbrace{\tan\delta}_{\text{loss tangent.}} = \frac{\hat{Y}_1''}{\hat{Y}_1'}$$

$$\underbrace{\hat{Y}_1(\omega)}_{\text{complex modulus.}} = \underbrace{Y_1(\infty)}_{\text{relaxation function in time domain}} + i\omega \int_0^\infty \left[Y_1(\eta) - Y_1(\infty)\right] e^{-i\omega\eta}\, d\eta$$

$\sigma = \sigma_0 e^{i\omega t}$.

long time shear strain.

$$\varepsilon = \sigma_0 \hat{C}_1(\omega) e^{i\omega t}.$$

creep modulus.

$$\hat{C}_1(\omega) = C_1(\infty) + \int_0^\infty \left[C_1(\eta) - C_1(\infty)\right] e^{-i\eta\omega}\, d\eta$$

$\varepsilon_0 e^{i\omega t} \longrightarrow \varepsilon_0 \hat{Y}_1(\omega) e^{i\omega t} = \sigma$.

$\sigma = \sigma_0 e^{i\omega t} \longrightarrow \sigma_0 \hat{C}_1(\omega) e^{i\omega t}$.

$\sigma_0 = \varepsilon_0 \hat{Y}_1(\omega)$

$\hookrightarrow \varepsilon_0 \underbrace{\hat{Y}_1(\omega)\hat{C}_1(\omega)}_{1} e^{i\omega t}$

$$\hat{Y}_1(\omega) = \frac{1}{\hat{C}_1(\omega)}.$$

$$\int_0^t C_1(t-\tau)\, Y_1(\tau)\, d\tau = t.$$

$$s^2 \tilde{C}_1 \tilde{Y}_1 = 1$$

$\hookrightarrow$ Laplace transform

$$\int_0^\infty e^{-st} C_1(t)\, dt = \tilde{C}_1(s)$$

<u>Standard model</u>

$$Y_1(t) = Y_\infty + (Y_0 - Y_\infty)\, e^{-t/t_R}$$

$Y_\infty \equiv Y_1(t \to \infty) \to$ long time shear modulus

$Y_0 = Y_1(t \to 0) \to$ short time

$$\begin{cases} \hat{Y}_1(\omega) = Y_\infty + \dfrac{i\omega(Y_0 - Y_\infty)}{t_R^{-2} + \omega^2} \\ \hat{Y}_1(\omega) = Y_\infty + \dfrac{\omega^2 t_R^2 (Y_0 - Y_\infty)}{1 + \omega^2 t_R^2} \\ \hat{Y}_1''(\omega) = \dfrac{\omega t_R (Y_0 - Y_\infty)}{1 + \omega^2 t_R^2} \end{cases}$$

$[\omega] \sim \frac{1}{[t]}$

▷ Time Temp. Superposition

ω : 10^{-2} Radium/s. + o 10^2 rad/s.

$$\begin{cases} Y_1'(\omega, T) = b(T, T_0)\, \phi_1(a(T, T_0)\,\omega) \\ Y_2'(\omega, T) = b(T, T_0)\, \phi_2(a(T, T_0)\,\omega) \end{cases}$$

Ref. Temp.

Shift Factors.

$$\frac{Y_1'}{b(t, t_0)} = \phi_1(a(t, t_0)\,\omega)$$

$\log \frac{\hat{Y}_1}{b(T, T_0)} = \log \hat{Y} - \log b(T - T_0)$

$\log[a(T, T_0)\,\omega]$

$= \log a(T, T_0) + \log(\omega)$

> 1, $T < T_0$

$a(T, T_0) = 1$, $T = T_0$

< 1, $T > T_0$

WLF - shift factor.

$$a(T, T_0) = \frac{-C_1(T_0)(T - T_0)}{C_2(T_0) + (T - T_0)}$$

Normally, $T_0 \to$ glass transition temp. of polymer

$$\hat{Y}_1(\omega) = Y(\infty) + i\omega \int_0^\infty [Y_1(\eta) - Y_1(\omega)]\, e^{i\omega\eta}\, d\eta$$

complex modulus.

Gwein.

time domain (relaxation fct.)

Y_1' & Y_2' are not independent to each other

if u know one, you know the other.

$$\frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{\hat{Y}_1'(r) - Y_1(\infty)}{r\omega}$$

↳ shear strain.
Engineering

usually, $\gamma = 2\varepsilon$.

$2G_1 \varepsilon = \tau$.

$\tau = G\gamma$,

$\Downarrow$

γ,

$e_{ij} = s Y_1^*$

$S_{ij} = s Y_1^* e_{ij}$

$\sigma_{kk} = s Y_2^* \varepsilon_{kk}$

$E = 2G(1+\nu)$.

Linear Elasticity.

G, ν.

Linear Viscoelasticity.

G, ν, E, K.

↑ bulk (relaxation).

OH: superposition principle.

want to stress in tension test.

$$\sigma(t) = \overset{\varepsilon}{\Psi}(0^+)\,Y(t) + \int_{0^+}^{t} \underbrace{Y(t-\tau)\,\frac{d\varepsilon}{d\tau}}_{\to 2a}\,d\tau$$

$$= \varepsilon(0^+)\,Y(t) + Y * \underbrace{\frac{d\varepsilon}{d\tau}}_{\text{strain history}}$$

$\Downarrow$

stress history

asked to evaluate the stress history.

load fast → strain rate → high →

End: should not have stress in Spring 1 & Spring 2.

After 2b, you should be able to see.

Q:

You have:

creep function $\boxed{C(t).\ Y(t).}$

find out equation of how

only one disp. field (r)

$\downarrow$

Strain (1→3)

ε_r & ε

$u_r \longrightarrow \varepsilon_{rr}, \varepsilon_{\theta\theta}, = \varepsilon_{\phi\phi}$

$\Downarrow$ Consti. model

$\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{\phi\phi}, \quad \phi,$

Equilibrium Eq.

Incompressible.

$$\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{\phi\phi} = 0$$

$\Downarrow$

pressure = const.

$r = r_0 \quad , \sigma_{rr} = -P.$

integrate incompressibility.

$$\sigma_{ij} = \pm 2G\varepsilon_{ij} + P\delta_{ij}$$

hydrostatic.

incompressibility

$\varepsilon_{kk} = 0$

$\sigma_{rr} = P_0 \sin\omega\theta.$

$$\sigma = \frac{(\cos\omega t)\, e^{t}}{(1+t^3)}.$$

only difference:

$\downarrow$

strain.

$\sigma = \sigma_0 \cos\omega t$

$\hookrightarrow \varepsilon = \dfrac{\sigma_0 \cos\omega t}{E}$

$\Downarrow$

elastic.

$$\varepsilon = \sigma(0^+)\, C(t) + \int_0^t C(t-\tau)\, \frac{\partial \sigma}{\partial \tau}\, d\tau$$

$\varepsilon = \sigma_1 / E_1$

$\sigma = \sigma_1 + \sigma_2$

$\sigma_2 = \sigma - \sigma_1$

$\dot{\varepsilon}$

$$\dot{\varepsilon} = \frac{\dot{\sigma}_2}{E_2} + \frac{\sigma_2}{\eta}$$

$$\int_0^\infty \frac{e^{-i\omega\eta}}{1 + \eta/t_R}\, d\eta \qquad \frac{G_0 - G_\infty}{\eta}$$

$$\frac{\eta}{t_R} = u.$$

Exponential Integral $= \underbrace{\int_0^\infty \frac{e^{-i\hat{\omega}\eta}\, d\eta}{1+u}}$

$$d\eta = u\, t_R$$

$$\hat{\omega} = \omega t_R$$

basic idea $\mapsto$ integrate this term.

Compute integral by picking ~~integral~~ infinite value

$$\int_0^\infty \frac{\cos \overset{\hat{\omega}}{\omega} u\, du}{(1+u)} - i \int_0^\infty \frac{\sin \omega u\, du}{(1+u)}$$

Riemann - Lamé theory.

$$\int_0^\infty \frac{\cos \omega x\, dx}{1+x}.$$

Nov. 22, Mon, 2021. Wk 14.

Linear visco-isotropic material.

$$e_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij} = \frac{1}{2G} S_{ij} \quad \text{elasticity.}$$

$$\varepsilon_{kk} = \frac{1}{K}\cdot\frac{\sigma_{kk}}{3}$$

bulk deformation.

$$\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

shear deformation

Visco.

$$e_{ij} = C_1(t)\, S_{ij}(0^+) + \underbrace{\int_{0^+}^{t} C(t-\tau)\frac{\partial S_{ij}}{\partial \tau}\, d\tau}_{C_1^* S_{ij}'}$$

$$\varepsilon_{kk} = C_2(t)\,\sigma_{kk}(t=0^+) + \underbrace{\int_{0^+}^{t} C_2(t-\tau)\frac{\partial \sigma_{kk}}{\partial \tau}\, d\tau}_{C_2^* \sigma_{kk}'}$$

transform variable.

$$\tilde{e}_{ij} = s\,\tilde{C}_1(s)\,\tilde{S}_{ij}$$

$$\tilde{\varepsilon}_{kk} = s\,\tilde{C}_2(s)\,\tilde{\sigma}_{kk}$$

$$\frac{1}{2G} \longleftrightarrow s\,\tilde{C}_1(s)$$

$$\frac{1}{3K} \longleftrightarrow s\,\tilde{C}_2(s).$$

$$\frac{s3\tilde{C}_2(\tilde{C}_1/2)}{\tilde{C}_2 + (\tilde{C}_1/2)}$$

$$E = \frac{9KG}{3K+G}.$$

$$= \frac{s3\tilde{C}_1\tilde{C}_2}{2\tilde{C}_2+\tilde{C}_1} = s\,\tilde{E}(s)$$

plane strain problem.

$$t_{22} = \frac{F}{A} \to 0.$$

> fundamental sol. for elas.

line force F.

(force per unit length).

in direction of blade.

$$F = \frac{P}{L}.$$

assume infinitely long.

F, G, r, θ. (go away)

$$\sigma = \frac{F}{r} f\left(\cancel{\frac{F}{Gr}}, \theta\right)$$

force per unit length.

$\curvearrowright \pi R$.

plane strain

solution has to be:

$$\sigma = \frac{F}{r} f(\theta) = 0$$

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \frac{1}{2E}(\sigma_{22} + \sigma_{33})$$

$$\frac{\sigma_{11}}{E} - \frac{1}{2E}(\sigma_{11} + \tfrac{1}{2}\sigma_{11}$$

$$= \cancel{\frac{\sigma_{22}}{E} - \frac{1}{2E}(\sigma_{11} + \tfrac{1}{2}\sigma_{22})}$$

$$\Downarrow$$

$$= \sigma_{11}\left[1 - \frac{1}{4}\right] = \frac{1}{2E}\cdot\left[\frac{3}{2}\sigma_{11}\right].$$

$$\frac{3}{4E} = \frac{1}{4G} = \frac{3\sigma_{11}}{4E} - \frac{3\sigma_{22}}{4E}.$$

$$= \frac{1}{4G}\left[\sigma_{11} - \sigma_{22}\right].$$

$\underbrace{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}}$

$$\operatorname{tr}(\varepsilon) = \underbrace{\varepsilon_{11} + \varepsilon_{22} = 0}$$

$\Leftarrow$ incompressibility

$$\underline{\sigma} = \frac{F}{r} \underline{f}(\theta).$$

$$\underline{\varepsilon} = \frac{F}{Gr} \underline{g}(\theta)$$

$$\Downarrow$$

displacement. $\underline{u}$.

$\sigma_{ij} = ?$

$\varepsilon_{ij} = ?$

$t_{22}(x)$

$dF(x') = t_{22}(x')\,dx'$.

$$\sigma_{ij}(x,y) = \frac{F}{\sqrt{x^2+y^2}} f_{ij}\left(\frac{y}{x}\right).$$

$$d\sigma_{ij}(x,y,x') = \frac{t(x')}{\sqrt{(x-x')^2 + y^2}} f_{ij}\left(\frac{y}{x-x'}\right)$$

$$\sigma_{ij}(x,y) = \int_{-\infty}^{\infty} \frac{t(x')\, f_{ij}\left(\frac{y}{x-x'}\right) dx'}{\sqrt{(x-x')^2 + y^2}}.$$

$\Rightarrow$ general solution. $\rightarrow$ you need to find

increase very slow.

$\sigma_{12} = \sigma_{22}$

$(|x| > a, y=0) = 0$

$w(|x| < a, y=0) = 0$

$$\frac{1}{2\pi G}\int_{-a}^{a}\frac{t(x')\,dx'}{x-x'} = 0$$

principal value integrate.

$$\nabla_x = -\frac{1}{2\pi G}\int_{-a}^{a}\frac{t(x')\,dx'}{x-x'} = 0.$$

gradient of disp.

SOLUTION: $t(x) = \dfrac{A}{\sqrt{a^2-x^2}} = \dfrac{F}{\pi\sqrt{a^2-x^2}}$.

HW 11. Q3.

First, ~~substitute~~ given the BCs:

$\sigma_{rr}(r=b) = -p$

Compatibility Eq.

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \quad \cdots (1)$$

Constitutive Eq.

$$\sigma_{ij} = 2G\varepsilon_{ij} + P\delta_{ij} \quad \cdots (2)$$

Equilibrium Eq.

$$\sigma_{ij,j} = 0 \quad \cdots (3)$$

Substitute the BCs into Eqs. (1), (2), (3), we can hence compute the stress (radial).

▷ Taking BG's advice, we take ~

from Eq. (2). $\varepsilon_{ij} = (\sigma_{ij} - P\delta_{ij})/2G$.

From compatibility Eq. (1):

$$\sigma_{rr,\theta\theta} - P\delta_{rr,\theta\theta} + \sigma_{\theta\theta,rr} - P\delta_{\theta\theta,rr} = \sigma_{r\theta,r\theta} - P\delta_{r\theta,r\theta}$$

▷ Equilibrium:

$$\sigma_{r\theta,\theta} + \sigma_{\theta\theta,r} + \sigma_{rr,r} + \sigma_{\theta\theta,\theta} = 0 \ldots (4)$$

▷ Modified compatibility

$$\sigma_{rr,\theta\theta} + \sigma_{\theta\theta,rr} - \sigma_{r\theta,r\theta} = P \ldots (5).$$

↳ ~~Now, Substitute BCs ▷ Nah~~

→ Equilibrium: $\int (\sigma_{\theta r} + \sigma_{rr})\,d\theta - \int (\sigma_{\theta\theta} + \sigma_{r\theta})\,d\theta = 0$

Laplace trans.

$$\begin{cases} \tilde{\sigma}_{ij,i} = 0 \\ \tilde{\varepsilon}_{ij} = (\tilde{u}_{i,j} + \tilde{u}_{j,i})/2. \\ \tilde{\varepsilon}_{kk} = s\tilde{C}_2\tilde{\sigma}_{kk}, \quad \tilde{e}_{ij} = s\tilde{C}_1\tilde{s}_{ij} \end{cases}$$

$$\tilde{\sigma}_{rr}(r=b) = -\frac{P}{s}.$$

$$\frac{d\sigma_{rr}}{dr} + \frac{2}{r}(\sigma_{rr} - \sigma) = 0.$$

Same in Laplace space:

$$\frac{d\tilde{\sigma}_{rr}}{dr} + \frac{2}{r}(\tilde{\sigma}_{rr} - \tilde{\sigma}) = 0.$$

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}.$$

$$\sigma_{ij} = 2G\varepsilon_{ij} + P\delta_{ij}$$

$$\frac{\partial^2 \varepsilon_{ii}}{\partial x_j^2} + \frac{\partial^2 \varepsilon_{jj}}{\partial x_i^2} = 2\frac{\partial^2 \varepsilon_{ij}}{\partial x_i \partial x_j}.$$

$$\begin{cases} \sigma_{ii} = 2G\varepsilon_{ii} + P\delta \\ \sigma_{jj} = 2G\varepsilon_{jj} + P\delta. \\ \sigma_{ij} = 2G\varepsilon_{ij} + P\delta \end{cases}$$

$$\Leftarrow \begin{cases} \varepsilon_{ii} = \dfrac{\sigma_{ii} - P\delta_{ii}}{2G} \\ \varepsilon_{jj} = \dfrac{\sigma_{jj} - P\delta_{jj}}{2G} \\ \varepsilon_{ij} = \dfrac{\sigma_{ij} - P\delta_{ij}}{2G}. \end{cases}$$

spherical: $\dfrac{\partial^2 \tilde{\sigma}_{rr}}{\partial \tilde{\theta}^2} + \dfrac{\partial^2 \tilde{\sigma}_{\theta\theta}}{\partial \tilde{r}^2} = \dfrac{\partial^2 \tilde{\sigma}_{r\theta}}{\partial \tilde{r} \partial \tilde{\theta}}$

$$\rightarrow \frac{\partial^2 \sigma_{ii}}{\partial x_j^2} + \frac{\partial^2 \sigma_{jj}}{\partial x_i^2} = \frac{\partial^2 \sigma_{ij}}{\partial x_i \partial x_j}.$$

in Laplace domain: $\dfrac{\partial^2 \tilde{\sigma}_{ii}}{\partial \tilde{x}_j^2} + \dfrac{\partial^2 \tilde{\sigma}_{jj}}{\partial \tilde{x}_i^2} = \dfrac{\partial^2 \tilde{\sigma}_{ij}}{\partial \tilde{x}_i \partial \tilde{x}_j}$

Nov. 29, Mon. Wk 15.

☆ Exam: Dec. 11. 9am – 9pm.

HW Review

no shear.
$\sigma_{12} = 0$, @ $|y| \to \infty$

$\sigma_{21} = 0$. @ $|x| \to \infty$

☆☆☆ Setup the traction free BCs.

traction free : $\underline{\sigma} \cdot \underline{n} = 0$

$\sigma_{12} = \sigma_{22} = 0$.

△

remember how the setup the traction free BCs

$\underline{n} = \cos\theta \, \underline{e}_1 + \sin\theta \, \underline{e}_2$

☆☆☆ $\underline{n}$ Should always normal to the surface BECAUSE it identifies the surface

$u_3(r < a, z = 0) = -\Delta$

traction free

☆☆☆ Defined in reference configuration.

~~$\sigma_{zr} = \sigma_{\theta r} = 0$.~~

$\sigma_{\theta z} = \sigma_{rz} = 0$.

$\rho = \sqrt{r^2 + z^2}$

$\underline{\sigma}(\rho \to \infty) \to 0$.

frictionless BCs: Cannot take any load.

⇓

☆☆☆ there are no shear

Next page

What is the displacement

What is the contact condition

BC $\sigma_{rz} = \sigma_{\theta z} = \sigma_{zz} = 0, \quad r > a, \; z = 0.$

traction free

inside the contact region: $\sigma_{rz} = \sigma_{\theta z} = 0,$

$r < a, \; z = 0$

$a = \cancel{\sqrt{2R\Delta}}$

$r^2 + (z-R)^2 = R^2$

$2Rz \cong r^2.$

$z \cong \frac{r^2}{2R}$

$z - \frac{r^2}{2R} = -u_z \; (r < a, z > 0)$

$(R-\Delta)^2 + a^2 = R^2$

$R^2 - 2R\Delta + \Delta^2 + a^2 = 0.$

$a^2 \approx 2R\Delta$

$\Delta = \frac{a^2}{2R}.$

wrong

⇓

like a fluid.

<u>Actual Hertz soln.</u> $\Delta = \frac{a^2}{R}.$

$\Delta = \frac{P}{G\pi a} \alpha$ numerical const.

← measure the load F

$$\boxed{\Delta = \left[\frac{9}{16R\,(4G)^2}\right] F^{2/3}}$$

⇓

incompressible solid.

$4G \longrightarrow E^* = \frac{E}{1-\nu^2}.$

shear modulus

$$\sigma_{ij} = \frac{P}{r^2} f_{ij}(\theta, \cancel{\phi})$$

independent of ϕ.

$$4\pi r^2 \sigma_{rr} \approx P$$

$$r \gg 1$$

$$u_i \approx \frac{P}{G r} \hat{u}_i(\theta)$$

Standard Bushness Soln.: $u_3 = \frac{P}{4\pi G} \cdot \frac{2(1-\nu)}{r}$ $(r, z=0)$

$r \to \sqrt{x^2+y^2}$

$$u_3(r, x_0, y_0) = \frac{P(x_0, y_0)}{4\pi G} \frac{2(1-\nu)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$$

line load

$$u_3(r, z=0) = \frac{P}{4\pi G} \frac{2(1-\nu)}{\sqrt{x^2+y^2}}$$

Superposition: $P(x_0, y_0) \Rightarrow p(x_0, y_0)\, dx_0\, dy_0$. distributed load

very small area

$$u_3 = \frac{(1-\nu)}{2\pi G} \iint_A \frac{p(x_0, y_0)\, dx_0\, dy_0}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}$$

wdl.

Nov. 24, Wed., 2021. Wk 13.

Introduction to anistropic Elasticity

Isotropic Elasticity: 2 materials consts

anisotropic:
▷ worst: 21 materials consts.

linear elasticity: $\boxed{\underline{\sigma} = \underline{\underline{K}}\,\underline{\varepsilon}.}$ (1)

Fourth order tensor (stiffness tensor).

$$\underline{\underline{K}} = k_{ijkl}\, \underline{e}_i\, \underline{e}_j\, \underline{e}_k\, \underline{e}_l$$

$$k_{ijkl} = k_{jikl} = k_{jilk}$$

(due to the symmetry of stress & strain tensors).

36 characteristics.

Existence of strain energy density W:

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}$$

$\underline{\underline{K}}$ has 21 independent consts.

$$\sigma_{ij} \longrightarrow \underline{\sigma} = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix}.$$

$$\underline{\sigma} = \sigma_{ij}\, \underline{e}_i\, \underline{e}_j.$$

$$\varepsilon_{ij} \longrightarrow \underline{\varepsilon} = \begin{pmatrix} \varepsilon_{11} = \varepsilon_1 \\ \varepsilon_{22} = \varepsilon_2 \\ \varepsilon_{33} = \varepsilon_3 \\ \varepsilon_{23} = \varepsilon_4 \\ \varepsilon_{13} = \varepsilon_5 \\ \varepsilon_{12} = \varepsilon_6 \end{pmatrix}$$

Eq. (1) $\Leftrightarrow$ $\underline{\sigma} = \underline{\underline{K}}\,\underline{\varepsilon}$

$$K = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ & & k_{33} & k_{34} & k_{35} & k_{36} \\ & & & k_{44} & \cdot & \cdot \\ & & & & k_{55} & \\ & & & & & k_{66} \end{pmatrix}$$

$$\sigma_{11} = k_{1111}\varepsilon_{11} + k_{1112}\varepsilon_{12} + k_{1113}\varepsilon_{13} + k_{1121}\varepsilon_{21} + k_{1122}\varepsilon_{22} + k_{1123}\varepsilon_{23} + k_{1131}\varepsilon_{31} + k_{1132}\varepsilon_{32} + k_{1133}\varepsilon_{33}.$$

$$\sigma_1 = k_{11}\varepsilon_1 + k_{16}\varepsilon_6 + k_{15}\varepsilon_5 + k_{14}\varepsilon_4 + k_{12}\varepsilon_2 + k_{13}\varepsilon_3 \implies \underline{\sigma} = K\underline{\varepsilon}.$$

<u>Plane of symmetry</u> : (material).

anisotropic

2 bases: $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\} = B_1$

$\{\underline{e}_1', \underline{e}_2', \underline{e}_3'\} = B_2$.

(Reflection).

$\underline{e}_1' \to \underline{e}_1$

$\underline{e}_2' = \underline{e}_2$

$\underline{e}_3' = -\underline{e}_3$

$\to k_{ijkl} \mapsto k'_{rstu}$

$\underline{\underline{K}}$ is the same for both basis. $(\underline{B}_0, \underline{B}_0')$.

transform into $\underline{B}'$ basis

$k'_{ijkl} = k_{ijkl}$.

$$k'_{rstu} = k_{ijkl}\underbrace{(\underline{e}_i \cdot \underline{e}_r')}_{P_{ri}}\underbrace{(\underline{e}_j \cdot \underline{e}_s')}_{P_{sj}}\underbrace{(\underline{e}_k \cdot \underline{e}_t')}_{P_{tk}}\underbrace{(\underline{e}_l \cdot \underline{e}_u')}_{P_{lu}}.$$

general transformation formula

P matrix is simple

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

$$k'_{1111} = k_{1111} = k_{ijkl}\underbrace{P_{1i}}_{\delta_{1i}}\underbrace{P_{1j}}_{\delta_{1j}}\underbrace{P_{1k}}_{\delta_{1k}}\underbrace{P_{1l}}_{\delta_{1l}}$$

(Hypothesis)

$$= k_{1111}$$

$$k'_{2222} = k_{2222} = k_{ijkl}P_{1i}P_{2j}P_{2k}P_{2l} = k_{2222}.$$

$$k'_{1122} = k_{ijkl}P_{1i}P_{1j}P_{2k}P_{2l}. = k_{1122}.$$

$$\delta_{1i}\,\delta_{1j}\,\delta_{2k}\,\delta_{2l}.$$

in the same way you can show for all.

$$k'_{1123} = k_{ijkl} P_{1i} P_{1j} P_{2k} P_{3l} = -k_{1123}$$

$\cancel{\delta_{1i}\,\delta_{1j}}$

$\delta_{1i}\,\delta_{1j}\,\delta_{2k}\,(-\delta_{3l})$

we know in prior: $k'_{1123} = k_{1123}$

Hence: $k_{1123} = -k_{1123}$.

$\therefore\ k_{1123} = 0$.

$k_{14} = 0$.

$k_{24} = k_{25} = k_{34} = k_{35} = k_{46} = k_{56} = 0$

reduce the num. of consts. to 8.

$$k = \begin{pmatrix} k_{11} & k_{12} & k_{13} & 0 & 0 & k_{16} \\ & k_{22} & k_{23} & 0 & 0 & k_{26} \\ & & k_{33} & 0 & 0 & k_{36} \\ & & & k_{44} & k_{45} & \cancel{k_{46}}\ 0 \\ & & & & k_{55} & \cancel{k_{56}}\ 0 \\ & & & & & k_{66} \end{pmatrix}$$

13 Mantterial constants.

I already have material plane of symmetry

(3)

$$\sigma_1 = k_{11}\varepsilon_1 + k_{12}\varepsilon_2 + k_{13}\varepsilon_3 + k_{16}\varepsilon_6.$$

$$\sigma_2 = k_{12}\varepsilon_1 + k_{22}\varepsilon_2 + k_{23}\varepsilon_3 + k_{26}\varepsilon_6$$

$$\sigma_3 = k_{13}\varepsilon_3 + k_{23}\varepsilon_2 + k_{33}\varepsilon_3 + k_{36}\varepsilon_6.$$

$$\sigma_4 = k_{44}\varepsilon_4 + k_{45}\varepsilon_5.$$

$$\sigma_5 = k_{54}\varepsilon_4 + k_{55}\varepsilon_5.$$

$$\sigma_6 = k_{61}\varepsilon_1 + k_{62}\varepsilon_2 + k_{63}\varepsilon_3 + k_{66}\varepsilon_6.$$

define a new basis:

$\underline{e}_1' = -\underline{e}_1$

$\underline{e}_2' = \underline{e}_2$.

$\underline{e}_3' = \underline{e}_3$

put an additional plane of symmetry.

$\sigma_{ij} \rightarrow \sigma'_{ij}$

$\varepsilon_{ij} \rightarrow \varepsilon'_{ij}$

$$[\sigma'_{ij}] = \begin{bmatrix} \sigma_{11} & -\sigma_{12} & -\sigma_{13} \\ -\sigma_{12} & \sigma_{22} & \sigma_{23} \\ -\sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}$$

$\sigma_4' = \sigma_4 = k_{44}\varepsilon_4 + k_{45}\varepsilon_5$

$\sigma_4 = k_{44}\varepsilon_4 - k_{45}\varepsilon_5$

$k_{45} = 0 \quad k_{36} = 0$

$k_{16} = 0 \quad k_{26} = 0$

$$K = \begin{pmatrix} k_{11} & k_{12} & k_{13} & 0 & 0 & 0 \\ & k_{22} & k_{23} & 0 & 0 & 0 \\ & & k_{33} & 0 & 0 & 0 \\ & & & k_{44} & 0 & 0 \\ & & & & k_{45} & 0 \\ & & & & & k_{66} \end{pmatrix}$$

HW 11

In torsion rheology test, circular cylinder

R. h $\quad \gamma(t) = \gamma_0 e^{i\omega t}$.

use cylinder coordinate,

only stress exist: $\sigma_{\theta z}$.

Initial condition: $\varepsilon_{ij} = \sigma_{ij} = 0$, $t=0$.

boundary condition:

$$\begin{cases} u_i(r, \theta, z=0, t>0) = 0, \\ u_r(r, \theta, z=h, t>0) = 0, \; u_z(r, \theta, z=h, t>0) = 0, \\ u_\theta(r \le R, \theta, z=h, t>0) = rh\gamma = rh\gamma_0 e^{i\omega t}. \\ \sigma_{rr}(r=R, \theta, 0<z<h, t>0) \; \ldots \\ = \sigma_{zr}(r=R, \theta, 0<z<h, t>0) = 0. \end{cases}$$

(traction free on side walls).

governing Eqs. for torsion:

$u_r = u_z = 0$. $\quad u_\theta = \gamma r z$.

the only non-vanishing strain: $\varepsilon_{z\theta} = \frac{r\gamma}{2}$

In cylindrical coor., all equilibrium satisfied!

* constitutive model: here linear viscoelasticity comes in

$$\sigma_{z\theta}(r, t) = 2G(t)\varepsilon_{z\theta}(r, t=0^+) + 2\int_{0^+}^{t} G(t-\tau)\frac{\partial \varepsilon_{z\theta}(r,\tau)}{\partial \tau}d\tau$$

$$\rightarrow \sigma_{z\theta}(r, t) = G(t)\, r\gamma(t=0^+) + r\int_{0^+}^{t} G(t-\tau)\frac{d\gamma(\tau)}{d\tau}d\tau$$

$$= G(t) r \gamma_0 + r \int_{0^+}^{t} G(t-\tau) \frac{d\gamma_0 e^{i\omega\tau}}{d\tau} d\tau.$$

$$= \left[G(t) + i\omega \int_{0^+}^{t} G(t-\tau) e^{i\omega\tau} d\tau \right] r \equiv \varphi(\omega, t) \gamma_0 r$$

the torque $M(t)$:

$$M(t) = 2\pi \int_0^R \sigma_{\theta z} r^2 dr = \pi \gamma_0 \varphi(\omega, t) \int_0^R r^3 dr$$

$$= \frac{\pi \varphi(\omega, t) R^4 \gamma_0}{2}.$$

1b. $\varphi(\omega, t) = G(t) + i\omega \int_{0^+}^{t} G(t-\tau) e^{i\omega\tau} d\tau.$

as $t \to \infty$,

integral term:

$$\int_{0^+}^{t} G(t-\tau) e^{i\omega\tau} d\tau = e^{i\omega t} \int_0^t G(t-\tau) e^{-i\omega(t-\tau)} d\tau$$

$$= e^{i\omega t} \int_0^t G(\eta) e^{-i\omega\eta} d\eta = e^{i\omega t} \left[\int_0^t [G(\eta) - G_\infty] e^{-i\omega\eta} d\eta + \int_0^t G_\infty e^{-i\omega\eta} d\eta \right]$$

$$= e^{i\omega t} \left[\int_0^t [G(\eta) - G_\infty] e^{-i\omega\eta} d\eta + \frac{G_\infty}{-i\omega} e^{-i\omega\eta} \Big|_0^t \right]$$

$$= e^{i\omega t} \int_0^t [G(\eta) - G_\infty] e^{-i\omega\eta} d\eta + \frac{G_\infty}{i\omega} (e^{i\omega t} - 1)$$

then we have φ:

$$\varphi(\omega, t) = G(t) + i\omega \left[e^{i\omega t} \int_0^t [G(\eta) - G_\infty] e^{-i\omega\eta} d\eta + \frac{G_\infty}{i\omega} (e^{i\omega t} - 1) \right]$$

$$= (G(t) - G_\infty) + G_\infty e^{i\omega t} + i\omega e^{i\omega t} \int_0^t [G(\eta) - G_\infty] e^{-i\omega\eta} d\eta$$

We already know:

$$\int_0^t [G(\eta) - G_\infty] e^{-i\omega\eta} d\eta$$

$$= \int_0^\infty [G(\eta) - G_\infty] e^{-i\omega\eta} d\eta - \int_t^\infty [G(\eta) - G_\infty] e^{-i\omega\eta} d\eta$$

$$\varphi(\omega, t) = \left[(G(t) - G_\infty) - i\omega e^{i\omega t} \int_t^\infty [G(\eta) - G_\infty] e^{-i\omega\eta} d\eta \right] + \left\{ G_\infty + i\omega \int_0^\infty [G(\eta) - G_\infty] e^{-i\omega\eta} d\eta \right\} e^{i\omega t}$$

since $G(\eta \to \infty) - G_\infty = 0$.

$$\varphi(\omega, t \to \infty) = \left\{ G_\infty + i\omega \int_0^\infty [G(\eta) - G_\infty] e^{-i\omega\eta} d\eta \right\} e^{i\omega t}$$

$$= \mu(\omega) e^{i\omega t}$$

$$M_{ss}(\omega) = \frac{\pi R^4 \gamma_0}{2} \mu(\omega) e^{i\omega t}.$$

Assuming $G(t) = G_\infty + \dfrac{G_0 - G_\infty}{(1+\frac{t}{t_R})^n}$, find storage & loss modulus.

- storage modulus:

$$\mu'(\omega) = Re[\mu(\omega)] = Re\left[G_\infty + i\omega\int_0^\infty \frac{G_0 - G_\infty}{(1+\eta/t_R)^n} e^{-i\omega\eta} d\eta\right]$$

$$= G_\infty + (G_0 - G_\infty)\,\omega\, Re\left[i\int_0^\infty (1+\eta/t_R)^{-n} e^{-i\omega\eta} d\eta\right]$$

- loss modulus:

$$\mu''(\omega) = Im[\mu(\omega)]$$

$$= (G_0 - G_\infty)\,\omega\, Im\left[i\int_0^\infty (1+\eta/t_R)^{-n} e^{-i\omega\eta} d\eta\right]$$

to evaluate the integrals, let $\eta/t_R = p$, so that

$$\int_0^\infty (1+\eta/t_R)^{-n} e^{-i\omega\eta}\, d\eta$$

$$= t_R\int_0^\infty (1+p)^{-n} e^{-i\Omega p}\, dp.$$

For our case, $n=1$.

$$\int_0^\infty (1+p)^{-1} e^{-i\Omega p}\, dp = \int_0^\infty (1+p)^{-1}\cos(\Omega p)\, dp$$

$$-i\int_0^\infty (1+p)^{-1}\sin(\Omega p)\, dp.$$

$$= \int_0^\infty (\Omega+p)^{-1}\cos q\, dq - i\int_0^\infty (\Omega+q)^{-1}\sin q\, dp.$$

$$= \{-Ci(\Omega)\cos\Omega - Si(\Omega)\sin\Omega\} - i\{Ci(\Omega)\sin\Omega - Si(\Omega)\cos\Omega\}.$$

Ci & Si : sine and cosine integrals.

normalize the storage & loss modulus:

$$\mu'(\omega) = 1 + \left(\frac{G_0}{G_\infty} - 1\right)\Omega\, Re\left[i\int_0^\infty (1+p)^{-n} e^{-i\Omega p}\, dp\right]$$

$$= 1 + \left(\frac{G_0}{G_\infty} - 1\right)\Omega\{Ci(\Omega)\sin\Omega - Si(\Omega)\cos\Omega\}.$$

$$\mu''(\omega) = \left(\frac{G_0}{G_\infty} - 1\right)\Omega\, Im\left[i\int_0^\infty (1+p)^{-n} e^{-i\Omega p}\, dp\right]$$

$$= -\left(\frac{G_0}{G_\infty} - 1\right)\Omega\{Ci(\Omega)\cos\Omega + Si(\Omega)\sin\Omega\}.$$

$$\therefore \tan\delta = \frac{-\left(\frac{G_0}{G_\infty} - 1\right)\Omega\{Ci(\Omega)\cos\Omega + Si(\Omega)\sin\Omega\}}{1 + \left(\frac{G_0}{G_\infty} - 1\right)\Omega\{Ci(\Omega)\sin\Omega - Si(\Omega)\cos\Omega\}}$$

Dec. 6., Mon, 2021. Wk 16.

orthotropic material. 9 constants.

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & 0 & 0 & 0 \\ & k_{22} & k_{23} & & & \\ & & k_{33} & & & \\ & & & k_{44} & & \\ & & & & k_{55} & \\ & & & & & k_{66} \end{bmatrix}$$

<u>Transversely Isotropic</u>

$\underline{e}_3$ is an axis of symmetry.

every plane containing this axis is a plane of reflection symmetry.

invariant to Rotation above this axis.

$$[\underline{\underline{R}}] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{cases} \underline{e}_1' = \cos\theta\, \underline{e}_1 + \sin\theta\, \underline{e}_2 \\ \underline{e}_2' = -\sin\theta\, \underline{e}_1 + \cos\theta\, \underline{e}_2 \\ \underline{e}_3' = \underline{e}_3 \end{cases}$$

$$[\underline{\underline{P}}] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

standard isotropic material.

$$\begin{cases} \sigma_1 = k_{11}\varepsilon_1 + k_{12}\varepsilon_2 + k_{13}\varepsilon_3 \\ \sigma_2 = k_{21}\varepsilon_1 + k_{22}\varepsilon_2 + k_{23}\varepsilon_3 \\ \sigma_3 = k_{31}\varepsilon_1 + k_{32}\varepsilon_2 + k_{33}\varepsilon_3 \\ \sigma_4 = k_{44}\varepsilon_4 \\ \sigma_5 = k_{55}\varepsilon_5 \\ \sigma_6 = k_{66}\varepsilon_6 \end{cases}$$

$$\underline{\underline{\sigma}}' = \underline{\underline{P}}\,\underline{\underline{\sigma}}\,\underline{\underline{P}}^T.$$

$$\underline{\underline{\varepsilon}}' = \underline{\underline{P}}\,\underline{\underline{\varepsilon}}\,\underline{\underline{P}}^T.$$

$$\begin{cases} \sigma_1' = \sigma_2 \quad \sigma_2' = \sigma_1 \quad \sigma_3' = \sigma_3 \\ \sigma_4' = -\sigma_5 \quad \sigma_5' = \sigma_4 \\ \sigma_6' = -\sigma_6 \end{cases}$$

~~Smat~~ same thing for strain.

$$\sigma_1' = k_{11}\varepsilon_1' + k_{12}\varepsilon_2' + k_{13}\varepsilon_3'$$

$$\Downarrow$$

$$\rightarrow \sigma_2 = k_{11}\varepsilon_2' + k_{12}\varepsilon_1 + k_{13}\varepsilon_3'$$

$$\sigma_2' = k_{21}\varepsilon_1' + k_{22}\varepsilon_2' + k_{23}\varepsilon_3$$

$$\sigma_2' = k_{21}\varepsilon_1' + k_{22}\varepsilon_2' + k_{23}\varepsilon_3'.$$

$$\sigma_1 = k_{21}\varepsilon_2 + k_{22}\varepsilon_1 + k_{23}\varepsilon_3$$

$\sigma_1 = k_{12}\varepsilon_1 + k_{22}\varepsilon_2 + k_{23}\varepsilon_3$

$k_{12}\varepsilon_2 + k_{22}\varepsilon_1 = k_{11}\varepsilon_1 + k_{12}\varepsilon_2.$

~~$k_{11}\varepsilon_2 = k_{22}\varepsilon_2$~~.

$k_{11}\varepsilon_2 + k_{13}\varepsilon_3 = k_{22}\varepsilon_2 + k_{23}\varepsilon_3.$

$(k_{11} - k_{22})\varepsilon_2 + (k_{13} - k_{23})\varepsilon_3 = 0.$

$k_{11} = k_{22}$

$k_{13} = k_{23}.$

$\sigma_4 = k_{44}\varepsilon_4.$

$\sigma_4' = k_{44}\varepsilon_4'$

$\sigma_5' = k_{55}\varepsilon_5'.$

$\sigma_4' = -\sigma_5. = -k_{44}\varepsilon_5'.$

$\vee$

$\sigma_5 = k_{44}\varepsilon_5'$

$k_{44} = k_{55}$

$k_{44} = k_{55}. \Rightarrow \left.\begin{array}{l} k_{11} = k_{22}. \\ k_{13} = k_{23}. \end{array}\right\}.$

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & 0 & 0 & 0 \\ & k_{11} & k_{13} & & & \\ & & k_{23} & & & \\ & & & k_{44} & & \\ & & & & k_{55} & \\ & & & & & k_{66} \end{bmatrix}$$

$\theta = 45^\circ \rightarrow \pi/4.$

$$[P] = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\sigma_{12}', \quad \varepsilon_{12}'.$

calculation. this has to be true.

$k_{66} = \frac{1}{2}(k_{11} - k_{12}).$

$\frac{1}{2}[k_{11} - k_{12}]$

$\underline{\underline{k}}^{-1} \cdot \underline{\sigma} = \underline{\underline{\varepsilon}}.$

S matrix compliance index

Poisson's ratio for anisotropic elastic material. Can have no bounds. Trig. TCT.

Linear viscoelasticity

- correspondence principle

↳ stress dependent of modulus

$$e_{ij} = S_{ij}(0^+)\, C_1(t) + \int_{0^+}^{t} C_1(t-\tau)\, \frac{\partial S_{ij}}{\partial \tau}\, d\tau .$$

$$\varepsilon_{kk} = \sigma_{kk}(0^+)\, C_2(t) + \int_{0^+}^{t} C_2(t-\tau)\, \frac{\partial \sigma_{kk}}{\partial \tau}\, d\tau .$$

creep modulus.

↳ remote hydrostatic tension.

$P_\infty(t<0) = 0$

Elastic problem:

Plane stress

$$\boxed{\begin{cases} \sigma_{rr} = \dfrac{A}{r^2} + P_\infty \\ \sigma_{\theta\theta} = -\dfrac{A}{r^2} + P_\infty . \qquad r>A \\ \sigma_{r\theta} = 0 \end{cases}}$$

$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{in}$.

$\Downarrow$ $r<A$.

Continuity of traction.

$$\frac{A}{r^2} + P_\infty = \sigma_{in}.$$

$A, P_\infty \to$ unknown.

$$3G_{out}\, \varepsilon_{\theta\theta} = \sigma_{\theta\theta} - \frac{1}{2}\sigma_{rr} . \quad r>a.$$

$$\underset{(r>a)}{\varepsilon_{\theta\theta}} = \frac{1}{3G_{out}}\left[\frac{-3A}{2r^2} + \frac{P_\infty}{2}\right].$$

$\Downarrow$

$$\varepsilon_{\theta\theta} = \frac{u}{r}$$

$r<a$, $\varepsilon_{in} = \varepsilon_{\theta\theta} = \varepsilon_{rr} = \dfrac{u}{r}$.

$$u = \varepsilon_{in}\, r$$

$$3G_{in}\,\varepsilon_{in} = \sigma_{\theta\theta} - \frac{1}{2}\sigma_{rr}.$$

$$= \sigma_{in}/2.$$

$$\therefore \varepsilon_{in} = \frac{\sigma_{in}}{6G_{in}} = \frac{u}{r}.$$

Continuity of Hooke's strain.

$$\frac{1}{2G_{out}}\left[\frac{-3A}{2a^2} + \frac{P_\infty}{2}\right] = \frac{\sigma_{in}}{6G_{in}}.$$

$$\boxed{\begin{aligned} \frac{A}{a^2} &= \frac{(\rho-1)P_\infty}{(1+3\rho)}, \quad \rho = \frac{G_{in}}{G_{out}}. \\ \sigma_{in} &= \frac{4\rho}{1+3\rho} P_\infty \end{aligned}}$$

$$\frac{G_{in}}{G_\infty} \to \infty \Rightarrow \underline{\frac{A}{a^2} = \frac{1}{3}P_\infty}$$

$$\sigma_{rr}(r = a+) = -\frac{2}{3}P_\infty.$$

$$G_{in} = \frac{1}{2} s \tilde{Y}_{in}(s). \quad ☆☆☆$$

$$G_{out} = \frac{1}{2} s \tilde{Y}_{out}(s).$$

$$\tilde{\sigma}_{in} = \frac{4\dfrac{\tilde{Y}_{in}(s)}{\tilde{Y}_{out}(s)}}{1 + 3\dfrac{\tilde{Y}_{in}(s)}{\tilde{Y}_{out}(s)}} \tilde{P}_\infty(s)$$

$$Y_{in}(t) = Y_{\infty in} + (Y_{0in} - Y_{\infty in})e^{-t/t_{in}}$$

$$\mathcal{L}(Y_{in}(t)) = \tilde{Y}_i(s) = \int_0^\infty e^{-st} Y_i(t)\,dt.$$

$$\tilde{Y}_{in}(s) = \frac{Y_{in\infty}}{s} + \frac{(Y_{in0} - Y_{in\infty})}{s + t/t_{in}}$$

$$\tilde{Y}_{out}(s) = \frac{Y_{out\infty}}{s} + \frac{Y_{out0} - Y_{out\infty}}{s + t/t_{out}}$$

$$\sigma_{in}(t) = \frac{1}{2\pi i}\int_{\delta-\infty}^{\delta+\infty} e^{st}\,\tilde{\sigma}_\infty(t)\,ds.$$

☆ Solve ODE with MATLAB.

HW 10. Review:

a. Problem formulation:

$\nabla^2 \phi = 0$. in $|x| < \frac{a}{2}$ & $|y| < \frac{b}{2}$.

BCs:

$\phi(x = \pm a/2, |y| < \frac{b}{2}) = \frac{1}{2}(\frac{a^2}{4} + y^2)$.

$\phi(|x| < \frac{a}{2}, y = \pm \frac{b}{2}) = \frac{1}{2}(\frac{b^2}{4} + x^2)$

b. $f = \nabla_{xx} \phi + 1$.

on the boundary $y = \pm \frac{b}{2}$,

$\Rightarrow f(|x| < a/2, y = \pm \frac{b}{2}) = 2$.

on the boundary $x = \pm \frac{a}{2}$.

we know $\partial_{xx} \phi = -\partial_{yy} \phi$.

$\Rightarrow f(x = \pm \frac{a}{2}, |y| < \frac{b}{2}) = 0$.

c. find f:

$f(x,y) = X(x) Y(y)$.

Substitute into $\nabla^2 f = 0$:

$\ddot{X}(x) Y(y) + X(x) \ddot{Y}(y) = 0$.

$\Rightarrow \frac{\ddot{X}(x)}{X(x)} = -\frac{\ddot{Y}(y)}{Y(y)} = C = -k^2$.

we look for solution: satisfy $x = \pm \frac{a}{2}$.

$$\begin{cases} \ddot{X}(x) + k^2 X(x) = 0. \\ \ddot{Y}(y) - k^2 Y(y) = 0 \end{cases}$$

$$\begin{cases} X(x) = B \sin kx + A \cos kx. \xrightarrow{x = \pm \frac{a}{2},\ X = 0} \\ Y(y) = C \cosh k_n y + D \sinh k_n y. \end{cases}$$

$f(x,y) = \sum_i a_n \cos(k_n x) \cosh(k_n y)$.

BCs: $f(x < \frac{a}{2}, y = \pm \frac{b}{2}) = 2$.

$\sum G_n \cos(k_n x) \cosh(k_n b/2) = 2$.

method of Fourier series:

$$a_n = \frac{2}{a \cosh(k_n b/2)} \int_{-a/2}^{a/2} 2 \cos(k_n x) dx$$

$$= \frac{8(-1)^n}{\pi(2n+1)\cosh(k_n b/2)}.$$

Thus:

$$f(x,y) = \frac{8}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cdot\frac{\cosh(k_n y)}{\cosh(k_n b/2)}\cos(k_n x),$$

$$k_n = \frac{2n+1}{a}\pi$$

d: find max stress:

Shear stresses:

$$\begin{cases} \sigma_{13} = G\gamma(-y + \phi_{,2}) \\ \sigma_{23} = G\gamma(x - \phi_{,1}) \\ \dfrac{\sigma_{13}}{G\gamma} + y = \phi_{,2} \\ -\dfrac{\sigma_{23}}{G\gamma} + x = \phi_{,1} \end{cases}$$

On the boundary $x = \pm\frac{a}{2}$, $\phi_{,2}\Big|_{x=\pm\frac{a}{2}} = y$.

$\sigma_{13} = 0$ on $x = \pm\frac{a}{2}$.

$\sigma_{23} = 0$ on $y = \pm\frac{b}{2}$.

$\phi_{,1}\Big|_{y=\pm\frac{b}{2}} = x$.