Review Questions: Money in Continuous Time

Econ720. Fall 2021. Prof. Lutz Hendricks

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1 Money in the production function

Consider a standard Sidrauski model with a single representative household who maximizes $\int_0^\infty e^{-\rho t} u(c_t, m_t)\, dt$ subject to the budget constraint $\dot{a}_t = f(k_t, m_t) - c_t + \tau_t - \pi_t m_t$. Here, $a_t = k_t + m_t$ denotes asset holdings, consisting of capital and real money balances, c_t is consumption, and $\pi_t = \dot{p}_t / p_t$ is the inflation rate. τ_t denotes lump-sum money transfers. Money transfers follow the rule $\tau_t = g\, m_t$ for some exogenous money growth rate g_t.

(a) Define a solution to the household problem.

(b) Define a competitive equilibrium.

(c) Provide a system of equations that characterizes the steady state. Is money generally super-neutral?

(d) Now assume that $u(c) = \ln(c) + \varphi \ln(m)$ and $f(k,m) = k^\alpha m^{1-\alpha}$. The model now features endogenous growth. How does faster money growth rate affect the balanced growth rate?

1.1 Answer: Money in the production function

(a) The current value Hamiltonian is

$$H = u(c,m) + \lambda \left\{ f(a-m, m) - c - \pi\, m + \tau \right\}$$

First-order conditions are

$$\begin{aligned} u_c &= \lambda \\ u_m &= \lambda \left\{ f_k - f_m + \pi \right\} \end{aligned}$$

The TVC is $\lim e^{-\rho t} u_c(c_t, m_t)\, a_t = 0$. A solution to the household problem consists of functions (c_t, m_t, a_t) that satisfy

$$\begin{aligned} u_m / u_c &= f_k - f_m + \pi & (1) \\ g(u_c) &= \rho - f_k \end{aligned}$$

together with the budget constraint and the TVC.

(b) A competitive equilibrium consists of functions $(c_t, m_t, a_t, p_t, \tau_t, k_t)$ that satisfy 3 household conditions, the definition of asset holdings $a = k + m$, the law of motion for money $\dot{m} = (g - \pi)\, m$, the transfer rule $\tau = g\, m$, and goods market clearing $f(k,m) = c + \dot{k}$.

(c) In general, a steady state is a vector (c, k, m, π) which satisfies $\pi = g$, $f(k,m) = c$, $f_k(k,m) = \rho$, and $u_m/u_c = \rho + g - f_m(k,m)$. Money is clearly not super-neutral.

(d) With this somewhat bizarre production function the economy exhibits endogenous growth. The balanced growth path is a vector $(c/k, m/k, \pi, \gamma)$ which satisfies

$$\begin{aligned} \gamma &= f_k - \rho = \alpha\, (k/m)^{\alpha - 1} - \rho & (2) \\ \gamma &= g - \pi & (3) \\ \gamma &= f(k,m)/k - c/k = (k/m)^{\alpha-1} - c/k & (4) \\ \frac{\varphi\, c}{m} &= \rho + g - f_m(k,m) = \rho + g - (1-\alpha)\,(k/m)^\alpha & (5) \end{aligned}$$

The second condition implies that m grows at rate γ. First solve (4) for

$$c/m = (k/m)^\alpha - \gamma\, k/m$$

Then substitute into (5):

$$\frac{c}{m} = (k/m)^\alpha - \gamma\, k/m = \left[\rho + g - (1-\alpha)\,(k/m)^\alpha\right] / \varphi$$

Replace $\gamma k/m$ using (2):

$$(k/m)^{\alpha}\,(1-\alpha)\,(1+1/\varphi)+\rho\,k/m=(\rho+g)/\varphi$$

It follows that a higher g implies a higher k/m and therefore slower growth. Higher inflation makes holding money more costly. But money in this model is very much like capital. So higher inflation is akin to taxing investment, which reduces growth.