Overlapping Generations Model: Equilibrium and Steady State

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Topics

We study the equilibrium of the OLG production economy

1. Dynamics of capital accumulation
2. Steady state
3. Dynamic efficiency

Competitive Equilibrium

Recall the equilibrium definition for the production economy:

An allocation: $\left(c_t^y, c_t^o, s_t, b_t, K_t, L_t\right)$

Prices: (q_t, r_t, w_t)

That satisfy:

- the household EE and budget constraints (3 equations)
- the firm's FOCs (2 equations)
- the market clearing conditions (4 equations)
- identity: $r = q - \delta$.

Saving Function and Dynamics

Saving Function and Dynamics

We need to describe how the economy evolves over time.

We derive a **difference equation** (a law of motion) for the economy's state variables.

What are the state variables?

- Variables carried over into the current period from the last period.
- Variables that are predetermind in the current period.

Here: the state variable is K_t.

More conveniently, we use $k_t = K_t/N_t$ as the state variable.

Saving Function and Dynamics

The evolution is k is characterized by the capital market clearing condition $K_{t+1} = N_t s_{t+1}$ or

$$\begin{aligned} K_{t+1}/N_{t+1} &= N_t/N_{t+1} \cdot s_{t+1} \\ (1+n)k_{t+1} &= s_{t+1} \end{aligned} \tag{1}$$

together with the household saving function

$$s_{t+1} = s(w_t, r_{t+1}) \tag{2}$$

Saving function

Start from the Euler equation

$$\beta(1+r_{t+1})u'(c^o_{t+1}) = u'(c^y_t)$$

Substitute in the budget constraints for both ages:

$$\beta(1+r_{t+1})u'([1+r_{t+1}]s_{t+1}) = u'(w_t - s_{t+1})$$

This implicitly defines a **saving function**

$$s_{t+1} = s(w_t, r_{t+1}) \tag{3}$$

Log utility example

$u(c) = \ln c$

$u'(c) = 1/c$

Euler equation:

$$1/c_t^y = \beta(1 + r_{t+1}) 1/c_{t+1}^o \tag{4}$$

Apply the budget constraints

$\frac{\beta(1+r_{t+1})}{(1+r_{t+1})s_{t+1}} = \frac{1}{w_t - s_{t+1}} \implies s_{t+1} = w_t \beta / (1 + \beta)$

Properties of the saving function

Totally differentiate

$$\beta(1+r_{t+1})u'([1+r_{t+1}]s_{t+1}) = u'(w_t - s_{t+1})$$

Higher endowments raise saving:

$$\frac{ds_{t+1}}{dw_t} = \frac{u''(c_t^y)}{\beta(1+r_{t+1})^2 u''(c_{t+1}^o) + u''(c_t^y)} > 0$$

Intuition...

Effect of the interest rate

$$\frac{\partial s_{t+1}}{\partial r_{t+1}} = -\frac{\beta u'(c^o_{t+1}) + \beta(1+r_{t+1})u''(c^o_{t+1})s_{t+1}}{\beta(1+r_{t+1})^2 u''(c^o_{t+1}) + u''(c^y_t)} \tag{5}$$

Simplify (details below):

$$\frac{\partial s_{t+1}}{\partial r_{t+1}} = -\frac{\beta u'(c^o_{t+1})(1-\sigma\left[c^o_{t+1}\right])}{\beta(1+r_{t+1})^2 u''(c^o_{t+1}) + u''(c^y_t)} \tag{6}$$

where

$$\sigma(c) \equiv -\frac{\partial u'}{\partial c}\frac{c}{u'} = -\frac{u''(c)c}{u'(c)} > 0 \tag{7}$$

is the elasticity of u' w.r.t c.

The point: The effect of the interest rate on saving is **ambiguous** and depends on σ.

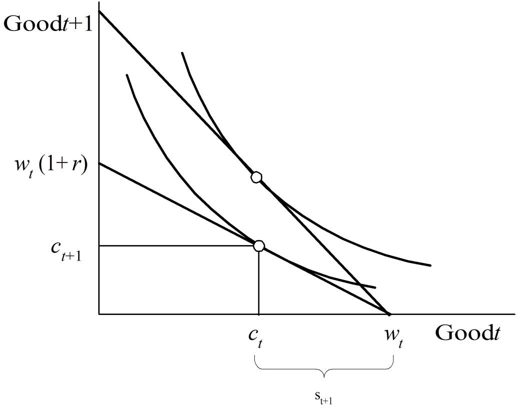
Derivation

Use the 2nd period budget constraint to replace $(1+r_{t+1})s_{t+1}$ by c^o_{t+1}.

$$\frac{\partial s_{t+1}}{\partial r_{t+1}} = -\frac{\beta u'(c^o_{t+1}) + \beta u''(c^o_{t+1})\ c^o_{t+1}}{\beta(1+r_{t+1})^2 u''(c^o_{t+1}) + u''(c^y_t)} \tag{8}$$

"Pull out" $u'\left(c^o_{t+1}\right)$.

Effect of a higher interest rate



The figure illustrates the case where income and substitution effect just cancel.

Note the role of the curvature of the ICs.

Effect of the interest rate

$$\sigma(c) \equiv -\frac{u''(c)c}{u'(c)} > 0 \tag{9}$$

measures the **curvature of the indifference curves**.

High $\sigma \implies$

- strong curvature of u and of indifference curves.
- small substitution effect.

σ is a key parameter in (macro) models

- it governs how easily people substitute consumption over time
- it also governs how willing people are to take risks (see later).

Elasticity of substitution

Fact

$1/\sigma$ is the elasticity of substitution between c_t and c_{t+1}.

Definition

The elasticity of substitution is given by

$$E = -\frac{d\ln\left(c_{t+1}/c_t\right)}{d\ln\left(u'\left(c_{t+1}\right)/u'\left(c_t\right)\right)} \tag{10}$$

Elasticity of substitution

Intuition:

- Consider moving along an indifference curve.
- The slope is $\beta u'(c_{t+1})/u'(c_t)$ – the marginal rate of substitution.
- Change the MRS by a given (percentage) amoung (log change)
- High elasticity $\implies$ c_{t+1}/c_t changes by a big amount
- The curvature of the indifference curve is small.

Elasticity of substitution (details) I

Consider an infinitesimal change of consumption from $c_t \rightarrow c_t(1+\varepsilon)$:

$$\ln(c_{t+1}/c_t) = \ln(1+\varepsilon) \approx \varepsilon \tag{11}$$

$$\begin{aligned} u'(c_{t+1}) &= u'(c_t(1+\varepsilon)) \approx u'(c_t) + u''(c_t)c_t\varepsilon && (12)\\ &= u'(c_t)\left[1 + \frac{u''(c_t)c_t}{u'(c_t)}\varepsilon\right] && (13)\\ &= u'(c_t)[1-\sigma\varepsilon] && (14) \end{aligned}$$

Elasticity of substitution (details) II

Hence

$$\ln\left(u'(c_{t+1})/u'(c_t)\right) = \ln(1-\sigma\varepsilon) \tag{15}$$

$$1/E = \frac{d\ln(1-\sigma\varepsilon)}{d\varepsilon} \tag{16}$$

$$= -\frac{\sigma}{1-\sigma\varepsilon} \tag{17}$$

Therefore: As $\varepsilon \to 0$, $1/E \to \sigma$.

CRRA Utility

In particular, for the popular CRRA utility function

$$u(c) = c^{1-\sigma}/(1-\sigma)$$

the $\sigma(c)$ is constant (namely σ, show this!).

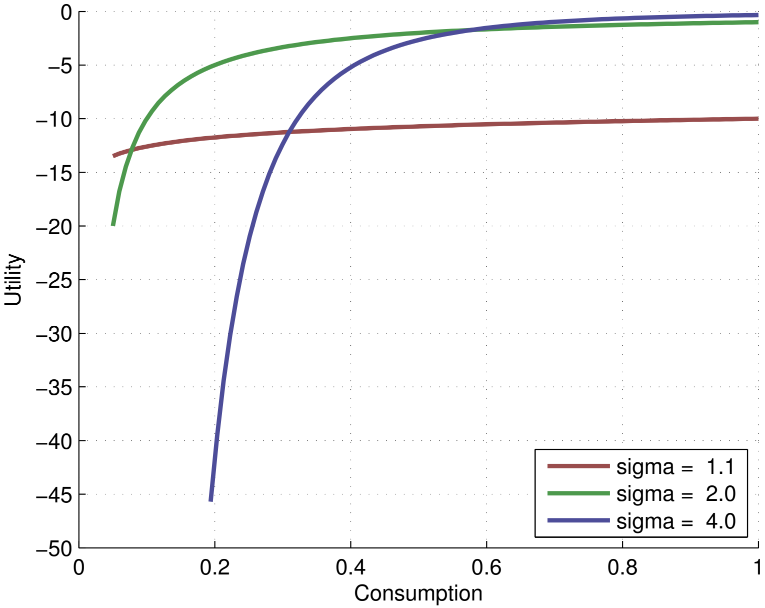
For $\sigma = 1$, this becomes log utility (and $s_r = 0$).

In the data, σ is most likely greater than one, although its value is highly controversial.

CRRA stands for "constant relative risk aversion."

- σ is the coefficient of relative risk aversion (see discussion of stochastic economies).

CRRA Utility



Law of motion for capital

Recall $(1+n)k_{t+1} = s(w_t, r_{t+1})$.

Use the firm FOCs to replace the prices:

$$(1+n)k_{t+1} = s(f(k_t) - f'(k_t)k_t,\ f'(k_{t+1}) - \delta)$$

This is a first order difference equation of the form

$$k_{t+1} = \phi(k_t)$$

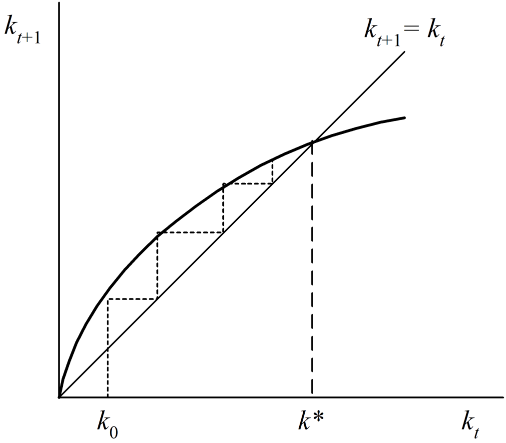
Implicitly differentiating yields

$$\frac{dk_{t+1}}{dk_t} = \frac{-s_w k_t f''(k_t)}{1+n-s_r f''(k_{t+1})} \tag{18}$$

This completely determines the behavior of the economy.

Concave law of motion

If ϕ is concave, we get simple dynamics.



From any initial condition (k_0) the economy converges monotonically to a unique steady state (k^*).

Properties of the law of motion

We know:

- $\phi(0) = 0$: $k = 0$ is a steady state.
- The derivative is

$$\frac{dk_{t+1}}{dk_t} = \frac{-s_w k_t f''(k_t)}{1 + n - s_r f''(k_{t+1})} \tag{19}$$

- A sufficient condition for $\phi' > 0$ is $s_r > 0$. Intuition: the supply of capital is upward sloping.

Otherwise, little can be said in general.

Log utility - Cobb Douglas example

The utility function is $u(c) = \ln(c)$.

Then the household saves a constant fraction of his earnings:

$$c_t^y = w_t/(1+\beta)$$

and therefore

$$s_{t+1} = w_t\beta/(1+\beta)$$

Log utility - Cobb Douglas example

Assume further that $f(k) = k^\theta$. Then

$$w = (1-\theta)k^\theta$$

The law of motion then becomes

$$(1+n)k_{t+1} = \frac{\beta}{1+\beta}(1-\theta)k_t^\theta$$

Because $s_r = 0$ and s_w is a constant, ϕ inherits the curvature of the production function.

A unique, stable steady state exists.

Log utility - Cobb Douglas example
Steady state

$$k^* = \left[\frac{1-\theta}{1+n}\frac{\beta}{1+\beta}\right]^{1/(1-\theta)}$$

Steady state interest rate:

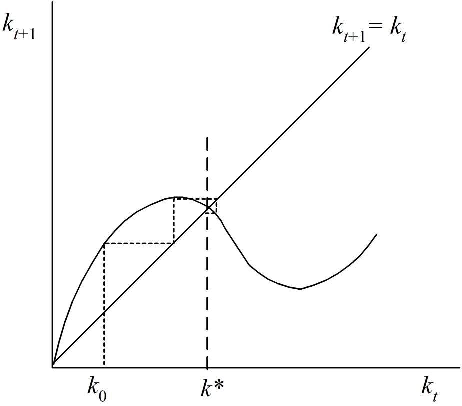
$$\begin{aligned} f'(k) &= \theta\ k^{\theta-1} \\ f'(k^*) &= \frac{\theta}{1-\theta}\ \frac{1+\beta}{\beta}\ (1+n) \\ r &= f'(k)-\delta \end{aligned}$$

Note: the steady state interest rate could be very small (low θ or high β) or very large.

Log utility - Cobb Douglas example

- The example provides a microfoundation for the Solow model.
- But it is a special case.

An ill behaved example



The economy osciallates towards the steady state.

Multiple steady states are possible.

An important insight: Even very simple models can have surprisingly complicated (and unpleasant) dynamics.

Steady State and Dynamic Efficiency

Steady State

Definition

A steady state is an equilibrium where all (per capita) variables are constant.

Note: Aggregates can grow $(K_t = k_t N_t)$, but per capita variables cannot (k_t).

The Golden Rule

Definition

The Golden Rule capital stock maximizes steady state consumption (per capita).

Consumption per young household is

$$c^y + c^o/(1+n) = f(k) + (1-\delta)k - (1+n)k'$$

Impose the steady state requirement $k' = k$ and maximize with respect to k:

$$f'(k_{GR}) = n + \delta \tag{20}$$

Intuition...

Dynamic Inefficiency

Definition

An allocation is dynamically efficient, if $k < k_{GR}$.

- $k > k_{GR}$ implies a Pareto inefficient allocation.
- By running down the capital stock, households at all dates could eat more.

Key point:

Nothing rules out a steady state that is dynamically inefficient.

Why is it surprising that the equilibrium can be Pareto inefficient?

Why Is Dynamic Inefficiency Possible?

- Vaguely, the **First Welfare Theorem** says:
  when all markets are competitive and some other conditions hold, every CE is Pareto Optimal.
- One of the "other conditions" comes in 2 flavors:
  1. there is a finite number of goods
  2. $\sum_{j=1}^{\infty} p_j < \infty$ where p_j are the CE (Arrow-Debreu) prices.
- Both conditions are violated in the OLG model.
- Acemoglu, ch. 9.1.

Intuition: Dynamic Inefficiency

- A **missing market**: the old must finance their consumption out of own saving, even if the rate of return is very low.
  - Suppose households value only c^o.
  - Then households save all income at rate of return $f'(k') - \delta$.
  - For high k', this can be negative.
- An alternative arrangement that makes everyone better off:
  - In each period, each young gives up 1 unit of consumption.
  - Each old gets to eat $1+n$ units.
  - If $n > f'(k) - \delta$, this makes everyone better off.
  - We will return to this idea in the section on "social security."

Summary

To characterize equilibrium, we typically derive a difference equation in the state variables.

- Here: $k_{t+1} = \phi(k_t)$

The properties of $\phi()$ depend on the saving function.

Even in this simple model, we cannot guarantee that we simple dynamics.

The steady state is an equlibrium where per capita variables are constant.

We cannot guarantee that the steady state is dynamically efficient.

- The potential problem is overaccumulation of capital.

Final Example: Government Bonds

We introduce harmless bonds into the model.

All the government does: issue new bonds to pay off the old ones.

Magical result: the steady state is at the golden rule.

One insight: introducing an infinitely lived asset fixes dynamic inefficiency

- actually, the assets here live for only one period
- but they serve the same function because there is now an infinitely lived agent who keeps trading the bonds

Environment

Demographics: $N_t = (1+n)^t$. Agents live for 2 periods.

Preferences:

$$(1-\beta)\ln(c_t^y) + \beta\ln(c_{t+1}^o)$$

Endowments:

- The initial old are endowed with s_0 units of capital.
- Each young is endowed with one unit of work time.

Environment

Technology:

$$C_t + K_{t+1} - (1-\delta)K_t = F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$$

Government: The government only rolls over debt from one period to the next:

$$B_{t+1} = R_t B_t$$

Markets: for goods, bonds, labor, capital rental.

Questions

1. Solve the household problem for a saving function.
2. Derive the FOCs for the firm.
3. Define a competitive equilibrium.
4. Derive the law of motion for the capital stock

$$k_{t+1}(1+n) = \beta(1-\alpha)k_t^{\alpha} - b_{t+1}(1+n) \tag{21}$$

   where $b = B/L$.
5. Derive the steady state capital stock for $b = 0$. Why does it not depend on δ?
6. Derive the steady state capital stock for $b > 0$.
7. Show that the capital stock is lower in the steady state with positive debt (crowding out).

Reading

- Acemoglu (2009), ch. 9.
- Krueger, "Macroeconomic Theory," ch. 8
- Ljungqvist and Sargent (2004), ch. 9 (without the monetary parts).
- McCandless and Wallace (1991)and De La Croix and Michel (2002) are book-length treatments of overlapping generations models.

References I

Acemoglu, D. (2009): *Introduction to modern economic growth*, MIT Press.

De La Croix, D. and P. Michel (2002): *A theory of economic growth: dynamics and policy in overlapping generations*, Cambridge University Press.

Ljungqvist, L. and T. J. Sargent (2004): *Recursive macroeconomic theory*, 2nd ed.

McCandless, G. T. and N. Wallace (1991): *Introduction to dynamic macroeconomic theory: an overlapping generations approach*, Harvard University Press.