The Solow Diagram

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Analyzing the Solow Model

What are the properties of the Solow model?

- Why do economies grow over time?
- Does the economy settle down in the long-run?
- What are the long-run and short-run effects of changes in behavior?

To answer that:

1. Study the steady state (where everything is constant over time).
2. Plot the law of motion for k.

The steady state

Definition

A **steady state** is a situation where all variables are constant over time (in per capita terms).

In the Solow model:

- Capital per worker is constant: $\dot{k} = 0$.

Law of motion:

$$\dot{k}(t) = s \underbrace{k(t)^{\alpha} A^{1-\alpha}}_{f(k)} - (n+\delta)\, k(t)$$

The steady state capital stock solves:

$$sf(k^*) = (n+\delta)k^* \tag{1}$$

Intuition?

The Steady State

With the Cobb-Douglas production function

$$sA^{1-\alpha}k^{\alpha} = (n+\delta)k \tag{2}$$

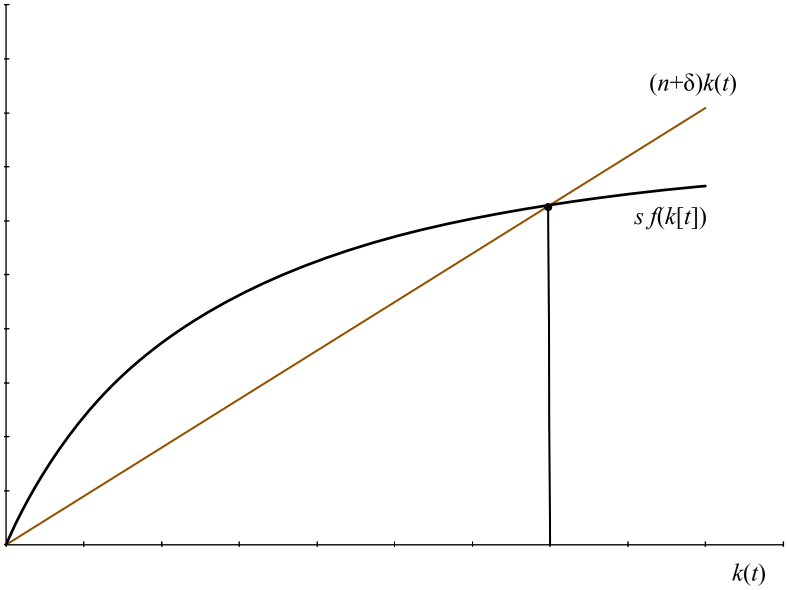
or

$$k^{1-\alpha} = \frac{sA^{1-\alpha}}{n+\delta} \tag{3}$$

Steady state **output** per worker

$$y = A^{1-\alpha}k^{\alpha} = A\left(\frac{s}{n+\delta}\right)^{\alpha/(1-\alpha)}$$

Steady state graph



Properties of the Steady State

Steady state output:

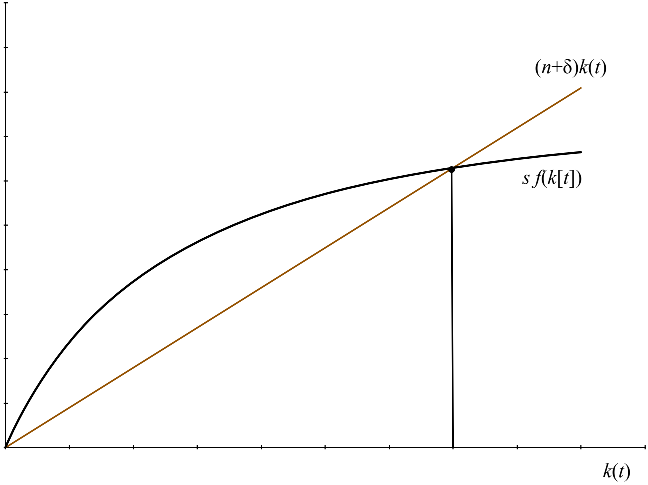
$$y = A \left(\frac{s}{n + \delta} \right)^{\alpha/(1-\alpha)}$$

1. Unique
2. Higher saving or productivity increase k and y
3. Higher depreciation or population growth reduce k and y

How big these effects are is governed by α.

- curvature of the production function
- more curvature $\implies$ smaller changes in y

Dynamics



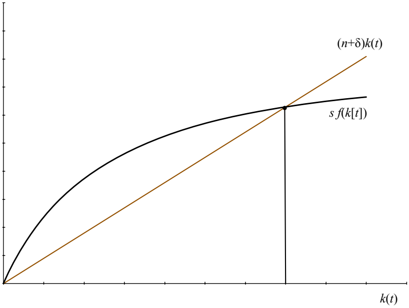
What can we say about the dynamics?

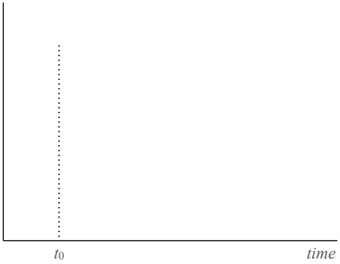
Key ideas

1. Growth is driven by investment > depreciation.
2. Low $k \implies$ high $MPK = f'(k) \implies$ saving generates a lot of output $\implies$ output grows
3. High $k \implies$ high depreciation $\implies$ output shrinks
4. Therefore, the economy always converges to a steady state where investment = depreciation

Comparative statics (or dynamics)

What happens if households save more?



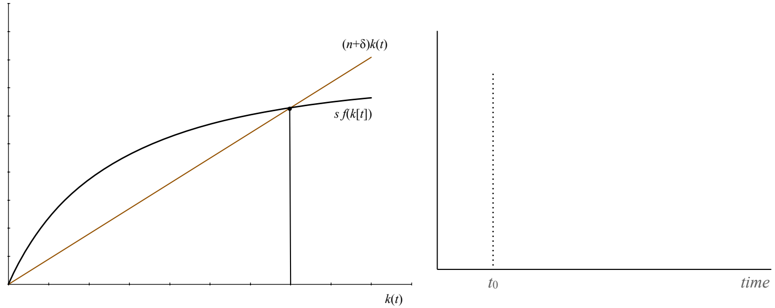


Plot the time paths of output and interest rates.

Reality Check

- The model says: more investment (or **lower consumption**) generates a period of **faster** growth.
- Isn't everybody saying: the U.S. is in a recession (slow growth) because consumption is too low?
- How does the contradiction get resolved?
- Where is the effect of lower consumption demand in the Solow model?
- Where is the demand side anyway?

What happens if there is a baby boom?



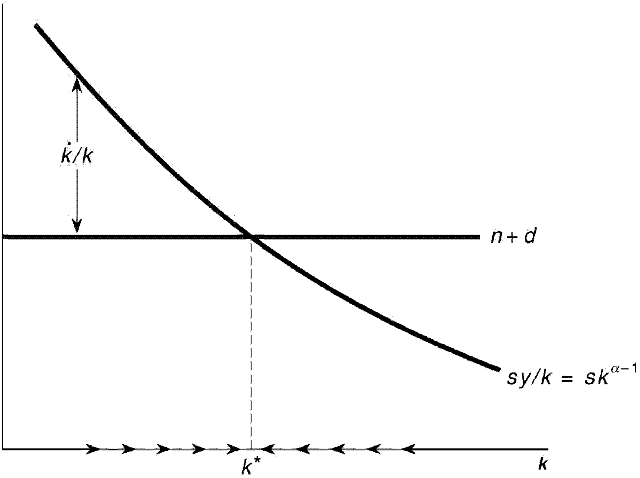
Economic Growth

- Why do countries grow?
- In the Solow model:
  - Growth can only occur along a **transition path**.
  - There is **no long-run** growth in GDP per worker $(y = Y/L)$.
- But growth slows as the economy approaches the steady state.
- To see this, write the law of motion for k as

$$\dot{k}/k = g(k) = sy/k - (n + \delta)$$

where $y/k = A^{1-\alpha}k^{\alpha-1}$ is declining in k. [Why?]

Economic Growth



The Principle of Transition Dynamics

Fact

In the Solow model, the farther away the economy is from its steady state, the faster it grows (or shrinks)

What is the intuition?

Why does investment not sustain growth?

- The problem is the diminishing MP_K.
- Giving up one unit of C today yields $MP_{K'} - \delta$ in additional output tomorrow.
- As k grows, MP_K eventually falls below δ:
  - Additional investment no longer even pays for its own depreciation.
- Sustained growth through capital accumulation requires that MP_K stays above δ, even as k grows without bounds.

Technical Change

Technical change

- To sustain long-run growth of y the Solow model requires **technical change**.
  - Technical change is modeled as shifting the production function up.
  - Productivity grows: $g(A) > 0$.
- Later, we treat A as the product of innovation.
- Here: A is exogenous.
- Assume that technical change is **labor augmenting**: $Y = F[K, AL]$.
  - Otherwise, the model is not consistent with the data
  - "Kaldor facts" (not obvious, but true).

Steady state?

Law of motion (unchanged):

$$\dot{k}(t) = sA(t)^{1-\alpha} k(t)^{\alpha} - (n+\delta) k(t) \tag{4}$$

But now A grows over time:

$$A(t) = A(0) e^{\gamma t} \tag{5}$$

Can we have a steady state?

It would imply

$$sA(t)^{1-\alpha} k^{\alpha} = (n+\delta) k \tag{6}$$

That can only work with constant A.

Growing A implies that k will grow forever.

Balanced growth path

We don't have a steady state, so we look for the next best thing.

Definition

A balanced growth path is an equilibrium where all variables grow at rates that are constant over time.

What are the balanced growth rates?

Write the law of motion as

$$g(k) = sy/k - (n + \delta)$$

Constant $g(k)$ requires constant y/k.
But

$$y/k = sA(t)^{1-\alpha} k(t)^{\alpha-1} \tag{7}$$

So we need constant $\bar{k} = k/A$.
Therefore: On the balanced growth path,

$$g(k) = g(y) = g(A) = \gamma \tag{8}$$

Law of motion

To analyze the dynamics: construct variables that are constant on the BGP

- $\bar{k} = k/A$, $\bar{y} = y/A$

We derive a low of motion for $\bar{k}$.

By the growth rate rule:

$$\begin{aligned} g\left(\bar{k}\right) &= g\left(k\right) - g\left(A\right) \\ &= sy/k - \left(n+\delta\right) - g\left(A\right) \end{aligned}$$

Note that $y/k = \bar{y}/\bar{k}$.

Law of motion:

$$d\bar{k}/dt = s\,\bar{y} - \left(n+\delta+g\left(A\right)\right)\bar{k} \tag{9}$$

with

$$\bar{y} = y/A = \frac{k^{\alpha}A^{1-\alpha}}{A} = \bar{k}^{\alpha} \tag{10}$$

What has changed?

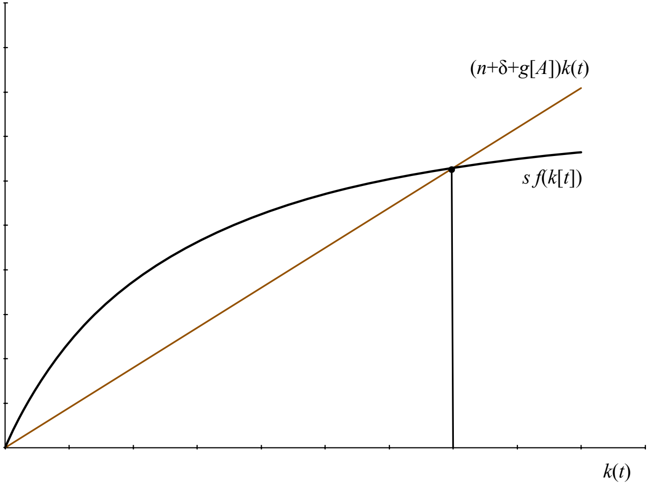
The model with technical change looks exactly like the previous model, except:

1. All variables are ”detrended” (divided) by AL.
2. The steady state has per capita variables growing at rate $g(A)$.
3. The law of motion contains an additional $g(A)$ term.

The model has a steady state in the "detrended" variables $(\bar{k}, \bar{y})$.

It has a balanced growth path in per capita variables (k, y).

The Solow diagram

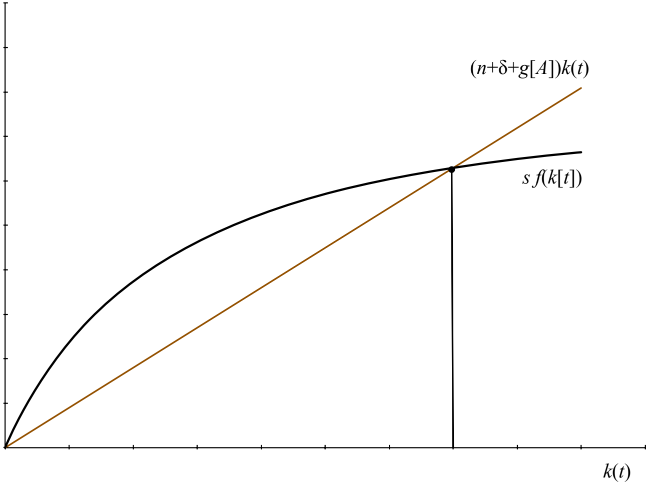


This is essentially the same diagram as without technical change, except:

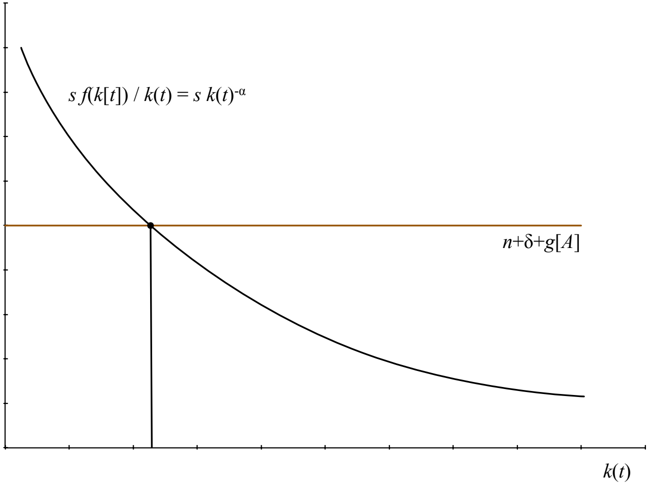
- variables are detrended.
- an additional g term appears in the straight line.

Comparative statics: higher saving rate

The Solow diagram is familiar:



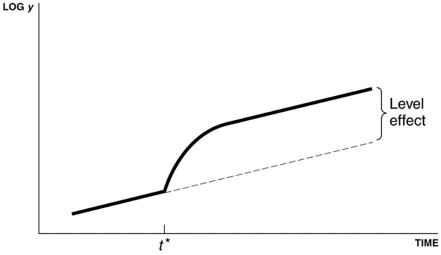
Transitional dynamics



Policies have level effects

A key implication of the Solow model: Policies, such as taxes, do not affect the long-run growth rate.

The growth rate rises on the transition to the new steady state, then levels off to $g(A)$.



Important Points

- The Solow model reveals how choices (saving, fertility) affect capital and output (levels and growth).
- Capital cannot sustain long-run growth.
  - the reason: diminishing returns
- Therefore policies have level effects.
- In the short run: countries grow fast when they are far below their steady states.
- In the long run: growth is determined by productivity improvements.

Final Example

Modify the Solow model by assuming that the production function is given by

$$Y_t = AK_t^{\alpha} L_t^{1-\alpha} - L_t X$$

where $X > 0$ is a constant.

1. How does the Solow diagram change?
2. How many steady states are there?
3. Which ones are stable?

Reading

- Jones (2013), ch. 2
- Blanchard and Johnson (2013), ch. 11

Further Reading:

- Romer (2011), ch. 1
- Barro and Sala-i Martin (1995), ch. 1.2

References I

Barro, R. and X. Sala-i Martin (1995): "Economic growth," *Boston, MA*.

Blanchard, O. and D. Johnson (2013): *Macroeconomics*, Boston: Pearson, 6th ed.

Jones, Charles; Vollrath, D. (2013): *Introduction To Economic Growth*, W W Norton, 3rd ed.

Romer, D. (2011): *Advanced macroeconomics*, McGraw-Hill/Irwin.