Review Questions: Money in discrete time

Econ720. Fall 2020. Prof. Lutz Hendricks

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1 Money and leisure

Consider the following version of a Sydrauski model. A representative household solves

$$\max \sum_{t=0}^{\infty} \beta^t \, u \left(c_t, m_{t+1}/p_t, 1 - n_t\right)$$

subject to the budget constraint

$$k_{t+1} + m_{t+1}/p_t + c_t = F(k_t, n_t) + m_t/p_t + x_t$$

where x_t is a (money) transfer from the government and n_t is labor time. The government hands out money according to the rule $x_t = g_t \, m_t/p_t$ where $\{g_t\}$ is an exogenous sequence. Assume that $F(k, n)$ has constant returns to scale and satisfies the usual regularity conditions.

(a) State the household's Bellman equation. Derive a system of equations that solves the household problem.

(b) Define a competitive equilibrium.

(c) Now assume that $g_t = g$ in all periods. Derive a system of equations that characterizes the steady state.

(d) Prove that money is super-neutral, if the utility function is of the form $u(c, m/p, 1-n) = U(c, 1-n) + W(m/p)$.

1.1 Money and leisure

(a) The household's Bellman equation is given by

$$V(k, m) = u\left(c, m'/p, 1 - n\right) + \beta \, V(k', m') + \lambda \left\{ F(k, n) + m/p + x - m'/p - c - k' \right\}$$

The first-order conditions are

$$
\begin{aligned}
u_c(t) - u_m(t) &= \beta \left(p_t/p_{t+1}\right) u_c(t + 1) \qquad &(1)\\
u_n(t) &= u_c(t) \, F_n(k_t, n_t)\\
u_c(t) &= \beta \, u_c(t + 1) \, F_k(k_t, n_t)
\end{aligned}
$$

A solution to the household problem is a set of sequences $\{c_t, m_t, n_t, k_t\}$ that solves the 3 FOCs and the budget constraint. The transversality conditions are $\lim_{t \to \infty} \beta^t \, u_c(t) \, (k_t + m_t/p_t) = 0$.

(b) A competitive equilibrium is a set of sequences $\{c_t, m_t, n_t, k_t, p_t, x_t\}$ that solve 4 household conditions, $x_t \, p_t = g_t \, m_t = m_{t+1} - m_t$, and goods market clearing: $c_t + k_{t+1} = F(k_t, n_t)$.

(c) A steady state is a set of scalars $(c, n, m/p, k, \pi, x)$ that solve $\pi = g$. The Euler equation implies $F_k(k, n) = 1/\beta$. Goods market clearing requires $c = F(k, n) - k$. The first-order condition (1) implies

$$u_m/u_c = 1 - \beta/(1 + \pi) \qquad (2)$$

Finally, $x \, p = g \, m$ and $u_c = u_n \, F_n(k, n)$. Note that (2) implies that $u_m = 0$, if the Friedman rule is followed. In that case the nominal interest rate equals zero, i.e. $(1 + \pi)/\beta = 1$.

(d) Clearly, money growth does not affect F_k and hence k/n. Therefore, $c/k = F(k, n)/k - 1$ is also unaffected by g.

If money is additively separable in the utility function, then money is super-neutral. The reason is that $u_c = u_n \, F_n(k, n)$ and $F_k(k, n) = 1/\beta$ together with $c = F(k, n) - k$ determine c, k, and n independently of g. However, if money is not separable in the utility function, money is not super-neutral. Not much more can be said without restrictions on the utility function.

2   Endogenous Velocity

Consider the following version of a standard CIA model. A single representative household solves

$$\max \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) + v(l_t) \right\}$$

subject to the budget constraint

$$k_{t+1} + c_t + m_t^d/p_t = f(k_t) + m_t/p_t$$

and the CIA constraint

$$m_t/p_t = g(c_t, l_t)$$

and the law of motion for the money stock

$$m_{t+1} = m_t^d + \tau_{t+1}.$$

Most of the notation is standard. The new feature is that the household can divide his time endowment between leisure l_t and "going to the bank" $(1 - l_t)$ which reduces the cash requirement for consumption. The transaction technology has the following properties. Consuming more requires additional cash ($g_c > 0$, $g_{cc} \geq 0$). Spending additional time on banking reduces the amount of cash needed to finance a given consumption stream ($g_l > 0$, $g_{ll} \geq 0$). τ_t denote lump-sum helicopter drops of money.

(a) State the household problem as a Dynamic Program. Derive a system of 4 equation that characterize the household's optimal choices of c, k, m, and l.

(b) Define a competitive equilibrium. Assume that the money handouts follow an exogenous sequence.

(c) Derive a recursive system of equations that characterizes the steady state values of $c, k, l, m/p$. By recursive I mean a system that can be solved sequentially for one variable at a time. Assume that the money growth rate is constant at γ.

(d) Determine how faster money growth affects the steady state variables. Show that a sufficient condition for higher inflation to increase efforts to economize on money holdings (i.e., $\gamma \uparrow \Rightarrow l \downarrow$) is $g_{cl} \leq 0$.

2.1   Answer: Endogenous Velocity

(a) The Bellman equation is

$$
\begin{aligned}
V(k, m) \quad = \quad & \max u(c) + v(l) + \beta V(k', m') \\
& + \lambda \left\{ f(k) + m/p - (m' - \tau')/p - c - k' \right\} \\
& + \mu \left\{ m/p - g(c, l) \right\}
\end{aligned}
$$

First order conditions are:

$$u'(c) = \lambda + \mu\, g_c; \quad v'(l) = \mu\, g_l; \quad \beta\, V_k(.') = \lambda; \quad \beta\, V_m(.') = \lambda/p$$

The envelope conditions are

$$V_k = \lambda\, f'(k); \quad V_m = (\lambda + \mu)/p.$$

Substituting out the value function terms yields

$$\lambda = \beta\, \lambda'\, f'(k') = \beta\, (\lambda' + \mu')\, p/p'$$

To simplify notation define

$$h(c, l) = u'(c) - v'(l)\, g_c/g_l$$

Next eliminate the multipliers to obtain

$$
\begin{aligned}
h(c,l) &= \beta\, f'(k')\, h(c',l') \\
&= \beta\, \{h(c',l') + v'(l')/g_c(c',l')\}\, p/p'
\end{aligned}
$$

These two equations together with the budget constraint and the CIA constraint solve for c,l,k,m.

(b) A competitive equilibrium is a set of sequences $(c_t, l_t, k_t, m_t, p_t)$ that satisfy:

- 4 household optimality conditions

- Goods market clearing: $f(k_t) = c_t + k_{t+1}$.

- The law of motion for the money stock $m_{t+1} = m_t + \tau_{t+1}$.

(c) The Euler equation implies $f'(k_{ss}) = 1/\beta$. A constant real money stock requires $(1 + \pi_{ss}) = p'/p = 1 + \gamma$. Goods market clearing allows us to solve for consumption: $c_{ss} = f(k_{ss}) - k_{ss}$. Next, the first-order condition

$$
h(c,l) = [h(c,l) + v'(l)/g_l]\, \beta/(1+\gamma) \tag{3}
$$

can be solved (in principle) for l. Finally, the CIA constraint yields $m/p = g(c,l)$.

(d) Clearly, the money growth rate has no effect on k and c. The intuition is that inflation does not distort the atemporal choice between leisure and conserving money. There are also no wealth effects as the utility function is separable and time is not used in production. The change in leisure is determined by (3), holding c fixed. Rewrite (3) as

$$
\frac{1+\gamma}{\beta} - 1 = \frac{v'(l)}{g_l(c,l)\, h(c,l)}
$$

and apply the Implicit Function Theorem (under what conditions is that legitimate here?). A higher l reduces the denominator of the RHS. It increases the denominator via the g_l term (recall that $g_{ll} \geq 0$). If $h_l > 0$, then the RHS is decreasing in l and $\partial l/\partial \gamma < 0$. Note that $h_l = -v''(l)\, g_c/g_l + v'(l)\, g_c/g_l^2\, g_{ll} - v'(l)\, g_{cl}/g_l$. The first two terms are positive, but the sign of the third term depends on the cross-derivative g_{cl}. A sufficient condition for higher inflation to reduce leisure (and increase efforts to economize on money holdings) is $g_{cl} \leq 0$.