MAE 6110: HW #9

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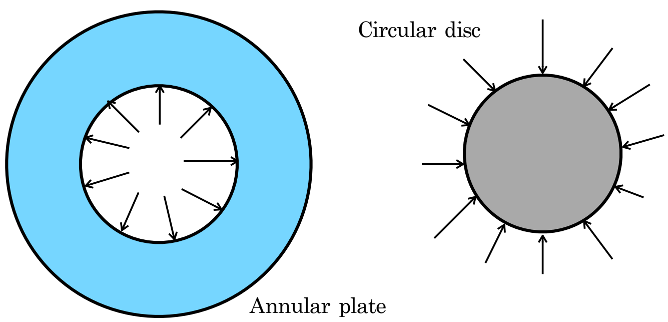
1. A residue stress problem (Shrink-fit problem):

The radius of a circular disc is $a + \epsilon$, where $\epsilon > 0$ and is much less than a. We want to fit this disc inside an annular plate with outer radius b and inner radius a. This can be done, for example, by cooling the disc so it shrink to radius a, for example, then slip the disc inside the hole in the plate). Assuming that this can be done, and the disc and the plate are now at the same temperature, what are the stresses in the plate and disc? You may assume that the plate and the disc are linearly elastic, isotropic and homogeneous, with Young's modulus and Poisson's ratio E_1, ν_1, for the plate and E_2, ν_2, for the disc respectively. Assume plane stress condition holds.

Hint 1: the problem has obvious symmetry; it is much easier to solve using polar coordinates (some strain components are zero). However, we have to write the strains and stress (including the equilibrium equations) in polar coordinates.

Hint 2: Think of two problems, one in which the disc is subjected to a pressure loading on its boundary, and the other the hole is subjected to pressure loading.



Solution: We first assume the outer radius of circular disc is u_c, and the inner radius of the annular plate is u_a; from the instructions, we can write out the **boundary conditions**:

$$\begin{aligned} u_{r(a)} - u_{r(c)} = \epsilon, &\quad \text{(in } r \text{ direction)} \\ \sigma_{rr} = 0, &\quad \text{(when } r = b) \end{aligned} \tag{1}$$

We can write out the contact pressure p in circular disc (stresses boundary conditions), taking the form:

$$(\sigma_{rr})_{r=a} = -p, \tag{2}$$

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With the two displacements, we can calculate the strains and stresses from **kinematics** and **constitutive laws** (*Hooke's law*)

$$\begin{aligned}\epsilon_{rr} &= \frac{\partial u_r}{\partial r} = \frac{1}{E}(\sigma_{rr} - \nu\sigma_{\theta\theta}) \\ \epsilon_{\theta\theta} &= \frac{u_r}{r} = \frac{1}{E}(\sigma_{\theta\theta} - \nu\sigma_{rr})\end{aligned} \tag{3}$$

Based on the linear elasticity constitutive model (*Hooke's law*) in plane stress assumption ($\sigma_{r\theta} = 0$), the stresses can be written as

$$\begin{aligned}\sigma_{rr} &= \frac{E}{1-\nu^2}\left(\frac{du_r}{dr} + \nu\frac{u_r}{r}\right) \\ \sigma_{\theta\theta} &= \frac{E}{1-\nu^2}\left(\frac{u_r}{r} + \nu\frac{du_r}{dr}\right)\end{aligned} \tag{4}$$

In the absence of radial body forces, polar **equation of equilibrium** reduces to

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \tag{5}$$

Substitute Equation (4) into Equation (5) we generate:

$$\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = 0 \tag{6}$$

where Equation (6) has the general solution

$$u_r = C_1 r + \frac{C_2}{r} \tag{7}$$

For the circular disc, as $r \to 0$, we should expect $u_r \to 0$, therefore the displacement function writes

$$u_{r(c)} = C_3 r \tag{8}$$

And for the annular plate

$$u_{r(a)} = C_1 r + \frac{C_2}{r} \tag{9}$$

In order to solve for Equations (8) and (9), we need three established equations, which are the displacement boundary condition on the contact interface, the traction free boundary condition on the outer plate surface and stresses boundary on the contact surfaces {Equation (1) *&* Equation (2)}. Substituting Equations (8), (9) to the BCs we can compute the three constants with three equations

$$\begin{aligned}C_1 a + \frac{C_2}{a} - C_3 a &= \epsilon \\ \frac{E_1}{1-\nu_1^2}\left(C_1 - \frac{C_2}{b^2} + \nu_1 C_3\right) &= 0 \\ \frac{E_1}{1-\nu_1^2}\left(C_1 - \frac{C_2}{a^2} + \nu_1 C_3\right) &= p \\ \frac{E_2}{1-\nu_2^2}\left(C_3 + \nu_2 C_3\right) &= -p\end{aligned} \tag{10}$$

With four given equations, we need to solve four parameters C_1, C_2, C_3, p. Solving these equations with MATLAB® we have:

```
clear;clc;
syms C1 C2 C3 E1 E2 v1 v2 a p b epsilon
Const1 = E1/(1 - v1^2);
Const2 = E2/(1 - v2^2);
eq1 = C1 * a + (C2/a) - C3 * a == epsilon;
eq2 = Const1 * (C1 - (C2/b^2) + v1*C3) == 0;
eq3 = Const1 * (C1 - (C2/a^2) + v1*C3) == p;
[C1, C2, C3] = solve([eq1,eq2,eq3],[C1,C2,C3])
eqn = Const2 * (C3 + v2*C3) == -p;
p = solve(eqn, p)
```

We eventually obtain the solution for p:

$$p = \frac{E_2 \left(\frac{E_1\, a^2\, \varepsilon - E_1\, b^2\, \varepsilon}{E_1\, (a^2-b^2)\,(a+a\,\nu_1)} + \frac{\nu_2 \left(E_1\, a^2\, \varepsilon - E_1\, b^2\, \varepsilon\right)}{E_1\, (a^2-b^2)\,(a+a\,\nu_1)}\right)}{({\nu_2}^2-1)\left(\frac{E_2\left(\frac{-a^3\,{\nu_1}^2+a^3-a\,b^2\,{\nu_1}^2+a\,b^2}{E_1\,(a^2-b^2)\,(a+a\,\nu_1)}+\frac{\nu_2\,(-a^3\,{\nu_1}^2+a^3-a\,b^2\,{\nu_1}^2+a\,b^2)}{E_1\,(a^2-b^2)\,(a+a\,\nu_1)}\right)}{{\nu_2}^2-1}-1\right)} \tag{11}$$

Further simplification we can write out p:

$$p = \frac{E_1 E_2 \epsilon (a^2 - b^2)(1+\nu_2)}{a E_2 (\nu_2 + 1)(1-\nu_1^2)(a^2+b^2) - (\nu_2^2 - 1) E_1 a (a^2 - b^2)(1+\nu_1)} \tag{12}$$

(*Obtained from* MATLAB *generated output*)

With this given p we can resolve this equation

```
clear;clc;
syms C1 C2 C3 E1 E2 v1 v2 a b epsilon
Const1 = E1/(1 - v1^2);
Const2 = E2/(1 - v2^2);
p =(E2*((E1*epsilon*a^2 - E1*epsilon*b^2)/(E1*(a^2 - b^2)*(a + a*v1))...
    + (v2*(E1*epsilon*a^2 - E1*epsilon*b^2))/(E1*(a^2 - b^2)*(a + a*v1))))/((v2^2 - 1)*((E2*((- a^3*v1^2 + a^3 - a*b^2*v1^2 + a*b^2)/(E1*(a^2 - b^2)*(a + a*v1)) + (v2*(- a^3*v1^2 + a^3 - a*b^2*v1^2 + a*b^2))/(E1*(a^2 - b^2)*(a + a*v1))))/(v2^2 - 1) - 1))
eq1 = C1 * a + (C2/a) - C3 * a == epsilon;
eq2 = Const1 * (C1 - (C2/b^2) + v1*C3) == 0;
eq3 = Const1 * (C1 - (C2/a^2) + v1*C3) == p;
[C1, C2, C3] = solve([eq1,eq2,eq3],[C1,C2,C3])
```

We therefore obtain the three constants C_1, C_2, C_3:

$$\begin{aligned} C_1 &= \frac{\varepsilon\,(E_2\,a^2 - E_2\,a^2\,{\nu_1}^2 + E_1\,a^2\,\nu_1 - E_1\,b^2\,\nu_1 - E_1\,a^2\,\nu_1\,\nu_2 + E_1\,b^2\,\nu_1\,\nu_2)}{(a + a\,\nu_1)\,(E_1\,a^2 + E_2\,a^2 - E_1\,b^2 + E_2\,b^2 - E_1\,a^2\,\nu_2 - E_2\,a^2\,\nu_1 + E_1\,b^2\,\nu_2 - E_2\,b^2\,\nu_1)} \\ C_2 &= -\frac{E_2\,a\,b^2\,\varepsilon\,(\nu_1 - 1)}{E_1\,a^2 + E_2\,a^2 - E_1\,b^2 + E_2\,b^2 - E_1\,a^2\,\nu_2 - E_2\,a^2\,\nu_1 + E_1\,b^2\,\nu_2 - E_2\,b^2\,\nu_1} \\ C_3 &= -\frac{\varepsilon\,(E_1\,a^2 - E_1\,b^2 - E_1\,a^2\,\nu_2 + E_1\,b^2\,\nu_2)}{(a + a\,\nu_1)\,(E_1\,a^2 + E_2\,a^2 - E_1\,b^2 + E_2\,b^2 - E_1\,a^2\,\nu_2 - E_2\,a^2\,\nu_1 + E_1\,b^2\,\nu_2 - E_2\,b^2\,\nu_1)} \end{aligned} \tag{13}$$

Substitute back the obtained C_1, C_2, C_3 (from MATLAB) into Equations (8) *&* (9); the stress fields can be solved, for circular disc:

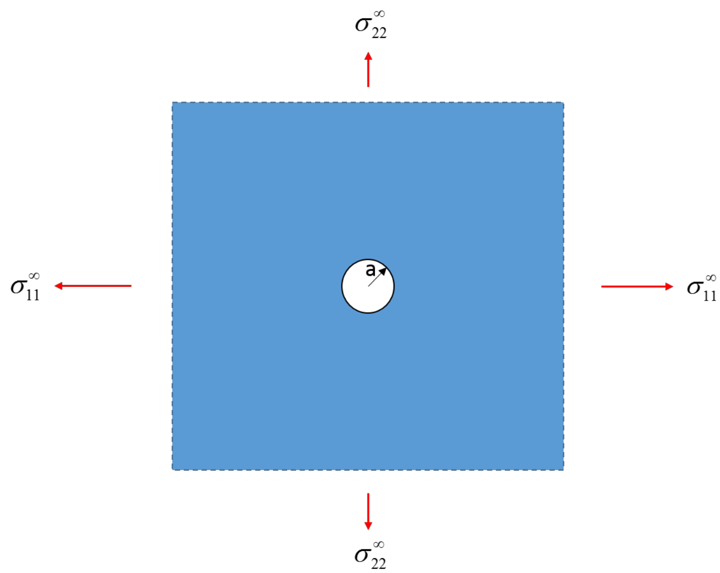
$$\begin{aligned} \sigma_{rr} &= \frac{E}{1-\nu_2^2}(1+\nu_2)C_3 \\ \sigma_{\theta\theta} &= \frac{E}{1-\nu_2^2}\,(1+\nu_2)\,C_3 \end{aligned} \tag{14}$$

For annular plate:

$$\begin{aligned}\sigma_{rr} &= C_1(1+\nu_1) + \frac{C_2}{r^2}(1-\nu_1)\\ \sigma_{\theta\theta} &= C_1(1+\nu_1) + \frac{C_2}{r^2}(1-\nu_1)\end{aligned} \tag{15}$$

With corresponding C_1 and C_2 as shown in Equation (13).

2. Make a model for the following problems using linear elasticity, write down the governing equation(s) you would solve and the appropriate boundary conditions. Use your intuition to guess at least one feature of the solution. DO NOT solve these equations.



a. A large plate with a hole of radius a loaded by remote biaxial tension, denoted by σ_{11}^{∞}, σ_{22}^{∞}, respectively.

Solution: In this problem, since the overall shape of the large plate is a square, therefore it is easier to formulate the problem under Cartesian coordinate. We first write out the **boundary conditions** on the edges, where there are bi-axial stresses loaded on the twos sides:

$$\begin{aligned}\sigma_{11} = \sigma_{11}^{\infty}, &\quad |x_1| \longrightarrow \infty\\ \sigma_{21} = 0, &\quad |x_1| \longrightarrow \infty\\ \sigma_{22} = \sigma_{22}^{\infty}, &\quad |x_2| \longrightarrow \infty\\ \sigma_{12} = 0, &\quad |x_2| \longrightarrow \infty\end{aligned} \tag{16}$$

We can also write out the **boundary conditions** on the hole, where the traction free conditions are obeyed:

$$\begin{aligned}\sigma_{rr}(\theta, r=a) &= \sigma_{11}^{\infty}\cos^2\theta + \sigma_{22}^{\infty}\sin^2\theta\\ \sigma_{\theta\theta}(\theta, r=a) &= \sigma_{11}^{\infty}\sin^2\theta + \sigma_{22}^{\infty}\cos^2\theta\\ \sigma_{r\theta}(\theta, r=a) &= (\sigma_{22}^{\infty} - \sigma_{11}^{\infty})\sin\theta\cos\theta\end{aligned} \tag{17}$$

The **constitutive model** (*Hooke's Law*) for linear elasticity in plane stress in matrix form:

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ 2\epsilon_{12}\end{bmatrix} \tag{18}$$

Kinematics tell us that

$$\epsilon_{ij} = \frac{u_{i,j} + u_{j,i}}{2}, \quad (i = 1, 2) \tag{19}$$

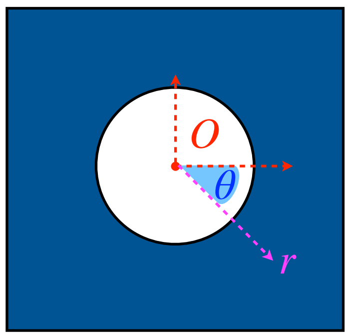
which can be further expanded to

$$\begin{aligned} \epsilon_{11} &= \frac{\partial u_1}{\partial x_1}, \ \epsilon_{22} = \frac{\partial u_2}{\partial x_2} \\ \epsilon_{12} &= \epsilon_{21} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) \end{aligned} \tag{20}$$

Since there are no body forces, the **equation of equilibrium** is $\sigma_{ij,j} = 0$, which can expanded to:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{22}}{\partial x_2} = 0 \tag{21}$$

Solving for Equations (18), (20) and (21), and substitute the terms into Equations (16) and (**??**) (BCs) we can thence solve the problem.



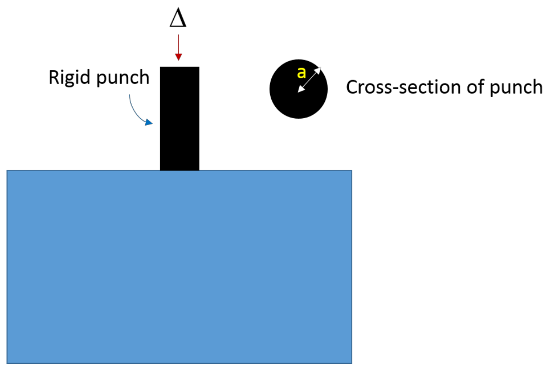
Here, if $\sigma_{11}^{\infty} \neq \sigma_{22}^{\infty}$, then this is not a asymmetric problem, then the effect of θ must be taken into consideration (above figure). Based on our boundary conditions in Equation (17), we can guess that the final solution of the stresses will look like some thing like

$$\begin{aligned} \sigma_{rr} &= f_1(r,\theta)\cos^2\theta + g_1(r,\theta)\sin^2\theta \\ \sigma_{\theta\theta} &= f_2(r,\theta)\cos^2\theta + g_2(r,\theta)\sin^2\theta \\ \sigma_{r\theta} &= f_3(r,\theta)\sin\theta\cos\theta \end{aligned} \tag{22}$$

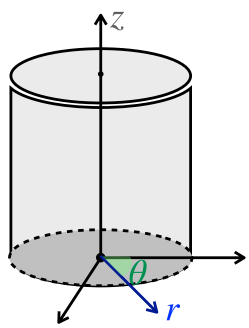
which can be further written to

$$\begin{aligned} \sigma_{rr} &= \cos 2\theta f(r,\theta) \\ \sigma_{\theta\theta} &= \cos 2\theta g(r,\theta) \\ \sigma_{r\theta} &= \sin 2\theta h(r,\theta) \end{aligned} \tag{23}$$

b. A large block of elastic solid indented by a rigid circular flat punch of radius a. Assume frictionless contact.



Solution: For this 3D problem, we first apply the cylindrical coordinates. Due to the symmetrical nature of the problem, we can further reduce this 3D problem to 2D: $(r, \theta, z) \Longrightarrow (r, z)$, as shown in the following figure.



In this case, the **equilibrium equation** under cylindrical coordinate turns to

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr}}{z} = 0 \\ \frac{\partial \sigma_{rz}}{\partial z} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0 \end{aligned} \tag{24}$$

On the boundary surface excluding the contacting area of the punch surface, the **boundary conditions** write:

$$\begin{aligned} \sigma_{zz}(r > a, z = 0) = 0,\ \sigma_{rr}(r > a, z = 0) = 0 \\ \sigma_{rz}(z \neq 0, r = a) = 0, \quad \text{(contact surface)} \\ -u_z(r, z = 0) = \Delta \end{aligned} \tag{25}$$

The **constitutive model** for elastic solid agrees with the Hooke's law:

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{rz} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{rr} \\ \epsilon_{zz} \\ 2\epsilon_{rz} \end{bmatrix} \tag{26}$$

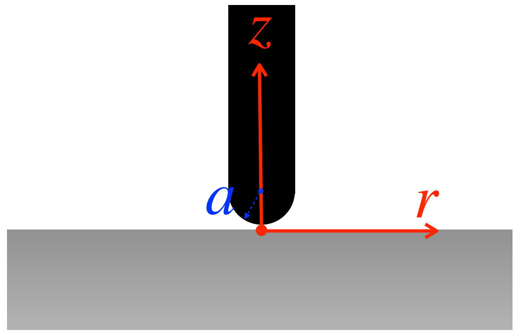
And **kinematics** tell us that

$$\begin{aligned} \epsilon_{rr} = \frac{\partial u_r}{\partial x_r},\ \epsilon_{zz} = \frac{\partial u_z}{\partial x_z} \\ \epsilon_{rz} = \epsilon_{zr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial x_z} + \frac{\partial u_z}{\partial x_r}\right) \end{aligned} \tag{27}$$

The solution of the stresses will possess the features of

$$\sigma_{ij} = \frac{G\Delta}{a} f(\frac{r}{a}, 0) \tag{28}$$

c. Same problem as 2b, but profile of punch is spherical, with radius a, assume contact is frictionless.



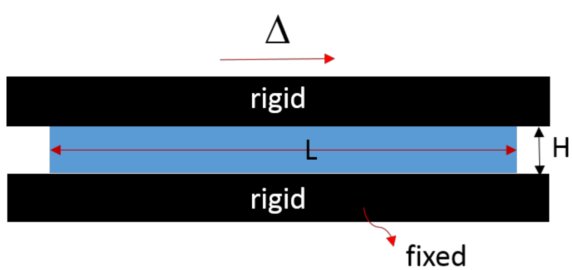
Solution: For this problem, most equations are same as 2b, expect the **boundary conditions** are slightly different:

$$\begin{aligned} \sigma_{zz}(r > a, z = 0) = 0, \;\; \sigma_{rr}(r > a, z = 0) = 0 \\ \sigma_{rz}(z > a, r = a) = 0, \quad \text{(contact surface)} \\ -u_z(r, z = 0) = \Delta \end{aligned} \tag{29}$$

We can hence make a guess of the stresses

$$\sigma_{ij} = G\Delta f(r) \tag{30}$$

d. A block of elastic solid subjected to shear, as shown in figure below. The depth of block in the out of plane direction can be considered as infinite in comparison with L and H.



Solution: For this problem, due to the rigid bodies are very long (vertical to the screen direction); we can take the **plane strain** assumption. We set the coordinate center at the exact center of the geometry before deformation.

We first write out the **constitutive equation** for plane strain:

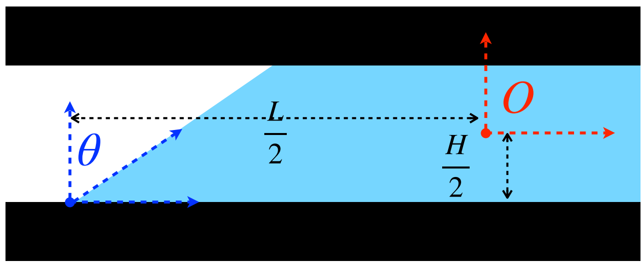
$$\begin{aligned} \epsilon_{11} = \frac{1}{E}\left((1-\nu^2)\sigma_{11} - \nu(1+\nu)\sigma_{22}\right) \\ \epsilon_{22} = \frac{1}{E}\left((1-\nu^2)\sigma_{22} - \nu(1+\nu)\sigma_{11}\right) \\ \epsilon_{12} = \frac{1}{E}\left((1+\nu)\sigma_{12}\right) = \epsilon_{21} \\ \sigma_{33} = \nu(\sigma_{11} + \sigma_{22}) \end{aligned} \tag{31}$$

The **equilibrium equation** for plane strain is

$$\begin{aligned}\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} = 0 \\ \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{12}}{\partial x_1} = 0\end{aligned} \tag{32}$$

And **kinematics** for plane strain problem turns into

$$\begin{aligned}\epsilon_{11} = \frac{\partial u_1}{\partial x_1},\ \epsilon_{22} = \frac{\partial u_2}{\partial x_2} \\ \epsilon_{12} = \epsilon_{21} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right)\end{aligned} \tag{33}$$



Now we can write out the **boundary conditions**; taking consideration of the coordinate established in the above figure, we have a shear angle θ, the BCs write:

$$\begin{aligned}\sigma_{ij} = 0, \quad (x_2, x_1 = \pm L/2;\ i, j = 1, 2, 3) \\ u_1 = u_2 = 0, \quad (x_1, x_2 = -H/2) \\ u_1 = \Delta, u_2 = u_1 \tan\theta \quad (x_2, x_1 = \pm L/2)\end{aligned} \tag{34}$$

where

$$\frac{x_2/2}{\Delta} = \tan\theta$$

Therefore we can guess that the solution will possess the form:

$$\sigma_{12} = \sigma_{21} = \frac{1}{\Delta} f(x_1, x_2) \tag{35}$$