Problem Set 1: OLG Models

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1 OLG with Arrow-Debreu

Consider the standard OLG production economy with Arrow-Debreu trading.

Demographics: N_t young are born in t. Each lives for 2 periods.

Endowments: Each young has 1 unit of work time. The initial old each have K_0/N_{-1} units of capital.

Preferences: $u\left(c_t^y\right) + \beta u\left(c_{t+1}^o\right)$.

Technology: $F\left(K_t, L_t\right) + (1 - \delta) K_t = K_{t+1} + N_t c_t^y + N_{t-1} c_t^o$.

Markets:

- Trading takes place in $t = 0$.

- Goods are traded at price p_t. Capital is rented at rate q_t (in units of account). Labor is rented at rate w_t (in units of account).

Questions:

1. Explain why the following is the correct budget constraint:

$$w_t + q_{t+1} s_{t+1} + (1 - \delta) p_{t+1} s_{t+1} = p_t c_t^y + p_{t+1} c_{t+1}^o + p_t s_{t+1} \tag{1}$$

2. Derive the household's first-order conditions.

3. Define a solution to the household problem.

4. What is the real interest rate in this economy?

5. Interpret the condition $p_t = (1 - \delta) p_{t+1} + q_{t+1}$.

6. State the firm's first-order conditions. Watch your units!

7. Define a competitive equilibrium.

8. Why don't we lose an equation due to Walras' law?

9. Where is the numeraire?

10. Define a steady state.

11. Under what condition do the Welfare theorems hold / fail? Recall that the Welfare theorems require $\lim_{t \to \infty} p_t = 0$.

1.1 Answer: OLG with Arrow-Debreu

Note: Notationally, it would be more convenient to denote the wage rate by $w_t p_t$ and the rental rate by $q_t p_t$.

1. Budget constraint: In date t market, rent out labor and earn w_t, buy capital s_{t+1} and consumption c_t^y. In date $t+1$ market, earn rental income $q_{t+1} s_{t+1}$, sell the undepreciated capital, and buy c_{t+1}^o.

2. Let λ_t be the Lagrange multiplier on the budget constraint.

$$u'\left(c_t^y\right) = \lambda_t p_t \tag{2}$$
$$\beta u'\left(c_{t+t}^o\right) = \lambda_t p_{t+1} \tag{3}$$
$$p_t = (1-\delta) p_{t+1} + q_{t+1} \tag{4}$$

3. Solution: $c_t^y, c_{t+1}^o, k_{t+1}$ that solve the Euler equation

$$u'\left(c_t^y\right) = \beta u'\left(c_{t+1}^o\right) p_t / p_{t+1} \tag{5}$$

the budget constraint and the no-arbitrage condition.

4. The household can move consumption between dates at the exchange rate p_t/p_{t+1}. This defines the interest rate.

5. The equation is a no-arbitrage condition. The household can also move consumption by buying capital and renting it out. Both approaches must yield the same rate of return.

6. Firm: $w_t/p_t = f(k_t) - f'(k_t) k_t$. $q_t/p_t = f'(k_t)$.

7. CE: $\{c_t^y, c_t^o, s_{t+1}, k_{t+1}\}$ and $\{p_t, q_t, w_t\}$ that solve: 3 household conditions, 2 firm first-order conditions, goods market clearing, identity $k_{t+1}(1+n) = s_{t+1}$.

8. Neither equation is redundant by Walras' law. We only lose an equation for one t.

9. By choosing units of account, we can make the price of, say, goods at any date t equal 1.

10. Steady state: $c^y, c^o, k, w/p, q/p, \pi$ that solve the equilibrium conditions without time subscripts. $\pi_{t+1} \equiv p_{t+1}/p_t$.

11. $1/\pi = (1-\delta) + f'(k)$. We have $p_t \to 0$ if $\pi < 1$ or $f'(k) < \delta$. This is related to dynamic inefficiency, though not as clearly as I would have thought.

2 OLG Model with Assets

Demographics: There are two types of households, indexed by h. In each period, a mass of 0.5 households is born of each type. Each person lives for 2 periods.

Endowments: Households receive endowments (e^y, e^o) when young and old, respectively.

Preferences: $\ln\left(c_{h,t}^y\right) + \beta_h \ln\left(c_{h,t+1}^o\right)$.

Technologies: None.

Markets: Households trade goods and one period bonds that are issued and purchased by households.

Questions:

1. Define a solution to the household problem. Solve for the household's bond supply function.

2. Solve for the equilibrium bond interest rate.

3. Your solution for R should reveal the following features: (i) If old endowments are larger, R is higher. (ii) If β_h increases, R decreases. (iii) R is time invariant. Provide intuition for these features.

4. Now add a durable good to the economy. It is in fixed supply, K. It pays a dividend d per period (in units of consumption goods). Households trade "shares" of this good in an asset market at price p_t, measured in units of consumption goods. Define a competitive equilibrium for this economy.

5. Why do you find that the number of equations equals the number of objects to be determined? Usually, we find that we have one additional equation, which is redundant by Walras' law.

6. Derive an equation that determines the equilibrium price sequence p_t.

2.1 Answer: OLG Model with Assets

1. Euler equation: $c^o_{h,t+1}/c^y_{h,t} = \beta_h R_{t+1}$. Budget constraint: $W_t = e^y + e^o/R_{t+1} = c^y_{h,t} + c^o_{h,t+1}/R_{t+1}$. Young consumption function: $c^y_{h,t} = W/[1+\beta]$. Bond supply from the young budget constraint:

$$b_{h,t+1} = e^y - c^y_{h,t} = e^y - [e^y + e^o/R_{t+1}]/[1+\beta_h]$$

Solution: $\left(c^y_{h,t}, c^o_{h,t+1}, b_{h,t+1}\right)$ that satisfy 2 policy functions and one budget constraint.

2. Bond market clearing: $\sum_h b_{h,t+1} = 0$.

$$2e^y = [e^y + e^o/R] \sum_h [1+\beta_h]^{-1}$$

$$R = \frac{e^o}{e^y} \left[\frac{2}{\sum_h (1+\beta_h)^{-1}} - 1\right]^{-1}$$

3. Intuition: (i) and (ii) High old endowments or low β: agents want to save less. But aggregate saving is fixed at 0. Need to adjust interest rate. (iii) Essentially a consequence of lack of intergenerational trade.

4. Nothing changes in the household problem, except that household asset holdings are $s_{h,t+1} = b_{h,t+1} + p_t k_{h,t+1}$ where $R_{t+1} = (d + p_{t+1})/p_t$ by no arbitrage. Goods market clearing requires $e^y + e^o + Kd = 0.5 \sum_h \left(c^y_{h,t} + c^o_{h,t+1}\right)$. Asset market clearing requires $\sum_h b_{h,t+1} = 0$ and $0.5 \sum_h k_{h,t+1} = K$. A CE consists of sequences $c^y_{h,t}, c^o_{h,t+1}, b_{h,t+1}, k_{h,t+1}, s_{h,t+1}, R_{t+1}, p_t$ (12 objects) which satisfy:

- Household: 2 policy functions and 1 budget constraint per type (6 equations)
- Market clearing: 3 equations
- Definition of s: 2 equations
- Definition of R: 1 equation.

5. We have only 12 equations because the model does not determine portfolio compositions for either household type.

6. Capital market clearing requires:

$$2p_t K = \sum p_t k_{h,t+1} = \sum p_t k_{h,t+1} + \sum b_{h,t+1} = \sum s_{h,t+1}$$

because from bond market clearing, aggregate saving equals aggregate wealth. As in the model without K, $s_{h,t+1} = e^y - c^y_{h,t+1} = e^y - W/(1 + \beta_h)$. Therefore

$$2pK = 2e^y - \left[e^y + e^o \frac{p}{p+d} \right] \sum (1 + \beta_h)^{-1}$$