Machine Learning (CS 181): 7. Probabilistic Classification

David Parkes and Sasha Rush

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1. Generative Probabilistic View
2. Discrete Features
3. Multinomial Naive Bayes
4. Discriminative Probabilistic View
5. Logistic Regression
6. Multiclass Classification

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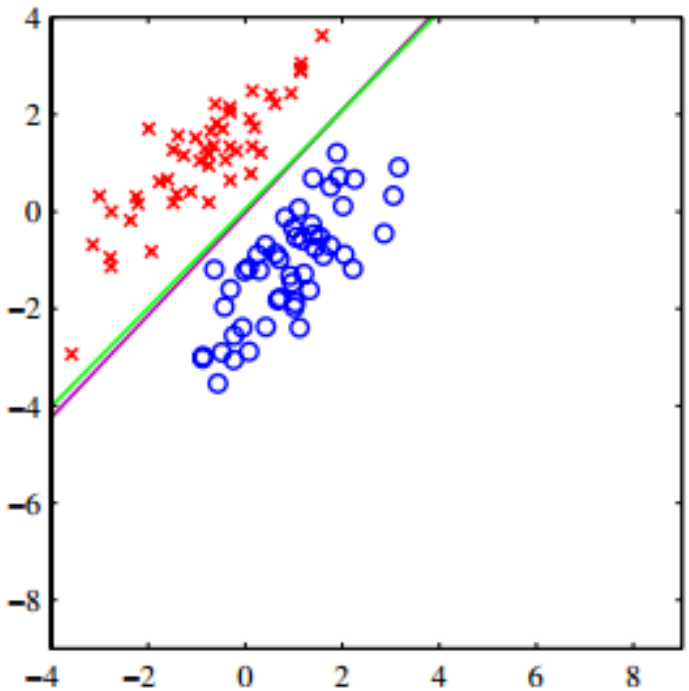
Last Class: Binary Classification

- Output space $\mathcal{Y}$ is a fixed set of classes.
- Simplest case $\mathcal{Y} = \{-1, 1\}$ (red/blue)
- Discriminant function:



Binary Classification

Generative Classification View

Model the joint probability of call y and data $\mathbf{x}$,

$$p(\mathbf{x}, y) = p(y)p(\mathbf{x}|y)$$

Note: Switch to different binary class representation,

$$\mathcal{Y} = \{0, 1\}$$

Process:

- Class is generated with probability $p(y)$
- Input $\mathbf{x}$ is generated conditional on class $p(\mathbf{x}|y)$

Class Probability

Choice of prior can depend on problem format,

- In binary case, we will use Bernoulli distribution,

$$p(y = 1; \theta) = \theta \quad \text{and} \quad p(y = 0; \theta) = 1 - \theta$$

or more compactly (motivates notation change),

$$p(y; \theta) = \theta^y (1 - \theta)^{1-y}$$

Class-Conditional Probability

- Choice depends on modeling assumptions
- Select parameteric model for data given class

$$p(\mathbf{x}|y, \mathbf{w}_0, \mathbf{w}_1)$$

- Use continuous data, use multivariate Gaussian (HW)

$$p(\mathbf{x}|y = 0; \boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0, \boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1) = \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$$

$$p(\mathbf{x}|y = 1; \boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0, \boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1) = \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$$

- Today's class, discrete inputs,

$$p(\mathbf{x}|y; \boldsymbol{\pi}_0, \boldsymbol{\pi}_1)$$

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- Today's class, discrete inputs,

$$p(\mathbf{x}|y; \boldsymbol{\pi}_0, \boldsymbol{\pi}_1)$$

Maximum Likelihood

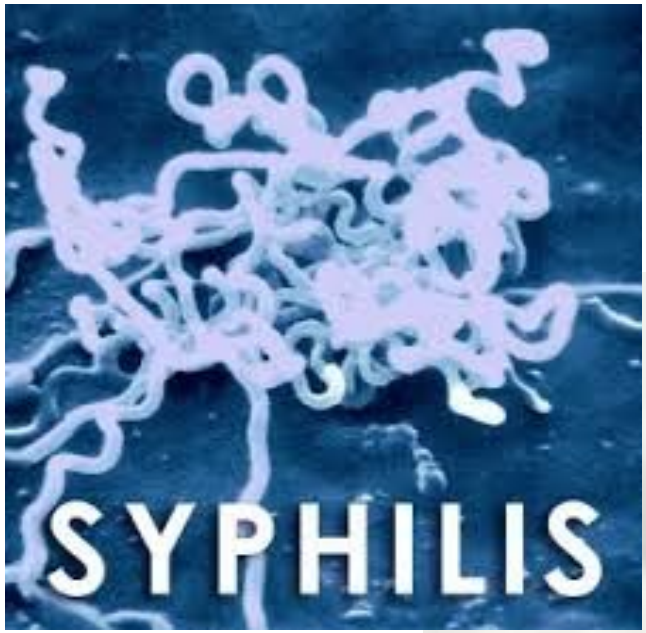
Same probabilistic approach as before,

1. Decide on generating process
2. Fix the parameterization of the model
3. Minimize negative log-likelihood of the data.

$$\min_{\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \theta} \mathcal{L}(\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \theta) = \min_{\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \theta} -\sum_{i=1}^{n} (\ln p(y_i; \theta) + \ln p(\mathbf{x}_i | y_i; \boldsymbol{\pi}_0, \boldsymbol{\pi}_1))$$

- Again, benefits will be having explicit probabilities for classes.
- Can also combine with Bayesian approaches from last class.

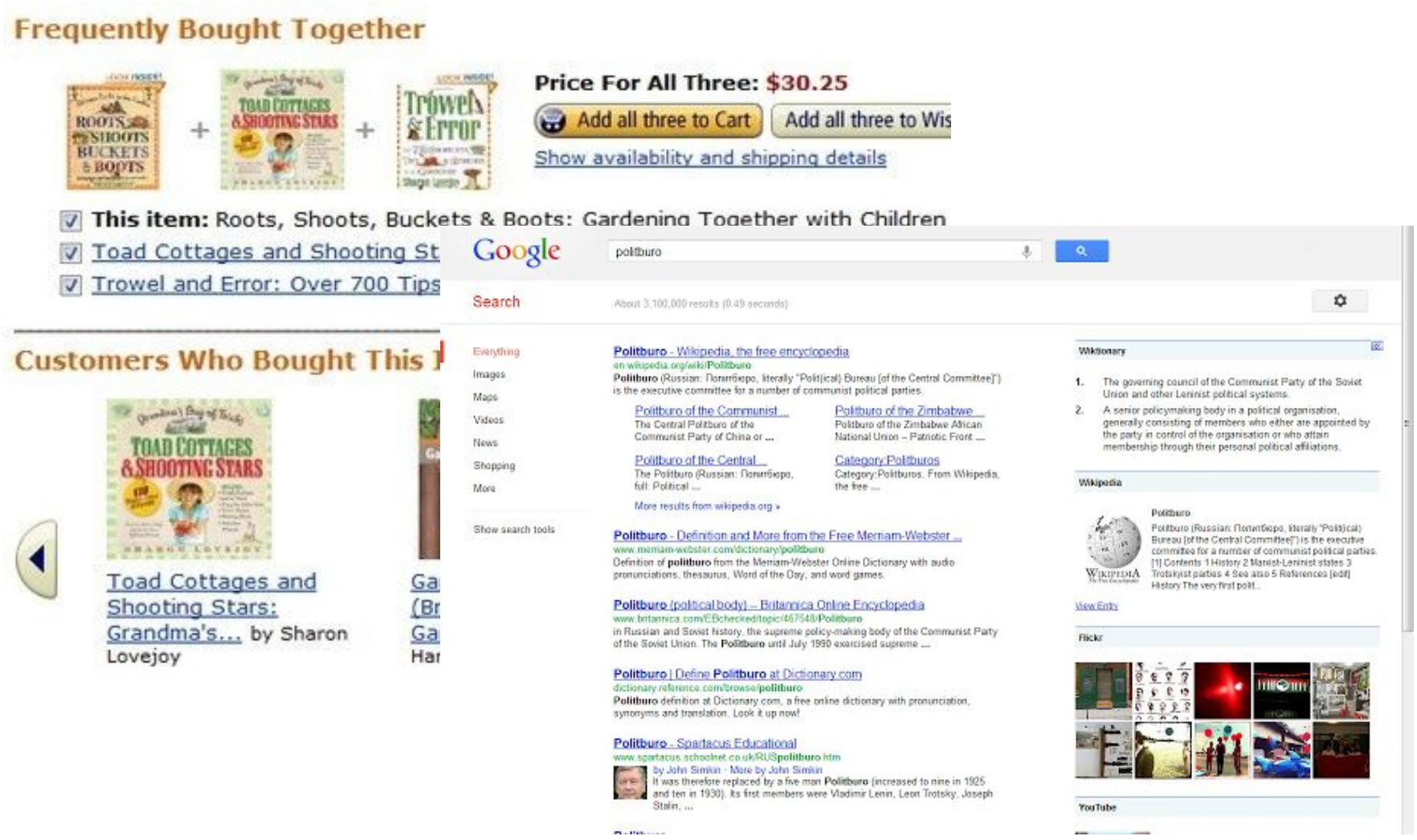
Probabilistic Classification



http://www.vancouversun.com/cms/binary/8997950.jp

http://www.health.pa.gov/My%20Health/D

Probabilistic Classification



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Discrete Features / Basis

- Features so far, mostly continuous:
  - revenue, distance, crime rate, etc.
- For many domains, discrete features are more natural.
- Features represent (sparse) indicators or counts
  - visits, responses, properties, etc

$$\mathbf{x} = [0; 0; 1; \ldots; 0; 10; 0]$$

- Want to model with discrete distributions as opposed to Gaussians.

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Sentiment

Good Sentences

- A thoughtful, provocative, insistently humanizing film.
- Occasionally melodramatic, it's also extremely effective.
- Guaranteed to move anyone who ever shook, rattled, or rolled.

Bad Sentences

- A sentimental mess that never rings true.
- This 100-minute movie only has about 25 minutes of decent material.
- Here, common sense flies out the window, along with the hail of bullets, none of which ever seem to hit Sascha.

Features 1: Sparse Bag-of-Words Features

Example: Movie review input,

$$\texttt{A sentimental mess}$$

$$\boldsymbol{\phi}(\mathbf{x}) = [\phi_{\texttt{word:A}}(\mathbf{x}); \ldots; \phi_{\texttt{word:sentimental}}(\mathbf{x}); \phi_{\texttt{word:mess}}(\mathbf{x}); \ldots]$$

$$\boldsymbol{\phi}(\mathbf{x}) = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{matrix} \texttt{word:A} \\ \vdots \\ \texttt{word:mess} \\ \texttt{word:sentimental} \\ \end{matrix}$$

Features 2: Sparse Word Properties

Example: Spam Email

`Your diploma puts a UUNIVERSITY JOB PLACEMENT COUNSELOR at your disposal.`

$$\boldsymbol{\phi}(\mathbf{x}) = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{matrix} \texttt{misspelling} \\ \vdots \\ \texttt{capital} \\ \texttt{word:diploma} \\ \end{matrix}$$

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Multinomial Distribution

Probability of histogram $\mathbf{x}$, assuming $\sum_{j=1}^{m} x_j$ trials, (exp trick)

$$p(\mathbf{x}; \boldsymbol{\pi}) = \frac{(\sum x_j)!}{\prod x_j!} \prod_{j=1}^{m} \pi_j^{x_j} \propto \prod_{j=1}^{m} \pi_j^{x_j}$$

Maximum likelihood estimate over $\mathbf{x}_1 \dots \mathbf{x}_n$, (log trick)

$$\arg\max_{\boldsymbol{\pi} \geq 0} \quad \prod_{i=1}^{n} p(\mathbf{x}_i; \boldsymbol{\pi}) =$$

$$\arg\max_{\boldsymbol{\pi} \geq 0} \quad \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} \ln \pi_j$$

$$\text{s.t.} \quad \sum_{j=1}^{m} \pi_j = 1$$

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\text{s.t.} \quad & \sum_{j=1}^{m} \pi_j = 1
\end{aligned}$$

Maximum Likelihood (1)

Lagrange multipliers:

$$L(\boldsymbol{\pi}, \lambda) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} \ln \pi_j + \lambda (\sum_{j=1}^{m} \pi_j - 1)$$

Solve π for all j,

$$\frac{\partial}{\partial \pi_j} L(\boldsymbol{\pi}, \lambda) = \sum_{i=1}^{n} x_{ij}/\pi_j + \lambda = 0$$

$$\pi_j = \sum_{i=1}^{n} x_{ij}/\lambda$$

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Maximum Likelihood (2)

Lagrange multipliers:

$$L(\boldsymbol{\pi}, \lambda) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} \ln \pi_j + \lambda (\sum_{j=1}^{m} \pi_j - 1)$$

$$\pi_j = \sum_{i=1}^{n} x_{ij} / \lambda$$

Solve λ,

$$\frac{\partial}{\partial \lambda} L(\boldsymbol{\pi}, \lambda) = \sum_{j=1}^{m} \pi_j = 1$$

$$\sum_{j=1}^{m} \sum_{i=1}^{n} x_{ij} / \lambda = 1$$

$$\lambda = \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij}$$

Maximum Likelihood (2)

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Solve λ,

$$\begin{aligned} \frac{\partial}{\partial \lambda} L(\boldsymbol{\pi}, \lambda) &= \sum_{j=1}^{m} \pi_j = 1 \\ \sum_{j=1}^{m} \sum_{i=1}^{n} x_{ij} / \lambda &= 1 \\ \lambda &= \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} \end{aligned}$$

Multinomial MLE Matrix Form

The maximum likelihood parameters become the counts over the data:

$$\hat{\boldsymbol{\pi}} = \frac{\sum_{i=1}^{n} \mathbf{x}_i}{\sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij}} = \frac{\mathbf{X}^\top \mathbf{1}}{\mathbf{1X1}}$$

Mulitinomial Binary-Class Naive Bayes

Assume each discrete features $\mathbf{x}$, model joint probability as

$$p(\mathbf{x}, y) = p(y; \theta)p(\mathbf{x}|y; \boldsymbol{\pi}_0, \boldsymbol{\pi}_1)$$

- Class probability, Bernoulli $p(y; \theta)$
- Features conditional Multinomial:

$$p(\mathbf{x}|y; \boldsymbol{\pi}_0, \boldsymbol{\pi}_1) \propto \prod_{j=1}^{m} \pi_{yj}^{x_j}$$

- Multinomial: each feature is generated independently (Naive)
- Conditional classification using Bayes rule (Bayes)

$$p(y|\mathbf{x}) \propto p(y; \theta)p(\mathbf{x}|y; \boldsymbol{\pi}_0, \boldsymbol{\pi}_1)$$

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Loss for Multinomial Naive Bayes

$$\max_{\theta,\boldsymbol{\pi}_0,\boldsymbol{\pi}_1} \sum_{i=1}^{n} \ln p(\mathbf{x}_i, y_i) = \max_{\boldsymbol{\pi}_1,\boldsymbol{\pi}_2} \sum_{i=1}^{n} \ln p(\mathbf{x}_i|y_i; \boldsymbol{\pi}_0, \boldsymbol{\pi}_1) + \max_{\theta} \sum_{i=1}^{n} \ln p(y_i; \theta)$$

- Class probability

$$\hat{\theta} = \frac{\mathbf{1}\mathbf{y}}{n}$$

- Class-Conditional probability

$$\hat{\boldsymbol{\pi}_0} = \frac{\sum_{i=1}^{n}(1-y_i)\mathbf{x}_i}{\sum_{i=1}^{n}\sum_{j=1}^{m}(1-y_i)x_{ij}}$$

$$\hat{\boldsymbol{\pi}_1} = \frac{\sum_{i=1}^{n} y_i\mathbf{x}_i}{\sum_{i=1}^{n}\sum_{j=1}^{m} y_i x_{ij}}$$

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Multinomial Naive Bayes (In Practice)

$$\ln p(y|\mathbf{x}) \propto \ln p(\mathbf{x}|y) + \ln p(y)$$

How do you decide what class? (Discriminant function)

$$\begin{aligned} h(\mathbf{x}) &= (\ln p(\mathbf{x}|y=1) + \ln p(y=1)) - (\ln p(\mathbf{x}|y=0) + \ln p(y=0)) \\ &= [\ln \prod_{j=1}^{m} \pi_{1j}^{x_j} - \ln \prod_{j=1}^{m} \pi_{0j}^{x_j}] + [\ln \theta - \ln(1-\theta)] \\ &= \sum_{j=1}^{m} x_j \ln \frac{\pi_{1j}}{\pi_{0j}} + \ln \frac{\theta}{1-\theta} = \mathbf{x}^\top (\ln \frac{\boldsymbol{\pi}_1}{\boldsymbol{\pi}_0}) + \ln \frac{\theta}{1-\theta} \end{aligned}$$

But this formulation is a linear model.

$$h(\mathbf{x}; \mathbf{w}) = \mathbf{x}^\top \mathbf{w} + w_0$$

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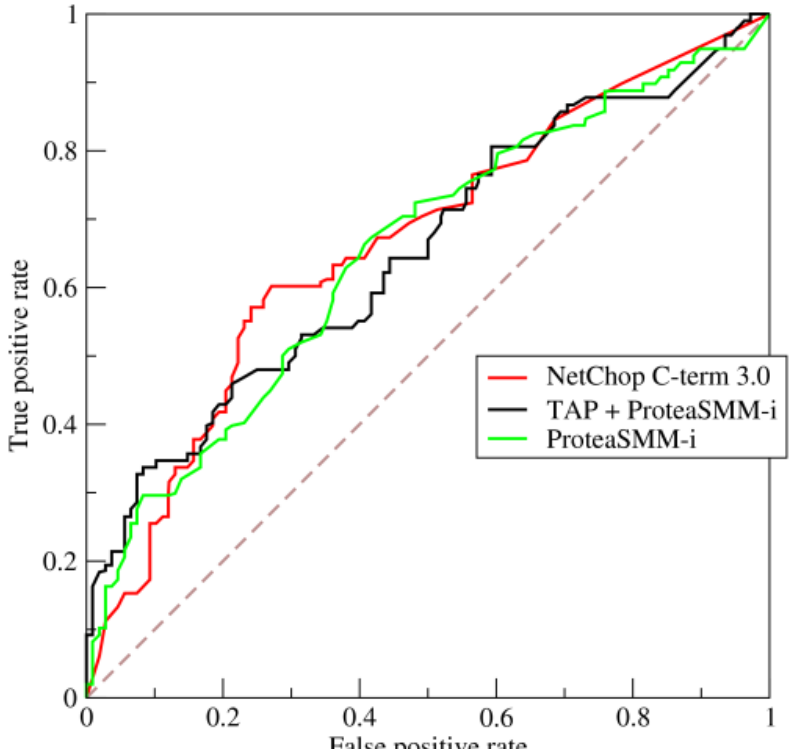
Role of Class Probabilities

$$\mathbf{x}^\top (\ln \frac{\boldsymbol{\pi}_1}{\boldsymbol{\pi}_0}) + \ln \frac{\theta}{1-\theta}$$

Class probabilities $p(y;\theta)$ determine the threshold.

$$h(\mathbf{x};\mathbf{w}) = \mathbf{x}^\top \mathbf{w} + w_0$$

Brings us back again to ROC curves.



Naive Bayes In Practice

- Super fast to train.
- Relatively interpretable.
- Performs quite well on small datasets.

Method	RT-s	MPQA	CR	Subj.
MNB-uni	77.9	85.3	79.8	**92.6**
MNB-bi	**79.0**	**86.3**	80.0	**93.6**
SVM-uni	76.2	86.1	79.0	90.8
SVM-bi	77.7	**86.7**	80.8	91.7
NBSVM-uni	**78.1**	85.3	80.5	92.4
NBSVM-bi	**79.4**	**86.3**	**81.8**	**93.2**
RAE	76.8	85.7	–	–
RAE-pretrain	77.7	**86.4**	–	–
Voting-w/Rev.	63.1	81.7	74.2	–

(RT-S [movie review], CR [customer reports], MPQA [opinion polarity], SUBJ [subjectivity])

Naive Bayes Example

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Generative versus Discriminative Model

- Generative models: Parameterize Joint Distribution

$$\arg\max_{\mathbf{w}} \prod_i p(\mathbf{x}_i, y_i; \mathbf{w})$$

- Discriminative model: Parameterize Conditional Distribution

$$\arg\max_{\mathbf{w}} \prod_i p(y_i | \mathbf{x}_i; \mathbf{w})$$

- Why does this matter?

Linear Discriminative Model

Set log prob to be proportional to some linear model, shorthand h

$$\ln p(y = 1|\mathbf{x}; \mathbf{w}) \propto \mathbf{w}^\top \mathbf{x} + w_0 = h$$

As before threshold at $h > 0$,

$$\ln p(y = 0|\mathbf{x}; \mathbf{w}) \propto 0$$

Now remove log and normalize,

$$p(y = 1|\mathbf{x}; \mathbf{w}) = \frac{\exp h}{\exp h + \exp 0} = (1 + \exp -h)^{-1}$$

$$p(y = 0|\mathbf{x}; \mathbf{w}) = \frac{\exp 0}{\exp h + \exp 0} = (1 + \exp h)^{-1}$$

Call this function the logistic sigmoid activation.

$$\sigma(h) = (1 + \exp -h)^{-1}$$

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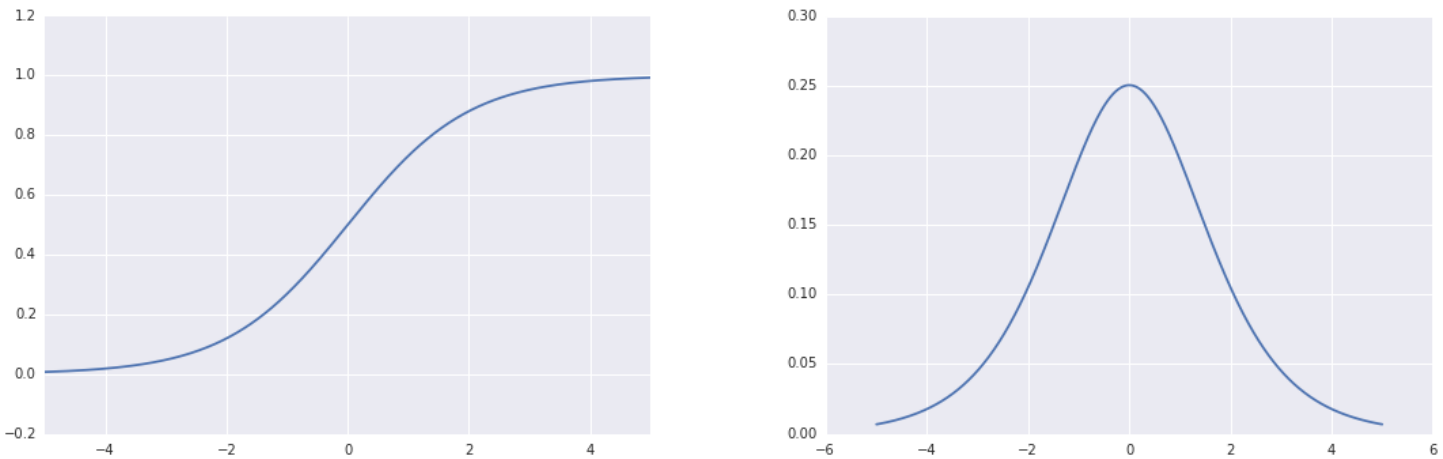
(Logistic) Sigmoid Activation

$$\sigma(h) = (1 + \exp(-h))^{-1}$$



Sigmoid Function and Derivative

- "Squashes" $\mathbb{R}$ to a probabilities.

Logistic Regression

Linear model converted to probability estimated by sigmoid

$$p(y = 1|\mathbf{x}; \mathbf{w}) = \sigma(\mathbf{w}^\top \mathbf{x} + w_0) = (1 + \exp(-h))^{-1}$$

$$p(y = 0|\mathbf{x}; \mathbf{w}) = 1 - \sigma(\mathbf{w}^\top \mathbf{x} + w_0) = (1 + \exp h)^{-1}$$

- Linear "Regression" transformed to probability estimate.
- Name is confusing, mostly used for *classification*.

Fitting Model

Reminder:

$$p(y = 1|\mathbf{x}; \mathbf{w}) = \sigma(\mathbf{w}^\top \mathbf{x} + w_0) = (1 + \exp(-h))^{-1}$$

$$p(y = 0|\mathbf{x}; \mathbf{w}) = 1 - \sigma(\mathbf{w}^\top \mathbf{x} + w_0) = (1 + \exp h)^{-1}$$

As this is now a probabilistic model, can fit with MLE.

$$\begin{aligned}
\mathcal{L}(\mathbf{w}) &= -\sum_{i=1}^{n} \ln p(y_i|\mathbf{x}_i; \mathbf{w}) = -\sum_{i=1}^{n} \ln \sigma(h)^{y_i} (1 - \sigma(h))^{1-y_i} \\
&= \sum_{i=1}^{n} y_i \ln(1 + \exp(-h)) + (1 - y_i) \ln(1 + \exp h)
\end{aligned}$$

Likelihood and Estimation

Reminder:

$$\begin{aligned} h &= \mathbf{w}^\top \mathbf{x}_i + w_0 \\ \mathcal{L}(\mathbf{w}) &= \sum_{i=1}^{n} y_i \ln(1 + \exp(-h)) + (1 - y_i) \ln(1 + \exp h) \end{aligned}$$

Take gradients:

$$\begin{aligned} \frac{\partial}{\partial \mathbf{w}} \ln(1 + \exp(-h)) &= -\mathbf{x}_i \frac{\exp -h}{1 + \exp(-h)} = -\mathbf{x}_i p(y_i = 0|\mathbf{x}) \\ \frac{\partial}{\partial \mathbf{w}} \ln(1 + \exp h) &= \mathbf{x}_i \frac{\exp h}{1 + \exp h} = \mathbf{x}_i p(y_i = 1|\mathbf{x}) \end{aligned}$$

$$\frac{\partial}{\partial \mathbf{w}} \mathcal{L}(\mathbf{w}) = \sum_{i=1}^{n} -y_i \mathbf{x}_i p(y_i = 0|\mathbf{x}_i) + (1 - y_i) \mathbf{x}_i p(y_i = 1|\mathbf{x}_i)$$

Likelihood and Estimation

Reminder:

$$\begin{aligned} h &= \mathbf{w}^\top \mathbf{x}_i + w_0 \\ \mathcal{L}(\mathbf{w}) &= \sum_{i=1}^{n} y_i \ln(1+\exp(-h)) + (1-y_i)\ln(1+\exp h) \end{aligned}$$

Take gradients:

$$\begin{aligned} \frac{\partial}{\partial \mathbf{w}} \ln(1+\exp(-h)) &= -\mathbf{x}_i \frac{\exp -h}{1+\exp(-h)} = -\mathbf{x}_i p(y_i = 0|\mathbf{x}) \\ \frac{\partial}{\partial \mathbf{w}} \ln(1+\exp h) &= \mathbf{x}_i \frac{\exp h}{1+\exp h} = \mathbf{x}_i p(y_i = 1|\mathbf{x}) \end{aligned}$$

$$\frac{\partial}{\partial \mathbf{w}} \mathcal{L}(\mathbf{w}) = \sum_{i=1}^{n} -y_i \mathbf{x}_i p(y_i = 0|\mathbf{x}_i) + (1-y_i)\mathbf{x}_i p(y_i = 1|\mathbf{x}_i)$$

Recall: Perceptron Algorithm

1. Iterate over the data:
   - If correct ($y_i = \hat{y}_i$), do nothing.
   - If incorrect, add/subtract $\eta \times \mathbf{x}_i$ to weights
2. If errors, repeat process.
3. Otherwise separator is found.

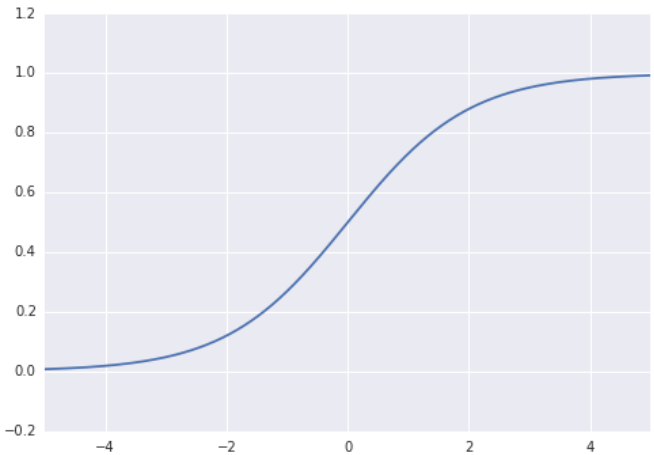
SGD on Logistic Regression

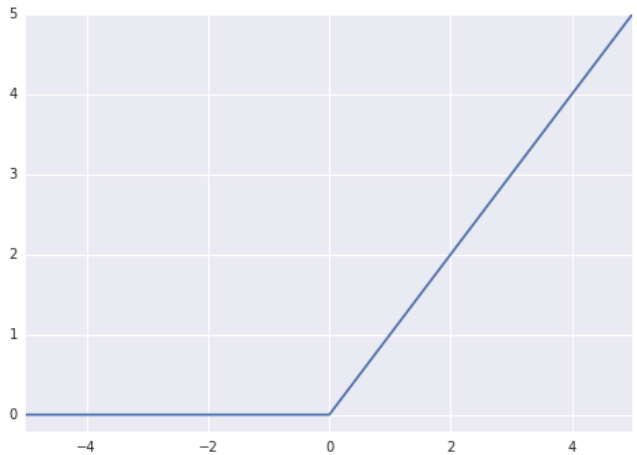
1. Iterate over the data:
   - Compute $p(y_i = 1|\mathbf{x}_i)$.
   - If $(y_i = 1)$, add $\eta \times \mathbf{x}_i p(y_i = 0|\mathbf{x}_i)$ to $\mathbf{w}$
   - If $(y_i = 0)$, add $\eta \times -\mathbf{x}_i p(y_i = 1|\mathbf{x}_i)$ to $\mathbf{w}$
2. Repeat until convergence.

Guaranteed to maximize conditional likelihood of data.

Algorithms Comes from Activations

- What is the difference between Perceptron and Logistic Regression?





Sigmoid Function versus Hinge

Contents

1 Generative Probabilistic View

2 Discrete Features

3 Multinomial Naive Bayes

4 Discriminative Probabilistic View

5 Logistic Regression

6 Multiclass Classification

Multiclass

- ★★★★

  I visited The Abbey on several occasions on a visit to Cambridge and found it to be a solid, reliable and friendly place for a meal.

- ★★

  However, the food leaves something to be desired. A very obvious menu and average execution

- ★★★★★

  Fun, friendly neighborhood bar. Good drinks, good food, not too pricey. Great atmosphere!

Multiclass Naive Bayes

Multiclass outputs: $\mathcal{Y} = \{C_1, \ldots C_c\}$

One-hot vectors

$$C_2 = [0; 1; 0; \ldots; 0]$$

- Class distribution uses categorical for each $k \in \{1 \ldots c\}$,

$$p(\mathbf{y} = C_k; \boldsymbol{\pi}) = \pi_k$$

- Class-Conditional uses separate parameters for each class

$$\{\mathbf{w}_\ell\}_{\ell=1}^c$$

$$p(\mathbf{x}|\mathbf{y}; \{\mathbf{w}_\ell\}_{\ell=1}^c)$$

In homework, these each parameterize a multivariate Gaussian.

Multiclass Logistic Regression

- Multiclass logistic regression uses a $\mathbf{w}_\ell$ for each class.
- Generalization of sigmoid is softmax function.

$$p(\mathbf{y} = C_k | \mathbf{x}; \{\mathbf{w}_\ell\}_{\ell=1}^c) = \frac{\exp(\mathbf{x}^\top \mathbf{w}_k)}{\sum_\ell \exp(\mathbf{x}^\top \mathbf{w}_\ell)}$$

(Derivation and exploration on homework.)

Neural Network Preview: Softmax In Action