The Growth Model In Continuous Time: Solow Model

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Econ720

October 1, 2020

Topics

- We study the standard growth model in continuous time.
- To solve it: Optimal Control
- To characterize it: phase diagrams

Continuous Time vs. Discrete Time

Continuous time

- So far, time was divided into discrete "periods."
- It is often more convenient to shrink the length of periods to 0.
- Difference equations then become differential equations.

Example: Law of motion for capital

Discrete time:

$$\underbrace{K_{t+1} - K_t}_{\text{change in stock}} = \underbrace{I_t - \delta K_t}_{\text{flow}} \tag{1}$$

Stocks:

- in units of goods
- e.g.: the capital stock = 100 machines

Flows:

- in units of goods per time period
- e.g.: investment = 10 machines per year

Change in stock = flow $\times$ duration.

Law of motion for capital

For any time period Δt:

$$\underbrace{K_{t+\Delta t} - K_t}_{\text{change in stock}} = \underbrace{[I_t - \delta K_t]}_{\text{flow}} \underbrace{\Delta t}_{\text{duration}} \tag{2}$$

Continuous time $(\Delta t \to 0)$:

$$\lim_{\Delta t \to 0} \frac{K_{t+\Delta t} - K_t}{\Delta t} = \dot{K}_t = I_t - \delta K_t \tag{3}$$

Notation: time derivative: $\dot{K} = dK/dt$.

Growth rates in continuous time

The growth rate of a variable is defined as

rate of change per period / level

ln

discrete time:

$$g(x) = \frac{x_{t+\Delta t} - x_t}{x_t \Delta t} \tag{4}$$

In continuous time:

$$g(x) = \frac{\dot{x}}{x} = \frac{d \ln x}{dt} \tag{5}$$

Growth rate rules

1. $g(xy) = g(x) + g(y)$.
2. $g(x/y) = g(x) - g(y)$.
3. $g(x^{\alpha}) = \alpha g(x)$.
4. $x(t) = e^{\gamma t} \Longrightarrow g(x) = \gamma$.

Exercise: Prove these using $g(x) = d \ln x / dt$.

Differential equations

Differential equations

Take a function of time:

$$x(t) = a + bt \tag{6}$$

There is another way of describing this function:

- Take the derivative:

$$\dot{x}(t) = dx(t)/dt = b \tag{7}$$

- Fix $x(0) = a$.
- The two pieces of information (the derivative and $x(0)$) completely describe $x(t)$.
- Only one function $x(t)$ satisfies both pieces.
  - But note that infinitely many functions satisfy the derivative!

Definition: Differential equation

A differential equation (DE) is a function of the form

$$\dot{x}(t) = f(x(t), t) \tag{8}$$

- This is actually a "first-order" DE.

Higher order DEs contain higher order derivatives of time.

- E.g.: A second order DE

$$\ddot{x}(t) + \dot{x}(t) = d^2x(t)/dt^2 + dx(t)/dt = a + bt \tag{9}$$

Together with a boundary condition, the DE can be solved for $x(t)$.

Solving DEs

- The bad news: There is no algorithm for solving DEs.
- But one look up solutions in tables.
- It is also easy to **verify** a solution one may guess.

Guess + Verify

Consider again

$$\dot{x}(t) = b \tag{10}$$
$$x(0) = a \tag{11}$$

Guess

$$x(t) = a + bt \tag{12}$$

Verify:

- Take the time derivative and find that it matches $\dot{x} = b$.
- Verify that $x(0) = a$.

Example: constant growth

$$\dot{x}(t) = b\,x(t) \tag{13}$$
$$x(0) = a \tag{14}$$

Guess:

$$x(t) = a\,e^{bt} \tag{15}$$

Verify: Take the derivative

$$\dot{x}(t) = b\,a\,e^{bt} = b\,x(t) \tag{16}$$
$$x(0) = a\,e^{0} = a \tag{17}$$

Boundary conditions

Boundary conditions can take many forms:

- $\int_a^b x(s)\,ds = 5$.
- $x(1.7) = 5$.
- $x(T) - x(T-2) = 5$.
- etc.

The Solow Model

The Solow Model - Structure

Modify the discrete time growth model in two ways:

1. Continuous time.
2. Fixed saving rate.

This is not an equilibrium model, but can be interpreted as one.

Model Elements

Demographics:

- households live forever
- the population growth rate is n:

$$L_t = e^{nt} \tag{18}$$

Preferences:

$$\int_{t=0}^{\infty} e^{-(\rho-n)t} u(c_t) dt \tag{19}$$

- discount factor: $\rho - n > 0$ (could be just ρ).

Model Elements

Endowments:

- at each moment, the household has 1 unit of work time
- at $t = 0$ he has K_0 goods

Technology:

$$F(K_t, L_t) - \delta K_t = \dot{K}_t + L_t c_t \tag{20}$$

- F: constant returns to scale

Markets:

- competitive markets for goods (numeraire), labor rental, capital rental
- this is still sequential trading; so we have a numeraire at each instant

Firms

The firm solves a static problem.

The same as in discrete time.

$$\max F(K,L) - wL - qK \tag{21}$$

FOC

$$q_t = F_K \tag{22}$$

$$w_t = F_L \tag{23}$$

Firms: Intensive Form

Define $k^F = K/L$ and

$$f\left(k^F\right) = F(K,L)/L = F\left(k^F,1\right) \tag{24}$$

The first order conditions are then

$$q = f'(k^F) \tag{25}$$

and

$$w = f(k^F) - f'(k^F)k^F \tag{26}$$

Households

Budget constraint

$$\dot{K}_t = w_t L_t + (q_t - \delta) K_t - L_t c_t$$

It is convenient to have everything per capita.

Define $k = K/L$ so that $g(k) = g(K) - n$.

Law of motion for k:

$$\begin{aligned} \dot{k}/k &= \dot{K}/K - n \\ &= w/k + (q - \delta) - c/k - n \end{aligned}$$

Or

$$\dot{k}_t = w_t + (q_t - \delta - n) k_t - c_t \tag{27}$$

Constant saving rate

- The modern way: Set up an optimization problem and derive the saving function.
- The Solow way: Assume that the saving rate is fixed:

$$c = (1-s)(w+qk) \tag{28}$$

- Therefore:

$$\dot{k} = s(w+qk) - (n+\delta)k \tag{29}$$

Market Clearing

Capital rental:

$$k = k^F \tag{30}$$

Goods market:

$$F(K_t, L_t) = C_t + \delta K_t + \dot{K}_t$$

or in per capita terms

$$\dot{k} = f(k) - (n+\delta)k - c \tag{31}$$

Equilibrium

An equilibrium is a collection of *functions* (of time)

$$c_t, k_t, k_t^F, w_t, q_t$$

that satisfy

1. the firm's first order conditions (2)
2. the household's budget constraint and the behavioral equation

$$\dot{k} = s(w + qk) - (n + \delta)k$$

3. market clearing (2)

Law of Motion

- The entire model boils down to to one key equation:

$$\dot{k}_t = sf(k_t) - (n+\delta)k_t \tag{32}$$

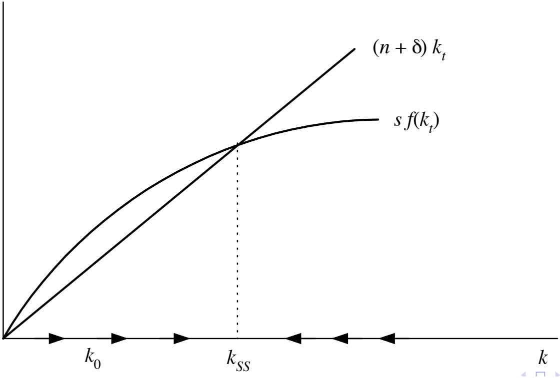
- This is simply the household's behavioral equation after applying $f(k) = w + qk$.

Steady state

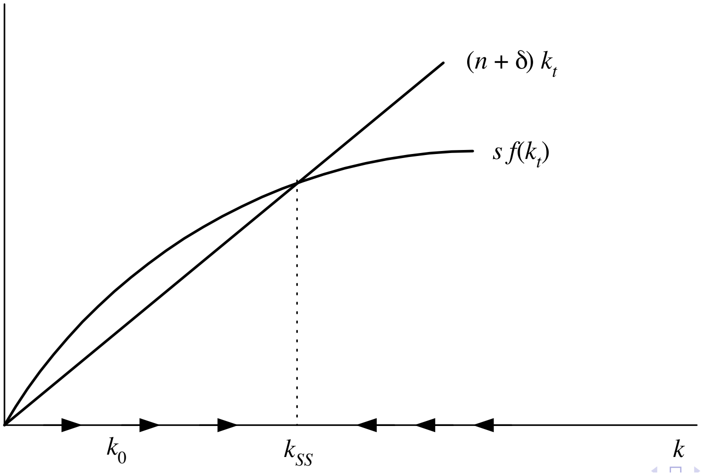
The steady state requires $\dot{k} = 0$ or

$$sf(k) = (n+\delta)k \tag{33}$$

With strictly concave f, there is a unique steady state with $k > 0$.



Dynamics



The steady state is *stable*.

Convergence is monotone.

Adding Technical Change

Adding Technical Change

The model does not have sustained growth in per capita income.

This requires technical change (A grows).

Assume exogenous growth in A:

$$A(t) = A(0)e^{\gamma t} \tag{34}$$

The general point: we learn how to analyze a growing model by detrending it.

Adding Technical Change

Assume that technical change takes the following form:

$$Y(t) = F(K(t), A(t)L(t)) \tag{35}$$

This type of technical change is called "labor-augmenting" or "Harrod-neutral."

This is the *only* form of technical change that is consistent with *balanced growth*.

Definition

A balanced growth path is a path along which all growth rates are constant.

How to analyze a growing model?

- Construct a **stationary transformation**.
- Divide each variable by its balanced growth factor:

$$\tilde{x}(t) = x(t) e^{-g_x t} \tag{36}$$

  where g_x is the **balanced growth rate** of x.
- Or take ratios of variables that grow at the same rate.
- The economy in transformed variables $(\tilde{x})$ has a steady state.

How to find the balanced growth rates?

- For equations that involve sums:

$$Y(t) = C(t) + I(t) + G(t) \tag{37}$$

  Constant growth (usually) requires that all summands grow at the same rate.

- For other equations: Try taking the growth rate of the whole equation.
- Example:

$$Y(t) = K(t)^{\alpha} \; [A(t) L(t)]^{1-\alpha} \tag{38}$$

  implies

$$g(Y) = \alpha g(K) + (1-\alpha)[g(A) + n] \tag{39}$$

Balanced growth path: Solow Model

Start from

$$\begin{aligned}\dot{K}(t) &= sF(K(t), A(t)L(t)) - \delta K(t) &(40)\\ g(K(t)) &= sF\left(1, \frac{A(t)L(t)}{K(t)}\right) - \delta &(41)\end{aligned}$$

Constant growth requires that

$$\bar{k}(t) = \frac{K(t)}{A(t)L(t)} \tag{42}$$

be constant over time. Thus, on a balanced growth path:

$$g(K) = \gamma + n \tag{43}$$

Balanced growth path

- Production function:

$$\bar{y}(t) = \frac{Y(t)}{A(t)L(t)} = F\left(\bar{k}(t), 1\right) \quad (44)$$

must be constant on a balanced growth path.

- Thus: The model has a steady state in $\left(\bar{k}, \bar{y}\right)$.

Law of motion

$$
\begin{aligned}
g\left(\bar{k}\right) &= g\left(K\right) - \gamma - n \\
&= sF\left(1, \frac{A\left(t\right)L\left(t\right)}{K\left(t\right)}\right) - \delta - \gamma - n \\
&= s\,f\left(\bar{k}\right)/\bar{k} - \delta - \gamma - n
\end{aligned}
$$

Or

$$\dot{\bar{k}}(t) = sf(\bar{k}(t)) - (n + \delta + \gamma)\bar{k}(t) \tag{45}$$

Nothing changes, except the added $\gamma\bar{k}$ in the law of motion.

Summary

1. We now have the basis for the growth model in continuous time.
   - we just need to add consumer optimization
2. The model looks like the discrete time model, except that all difference equations become differential equations.
3. If there is growth, we detrend all variables by their balanced growth rates.
4. To find the balanced growth rates, we look for relationships among growth rates implied by each model equation.

Reading

- Acemoglu (2009), ch. 2 covers the Solow model and stationary transformations of growing economies.
- Barro and Martin (1995), ch. 1
- Romer (2011), ch. 1
- Krusell (2014) ch. 2 discusses some insights that might be gained from the Solow model (and its limitations).

References I

Acemoglu, D. (2009): *Introduction to modern economic growth*, MIT Press.

Barro, R. and S.-i. Martin (1995): "X., 1995. Economic growth," *Boston, MA*.

Krusell, P. (2014): "Real Macroeconomic Theory," Unpublished.

Romer, D. (2011): *Advanced macroeconomics*, McGraw-Hill/Irwin.