MAE 6110: HW #7

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1. Derive equations (20) – (23) in class notes.
You can use the result $\frac{\partial \det(\mathbf{A})}{\partial A_{ij}} = \det(\mathbf{A}) A_{ji}^{-1}$ derived in class.

Solution: For equation (20), recall the definition of strain energy function: $\hat{W}(\mathbf{C}) = W(\mathbf{F})$ and applies the partial derivative on W with respect to $\mathbf{F}$, and apply the chain rule:

$$\begin{aligned}
\frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} &= \frac{\partial \hat{W}(\mathbf{F}^T\mathbf{F})}{\partial \mathbf{C}} \frac{\partial \mathbf{C}}{\partial \mathbf{F}} \\
&= \frac{\partial \hat{W}(\mathbf{C})}{\partial \mathbf{C}} \frac{\partial \mathbf{F}^T\mathbf{F}}{\partial \mathbf{F}} \\
&= \frac{\partial \hat{W}(\mathbf{C})}{\partial \mathbf{C}} \left(\mathbf{F} \frac{\partial \mathbf{F}^T}{\partial \mathbf{F}} + \mathbf{F} \frac{\partial \mathbf{F}^T}{\partial \mathbf{F}} \right) \\
&= \frac{\partial \hat{W}(\mathbf{C})}{\partial \mathbf{C}} (2\mathbf{F}) \\
\text{Therefore we have}: \quad \frac{\partial W}{\partial F_{ij}} &= 2F_{ik} \frac{\partial \hat{W}}{\partial C_{kj}}
\end{aligned}$$

For equation (21), we substitute $\hat{W}(\mathbf{C}) = \Phi(I_1,\ I_2,\ I_3)$ into equation $\frac{\partial \hat{W}}{\partial \mathbf{C}}$:

$$\begin{aligned}
\frac{\partial \hat{W}(\mathbf{C})}{\partial \mathbf{C}} &= \frac{\partial \Phi}{\partial I_1}\frac{\partial I_1}{\partial \mathbf{C}} + \frac{\partial \Phi}{\partial I_2}\frac{\partial I_2}{\partial \mathbf{C}} + \frac{\partial \Phi}{\partial I_3}\frac{\partial I_3}{\partial \mathbf{C}} \\
&= \frac{\partial \Phi}{\partial I_1}\frac{\partial \mathrm{tr}\mathbf{C}}{\partial \mathbf{C}} + \frac{1}{2}\frac{\partial \Phi}{\partial I_2}\frac{\partial \left((\mathrm{tr}\mathbf{C})^2 - \mathrm{tr}\mathbf{C}^2 \right)}{\partial \mathbf{C}} + \frac{\partial \Phi}{\partial I_3}\frac{\partial \det \mathbf{C}}{\partial \mathbf{C}} \\
&= \frac{\partial \Phi}{\partial I_1}\mathbf{I} + \frac{1}{2}\frac{\partial \Phi}{\partial I_2}\frac{\partial I_1^2}{\partial I_1}\frac{\partial I_1}{\partial \mathbf{C}} - \frac{1}{2}\frac{\partial \Phi}{\partial I_2}\frac{\partial \mathrm{tr}\mathbf{C}^2}{\partial \mathbf{C}} + \frac{\partial \Phi}{\partial I_3}\det(\mathbf{C})\mathbf{C}^{-1} \\
&= \frac{\partial \Phi}{\partial I_1}\mathbf{I} + \frac{\partial \Phi}{\partial I_2} I_1 \mathbf{I} - \frac{1}{2}\frac{\partial \Phi}{\partial I_2}\frac{\partial \mathrm{tr}\mathbf{C}^2}{\partial \mathbf{C}^2}\frac{\partial \mathbf{C}^2}{\partial \mathbf{C}} + \frac{\partial \Phi}{\partial I_3}\det(\mathbf{C})\mathbf{C}^{-1} \\
&= \frac{\partial \Phi}{\partial I_1}\mathbf{I} + \frac{\partial \Phi}{\partial I_2} I_1 \mathbf{I} - \frac{1}{2}\frac{\partial \Phi}{\partial I_2} 2\mathbf{C} + \frac{\partial \Phi}{\partial I_3}\det(\mathbf{C})\mathbf{C}^{-1} \\
&= \frac{\partial \Phi}{\partial I_1}\mathbf{I} + \frac{\partial \Phi}{\partial I_2}(I_1\mathbf{I} - \mathbf{C}) + \frac{\partial \Phi}{\partial I_3} I_3 \mathbf{C}^{-1}
\end{aligned} \tag{1}$$

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For equation (22), substituting equation (1):

$$
\begin{aligned}
\mathbf{P} = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} &= 2\mathbf{F}\frac{\partial \hat{W}}{\partial \mathbf{C}} \\
&= 2\mathbf{F}\left(\frac{\partial \Phi}{\partial I_1}\mathbf{I} + \frac{\partial \Phi}{\partial I_2}\left(I_1\mathbf{I} - \mathbf{C}\right) + \frac{\partial \Phi}{\partial I_3} I_3 \mathbf{C}^{-1}\right) \\
&= 2\left(\frac{\partial \Phi}{\partial I_1}\mathbf{F} + \frac{\partial \Phi}{\partial I_2} I_1 \mathbf{F} - \frac{\partial \Phi}{\partial I_2}\mathbf{F}\mathbf{C} + I_3\frac{\partial \Phi}{\partial I_3}\mathbf{F}\mathbf{C}^{-1}\right) \\
&= 2\left(\frac{\partial \Phi}{\partial I_1}\mathbf{F} + \frac{\partial \Phi}{\partial I_2} I_1 \mathbf{F} - \frac{\partial \Phi}{\partial I_2}\mathbf{F}\mathbf{C} + I_3\frac{\partial \Phi}{\partial I_3}\mathbf{F}(\mathbf{F}^T\mathbf{F})^{-1}\right) \\
&= 2\left(\frac{\partial \Phi}{\partial I_1}\mathbf{F} + \frac{\partial \Phi}{\partial I_2} I_1 \mathbf{F} - \frac{\partial \Phi}{\partial I_2}\mathbf{F}\mathbf{C} + I_3\frac{\partial \Phi}{\partial I_3}\mathbf{F}^{-1}\right) \\
&= 2\left(\left(\frac{\partial \Phi}{\partial I_1} + \frac{\partial \Phi}{\partial I_2} I_1\right)\mathbf{F} - \frac{\partial \Phi}{\partial I_2}\mathbf{F}\mathbf{C} + I_3\frac{\partial \Phi}{\partial I_3}\mathbf{F}^{-1}\right)
\end{aligned} \tag{2}
$$

For equation (23), substituting equation (2):

$$
\begin{aligned}
\sigma = \frac{1}{J}\frac{\partial W(\mathbf{F})}{\partial \mathbf{F}}\mathbf{F}^T &= \frac{2}{J}\left(\left(\frac{\partial \Phi}{\partial I_1} + \frac{\partial \Phi}{\partial I_2} I_1\right)\mathbf{F} - \frac{\partial \Phi}{\partial I_2}\mathbf{F}\mathbf{C} + I_3\frac{\partial \Phi}{\partial I_3}\mathbf{F}^{-1}\right)\mathbf{F}^T \\
&= \frac{2}{\det \mathbf{F}}\left(\left(\frac{\partial \Phi}{\partial I_1} + \frac{\partial \Phi}{\partial I_2} I_1\right)\mathbf{F}\mathbf{F}^T - \frac{\partial \Phi}{\partial I_2}\mathbf{F}\mathbf{C}\mathbf{F}^T + I_3\frac{\partial \Phi}{\partial I_3}\mathbf{F}^{-1}\mathbf{F}^T\right) \\
&= \frac{2}{\det \mathbf{F}}\left(\left(\frac{\partial \Phi}{\partial I_1} + \frac{\partial \Phi}{\partial I_2} I_1\right)\mathbf{b} - \frac{\partial \Phi}{\partial I_2}\mathbf{b} + I_3\frac{\partial \Phi}{\partial I_3}\mathbf{I}\right)
\end{aligned} \tag{3}
$$

2. A simple model for a compressible elastic solid is given by equation (24) in class notes, that is

$$
W = c_1(I_1 - 3 - 2\log J) + c_2\left(\log J\right)^2
$$

Use this model to predict the stress (both True and Nominal stress) versus stretch behavior in a uniaxial tension/compression test. In doing so find the physical meaning of c_1, c_2. Find all the stresses and the displacement fields. Make a plot of the stress versus stretch behavior in the loading direction (both true stress versus stretch and nominal stress versus stretch, put these plots in the same figure). Make sure all the governing equations (equilibrium etc) are satisfied.

Solution: Considering a uniaxial tension test on the $\mathbf{e}_3$ direction, the *Deformation Gradient Tensor* writes:

$$
\mathbf{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}
$$

Recall equations (22) and (23) for first Piola and Cauchy stresses and substituting the given model for elastic solid as

$$
W(\mathbf{F}) = c_1\left(\mathrm{tr}(\mathbf{F}^T\mathbf{F}) - 3 - 2\log(\det \mathbf{F})\right) + c_2\left(\log(\det \mathbf{F})\right)^2
$$

to the two stresses, since this is a simple tensile test, the shear component of $\mathbf{F}$ are zero, therefore $\mathrm{tr}(\mathbf{F}^T\mathbf{F}) = \mathrm{tr}(\mathbf{F}^2)$, hence the two stresses writes:

$$
\begin{aligned}
\mathbf{P} = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}} &= c_1 \frac{\partial \mathrm{tr}\mathbf{F}^T\mathbf{F}}{\partial \mathbf{F}^T\mathbf{F}} \frac{\partial \mathbf{F}^T\mathbf{F}}{\partial \mathbf{F}} - 2c_1 \frac{\partial \log\det \mathbf{F}}{\partial \det \mathbf{F}} \frac{\partial \det \mathbf{F}}{\partial \mathbf{F}} + 2c_2 \frac{\partial \log\det \mathbf{F}}{\partial \det \mathbf{F}} \frac{\partial \det \mathbf{F}}{\partial \mathbf{F}} \\
&= 2c_1(\mathbf{F} - \mathbf{F}^{-1}) + 2c_2\mathbf{F}^{-1} \\
\sigma &= \frac{1}{J}\frac{\partial W(\mathbf{F})}{\partial \mathbf{F}}\mathbf{F}^T = \frac{1}{J}\left(2c_1(\mathbf{F} - \mathbf{F}^{-1}) + 2c_2\mathbf{F}^{-1}\right)\mathbf{F}^T \\
&= \frac{1}{\det \mathbf{F}}(2c_1(\mathbf{F}\mathbf{F}^T - \mathbf{F}^{-1}\mathbf{F}^T) + 2c_2\mathbf{F}^{-1}\mathbf{F}^T)
\end{aligned} \tag{4}
$$

To derive a relation between λ_1 and λ_3, we pick a Poisson's ratio

$$
\begin{aligned}
\nu = -\frac{\epsilon_{trans}}{\epsilon_{axial}} &= -\frac{\lambda_1 - 1}{\lambda_3 - 1} \\
\implies \lambda_1 &= (1+\nu) - \nu\lambda_3
\end{aligned} \tag{5}
$$

We can also know write out the terms related to the gradient deformation tensor:

$$
\mathbf{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad \mathbf{F}^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & \frac{1}{\lambda_1} & 0 \\ 0 & 0 & \frac{1}{\lambda_3} \end{bmatrix} \tag{6}
$$

We can further reduce Equation 4 by substituting c_1 and c_2, based on the lecture note:

$$
c_1 = \frac{\mu}{2}, \; c_2 = \frac{\Lambda}{2} \tag{7}
$$

where c_1 and c_2 stands for one half of the Lamé's first parameter Λ and Shear modulus μ.

Before continuing the rest part of calculating the two stresses, we here take a small strain assumption for a better physical understanding of c_1 and c_2.

Under small strain in a simple tension test, the deformation gradient tensor writes $\mathbf{F} = \mathbf{I} + \epsilon = \mathbf{I} + \nabla u$ and $\mathbf{F}^T = \mathbf{I} - \epsilon = \mathbf{I} - \nabla u$. The inverse of the deformation gradient tensor takes the form:

$$
\mathbf{F}^{-1} = \frac{1}{\mathbf{I} + \epsilon}
$$

and

$$
\mathbf{F}^{-1} = \frac{1}{\mathbf{I} - \epsilon}
$$

Substitute this condition and represent the form of stretch we have:

$$
\lambda_i = 1 + \epsilon_{ii} \;\&\; \frac{1}{\lambda_i} = 1 - \epsilon_{ii} \tag{8}
$$

In the simple tension test, we know that on the lateral sides, the stresses obey the traction free condition, therefore $P_{11} = P_{22} = \sigma_{11} = \sigma_{22} = 0$.

According to traction free boundary conditions: $\mathbf{P} \cdot \mathbf{n} = 0$.

Substituting these conditions into Equation 4, we have:

$$
\sigma_{11} = \sigma_{22} = \frac{1}{\lambda_1^2\lambda_3}\left(2c_1(\lambda_1^2 - 1) + 2c_2\log(\lambda_1^2\lambda_3)\right) = 0 \tag{9}
$$

To establish Equation 9, we need to let

$$(\lambda_1^2 - 1) + \frac{c_1}{c_2}\log(\lambda_1^2\lambda_3) = 0 \Longrightarrow (\lambda_1^2 - 1) + \frac{\mu}{\Lambda}\log(\lambda_1^2\lambda_3) = 0 \tag{10}$$

Before continuing deriving the equations, we substitute our previously derived conditions (Equation 8) and get:

$$c_1 = c_2\left(\frac{1}{2\nu} - 1\right) \tag{11}$$

which physically connects c_1 and c_2.

Due to $\Lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$, and $\mu = \frac{E}{2(1+\nu)}$, substituting these, and from Equation 10 we can conclude:

$$\begin{aligned}(\lambda_1^2 - 1) &= -\frac{\mu}{\Lambda}\log(\lambda_1^2\lambda_3) \\ \Longrightarrow \lambda_3 &= \frac{1}{\lambda_1^2 e^{\frac{1-2\nu}{2\nu}(\lambda_1^2-1)}}\end{aligned} \tag{12}$$

where ν is the Poisson's ratio.

The form of 1st Piola stress can thence be written out:

$$P_{33} = \frac{E}{2(1+\nu)}(\lambda_3 - \frac{1}{\lambda_3}) + \frac{E\nu}{(1+\nu)(1-2\nu)}\frac{1}{\lambda_3} \tag{13}$$

We can also write out the form of σ_{33}:

$$\begin{aligned}\sigma_{33} &= \frac{1}{\lambda_1^2\lambda_3}\left(2c_1(\lambda_3^2 - 1) + 2c_2\log(\lambda_3^2\lambda_1)\right) \\ &= \frac{1}{\lambda_1^2\lambda_3}\left(\mu(\lambda_3^2 - 1) + \Lambda\log(\lambda_3^2\lambda_1)\right) \\ &= \frac{1}{\lambda_1^2\lambda_3}\left(\frac{E}{2(1+\nu)}(\lambda_3^2 - 1) + \frac{E\nu}{(1+\nu)(1-2\nu)}\log(\lambda_3^2\lambda_1)\right)\end{aligned} \tag{14}$$

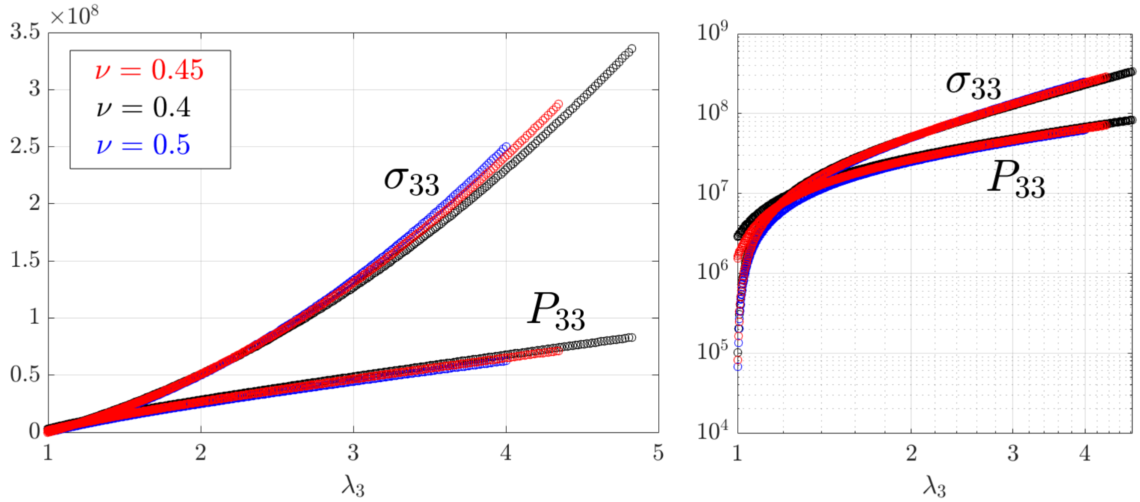
Substituting all these to a MATLAB® function `calcustress`:

```
function calstress(nu,E)
    for i = 1:-0.001:0.5
        lambda1 = i;
        lambda3 = 1./(lambda1.^2 .* exp(((1 - 2.*nu)/(2.*nu)).*(lambda1.^2 - 1)));
        constA = E./(2.*(1+nu));
        constB = (E.*nu)./(1+nu).*(1-2.*nu);
        sigma = (1/(lambda1.^2.*lambda3)).*(constA.*(lambda3.^2 - 1) + constB.*log(lambda3.^2.*lambda1)
    );
        piola = constA.*(lambda3 - (1./lambda3)) + constB./lambda3;
        scatter(lambda3,sigma,30,'red'); hold on;scatter(lambda3,piola,30,'red')
    end
end
```

And running a simple command:

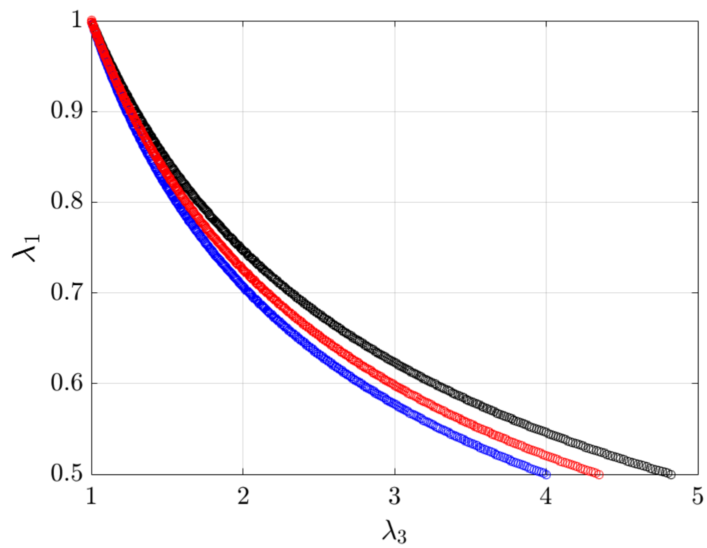
```
>>  calstress(.5,.05e9); % in this case Poisson's ratio = 0.5, elastic modulus 0.05e9Pa
```

Then we will generate the stress-stretch diagram as shown in the following figure.



To illustrate the results more comprehensively, we set three different cases: Let the Poisson's ratio equals to 0.4, 0.45, and 0.5, respectively; and plotted the the results of 1st Piola stress $\mathbf{P}$ and Cauchy stress σ as shown beyond. To better 'feel' the relation between the two stresses as we derived in the previous Assignments we also did a logarithmic plot to see "*what's going at the small strain regime*" as shown in the right sub figure.

From this figure we can deduce that as the Poisson's ratio increases, the Cauchy stress increases and the 1st Piola stress decreases.



Also, in a similar way, we can also plot the relations between the two stretches λ_1 and λ_3.