Stanford AA274A Autumn 2020 HW1

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1. Trajectory Generation via Differential Flatness

(i). Write a set of linear equations in the coefficients x_i, y_i.

Similar to lecture notes, the set of equations can be written as:

$$\begin{bmatrix} \psi_1(t_0) & \psi_2(t_0) & \psi_3(t_0) & \psi_4(t_0) \\ \dot\psi_1(t_0) & \dot\psi_2(t_0) & \dot\psi_3(t_0) & \dot\psi_4(t_0) \\ \psi_1(t_f) & \psi_2(t_f) & \psi_3(t_f) & \psi_4(t_f) \\ \dot\psi_1(t_f) & \dot\psi_2(t_f) & \dot\psi_3(t_f) & \dot\psi_4(t_f) \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} = \begin{bmatrix} x(t_0) & y(t_0) \\ \dot x(t_0) & \dot y(t_0) \\ x(t_f) & y(t_f) \\ \dot x(t_f) & \dot y(t_f) \end{bmatrix}$$

Given $\psi_1(t) = 1, \psi_2(t) = t, \psi_3(t) = t^2, \psi_4(t) = t^3$, and $\dot x(t) = V(t)\cos(\theta(t)), \dot y(t) = V(t)\sin(\theta(t))$, the matrix equation becomes:

$$\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 0 & 1 & 2t_0 & 3t_0^2 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} = \begin{bmatrix} x(t_0) & y(t_0) \\ V(t_0)\cos(\theta(t_0)) & V(t_0)\sin(\theta(t_0)) \\ x(t_f) & y(t_f) \\ V(t_f)\cos(\theta(t_f)) & V(t_f)\sin(\theta(t_f)) \end{bmatrix}$$

Plug in with the following:

$$\begin{aligned} t_0 &= 0 \\ x(0) &= 0 \\ y(0) &= 0 \\ V(0) &= 0.5 \\ \theta(0) &= -\frac{\pi}{2} \\ t_f &= 15 \\ x(t_f) &= 5 \\ y(t_f) &= 5 \\ V(t_f) &= 0.5 \\ \theta(t_f) &= -\frac{\pi}{2} \end{aligned}$$

The final solution for the equations is:

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 15 & 225 & 3375 \\ 0 & 1 & 30 & 675 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & -0.5 \\ 5 & 5 \\ 0 & -0,5 \end{bmatrix}$$

The coefficients for the equations can be calculated with python numpy:

In [1]:

```
import numpy as np
A=np.array([[1,0,0,0],[0,1,0,0],[1,15,225,3375],[0,1,30,675]])
b=np.array([[0,0],[0,-0.5],[5,5],[0,-0.5]])
coeffs = np.dot(np.linalg.inv(A),b)
coeffs
```

Out[1]:

```
array([[ 0.        ,  0.        ],
       [ 0.        , -0.5       ],
       [ 0.06666667,  0.16666667],
       [-0.00296296, -0.00740741]])
```

(ii). Why can't we set $V(t_f) = 0$?

When we set $V(t_f) = 0$, the system is no longer differential flat.
In particular, the following equation is no longer solvable because of a singular transforming matrix.

$$\begin{bmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} \cos\theta & -V\sin\theta \\ \sin\theta & V\cos\theta \end{bmatrix} \begin{bmatrix} \alpha \\ \omega \end{bmatrix}$$

(iii). Coding

(iv). Coding

(v). differential flatness plot

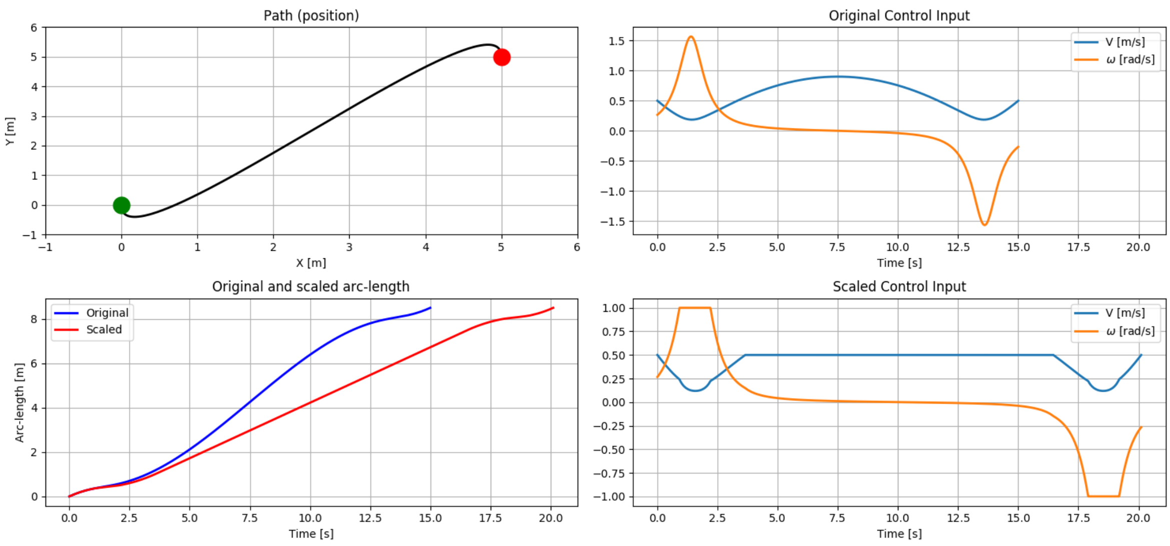
In [2]:

```
from IPython.display import Image
```

In [3]:

```
Image(filename='HW1/plots/differential_flatness.png')
```

Out[3]:

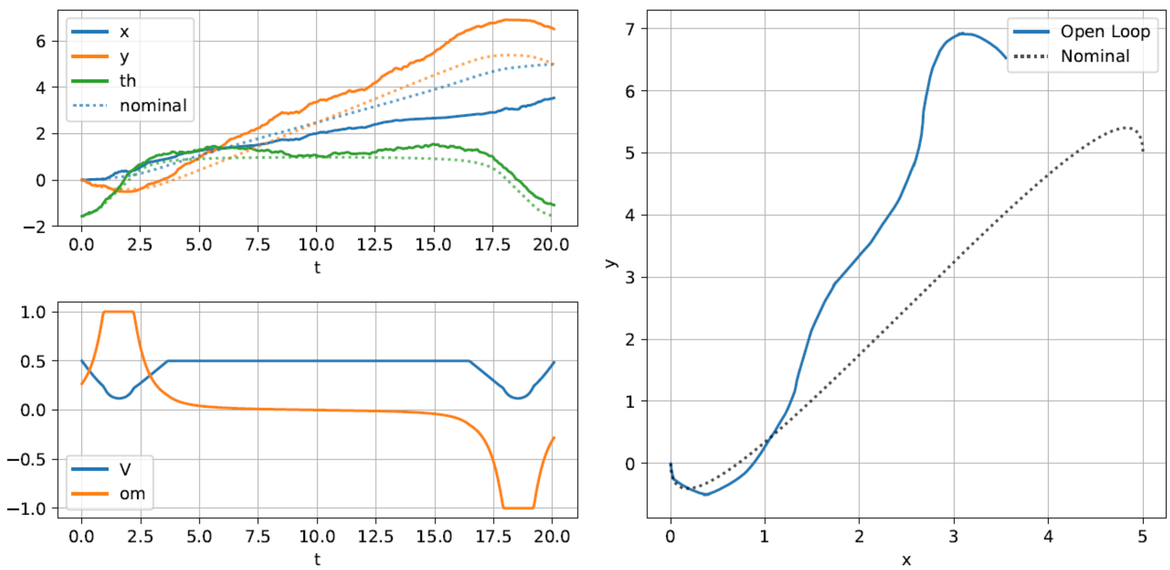


(vi). sim_traj_openloop plot

In [4]:

```
#PDF('HW1/plots/sim_traj_openloop.pdf',size=(1000,600))
Image(filename='HW1/plots/sim_traj_openloop.PNG')
```

Out[4]:



2. Pose Stabilization

(i). Coding

(ii). Test result (no write up)

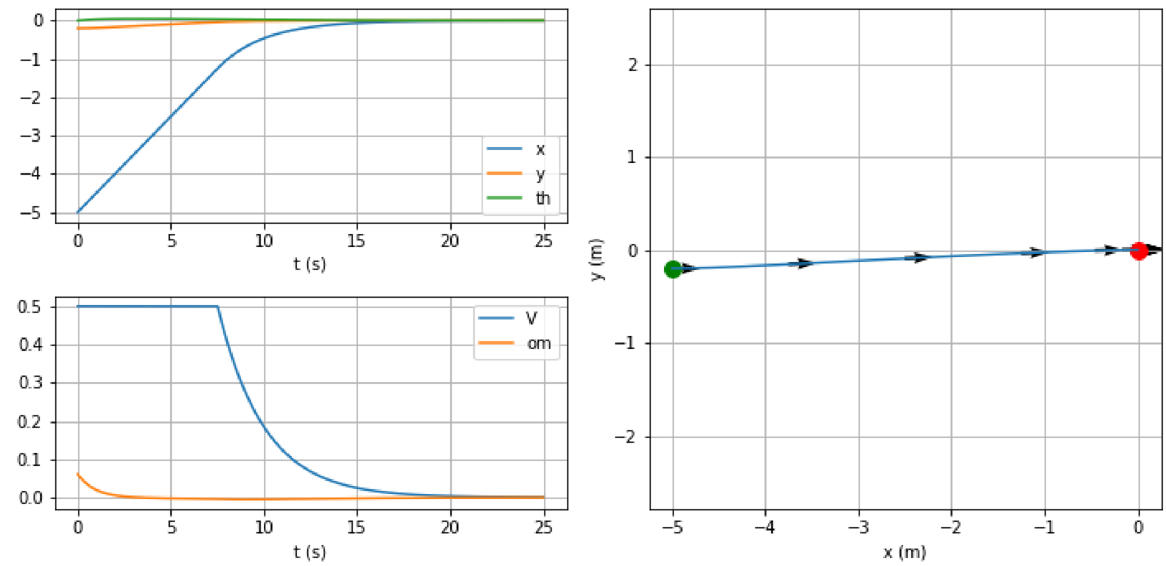
(iii). sim_parking_[parking-type] plot

sim_parking_forward.png:

In [5]:

```
Image(filename='HW1/plots/sim_parking_forward.png')
```

Out[5]:

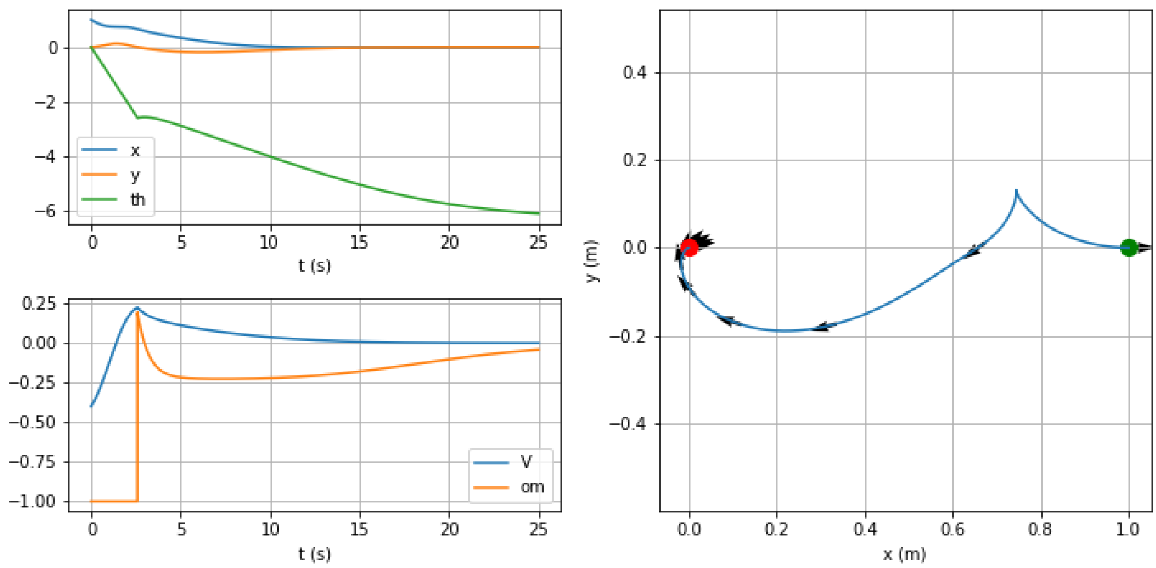


sim_parking_reverse.png:

In [6]:

```
Image(filename='HW1/plots/sim_parking_reverse.png')
```

Out[6]:

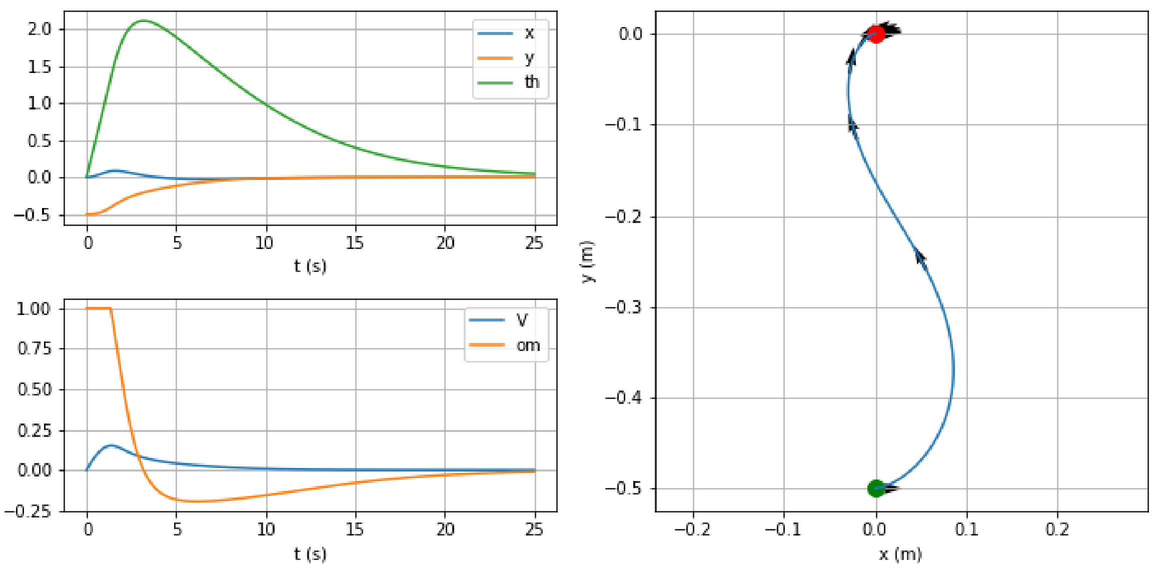


sim_parking_parallel.png:

In [7]:

```
Image(filename='HW1/plots/sim_parking_parallel.png')
```

Out[7]:



3. Trajectory Tracking

(i). Write down a system of equations for computing the true control inputs (V, ω) in terms of the virtual controls $(u_1, u_2) = \ddot{x}, \ddot{y}$ and the vehicle state.

From the kinematic model:

$$\dot{x}(t) = V\cos\theta(t)$$
$$\dot{y}(t) = V\sin\theta(t)$$

Take the derivative w.r.t. $\dot{x}$, $\dot{y}$ we get:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} \cos\theta & -V\sin\theta \\ \sin\theta & V\cos\theta \end{bmatrix} \begin{bmatrix} \alpha \\ \omega \end{bmatrix}$$

where $\alpha(t) = \dot{V}(t)$ is the acceleration, $\omega(t) = \dot{\theta}(t)$ is the torque.
Solve the above linear equation we get the acceleration and torque:

$$\begin{bmatrix} \alpha \\ \omega \end{bmatrix} = \begin{bmatrix} \cos\theta & -V\sin\theta \\ \sin\theta & V\cos\theta \end{bmatrix}^{-1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

and from the acceleration $\alpha(t)$ we integrate to the current speed:

$$V = \int_0^t \alpha(t')dt'$$

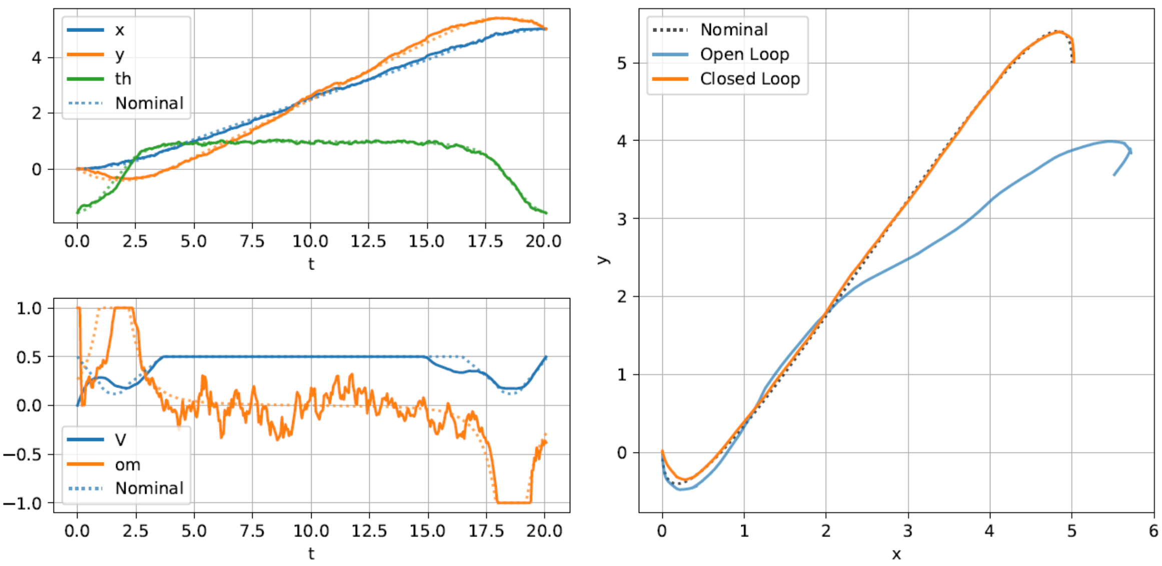
(ii). Coding

(iii) Validate and plot.

In [8]:

```
Image(filename='HW1/plots/sim_traj_closedloop.PNG')
```

Out[8]:



4. Extra. Optimal Control and Trajectory Optimization

(i). Derive the Hamiltonian and NOC and formulate the problem as a 2P-BVP.

The target function contains only a running cost, the termination cost is hence 0; we have:

$$h(x(t), t) = 0$$

$$g(x(t), u(t), t) = \lambda + V(t)^2 + \omega(t)^2$$

$$a(x(t), u(t), t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{\theta}(t) \end{bmatrix} = \begin{bmatrix} V(t)\cos\theta(t) \\ V(t)\sin\theta(t) \\ \omega(t) \end{bmatrix}$$

And it has no state/control constraints, so we follow lecture 5 slide 6 to form the Halmitonianas following:

$$\begin{aligned} H(x^*(t), u^*(t), p^*(t), t)) &= g(x(t), u(t), t) + p^T(t)[a(x(t), u(t), t)] \\ &= \lambda + V(t)^2 + \omega(t)^2 + p_1(t)V(t)\cos\theta(t) + p_2(t)V(t)\sin\theta(t) + p_3(t)\omega(t) \end{aligned}$$

The optimal solution satisfies the following Hamiltonian equations. The superscript * denotes this is the optimal value of the OCP problem.

$$\dot{x}^* = \frac{\partial H}{\partial p_1} = V^*(t)\cos\theta^*(t) \tag{1}$$
$$\dot{y}^* = \frac{\partial H}{\partial p_2} = V^*(t)\sin\theta^*(t) \tag{2}$$
$$\dot{\theta}^* = \frac{\partial H}{\partial p_3} = \omega^*(t) \tag{3}$$
$$\dot{p}_1^* = -\frac{\partial H}{\partial x} = 0 \tag{4}$$
$$\dot{p}_2^* = -\frac{\partial H}{\partial y} = 0 \tag{5}$$
$$\dot{p}_3^* = -\frac{\partial H}{\partial \theta} = p_1^*(t)V^*(t)\sin\theta^*(t) - p_2^*(t)V^*(t)\cos\theta^*(t) \tag{6}$$
$$0 = \frac{H}{V} = 2V^*(t) + p_1^*(t)\cos\theta^*(t) + p_2^*(t)\sin\theta^*(t) \tag{7}$$
$$0 = \frac{H}{w} = 2\omega^*(t) + p_3^*(t) \tag{8}$$

The boundary conditions are:

$$x(0) = 0$$
$$y(0) = 0$$
$$\theta(0) = -\frac{\pi}{2}$$
$$x(t_f) = 5$$
$$y(t_f) = 5$$
$$\theta(t_f) = -\frac{\pi}{2}$$

And since t_f is free, $x(t_f)$ is fixed, from lexture 5 lecture slide 8 we got another boundary condition:

$$H(x^*(t_f), u^*(t_f), p^*(t_f), t_f)) + \frac{\partial h}{\partial t}(x^*(t_f), t_f) = 0$$
$$\Rightarrow \lambda + V^*(t_f)^2 + \omega^*(t_f)^2 + p_1^*(t_f)V^*(t_f)\cos\theta^*(t_f) + p_2^*(t_f)V^*(t_f)\sin\theta^*(t_f) + p_3^*(t_f)\omega^*(t_f) = 0$$

For free t_f problem we introduce a new dummy state r that corresponds to t_f with dynamics $\dot{r} = 0$ as the last ODE of the BVP problem:

$$\dot{r} = 0 \tag{9}$$

the BVP state is hence $z = [x, y, \theta, p_1, p_2, p_3, r]$. Replace all occurrances of t_f with 1 and rewrite the BVP's boundary conditions as:

$$x(0) = 0 \tag{10}$$
$$y(0) = 0 \tag{11}$$
$$\theta(0) + \frac{\pi}{2} = 0 \tag{12}$$
$$x(1) - 5 = 0 \tag{13}$$
$$y(1) - 5 = 0 \tag{14}$$
$$\theta(1) + \frac{\pi}{2} = 0 \tag{15}$$
$$\begin{aligned} H(x^*(1), u^*(1), p^*(1), 1) &= \lambda + V^*(1)^2 + \omega^*(1)^2 + p_1^*(1)V^*(1)\cos\theta^*(1) \\ &+ p_2^*(1)V^*(1)\sin\theta^*(1) + p_3^*(1)\omega^*(1) = 0 \end{aligned} \tag{16}$$

Now the set of ODE for the 2P-BVP is the above Hamiltonian equations (1)-(6) and equation (9), and the boundary conditions for the 2P-BVP is the aboive equation (10)-(16).

The Hamiltonian equations (7), (8) also gives the solution of the input $[V, \omega]$ of the dynamics.

(ii). Coding

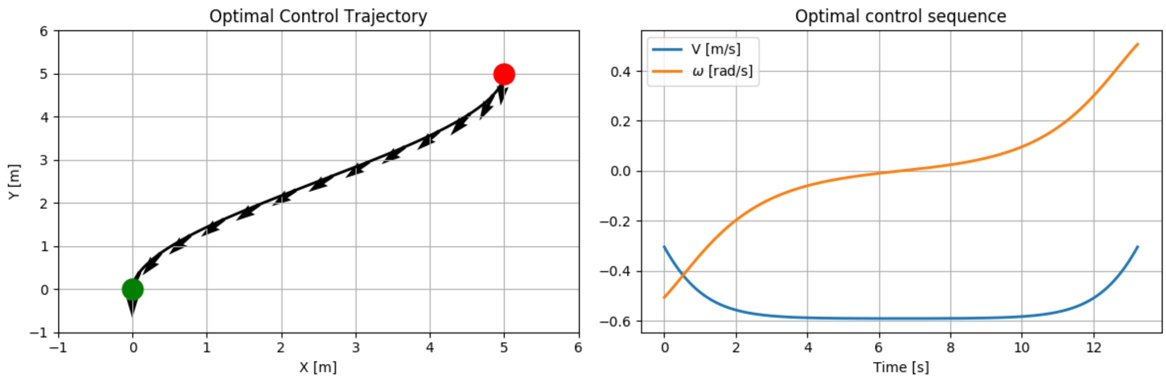
(iii). plot

Solution 1

In [9]:

```
Image(filename='HW1/plots/optimal_control.png')
```

Out[9]:

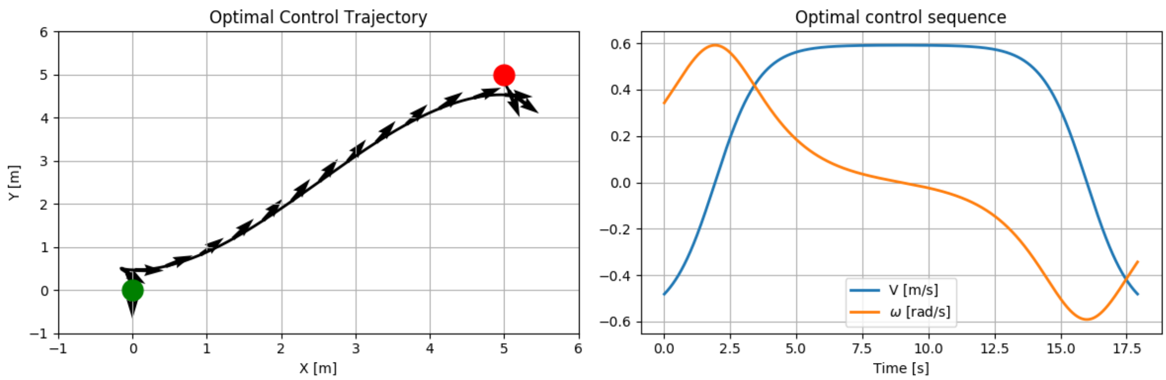


Solution 2

In [10]:

```
Image(filename='HW1/plots/optimal_control-1.png')
```

Out[10]:



(iv). Explain the significance of using the largest feasible λ

The choose of largest feasible λ will allow the solution to be able to reach the upper bound of input V and ω that is within the input constraints, and find a fastest solution. In the following cost function:

$$J = \lambda t_f + \int_0^{t_f} V^2 + \omega^2 dt$$

In order to minimize J, the larget the value λ, the smaller the time is driven, and hence pushing speed V and ω to reach their upper bounds.

(v). Validate and plot

By randomly sample a initial guess, I came up with two different solutions. Further step is to pick one from them and use the one with smaller cost. Here I just include both these solutions.

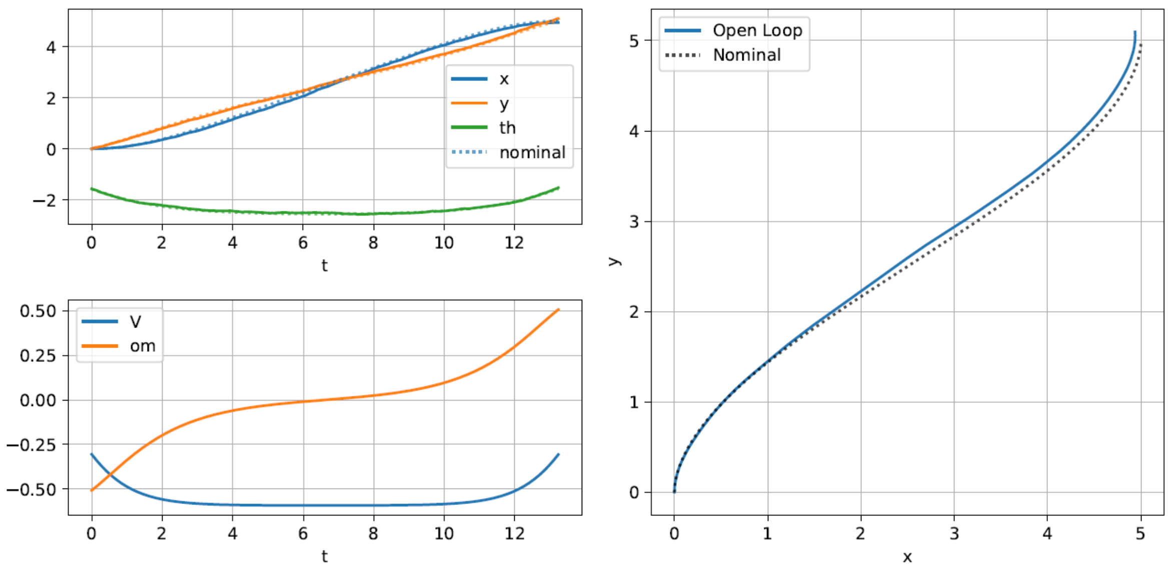
Solution 1

initial_guess=(3.24207804e+00,3.33710509e+00,-3.14159265e+00,1.86043668e+00,8.46899596e-01,2.05139181e+00,2.00000000e+03)

In [11]:

```
Image(filename='HW1/plots/sim_traj_optimal_control.PNG')
```

Out[11]:



Solution 2:

initial_guess = (2.56992092e-01,9.49860400e-01,-3.14159265e+00,5.06923249e-01,2.80611260e+00,2.59356897e+00,2.00000000e+03)

In [12]:

```
Image(filename='HW1/plots/sim_traj_optimal_control-1.PNG')
```

Out[12]: