MAE 6110: HW #4

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1. The motion of a solid body is given by

$$x_1 = X_1(1 + a^2t^2),\ x_2 = X_3,\ x_3 = X_3$$

Find the velocity and the accelerations in both material and spatial descriptions.

Solution: To calculate the velocity, we first take the derivative of $\mathbf{x}$:

$$\mathbf{v} = \frac{\partial \mathbf{x}}{\partial t} = \begin{bmatrix} 2X_1a^2t \\ 0 \\ 0 \end{bmatrix}$$

We can then compute the acceleration $\mathbf{a}$ (in the material description) as

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} = \begin{bmatrix} 2X_1a^2 \\ 0 \\ 0 \end{bmatrix}$$

Substitute back $X_1 = x_1/(1 + a^2t^2)$ we have:

$$\mathbf{V} = \begin{bmatrix} \frac{2x_1a^2t}{(1+a^2t^2)} \\ 0 \\ 0 \end{bmatrix} \ \&\ \mathbf{A} = \begin{bmatrix} \frac{2x_1a^2}{(1+a^2t^2)} \\ 0 \\ 0 \end{bmatrix}$$

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2. A continuum motion is defined by the velocity components in spatial description:

$$v_1 = \frac{3x_1}{1+t},\ v_2 = \frac{x_2}{1+t},\ v_3 = \frac{5x_3^2}{1+t}$$

Assume the reference configuration is at $t = 0$, with the consistency condition $\mathbf{X} = \mathbf{x}$.

2a. Find the motion $\mathbf{x} = \chi(\mathbf{X}, t)$

Solution: We know that the transformation relation between the material and spatial description is the motion $\mathbf{x} = \chi(\mathbf{X}, t)$, therefore the coordinates of reference configuration: $\mathbf{X} = \chi^{-1}(\mathbf{x}, t)$. With the given condition of $\mathbf{v} = [v_1, v_2, v_3]^T$ we can expand:

$$v_1 = \frac{\partial x_1}{\partial t}|_{X_1} = \frac{3x_1}{1+t},\ v_2 = \frac{\partial x_2}{\partial t}|_{X_2} = \frac{x_2}{1+t},\ v_3 = \frac{\partial x_3}{\partial t}|_{X_3} = \frac{5x_3^2}{1+t},$$

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Therefore we have

$$\log x_1|_{X_1}^{x_1} = 3\log(1+t)|_0^t,\ \log x_2|_{X_2}^{x_2} = \log(1+t)|_0^t,\ -\frac{1}{x_3}|_{X_3}^{x_3} = 5\log(1+t)|_0^t,$$

Solving the above equations we have:

$$x_1 = X_1(1+t)^3,\ x_2 = X_2(1+t),\ x_3 = \frac{X_3}{1 - X_3\log(1+t)^5}$$

We therefore write:

$$\mathbf{x} = \chi(\mathbf{X}, t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} X_1(1+t)^3 \\ X_2(1+t) \\ \frac{X_3}{1-X_3\log(1+t)^5} \end{bmatrix}$$

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2b. Find the velocity in material description and the acceleration in material and spatial description.

Solution: The velocity in material description:

$$\mathbf{V}(\mathbf{X}, t) = \frac{\partial \chi(\mathbf{X}, t)}{\partial t} = \begin{bmatrix} 3X_1(1+t)^2 \\ X_2 \\ \frac{5\,{X_3}^2}{(5\,X_3\ \log(t+1)-1)^2\,(t+1)} \end{bmatrix}$$

and acceleration in material description:

$$\mathbf{A}(\mathbf{X}, t) = \frac{\partial \mathbf{V}(\mathbf{X}, t)}{\partial t} = \begin{bmatrix} 6X_1(1+t) \\ 0 \\ \frac{5\,{X_3}^2}{(5\,X_3\ \log(t+1)-1)^2\,(t+1)} \end{bmatrix}$$

and acceleration in spatial description:

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} = \begin{bmatrix} -\frac{3x_1}{(1+t)^2} \\ -\frac{x_2}{(1+t)^2} \\ -\frac{5x_3^2}{(1+t)^2} \end{bmatrix}$$

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3. A body undergoes the homogenous deformation

$$x_1 = \sqrt{2}X_1 + \frac{3}{4}\sqrt{2}X_2,\ x_2 = -X_1 + \frac{3}{4}X_2 + \frac{\sqrt{2}}{4}X_3,\ x_3 = X_1 - \frac{3}{4}X_2 + \frac{\sqrt{2}}{4}X_3$$

3a. Find the direction after the deformation of a line element with direction ratios 1:1:1 in the reference configuration.

Solution: We can first write out the gradient deformation tensor:

$$\mathbf{F} = \begin{bmatrix} \sqrt{2} & \frac{3\sqrt{2}}{4} & 0 \\ -1 & \frac{3}{4} & \frac{\sqrt{2}}{4} \\ 1 & -\frac{3}{4} & \frac{\sqrt{2}}{4} \end{bmatrix}$$

Given the condition, we first take the vector in the reference configuration $\mathbf{X} = (X_1, X_2, X_3)$, and obtain the new line element: $d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}$

$$d\mathbf{x} = \begin{bmatrix} \sqrt{2} & \frac{3\sqrt{2}}{4} & 0 \\ -1 & \frac{3}{4} & \frac{\sqrt{2}}{4} \\ 1 & -\frac{3}{4} & \frac{\sqrt{2}}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{7\sqrt{2}}{4} \\ \frac{-1+\sqrt{2}}{4} \\ \frac{1+\sqrt{2}}{4} \end{bmatrix}$$

such direction element can be further reduced to $d\mathbf{x} = [7\sqrt{2}, \sqrt{2}-1, \sqrt{2}+1]^{\mathrm{T}}$

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3b. Find the stretch ratio of this line element.

Solution: Recall the original form of $d\mathbf{x}$:

$$d\mathbf{x} = \left[\frac{7\sqrt{2}}{4}, \frac{-1+\sqrt{2}}{4}, \frac{1+\sqrt{2}}{4}\right]^{\mathrm{T}}$$

Then we can compute the stretch ratio:

$$\begin{aligned} \lambda &= \sqrt{\frac{1}{3}\left((dx_1)^2 + (dx_2)^2 + (dx_3)^2\right)} \\ &= \sqrt{\frac{1}{3}\left(\frac{98}{16} + \frac{6}{16}\right)} \\ &= \sqrt{\frac{13}{6}} \end{aligned}$$

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4. Find the tensor (denote by $\mathbf{b^{-1}}$) such that

$$d\mathbf{X} \cdot d\mathbf{X} = d\mathbf{x} \cdot \mathbf{b}^{-1} \cdot d\mathbf{x}$$

4a. In particular, show $\mathbf{b^{-1}} = \mathbf{F^{-T}F^{-1}}$. What is the inverse of $\mathbf{b^{-1}}$ (it is called the Left Cauchy-Green tensor)?

Solution: Based on the definition of the deformation gradient tensor:

$$\because \mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \ \therefore d\mathbf{X} = \mathbf{F}^{-1} \cdot d\mathbf{x}$$

Therefore we write:

$$\begin{aligned} d\mathbf{X} \cdot d\mathbf{X} &= \mathbf{F}^{-1} \cdot d\mathbf{x} \cdot \mathbf{F}^{-1} \cdot d\mathbf{x} \\ &= d\mathbf{x} \cdot \mathbf{F}^{-\mathrm{T}} \cdot \mathbf{F}^{-1} \cdot d\mathbf{x} \\ &= d\mathbf{x} \cdot \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1} \cdot d\mathbf{x} \end{aligned}$$

we thence proved $\mathbf{b}^{-1} = \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1}$.

Calculating the inverse of $\mathbf{b^{-1}}$:

$$\begin{aligned}\mathbf{b} &= (\mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1})^{-1} \\ &= \mathbf{F}\mathbf{F}^{\mathrm{T}}\end{aligned}$$

Due to $\mathbf{F}^{-1}\mathbf{F}\mathbf{F}^{T} = \mathbf{F}^{T}\mathbf{F}\mathbf{F}^{-1}$, we deduce the relation between $\mathbf{b}$ and right Cauchy-Green Tensor $\mathbf{C}$:

$$\mathbf{b} = \mathbf{F}\mathbf{C}\mathbf{F}^{-1}$$

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4b. Show that $\mathbf{b}$ and $\mathbf{b^{-1}}$ are symmetric. Define a tensor $\mathbf{e} \equiv \frac{\mathbf{I}-\mathbf{b}^{-1}}{2}$. This tensor is called the Eulerian strain tensor. What is $d\mathbf{x}\cdot\mathbf{e}\cdot d\mathbf{x}$?

Solution: To show $\mathbf{b}$ is symmetric, we first expand $\mathbf{b^T}$:

$$\mathbf{b}^{T} = (\mathbf{F}\mathbf{F}^{T})^{T} = (\mathbf{F}^{T})^{T}\mathbf{F}^{T} = \mathbf{F}\mathbf{F}^{T} = \mathbf{b}$$

For $\mathbf{b}^{-1}$, we also expand $(\mathbf{b}^{-1})^{T}$:

$$(\mathbf{b}^{-1})^{T} = (\mathbf{F}^{-T}\mathbf{F}^{-1})^{T} = (\mathbf{F}^{-1})^{T}(\mathbf{F}^{-T})^{T} = \mathbf{F}^{-T}\mathbf{F}^{-1}$$

Expanding the term $\mathbf{e}$:

$$\mathbf{e} = \frac{\mathbf{I}-\mathbf{F}^{-T}\mathbf{F}^{-1}}{2}$$

Calculating $d\mathbf{x}\cdot\mathbf{e}\cdot d\mathbf{x}$:

$$\begin{aligned}d\mathbf{x}\cdot\mathbf{e}\cdot d\mathbf{x} &= d\mathbf{x}\cdot\frac{1}{2}(\mathbf{I}-\mathbf{F}^{-T}\mathbf{F}^{-1})\cdot d\mathbf{x} \\ &= d\mathbf{x}\cdot\frac{1}{2}(\mathbf{I}-\mathbf{b}^{-1})\cdot d\mathbf{x} \\ &= \frac{1}{2}\left((d\mathbf{x})^2 - d\mathbf{x}\cdot\mathbf{b}^{-1}\cdot d\mathbf{x}\right)\end{aligned}$$

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4c. Derive the expression for Lagrangian strain tensor in class notes.

Solution: Based on the lecture note:

$$\mathbf{E} = \frac{1}{2}(\mathbf{C}-\mathbf{I}) = \frac{1}{2}(\mathbf{F}^{T}\mathbf{F}-\mathbf{I})$$

Due to $\mathbf{b} = \mathbf{F}\mathbf{F}^{T}$, so that $\mathbf{C} = \mathbf{F}^{-1}\mathbf{b}\mathbf{F}$.

Therefore the Lagrangian strain writes:

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^{-1}\mathbf{b}\mathbf{F}-\mathbf{I})$$

Also we know, $\mathbf{F} = \mathbf{I} + \nabla_X\mathbf{u}$, therefore $\mathbf{F}^{T} = \mathbf{I} + (\nabla_X\mathbf{u})^{T}$

Then

$$\begin{aligned}\mathbf{F}^{T}\mathbf{F} &= \left(\mathbf{I} + (\nabla_X\mathbf{u})^{T}\right)\cdot(\mathbf{I}+\nabla_X\mathbf{u}) \\ &= \mathbf{I} + \nabla_X\mathbf{u} + (\nabla_X\mathbf{u})^{T} + (\nabla_X\mathbf{u})^{T}\nabla_X\mathbf{u}\end{aligned}$$

$$= \mathbf{I} + \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}$$

Substitute into $\mathbf{E}$ we have

$$\mathbf{E} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right)$$

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4d. Find the components of $\mathbf{e}$ in terms of the displacements expressed in current coordinates and compare your answer with the components of the Lagrangian strain tensor.

Solution: We first expand the terms of $\mathbf{F}$:

$$\mathbf{F} = \mathbf{I} + \nabla_X \mathbf{u} = \mathbf{I} + \frac{\partial u_i}{\partial X_j}\mathbf{e}_i\mathbf{e}_j$$

We thence deduce $\mathbf{F}^T$:

$$\mathbf{F}^T = \left(\mathbf{I} + \frac{\partial u_i}{\partial X_j}\mathbf{e}_i\mathbf{e}_j\right)^T = \mathbf{I} + \frac{\partial u_j}{\partial X_i}\mathbf{e}_i\mathbf{e}_j$$

We begin with $\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$, then $d\mathbf{X} = \mathbf{F}^{-1}d\mathbf{x}$, or $\mathbf{F}^{-1} = \frac{\partial \mathbf{X}}{\partial \mathbf{x}}$

From $\mathbf{X} - \mathbf{x} = \mathbf{u}$, we also know that $\nabla_x \mathbf{X} = \nabla_x \mathbf{x} + \nabla_x \mathbf{u} = \mathbf{I} + \nabla_x \mathbf{u} = \mathbf{F}^{-1}$

and

$$\mathbf{F}^{-T} = \mathbf{I} + (\nabla_x \mathbf{u})^T$$

We thence know

$$\mathbf{F}^{-T}\mathbf{F}^{-1} = (\mathbf{I} + (\nabla_x \mathbf{u})^T) \cdot (\mathbf{I} + (\nabla_x \mathbf{u}))$$
$$= \mathbf{I} + (\nabla_x \mathbf{u})^T + \nabla_x \mathbf{u} + (\nabla_x \mathbf{u})^T \nabla_x \mathbf{u}$$

Substitute such in the matrix $\mathbf{e}$,

$$\mathbf{e} = \frac{\mathbf{I} - \mathbf{F}^{-T}\mathbf{F}^{-1}}{2} = \frac{1}{2}((\nabla_x \mathbf{u})^T + \nabla_x \mathbf{u} + (\nabla_x \mathbf{u})^T \nabla_x \mathbf{u}))$$
$$= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$$

Compare $\mathbf{E}$ and $\mathbf{e}$ for small deformation we can observe they are just gradients of displacements in different configurations. And they are written in similar forms in different configurations.