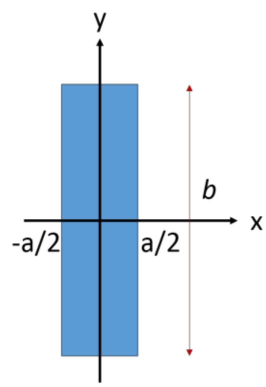
MAE 6110: HW #10

Hanfeng Zhai*

November 18, 2021

1. Consider the torsion of a bar of rectangular cross-section, as shown in Figure 1 below.



It the following, assume $b \geq a$ and the z axis passes through the center of the section.

a. Formulate the torsion problem using the ψ function, that is, state the PDE and the boundary conditions.

Solution: In the torsion of the rectangular cross-section, adopting the semi-inverse method, the displacements take the form:

$$u_1 = -\alpha x_2 x_3,\ u_2 = \alpha x_1 x_3,\ u_3 = \alpha w(x_1, x_2) \tag{1}$$

Note that when plane's cross-sections are not circular, w is not zero (*warping function*).

Now, we can write out the strain tensor under Cartesian coordinate:

$$\epsilon_{11} = u_{1,1},\ \epsilon_{12} = u_{1,2},\ \epsilon_{22} = u_{2,2},\ \epsilon_{13} = u_{1,3},\ \epsilon_{23} = u_{2,3},\ \epsilon_{33} = u_{3,3} \tag{2}$$

With consideration we can deduce that $u_3 = 0$,

Considering the **boundary conditions**, the traction free boundary conditions holds on the side surfaces:

$$\sigma_{ij} n_j = 0$$

If the coordinate writes $x = x_1$ *&* $y = x_2$; then the BCs takes the form

$$\sigma_{21} n_1 \left(x_1 = \pm\frac{a}{2}, x_2 \right) = 0,\ \sigma_{12} n_2 \left(x_2 = \pm\frac{b}{2}, x_1 \right) = 0$$

$$\sigma_{11}(x_1, x_2) = \sigma_{22}(x_1, x_2) = 0$$

* [redacted] www.hanfengzhai.net
Sibley School of Mechanical and Aerospace Engineering, Cornell University

Recalling lecture note, the harmonic conjugate function ψ is related to w:

$$\psi_{,1} = -w_{,2}, \;\; \psi_{,2} = -w_{,1} \tag{3}$$

The alternate formulation of the torsion problem writes

$$\nabla^2 \psi = 0 \tag{4}$$

The BCs then turned into

$$\begin{aligned} \psi(x_1, x_2)\left(x_1 = \pm\frac{a}{2}, x_2 = \pm\frac{b}{2}\right) &= \frac{1}{2}(x_1^2 + x_2^2), \\ f(x_1, x_2 = \pm\frac{b}{2}) &= 2, \\ f(x_1 = \pm\frac{a}{2}, x_2) &= 0 \end{aligned} \tag{5}$$

b. Instead of solving (a) directly, Define a new function $f \equiv \frac{\partial^2 \psi}{\partial x^2} + 1$. Show that f is harmonic, what is the boundary conditions satisfy by f?

Solution: To show f is harmonic, we first need to compute the Laplacian

$$\Delta f = \partial_{11} f + \partial_{22} f = \frac{\partial^4 \psi}{\partial x_1^4} + \frac{\partial^4 \psi}{\partial x_1^2 x_2^2} \tag{6}$$

further derivation:

$$\Delta f = \frac{\partial^2 \psi}{\partial x_1^2}\left(\frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2}\right) \tag{7}$$

Since we already know ψ is harmonic $\Longrightarrow \Delta\psi = 0$. Therefore it is obvious that $\Delta f = \partial_{11}(0) = 0 \Longrightarrow$ f is harmonic.

Due to ψ is harmonic: $\nabla^2\psi = 0 \Longrightarrow \partial_{11}\psi = -\partial_{22}\psi$. Substituting this into f we know that $f = -\partial_{22}\psi + 1$; we hence deduce that on the left and right sides $\partial_{22}\psi = 1$: function f taking the form: $f = 0$. Recalling the BCs

$$\begin{aligned} f(x_1, x_2) = 2, &\quad y = \pm\frac{b}{2} \\ f(x_1, x_2) = 0, &\quad x = \pm\frac{a}{2} \end{aligned} \tag{8}$$

c. Find f using separation of variables.

Solution: The cross-section is enclosed by the rectangle, as described:

$$\left(x_1^2 - \frac{a^2}{4}\right)\left(x_2^2 - \frac{b^2}{4}\right) = 0, \quad -\frac{a}{2} \le x \le \frac{a}{2}, \& -\frac{b}{2} \le y \le \frac{b}{2} \tag{9}$$

Using separation of variables, we can write a general form of f:

$$f(x_1, x_2) = f_1(x_1) f_2(x_2) \tag{10}$$

The aim of the problem turned into solving the Laplacian equation $\nabla^2 f = 0$, which turned into solving

$$\nabla^2 f = f_2 \partial_{11} f_1 + f_1 \partial_{22} f_2 \tag{11}$$

To solve this Laplace equation, we first taking the hint from office hours, assuming

$$\frac{\partial_{11} f_1(x)}{f_1(x)} = -\frac{\partial_{22} f_2(x)}{f_2(x)} = -K^2 \tag{12}$$

which can be written into

$$\begin{aligned} \partial_{11} f_1(x) &= -K^2 f_1 \\ \partial_{22} f_2(x) &= K^2 f_2 \end{aligned} \tag{13}$$

The general solution of these ordinary differential equations are:

$$\begin{aligned} f_1 &= A \cosh K x_2 + B \sinh K x_2 \\ f_2 &= C \cos K x_1 + D \sin K x_1 \end{aligned} \tag{14}$$

Due to the symmetry nature of the geometry, we know that $B = D = 0$. Hence

$$f_1 = A \cosh K x_2 \quad \& \quad f_2 = C \cos K x_1 \Longrightarrow f = A \cosh K x_2 C \cos K x_1 \tag{15}$$

However, if we go back to calculus and review the solution of the Laplace equation (which I already forgot), and we know

$$K = \kappa(2n+1)\pi$$

where κ is a random constant, we hence find out the expression of f contains a series expansion, written as:

$$f(x_1, x_2) = \sum_{n=0}^{\infty} C_n \cosh\left(\kappa(2n+1)\pi x_2\right) \cos\left(\kappa(2n+1)\pi x_1\right) \tag{16}$$

Now, we need to deduce the constant κ from the boundary conditions. Substitute the two BCs we have, and assume $n = 0$[1]:

$$\begin{aligned} f(x_1 = \frac{a}{2}, x_2) &= \sum_{n=0}^{\infty} C_n \cosh\left(\kappa(2n+1)\pi x_2\right) \cos\left(\kappa(2n+1)\pi x_1\right) = 0 \\ &= C_0 \cosh\left(\kappa\pi\frac{a}{2}\right) \cos\left(\kappa\pi\frac{a}{2}\right) = 0 \end{aligned} \tag{17}$$

To let Equation (17) establish, since we already know $\cos(\frac{\pi}{2}) = 0$, we can deduce $\kappa = \frac{1}{a}$.
Equation (16) then turned into

$$f(x_1, x_2) = \sum_{n=0}^{\infty} C_n \cosh\left(\frac{(2n+1)\pi}{a} x_2\right) \cos\left(\frac{(2n+1)\pi}{a} x_1\right) \tag{18}$$

Now we substitute the second BCs:

$$f(x_1, x_2 = \frac{b}{2}) = \sum_{n=0}^{\infty} C_n \cosh\left(\frac{(2n+1)\pi}{a}\frac{b}{2}\right) \cos\left(\frac{(2n+1)\pi}{a} x_1\right) = 2 \tag{19}$$

[1] because this equation of series should be established for any n values

Now, to solve this equation, we need apply the Fourier's trick:

$$C_n \cosh\left(\frac{(2n+1)\pi}{a}\frac{b}{2}\right)\int_{-\frac{a}{2}}^{\frac{a}{2}} \cos^2\left(\frac{(2n+1)\pi}{a}x_1\right)dx_1 = 2\int_{-\frac{a}{2}}^{\frac{a}{2}} \cos\left(\frac{(2n+1)\pi}{a}x_1\right)dx_1 \tag{20}$$

Therefore we can derive that

$$C_n = \frac{2\int_{-\frac{a}{2}}^{\frac{a}{2}} \cos\left(\frac{(2n+1)\pi}{a}x_1\right)dx_1}{\cosh\left(\frac{(2n+1)\pi}{a}\frac{b}{2}\right)\int_{-\frac{a}{2}}^{\frac{a}{2}} \cos^2\left(\frac{(2n+1)\pi}{a}x_1\right)dx_1} \tag{21}$$

This equation seems not trivial for me to solve, so I decide to use MATLAB and generate the following code:

```
syms n a x1 b
nom = 2*int(cos(((2*n+1)*pi*x1)/a),x1, -a/2,a/2)
den = cosh(((2*n+1)*pi*b)/(2*a)) * int(cos(((2*n+1)*pi*x1)/a)^2,x1, -a/2,a/2)
Cn = nom/den
simplify(Cn)
```

Then we generate the constant C_n:

$$C_n = \frac{8\sin\left(\frac{\pi(2n+1)}{2}\right)}{\cosh\left(\frac{\pi b(2n+1)}{2a}\right)\left(\pi - \sin(2\pi n) + 2\pi n\right)} \tag{22}$$

Further simplication we have:

$$C_n = \frac{8(-1)^n}{\cosh\left(\frac{\pi b(2n+1)}{2a}\right)(2n+1)\pi} \tag{23}$$

Plug in Equation (23), we can write out the full form of the form of f:

$$f(x_1,x_2) = \sum_{n=0}^{\infty} \frac{8(-1)^n}{\cosh\left(\frac{\pi b(2n+1)}{2a}\right)(2n+1)\pi}\cosh\left(\frac{(2n+1)\pi}{a}x_2\right)\cos\left(\frac{(2n+1)\pi}{a}x_1\right) \tag{24}$$

d. Find the shear stresses (*Hint: what are the shear stresses on the boundaries?*). Where is the maximum stress, show that the maximum shear stress σ_{max} is well approximated by

$$\sigma_{max} \approx G\alpha a\left[1 - \frac{8}{\pi^2}\operatorname{sech}\frac{\pi b}{2a}\right]$$

Solution: Based on the lecture note, we know that the shear stresses take the form (Eq. 3.14c,d):

$$\begin{aligned}\sigma_{13} &= G\alpha(-x_2 + w_{,1})\\ \sigma_{23} &= G\alpha(x_1 + w_{,2})\end{aligned} \tag{25}$$

Since we already know

$$\begin{aligned}\psi_{,1} &= -w_{,2}\\ \psi_{,2} &= w_{,1}\end{aligned} \tag{26}$$

We further have

$$\begin{aligned}\sigma_{13} &= G\alpha(-x_2 + \psi_{,2}) \\ \sigma_{23} &= G\alpha(x_1 - \psi_{,1})\end{aligned} \tag{27}$$

Now, if we plug in f and compute all the shear stresses:

$$\begin{aligned}
\psi_{,1} &= \int_{-\frac{a}{2}}^{\frac{a}{2}} (f-1)dx_1 \\
&= \left[\sum_{n=0}^{\infty} \frac{8(-1)^n}{(2n+1)\pi} \operatorname{sech}\left(\frac{\pi b(2n+1)}{2a}\right) \cosh\left(\frac{(2n+1)\pi}{a}x_2\right) \sin\left(\frac{(2n+1)\pi}{a}x_1\right) \frac{a}{(2n+1)\pi} - x_1\right]_{-\frac{a}{2}}^{\frac{a}{2}} \\
\psi_{,2} &= -\int_{-\frac{b}{2}}^{\frac{b}{2}} \psi_{,11} dx_2 = -\int_{-\frac{b}{2}}^{\frac{b}{2}} (f-1)dx_2 \\
&= \left[\sum_{n=0}^{\infty} \frac{8(-1)^n}{(2n+1)\pi} \operatorname{sech}\left(\frac{\pi b(2n+1)}{2a}\right) \sinh\left(\frac{(2n+1)\pi}{a}x_2\right) \cos\left(\frac{(2n+1)\pi}{a}x_1\right) \frac{a}{(2n+1)\pi} + x_2\right]_{-\frac{b}{2}}^{\frac{b}{2}}
\end{aligned} \tag{28}$$

We therefore write out the two shear stresses (based on lecture notes):

$$\begin{aligned}
\sigma_{13} &= -G\alpha(-x_2 + \psi_{,2}) \\
&= -G\alpha\left(\sum_{n=0}^{\infty} \frac{8(-1)^n}{(2n+1)\pi} \operatorname{sech}\left(\frac{\pi b(2n+1)}{2a}\right) \sinh\left(\frac{(2n+1)\pi}{a}x_2\right) \cos\left(\frac{(2n+1)\pi}{a}x_1\right) \frac{a}{(2n+1)\pi}\right) \\
\sigma_{23} &= G\alpha(x_1 - \psi_{,1}) \\
&= G\alpha\left(2x_1 - \sum_{n=0}^{\infty} \frac{8(-1)^n}{(2n+1)\pi} \operatorname{sech}\left(\frac{\pi b(2n+1)}{2a}\right) \cosh\left(\frac{(2n+1)\pi}{a}x_2\right) \sin\left(\frac{(2n+1)\pi}{a}x_1\right) \frac{a}{(2n+1)\pi}\right)
\end{aligned} \tag{29}$$

Taking the hint, we can first compute the maximum shear stress occurs on the right boundary of the rectangle:

$$\begin{aligned}
&\sigma_{23}\left(x_1 = \frac{a}{2}, x_2\right) \\
&= G\alpha\left(2x_1 - \sum_{n=0}^{\infty} \frac{8(-1)^n}{(2n+1)\pi} \operatorname{sech}\left(\frac{\pi b(2n+1)}{2a}\right) \cosh\left(\frac{(2n+1)\pi}{a}x_2\right) \sin\left((2n+1)\pi\right) \frac{a}{(2n+1)\pi}\right)_{x_1=\frac{a}{2}} \\
&= G\alpha a\left(1 - \frac{8}{\pi^2} \operatorname{sech}\frac{\pi b}{2a}\right) \quad \text{(taking } n=0\text{)}
\end{aligned} \tag{30}$$

e. Show that the bending moment (M) is given by

$$M = \frac{G\alpha b a^3}{3} - \frac{64 G\alpha a^4}{\pi^5} \sum_{n=0}^{\infty} \frac{\tanh\left(\frac{\pi b}{a}\frac{2n+1}{2}\right)}{(2n+1)^5}$$

Hint: you may want to know that series of the form $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^m}$ *for positive integers* $m > 1$ *can be summed exactly.*

Solution: Going back to the lecture note we can write out the form of moment M:

$$\begin{aligned} M &= \int\int_{\Omega} (x_1\sigma_{23} - x_2\sigma_{13}) dA \\ &= \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (x_1\sigma_{23} - x_2\sigma_{13}) dx_1 dx_2 \end{aligned} \tag{31}$$

Note that if we recall Equation (29) and plug in the σ terms this equation would be turned into a extremely complicated and dirty form. Hence, we ask MATLAB for help, and generate the following code:

And with the newly `Livescript`® interface of MATLAB we can directly output the results in the form of LaTeX:

```
clear;clc;close
syms G alpha x1 x2 n a b

sigma13 = -G*alpha*...
    ( ( (8*(-1)^n)/((2*n+1)*pi) )...
    * sech( (pi*b*(2*n+1))/(2*a) )...
    *sinh( ((2*n+1)*pi*x1)/a )...
    * cos( (2*n+1)*pi/a) ...
    *( a/((2*n+1)*pi) ) );

sigma23 = G*alpha*...
    ( 2*x1 - ...
    ( (8*(-1)^n)/((2*n+1)*pi) )...
    * sech( (pi*b*(2*n+1))/(2*a) )...
    *sinh( ((2*n+1)*pi*x1)/a )...
    * cos( (2*n+1)*pi/a) ...
    *( a/((2*n+1)*pi) ));

M = int(int(x1*sigma23 - x2*sigma13, x1, -a/2, a/2), x2, -b/2, b/2)
```

And we generate the form of moment M:

$$M = \frac{G\, a^3\, \alpha\, b}{6} + \frac{8\, (-1)^n\, G\, a\, \alpha\, b\, \cos\left(\frac{\sigma_1}{a}\right) \left(\frac{2\, a^2\, \sinh\left(\frac{\sigma_1}{2}\right)}{(\pi + 2\,\pi\, n)^2} - \frac{a^2\, \cosh\left(\frac{\sigma_1}{2}\right)}{\sigma_1}\right)}{\pi^2\, \cosh\left(\frac{\pi\, b\, (2\, n+1)}{2\, a}\right)\, (2\, n+1)^2} \tag{32}$$

Further simplification we have

$$M = \frac{G\alpha b a^3}{6} - \frac{64 G\alpha a^4}{\pi^5} \sum_{n=0}^{\infty} \frac{\tanh\left(\frac{\pi b}{a}\frac{2n+1}{2}\right)}{(2n+1)^5} \tag{33}$$

f. Show that (you can use Matlab) that a good approximation for the torsion stiffness K is

$$K \approx \frac{Gba^3}{3} - \frac{64 G\alpha a^4}{\pi^5} \tanh\left(\frac{\pi b}{2a}\right)$$

Solution: Due to we know a good approximation for torsion stiffness K can be written into $M \approx K\alpha$, further expansion we have

$$K \approx \frac{M}{\alpha} = \frac{Gba^3}{6} - \frac{64Ga^4}{\pi^5} \sum_{n=0}^{\infty} \frac{\tanh\left(\frac{\pi b}{a}\frac{2n+1}{2}\right)}{(2n+1)^5} \tag{34}$$

Now, we can take $n = 0$, Equation (34) hence turned into:

$$K \approx \frac{M}{\alpha} = \frac{Gba^3}{6} - \frac{64Ga^4}{\pi^5} \sum_{n=0}^{\infty} \tanh \frac{\pi b}{2a} \tag{35}$$

g. Find the warping function w.

Solution: Now, recall the lecture note, we have

$$\psi_{,2} = w_{,1} \quad \& \quad \psi_{,1} = -w_{,2} \tag{36}$$

Recall Equation (28) and plug in the ψ terms, we hence have

$$w_{,1} = -\sum_{n=0}^{\infty} \frac{8(-1)^n}{(2n+1)\pi} \operatorname{sech}\left(\frac{\pi b(2n+1)}{2a}\right) \sinh\left(\frac{(2n+1)\pi}{a}x_2\right) \cos\left(\frac{(2n+1)\pi}{a}x_1\right) \frac{a}{(2n+1)\pi} + x_2 \tag{37}$$

we can therefore generate

$$w = \sum_{n=0}^{\infty} \frac{8(-1)^n}{(2n+1)\pi} \operatorname{sech}\left(\frac{\pi b(2n+1)}{2a}\right) \sinh\left(\frac{(2n+1)\pi}{a}x_2\right) \sin\left(\frac{(2n+1)\pi}{a}x_1\right) \left(\frac{a}{(2n+1)\pi}\right)^2 + x_1x_2 \tag{38}$$