Discrete Mathematics and Graph Theory (MAT2002)

Google Classroom Code:- imrxec

Module-1

Set Theory and Boolean Algebra

Relations and Functions, Partial Order Relations, Lattices, Boolean Algebra, Laws of Boolean Algebra, Boolean Functions- Normal Forms, Application of Boolean Algebra to Switching Circuits.

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Class Assessment Marks

Quiz (10 Marks)

- Min 4 Quiz, Max 6 Quiz
- Best of 3, average marks to be taken
- Quiz must be of 15 min duration

10 marks per quiz

Tutorial (10 Marks)

- Min 5
- Average mark of the tutorials
- 1 full session per tutorial
- 10Marks/Tutorial
- Printed tutorial sheets

Returned at the end of the session

Group Activity (5 Marks)

- Minimum 3 activity
- Duration 30 min each
- 5 Marks / Activity

Average marks to be taken

Assignment (10 Marks)

2 assignments each 5 marks

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Importance of this course

- Beneficial for GATE and other competitive examinations
- It develops mathematical thinking.
- Improves problem solving ability.
- If you are computer science students, then no need to go anywhere because discrete mathematics is for you.
- Discrete Mathematics is important to survive in subjects like: compiler design, databases, computer security, operating system, automata theory etc.

Problems that can be solved using knowledge of Discrete Mathematics

1. Sorting the list of integers.
2. Finding shortest path from your home to your friend’s home.
3. Drawing graph with two conditions:
   a. You are not allowed to lift your pen.
   b. You are not allowed to repeat edges.

4. How many different combinations of passwords are possible with just 8 alphanumeric characters.

5. Encrypt a message and deliver it to your friend and you don’t want anybody to read that message except your friend.

What is Discrete Mathematics?

- Discrete Mathematics is the study of discrete objects.

  Discrete means "distinct" or "not connected".
- Mathematics can be broadly classified into two categories −
- ***Continuous Mathematics*** − It is based upon continuous number line or the real numbers. It is characterized by the fact that between any two numbers, there are almost always an infinite set of numbers. For example, a function in continuous mathematics can be plotted in a smooth curve without breaks.
- ***Discrete Mathematics*** − It involves distinct values; i.e. between any two points, there are a countable number of points. For example, if we have a finite set of objects, the function can be defined as a list of ordered pairs having these objects, and can be presented as a complete list of those pairs.

Set Theory

- German Mathematician **G. Cantor** introduced the concept of sets. He had defined a set as a collection of definite and distinguishable objects selected by the means of certain rules or description.
- **Set** theory forms the basis of several other fields of study like counting theory, relations, graph theory and finite state machines. In this chapter, we will cover the different aspects of **Set Theory**.
- ***Definition:*** A set is an unordered collection of different elements. A set can be written explicitly by listing its elements using set bracket. If the order of the elements is changed or any element of a set is repeated, it does not make any changes in the set.

Some examples of sets

- A set of all positive integers
- A set of all the planets in the solar system
- A set of all the states in India
- A set of all the lowercase letters of the alphabet
- Many more…

Representation of a Set

Sets can be represented in two ways −

- Roster or Tabular Form
- Set Builder Notation

Roster or Tabular Form

The set is represented by listing all the elements comprising it. The elements are enclosed within braces and separated by commas.

Example 1 − Set of vowels in English alphabet, A = {a,e,i,o,u}.

Example 2- Set of odd numbers less than 10, B= {1,3,5,7,9}

Set Builder Notation

The set is defined by specifying a property that elements of the set have in common. The set is described as A={x: p(x)}.

Example 1 − The set {a,e,i,o,u} is written as A = {x: x is a vowel in a English Alphabet}

- If an element x is a member of any set S, it is denoted by $x \in S$ and if an element y is not a member of set S, it is denoted by $y \notin S$

Example – If $S = \{1, 1.2, 1.7, 2\}, 1 \in S$ but $1.5 \notin S$

Some Important Sets

N – the set of all natural numbers = $\{1, 2, 3, 4, \ldots\ldots\}$

Z – the set of all integers = $\{\ldots\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\ldots\}$

Z⁺ – the set of all positive integers

Q – the set of all rational numbers

R – the set of all real numbers

W – the set of all whole numbers

Cardinality of a Set

Cardinality of a set S, denoted by $|S|$, is the number of elements of the set. The number is also referred as the cardinal number. If a set has an infinite number of elements, its cardinality is ∞.

Example − $|\{1, 4, 3, 5\}| = 4, |\{1, 2, 3, 4, 5, \ldots\}| = \infty$

If there are two sets X and Y,

- $|X| = |Y|$ denotes two sets X and Y having same cardinality. It occurs when the number of elements in X is exactly equal to the number of elements in Y. In this case, there exists a bijective function 'f' from X to Y.
- $|X| \le |Y|$ denotes that set X's cardinality is less than or equal to set Y's cardinality. It occurs when number of elements in X is less than or equal to that of Y. Here, there exists an injective function 'f' from X to Y.
- $|X| < |Y|$ denotes that set X's cardinality is less than set Y's cardinality. It occurs when number of elements in X is less than that of Y. Here, the function 'f' from X to Y is injective function but not bijective.
- $If\ |X| \le |Y|$ and $|X| \ge |Y|$ then $|X| = |Y|$. The sets X and Y are commonly referred as equivalent sets.

Types of Sets

Sets can be classified into many types. Some of which are finite, infinite, subset, universal, proper, singleton set, etc.

Finite Set

A set which contains a definite number of elements is called a finite set.

Example − $S = \{x \mid x \in N$ and $70 > x > 50\}$

Infinite Set

A set which contains infinite number of elements is called an infinite set.

Example − $S = \{x \mid x \in N$ and $x > 10\}$

Subset

A set X is a subset of set Y (Written as $X \subseteq Y$) if every element of X is an element of set Y.

Example 1 − Let, $X = \{1, 2, 3, 4, 5, 6\}$ and $Y = \{1, 2\}$. Here set Y is a subset of set X as all the elements of set Y is in set X. Hence, we can write $Y \subseteq X$.

Proper Subset

The term "proper subset" can be defined as "subset of but not equal to". A Set X is a proper subset of set Y (Written as $X \subset Y$) if every element of X is an element of set Y and $|X| < |Y|$.

Example – Let, $X = \{1, 2, 3, 4, 5, 6\}$ and $Y = \{1, 2\}$. Here set $Y \subset X$ since all elements in Y are contained in X too and X has at least one element is more than set Y .

Universal Set

It is a collection of all elements in a particular context or application. All the sets in that context or application are essentially subsets of this universal set. Universal sets are represented as U .

Empty Set or Null Set

An empty set contains no elements. It is denoted by $\emptyset$. As the number of elements in an empty set is finite, empty set is a finite set. The cardinality of empty set or null set is zero.

Example – $S = \{x \mid x \in N$ and $7 < x < 8\} = \emptyset$

Singleton Set or Unit Set

Singleton set or unit set contains only one element. A singleton set is denoted by $\{s\}$.

Example – $S = \{x \mid x \in N,\ 7 < x < 9\}$ = $\{8\}$

Equal Set

If two sets contain the same elements they are said to be equal.

Example − If $A = \{1, 2, 6\}$ and $B = \{6, 1, 2\}$, they are equal as every element of set A is an element of set B and every element of set B is an element of set A.

Equivalent Set

If the cardinalities of two sets are same, they are called equivalent sets.

Example − If $A = \{1, 2, 6\}$ and $B = \{16, 17, 22\}$, they are equivalent as cardinality of A is equal to the cardinality of B. i.e. $|A| = |B| = 3$

Overlapping Set

Two sets that have at least one common element are called overlapping sets.

In case of overlapping sets −

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$
- $n(A) = n(A - B) + n(A \cap B)$
- $n(B) = n(B - A) + n(A \cap B)$

Example − Let, $A = \{1, 2, 6\}$ and $B = \{6, 12, 42\}$. There is a common element '6', hence these sets are overlapping sets.

Disjoint Set

Two sets A and B are called disjoint sets if they do not have even one element in common. Therefore, disjoint sets have the following properties −

- $n(A \cap B) = \emptyset$

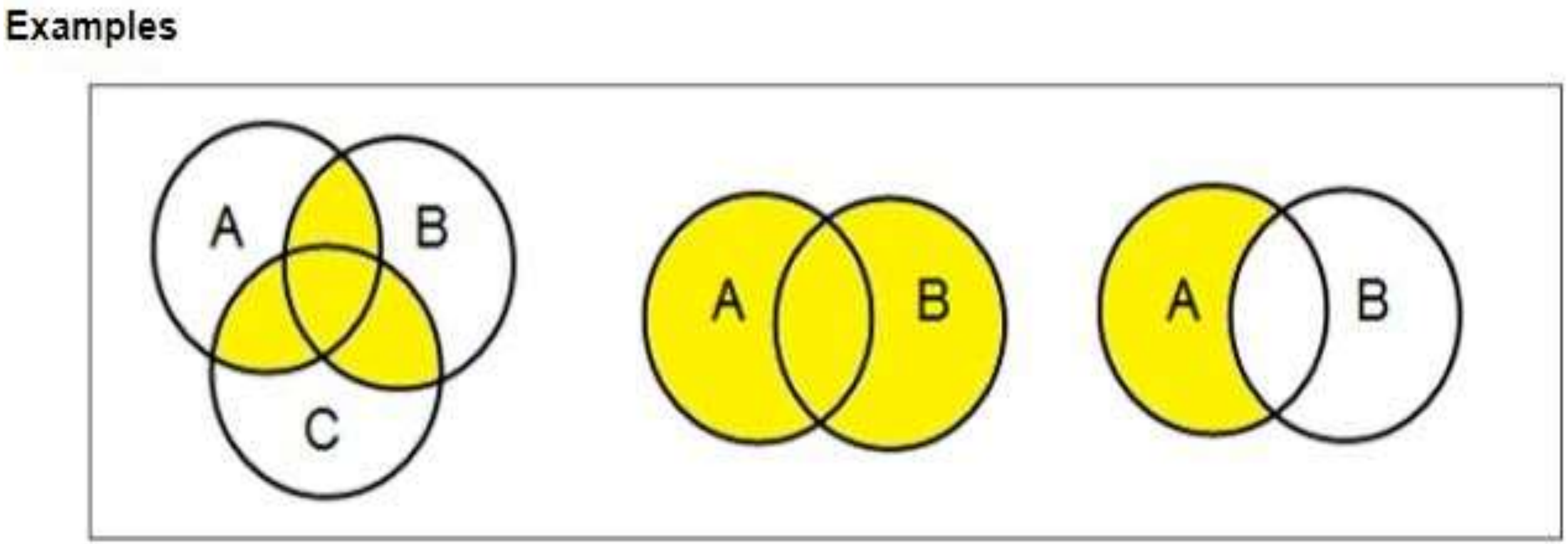
- $n(A \cup B) = n(A) + n(B)$

Example − Let, $A = \{1, 2, 6\}$ and $B = \{7, 9, 14\}$, there is not a single common element,

hence these sets are overlapping sets.

- Venn Diagrams

Venn diagram, invented in 1880 by John Venn, is a schematic diagram that shows all possible logical relations between different mathematical sets.



Set Operations

Set Operations include Set Union, Set Intersection, Set Difference, Complement of Set, and Cartesian Product.

Set Union

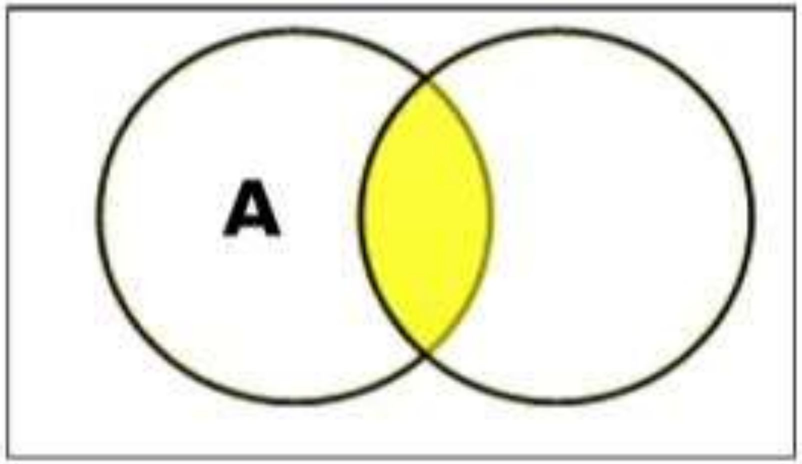
The union of sets A and B (denoted by $A \cup B$) is the set of elements which are in A, in B, or in both A and B. Hence, $A \cup B = \{x \mid x \in A \; OR \; x \in B\}$.

Example – If $A = \{10, 11, 12, 13\}$ and B = $\{13, 14, 15\}$, then $A \cup B = \{10, 11, 12, 13, 14, 15\}$. (The common element occurs only once)

Set Intersection

The intersection of sets A and B (denoted by $A \cap B$) is the set of elements which are in both A and B. Hence, $A \cap B = \{x \mid x \in A \; AND \; x \in B\}$.

Example – If $A = \{11, 12, 13\}$ and $B = \{13, 14, 15\}$, then $A \cap B = \{13\}$.



Set Difference/ Relative Complement

The set difference of sets A and B (denoted by $A - B$) is the set of elements which are only in A but not in B. Hence, $A - B = \{x \mid x \in A \; AND \; x \notin B\}$.

Example – If $A = \{10, 11, 12, 13\}$ and $B = \{13, 14, 15\}$, then $(A - B) = \{10, 11, 12\}$ and $(B - A) = \{14, 15\}$. Here, we can see $(A - B) \neq (B - A)$

Complement of a Set

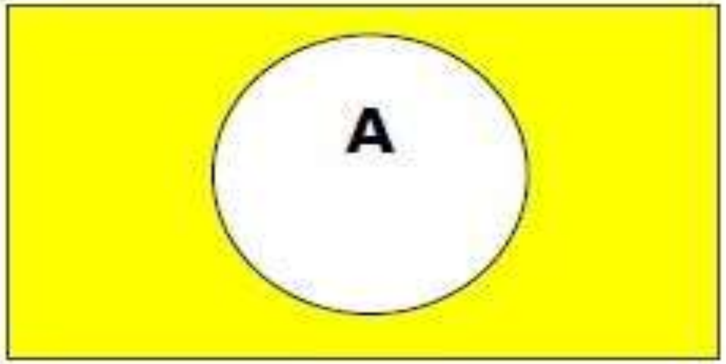
The complement of a set A (denoted by A') is the set of elements which are not in set A. Hence,

$$A' = \{x | x \notin A\}$$

More specifically, $A' = (U - A)$ where U is a universal set which contains all objects.

Example – If $A = \{x \mid x \ \ belongs\ to\ set\ of\ odd\ integers\}$ then

$$A' = \{y \mid y \ \ does\ not\ belong\ to\ set\ of\ odd\ integers\}$$



Cartesian Product / Cross Product

The Cartesian product of n number of sets $A_1, A_2, \ldots A_n$ denoted as $A_1 \times A_2 \cdots \times A_n$ can be defined as all possible ordered pairs $(x_1, x_2, \ldots x_n)$ where $x_1 \in A_1, x_2 \in A_2, \ldots x_n \in A_n$

Example – If we take two sets $A = \{a, b\}$ and $B = \{1, 2\}$,

The Cartesian product of A and B is written as – $A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$

The Cartesian product of B and A is written as – $B \times A = \{(1, a), (1, b), (2, a), (2, b)\}$

Power Set

Power set of a set S is the set of all subsets of S including the empty set. The cardinality of a power set of a set S of cardinality n is 2^n . Power set is denoted as $P(S)$.

Example –

For a set $S = \{a, b, c, d\}$ let us calculate the subsets –

- Subsets with 0 elements – $\{\emptyset\}$ (the empty set)
- Subsets with 1 element – $\{a\}, \{b\}, \{c\}, \{d\}$
- Subsets with 2 elements – $\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$
- Subsets with 3 elements – $\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$

- Subsets with 4 elements – $\{a, b, c, d\}$

Hence, $P(S) =$

$$\{ \quad \{\emptyset\}, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\} \quad \}$$

$$|P(S)| = 2^4 = 16$$

Note – The power set of an empty set is also an empty set.

$$|P(\{\emptyset\})| = 2^0 = 1$$

Partitioning of a Set

Partition of a set, say *S*, is a collection of *n* disjoint subsets, say $P_1, P_2, \ldots P_n$ that satisfies the following three conditions −

- P_i does not contain the empty set.

$$[P_i \neq \{\emptyset\} \ for \ all \ 0 < i \leq n]$$

- The union of the subsets must equal the entire original set.

$$[P_1 \cup P_2 \cup \cdots \cup P_n = S]$$

- The intersection of any two distinct sets is empty.

$$[P_a \cap P_b = \{\emptyset\}, \ for \ a \neq b \ where \ n \geq a, \ b \geq 0]$$

Example

Let $S = \{a, b, c, d, e, f, g, h\}$

One probable partitioning is $\{a\}, \{b, c, d\}, \{e, f, g, h\}$

Another probable partitioning is $\{a, b\}, \{c, d\}, \{e, f, g, h\}$

Relations

- Whenever sets are being discussed, the relationship between the elements of the sets is the next thing that comes up. **Relations** may exist between objects of the same set or between objects of two or more sets.

Definition and Properties

A binary relation R from set x to y (written as xRy or $R(x, y)$) is a subset of the Cartesian product $x \times y$. If the ordered pair of G is reversed, the relation also changes.

Generally an n-ary relation R between sets $A_1, \ldots, \; and \; A_n$ is a subset of the n-ary product $A_1 \times \cdots \times A_n$. The minimum cardinality of a relation R is Zero and maximum is n^2 in this case.

A binary relation R on a single set A is a subset of $A \times A$.

For two distinct sets, A and B, having cardinalities *m* and *n* respectively, the maximum cardinality of a relation R from A to B is *mn*.

Domain and Range

If there are two sets A and B, and relation R have order pair (x, y), then −

- The **domain** of R, Dom(R), is the set $\{x \mid (x, y) \in R \; for \; some \; y \; in \; B\}$
- The **range** of R, Ran(R), is the set $\{y \mid (x, y) \in R \; for \; some \; x \; in \; A\}$

Examples

Let, $A = \{1, 2, 9\}$ and $B = \{1, 3, 7\}$

- Case 1 − If relation R is 'equal to' then $R = \{(1, 1), (3, 3)\}$

  Dom(R) = $\{1, 3\}, Ran(R) = \{1, 3\}$

- Case 2 − If relation R is 'less than' then $R = \{(1, 3), (1, 7), (2, 3), (2, 7)\}$

  Dom(R) = $\{1, 2\}, Ran(R) = \{3, 7\}$

- Case 3 − If relation R is 'greater than' then $R = \{(2, 1), (9, 1), (9, 3), (9, 7)\}$

  Dom(R) = $\{2, 9\}, Ran(R) = \{1, 3, 7\}$

Types of Relations

- A relation R on set A is called **Reflexive** if $\forall a \in A$ is related to a (aRa holds)

Example – The relation $R = \{(a,a),(b,b)\}$ on set $X = \{a,b\}$ is reflexive.

A relation R on set A is called **Symmetric** if xRy implies yRx, $\forall x \in A$ and $\forall y \in A$.

Example – The relation $R = \{(1,2),(2,1),(3,2),(2,3)\}$ on set $A = \{1,2,3\}$ is symmetric.

- A relation R on set A is called **Transitive** if xRy and yRz implies $xRz, \forall x, y, z \in A$.

  Example − The relation $R = \{(1,2), (2,3), (1,3)\}$ on set $A = \{1,2,3\}$ is transitive.

- A relation is an **Equivalence Relation** if it is reflexive, symmetric, and transitive.

  Example − The relation $R = \{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2), (1,3), (3,1)\}$ on set $A = \{1,2,3\}$ is an equivalence relation since it is reflexive, symmetric, and transitive.

Functions

- ***Definition:*** A **Function** assigns to each element of a set, exactly one element of a related set.
- Functions find their application in various fields like representation of the computational complexity of algorithms, counting objects, study of sequences and strings, to name a few.

Function - Definition

A function or mapping (Defined as $f: X \to Y$) is a relationship from elements of one set X to elements of another set Y (X and Y are non-empty sets). X is called Domain and Y is called Codomain of function 'f'.

Function 'f' is a relation on X and Y such that for each $x \in X$, there exists a unique $y \in Y$ such that $(x, y) \in R$. 'x' is called pre-image and 'y' is called image of function f.

A function can be one to one or many to one but not one to many.

Injective / One-to-one function

A function $f : A \to B$ is injective or one-to-one function if for every $b \in B$, there exists at most one $a \in A$ such that $f(s) = t$.

This means a function **f** is injective if $a_1 \neq a_2$ implies $f(a1) \neq f(a2)$.

Example

- $f : N \to N, f(x) = 5x$ is injective.
- $f : N \to N, f(x) = x^2$ is injective.
- $f : R \to R, f(x) = x^2$ is not injective as $(-x)^2 = x^2$

Surjective / Onto function

A function $f : A \to B$ is surjective (onto) if the image of f equals its range. Equivalently, for every $b \in B$, there exists some $a \in A$ such that $f(a) = b$. This means that for any y in B, there exists some x in A such that $y = f(x)$.

Example

- $f : N \to N, f(x) = x + 2$ is surjective.
- $f : R \to R, f(x) = x^2$ is not surjective since we cannot find a real number whose square is negative.

Bijective / One-to-one Correspondent

A function $f : A \rightarrow B$ is bijective or one-to-one correspondent if and only if **f** is both injective and surjective.

Problem

Prove that a function $f : R \rightarrow R$ defined by $f(x) = 2x - 3$ is a bijective function.

Explanation − We have to prove this function is both injective and surjective.

If $f(x_1) = f(x_2)$, then $2x_1 - 3 = 2x_2 - 3$ and it implies that $x_1 = x_2$.

Hence, f is **injective**.

Here, $2x - 3 = y$

So, $x = (y + 5)/3$ which belongs to R and $f(x) = y$.

Hence, f is **surjective**.

Since **f** is both **surjective** and **injective**, we can say **f** is **bijective**.

Composition of Functions

Two functions $f : A \to B$ and $g : B \to C$ can be composed to give a composition gof .

This is a function from A to C defined by $(gof)(x) = g(f(x))$

Example

Let $f(x) = x + 2$ and $g(x) = 2x + 1$, find $(fog)(x)$ and $(gof)(x)$.

Solution

$$(fog)(x) = f(g(x)) = f(2x + 1) = 2x + 1 + 2 = 2x + 3$$

$$(gof)(x) = g(f(x)) = g(x + 2) = 2(x + 2) + 1 = 2x + 5$$

Questions on Set theory, Relations, Functions

1. Which of the following collections are sets ?

   (i) The collection of days in a week starting with S.

   (ii) The collection of natural numbers upto fifty.

   (iii) The collection of poems written by Tulsidas.

   (iv) The collection of fat students of your school.

(2) Insert the appropriate symbol in blank spaces.

If $A = \{1,2,3\}$.

(i) 1..........A (ii) 4........A.

3. Write each of the following sets in the Roster form :

(i) $A = \{x : x \in z \text{ and } -5 \leq x \leq 0\}$.

(ii) $B = \{x : x \in R \text{ and } x^2 - 1 = 0\}$.

(iii) $C = \{x : x$ is a letter of the word banana$\}$.

(iv) $D = \{x : x$ is a prime number and exact divisor of $60\}$.

4. Write each of the following sets are in the set builder form ?

(i) $A = \{2, 4, 6, 8, 10\}$ (ii) $B = \{3, 6, 9, \ldots\ldots \infty\}$

(iii) $C = \{2, 3, 5, 7\}$ (iv) $D = \{-\sqrt{2}, \sqrt{2}\}$

Are A and B disjoints sets ?

5. Which of the following sets are finite and which are infinite ?

(i) Set of lines which are parallel to a given line.

(ii) Set of animals on the earth.

(iii) Set of Natural numbers less than or equal to fifty.

(iv) Set of points on a circle.

6. Which of the following are null set or singleton ?

(i) $A = \{x : x \in R \text{ and } x \text{ is a solution of } x^2 + 2 = 0\}$.

(ii) $B = \{x : x \in Z \text{ and } x \text{ is a solution of } x - 3 = 0\}$.

(iii) $C = \{x : x \in Z \text{ and } x \text{ is a solution of } x^2 - 2 = 0\}$.

(iv) $D = \{x : x$ is a student of your school studying in both the classes XI and XII $\}$

7. In the following check whether $A = B$ or $A \approx B$.

(i) $A = \{a\}$, $B = \{x : x$ is an even prime number$\}$.

(ii) $A = \{1, 2, 3, 4\}$, $B = \{x : x$ is a letter of the word guava$\}$.

(iii) $A = \{x : x \text{ is a solution of } x^2 - 5x + 6 = 0\}, B = \{2, 3\}$.

1. Insert the appropriate symbol in the blank spaces, given that

$A=\{1, 3, 5, 7, 9\}$

(i) ϕ ....A (ii) $\{2, 3, 9\}$.........A

(iii) 3...........A (iv) 10..................A

2. Given that $A = \{a, b\}$, how many elements P(A) has ?

3. Let $A = \{\phi, \{1\}, \{2\}, \{1,2\}\}$

Which of the following is true or false ?

(i) $\{1,2\} \subset A$ (ii) $\phi \in A$.

4. Which of the following statements are true or false ?

(i) Set of all boys, is contained in the set of all students of your school.

(ii) Set of all boy students of your school, is contained in the set of all students of your school.

(iii) Set of all rectangles, is contained in the set of all quadrilaterals.

(iv) Set of all circles having centre at origin is contained in the set of all ellipses having centre at origin.

1. Which of the following pairs of sets are disjoint and which are not ?

(i) $\{x : x$ is an even natural number$\}$, $\{y : y$ is an odd natural number$\}$

(ii) $\{x : x$ is a prime number and divisor of $12\}$, $\{y : y \in N$ and $3 \leq y \leq 5\}$

(iii) $\{x : x$ is a king of 52 playing cards$\}$, $\{y : y$ is a diamond of 52 playing cards$\}$

(iv) $\{1, 2, 3, 4, 5\}$, $\{a, e, i, o, u\}$

2. Find the intersection of A and B in each of the following :

(i) $A = \{x : x \in Z\}$, $B = \{x : x \in N\}$ (ii) $A =$ {Ram, Rahim, Govind, Gautam}

$B =$ {Sita, Meera, Fatima, Manprit}

3. Given that $A = \{1, 2, 3, 4, 5\}$, $B = \{5, 6, 7, 8, 9, 10\}$

find (i) $A \cup B$ (ii) $A \cap B$.

4. If $A = \{x : x \in N\}$, $B = \{y : y \in z$ and $-10 \leq y \leq 0\}$ find $A \cup B$ and write your answer in the Roster form as well as set-builder form.

5. If $A = \{2, 4, 6, 8, 10\}$, B $\{8, 10, 12, 14\}$, $C = \{14, 16, 18, 20\}$.

Find (i) $A \cup (B \cup C)$ (ii) $A \cap (B \cap C)$.

6. Let $U = \{1, 2, 3, ...10\}$, $A = \{2, 4, 6, 8, 10\}$, $B = \{1, 3, 5, 7, 9, 10\}$

Find (i) $(A \cup B)'$ (ii) $(A \cap B)'$ (iii) $(B')'$ (iv) $(B - A)'$.

8. Draw Venn diagram for each the following :

(i) $A \cup B$ when $A \subset B$. (ii) $A \cup B$ when A and B are disjoint sets.

(iii) $A \cup B$ when A and B are neither subsets of each other nor disjoint sets.

9. Draw Venn diagram for each the following :

(i) $A - B$ and $B - A$ when $A \subset B$.

(ii) $A - B$ and $B - A$ when A and B are disjoint sets.

(iii) $A - B$ and $B - A$ when A and B are neither subsets of each other nor disjoint sets.

Questions on Relations

1. Given that $A = \{4, 5, 6, 7\}$, $B = \{8, 9\}$, $C = \{10\}$

   Verify that $A \times (B - C) = (A \times B) - (A \times C)$.

2. If U is a universal set and A, B are its subsets.

   Where $U = \{1, 2, 3, 4, 5\}$.

   $A = \{1,3,5\}$, $B = \{x : x \text{ is a prime number}\}$ find $A' \times B'$

3. If $A = \{4, 6, 8, 10\}$, $B = \{2, 3, 4, 5\}$

   R is a relation defined from A to B where

   $R = \{(a, b) : a \in A,\ b \in B \text{ and a is a multiple of b}\}$

   find (i)R in the Roster form (ii) Domain of R (iii) Range of R .

4. If R be a relation from N to N defined by

   $R = \{(x,y) : 4x + y = 12,\ x, y \in N\}$

   find (i) R in the Roster form (ii) Domain of R (iii) Range of R.

5. If R be a relation on N defined by

   $R = \{(x, x^2) : x \text{ is a prime number less than } 15\}$

Find (i) R in the Roster form (ii) Domain of R (iii) Range of R

6. If R be a relation on set of real numbers defined by $R=\{(x,y) : x^2 + y^2 = 0\}$ Find

(i) R in the Roster form (ii) Domain of R (iii) Range of R.

Questions on Functions

Example 15.20 Which of the following relations are functions from A to B. Write their domain and range. If it is not a function give reason ?

(a) $\{(1,-2),(3,7),(4,-6),(8,1)\}$, $A=\{1,3,4,8\}$, $B=\{-2,7,-6,1,2\}$

(b) $\{(1,0),(1-1),(2,3),(4,10)\}$, $A=\{1,2,4\}$, $B=\{0,-1,3,10\}$

(c) $\{(a,b),(b,c),(c,b),(d,c)\}$, $A=\{a,b,c,d,e\}$ $B=\{b,c\}$

(d) $\{(2,4),(3,9),(4,16),(5,25),(6,36\}$, $A=\{2,3,4,5,6\}$, $B=\{4,9,16,25,36\}$

(e) $\{(1,-1),(2,-2),(3,-3),(4,-4),(5,-5)\}$, $A=\{0,1,2,3,4,5\}$,
$B=\{-1,-2,-3,-4,-5\}$

(f) $\left\{\left(\sin\frac{\pi}{6},\frac{1}{2}\right),\left(\cos\frac{\pi}{6},\frac{\sqrt{3}}{2}\right),\left(\tan\frac{\pi}{6},\frac{1}{\sqrt{3}}\right),\left(\cot\frac{\pi}{6},\sqrt{3}\right)\right\}$,

$$A=\left\{\sin\frac{\pi}{6},\cos\frac{\pi}{6},\tan\frac{\pi}{6},\cot\frac{\pi}{6}\right\}\quad B=\left\{\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{\sqrt{3}},\sqrt{3},1\right\}$$

Example 15.21 State whether each of the following relations represent a function or not.

(a)

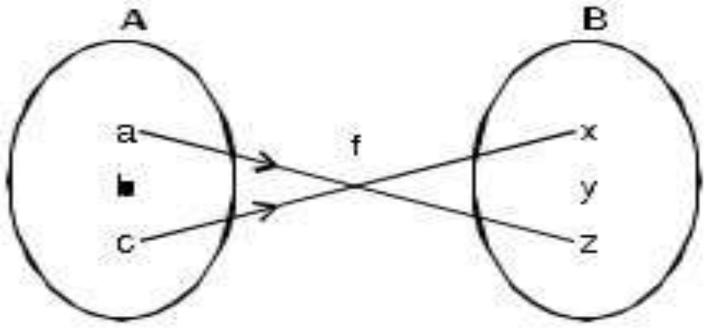


Fig.15.14

(b)

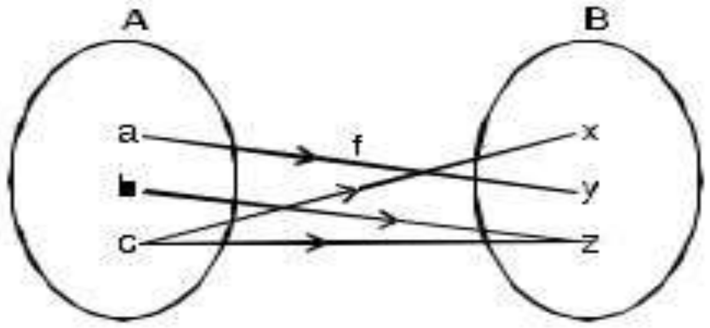


Fig.15.15

(c)

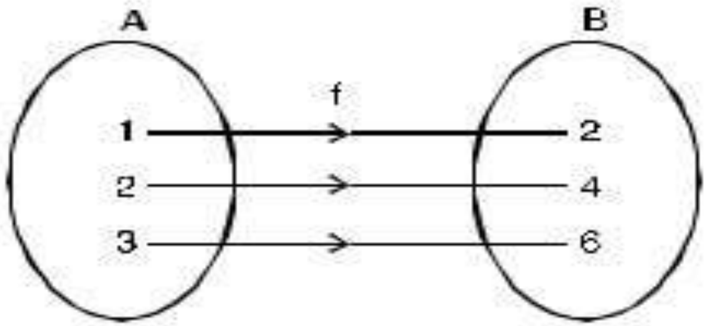


Fig. 15.16

(d)

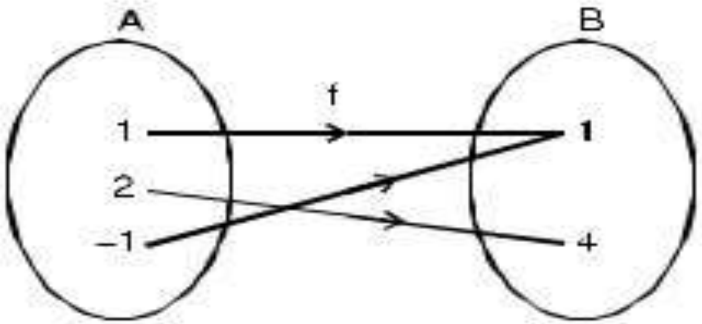


Fig.15.17

1. Find the domain of each of the following functions $x \in R$:

(a) (i) $y = 2x$ (ii) $y = 9x + 3$ (iii) $y = x^2 + 5$

(b) (i) $y = \frac{1}{3x - 1}$ (ii) $y = \frac{1}{(4x + 1)(x - 5)}$

(iii) $y = \frac{1}{(x - 3)(x - 5)}$ (iv) $y = \frac{1}{(3 - x)(x - 5)}$

(c) (i) $y = \sqrt{6 - x}$ (ii) $y = \sqrt{7 + x}$

(iii) $y = \sqrt{3x + 5}$

(d) (i) $y = \sqrt{(3 - x)(x - 5)}$ (ii) $y = \sqrt{(x - 3)(x + 5)}$

(iii) $y = \frac{1}{\sqrt{(3 + x)(7 + x)}}$ (iv) $y = \frac{1}{\sqrt{(x - 3)(7 + x)}}$

2. Find the range of the function, given its domain in each of the following cases.

(a) (i) $f(x) = 3x + 10,$ $x \in \{1, 5, 7, -1, -2\}$

(ii) $f(x) = 2x^2 + 1,$ $x \in \{-3, 2, 4, 0\}$

(iii) $f(x) = x^2 - x + 2,$ $x \in \{1, 2, 3, 4, 5\}$

(i) Does the graph represent a function?

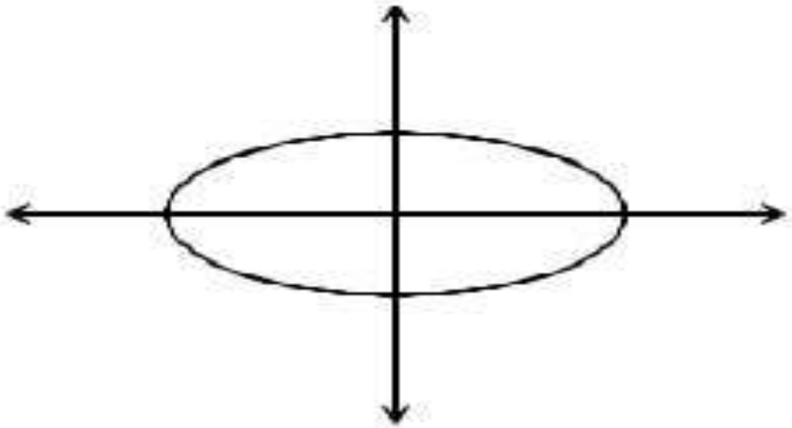


Fig. 15.36

(ii) Does the graph represent a function ?

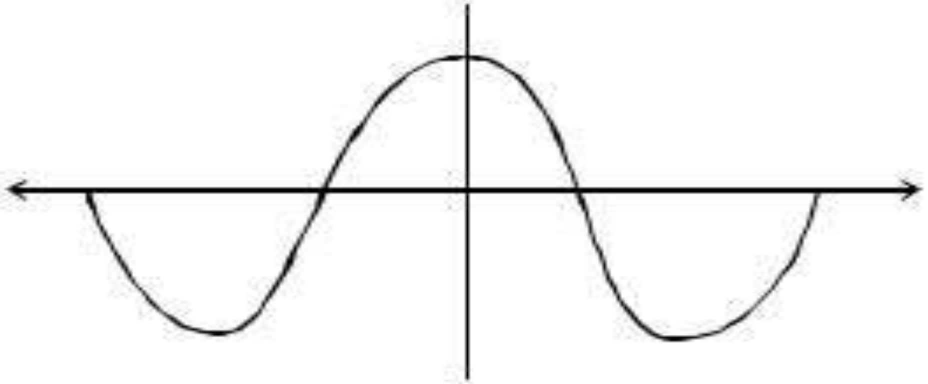


Fig. 15.37

4. Which of the following functions are one-to-one functions ?

(a) $f : \{20,21,22\} \to \{40,42,44\}$ defined as $f(x) = 2x$

(b) $f : \{7,8,9\} \to \{10\}$ defined as $f(x) = 10$

(c) $f : I \to R$ defined as $f(x) = x^3$

(d) $f : R \to R$ defined as $f(x) = 2 + x^4$

(d) $f : N \to N$ defined as $f(x) = x^2 + 2x$

5. Which of the following functions are many-to-one functions ?

(a) $f : \{-2,-1,1,2\} \to \{2,5\}$ defined as $f(x) = x^2 + 1$

(b) $f : \{0,1,2\} \to \{1\}$ defined as $f(x) = 1$

(c)

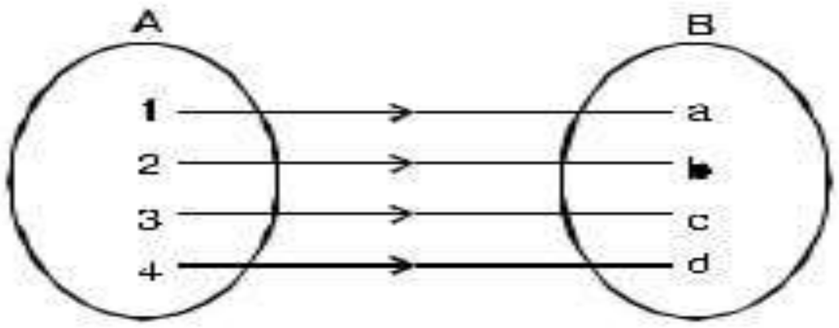


Fig.15.39

1. Find *fog*, *gof*, *fof* and *gog* for the following functions :

$$f(x) = x^2 + 2, \quad g(x) = 1 - \frac{1}{1-x}, \; x \neq 1.$$

2. For each of the following functions write *fog*, *gof*, *fof* and *gog*.

   (a) $f(x) = x^2 - 4, \; g(x) = 2x + 5$

   (b) $f(x) = x^2, \; g(x) = 3$

   (c) $f(x) = 3x - 7, \; g(x) = \frac{2}{x}, \; x \neq 0$

3. Let $f(x) = |x|, \; g(x) = [x]$. Verify that $fog \neq gof$.

4. Let $f(x) = x^2 + 3, \; g(x) = x - 2$

   Prove that $fog \neq gof$ and $f\left(f\left(\frac{3}{2}\right)\right) = g\left(f\left(\frac{3}{2}\right)\right)$

5. If $f(x) = x^2, \; g(x) = \sqrt{x}$. Show that $fog = gof$.

6. Let $f(x) = |x|, \; g(x) = (x)^{\frac{1}{3}}, \; h(x) = \frac{1}{x}; \; x \neq 0$.

   Find (a) *fog* (b) *goh* (c) *foh* (d) *hog* (e) *fogoh*

Partially Ordered Set (POSET)

A partially ordered set consists of a set with a binary relation which is reflexive, antisymmetric and transitive. "Partially ordered set" is abbreviated as POSET.

Examples

- The set of real numbers under binary operation less than or equal to $(\leq)$ is a poset.

  Let the set $S = \{1, 2, 3\}$ and the operation is $\leq$

  The relations will be $\{(1,1),(2,2),(3,3),(1,2),(1,3),(2,3)\}$

This relation R is reflexive as $\{(1,1),(2,2),(3,3)\} \in R$

This relation R is anti-symmetric, as

$$\{(1,2),(1,3),(2,3)\} \in R \; and \; \{(1,2),(1,3),(2,3)\} \notin R$$

This relation R is also transitive as $\{(1,2),(2,3),(1,3)\} \in R$.

Hence, it is a **poset**.

Examples of POSET

1. Let $A = \{a, b, c, d\}$, and let R be the relation defined as follows:

$$R = \{(a,a),(b,b),(c,c),(d,d),(c,a),(a,d),(c,d),(b,c),(b,d),(b,a)\}.$$

2. Let $\mathbb{R}$ be the set of all real numbers, and define a binary relation R on $\mathbb{R} \times \mathbb{R}$: for all $(a,b),(c,d) \in \mathbb{R} \times \mathbb{R}$, $(a,b)R(c,d)$ iff either $a < c$ or both $a = c$ and $b \leq d$. Prove that R is a partial order relation.

Hasse Diagram

- A **Hasse diagram** is a *graphical representation* of the relation of elements of a **partially ordered set (poset)** with an implied *upward orientation*. A point is drawn for each element of the partially ordered set (poset) and joined with the line segment according to the following rules:
- If $p<q$ in the poset, then the point corresponding to *p appears lower* in the drawing than the point corresponding to q.
- The two points p and q will be joined by line segment *iff p is related to q*

Description of Hasse Diagram

Hasse Diagrams :

A partial order, being a relation, can be represented by a di-graph. But most of the edges do not need to be shown since it would be redundant.

For instance, we know that every partial order is reflexive, so it is redundant to show the self-loops on every element of the set on which the partial order is defined.

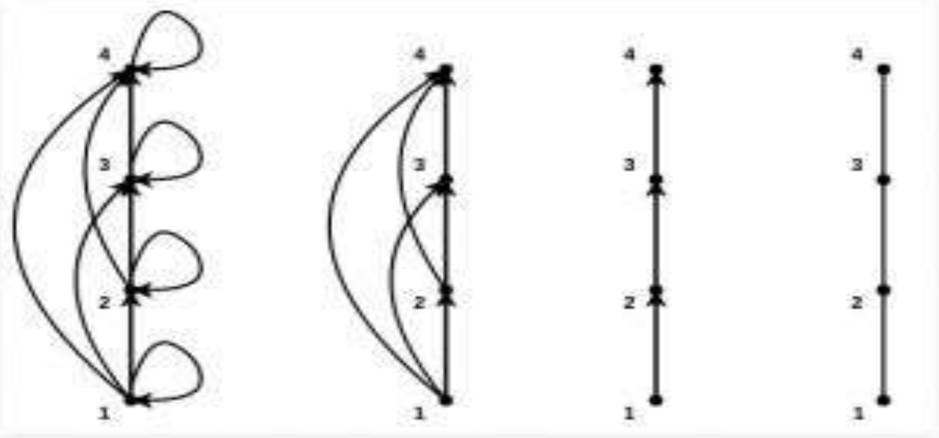
Every partial order is transitive, so all edges denoting transitivity can be removed.

The directions on the edges can be ignored if all edges are presumed to have only one possible direction, conventionally upwards.

In general, a partial order on a finite set can be represented using the following procedure –

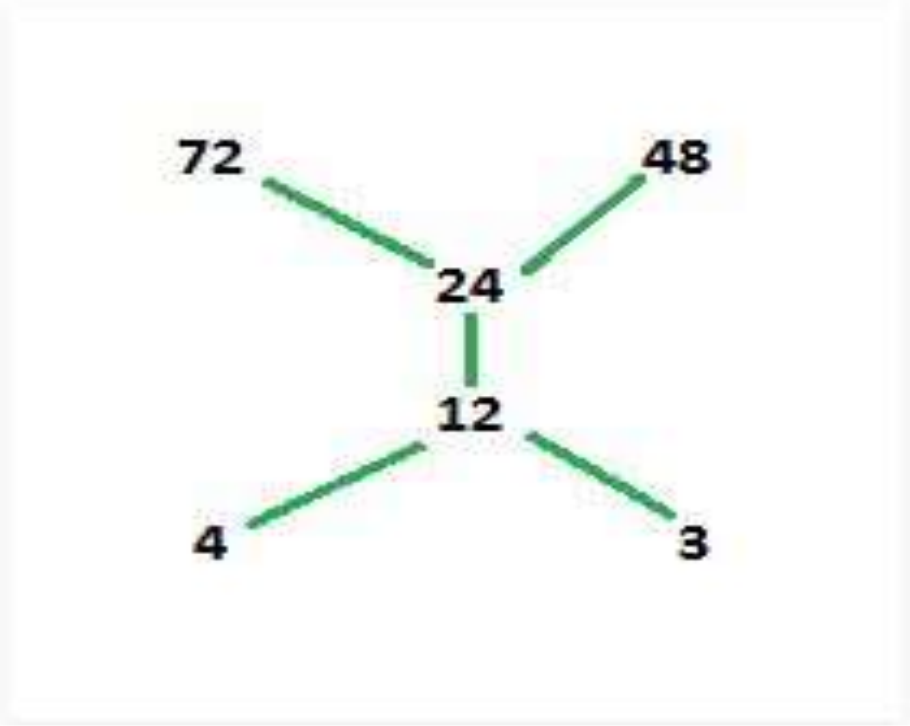
1. Remove all self-loops from all the vertices. This removes all edges showing reflexivity.
2. Remove all edges which are present due to transitivity i.e. if (a, b) and (b, c) are in the partial order, then remove the edge (a, c). Furthermore if (c, d) is in the partial order, then remove the edge (a, d).
3. Arrange all edges such that the initial vertex is below the terminal vertex.
4. Remove all arrows on the directed edges, since all edges point upwards.

For example, the poset $(\{1, 2, 3, 4\}, \leq)$ would be converted to a Hasse diagram like –



The last figure in the above diagram contains sufficient information to find the partial ordering. This diagram is called a **Hasse Diagram**.

- To draw a Hasse diagram, provided set must be a poset.
- **Example-1:** Draw Hasse diagram for ({3, 4, 12, 24, 48, 72}, |)


72
48
24
12
4
3

Description of Hasse Diagram in example-1

- In above diagram, 3 and 4 are at same level because they are not related to each other and they are smaller than other elements in the set. The next succeeding element for 3 and 4 is 12 i.e., 12 is divisible by both 3 and 4. Then 24 is divisible by 3, 4 and 12. Hence, it is placed above 12. 24 divides both 48 and 72 but 48 does not divide 72. Hence 48 and 72 are not joined.
We can see transitivity in our diagram as the level is increasing.

Example 2

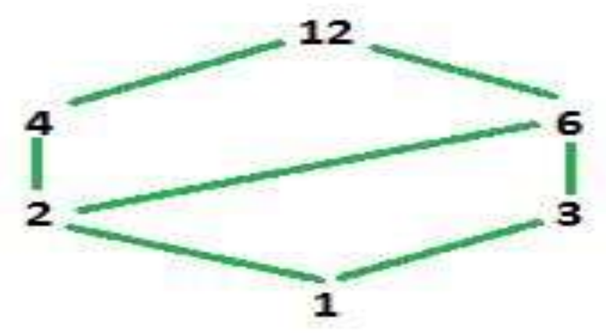
Draw Hasse diagram for (D_{12}, /)

– Here, D_{12} means set of positive integers divisors of 12.

So, D_{12}={1, 2, 3, 4, 6, 12}

poset A = {($1 \prec 2$), ($1 \prec 3$), ($1 \prec 4$), ($1 \prec 6$), ($1 \prec 12$), ($2 \prec 4$), ($2 \prec 6$), ($2 \prec 12$), ($3 \prec 6$), ($3 \prec 12$), ($4 \prec 12$), ($6 \prec 12$)}

So, now the Hasse diagram will be-



Explanation of example 2

In above diagram, 1 is the only element that divides all other elements and smallest. Hence, it is placed at the bottom. Then the elements in our set are 2 and 3 which do not divide each other hence they are placed at same level separately but divisible by 1 (both joined by 1). 4 is divisible by 1 and 2 while 6 is divisible by 1, 2 and 3 hence, 4 is joined by 2 and 6 is joined by 2 and 3. 12 is divisible by all the elements hence, joined by 4 and 6 not by all elements because we have already joined 4 and 6 with smaller elements accordingly.

Maximal and Minimal Elements

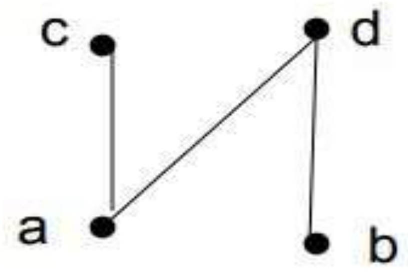
- Elements of poset have certain extremal properties are important for many applications.
- An element of poset is called maximal if it is not related to any element of poset.
- Similarly, an element is called minimal if no element of the poset is related to it.
- Maximal and Minimal elements are easy to spot using a Hasse Diagram. They are the top and bottom elements in the diagram respectively.

Greatest and Least element of Poset

- If the maximal element of poset is unique, then it is also the greatest element of the poset. Otherwise, greatest/Maximum element do not exist.
- If the minimal element of poset is unique, then it is also the least element of the poset. Otherwise, least/Minimum element do not exist.

Important Note : If the maximal or minimal element is unique, it is called the **greatest** or **least** element of the poset respectively.

Maximal and Minimal Element



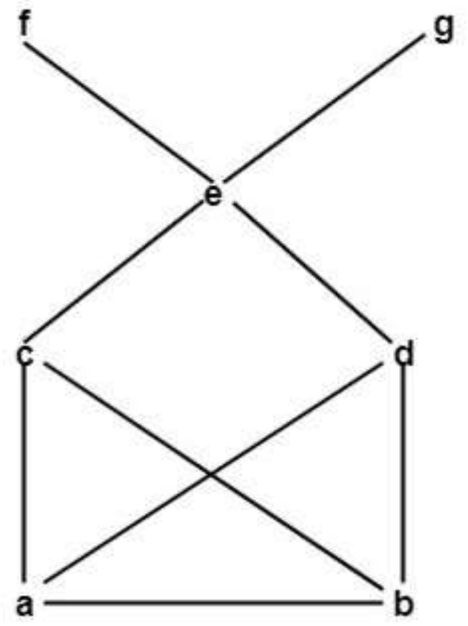
What are the minimal, maximal, minimum, maximum elements?

- Minimal: {a,b}
- Maximal: {c,d}
- There are no unique minimal or maximal elements, thus no minimum or maximum

Upper Bound: Consider B be a subset of a partially ordered set A. An element $x \in A$ is called an upper bound of B if $y \leq x$ for every $y \in B$.

Lower Bound: Consider B be a subset of a partially ordered set A. An element $z \in A$ is called a lower bound of B if $z \leq x$ for every $x \in B$.

Example: Consider the poset $A = \{a, b, c, d, e, f, g\}$ be ordered shown in fig. Also let $B = \{c, d, e\}$. Determine the upper and lower bound of B.



Solution: The upper bound of B is e, f, and g because every element of B is '≤' e, f, and g.

The lower bounds of B are a and b because a and b are '≤' every elements of B.

Least Upper Bound (SUPREMUM):

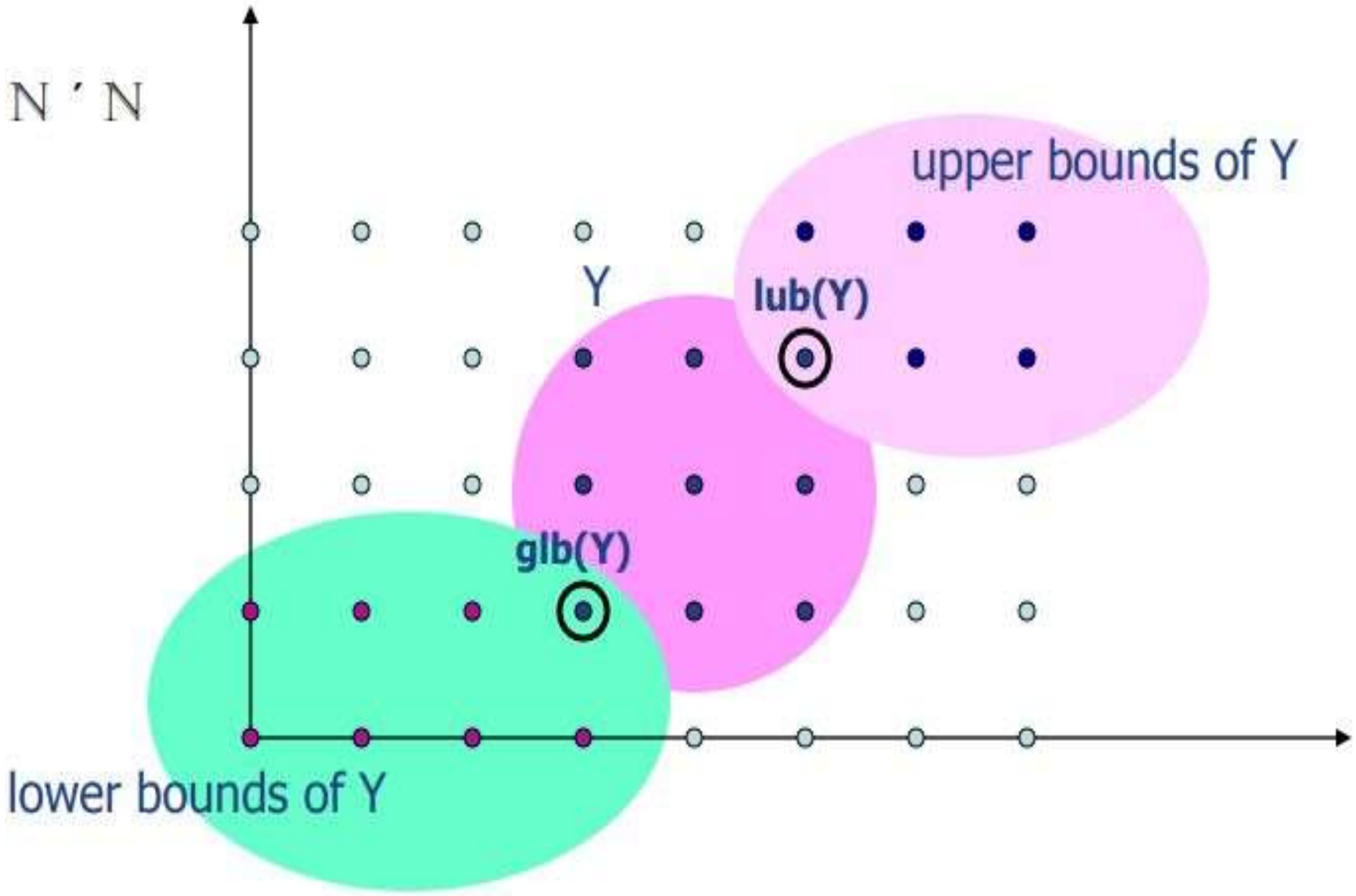
Let A be a subset of a partially ordered set S. An element M in S is called an upper bound of A if M succeeds every element of A, i.e. if, for every x in A, we have x <=M

If an upper bound of A precedes every other upper bound of A, then it is called the supremum of A and is denoted by Sup (A)

Greatest Lower Bound (INFIMUM):

An element m in a poset S is called a lower bound of a subset A of S if m precedes every element of A, i.e. if, for every y in A, we have m <=y

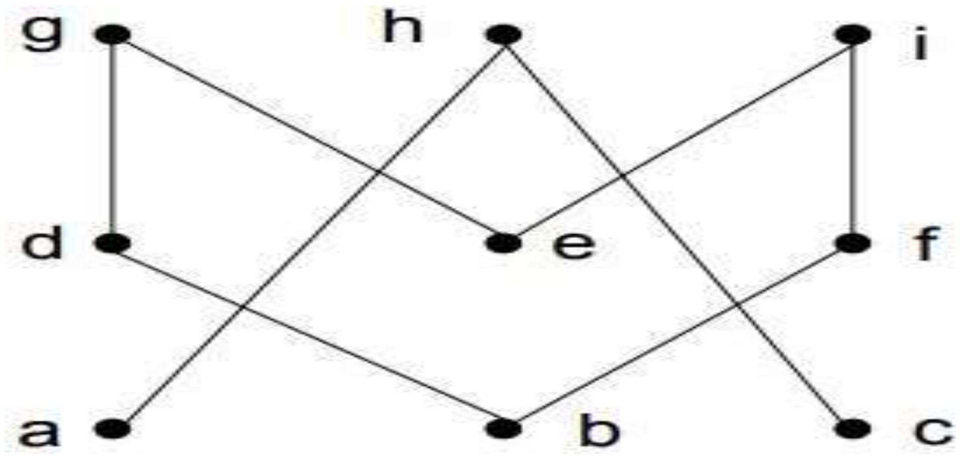
If a lower bound of A succeeds every other lower bound of A, then it is called the infimum of A and is denoted by Inf (A)


N ′ N
upper bounds of Y
Y
lub(Y)
glb(Y)
lower bounds of Y

Example-1

Give lower/upper bounds & glb/lub of the sets:

{d,e,f}, {a,c} and {b,d}



Solution

{d,e,f}

- Lower bounds: $\varnothing$, thus no glb
- Upper bounds: $\varnothing$, thus no lub

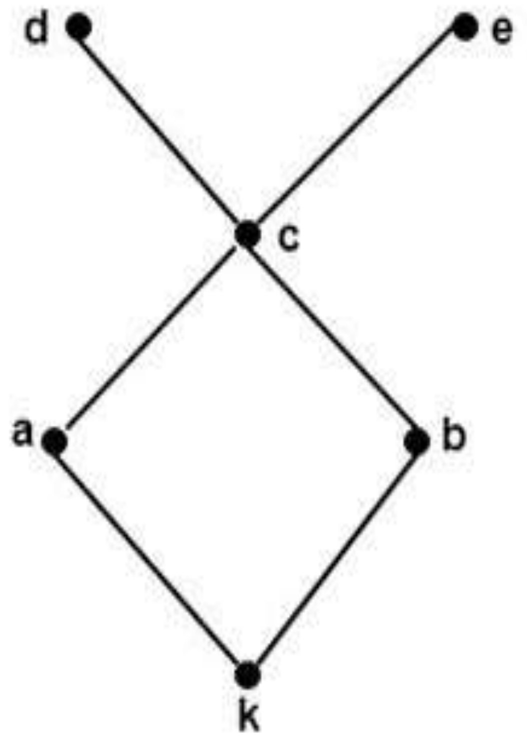
{a,c}

- Lower bounds: $\varnothing$, thus no glb
- Upper bounds: {h}, lub: h

{b,d}

- Lower bounds: {b}, glb: b
- Upper bounds: {d,g}, lub: d because $d \prec g$

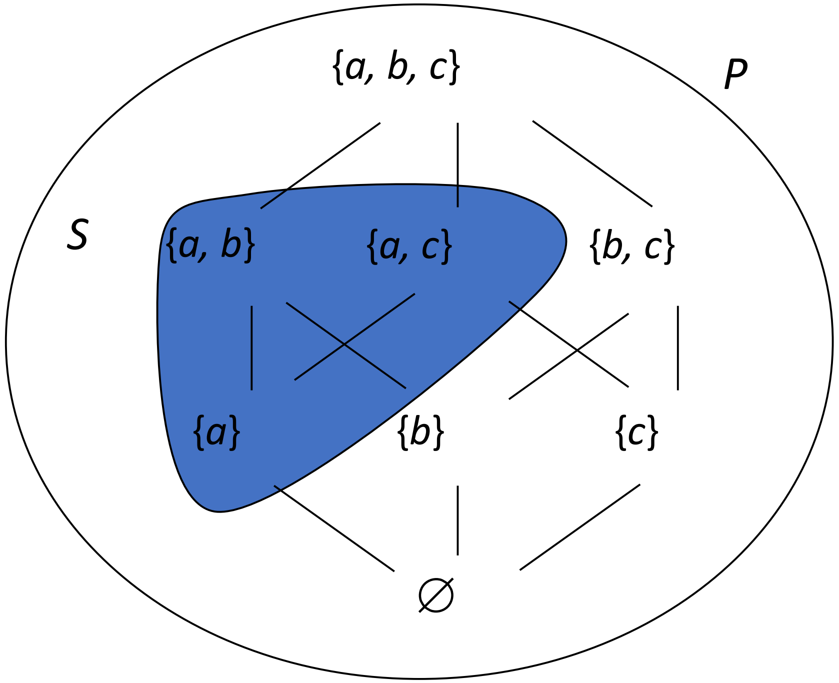
Example: Determine the least upper bound and greatest lower bound of B = {a, b, c} if they exist, of the poset whose Hasse diagram is shown in fig:



Solution: The least upper bound is c.

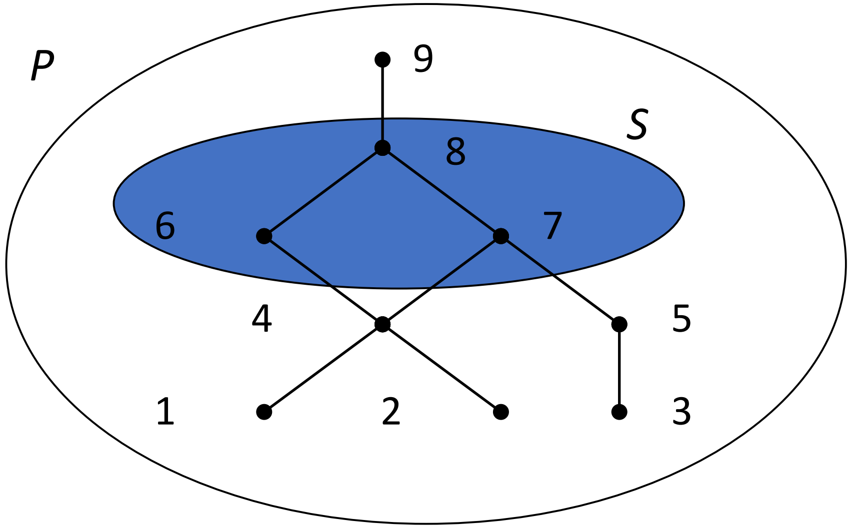
The greatest lower bound is k.

Example. Let P = power($\{a, b, c\}$) shown in the picture and let $S = \{\{a\}, \{a, b\}, \{a, c\}\}$. Find the minimal, maximal, least, greatest, upper bounds, least upper bounds, lower bounds and greatest upper bound.



{a} is the only minimal element of S, so $\{a\}$ is the least element of S. The lower bounds of S are $\{a\}$ and $\varnothing$, with glb(S) = $\{a\}$. The maximal elements of S are $\{a, b\}$ and $\{a, c\}$, but there is no greatest element of S. The only upper bound of S is $\{a, b, c\}$, so lub(S) = $\{a, b, c\}$.

Quiz (2 minutes). Find the minima, maxima, and bounds for the subset S ={6, 7, 8} of the poset P pictured poset diagram.



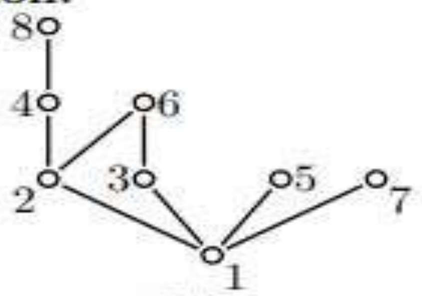
Solution. The minimal elements of S are 6 and 7, but there is no least element of S. The lower bounds of S are 1, 2, 4 with glb(S) = 4. The only maximal element of S is 8, so 8 is also the greatest element of S. The upper bounds of S are 8 and 9, with lub(S) = 8.

Exercise: Draw a Hasse diagram for $(A,)$ (divisibility relation), where
(i) $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$; (iii) $A = \{2, 3, 4, 5, 6, 30, 60\}$; (v) $A = \{1, 2, 4, 8, 16, 32, 64\}$;
(ii) $A = \{1, 2, 3, 5, 11, 13\}$; (iv) $A = \{1, 2, 3, 6, 12, 24\}$; (vi) $A = \{2, 4, 6, 12, 24, 36\}$.

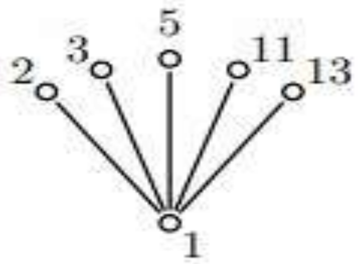
Exercise: Consider the poset $(\{3, 5, 9, 15, 24, 45\},)$, that is, the divisibility relation.
(i) Draw its Hasse diagram.
(ii) Find its maxima, minima, greatest and least elements when they exist.
(iii) Find maxima, minima, greatest and least elements of the set $M = \{3, 9, 15\}$, when they exist.

Solution:

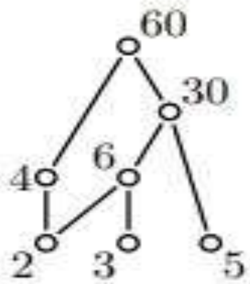
(i):



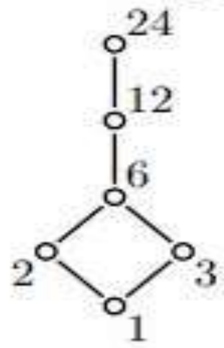
(ii):



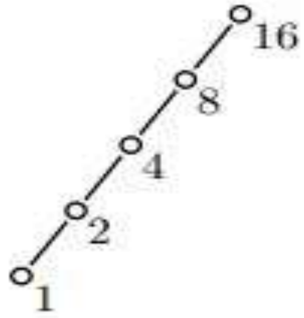
(iii):



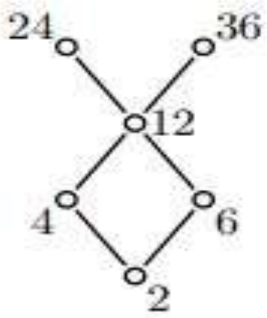
(iv):



(v):



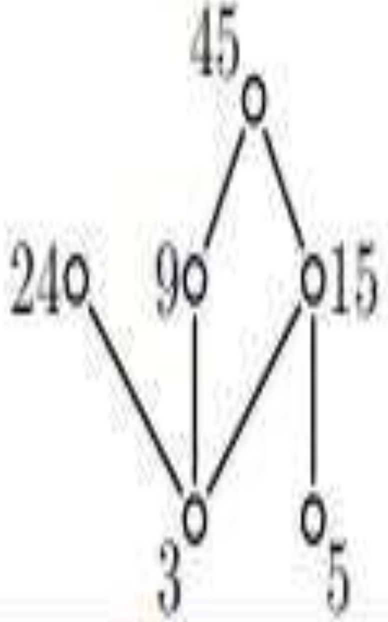
(vi):



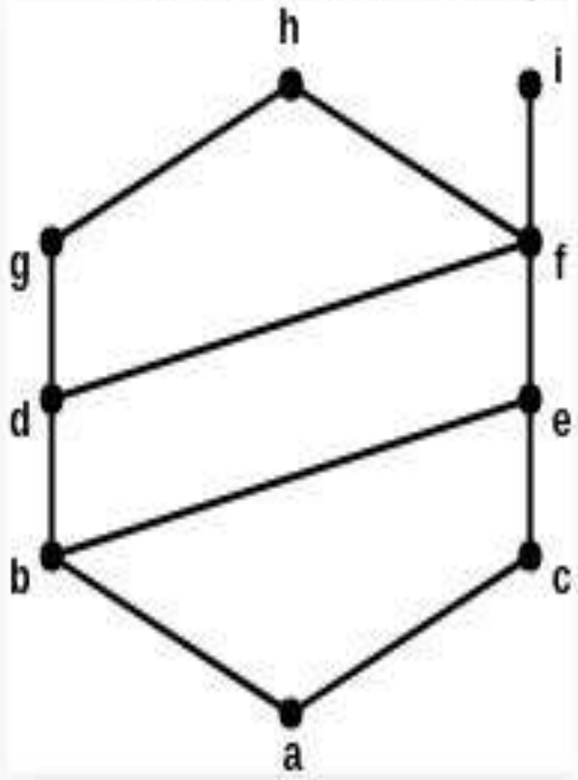
Solution:

(ii): Max 24,45, greatest DNE, min 3,5, least DNE.

(iii): Max 9,15, greatest DNE, min 3, least 3.



- **Example –** Find the least upper bound and greatest lower bound of the following subsets- $\{b, c\}$, $\{g, e, a\}$, $\{e, f\}$.



- **Solution –** For the set $\{b, c\}$

  The upper bounds are – e, f, h, i. So the least upper bound is e.

  The lower bounds are – a. So the greatest lower bound is a.

  For the set $\{g, e, a\}$

  The upper bounds are – h. So the least upper bound is h.

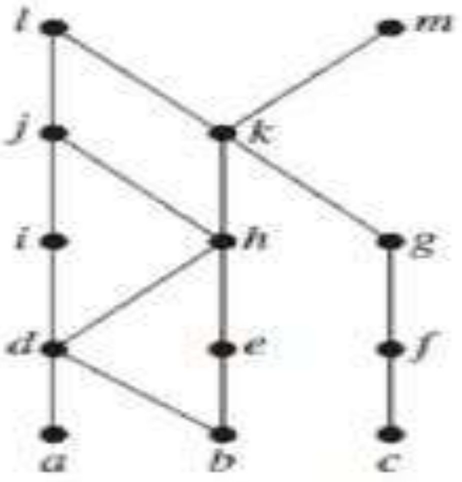
  The lower bounds are – a. So the greatest lower bound is a.

  For the set $\{e, f\}$

  The upper bounds are – f, h, i. So the least upper bound is f.

  The lower bounds are – e, c, b, a. So the greatest lower bound is e.

1. For the Hasse diagram given below; find maximal, minimal, greatest, least, LB, glb, UB, lub for the subsets;
   (i) $\{d, k, f\}$

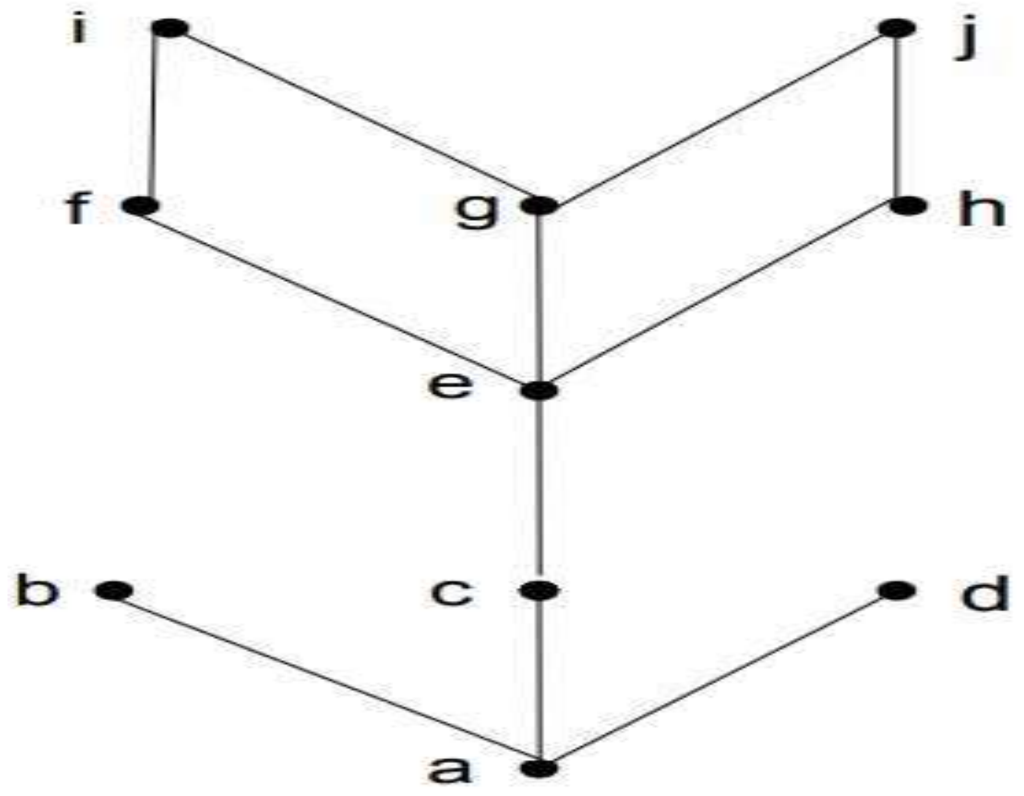


   (ii) $\{b, h, f\}$
   (iii) $\{d\}$
   (iv) $\{a, b, c\}$
   (v) $\{l, m\}$

Solution

Set	Greatest	Least	Maximum	Minimum	UB	LB	LUB	GLB
$\{d,k,f\}$	$\{k\}$	NIL	$\{k\}$	$\{d,f\}$	$\{k,l,m\}$	NIL	$\{k\}$	NIL
$\{b,h,f\}$	NIL	NIL	$\{h,f\}$	$\{b,f\}$	$\{l,m\}$	NIL	$\{k\}$	NIL
$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d,h,i,j,k,l,m\}$	$\{d,a,b\}$	$\{d\}$	$\{d\}$
$\{a,b,c\}$	NIL	NIL	$\{a,b,c\}$	$\{a,b,c\}$	$\{k,l,m\}$	NIL	$\{k\}$	NIL
$\{l,m\}$	NIL	NIL	$\{l,m\}$	$\{l,m\}$	NIL	$\{a,b,c,d,e,f,g,h,k\}$	NIL	$\{k\}$

Practice Question-1: With Solution



- Minimal/Maximal elements?
  - Minimal & Minimum element: a
  - Maximal elements: b,d,i,j
- Bounds, glb, lub of {c,e}?
  - Lower bounds: {a,c}, thus glb is c
  - Upper bounds: {e,f,g,h,i,j}, thus lub is e
- Bounds, glb, lub of {b,i}?
  - Lower bounds: {a}, thus glb is c
  - Upper bounds: ∅, thus lub DNE

Lattices

Definition: A partially ordered set in which every pair of elements has both a least upper bound and greatest lower bound is called a lattice.

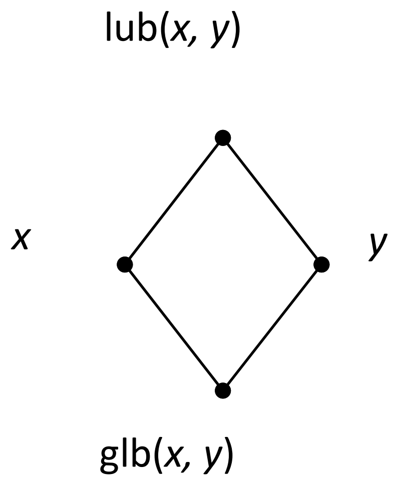
- Lattices are used in many different applications such as models of information flow and also play am important role in Boolean Algebra.

There are two binary operations defined for lattices –

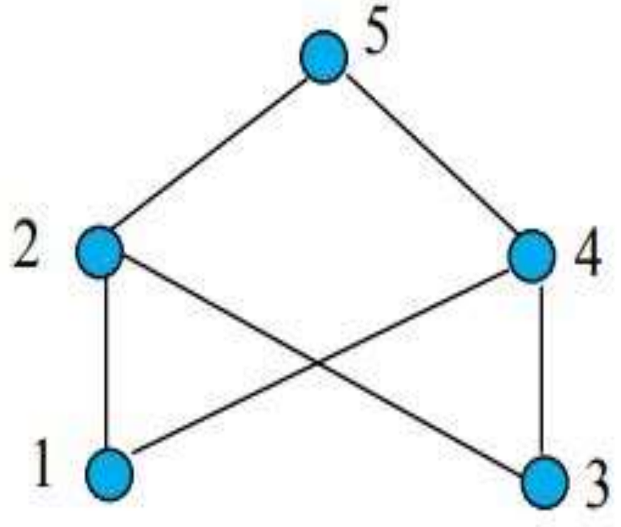
1. **Join** – The join of two elements is their least upper bound. It is denoted by $\vee$, not to be confused with disjunction.
2. **Meet** – The meet of two elements is their greatest lower bound. It is denoted by $\wedge$, not to be confused with conjunction.

Example. For any set S the poset $\langle$ power(S), $\subseteq \rangle$ is a lattice because for any sets A and B, we have glb(A, B) = $A \cap B$ and lub(A, B) = $A \cup B$.

Example/Quiz. Is $\langle$ {1, 2, 3, 4, 5, 6}, | $\rangle$ a lattice?
Answer: No. For example, there is no lub for 2 and 5.
Quiz. Is $\langle$ {1, 2, 3, 6, 12}, | $\rangle$ a lattice?
Answer: Yes.



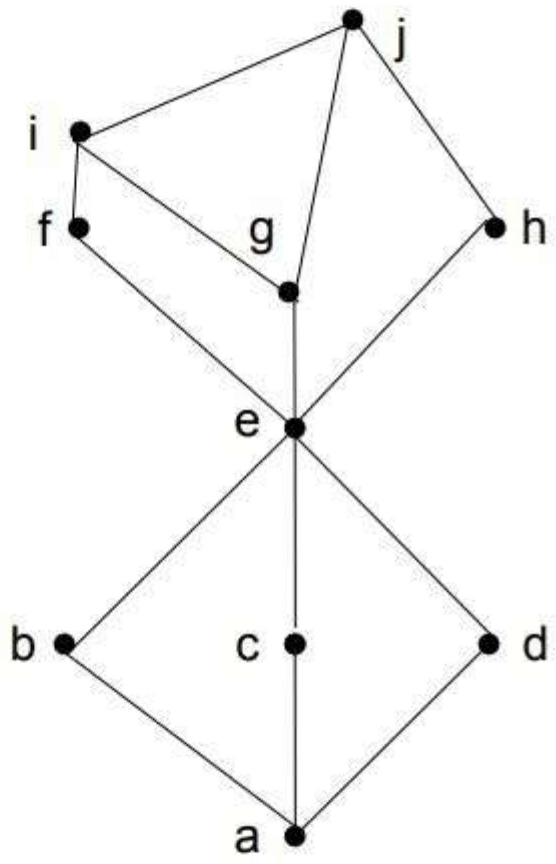
Example-1



- 2, 4 and 5 are upper bounds for the pair 1 and 3.
- There is no lub since
    - 2 is not related to 4
    - 4 is not related to 2
    - 2 and 4 are both related to 5.
- There is no glb either.

The poset is <u>not</u> a lattice.

Lattices: Example 2



A Lattice Or Not a Lattice?

- To show that a partial order is not a lattice, it suffices to find a pair that does not have an lub or a glb (i.e., a counter-example)
- For a pair not to have an lub/glb, the elements of the pair must first be <u>incomparable</u> (Why?)
- You can then view the upper/lower bounds on a pair as a sub-Hasse diagram: If there is no maximum/minimum element in this sub-diagram, then it is not a lattice