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Fundementals of Quantum Computing

What is quantum computing?

– When computation is done using the principle of quantum mechanics it is called quantum computing.

Review of linear algebra

An Hilbert space H is a normed vector space over the complex number $\mathbb{C}$.

$|\psi\rangle$ 'ket'

$\langle\psi|$ "bra"

$|a\rangle \rightarrow$ (ket notation)

Dirac's bracket Notation:

z^* – complex conjugate if $z = a+ib$, then $z^* = a - ib$

The ket $|\psi\rangle$ corresponds to a column vector ψ.

ψ^+ is a conjugate transpose of a vector ψ

Ex: $\psi = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ $\quad \psi^+ = (\bar{a}_1, \bar{a}_2 \cdots \bar{a}_n)$

In Dirac's notation, the conjugate transpose of a ket $|\psi\rangle$ is called a "bra" and is written as $\langle\psi|$

hence

$|\psi\rangle = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ and $\langle\psi| = (\bar{a}_1, \bar{a}_2 \cdots \bar{a}_n)$

A bra $\langle\psi|$ corresponds to a row vector ψ^+

Inner product between vectors $|\psi\rangle$ and $|\phi\rangle$

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix} \quad \text{and} \quad |\phi\rangle = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{pmatrix}$$

The inner product $\langle\psi|\phi\rangle$ is defined to be scalar obtained by multiplying the conjugate transpose $\langle\psi| = (\bar{\psi}_1 \dots \bar{\psi}_n)$ with $|\phi\rangle$

$$\langle\psi|\phi\rangle = (\bar{\psi}_1 \dots \bar{\psi}_n)\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} = \sum_{i=1}^{n} \bar{\psi}_i \phi_i$$

Example:

$$|\psi\rangle = \begin{bmatrix} 2 \\ 6i \end{bmatrix} \quad |\phi\rangle = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\langle\psi|\phi\rangle = [2 \; -6i] \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= 6 - 24i$$

Tensor product of $|\psi\rangle$ and $|\phi\rangle$

It is given by

$$|\psi\rangle \otimes |\phi\rangle = |\psi\rangle|\phi\rangle$$

Ex: $|\psi\rangle|\phi\rangle = \begin{bmatrix} 2 \\ 6i \end{bmatrix} \otimes \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$$= \begin{bmatrix} 2 \times 3 \\ 2 \times 4 \\ 6i \times 3 \\ 6i \times 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 18i \\ 24i \end{bmatrix}$$

ψ^{+} – Hermition conjugate (adjoint) of matrix ψ

if $\psi = \begin{bmatrix} 1 & 6i \\ 3i & \cancel{24}\ 2+4i \end{bmatrix}$ then $\psi^{+} = \begin{bmatrix} 1 & -3i \\ -6i & 2-4i \end{bmatrix}$

Note: $\psi^{+} = (\bar{\psi})^{T}$

$\| |\psi\rangle \|$ – norm of vector $|\psi\rangle$

$$\| |\psi\rangle \| = \sqrt{\langle \psi | \psi \rangle}$$

Normalization of $|\psi\rangle$ is given by $\dfrac{|\psi\rangle}{\| |\psi\rangle \|}$

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Find norm $|\psi\rangle = \begin{pmatrix} 1 \\ 2 \\ -i \end{pmatrix}$

$\rightarrow$ $\| |\psi\rangle \| = \sqrt{\langle \psi | \psi \rangle}$ $\quad$ $\langle \psi | \psi \rangle = (1 \ 2 \ +i) \begin{pmatrix} 1 \\ 2 \\ -i \end{pmatrix}$

$\underline{\underline{\sqrt{6}}}$ $\quad$ $= 1+4+1 = \underline{\underline{6}}$

normalization of $|\psi\rangle = \dfrac{|\psi\rangle}{\| |\psi\rangle \|} = \dfrac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -i \end{pmatrix}$

$$\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{i}{\sqrt{6}}\right) \begin{pmatrix} \frac{1}{\sqrt{6}} \\ 2/\sqrt{6} \\ -i/\sqrt{6} \end{pmatrix}$$

$i^2 = -1$

$i = \sqrt{-1}$

$$= \frac{1}{6} + \frac{4}{6} + 1/6$$

$$= 6/6 = \underline{\underline{1}}$$

Unit vector: A normalized vector or unit vector $|\psi\rangle$ is a vector whose norm is 1.

Orthogonal vectors: Two vectors $|\psi\rangle$ & $|\phi\rangle$ is said to be orthogonal if their inner product is zero.

$$\langle\psi|\phi\rangle = 0$$

Ex: $|\psi\rangle = \begin{pmatrix} i \\ i \end{pmatrix}$ & $\begin{pmatrix} 1 \\ -1 \end{pmatrix} = |\phi\rangle$

$$\langle\psi|\phi\rangle = (-i \;\; -i)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= -i + i = \underline{\underline{0}}$$

Problems: If $|x\rangle = \begin{pmatrix} 1 \\ i \\ 2+i \end{pmatrix}$ $\quad |y\rangle = \begin{pmatrix} 2-i \\ 1 \\ 2+i \end{pmatrix}$

Find $\| |x\rangle \|$, $\langle x|y\rangle$, $\langle y|x\rangle$

prove that $\langle x|y\rangle = \langle y|x\rangle^*$

$\rightarrow \| |x\rangle \| = \sqrt{\langle x|x\rangle}$

$$\langle x|x\rangle = (1 \;\; -i, \;\; 2-i)\begin{pmatrix} 1 \\ i \\ 2+i \end{pmatrix}$$

$(2-i)(2+i)$

$4+\cancel{2i}-\cancel{2i}+1$

$= \underline{\underline{5}}$

$$= 1+1+5 = \underline{\underline{7}}$$

$\cancel{\langle y|x\rangle}$ $\langle x|y\rangle = (1, -i, \;\; 2-i)\begin{pmatrix} 2-i \\ 1 \\ 2+i \end{pmatrix}$

$$= \underline{7-2i}$$

$$\langle y|x\rangle = (2+i, \; 1, \; 2-i)\begin{pmatrix} 1 \\ i \\ 2+i \end{pmatrix}$$

$$= \underline{\underline{7+2i}}$$

$$\langle x|y\rangle = \langle y|x\rangle^*$$

$$7-2i = (7+2i)^* = 7-2i$$

Linearly independent vectors

The set of vectors $|\varphi_1\rangle, \dots\dots |\varphi_n\rangle$ in $\mathbb{C}^n$ are said to be linearly independent if these exists scales $a_1 \dots a_n$ such that

$$a_1|\varphi_1\rangle + a_2|\varphi_2\rangle + \dots\dots + a_n|\varphi_n\rangle = 0$$

$$\Rightarrow a_1 = a_2 = \dots\ a_n = 0$$

Ex: $|\varphi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $|\varphi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ linearly independent.

$$a_1|\varphi_1\rangle + a_2|\varphi_2\rangle = 0$$

$$a_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \qquad \begin{pmatrix} a_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\cancel{a_1 + a_2 = 0} \qquad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0.$$

$\Rightarrow$ if $a_1 = 0,\ a_2 = 0$

Linearly dependent vectors

The set of vectors $|\varphi_1\rangle, \dots\dots |\varphi_n\rangle$ in $\mathbb{C}^n$ are said to be linearly dependent if these exist scalars $a_1 \dots a_n$ not all of them are 0, such that

$$a_1|\varphi_1\rangle + a_2|\varphi_2\rangle + \dots + a_n|\varphi_n\rangle = 0.$$

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \ c^2 \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

basis

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \psi_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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Ex: The vectors $|\varphi_1\rangle = \begin{pmatrix} 1 \\ 2 \\ -9 \end{pmatrix}$ and $|\psi\rangle = \begin{pmatrix} -2 \\ -4 \\ 18 \end{pmatrix}$ are linearly dependent in $\mathbb{C}^3$.

$$a_1|\psi_1\rangle + a_2|\psi_2\rangle = 0$$

$$a_1\begin{pmatrix}1\\2\\-9\end{pmatrix} + a_2\begin{pmatrix}-2\\-4\\18\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$a_1 - 2a_2 = 0$$
$$2a_1 - 4a_2 = 0.$$

$$a_1 - 2a_2 = 0$$
$$a_1 - 2a_2 = 0 \qquad \underline{a_1 = 2 \quad a_2 = 1}$$

Ex: The vectors $|\psi_1\rangle = \begin{pmatrix}1\\2\\3\end{pmatrix}$ $|\psi_2\rangle = \begin{pmatrix}2\\5\\7\end{pmatrix}$ $|\psi_3\rangle = \begin{pmatrix}1\\3\\5\end{pmatrix}$ are linearly independent.

$$\Rightarrow a_1|\psi_1\rangle + a_2|\psi_2\rangle + a_3|\psi_3\rangle = 0$$

$$a_1\begin{pmatrix}1\\2\\3\end{pmatrix} + a_2\begin{pmatrix}2\\5\\7\end{pmatrix} + a_3\begin{pmatrix}1\\3\\5\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}a_1 + 2a_2 + a_3\\2a_1 + 5a_2 + 3a_3\\3a_1 + 7a_2 + 5a_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{matrix}a_1 + 2a_2 + a_3 = 0\\2a_1 + 5a_2 + 3a_3 = 0\\3a_1 + 7a_2 + 5a_3 = 0\end{matrix}$$

$\Rightarrow a_1 = 0,\ a_2 = 0,\ a_3 = 0.$ only possible soln.

__Basis__: The set of vectors $|\psi_1\rangle, |\psi_2\rangle, \ldots |\psi_n\rangle$ is said to be basis for $\mathbb{C}^n$, then every vector $|v\rangle$ can be uniquely expressed as a linear combination of $|\psi_1\rangle, |\psi_2\rangle, \ldots |\psi_n\rangle$

ie $|v\rangle = a_1|\psi_1\rangle + a_2|\psi_2\rangle + \ldots a_n|\psi_n\rangle$ where $a_1 \ldots a_n$ are scalar.

Ex: The vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ is basis in C^2

$|v\rangle = \begin{pmatrix}a\\b\end{pmatrix}$ $\qquad \begin{pmatrix}a\\b\end{pmatrix} = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}$.

<u>Note</u>: A basis for C^n consists of exactly n linearly independent vectors

<u>Orthonormal basis</u>: A basis said to be orthonormal if each vector has norm 1 and each pair of vectors are orthogonal.

<u>Ex</u>: The vectors $|\psi_1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|\psi_2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ are orthonormal basis in C^2.

$$\| |\psi_1\rangle \| = \sqrt{\langle \psi_1 | \psi_1 \rangle} = \sqrt{1} = 1$$

$$\langle \psi_1 | \psi_1 \rangle = (1\ 0)\begin{pmatrix}0\\1\end{pmatrix}$$

$$= 1$$

$$\langle \psi_2 | \psi_2 \rangle = (0\ 1)\begin{pmatrix}0\\1\end{pmatrix}$$

$$= 1$$

$$\| |\psi_2\rangle \| = \sqrt{\langle \psi_2 | \psi_2 \rangle} = \sqrt{1} = 1$$

$$\langle \psi_1 | \psi_2 \rangle = (1\ 0)\begin{pmatrix}0\\1\end{pmatrix} = 0$$

<u>Ex</u>: $|\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $|\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

$$\| |\phi_1\rangle \| = \sqrt{\langle \phi_1 | \phi_1 \rangle}$$

$$\langle \phi_1 | \phi_1 \rangle = \left(1/\sqrt{2}\ \ 1/\sqrt{2}\right)\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$$

$$= 1/2 + 1/2 = 1$$

$$\langle \phi_2 | \phi_2 \rangle = \left(\frac{1}{\sqrt{2}}\ \ -\frac{1}{\sqrt{2}}\right)\begin{pmatrix}\frac{1}{\sqrt{2}}\\-1/\sqrt{2}\end{pmatrix}$$

$$1/2 + 1/2 = 1$$

$$\langle \phi_1 | \phi_2 \rangle = \left(1/\sqrt{2} \quad 1/\sqrt{2} \right) \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$
$$= \underline{\underline{0}}$$

The vectors are orthonormal basis in C^2

2) The vector $|v_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$ $|v_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$ are orthonormal basis in C^2

$$\langle v_1 | v_2 \rangle = \left(1/\sqrt{2}, -i/\sqrt{2} \right) \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}$$
$$1/2 - 1/2 = \underline{\underline{0}}$$

$$\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} = 1/\sqrt{5} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cancel{2/\sqrt{5}} \; 2/\sqrt{5} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$= 1/\sqrt{5}\, |0\rangle + 2/\sqrt{5}\, |1\rangle$$

$$\| |v_1\rangle \| = \sqrt{\langle v_1 | v_1 \rangle}$$
$$\langle v_1 | v_1 \rangle = \left(1/\sqrt{2}, \frac{-i}{\sqrt{2}} \right) \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}$$
$$= \underline{\underline{1/2 + 1/2 = 1}}$$

$$\| |v_2\rangle \| = \sqrt{\langle v_2 | v_2 \rangle}$$
$$\langle v_2 | v_2 \rangle = \left(1/\sqrt{2}, i/\sqrt{2} \right) \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}$$
$$= \underline{\underline{1}}$$

C bit	Q bit
0, 1	$|0\rangle, |1\rangle$
OFF, ON	$|\psi\rangle = \beta_1 |0\rangle + \beta_2 |1\rangle$
	$= \beta_1$ OFF $+ \beta_2$ ON

$$\boxed{\begin{array}{l} |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{array}}$$

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Postulates of Quantum mechanics

Postulate 1: Definition of quantum bit or qubit

postulate 2: How qubit(s) transform (evolve)

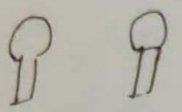
postulate 3: The effect of measurement.

postulate : How qubits combine together into systems of qubits

Classical bit

0 or 1

light



OFF ON

0 1

qubit

$|0\rangle$ or $|1\rangle$

OFF ON

$|\psi\rangle = \alpha\, OFF + \beta\, ON$

$= \alpha|0\rangle + \beta|1\rangle$

<u>qubit</u> : In Quantum mechanics two possible states for a qubit are $|0\rangle$ & $|1\rangle$

These qubits can be expressed as orthonormal basic vector in $\mathbb{C}^2$ (two dimensional complex space)

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

A generic qubit ~~stable~~ state $|\psi\rangle$ is represented by linear combination (or superposition) of $|0\rangle$ & $|1\rangle$

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$

where α & β are complex numbers with $|\alpha|^2$

when we measure qubit we get 0 with probability $|\alpha|^2$

or we get 1 with probability $|\beta^2|$.

<u>Example</u>

$|\phi\rangle = \frac{1+i}{2}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

length of complex no $= a^2 + b^2$

After measurement we get 0 with probability. 1/2 or we get 1 with prob 1/2.

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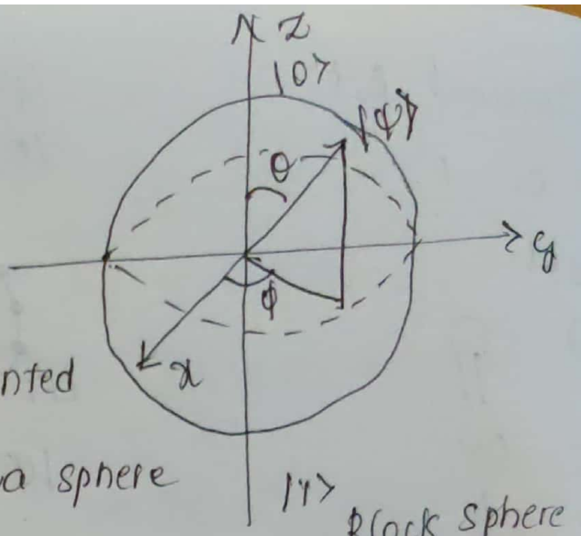
$|\alpha|^2 = 1/4 + 1/4 = 1/2$

$|\beta|^2 = 1/2$

Bloch sphere :



Block sphere or unit sphere

$x^2+y^2+z^2=1$

The state $|\psi\rangle$ of a qubit is represented by a ~~set~~ point on the surface of a sphere of unit radius called block sphere.

~~$|\psi\rangle = \cos \phi/2$~~

$$|\psi\rangle = \cos \theta/2\,|0\rangle + e^{i\phi} \sin \theta/2\,|1\rangle$$

$\alpha = \cos \theta/2$

$\beta = e^{i\phi} \sin \theta/2$, $|\alpha|^2+|\beta|^2=1$

multiple qubits

2-qubits system has 4 possible states.

$|00\rangle, |01\rangle, |10\rangle, |11\rangle$

A generic qubit state describing the 2-qubits is given by,

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} = (1000)^T$$

$$|01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = (0\ 1\ 0\ 0)^T$$

$$|10\rangle = |1\rangle \otimes |0\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = (0\ 0\ 1\ 0)^T$$

$$|11\rangle = |1\rangle \otimes |1\rangle = \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = (0\ 0\ 0\ 1)^T$$

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$$(\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\alpha_2|0\rangle + \beta_2|1\rangle)$$

$$= \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \otimes \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_1\alpha_2 \\ \alpha_1\beta_2 \\ \beta_1\alpha_2 \\ \beta_1\beta_2 \end{pmatrix} = \alpha_1\alpha_2\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_1\beta_2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta_1\alpha_2\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \beta_1\beta_2\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle$$

Entagled States: States that cannot be written or the tensor product of n single qubit states are ~~st~~ entagled states.

Ex: consider the state

$r_1|00\rangle + r_2|11\rangle$, with $r_1 \neq 0$ and $r_2 \neq 0$

this cannot be written as

$(\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\alpha_2|0\rangle + \beta_2|1\rangle)$

The entagled states known as Bell states or EPR pairs. [Einstein, podolsky, Rosen)

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Examples: The following two-qubit states are

$|\phi+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$|\phi-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

$|\psi+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

$|\psi-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ are entagled.

P.T $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ in an entagled state

→ It is impossible to find $\alpha_1 \beta_1$ $\alpha_2 \beta_2$ such that

$$(\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\alpha_2|0\rangle + \beta_2|1\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Since we have,

$$(\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\alpha_2|0\rangle + \beta_2|1\rangle) = \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle$$

If $\alpha_1\beta_2 = 0$ implies $\alpha_1\alpha_2 = 0$ or $\beta_1\beta_2 = 0$ this is impossible.

2) PT $\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$ is not entagled state.

→ $|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $|01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$$\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$= |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

Since it can be written as single qubit state it is not entagled.

PT $\frac{1}{\sqrt{2}}(|00\rangle - |01\rangle)$ is not entagled state.

$|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $|01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$

$\frac{1}{\sqrt{2}}(|00\rangle - |01\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\\0\end{pmatrix}$

$= \begin{pmatrix}1\\0\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

$= \begin{pmatrix}1\\0\end{pmatrix} \times \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\right)$

$|0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

Problems :

① show the following 2-qubit states are entagled

a) $\frac{1}{\sqrt{2}}(|00\rangle - i|11\rangle)$

b) $\frac{i}{10}|00\rangle + \frac{\sqrt{99}}{10}|11\rangle$

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3-qubit entagled state (GHZ state)

The state

$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ is 3-qubit entagled state. It is called Greenberger - Horne - Zeilinger (GHZ) state.

face flip gate

$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$
$\|0\rangle$	$\|0\rangle$
$\|1\rangle$	$-\|1\rangle$

Hermitian Matrix :

A complex matrix A is said to be Hermition if

$A^{+} = A \quad (A^{+} = (\bar{A})^{T}$ = conjugate transpose of $A)$

Unitary matrix : A complex matrix A is said to be unitary if $A^{+}A = I$ or $A^{+} = A^{-1}$

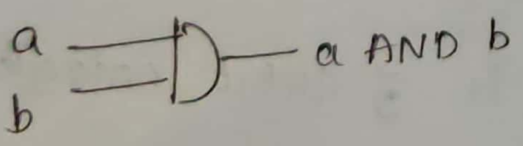
Eigen value & Eigenvectors : Let A be any complex square matrix. A scalar λ is called an eigen value of A if there exists a non-zero column vector $|v\rangle \in \mathbb{C}^{n}$ such that

$A|v\rangle = \lambda|v\rangle, \quad \lambda \in C$

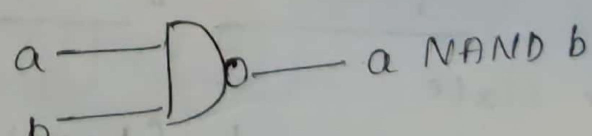
The vector $|v\rangle$ is called an eigen vector of A corresponding to λ.

Quantum Computation : A quantum computes is built from quantum circuit containing wires and elementory quantum gates to carry around & manipulate quantum information

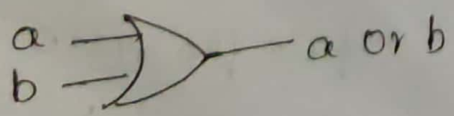
AND gate



NAND gate

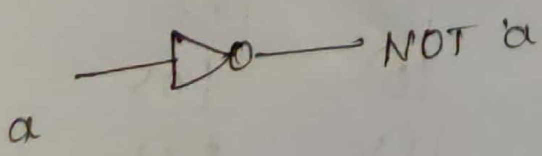


OR gate



NOR gate

NOT gate



Single qubit Quantum gate

1. Quantum NOT gate (X gate or bit flip gate) we define a matrix X to represent the quantum NOT gate by

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

$$X|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

$$X(\alpha|0\rangle + \beta|1\rangle) = X\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

$$\alpha|0\rangle + \beta|1\rangle \;—\boxed{X}—\; \beta|0\rangle + \alpha|1\rangle$$

X is unitary. $X^t X = I$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Single qubit Quantum gate

2. Z-gate (phase flip gate)

Z-gate is defined by the matrix

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Z|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

$$Z|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -|1\rangle$$

$$Z(\alpha|0\rangle + \beta|1\rangle) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}$$

$\alpha|0\rangle + \beta|1\rangle$ —$\boxed{Z}$— $\alpha|0\rangle - \beta|1\rangle$

Hadamard gate (H-gate): The H-gate is defined by the matrix.

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$\alpha|0\rangle + \beta|1\rangle$ —$\boxed{H}$— $\alpha\frac{|0\rangle + |1\rangle}{\sqrt{2}} + \beta\frac{|0\rangle - |1\rangle}{\sqrt{2}}$

$$H\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

X-gate	Z-gate	H-gate
$\|0\rangle - \|1\rangle$	$\|0\rangle - \|0\rangle$	$\|0\rangle \to \frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$
$\|1\rangle - \|0\rangle$	$\|1\rangle - -\|1\rangle$	$\|1\rangle \to \frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$
$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$X^{\dagger}X = I$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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$Z^{\dagger} Z = I$ $\quad H^{\dagger} H = I$

(conjugate transpose)

Y-gate

$|0\rangle - i|1\rangle$

$|1\rangle - -i|0\rangle$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Y|0\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix} = i\begin{pmatrix} 0 \\ 1 \end{pmatrix} = i|1\rangle$$

$$Y|1\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} = -i\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -i|0\rangle$$

$$\alpha|0\rangle + \beta|1\rangle \;-\!\!\boxed{Y}\!\!-\!\!-\; i(\alpha|1\rangle - \beta|0\rangle)$$

$$Y^{\dagger} Y = I$$

Phase-shift gate

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\theta = \pi$$

$$e^{i\pi} = -1$$

$|0\rangle \rightarrow |0\rangle$

$|1\rangle \rightarrow e^{i\theta}|1\rangle$

$$R_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} \quad \theta \text{ is any value}$$

$$R_\theta|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

$$R_\theta|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ e^{i\theta} \end{pmatrix} = e^{i\theta}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= e^{i\theta}|1\rangle$$

when $\theta = \pi$ we get Z-gate.

Problems:

1. write following quantum gate in bra|ket notation.

1) X, 2. Z, 3. H, 4. Y, 5. R_θ

1) $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix}$

$= |1\rangle\langle 0| + |0\rangle\langle 1|$

2) $Z = \cancel{\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix}$

$= |0\rangle\langle 0| - |1\rangle\langle 1|$

$|00\rangle \quad |01\rangle \quad |10\rangle \quad |11\rangle$

$|00\rangle = |0\rangle \otimes |0\rangle$

$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$|01\rangle = |0\rangle \otimes |1\rangle$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

3 qubit entagled state (GHZ state)

This state

$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ is 3 qubit entagled state.

It is called Greenberger Horne Zeilinger (GHZ) state.

Hermition Matrix :

A complex matrix A is called to be Hermition if $A^T = A$.

$A^T = A$ $(A^T = (\bar{A})^T$ = conjugate transpose of A).

Unitary Matrix :

A complex matrix A is said to be unitary of $A^T A = I$ or

$A^T = A^{-1}$.

Eigen value and Eigen vector

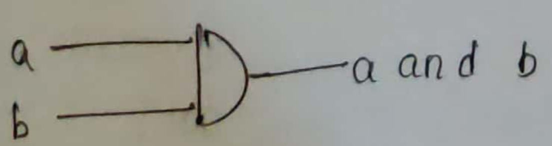
If A be any complex square matrix. A scalar λ is called an eigen value of A. If there exists a non-zero column vector $|v\rangle \in \mathbb{C}^n$ such that $A|v\rangle = \lambda|v\rangle$, $\lambda \in \mathbb{C}$.

The vector $|v\rangle$ is called an eigen vector of A corresponding to λ.

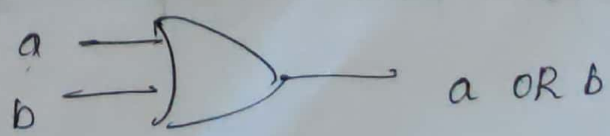
Quantum Competation : A quantum computers is built from quantum circuit containing wires and elementary quantum gates to carry around and manipulate quantum info.

Logical gates

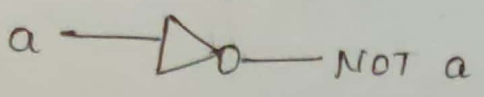
AND gate



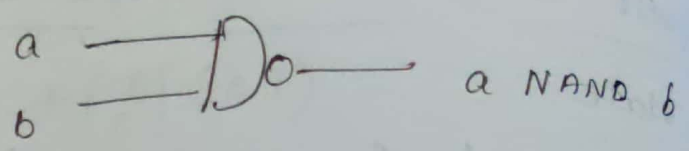
OR gate



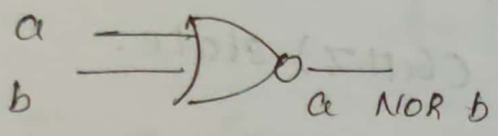
NOT gate



NAND gate



NOR gate



$$X(\alpha|0\rangle + \beta|1\rangle) = X\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\beta\\ \alpha\end{pmatrix}$$

$$\alpha|0\rangle + \beta|1\rangle \; \boxed{X} \longrightarrow \beta|0\rangle + \alpha|1\rangle$$

X is unitary $X^{\dagger}X = I$

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = I$$

<u>Single Qubit Quantum Gate</u> :

Z-gate (phase flip gate)

Z gate is designed by the matrix

$$Z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$Z|0\rangle = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix} = |0\rangle$$

$$Z|1\rangle = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}0\\ -1\end{pmatrix} = -|1\rangle$$

$$Z(\alpha|0\rangle + \beta|1\rangle) = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\alpha\\ -\beta\end{pmatrix}$$

$$\alpha|0\rangle + \beta|1\rangle - \boxed{Z} - \alpha|0\rangle - \beta|1\rangle$$

Hadamard gate (H-gate) : The H-gate is defined by the matrix

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

$$H|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

$$= \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$$

$$\alpha|0\rangle + \beta|1\rangle \longrightarrow \boxed{H} \quad \alpha \frac{|0\rangle + |1\rangle}{\sqrt{2}} + \beta \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$H \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

The matrix from this is

$$C_{NOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

C_{NOT} is unitary

$$C_{NOT} C_{NOT}^{\dagger} = I$$

C_{NOT} is a generalization of the classical XOR gate because

$$|AB\rangle \longrightarrow |A B \oplus A\rangle$$

The CNOT gate cannot be decomposed into a

Two qubit controlled gates

Controlled NOT gate. It flips the second bit in 1 does nothing otherwise

C_{NOT} $|00\rangle - |00\rangle$
$|01\rangle - |01\rangle$
$|10\rangle - |11\rangle$
$|11\rangle - |10\rangle$

Tensor product of 2 single Qubit states

C_{NOT} takes the unentagled qubit state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$ to the entagled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

$C_{NOT}\left(\frac{1}{\sqrt{2}}|0\rangle + |1\rangle\right) \otimes |0\rangle = C_{NOT}\left(\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)\right)$

$$= C_{NOT}\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\right)$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right)$$

$$= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

The No-Cloning principle states that we cannot copy or clone can unknown qubit.

Let U is a unitary transformation that clone ie

$U(|a0\rangle) = |aa\rangle$ for all quantum state a

Let $|a\rangle$ and $|b\rangle$ be 2 orthonormal quantum state,

consider $|c\rangle = \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle)$

$$U(|c0\rangle = U(\frac{1}{\sqrt{2}}(|a0\rangle + |b0\rangle))$$

$$= \frac{1}{\sqrt{2}}(|aa\rangle + |bb\rangle)$$

But $U(|c0\rangle) = |cc\rangle$

$$= |c\rangle \otimes |c\rangle$$

$$= \frac{1}{2}(|aa\rangle + |ab\rangle + |ba\rangle + |bb\rangle) \neq$$

$$\frac{1}{\sqrt{2}}(|aa\rangle + |bb\rangle)$$

$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ $\quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$

$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$

C_{NOT} $\quad |A\rangle$ —•— $|A\rangle$

$|B\rangle$ —⊕— $|B \oplus A\rangle$

23/1/2020

SWAP gate

Simply exchanges bit values

$|00\rangle \rightarrow |00\rangle$
$|01\rangle \rightarrow |10\rangle$
$|10\rangle \rightarrow |01\rangle$
$|11\rangle \rightarrow |11\rangle$

In matrix form

$$SWAP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Tensor product for matrices

Consider 2 vectors ~~$|v\rangle$ & $|w\rangle$~~ $|v\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $|w\rangle = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$$|v\rangle \otimes |w\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \otimes \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ v_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} v_1 w_1 \\ v_1 w_2 \\ v_2 w_1 \\ v_2 w_2 \end{pmatrix}$$

Suppose we have 2 matrices

$$M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}_{2\times 2} \quad N = \begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix}_{2\times 2}$$

The tensor product of M and N is defined as

$$M \otimes N = \begin{pmatrix} m_1 \cdot \begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix} & m_2 \cdot \begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix} \\ m_3 \cdot \begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix} & m_4 \begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix} \end{pmatrix}$$

$$\begin{pmatrix} m_1 n_1 & m_1 n_2 & m_2 n_1 & m_2 n_2 \\ m_1 n_3 & m_1 n_4 & m_2 n_3 & m_2 n_4 \\ m_3 n_1 & m_3 n_2 & m_4 n_1 & m_4 n_2 \\ m_3 n_3 & m_3 n_4 & m_4 n_3 & m_4 n_4 \end{pmatrix}_{4\times 4}$$

1. Find the tensor product of X-gate and Z-gate

$$X \otimes Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Quantum Measurement

Quantum state : The collection of all relevant physical properties of quantum system. (for example position, momentum, spin, polarization) is known as the state of the system.

physical support	Name	Info support	$\|0\rangle$	$\|1\rangle$
photon	polarization	polarization	horizontal ~~vertical~~	vertical ↑ ~~horizontal~~
Electron	Electronic spin	spin	up ↑	down ↓

For example, if we use the energy of an electron as our qubits $|0\rangle$ and $|1\rangle$, we could say that the ground state (lowest energy) in our qubit $|0\rangle$ and excited state (higher energy) in our qubit $|1\rangle$.

ground state $= |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

excited state $= |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

We can define state $|+\rangle$ and $|-\rangle$ with the vectors.

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

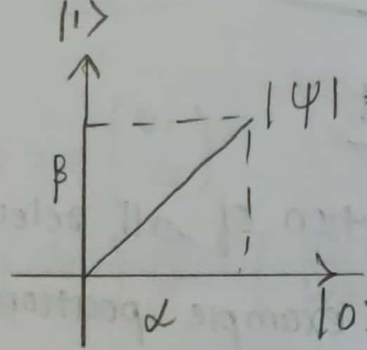
$$|+\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\right) = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

24/1/2020

Physical support



$|\psi| = \alpha|0\rangle + \beta|1\rangle$

$|0\rangle = OFF$

$|1\rangle = ON$

$|\psi| = \alpha\, OFF + \beta\, ON$

Quantum mechanics describe the behaviour of systems such as photons, molecules, electron. we use mathematics to model these physical phenomena.

Consider general quantum state.

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

where $\alpha, \beta \in C$ and $|\alpha|^2 + |\beta|^2 = 1$

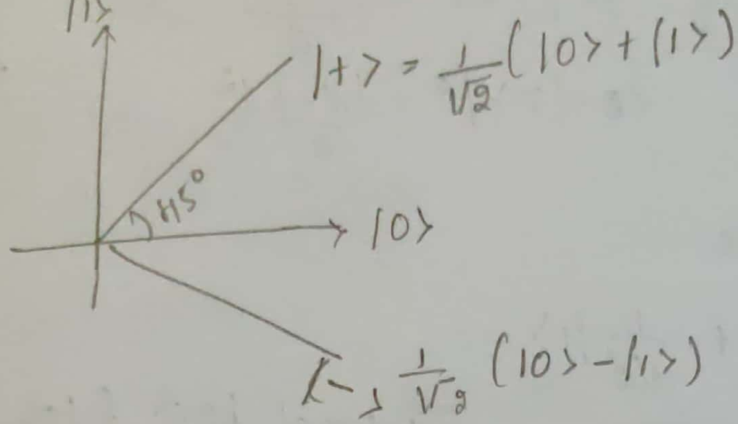
when we measure Quantum state $|\psi\rangle$ we get either the result $|0\rangle$ with probability $|\alpha|^2$ or we get result $|1\rangle$ with probability $|\beta|^2$

Definition : Born's rule.

Suppose we have a quantum state $|\psi\rangle$ and orthonormal basis $\{|\phi_1\rangle, \ldots |\phi_n\rangle\}$. Then we measure $|\psi\rangle$ w.r.to this orthonormal basis. ie we ask the quantum system which one of these state it is in.

$\{|0\rangle, |1\rangle\}$ $\qquad$ $\{|+\rangle, |-\rangle\}$

$\left\{\begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix}\right\}$ $\qquad$ $\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right\}$



$|\uparrow\rangle$

$|\downarrow\rangle$

The probability of measuring state $|\phi_i\rangle$ is given by

$$\boxed{P(\phi_i) = |\langle\phi_i|\psi\rangle|^2}$$

After the measurement the original state collapses in the measured state. i.e we are left with one of the states

$|\phi_1\rangle, \dots |\phi_n\rangle$

<u>Example:</u> Given a quantum state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$$

we measure $|\psi\rangle$ is standard computational basis $\{|0\rangle, |1\rangle\}$

then outcome state will be

$$\begin{cases} |0\rangle \text{ with probability } |\langle 0|\psi\rangle|^2 = |\alpha|^2 \\ |1\rangle \text{ with } \text{"} \ |\langle 1|\psi\rangle|^2 = |\beta|^2 \end{cases}$$

$$|\langle 0|\psi\rangle|^2 = \left|(1,0)\begin{pmatrix}\alpha\\\beta\end{pmatrix}\right|^2 = \underline{|\alpha|^2}$$

$$|\langle 1|\psi\rangle|^2 = \left|(0,1)\begin{pmatrix}\alpha\\\beta\end{pmatrix}\right|^2 = \underline{|\beta|^2}$$

Example: what will be outcome if we measure in $\{|+\rangle, |-\rangle\}$

$\begin{cases} |+\rangle \text{ with probability } & |\langle + | \psi \rangle|^2 = \frac{|\alpha+\beta|^2}{2} \quad \frac{1}{\sqrt{2}}(1,1)\begin{pmatrix}\alpha \\ \beta\end{pmatrix} \\ |-\rangle \text{ with } \quad " & |\langle - | \psi \rangle|^2 = \frac{|\alpha-\beta|^2}{2} \end{cases}$

$\left|\frac{\alpha+\beta}{\sqrt{2}}\right|^2 = \frac{|\alpha+\beta|^2}{2}$

Given a quantum state

$$|\psi\rangle = \frac{1+i}{2}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

We measure $|\psi\rangle$ in standard computational basis $\{|0\rangle, |1\rangle\}$

then outcome state will be

$\begin{cases} |0\rangle \text{ with probability } 1/2 \\ |1\rangle \quad\quad\quad\quad 1/2 \end{cases}$

① Given a Quantum state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 \\ i\end{pmatrix}$$

What is the prob of measuring it in the state $|+\rangle$

$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 \\ 1\end{pmatrix}$

$$P(+) = |\langle + | \psi \rangle|^2$$

$$= \left|\frac{1}{2}(1,1)\begin{pmatrix}1 \\ i\end{pmatrix}\right|^2$$

$|z|^2 = z\bar{z}$

$$= \frac{1}{4}|1+i|^2$$

$$= \frac{1}{4}(1+i)(1-i)$$

$$= 2/4 = \underline{1/2}$$

28/1/2020

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1$

$|\psi\rangle = \frac{3i}{\sqrt{5}}|0\rangle + \frac{1}{\sqrt{5}}|1\rangle$

$|\alpha|^2 + |\beta|^2 \neq 1$

$$= \left|\frac{3i}{\sqrt{5}}\right|^2 + \left|\frac{1}{\sqrt{5}}\right|^2$$

$= \frac{-9}{5} + 1/5 = -8/5 \neq 1$

Normalize and get it

$|v\rangle = \frac{|v\rangle}{||v\rangle|}$

$z = a + ib$

$|z| = \sqrt{a^2+b^2}$

std bases

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

b) $|\varphi\rangle = \frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle$

$|\alpha|^2 + |\beta|^2 = 1$

$\left|\frac{1}{\sqrt{5}}\right|^2 + \left|\frac{2}{\sqrt{5}}\right|^2 = 1/5 + 4/5 = 1$

c) $|\varphi\rangle = \frac{1}{3}|0\rangle + \frac{2\sqrt{2}}{3}|1\rangle$

$|\alpha|^2 + |\beta|^2 = 1$

$1/9 + 8/9 = 1$

$|0\rangle$ prob $1/9$

$|1\rangle$ prob $8/9$

Measuring Two-qubits :

The general state of a 2-qubit system is written as $|\psi\rangle = \alpha_1|00\rangle + \beta_1|01\rangle + \alpha_2|10\rangle + \beta_2|11\rangle$ where $\alpha_1, \beta_1, \alpha_2$, and $\beta_2 \in C$ & $|\alpha_1|^2 + |\beta_1|^2 + |\alpha_2|^2 + |\beta_2|^2 = 1$

Then we have following measurement rule.

$$\begin{cases} |00\rangle : \text{ with prob } |\alpha_1|^2 \\ |01\rangle : \quad \text{"} \quad \text{"} \quad |\beta_1|^2 \\ |10\rangle : \quad \text{"} \quad \text{"} \quad |\alpha_2|^2 \\ |11\rangle : \quad \text{"} \quad \text{"} \quad |\beta_2|^2 \end{cases}$$

- But we don't have to measure both qubits at once.

Let us say we measure only the first qubit & get result $|0\rangle$.

The only way we could get this is from the terms $|00\rangle$ & $|01\rangle$

$\{|0\rangle$: with probability $|\alpha_1|^2 + |\beta_1|^2$

The new state is

$$|\psi'\rangle = \frac{\alpha_1|00\rangle + \beta_1|01\rangle}{\sqrt{|\alpha_1|^2 + |\beta_1|^2}}$$

We now measure the second qubit. From the coefficient in the new state, we must have

$$\begin{cases} |0\rangle \text{ with prob } \dfrac{|\alpha_1|^2}{|\alpha_1|^2 + |\beta_1|^2} \\ |1\rangle \text{ with prob } \dfrac{|\beta_1|^2}{|\alpha_1|^2 + |\beta_1|^2} \end{cases}$$

The new state will be either $|00\rangle$ or $|01\rangle$ depending on whether the result was $|0\rangle$ or $|1\rangle$

Measuring multi-qubit state

i) consider 2 qubit state

$$|\psi\rangle = \frac{1}{3}|00\rangle + 2/3\,|10\rangle - 2/3\,|11\rangle$$

$$= (1/3)^2 + (2/3)^2 + (-2/3)^2$$

$$= \frac{1}{9} + \frac{4}{9} + 4/9 = 9/9 = \underline{1}$$

a) What is the prob that the result is $|0\rangle$?

b) What is the prob that the result is $|1\rangle$.

c) For each possible write down the post measurement state.

d) Calculate the prob that a measurement of the second qubit will give 0 & 1.

e) write down the state after the 2nd measurement.

a→ $1/9$ $\quad |\psi'\rangle = |00\rangle$

Subsequent measurement of 2nd qubit gives $|0\rangle$ with prob 1. or $|1\rangle$ with prob 0.

post measurement state is $|00\rangle$

b→ $8/9$ $\quad |\psi'\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |11\rangle)$

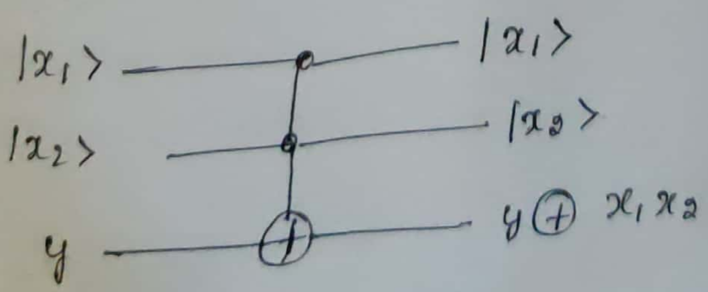
Subsequent measurement of 2nd qubit gives $|0\rangle$ with prob $1/2$ or $|1\rangle$ with prob $1/2$.

with post measurement state $|10\rangle$ or $|11\rangle$

6/9/19

~~NAND~~

CCNOT



$$y \oplus x_1 x_2 = \begin{cases} AND(x_1, x_2) & \text{if } y = 0 \\ XOR(y, x_1) & \text{if } x_2 = 0 \\ NOT\ y & \text{if } x_1 = x_2 = 1 \\ NAND(x_1 x_2) & \text{if } y = 1 \\ x_1 & \text{if } y = 0 \ \& \ x_2 = 1 \end{cases}$$

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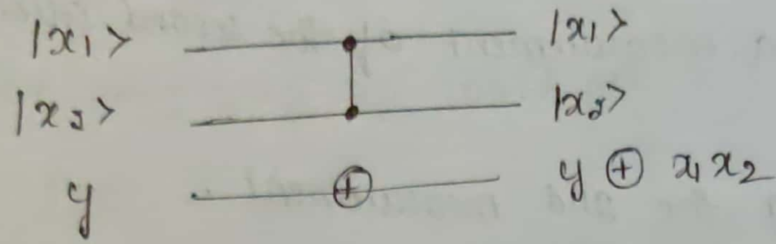
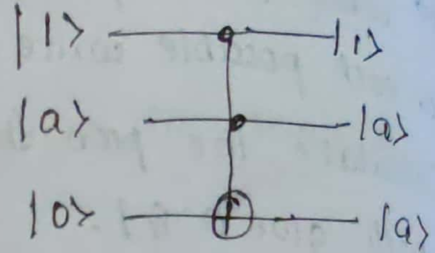
Copy operation using CCNOT





Fredkin gate (controlled - swap gate)

It swaps the values of the 2nd and 3rd bits if and only if, the first bit is set to 1.

$|000\rangle \rightarrow |000\rangle$

$|001\rangle \rightarrow |001\rangle$

$|010\rangle \rightarrow |010\rangle$

$|011\rangle \rightarrow |011\rangle$

$|100\rangle \rightarrow |100\rangle$

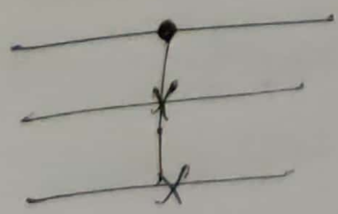
$|101\rangle \rightarrow |110\rangle$

$|110\rangle \rightarrow |101\rangle$

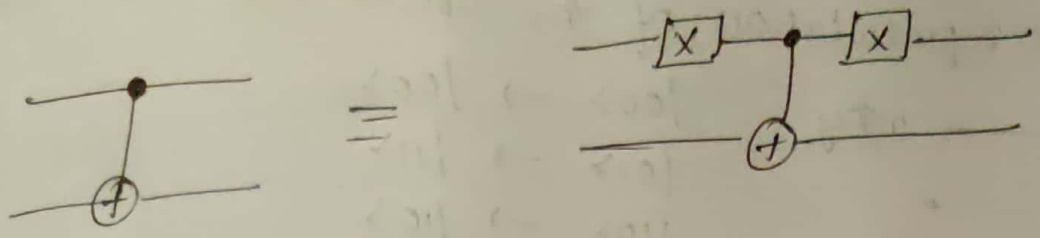
$|111\rangle \rightarrow |111\rangle$

$$CSWAP = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

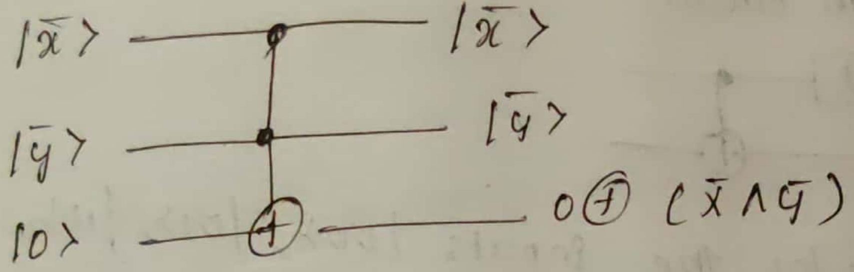
C-SWAP circuit



1. Show that the two circuit below are equivalent

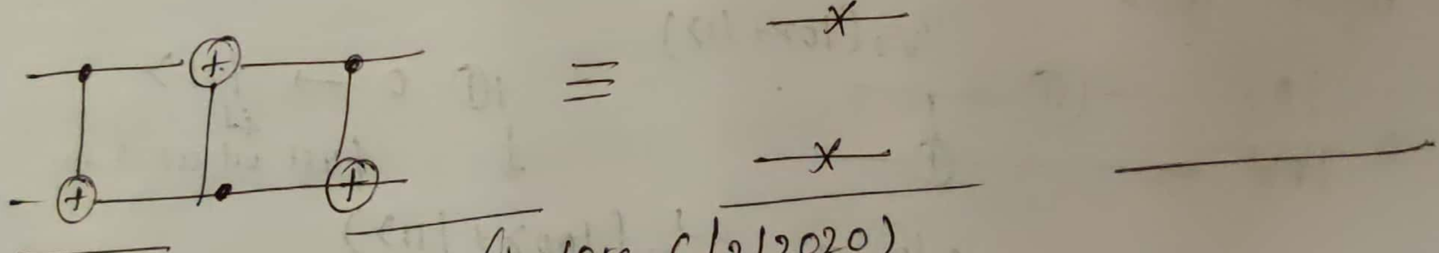


<u>NOR gate</u>



x	y	output
0	0	1
0	1	0
1	0	0
1	1	0.

<u>Swap operation</u>



(before 6/2/2020)

<u>CNOT</u>

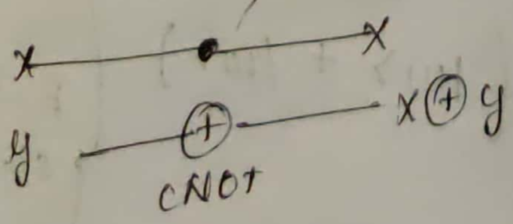
$|00\rangle \rightarrow |00\rangle$

$|01\rangle \rightarrow |01\rangle$

$|10\rangle \rightarrow |11\rangle$

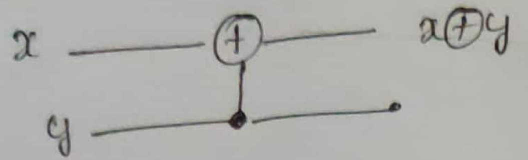
$|11\rangle \rightarrow |10\rangle$

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



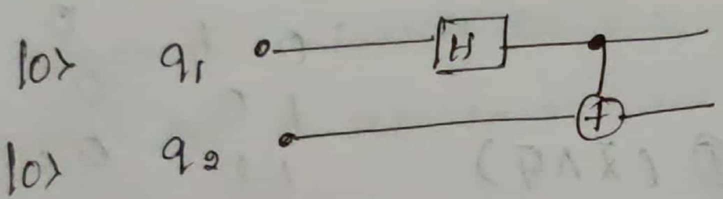
Problem :

Find the matrix representation of the upside down CNOT - ga



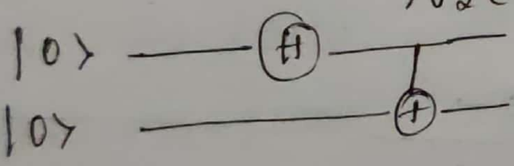
$|00\rangle \to |00\rangle$
$|01\rangle \to |11\rangle$
$|10\rangle \to |10\rangle$
$|11\rangle \to |01\rangle$

2) Consider the Quantum circuit



what are the outputs for the inputs $|00\rangle, |01\rangle, |10\rangle,$ and $|11\rangle$

Input $|00\rangle$



first bit as it is

$0 \oplus 0 \to |00\rangle$

$1 \oplus 0 \to |11\rangle$

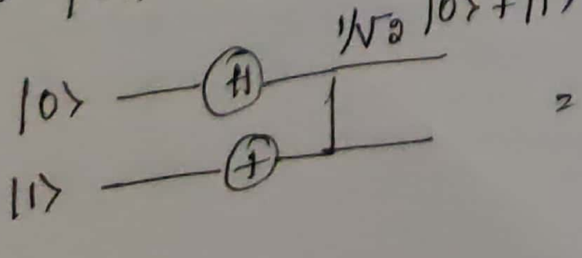
first bit as it is

$$O/P = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

O/P $|01\rangle \to$



O/P

$$= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

Bell state

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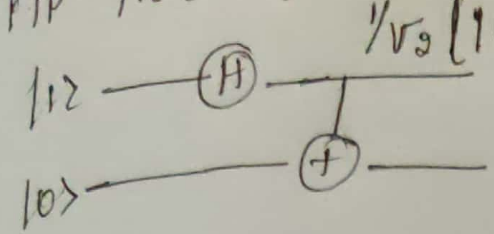
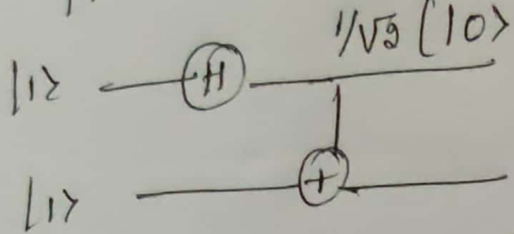
I/P $|10\rangle \rightarrow$

O/P



$$= \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$$

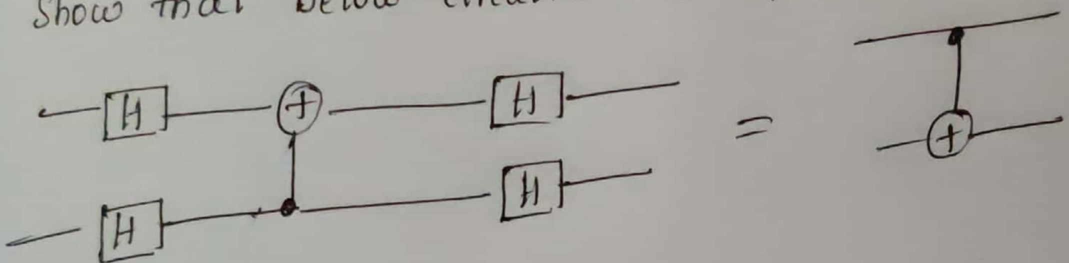
I/P $|11\rangle \longrightarrow$



$$= \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

Bell states

Show that below circuits are equivalent



3 qubit gates

1. Toffoli gate (controlled - controlled - NOT or CCNOT)

The third input bit is flipped, if and only if the first 2 input bits are both 1.

$|000\rangle \longrightarrow |000\rangle$
$|001\rangle \longrightarrow |001\rangle$
$|010\rangle \longrightarrow |010\rangle$
$|011\rangle \longrightarrow |011\rangle$

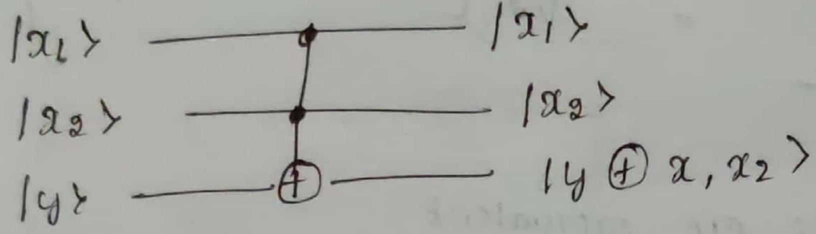
$|100\rangle \rightarrow |100\rangle$
$|101\rangle \rightarrow |101\rangle$
$|110\rangle \rightarrow |111\rangle$
$|111\rangle \rightarrow |110\rangle$

Third bit is flipped when first 2 bits are 1

$$|000\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

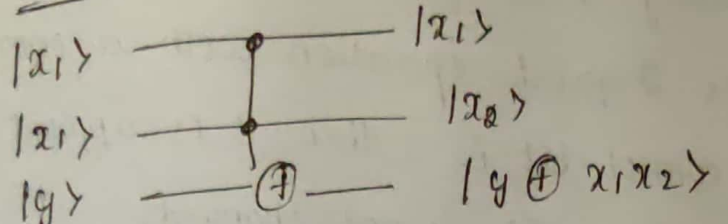
$$\underline{CCNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

CCNOT circuit



7/2/2020

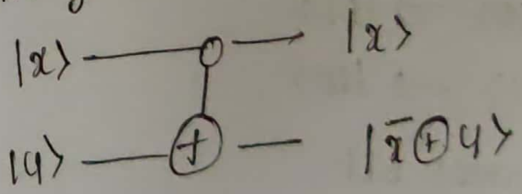
CCNOT



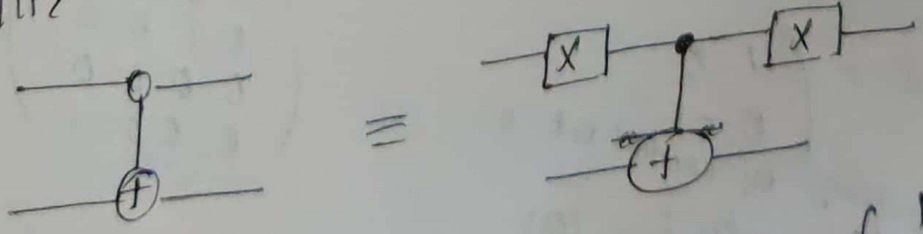
$$y \oplus x_1 x_2 = \begin{cases} AND(x_1, x_2) & \text{if } y = 0 \\ XOR(y, x_2) & \text{if } x_2 = 0 \\ NOT\ y & \text{if } x_1 = x_2 = 1 \\ NAND(x_1, x_2) & \text{if } y = 1 \\ OR(x_1, x_2) & \text{if } x_1 = \bar{x}_1,\ x_2 = \bar{x}_2\ \&\ y = 1 \\ NOR(x_1, x_2) & \text{if } x_1 = \bar{x}_1\ \ x_2 = x\ \ y = 0 \end{cases}$$

2-qubit gate

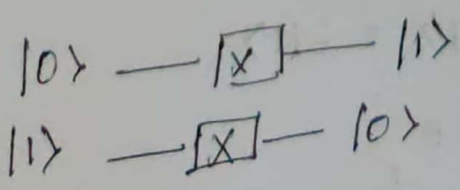
Suppose we want to implement a 2 qubit gate in which the second (target) qubit is flipped, if first qubit (control) is set to "0", otherwise no change.



$|00\rangle \rightarrow |01\rangle$
$|01\rangle \rightarrow |00\rangle$
$|10\rangle \rightarrow |10\rangle$
$|11\rangle \rightarrow |11\rangle$



X gate



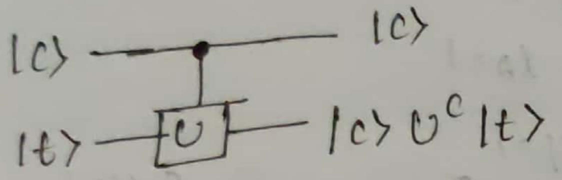
$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

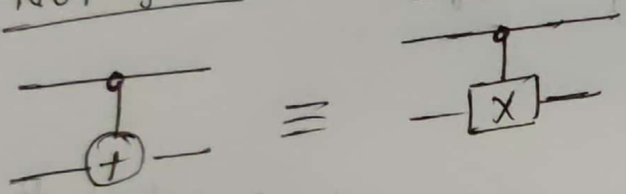
$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Controlled U gate

Suppose U is an arbitrary single qubit unitary operation. A controlled-U operation is 2 qubit operation with a control and a target qubit. If the control bit is 1 then U is applied to the target bit, otherwise target bit is not changed.



C-NOT gate is controlled X-gate



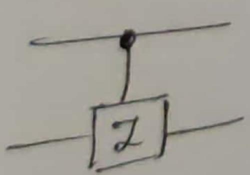
$|00\rangle \to |00\rangle$
$|01\rangle \to |01\rangle$
$|10\rangle \to |11\rangle$
$|11\rangle \to |10\rangle$
control bit

≡

X gate

$|00\rangle \to |00\rangle$
$|01\rangle \to |01\rangle$
$|10\rangle \to |11\rangle$
$|11\rangle \to |10\rangle$

Controlled Z gate



$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

controlled Z

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\left.\begin{matrix} |00\rangle \to |00\rangle \\ |01\rangle \to |01\rangle \end{matrix}\right\}$ No change by control bit 0.

$|0\rangle$ —$\boxed{Z}$— $|0\rangle$

$|1\rangle$ —$\boxed{Z}$— $|-1\rangle = -|1\rangle$

$\left.\begin{matrix} |10\rangle \to |10\rangle \\ |11\rangle \to -|11\rangle \end{matrix}\right\}$ changes

$0 \to 0$
$1 \to -1$

1. Is the following circuit are equivalent.

they are equal.

2. Is the following circuit are equivalent