AA274A PS1

1. Trajectory Generation via Differential Flatness

(i). Write a set of linear equations in the coefficients x_i, y_i.

Similar to lecture notes, the set of equations can be written as:

$$\begin{bmatrix} \psi_1(t_0) & \psi_2(t_0) & \psi_3(t_0) & \psi_4(t_0) \\ \dot{\psi}_1(t_0) & \dot{\psi}_2(t_0) & \dot{\psi}_3(t_0) & \dot{\psi}_4(t_0) \\ \psi_1(t_f) & \psi_2(t_f) & \psi_3(t_f) & \psi_4(t_f) \\ \dot{\psi}_1(t_f) & \dot{\psi}_2(t_f) & \dot{\psi}_3(t_f) & \dot{\psi}_4(t_f) \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} = \begin{bmatrix} x(t_0) & y(t_0) \\ \dot{x}(t_0) & \dot{y}(t_0) \\ x(t_f) & y(t_f) \\ \dot{x}(t_f) & \dot{y}(t_f) \end{bmatrix}$$

Given $\psi_1(t) = 1, \psi_2(t) = t, \psi_3(t) = t^2, \psi_4(t) = t^3$, and $\dot{x}(t) = V(t)\cos(\theta(t)), \dot{y}(t) = V(t)\sin(\theta(t))$, the matrix equation becomes:

$$\begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 0 & 1 & 2t_0 & 3t_0^2 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} = \begin{bmatrix} x(t_0) & y(t_0) \\ V(t_0)\cos(\theta(t_0)) & V(t_0)\sin(\theta(t_0)) \\ x(t_f) & y(t_f) \\ V(t_f)\cos(\theta(t_f)) & V(t_f)\sin(\theta(t_f)) \end{bmatrix}$$

Plug in with the following:

$$\begin{aligned} t_0 &= 0 \\ x(0) &= 0 \\ y(0) &= 0 \\ V(0) &= 0.5 \\ \theta(0) &= -\frac{\pi}{2} \\ t_f &= 15 \\ x(t_f) &= 5 \\ y(t_f) &= 5 \\ V(t_f) &= 0.5 \\ \theta(t_f) &= -\frac{\pi}{2} \end{aligned}$$

The final solution for the equations is:

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 15 & 225 & 3375 \\ 0 & 1 & 30 & 675 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & -0.5 \\ 5 & 5 \\ 0 & -0,5 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

The coefficients for the equations can be calculated with python numpy:

In [1]:
```
import numpy as np
A=np.array([[1,0,0,0],[0,1,0,0],[1,15,225,3375],[0,1,30,675]])
b=np.array([[0,0],[0,-0.5],[5,5],[0,-0.5]])
coeffs = np.dot(np.linalg.inv(A),b)
coeffs
```

Out[1]:
```
array([[ 0.        ,  0.        ],
       [ 0.        , -0.5       ],
       [ 0.06666667,  0.16666667],
       [-0.00296296, -0.00740741]])
```

(ii). Why can't we set $V(t_f) = 0$?

When we set $V(t_f) = 0$, the system is no longer differential flat.
In particular, the following equation is no longer solvable because of a singular transforming matrix.

$$\begin{bmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} \cos\theta & -V\sin\theta \\ \sin\theta & V\cos\theta \end{bmatrix} \begin{bmatrix} \alpha \\ \omega \end{bmatrix}$$

(iii). Coding

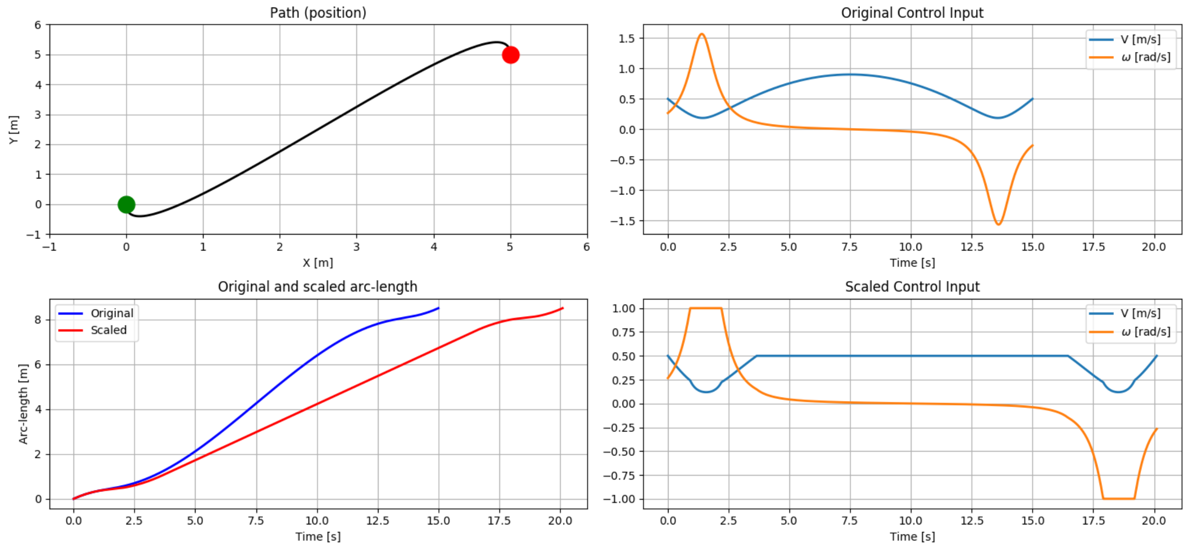
(iv). Coding

(v). differential flatness plot

In [2]:
```
from IPython.display import Image
```

In [3]:
```
Image(filename='HW1/plots/differential_flatness.png')
```

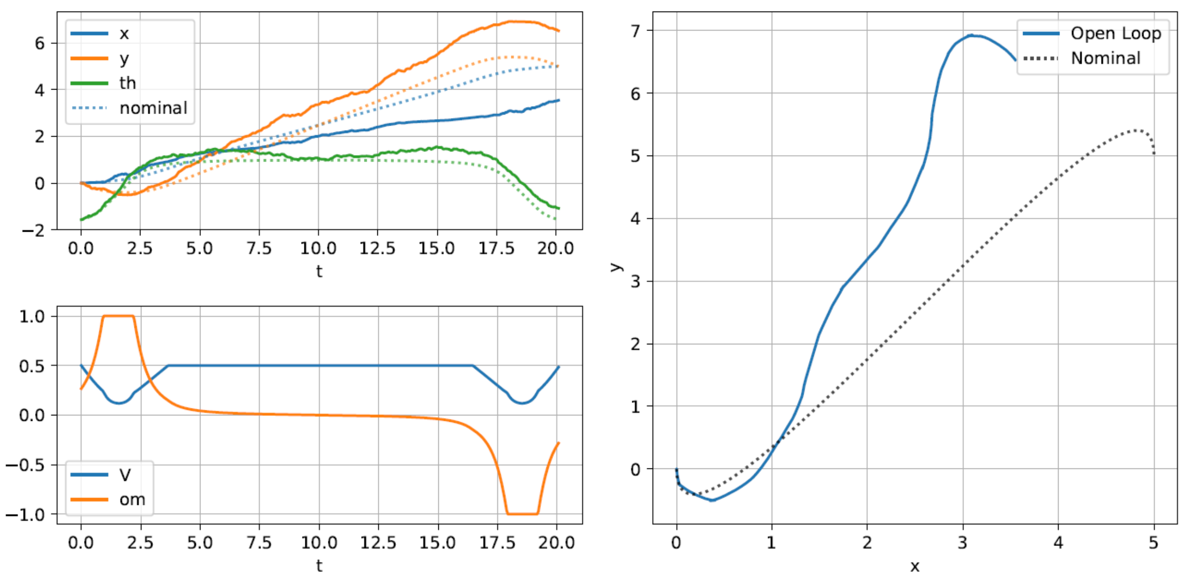
Out[3]:



(vi). sim_traj_openloop plot

In [4]:
```
#PDF('HW1/plots/sim_traj_openloop.pdf',size=(1000,600))
Image(filename='HW1/plots/sim_traj_openloop.PNG')
```

Out[4]:



2. Pose Stabilization

(i). Coding

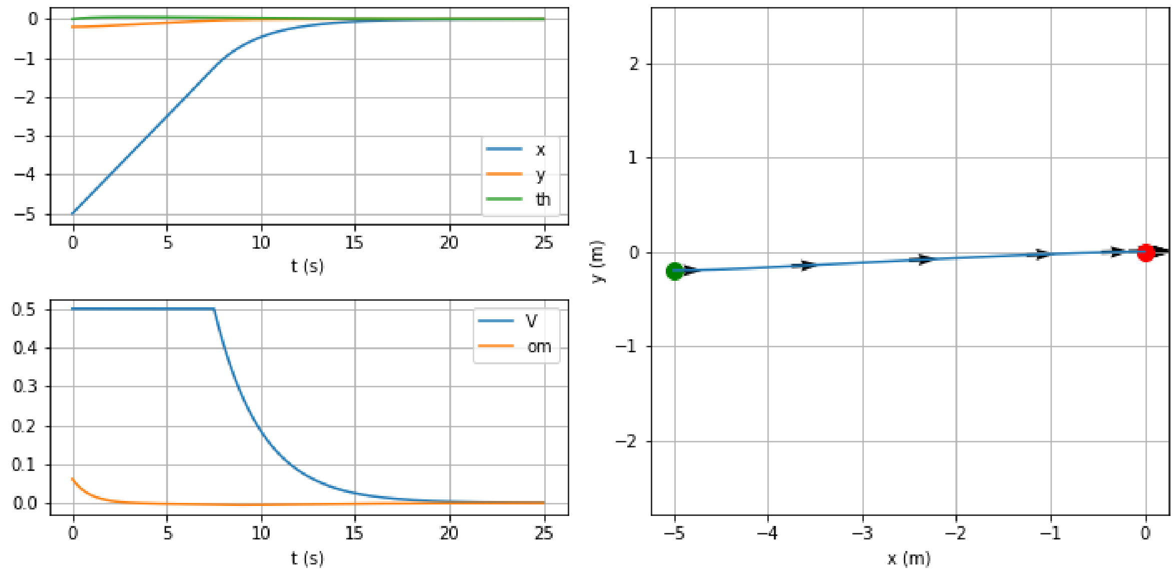
(ii). Test result (no write up)

(iii). simparking[parking-type] plot

sim_parking_forward.png:

In [5]:
```
Image(filename='HW1/plots/sim_parking_forward.png')
```

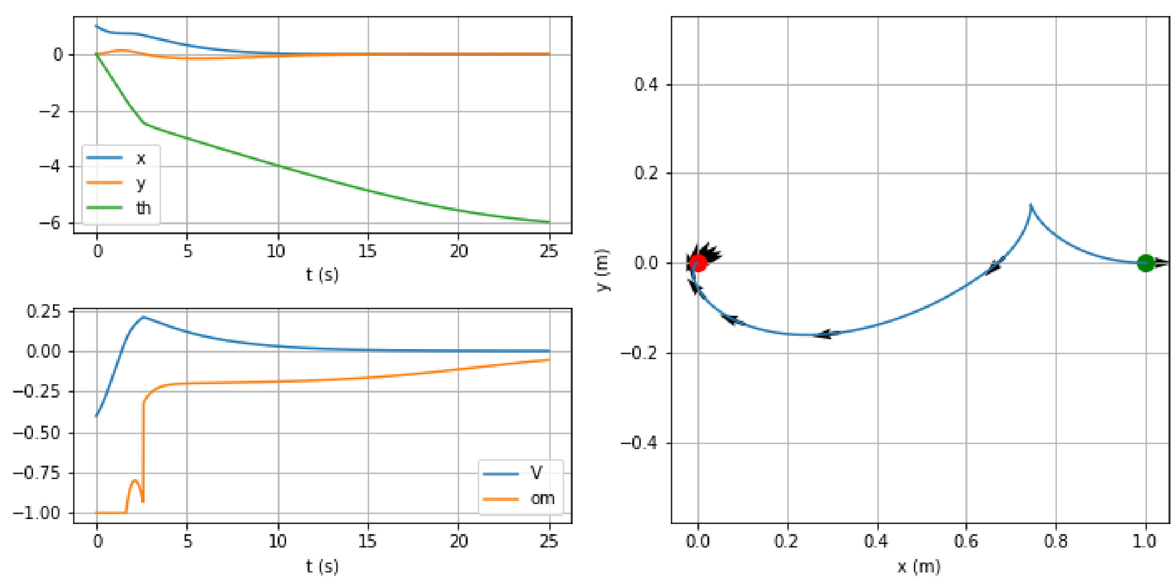
Out[5]:



sim_parking_reverse.png:

In [6]:
```
Image(filename='HW1/plots/sim_parking_reverse.png')
```

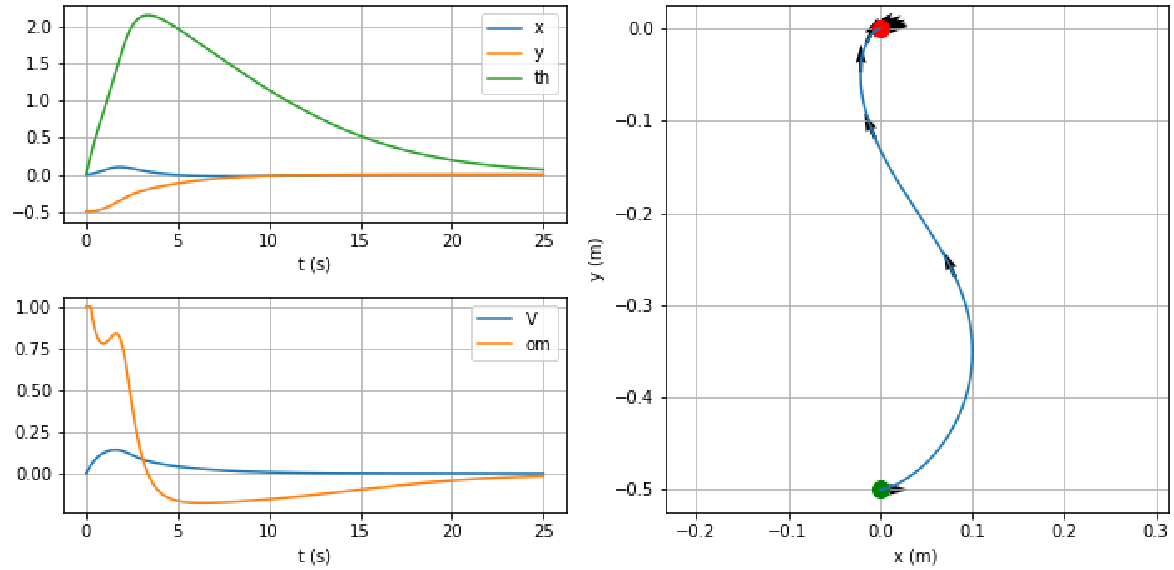
Out[6]:



sim_parking_parallel.png:

In [7]:
```
Image(filename='HW1/plots/sim_parking_parallel.png')
```

Out[7]:



3. Trajectory Tracking

(i). Write down a system of equations for computing the true control inputs (V, ω) in terms of the virtual controls $(u_1, u_2) = \ddot{x}, \ddot{y}$ and the vehicle state.

From the kinematic model:

$$\dot{x}(t) = V\cos\theta(t)$$
$$\dot{y}(t) = V\sin\theta(t)$$

Take the derivative w.r.t. $\dot{x}, \dot{y}$ we get:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} \cos\theta & -V\sin\theta \\ \sin\theta & V\cos\theta \end{bmatrix} \begin{bmatrix} \alpha \\ \omega \end{bmatrix}$$

where $\alpha(t) = \dot{V}(t)$ is the acceleration, $\omega(t) = \dot{\theta}(t)$ is the torque.
Solve the above linear equation we get the acceleration and torque:

$$\begin{bmatrix} \alpha \\ \omega \end{bmatrix} = \begin{bmatrix} \cos\theta & -V\sin\theta \\ \sin\theta & V\cos\theta \end{bmatrix}^{-1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

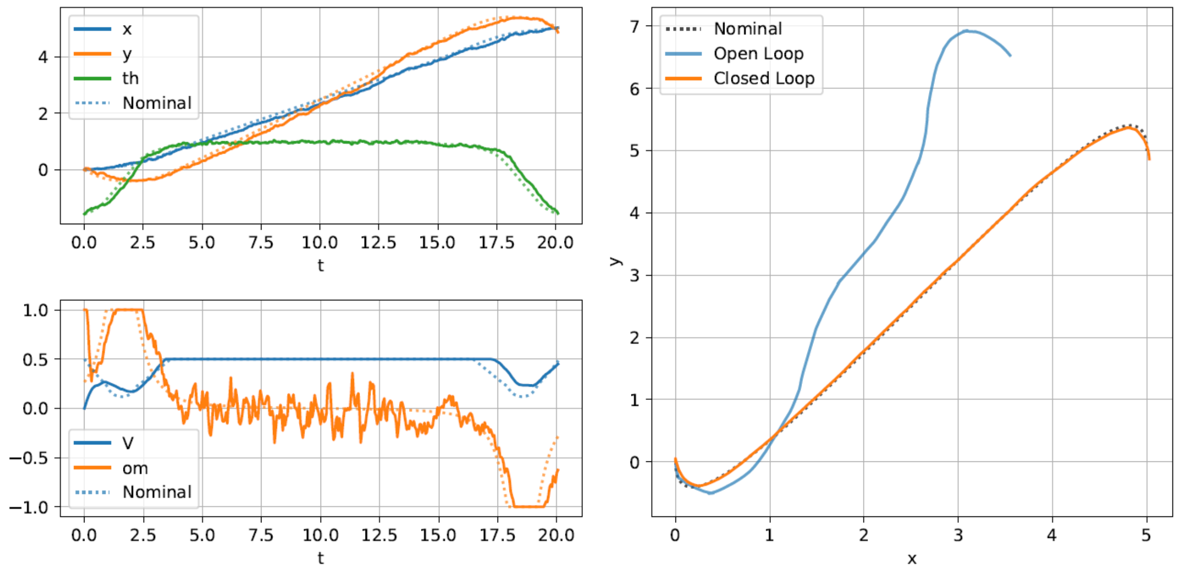
and from the acceleration $\alpha(t)$ we integrate to the current speed:

$$V = \int_0^t \alpha(t')dt'$$

(ii). Coding

(iii) Validate and plot.

In [8]:
```
Image(filename='HW1/plots/sim_traj_closedloop.PNG')
```
Out[8]:



4. Extra. Optimal Control and Trajectory Optimization

(i). Derive the Hamiltonian and NOC and formulate the problem as a 2P-BVP.

The target function contains only a running cost, the termination cost is hence 0; we have:

$$
\begin{aligned}
h(x(t), t) &= 0 \\
g(x(t), u(t), t) &= \lambda + V(t)^2 + \omega(t)^2 \\
a(x(t), u(t), t) &= \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{\theta}(t) \end{bmatrix} = \begin{bmatrix} V(t)\cos\theta(t) \\ V(t)\sin\theta(t) \\ \omega(t) \end{bmatrix}
\end{aligned}
$$

And it has no state/control constraints, so we follow lecture 5 slide 6 to form the Halmitonianas following:

$$
\begin{aligned}
H(x^*(t), u^*(t), p^*(t), t)) &= g(x(t), u(t), t) + p^T(t)[a(x(t), u(t), t)] \\
&= \lambda + V(t)^2 + \omega(t)^2 + p_1(t)V(t)\cos\theta(t) + p_2(t)V(t)\sin\theta(t) + p_3(t)\omega(t)
\end{aligned}
$$

The optimal solution satisfies the following Hamiltonian equations. The superscript denotes this is the optimal value of the OCP problem. $$ \begin{align} \dot x^&=\frac{\partial H}{\partial p_1} = V^(t) \cos \theta^(t) \tag 1 \ \dot y^&=\frac{\partial H}{\partial p_2} = V^(t) \sin \theta^(t) \tag 2 \ \dot \theta^&=\frac{\partial H}{\partial p_3} = \omega^(t) \tag 3 \ \dot p_1^&=-\frac{\partial H}{\partial x} = 0 \tag 4 \ \dot p_2^&=-\frac{\partial H}{\partial y} = 0 \tag 5 \ \dot p_3^&=-\frac{\partial H}{\partial \theta} = p_1^(t) V^(t) \sin \theta^(t) - p_2^(t) V^(t) \cos \theta^(t) \tag 6 \ 0 &= \frac{H}{V} = 2 V^(t) + p_1^(t) \cos \theta^(t) + p_2^(t) \sin \theta^(t) \tag 7 \ 0 &= \frac{H}{w} = 2 \omega^(t) + p_3^(t) \tag 8 \end{align*} $$

The boundary conditions are:

$$\begin{aligned} x(0) &= 0 \\ y(0) &= 0 \\ \theta(0) &= -\frac{\pi}{2} \\ x(t_f) &= 5 \\ y(t_f) &= 5 \\ \theta(t_f) &= -\frac{\pi}{2} \end{aligned}$$

And since t_f is free, $x(t_f)$ is fixed, from lexture 5 lecture slide 8 we got another boundary condition:

$$\begin{aligned} &H(x^*(t_f), u^*(t_f), p^*(t_f), t_f)) + \frac{\partial h}{\partial t}(x^*(t_f), t_f) = 0 \\ &\Rightarrow \lambda + V^*(t_f)^2 + \omega^*(t_f)^2 + p_1^*(t_f)V^*(t_f)\cos\theta^*(t_f) + p_2^*(t_f)V^*(t_f)\sin\theta^*(t_f) + p_3^*(t_f)\omega^*(t_f) = 0 \end{aligned}$$

For free t_f problem we introduce a new dummy state r that corresponds to t_f with dynamics $\dot{r} = 0$ as the last ODE of the BVP problem:

$$\dot{r} = 0 \tag{9}$$

the BVP state is hence $z = [x, y, \theta, p_1, p_2, p_3, r]$. Replace all occurrances of t_f with 1 and rewrite the BVP's boundary conditions as:

$$\begin{aligned} x(0) &= 0 \\ y(0) &= 0 \\ \theta(0) + \frac{\pi}{2} &= 0 \\ x(1) - 5 &= 0 \\ y(1) - 5 &= 0 \\ \theta(1) + \frac{\pi}{2} &= 0 \\ H(x^*(1), u^*(1), p^*(1), 1) &= \lambda + V^*(1)^2 + \omega^*(1)^2 + p_1^*(1)V^*(1)\cos\theta^*(1) + p_2^*(1)V^*(1)\sin\theta^*(\end{aligned}$$

Now the set of ODE for the 2P-BVP is the above Hamiltonian equations (1)-(6) and equation (9), and the boundary conditions for the 2P-BVP is the aboive equation (10)-(16).

The Hamiltonian equations (7), (8) also gives the solution of the input $[V, \omega]$ of the dynamics.

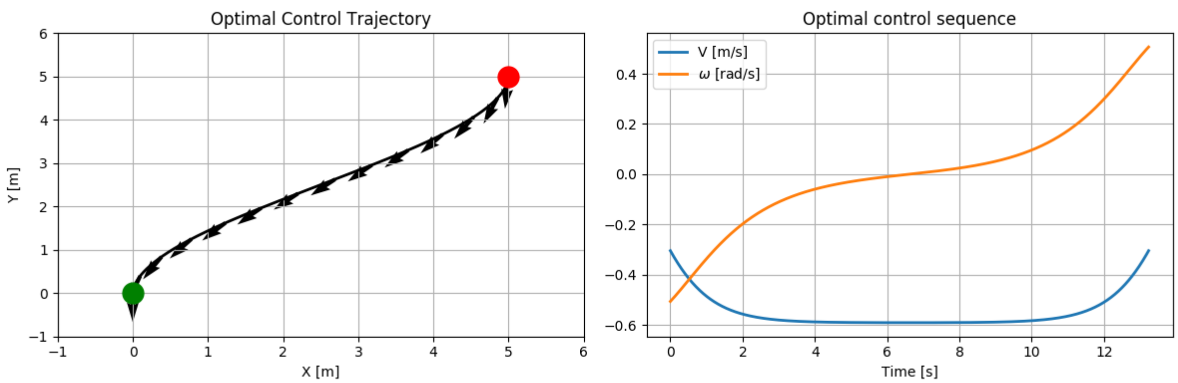
(ii). Coding

(iii). plot

Solution 1

In [9]:
```
Image(filename='HW1/plots/optimal_control.png')
```

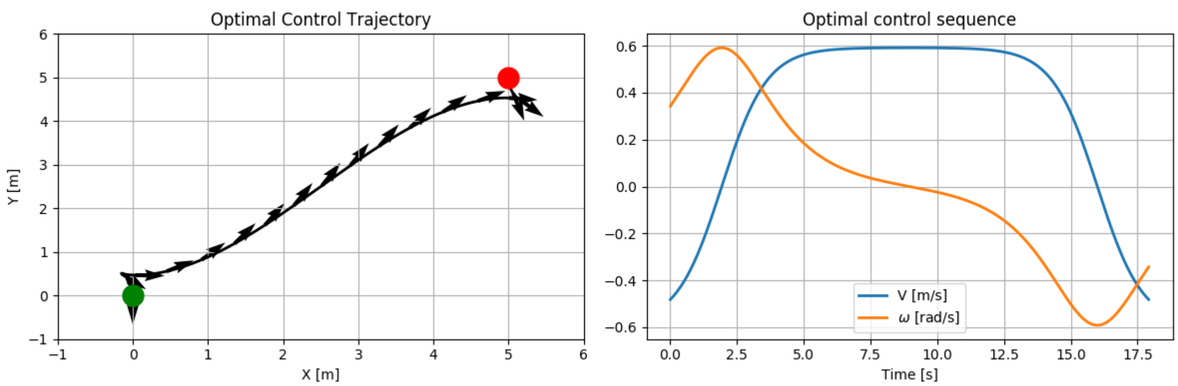
Out[9]:



Solution 2

In [10]:
```
Image(filename='HW1/plots/optimal_control-1.png')
```

Out[10]:



(iv). Explain the significance of using the largest feasible λ

The choose of largest feasible λ will allow the solution to be able to reach the upper bound of input V and ω that is within the input constraints, and find a fastest solution. In the following cost function:

$$J = \lambda t_f + \int_0^{t_f} V^2 + \omega^2 dt$$

In order to minimize J, the larget the value λ, the smaller the time is driven, and hence pushing speed V and ω to reach their upper bounds.

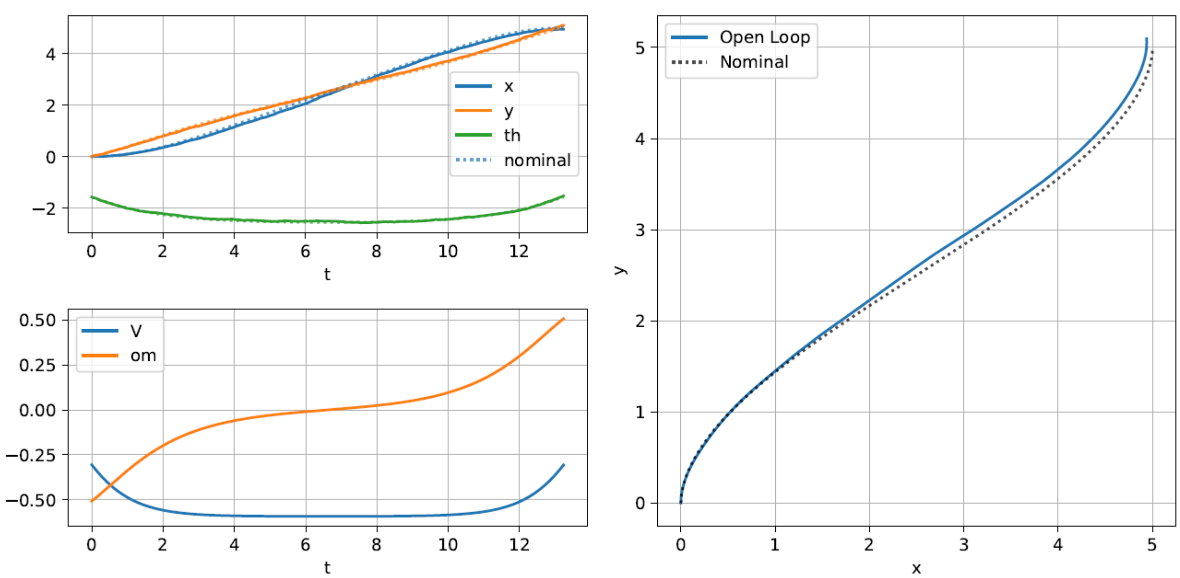
(v). Validate and plot

Solution 1

initial_guess=(3.24207804e+00,3.33710509e+00,-3.14159265e+00,1.86043668e+00,8.46899596e-01,2.05139181e+00,2.00000000e+03)

In [11]:
```
#PDF('HW1/plots/sim_traj_optimal_control.pdf', size=(1000,600))
Image(filename='HW1/plots/sim_traj_optimal_control.PNG')
```

Out[11]:



Solution 2:

initial_guess = (2.56992092e-01,9.49860400e-01,-3.14159265e+00,5.06923249e-01,2.80611260e+00,2.59356897e+00,2.00000000e+03)

In [12]:
```
#PDF('HW1/plots/sim_traj_optimal_control-1.pdf', size=(1000,600))
Image(filename='HW1/plots/sim_traj_optimal_control-1.PNG')
```

Out[12]: