Problem Set 2: OLG Models with Money

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1 Money and Heterogeneity

Consider a two-period OLG model with fiat money.

Demographics: In each period $N_t = (1+n)^t$ persons are born. Each lives for 2 periods. Half of the agents are of type I, the other half of type II.

Endowments: The initial old hold M_0 units of money, evenly distributed across agents. Each person is endowed with (e_i^y, e_i^o) units of consumption when (young, old).

Preferences: $\ln(c_t^y) + \beta \ln(c_{t+1}^o)$.

Technology: Goods can only be eaten the day they drop from the sky.

Government: The government pays a lump-sum transfer of $x_t p_t$ units of money to each old person: $M_t = M_{t-1} + N_{t-1} x_t p_t$. The aggregate money supply grows at the constant rate μ: $M_{t+1} = (1+\mu) M_t$.

Markets: In each period, agents buy/sell goods and money in spot markets.

Questions:

1. Define a competitive equilibrium.
2. Derive the household consumption function.
3. Derive a difference equation for the equilibrium interest rate when $\mu = 0$.
4. Is the monetary steady state dynamically efficient?

1.1 Answer: Money and Heterogeneity

It is important to be clear about the timing. At the start of period t, the money stock is M_{t-1}. Then the old receive the transfer x_t, which raises the money stock to M_t. Let $m_{i,t}^d$ be the real, per capita money holdings of the young in t (and thus of the old at the start of $t+1$).

1. Equilibrium:

   Objects: $\left\{c_{i,t}^y, c_{i,t}^o, m_{i,t}^d, x_t, p_t, M_t\right\}$

   Equations:

   - household: Euler equation and 2 budget constraints. Young: $e_i^y = c_{i,t}^y + m_{i,t}^d$. Old: $e_i^o + m_{i,t}^d p_t / p_{t+1} + x_{t+1}$.
   - government: 2 laws of motion for M_t
   - market clearing:
     - goods: $\sum_i e_i^y + e_i^o / (1+n) = \sum_i c_{i,t}^y + c_{i,t}^o / (1+n)$

– money: $M_t = p_t \sum_i N_{i,t} m^d_{i,t}$ (here $N_{i,t} = N_t/2$).

2. Consumption function: Euler + lifetime budget constraint implies that

$$c^y_{it}(1+\beta) = W_{it} = e^y_i + (e^o_i + x_{t+1})/R_{t+1} \tag{1}$$

and

$$c^o_{it} = \frac{\beta}{1+\beta} W_{it} R_{t+1} \tag{2}$$

where $R_{t+1} = 1/(1+\pi_{t+1})$ and $1+\pi_{t+1} = p_{t+1}/p_t$.

3. Implicit solution for the equilibrium interest rate when $x = 0$: Start from goods market clearing

$$\sum_i W_{it}/(1+\beta) + W_{i,t-1} \frac{\beta R_t}{(1+\beta)(1+n)} = E = \sum_i e^y_i + e^o_i/(1+n) \tag{3}$$

Now sub in the definitions for $W_{i,t}$:

$$\sum_i \frac{e^y_i + e^o_i/R_{t+1}}{1+\beta} + \sum_i \frac{e^y_i + e^o_i/R_t}{1+n} \frac{\beta R_t}{1+\beta} = E \tag{4}$$

This produces a difference equation in R_t.

4. Steady state with money:

We need $m = M/(NP)$ to be constant over time. Therefore, $1+\pi = (1+\mu)/(1+n)$. With $\mu > 0$, the monetary economy must be dynamically *in*efficient. With $\mu = 0$, we get the Golden Rule and the economy is efficient. The same could be derived by setting R constant in the difference equation above.

2 Money and Storage

Consider a two-period OLG model with fiat money and a storage technology.

Demographics: In each period $N_t = (1+n)^t$ persons are born. Each lives for 2 periods.

Endowments: The initial old hold capital K_0 and money M_0. Each young person is endowed with a e units of the good.

Preferences: $u(c^y_t) + \beta u(c^o_{t+1})$.

Technology: Storing k_t units of the good in t yields $f(k_t)$ units in $t+1$. f obeys Inada conditions. The resource constraint is $N_t k_{t+1} = N_t e + N_{t-1} f(k_t) - C_t$ where $C_t = N_t\, c^y_t + N_{t-1}\, c^o_t$.

Government: The government pays a lump-sum transfer of $x_t p_t$ units of money to each old person: $M_{t+1} = M_t + N_{t-1} x_t p_t$. The aggregate money supply grows at the constant rate μ: $M_{t+1} = (1+\mu) M_t$.

Markets: In each period, agents buy/sell goods and money in spot markets.

The timing in period t is a follows:

- The old enter period t holding aggregate capital $K_t = N_{t-1} k_t$ and nominal money balances of $M_t = m_t\, N_{t-1}$.

- The old receive the money transfer from the government and now hold M_{t+1}.
- Each old person produces $f(k_t)$.
- The young buy money (m_{t+1}/p_t) from the old, consume c_t^y and save k_{t+1}.
- The old consume their income.

Questions:

1. State the household's budget constraints when young and old.
2. Derive the household's optimality conditions. Define a solution to the household problem.
3. Define a competitive equilibrium.
4. Does an equilibrium with positive inflation exist? Intuition?
5. Define a steady state as a system of 6 equations in 6 unknowns.
6. Find the money growth rate (μ) that maximizes steady state consumption per young person, $(N_t\, c_t^y + N_{t-1}\, c_t^o)/N_t$.

2.1 Answer: Money and Storage

1. Young: $e_t = c_t^y + k_{t+1} + m_{t+1}/p_t$. Old: $c_{t+1}^o = f(k_{t+1}) + m_{t+1}/p_{t+1} + x_{t+1}$.
2. The household solves

$$\max\, u\left(e_t - k_{t+1} - m_{t+1}/p_t\right) + \beta\, u\left(f(k_{t+1}) + m_{t+1}/p_{t+1} + x_{t+1}\right)$$

   First-order conditions are

$$\begin{aligned} u'(c_t^y) &= \beta\, u'(c_{t+1}^o)\, f'(k_{t+1}) \\ &= \beta\, u'(c_{t+1}^o)\, p_t/p_{t+1} \end{aligned}$$

   A solution is a vector $(c_t^y, c_{t+1}^o, k_{t+1}, m_{t+1})$ that solves the 2 first-order conditions and 2 budget constraints. From the household first-order conditions, money and capital must have the same rate of return: $f'(k_{t+1}) = p_t/p_{t+1}$.

3. A CE consists of sequences $(c_t^y, c_{t+1}^o, k_{t+1}, m_{t+1}, M_t, p_t, x_t)$ that solve 4 household conditions, 2 government conditions, the definition $M_t = m_t\, N_{t-1}$ and goods market clearing (same as resource constraint). In per capita terms: $k_{t+1} = e_t + f(k_t)/(1+n) - c_t^y - c_t^o/(1+n)$. Transfer payments equal new money issues: $N_{t-1}\, p_t\, x_t = \mu\, M_t$.
4. There is no equilibrium with positive inflation. That would imply rate of return dominance, and nobody would hold money.
5. A steady state consists of constants $(c^y, c^o, k, m/p, \pi, x)$ that satisfy:
   (a) Constant m/p requires $(1+\mu) = (1+n)(1+\pi)$. This determines π.

(b) $f'(k) = (1+\pi)^{-1}$ determines k.

(c) Both consumption levels are determined by the Euler equation and goods market clearing.

(d) m/p is determined from the young budget constraint.

(e) $x = \mu(m/p)$ from the law of motion for M.

6. Maximizing steady state consumption requires $f'(k) = 1+n$. Therefore, $\mu = 0$. The intuition is that with a constant nominal money supply the per capita money supply shrinks at rate n. For the real money supply to be constant, inflation at rate $-n$ is needed, which satisfies the Golden Rule.
