Hazelnut Live Dynamics

2/23/18

metavariables (hole names)

HTyp $\tau ::= b \mid \tau \to \tau \mid \llparenthesis\rrparenthesis$

HExp $e ::= c \mid x \mid \lambda x{:}\tau.e \mid e(e) \mid \llparenthesis\rrparenthesis^u \mid \llparenthesis e \rrparenthesis^u \mid \lambda x.e \mid e : \tau$

DHExp $d ::= c \mid x \mid \lambda x{:}\tau.d \mid d(d) \mid \llparenthesis\rrparenthesis^u_\sigma \mid \llparenthesis d \rrparenthesis^u_\sigma \mid d\langle \tau \Rightarrow \tau \rangle$

$\mid \; d\langle \tau \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau_2 \rangle \; \$$

n-ary substitutions (environments)

Shorthand: $d\langle \tau_1 \Rightarrow \tau_2 \Rightarrow \tau_3 \rangle \equiv d\langle \tau_1 \Rightarrow \tau_2 \rangle\langle \tau_2 \Rightarrow \tau_3 \rangle$

$\boxed{\Gamma \vdash e \Rightarrow \tau \leadsto d}$ Synthetic Expansion

$\boxed{\tau \blacktriangleright_{\to} \tau_1 \to \tau_2}$

$$\frac{}{\tau_1 \to \tau_2 \blacktriangleright_{\to} \tau_1 \to \tau_2} \qquad \frac{}{\llparenthesis\rrparenthesis \blacktriangleright_{\to} \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis}$$

ES-CONST

$$\frac{}{\Gamma \vdash c : b \leadsto c \dashv \cdot}$$

ES-VAR

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x \Rightarrow \tau \leadsto x \dashv \cdot}$$

ES-LAM

$$\frac{\Gamma, x : \tau_1 \vdash e \Rightarrow \tau_2 \leadsto d \dashv \Delta}{\Gamma \vdash \lambda x{:}\tau_1.e \Rightarrow \tau_1 \to \tau_2 \leadsto \lambda x{:}\tau_1.d \dashv \Delta}$$

ES-AP

$$\frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \blacktriangleright \tau_2 \to \tau \qquad \Gamma \vdash e_1 \Leftarrow \tau_2 \to \tau \leadsto d_1 : \tau_1' \dashv \Delta_1 \qquad \Gamma \vdash e_2 \Leftarrow \tau_2 \leadsto d_2 : \tau_2' \dashv \Delta_2}{\Gamma \vdash e_1(e_2) \Rightarrow \tau \leadsto (d_1\langle \tau_1' \Rightarrow \tau_2 \to \tau \rangle)(d_2\langle \tau_2' \Rightarrow \tau_2 \rangle) \dashv \Delta_1 \cup \Delta_2}$$

ES-EHOLE

$$\frac{}{\Gamma \vdash \llparenthesis\rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis \leadsto \llparenthesis\rrparenthesis^u_{id(\Gamma)} \dashv u :: [\Gamma]\llparenthesis\rrparenthesis}$$

ES-NEHOLE

$$\frac{\Gamma \vdash e \Rightarrow \tau \leadsto d \dashv \Delta}{\Gamma \vdash \llparenthesis e \rrparenthesis^u \Rightarrow \llparenthesis\rrparenthesis \leadsto \llparenthesis d \rrparenthesis^u_{id(\Gamma)} \dashv \Delta, u :: [\Gamma]\llparenthesis\rrparenthesis}$$

ES-ASC

$$\frac{\Gamma \vdash e \Leftarrow \tau \leadsto d : \tau' \dashv \Delta}{\Gamma \vdash (e : \tau) \Rightarrow \tau \leadsto d\langle \tau' \Rightarrow \tau \rangle \dashv \Delta}$$

$\boxed{\Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau \dashv \Delta}$ Analytic Expansion

EALAM

$$\frac{\tau \blacktriangleright_{\to} \tau_1 \to \tau_2 \qquad \Gamma, x:\tau_1 \vdash e \Leftarrow \tau_2 \rightsquigarrow d : \tau_2' \dashv \Delta}{\Gamma \vdash \lambda x.e \Leftarrow \tau \rightsquigarrow \lambda x:\tau_1.d : \tau_1 \to \tau_2' \dashv \Delta}$$

EAEHOLE

$$\frac{}{\Gamma \vdash (\!|\,|\!)^u \Leftarrow \tau \rightsquigarrow (\!|\,|\!)^u_{id(\Gamma)} : \tau \dashv u::[\Gamma]\tau}$$

EASUBSUME

$$\frac{e \neq (\!|\,|\!)^u \qquad e \neq (\!|e'|\!)^u \qquad \Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta \qquad \tau \sim \tau'}{\Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta}$$

EANEHOLE

$$\frac{\Gamma \vdash e \Rightarrow \tau' \rightsquigarrow d \dashv \Delta}{\Gamma \vdash (\!|e|\!)^u \Leftarrow \tau \rightsquigarrow (\!|d|\!)^u_{id(\Gamma)} : \tau \dashv \Delta, u::[\Gamma]\tau}$$

$\boxed{\Delta;\Gamma \vdash d : \tau}$ Type Assignment

TACONST

$$\frac{}{\Delta;\Gamma \vdash c : b}$$

TAVAR

$$\frac{x:\tau \in \Gamma}{\Delta;\Gamma \vdash x : \tau}$$

TALAM

$$\frac{\Delta;\Gamma, x:\tau_1 \vdash d : \tau_2}{\Delta;\Gamma \vdash \lambda x:\tau_1.d : \tau_1 \to \tau_2}$$

TAAP

$$\frac{\Delta;\Gamma \vdash d_1 : \tau_2 \to \tau \qquad \Delta;\Gamma \vdash d_2 : \tau_2}{\Delta;\Gamma \vdash d_1(d_2) : \tau}$$

TAEHOLE

$$\frac{u::[\Gamma']\tau \in \Delta \qquad \Delta;\Gamma \vdash \sigma : \Gamma'}{\Delta;\Gamma \vdash (\!|\,|\!)^u_\sigma : \tau}$$

TANEHOLE

$$\frac{\Delta;\Gamma \vdash d : \tau' \qquad u::[\Gamma']\tau \in \Delta \qquad \Delta;\Gamma \vdash \sigma : \Gamma'}{\Delta;\Gamma \vdash (\!|d|\!)^u_\sigma : \tau}$$

TACAST

$$\frac{\Delta;\Gamma \vdash d : \tau_1 \qquad \tau_1 \sim \tau_2}{\Delta;\Gamma \vdash d\langle\tau_1 \Rightarrow \tau_2\rangle : \tau_2}$$

$\Delta;\Gamma \vdash \sigma : \Gamma'$ iff $dom(\sigma) = dom(\Gamma)$ and for each $d/x \in \sigma$, we have $x:\tau \in \Gamma'$ and $\Delta;\Gamma \vdash d : \tau$.

TAFAILEDCAST

$$\frac{\Delta;\Gamma \vdash d : \tau_1 \qquad \tau_1 \text{ ground} \qquad \tau_2 \text{ ground} \qquad \tau_1 \neq \tau_2}{\Delta;\Gamma \vdash d\langle\tau_1 \Rightarrow (\!|\,|\!) \nRightarrow \tau_2\rangle : \tau_2}$$

$\boxed{d \text{ val}}$ d is a value

VCONST
$$\frac{}{c \text{ val}}$$

VLAM
$$\frac{}{\lambda x{:}\tau.d \text{ val}}$$

$\boxed{\tau \text{ ground}}$

BGROUND
$$\frac{}{b \text{ ground}}$$

ARR-HOLE-GROUND
$$\frac{}{\llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis \text{ ground}}$$

$\boxed{d \text{ boxedval}}$ d is a possibly-boxed value

VALS-ARE-BOXED
$$\frac{d \text{ val}}{d \text{ boxedval}}$$

ARROW-CAST-BOXED
$$\frac{\tau_1 \to \tau_2 \neq \tau_3 \to \tau_4 \qquad d \text{ boxedval}}{d\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4 \rangle \text{ boxedval}}$$

HOLE-CAST-BOXED
$$\frac{d \text{ boxedval} \qquad \tau \text{ ground}}{d\langle \tau \Rightarrow \llparenthesis\rrparenthesis \rangle \text{ boxedval}}$$

$\boxed{d \text{ indet}}$ d is indeterminate

IEHOLE
$$\frac{}{\llparenthesis\rrparenthesis^u_\sigma \text{ indet}}$$

INEHOLE
$$\frac{d \text{ final}}{\llparenthesis d \rrparenthesis^u_\sigma \text{ indet}}$$

IAP
$$\frac{d_1 \neq d_1'\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4 \rangle \qquad d_1 \text{ indet} \qquad d_2 \text{ final}}{d_1(d_2) \text{ indet}}$$

ICASTARR
$$\frac{\tau_1 \to \tau_2 \neq \tau_3 \to \tau_4 \qquad d \text{ indet}}{d\langle \tau_1 \to \tau_2 \Rightarrow \tau_3 \to \tau_4 \rangle \text{ indet}}$$

ICASTGROUNDHOLE
$$\frac{d \text{ indet} \qquad \tau \text{ ground}}{d\langle \tau \Rightarrow \llparenthesis\rrparenthesis \rangle \text{ indet}}$$

ICASTHOLEGROUND
$$\frac{d \neq d'\langle \tau' \Rightarrow \llparenthesis\rrparenthesis \rangle \qquad d \text{ indet} \qquad \tau \text{ ground}}{d\langle \llparenthesis\rrparenthesis \Rightarrow \tau \rangle \text{ indet}}$$

* IFAILEDCAST
$$\frac{d \text{ final} \qquad \tau_1 \text{ ground} \qquad \tau_2 \text{ ground} \qquad \tau_1 \neq \tau_2}{d\langle \tau_1 \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau_2 \rangle \text{ indet}}$$

$\boxed{d \text{ final}}$ d is final

FBOXEDVAL
$$\frac{d \text{ boxedval}}{d \text{ final}}$$

FINDET
$$\frac{d \text{ indet}}{d \text{ final}}$$

★ $

EvalCtx $\mathcal{E} ::= \circ \mid \mathcal{E}(d) \mid d(\mathcal{E}) \mid (\!|\mathcal{E}|\!)^u_\sigma \mid \mathcal{E}\langle\tau \Rightarrow \tau\rangle \mid \mathcal{E}\langle\tau_1 \Rightarrow (\!|\,|\!) \not\Rightarrow \tau_2\rangle$

$\boxed{\mathcal{E} \text{ evalctx}}$ $\mathcal{E}$ is an evaluation context

ECDOT
$$\frac{}{\circ \text{ evalCtx}}$$

ECAP1 (switched w/ EC-AP2 in PDF)
$$\frac{\mathcal{E} \text{ evalctx}}{\mathcal{E}(d) \text{ evalctx}}$$

ECAP2
$$\frac{[d \text{ final}] \quad \mathcal{E} \text{ evalctx}}{d(\mathcal{E}) \text{ evalctx}}$$

ECNEHOLE
$$\frac{\mathcal{E} \text{ evalctx}}{(\!|\mathcal{E}|\!)^u_\sigma \text{ evalctx}}$$

ECCAST
$$\frac{\mathcal{E} \text{ evalctx}}{\mathcal{E}\langle\tau_1 \Rightarrow \tau_2\rangle \text{ evalctx}}$$

★
$$\frac{\mathcal{E} \text{ evalctx}}{\mathcal{E}\langle\tau_1 \Rightarrow (\!|\,|\!) \not\Rightarrow \tau_2\rangle \text{ evalctx}}$$ $

$\boxed{d = \mathcal{E}\{d'\}}$ d is the result of filling the focus in $\mathcal{E}$ with d'

FHOUTER
$$\frac{}{d = \circ\{d\}}$$

FHAP1 (also switched in PDF)
$$\frac{d_1 = \mathcal{E}\{d_1'\}}{d_1(d_2) = \mathcal{E}(d_2)\{d_1'\}}$$

FHAP2
$$\frac{[d_1 \text{ final}] \quad d_2 = \mathcal{E}\{d_2'\}}{d_1(d_2) = d_1(\mathcal{E})\{d_2'\}}$$

FHNEHOLEINSIDE
$$\frac{d = \mathcal{E}\{d'\}}{(\!|d|\!)^u_\sigma = (\!|\mathcal{E}|\!)^u_\sigma\{d'\}}$$

FHCASTINSIDE
$$\frac{d = \mathcal{E}\{d'\}}{d\langle\tau_1 \Rightarrow \tau_2\rangle = \mathcal{E}\langle\tau_1 \Rightarrow \tau_2\rangle\{d'\}}$$

★ FHFAILEDCAST $
$$\frac{d = \mathcal{E}\{d'\}}{d\langle\tau_1 \Rightarrow (\!|\,|\!) \not\Rightarrow \tau_2\rangle = \mathcal{E}\langle\tau_1 \Rightarrow (\!|\,|\!) \not\Rightarrow \tau_2\rangle\{d'\}}$$

THEOREM (Focus Formation) If $d = \mathcal{E}\{d'\}$ then $\mathcal{E}$ evalctx.

$\boxed{d \mapsto d'}$ d steps to d'

STEP
$$\frac{d = \mathcal{E}\{d_0\} \quad d_0 \longrightarrow d_0' \quad d' = \mathcal{E}\{d_0'\}}{d \mapsto d'}$$

$\boxed{d \longrightarrow d'}$ d takes an instruction transition to d'

ITBETA (ITLAM in PDF)

$$\frac{[d_2 \text{ final}]}{(\lambda x{:}\tau.d_1)\, d_2 \longrightarrow [d_2/x]d_1}$$

ITCASTID

$$\frac{[d \text{ final}]}{d\langle \tau \Rightarrow \tau \rangle \longrightarrow d}$$

ITCASTSUCCEED

$$\frac{[d \text{ final}] \quad \tau \text{ ground}}{d\langle \tau \Rightarrow (\!|\!) \Rightarrow \tau \rangle \longrightarrow d}$$

ITCASTFAIL

$$\frac{[d \text{ final}] \quad \tau_1 \text{ ground} \quad \tau_2 \text{ ground} \quad \tau_1 \neq \tau_2}{d\langle \tau_1 \Rightarrow (\!|\!) \Rightarrow \tau_2 \rangle \longrightarrow d\langle \tau_1 \Rightarrow (\!|\!) \not\Rightarrow \tau_2 \rangle}$$

ITAPCAST

$$\frac{[d_1 \text{ final}] \quad [d_2 \text{ final}]}{(d_1\langle \tau_1 \to \tau_2 \Rightarrow \tau_1' \to \tau_2' \rangle)\, d_2 \longrightarrow (d_1\, (d_2\langle \tau_1' \Rightarrow \tau_1 \rangle))\langle \tau_2 \Rightarrow \tau_2' \rangle}$$

ITGROUND

$$\frac{[d \text{ final}] \quad \llcorner \tau \lrcorner = \tau'}{d\langle \tau \Rightarrow (\!|\!) \rangle \longrightarrow d\langle \tau \Rightarrow \tau' \Rightarrow (\!|\!) \rangle}$$

ITEXPAND

$$\frac{[d \text{ final}] \quad \llcorner \tau \lrcorner = \tau'}{d\langle (\!|\!) \Rightarrow \tau \rangle \longrightarrow d\langle (\!|\!) \Rightarrow \tau' \Rightarrow \tau \rangle}$$

$\boxed{\llcorner \tau \lrcorner = \tau'}$ τ has matched ground type τ'

$$\frac{\tau_1 \to \tau_2 \neq (\!|\!) \to (\!|\!)}{\llcorner \tau_1 \to \tau_2 \lrcorner = (\!|\!) \to (\!|\!)}$$

Theorem (Matched Ground Type Invariant)

If $\llcorner \tau \lrcorner = \tau'$ then τ' ground and $\tau \sim \tau'$ and $\tau \neq \tau'$.

Metatheory

Theorem (Expandability).

(1) If $\Gamma \vdash e \Rightarrow \tau$ then $\Gamma \vdash e \Rightarrow \tau \leadsto d \dashv \Delta$ for some d and Δ.

(2) If $\Gamma \vdash e \Leftarrow \tau$ then $\Gamma \vdash e \Leftarrow \tau \leadsto d : \tau' \dashv \Delta$ for some d and τ' and Δ.

Theorem (Correspondence).

(1) If $\Gamma \vdash e \Rightarrow \tau \leadsto d \dashv \Delta$ then $\Gamma \vdash e \Rightarrow \tau$.

(2) If $\Gamma \vdash e \Leftarrow \tau \leadsto d : \tau' \dashv \Delta$ then $\Gamma \vdash e \Leftarrow \tau$.

Theorem (Typed Expansion).

(1) If $\Gamma \vdash e \Rightarrow \tau \leadsto d \dashv \Delta$ then $\Delta \, \Gamma \vdash d : \tau$.

(2) If $\Gamma \vdash e \Leftarrow \tau \leadsto d : \tau' \dashv \Delta$ then $\tau \sim \tau'$ and $\Delta \, \Gamma \vdash d : \tau'$.

Theorem (Expansion Unicity).

(1) If $\Gamma \vdash e \Rightarrow \tau \leadsto d \dashv \Delta$ and $\Gamma \vdash e \Rightarrow \tau' \leadsto d' \dashv \Delta$ then $\tau = \tau'$ and and $d = d'$ and $\Delta = \Delta'$.

(2) If $\Gamma \vdash e \Leftarrow \tau_1 \leadsto d : \tau_2 \dashv \Delta$ and $\Gamma \vdash e \Leftarrow \tau_1 \leadsto d' : \tau_2' \dashv \Delta'$ then $d = d'$ and $\tau_2 = \tau_2'$ and $\Delta = \Delta'$.

Theorem (Type Assignment Unicity).

If $\Delta \, \Gamma \vdash d : \tau$ and $\Delta \, \Gamma \vdash d : \tau'$ then $\tau = \tau'$.

Theorem (Preservation)

If $\Delta;\emptyset \vdash d : \tau$ and $d \mapsto d'$ then $\Delta;\emptyset \vdash d : \tau$.

Theorem (Progress)

If $\Delta;\emptyset \vdash d : \tau$ then either

(1) $d \mapsto d'$ or

~~(2) d castErr~~ ~~or~~

(3) d indet or

(4) d boxedval.

Theorem (Canonical Value Forms)

If $\Delta \emptyset : d : \tau$ and d val then $\tau \neq ⦇⦈$ and

a) If $\tau = b$ then $d = c$.

b) If $\tau = \tau_1 \to \tau_2$ then $d = \lambda x : \tau_1 . d'$ where $\Delta; x : \tau_1 \vdash d : \tau_2$.

Theorem (Canonical Boxed Forms)

If $\Delta \emptyset \vdash d : \tau$ and d boxedval then

a) If $\tau = b$, then $d = c$.

b) If $\tau = \tau_1 \to \tau_2$, then either

i. $d = \lambda x : \tau_1 . d'$ where $\Delta;\ x : \tau_1 \vdash d : \tau_2$ or

ii. $d = d' \langle \tau_1' \to \tau_2' \Rightarrow \tau_1 \to \tau_2 \rangle$ where $\tau_1' \to \tau_2' \neq \tau_1 \to \tau_2$ and $\Delta \emptyset \vdash d' : \tau_1' \to \tau_2'$.

c) If $\tau = ⦇⦈$, then $d = d' \langle \tau' \Rightarrow ⦇⦈ \rangle$ where τ' ground and $\Delta \emptyset \vdash d' : \tau'$.

Theorem (Canonical Indeterminate Forms)

If $\Delta\,\emptyset \vdash d : \tau$ and d indet then

1. If $\tau = b$ then either

   a) $d = \llparenthesis\rrparenthesis^u_\sigma$ and $u : [\Gamma']\,b \in \Delta$ or

   b) $d = \llparenthesis d' \rrparenthesis^u_\sigma$ and d' final and $\Delta\,\emptyset \vdash d' : \tau'$ and $u : [\Gamma']\,b \in \Delta$ or

   c) $d = d_1(d_2)$ and $\Delta\,\emptyset \vdash d_1 : \tau_2 \to b$ and $\Delta\,\emptyset \vdash d_2 : \tau_2$ and d_1 indet and d_2 final and $d_1 \neq \langle \tau_3 \to \tau_4 \Rightarrow \tau_3' \to \tau_4' \rangle$.

   d) $d = d'\langle \llparenthesis\rrparenthesis \Rightarrow b \rangle$ and d' indet and $d' \neq d''\langle \tau' \Rightarrow \llparenthesis\rrparenthesis \rangle$

   ☆ e) $d = d'\langle \tau' \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow b \rangle$ and τ' ground and $\tau' \neq b$ and $\Delta\,\emptyset \vdash d' : \tau'$.

2. If $\tau = \tau_{11} \to \tau_{12}$ then either

   a) $d = \llparenthesis\rrparenthesis^u_\sigma$ and $u : [\Gamma']\,\tau_{11} \to \tau_{12} \in \Delta$ or

   b) $d = \llparenthesis d' \rrparenthesis^u_\sigma$ and d' final and $\Delta\,\emptyset \vdash d' : \tau'$ and $u : [\Gamma']\,\tau_{11} \to \tau_{12} \in \Delta$ or

   c) $d = d_1(d_2)$ and $\Delta\,\emptyset \vdash d_1 : \tau_2 \to (\tau_{11} \to \tau_{12})$ and $\Delta\,\emptyset \vdash d_2 : \tau_2$ and d_1 indet and d_2 final and $d_1 \neq d_1'\langle \tau_3 \to \tau_4 \Rightarrow \tau_3' \to \tau_4' \rangle$

   d) $d = d'\langle \tau_1 \to \tau_2 \Rightarrow \tau_{11} \to \tau_{12} \rangle$ and d' indet and $\tau_1 \to \tau_2 \neq \tau_{11} \to \tau_{12}$

   e) $\tau_{11} = \llparenthesis\rrparenthesis$ and $\tau_{12} = \llparenthesis\rrparenthesis$ and $d = d'\langle \llparenthesis\rrparenthesis \Rightarrow \llparenthesis\rrparenthesis \to \llparenthesis\rrparenthesis \rangle$ and d' indet and $d' \neq d''\langle \tau' \Rightarrow \llparenthesis\rrparenthesis \rangle$.

   f) $\tau_{11} = \llparenthesis\rrparenthesis$ and $\tau_{12} = \llparenthesis\rrparenthesis$ and $d = d'\langle \tau' \Rightarrow \llparenthesis\rrparenthesis \not\Rightarrow \tau_{11} \to \tau_{12} \rangle$ and $\tau' \neq \tau$ and τ' ground and d' indet and $\Delta\,\emptyset \vdash d' : \tau'$.

3. If $\tau = \llparenthesis\rrparenthesis$ then either

   a) $d = \llparenthesis\rrparenthesis^u_\sigma$ and $u : [\Gamma']\,\llparenthesis\rrparenthesis \in \Delta$ or

   b) $d = \llparenthesis d' \rrparenthesis^u_\sigma$ and d' final and $\Delta\,\emptyset \vdash d' : \tau'$ and $u : [\Gamma']\,\llparenthesis\rrparenthesis \in \Delta$ or

   c) $d = d_1(d_2)$ and $\Delta\,\emptyset \vdash d_1 : \tau_2 \to \llparenthesis\rrparenthesis$ and $\Delta\,\emptyset \vdash d_2 : \tau_2$ and d_1 indet and d_2 final and $d_1 \neq \langle \tau_3 \to \tau_4 \Rightarrow \tau_3' \to \tau_4' \rangle$

   d) $d = d'\langle \tau' \Rightarrow \llparenthesis\rrparenthesis \rangle$ and τ' ground and d' indet.

$\boxed{\tau \text{ complete}}$

B-COMPLETE
$$\frac{}{b \text{ complete}}$$

ARR-COMPLETE
$$\frac{\tau_1 \text{ complete} \quad \tau_2 \text{ complete}}{\tau_1 \to \tau_2 \text{ complete}}$$

$\boxed{d \text{ complete}}$

DVAR-COMPLETE
$$\frac{}{x \text{ complete}}$$

DCONST-COMPLETE
$$\frac{}{c \text{ complete}}$$

DLAM-COMPLETE
$$\frac{\tau \text{ complete} \quad d \text{ complete}}{\lambda x{:}\tau.d \text{ complete}}$$

DAP-COMPLETE
$$\frac{d_1 \text{ complete} \quad d_2 \text{ complete}}{d_1(d_2) \text{ complete}}$$

DCAST-COMPLETE
$$\frac{d \text{ complete} \quad \tau_1 \text{ complete} \quad \tau_2 \text{ complete}}{d\langle \tau_1 \Rightarrow \tau_2 \rangle \text{ complete}}$$

$\boxed{e \text{ complete}}$

EVAR-COMPLETE
$$\frac{}{x \text{ complete}}$$

ECONST-COMPLETE
$$\frac{}{c \text{ complete}}$$

ESYNLAM-COMPLETE
$$\frac{\tau \text{ complete} \quad e \text{ complete}}{\lambda x{:}\tau.e \text{ complete}}$$

E-ANALAM-COMPLETE
$$\frac{e \text{ complete}}{\lambda x.e \text{ complete}}$$

EAP-COMPLETE
$$\frac{e_1 \text{ complete} \quad e_2 \text{ complete}}{e_1(e_2) \text{ complete}}$$

EASC-COMPLETE
$$\frac{e \text{ complete} \quad \tau \text{ complete}}{e : \tau \text{ complete}}$$

Theorem (Complete Progress)
If $\Delta\,\emptyset \vdash d : \tau$ and d complete then either $d \mapsto d'$ or d val.

by Preservation ✓

Theorem (Complete Preservation)
If $\Delta\,\emptyset \vdash d : \tau$ and d complete and $d \mapsto d'$ then $\Delta\,\emptyset \vdash d' : \tau$ and d' complete.

Theorem (Complete Expansion)
1. If e complete and $\Gamma \vdash e \Rightarrow \tau \rightsquigarrow d \dashv \Delta$ then d complete.
2. If e complete and $\Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta$ then d complete.

$\boxed{d \mapsto^* d}$

MULTISTEP-REFL

$$\frac{}{d \mapsto^* d}$$

MULTISTEP-STEP

$$\frac{d \mapsto d' \qquad d' \mapsto^* d''}{d \mapsto^* d''}$$

$\boxed{[\![d/u]\!]d' = d''}$

$$[\![d/u]\!]c = c$$

$$[\![d/u]\!]x = x$$

$$[\![d/u]\!]\lambda x{:}\tau.d' = \lambda x{:}\tau.[\![d/u]\!]d'$$

$$[\![d/u]\!]d_1(d_2) = ([\![d/u]\!]d_1)([\![d/u]\!]d_2)$$

$$[\![d/u]\!](\!|\,|\!)^u_\sigma = [\sigma]d$$

$$[\![d/u]\!](\!|d'|\!)^u_\sigma = [\sigma]d$$

$$[\![d/u]\!]d'\langle\tau \Rightarrow \tau'\rangle = ([\![d/u]\!]d')\langle\tau \Rightarrow \tau'\rangle$$

$$[\![d/u]\!]d'\langle\tau_1 \Rightarrow (\!|\,|\!) \not\Rightarrow \tau_2\rangle = ([\![d/u]\!]d')\langle\tau_1 \Rightarrow (\!|\,|\!) \not\Rightarrow \tau_2\rangle \quad \$$$

Theorem (Instantiation)

If $\Delta;\Gamma \vdash d:\tau$ and $u::[\Gamma']\tau' \in \Delta$ and $\Delta;\Gamma' \vdash d':\tau'$ then $\Delta;\Gamma \vdash [\![d'/u]\!]d : \tau$.

Lemma (Finality)

If d final and $d \mapsto^* d'$ then $d = d'$.

Theorem (Commutativity)

If $d_0 \mapsto^* d_1$ then $[\![d/u]\!]d_0 \mapsto^* [\![d/u]\!]d_1$.

Theorem (Confluence)

If $d \mapsto^* d_1$ and $d \mapsto^* d_2$ then there exists d' such that $d_1 \mapsto^* d'$ and $d_2 \mapsto^* d'$.

Corollary (Final Confluence)

If $d \mapsto^* d_1$ and d_1 final and $d \mapsto^* d_2$ then $d_2 \mapsto^* d_1$.

Pf By confluence and finality.

Theorem (Resumption)

If $d_1 \mapsto^* d_2$ and d_2 final and $[\![d_3/u]\!]d_1 \mapsto^* d_4$ and d_4 final then $[\![d_3/u]\!]d_2 \mapsto^* d_4$.

Pf ① By commutativity, $[\![d_3/u]\!]d_1 \mapsto^* [\![d_3/u]\!]d_2$

② By final confluence, we can conclude.

(Some of these theorems might need additional typing premises)

Extension: Sum Types

HTyp $\tau ::= \ldots \mid \tau + \tau$

HExp $e ::= \ldots \mid \text{inl } e \mid \text{inr } e \mid \text{case } e \text{ of } \text{inl}(x) \Rightarrow e \mid \text{inr}(x) \Rightarrow e$

DHExp $d ::= \ldots \mid \text{inl}_\tau d \mid \text{inr}_\tau d \mid \text{case } d \text{ of } \text{inl}(x) \Rightarrow d \mid \text{inr}(x) \Rightarrow d$

$\boxed{\Gamma \vdash e \Leftarrow \tau \rightsquigarrow d : \tau' \dashv \Delta}$

$\boxed{\tau \blacktriangleright_+ \tau_1 + \tau_2}$

$$\frac{}{\tau_1 + \tau_2 \blacktriangleright_+ \tau_1 + \tau_2}$$

$$\frac{}{(\!|\,|\!) \blacktriangleright_+ (\!|\,|\!) + (\!|\,|\!)}$$

$$\frac{\tau \blacktriangleright_+ \tau_1 + \tau_2 \qquad \Gamma \vdash e \Leftarrow \tau_1 \rightsquigarrow d : \tau_1' \dashv \Delta}{\Gamma \vdash \text{inl } e \Leftarrow \tau \rightsquigarrow \text{inl}_{\tau_2} d : \tau_1' + \tau_2 \dashv \Delta}$$

$$\frac{\tau \blacktriangleright_+ \tau_1 + \tau_2 \qquad \Gamma \vdash e \Leftarrow \tau_2 \rightsquigarrow d : \tau_2' \dashv \Delta}{\Gamma \vdash \text{inr } e \Leftarrow \quad {}_2 \rightsquigarrow \text{inr}_{\tau_1} d : \tau_1 + \tau_2' \dashv \Delta}$$

$$\frac{\begin{array}{c}\Gamma \vdash e_1 \Rightarrow \tau_1 \rightsquigarrow d_1 \dashv \Delta_1 \qquad \tau_1 \blacktriangleright_+ \tau_{11} + \tau_{12} \qquad \text{join}(\tau_2, \tau_3) = \tau' \\ \Gamma, x : \tau_{11} \vdash e_2 \Leftarrow \tau \rightsquigarrow d_2 : \tau_2 \dashv \Delta_2 \qquad \Gamma, x : \tau_{12} \vdash e_3 \Leftarrow \tau \rightsquigarrow d_3 : \tau_3 \dashv \Delta_3\end{array}}{\begin{array}{l}\Gamma \vdash \text{case } e_1 \text{ of } \text{inl}(x) \Rightarrow e_2 \mid \text{inr}(x) \Rightarrow e_3 \Leftarrow \tau \rightsquigarrow \\ \quad \text{case } d_1 \langle \tau_1 \Rightarrow \tau_{11} + \tau_{12} \rangle \text{ of } \text{inl}(x) \Rightarrow d_2 \langle \tau_2 \Rightarrow \tau' \rangle \mid \text{inr}(x) \Rightarrow d_3 \langle \tau_3 \Rightarrow \tau' \rangle : \\ \quad \tau' \dashv \Delta_1 \cup \Delta_2 \cup \Delta_3\end{array}}$$

$\boxed{\text{join } \tau_1\ \tau_2 = \tau}$ the join of τ_1 and τ_2 is τ

$\text{join } \tau \quad \tau = \tau$

$\text{join } (\!|\,|\!) \quad \tau = \tau$

$\text{join } \tau \quad (\!|\,|\!) = \tau$

$\text{join } \tau_1 \to \tau_2 \quad \tau_1' \to \tau_2' = (\text{join } \tau_1\ \tau_1') \to \text{join}(\tau_2\ \tau_2')$

$\text{join } \tau_1 + \tau_2 \quad \tau_1' + \tau_2' = (\text{join } \tau_1\ \tau_1') + \text{join}(\tau_2\ \tau_2')$

Theorem (Joins)

If $\text{join } \tau_1\ \tau_2 = \tau$ then $\tau_1 \sim \tau_2$ and $\tau_1 \sim \tau$ and $\tau_2 \sim \tau_2$.

$\boxed{\Delta;\Gamma \vdash d : \tau}$

$$\frac{\Delta;\Gamma \vdash d : \tau_1}{\Delta;\Gamma \vdash \mathrm{inl}_{\tau_2}\, d : \tau_1 + \tau_2} \qquad \frac{\Delta;\Gamma \vdash d : \tau_2}{\Delta;\Gamma \vdash \mathrm{inr}_{\tau_1}\, d : \tau_1 + \tau_2}$$

$$\frac{\Delta;\Gamma \vdash d_1 : \tau_1 + \tau_2 \qquad \Delta;\Gamma, x:\tau_1 \vdash d_2 : \tau \qquad \Delta;\Gamma, x:\tau_2 \vdash d_3 : \tau}{\Delta;\Gamma \vdash \mathrm{case}\ d_1\ \mathrm{of}\ \mathrm{inl}(x) \Rightarrow d_2 \mid \mathrm{inr}(x) \Rightarrow d_3 : \tau}$$

$\boxed{d\ \mathrm{val}}$ $$\frac{d\ \mathrm{val}}{\mathrm{inl}_\tau\, d\ \mathrm{val}} \qquad \frac{d\ \mathrm{val}}{\mathrm{inr}_\tau\, d\ \mathrm{val}}$$

$\boxed{\tau\ \mathrm{ground}}$ $$\frac{}{(\!|\,|\!) + (\!|\,|\!)\ \mathrm{ground}}$$

$\boxed{d\ \mathrm{boxedval}}$ $$\frac{d\ \mathrm{boxedval}}{\mathrm{inl}_\tau\, d\ \mathrm{boxedval}} \qquad \frac{d\ \mathrm{boxedval}}{\mathrm{inr}_\tau\, d\ \mathrm{boxedval}} \qquad \frac{\tau_1 + \tau_2 \neq \tau_1' + \tau_2' \qquad d\ \mathrm{boxedval}}{d\langle \tau_1 + \tau_2 \Rightarrow \tau_1' + \tau_2' \rangle\ \mathrm{boxedval}}$$

$\boxed{d\ \mathrm{indet}}$ $$\frac{d\ \mathrm{indet}}{\mathrm{inl}_\tau\, d\ \mathrm{indet}} \qquad \frac{d\ \mathrm{indet}}{\mathrm{inr}_\tau\, d\ \mathrm{indet}} \qquad \frac{\tau_1 + \tau_2 \neq \tau_1' + \tau_2' \qquad d\ \mathrm{indet}}{d\langle \tau_1 + \tau_2 \Rightarrow \tau_1' + \tau_2' \rangle\ \mathrm{indet}}$$

$$\frac{d_1 \neq \mathrm{inl}_\tau(d_1') \qquad d_1 \neq \mathrm{inr}_\tau(d_1') \qquad d_1 \neq d_1'\langle \tau_1 + \tau_2 \Rightarrow \tau_1' + \tau_2' \rangle \qquad d_1\ \mathrm{indet}}{\mathrm{case}\ d_1\ \mathrm{of}\ \mathrm{inl}(x) \Rightarrow d_2 \mid \mathrm{inr}(x) \Rightarrow d_3\ \mathrm{indet}}$$

$$\mathcal{E} ::= \ldots \mid \mathrm{inl}_\tau(\mathcal{E}) \mid \mathrm{inr}_\tau(\mathcal{E}) \mid \mathrm{case}\ \mathcal{E}\ \mathrm{of}\ \mathrm{inl}(x) \Rightarrow d_2 \mid \mathrm{inr}(x) \Rightarrow d_3$$

$\boxed{\mathcal{E}\ \mathrm{evalctx}}$ $$\frac{\mathcal{E}\ \mathrm{evalctx}}{\mathrm{inl}_\tau(\mathcal{E})\ \mathrm{evalctx}} \qquad \frac{\mathcal{E}\ \mathrm{evalctx}}{\mathrm{inr}_\tau(\mathcal{E})\ \mathrm{evalctx}} \qquad \frac{\mathcal{E}\ \mathrm{evalctx}}{\mathrm{case}\ \mathcal{E}\ \mathrm{of} \ldots\ \mathrm{evalctx}}$$

$\boxed{d = \mathcal{E}\{d'\}}$ $$\frac{d = \mathcal{E}\{d'\}}{\mathrm{case}\ d\ \mathrm{of} \ldots = \mathrm{case}\ \mathcal{E}\ \mathrm{of} \ldots \{d'\}}$$

$\boxed{d \longrightarrow d'}$

$$\frac{[d_1 \text{ final}]}{\text{case } \mathrm{inl}_\tau(d_1) \text{ of } \mathrm{inl}(x) \Rightarrow d_2 \mid \mathrm{inr}(y) \Rightarrow d_3 \longrightarrow [d_1/x]d_2}$$

$$\frac{[d_1 \text{ final}]}{\text{case } \mathrm{inr}_\tau(d_1) \text{ of } \mathrm{inl}(x) \Rightarrow d_2 \mid \mathrm{inr}(y) \Rightarrow d_3 \longrightarrow [d_1/y]d_3}$$

$$\frac{[d_1 \text{ final}]}{\begin{array}{l}\text{case } d_1\langle \tau_1+\tau_2 \Rightarrow \tau_1'+\tau_2'\rangle \text{ of } \mathrm{inl}(x) \Rightarrow d_2 \mid \mathrm{inr}(y) \Rightarrow d_3 \longrightarrow \\ \quad \text{case } d_1 \text{ of } \mathrm{inl}(x) \Rightarrow [x\langle \tau_1 \Rightarrow \tau_1'\rangle/x]d_2 \mid \mathrm{inr}(y) \Rightarrow [y\langle \tau_2 \Rightarrow \tau_2'\rangle/y]d_3\end{array}}$$

$\boxed{\lfloor \tau \rfloor = \tau'}$

$$\frac{\tau_1 + \tau_2 \neq ⦇⦈ + ⦇⦈}{\lfloor \tau_1 + \tau_2 \rfloor = ⦇⦈ + ⦇⦈}$$

Theorem (Canonical Value Forms - Sums)

If $\Delta \emptyset \vdash d : \tau_1 + \tau_2$ and d val then either

1. $d = \mathrm{inl}_{\tau_2}\, d'$ where d' val and $\Delta \emptyset \vdash d' : \tau_1$ or
2. $d = \mathrm{inr}_{\tau_1}\, d'$ where d' val and $\Delta \emptyset \vdash d : \tau_2$.

Theorem (Canonical Boxed Value Forms - Sums)

If $\Delta \emptyset \vdash d : \tau_1 + \tau_2$ and d boxed val then either

1. $d = \mathrm{inl}_{\tau_2}\, d'$ where d' boxedval and $\Delta \emptyset \vdash d' : \tau_1$ or
2. $d = \mathrm{inl}_{\tau_1}\, d'$ where d' boxedval and $\Delta \emptyset \vdash d' : \tau_2$ or
3. $d = d'\langle \tau_1' + \tau_2' \Rightarrow \tau_1 + \tau_2\rangle$ where $\tau_1 + \tau_2' \neq \tau_1 + \tau_2'$ and d' boxedval and $\Delta \emptyset \vdash d' : \tau_1' + \tau_2'$.

Theorem (Canonical Indeterminate Forms)

→ have to add clauses about indeterminate case expressions to each clause in previous theorem

(TODO - think about how to state this theorem more conservatively)