MA0301
ELEMENTARY DISCRETE MATHEMATICS
NTNU, SPRING 2022

Solutions Set 2

Exercise 1. *Lewis, Zax: Exercise 12.1*

Solution. In the universe of individual persons define the predicate:

$$L(x,y): \quad x \text{ loves } y$$

Then we get

a) $\exists x \forall y L(x,y)$
a) $\exists x \forall y \neg L(x,y)$
a) $\neg(\exists x \forall y L(x,y) \equiv \forall x \exists y \neg L(x,y)$

Exercise 2. *Consider the following predicates in the universe of real numbers:*

(1) F(x,y) : x is smaller then y
(2) G(x,y) : x is equal to y

Translate the following statements into an English sentence and then comment on whether they should be true or false.

$$a) \forall x \exists y F(x,y)$$
$$b) \exists x \forall y G(x,y)$$
$$c) \forall x \forall y \forall z F(x,y) \wedge F(y,z) \Rightarrow F(x,z)$$
$$d) \forall x \forall y \forall z [F(x,y) \vee G(x,y)] \wedge [F(y,z) \vee G(y,z)] \wedge G(x,z) \Rightarrow G(x,y)$$

Compare the statements in a) and b) to statements a) and c) of exercise 3 in Set 1. What is the difference?

Solution.

a) For all real numbers x it exists a real number y such that x is smaller then y. This statement should be true.
b) It exista a real number x such that for all all real numbers y the number x is equal to the number y. This statement should be false.
c) For all real numbers x, y and z if x is smaller then y and y is smaller then z, then x is smaller then z.
d) For all real numbers x, y and z if x is smaller or equal to y and y is smaller or equal to z and x is equal to z, then x is equal to y. This statement should be true.

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Exercise 3. *Which of the following four quantificational formulas are logically equivalent? Justify your answer.*

$$
\begin{aligned}
&a)\neg[\forall x \exists y F(x,y) \Rightarrow F(y,x)]\\
&b)\exists x \forall y \neg F(x,y) \wedge F(y,x)\\
&c)\exists x \forall y F(x,y) \wedge \neg F(y,x)\\
&d)\exists y \forall x \neg F(x,y) \wedge F(y,x)
\end{aligned}
$$

Solution. We simplify the expression in a)

$$
\begin{aligned}
&\neg[\forall x \exists y F(x,y) \Rightarrow F(y,x)]\\
=&\neg[\forall x \exists y F(x,y)) \Rightarrow F(y,x)]\\
=&\exists x \neg[\exists y F(x,y) \Rightarrow F(y,x)]\\
=&\exists x \forall y \neg[F(x,y) \Rightarrow F(y,x)]\\
=&\exists x \forall y \neg[\neg F(x,y) \vee F(y,x)]\\
=&\exists x \forall y F(x,y) \wedge \neg F(y,x)
\end{aligned}
$$

We see that a) and c) are logically equivalent. By interchanging the variables x and y we further see that d) is logically equivalent to a) and c).

Exercise 4. *Translate the following English sentences into predicate logic by defining predicates in an appropriate universe and forming statements with them:*

a) All apples are either red or green.
b) All fruits are red or green or not apples
c) It does not exist an apple that is neither red nor green

Solution. We work in the universe of all fruits. This means our variables stand for fruits (not necessarily apples). Then we define the following predicates:

- $A(x)$: x is an apple
- $R(x)$: x is red
- $G(x)$: x is green

Now we can form the appropriate statements.

a) $\forall x A(x) \Rightarrow R(x) \oplus G(x)$
b) $\forall x (R(x) \vee G(x)) \vee \neg A(x)$ note that because of $a \Rightarrow b \equiv \neg a \vee b$ this statement is actually equivalent to $\forall x A(x) \Rightarrow R(x) \vee G(x)$. So it's almost the same as in a). But now apples can be red and green at the same time.
c) $\neg \exists x A(x) \wedge (\neg R(x) \wedge \neg G(x)) \equiv \forall x \neg A(x) \vee (R(x) \vee G(x))$. This statement is equivalent to the one in b).

Exercise 5. *Lewis, Zax: Exercise 12.3 a),b),c)*

Solution. We translate the statements to English sentences and then decide whether they should be true.

a) There exist natural numbers x, y and z such that x is smaller then y and z is smaller then y and x is smaller then z and z is not smaller then x. This is true (for example with $x = 1$, $z = 2$ and $y = 3$).
b) There exists natural numbers x, y and z such that $x + 1 = y$ and $z + 1 = y$ and $x + 1 = z$ and $z + 1 \neq x$. This can not be true for $x + 1 = y$ and $z + 1 = y$ implies $x = z$.
c) There exist bitstrings x, y and z such that x comes lexicographically earlier then y and z comes lexicographically earlier then y, and x comes lexicographically earlier then z and z comes not lexicographically earlier then x. This is true (for example with $x = 00$, $z = 01$ and $y = 11$). .

Exercise 6. *Lewis, Zax: Exercise 12.6*

Solution.

a) The formula is unsatisfiable since for $u = v$ the statement $P(u, u, v) \Leftrightarrow P(v, v, u)$ is contradictory.
b) Let the universe be $U = 0$. The set just containing the natural number 1. Then define P to be the predicate $P(x, y) : \quad x \neq y$. This predicate evaluates to false for all x and all y. Thus the model satisfies the formula.