# Stochastic Optimization

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# Stochastic Optimization

- We add shocks to the growth model.
- Recursive methods are needed.
- The resulting model is used to study
  - business cycles
  - asset pricing

# Model

Demographics:

- 1 representative household lives forever

Preferences: expected utility

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

- $\mathbb{E}_0$ denotes the expectation given information at date 0.
- The household needs to know the probability distribution of $c_t$ implied by his choices

Endowments: $k_0$ at $t = 0$

# Technology

$$k_{t+1} = f(k_t, \theta_t) - c_t \tag{2}$$

$\theta_t$: productivity shock

a Markov process with $N$ (finite) discrete values

$$\theta_t \in \{\theta^1, ..., \theta^N\} \tag{3}$$

and transition matrix

$$\Pr(\theta_{t+1} = \theta^j \,|\, \theta_t = \theta^i) = \Omega_{ij}$$

# Planner's Problem

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{4}$$

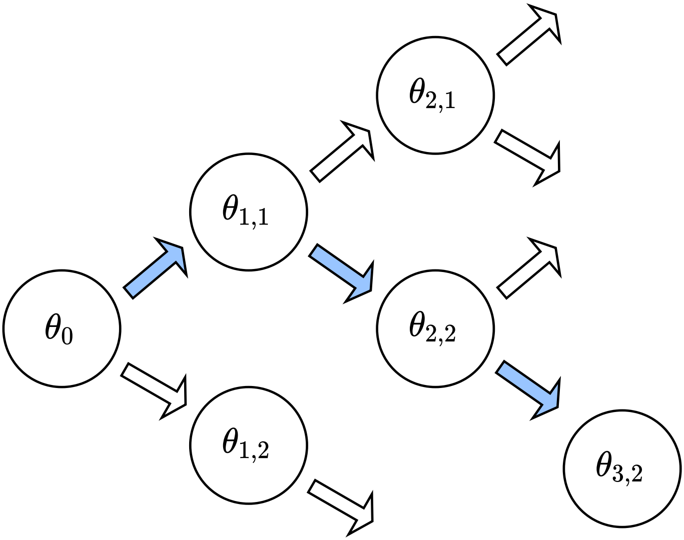
subject to

$$k_{t+1} = f(k_t, \theta_t) - c_t \tag{5}$$

Key problem: **what does the planner choose?**

- He cannot choose *sequences* $\{c_t, k_t\}$ because we don't know the realizations of $\theta_t$.
- He must choose a sequence $\{c_t, k_t\}$ for *every possible history* of shocks $(\theta_0, \theta_1, \ldots)$.
- *state contingent plans*.

# State Contingent Plans



# Notation

Define a history up to date $t$ as

$$s^t = (\theta_0, ..., \theta_t) \tag{6}$$

A state contingent plan is a mapping from histories to choices:

$$c_t = c(s^t) \tag{7}$$

and

$$k_{t+1} = \kappa(s^t) \tag{8}$$

# Two period example

# Two period example

The problem is

$$\max \mathbb{E}_1 \sum_{t=1}^{2} \beta^{t-1} u(c_t)$$

subject to

$$\begin{aligned} k_2 &= f(k_1, \theta_1) - c_1 \\ c_2 &= f(k_2, \theta_2) \\ & \theta_1 \text{ known} \end{aligned}$$

# Two period example

Write out the expectation explicitly:

$$\begin{aligned}
&\max u(c_1) + \sum_{j=1}^{N} \Pr\left(\theta_2 = \theta^j\right) \beta \ u\left(c_2\left(\theta^j\right)\right) \\
&+\lambda_1 \left[f\left(k_1, \theta_1\right) - c_1 - k_2\right] \\
&+\sum_{j=1}^{N} \lambda_2\left(\theta^j\right) \left[f\left(k_2, \theta^j\right) - c_2\left(\theta^j\right)\right]
\end{aligned}$$

The household chooses $c_1, k_2$ and $c_2\left(\theta^j\right)$.

The budget constraint must hold in every history.

# First-order conditions

$$
\begin{aligned}
c_1 &: \quad u'(c_1) = \lambda_1 \\
k_2 &: \quad \lambda_1 = \sum \lambda_2\left(\theta^j\right) f_k\left(k_2, \theta^j\right) \\
c_2\left(\theta^j\right) &: \quad \mathsf{Pr}\left(\theta_2 = \theta^j\right) \beta u'\left(c_2\left(\theta^j\right)\right) = \lambda_2\left(\theta^j\right)
\end{aligned}
$$

Euler equation:

$$
\begin{aligned}
u'(c_1) &= \beta \sum \mathsf{Pr}\left(\theta_2 = \theta^j\right) u'\left(c_2(\theta^j)\right) f_k\left(k_2, \theta^j\right) \\
&= \beta \; E\left\{u'(c_2) \; f_k(k_2, \theta_2) | \theta_1\right\}
\end{aligned}
$$

Interpretation ...

# Notes

1. $\mathbb{E}\{u'(c_2(\theta_2))f_k(k_2,\theta_2)\} \neq \mathbb{E}u'(c_2(\theta_2)) \times \mathbb{E}f_k(k_2,\theta_2)$
2. Naively maximizing

$$u(c_1) + \mathbb{E}\beta u(f(k_1;\theta_1) - c_1) \tag{9}$$

treating $\mathbb{E}$ as a number (!)
would have produced the right answer.

# Many periods

# Many periods

A history of length $t$ is $s^t$.

- e.g., $(\theta^2, \theta^6, ..., \theta_t)$

The household chooses $c(s^t)$ and $k(s^t)$ to maximize

$$\sum_t \sum_{s^t} p(s^t) \beta^t u(c(s^t))$$

subject to

$$\begin{aligned} x(s^t) + c(s^t) &= f(k(s^t), \theta(s^t)), \ \forall s^t \\ k(s_{t+1}, s^t) &= x(s^t), \ \forall s^t, s_{t+1} \end{aligned}$$

# Important point

You need the constraint

$$k\left(s_{t+1}, s^t\right) = x\left(s^t\right),\ \forall s^t, s_{t+1} \tag{10}$$

that ensures $k(s^{t+1})$ is the same for all $s^t$!

The following would be **wrong**:

$$k\left(s^{t+1}\right) + c\left(s^t\right) = f\left(k\left(s^t\right), \theta\left(s^t\right)\right) \tag{11}$$

Try to write down a Lagrangian and take FOCs - it does not work.

# Lagrangian

$$\sum_t \sum_{s^t} p(s^t)\beta^t u(f(k(s^t),\theta(s^t)) - x(s^t))$$
$$+\sum_t \sum_{s^t} \sum_{s_{t+1}} \varphi(s_{t+1},s^t)[k(s_{t+1},s^t) - x(s^t)]$$

FOC:

$$\begin{aligned} x(s^t) &: \beta^t p(s^t) u'(s^t) = \sum_{s_{t+1}} \varphi(s_{t+1},s^t) \\ k(s^t) &: \beta^t p(s^t) u'(s^t) f_k(s^t) = \varphi(s^t) \end{aligned}$$

What do these say in words?

# Euler equation

$$\beta^t p(s^t) u'(s^t) = \sum_{s_{t+1}} f_k(s_{t+1}, s^t) \beta^{t+1} p(s_{t+1,} s^t) u'(s_{t+1,} s^t) \qquad (12)$$

Divide by $\beta^t p(s^t)$ and note that

$$p(s_{t+1,} s^t) / p(s^t) = \Pr(s_{t+1} | s^t) \qquad (13)$$

This yields the usual Euler equation:

$$\begin{aligned} u'(s^t) &= \beta \sum_{s^{t+1}} \Pr(s_{t+1} | s^t) f_k(s_{t+1}, s^t) u'(s_{t+1}, s^t) && (14) \\ &= \beta \mathbb{E} \left\{ f_k(t+1) u'(c(t+1)) \mid s^t \right\} && (15) \end{aligned}$$

# Euler equation

- Be careful with notation.
- It would be wrong to write

$$\beta^t p(s^t) u'(s^t) = \sum_{s^{t+1}} f_k\left(s^{t+1}\right) \beta^{t+1} p\left(s^{t+1}\right) u'\left(s^{t+1}\right) \tag{16}$$

- Why is this wrong?

# The point

- With uncertainty, the sequence approach is a mess.
- Two solutions:
  1. Recursive methods.
  2. A shortcut: Maximize as if one could choose sequences.

# A shortcut

Let's proceed mechanically as if we were choosing sequences $\{c_t, k_t\}$:

$$\begin{aligned}\Gamma &= E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \\ &\quad + E_0 \sum_{t=0}^{\infty} \lambda_t \left[ f(k_t, \theta_t) - c_t - k_{t+1} \right]\end{aligned}$$

This is not quite right: the budget constraint should bind state by state, not just in expectation.

- See the 2 period example.

# A shortcut

- First-order conditions

$$\begin{aligned} E_0\,\beta^t u'(c_t) &= E_0\,\lambda_t \\ E_0\,\lambda_{t-1} &= E_0\lambda_t f_k(k_t,\theta_t) \end{aligned}$$

- Euler equation:

$$E_0\,u'(c_t) = \beta\,E_0\,u'(c_{t+1}) f_k(k_{t+1},\theta_{t+1})$$

- When date $t$ arrives:

$$u'(c_t) = \beta\,E_t\,u'(c_{t+1}) f_k(k_{t+1},\theta_{t+1})$$

- The point: Treating the $E$ as a constant in the optimization problem actually yields the right result!

# A shortcut

This approach is advocated by Chow (1997)

Why does the shortcut work, even though it is entirely wrong?

- One reason: $\mathbb{E}$ is linear: $\mathbb{E}(x) = \sum p(x_i) x_i$.
- The recursive approach makes this clearer...

# Recursive Approach

# Recursive Approach

- We generalize the DP approach introduced for deterministic problems.
- Nothing of substance changes, except there is an $E$ in front of each equation.
- Why does nothing change?
  - Because $E$ is a linear operator - just the sum of $\Pr(s'|s) \times outcome(s')$.
- We start by assuming that stochastic DP works as expected.
- Then we state the conditions under which it works.

# Recursive Approach to the Growth Model

State vector: $s_t = (k_t, \theta_t)$.

Bellman equation:

$$\begin{aligned} V(k,\theta) &= \max u(c) + \beta\, E\, V\left(f(k,\theta) - c, \theta'\right) \\ &= \max u(c) + \beta \sum_{\theta'} \Pr(\theta' \,|\, \theta)\, V\left(f(k,\theta) - c, \theta'\right) \end{aligned}$$

First-order conditions:

$$u'(c) = \beta\, E\, V_k(k', \theta')$$

Envelope condition:

$$V_k(k,\theta) = \beta\, E\, V_k(k', \theta') f_k(k,\theta)$$

Now we can see why the shortcut works.

# Euler Equation

$$
\begin{aligned}
u'(c) &= \beta \sum \Pr(\theta'|\theta) V_k(k', \theta') && (17) \\
&= \beta \sum \Pr(\theta'|\theta) f_k(k', \theta') \underbrace{\left[\beta \sum \Pr(\theta''|\theta') V_k(k'', \theta'')\right]}_{u'(c')} && (18)
\end{aligned}
$$

Or

$$
u'(c) = \beta E\left\{u'(c') f_k(k', \theta')\right\} \qquad (19)
$$

# Recursive Solution

Solution: $V(k, \theta), c(k, \theta)$ that satisfy:

1. Given $V$, $c(k, \theta)$ maximizes the right-hand-side of the Bellman equation.
2. $V$ is a fixed point of the Bellman operator:

$$V(k, \theta) = u(c[k, \theta]) + \beta E V(f[k, \theta] - c[k, \theta], \theta') \tag{20}$$

# Continuous state Markov chains

- What if the random vector takes on a continuum of values?
- Simply replace sums with integrals when calculating expectations.

# Continuous state Markov chains

- Assume that $\theta \in \Theta = \left[\underline{\theta}, \bar{\theta}\right]$.
- The evolution of $\theta$ is governed by a **transition function**:

$$\Pr\left(\theta' \leq x | \theta\right) = \Pi\left(x, \theta\right) \tag{21}$$

  - This is really a cdf conditional on $\theta$.
- The transition density for this CDF is $\pi$ with

$$\int_{\underline{\theta}}^{x} \pi\left(\theta', \theta\right) d\theta' = \Pi\left(x, \theta\right) \tag{22}$$

- This is the analogue to the transition matrix $\Pr\left(\theta' | \theta\right)$ in the discrete case.

# Continuous state Markov chains

- The conditional expectation of $f$ is then

$$\begin{aligned} E\left[f\left(\theta'\right)|\theta\right] &= \int_{\underline{\theta}}^{\bar{\theta}} f\left(\theta'\right)\ \pi\left(\theta',\theta\right) d\theta' \\ &= \int_{\underline{\theta}}^{\bar{\theta}} f\left(\theta'\right)\ \Pi\left(d\theta',\theta\right) \end{aligned}$$

- The point: In the continuous case, simply replace all the $\sum_{\theta'} \Pr(\theta'|\theta)$ with $\int \pi(\theta',\theta)\, d\theta'$.

# Reading

- Acemoglu (2009), ch. 16.1-16.2.
- Krusell (2014), ch. 6.
- Stokey et al. (1989) discuss the technical details of stochastic Dynamic Programming.
- Ljungqvist and Sargent (2004), ch. 2 talk about Markov chains.

# References I

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Chow, G. C. (1997): *Dynamic Economics: Optimization by the Lagrange Method*, Oxford University Press, USA.

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