Arrow-Debreu and Sequential Trading

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Introduction

Macro models are dynamic (have many periods).

Agents are forward looking; **make all choices at date 0**

- It is then natural to open markets for all goods at date 0
- **Arrow-Debreu** trading

But more "realistic"

- **Sequential trading**: markets open in each period
- Not better; just easier to map into data

Both setups produce the same equilibrium allocations.

This is where the details matter (units of account, Walras' law, ...)

Two Period Example

Demographics:

- N identical households live for 2 periods, $t = 1, 2$.

Commodities:

- there is one good in each period

Preferences: $u(c_1, c_2)$

Endowments: e_t

"Technology": $Nc_t = Ne_t$

Markets

Now we have a choice between 2 equivalent arrangements

- Arrow-Debreu: all trades take place at $t = 1$
- Sequential trading: markets open in each period

2. Arrow Debreu Trading

Arrow-Debreu Trading

The arrangement:

- All trades take place at $t = 1$
- Agents can buy and sell goods for delivery at any date t
- Prices are p_t

Can we normalize prices to 1?

Surprise:
If we write out this model, it **looks exactly like a static 2 good model**.

Household problem

Household budget constraint:

$$\sum_t p_t e_t = \sum_t p_t c_t \tag{1}$$

Interpretation:

The household sells e_t to and buys c_t from the Walrasian auctioneer at a single trading date.

Household problem

Lagrangian:

$$\max_{c_1,c_2} u(c_1,c_2) + \lambda \left[\sum_t p_t e_t - \sum_t p_t c_t \right] \tag{2}$$

FOCs:

$$u_t(c_1,c_2) = \lambda p_t \tag{3}$$

Or:

$$\frac{u_1}{u_2} = \frac{p_1}{p_2} \tag{4}$$

Arrow-Debreu Equilibrium

Market clearing:

$$e_t = c_t \tag{5}$$

- Again the same as resource constraints.

Equilibrium

Objects: c_t, p_t, $t = 1, 2$

Equations:

- Household FOC
$$\frac{u_1}{u_2} = \frac{p_1}{p_2} \tag{6}$$
  and budget constraint
- Market clearing: $e_t = c_t$

Notes:

- only p_2/p_1 is determined in equilibrium (choice of unit of account)
- only **one** equation is redundant by Walras' law (why?)

Equivalence of Dates and Goods

Fact

A model with T goods is equivalent to a model with T periods.

Why is this true? What is a "good?"

This is only true under "**complete markets**"

- roughly: there are markets that allow agents to trade goods across all periods and states of the world
- we will talk about details later

3. Sequential Trading

Sequential Trading

An alternative trading arrangement.

Markets open at each date.

Only the date t good can be purchased in the period t market.

Now we have **one numeraire for each trading period**: $p_t = 1$.

We need assets to transfer resources between periods.

Markets

At each date we have

1. a market for goods $(p_t = 1)$;
2. a market for 1 period discount bonds (price q_t)

A discount bond pays 1 unit of $t+1$ consumption.

Digression: Modeling bonds

Definition

A one period bond promises to pay one unit of consumption in $t+1$.

Call its price q_t.

Then the real interest rate is: $R_{t+1} = 1/q_t$.

Econ101 question

What is a real interest rate?

Alternative normalization:

- set $q_t = 1$ and let each bond pay R_{t+1} units of consumption
- why can I do this?

Household problem

Now we have one budget constraint per period:

$$e_t + b_{t-1} = c_t + b_t q_t \tag{7}$$

with $b_0 = 0$.

Fact

Agents have one budget constraint for each trading period

Household solves:

$$\max_{b_1} u(e_1 - b_1 q_1, e_2 + b_1) \tag{8}$$

Household solution

FOC:

$$u_1 q_1 = u_2 \tag{9}$$

q_1 is the relative price of period 2 consumption.

Give up 1 unit of c_1 and get $1/q_1$ units of c_2.

Solution: c_1, c_2, b_1 that solve FOC and 2 budget constraints.

Market Clearing

- Goods: $e_t = c_t$
- Bonds: $b_t = 0$

Why does bond market clearing look so odd?

Equivalence

Note that the relative price is the same under both trading arrangements:

$$p = q = u_2/u_1 \tag{10}$$

Fact

When markets are complete, Arrow-Debreu and sequential trading equilibria are identical.

Why two trading arrangements?

When building a model, choose the most convenient trading arrangement.

In many cases, it does not matter

- equilibrium allocations are the same
- both are similarly convenient

It matters when there is uncertainty.

- with complete markets, Arrow-Debreu is much simpler
- with incomplete markets, sequential trading makes it easy to specify which assets exist

Summary

Macro is micro

- The method outlined here is central to all of (macro) economics.
- Being able to translate a description of an economy into the definition of a competitive equilibrium is an important skill.

Final example

Demographics: There are N households. Each lives for $T > 1$ periods.

Preferences: $\sum_{t=1}^{T} u(c_{1,t}, \ldots, c_{J,t})$ where J is the number of goods available in each period.

Endowments: Household i receives $e_{i,j,t}$.

Technologies: Endowments can only be eaten in the period they are received.

- Resource constraint:

Markets:

- Sequential trading: there are competitive markets for the J goods; there are one period discount bonds in each period.
- Arrow-Debreu: the $J \times T$ goods are traded in $t = 1$.

Final example: Arrow-Debreu Equilibrium

Household problem:

FOCs:

Household solution:

Equilibrium:

Recap Questions

1. How often does the Walrasian auctioneer show up?
2. How many prices can be normalized to 1?
3. How many budget constraints does the household have?
4. How does the household move goods between dates?
5. Why are there no financial assets with Arrow-Debreu trading?

Reading

Krusell (2014), ch. 5 talks about Arrow-Debreu versus sequential trading.

References

Krusell, P. (2014): "Real Macroeconomic Theory," Unpublished.