

# Asset Pricing

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# Topics

1. What determines the rates of return / prices of various assets?
2. How can risk be measured and priced?
  - ▶ We use the Lucas (1978) fruit tree model.
  - ▶ The implications are far more general than the simple model.
  - ▶ The model forms the basis for the CAPM and the  $\beta$  risk measure.

# The Lucas (1978) Fruit Tree Model

- ▶ **Agents:**

- ▶ A single representative household.

- ▶ **Preferences:**

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \quad (1)$$

- ▶  $E_0$  is the expectation as of time  $t = 0$ .

# Technology

- ▶ This is an endowment economy.
- ▶ There are  $K$  identical fruit trees.
- ▶ Each tree yields  $d_t$  units of consumption goods in period  $t$ .
- ▶  $d_t$  is random and the same for all trees.
- ▶ Trees cannot be produced.
- ▶ Fruits cannot be stored.

- ▶ The aggregate resource constraint:

$$c_t = Kd_t \quad (2)$$

- ▶ Assume that  $d$  is a finite Markov chain with transition matrix  $\pi(d', d)$ .
- ▶ An important feature: All uncertainty is **aggregate**.
- ▶ There are no opportunities for households to insure each other.
- ▶ This is why we can work with a representative household.

# Markets

- ▶ There are markets for fruits and for trees.
- ▶ There is also a one period bond, issued by households (in zero net supply).
  - ▶ Its purpose is to determine a risk-free interest rate.
- ▶ Digression: Why this is a good model...

# Household problem

- ▶ The household starts out with bonds ( $b_0$ ) and shares ( $k_0$ ).
- ▶ At each date, he chooses  $c_t, b_{t+1}, k_{t+1}$ .
- ▶ The **budget constraint** is

$$p_t k_{t+1} + b_{t+1} = R_t b_t + (p_t + d_t) k_t - c_t \quad (3)$$

- ▶ Notation:
  - ▶  $p$ : the price of trees. Suppressing dependence on the state.
  - ▶  $R$ : the real interest rate on bonds.
  - ▶ the price of bonds is normalized to 1 (how?).

## Household problem

$$V(k, b, d) = \max u(c) + \beta EV(k', b', d') \quad (4)$$

subject to

$$Rb + (p + d)k - c + pk' - b' = 0 \quad (5)$$



# Household problem

First-order conditions:

$$c : u'(c) = \lambda$$

$$k' : \lambda p = EV_k(k', b', d')$$

$$b' : \lambda = EV_b(k', b', d')$$

Envelope:

$$V_k = \lambda(p + d)$$

$$V_b = \lambda R$$

## Euler equations

$$\begin{aligned}u'(c_t) &= \beta E_t \{ u'(c_{t+1}) R_{t+1} \} \\ &= \beta E_t \{ u'(c_{t+1}) \underbrace{\frac{p_{t+1} + d_{t+1}}{p_t}}_{R_{t+1}^S} \}\end{aligned}$$

This is very general - holds for any number of assets / for any type of asset.

**Question:** Why doesn't the correlation of asset returns show up anywhere?

# Solution

- ▶ A solution consists of state contingent plans  $\{c(d^t), k(d^t), b(d^t)\}$  for all histories  $d^t$ .
- ▶ These satisfy:
  - ▶ 2 Euler equations
  - ▶ 1 budget constraint.
  - ▶  $b_0$  and  $k_0$  given.
  - ▶ Transversality:  $\lim_{t \rightarrow \infty} E_0 \beta^t u'(c_t) [b_t + p_t k_t] = 0$ .

## Market clearing

For every history we need:

Bonds:

$$b_t = 0$$

Trees:

$$k_t = K_t$$

Goods:

$$c_t = K_t d_t$$

There is no trade in equilibrium!

# Competitive Equilibrium

- ▶ A CE consists of:
  1. an allocation:  $\{c(d^t), b(d^t), k(d^t)\}$ .
  2. a price system:  $\{p(d^t), R(d^t)\}$
- ▶ These satisfy:
  1. household: 2 Euler equations and 1 budget constraint.
  2. 3 market clearing conditions.

# Recursive Competitive Equilibrium

Objects:

- ▶ Solution to the household problem:  $V(k, b, d)$  and  $c(k, b, d)$ ,  $k' = \kappa(k, b, d)$ ,  $b' = B(k, b, d)$ .
- ▶ Price functions:  $p(d), R(d)$ .

Equilibrium conditions:

- ▶ Household: 4
- ▶ Market clearing: 3
- ▶ No need for consistency: law of motion of the aggregate state is exogenous.

## Consumption smoothing

The Euler equation implies (for any asset):

$$E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} R_{t+1} \right\} = 1 \quad (6)$$

Define: Marginal rate of substitution:

$$MRS_{t+1} = \beta u'(c_{t+1})/u'(c_t) \quad (7)$$

$MRS_{t+1}$  is inversely related to consumption growth.

## Consumption smoothing

With  $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ :

$$u'(c) = c^{-\sigma} \quad (8)$$

$$MRS_{t+1} = \beta (c_{t+1}/c_t)^{-\sigma} \quad (9)$$

Euler equation:

$$\beta \mathbb{E}_t \{ (1 + g(c_{t+1}))^{-\sigma} R_{t+1} \} = 1 \quad (10)$$

High interest rate

⇒ reward for postponing consumption

⇒ high consumption growth

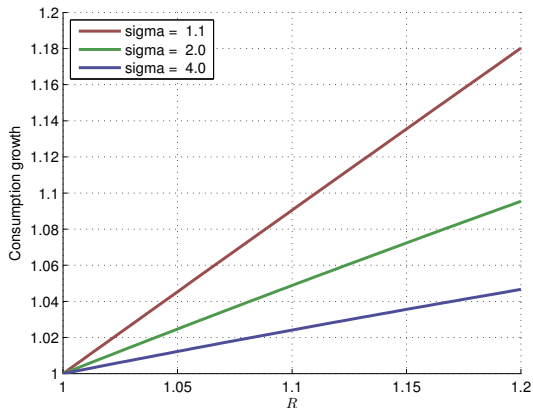
The **coefficient of relative risk aversion** ( $\sigma$ ) determines how much consumption growth responds to interest rates.



# Consumption smoothing

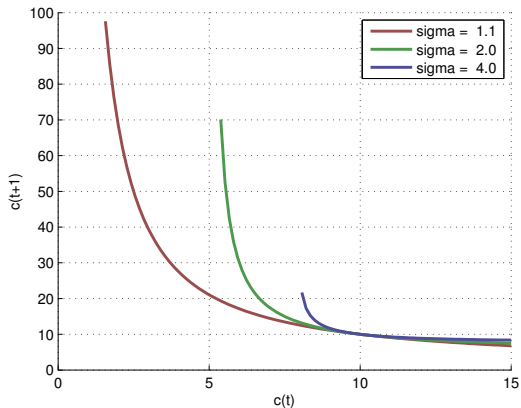
High  $\sigma$  implies that the household chooses smooth consumption.

Illustration for the deterministic case:



# Consumption smoothing

With high  $\sigma$ , marginal utility changes a lot when  $c$  changes.  
The household then keeps  $c$  smooth.



# Asset Prices

# Asset pricing implications

- ▶ We will now derive the famous **Lucas asset pricing equation**.
- ▶ Define: Rate of return on trees:  $R_{t+1}^S = (p_{t+1} + d_{t+1}) / p_t$ .
- ▶ Directly from the 2 Euler equations:

$$E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} R_{t+1} \right\} = E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} R_{t+1}^S \right\} = 1$$

- ▶ Or

$$E \{ MRS_{t+1} R_{t+1} \} = E \{ MRS_{t+1} R_{t+1}^S \} = 1 \quad (11)$$

## When does an asset pay a high expected return?

Re-write asset pricing equation using

$$Cov(x, y) = E(xy) - E(x)E(y)$$

as

$$1 = E\{MRS\} E\{R\} + Cov(MRS, R) \quad (12)$$

$$E(R) = \frac{1 - Cov(MRS, R)}{E(MRS)} \quad (13)$$

## When do assets pay high returns?

$$\mathbb{E}(R) = \frac{1 - \text{Cov}(MRS, R)}{\mathbb{E}(MRS)} \quad (14)$$

- ▶ Take a “safe” asset with fixed  $R$ .
  - ▶  $\text{Cov}(MRS, R) = 0$
  - ▶  $\mathbb{E}(R) = 1/\mathbb{E}(MRS)$ .
- ▶ If  $\text{Cov}(MRS, R) < 0$ : the asset pays higher return than the safe asset
  - ▶ a **risk premium**
- ▶ If  $\text{Cov}(MRS, R) > 0$ : the asset pays **lower** return than the safe asset
  - ▶ important point: an asset return can have lots of volatility, but pay a lower return than a t-bill
  - ▶ examples?

## When do assets pay high returns?

High returns require low / negative  $Cov(MRS, R)$ .

Example: log utility

- ▶  $u'(c) = 1/c$
- ▶  $MRS = \beta u'(c_{t+1})/u'(c_t) = \beta c_t/c_{t+1}$ .

High  $MRS$  means low consumption growth.

### Key implication

Assets are **risky** if their returns are **positively correlated with consumption growth**.

Then they have high **expected returns**.

Note: Uncertainty about  $R$  by itself is not priced.

- ▶ Only the part that is correlated with consumption growth is priced.

# Intuition

- ▶ Imagine there are good times (high  $c$ ) and bad times (low  $c$ ).
- ▶ There are 2 assets: A pays dividends in good times, B pays in bad times.
- ▶ The value of the dividend is  $u'(c)$ .
- ▶ Assets that pay in good times are not valuable:  $u'(c)$  is low.
- ▶ Assets that pay in bad times provide insurance - they are valuable (have low expected returns).



## Risk (premium)

- ▶ The "risk free" assets has expected return

$$E(R_f) = \frac{1}{E(MRS)} \quad (15)$$

- ▶ A "risky" asset has expected return

$$E(R) = \frac{1 - Cov(MRS, R)}{E(MRS)} \quad (16)$$

- ▶ The **risk premium** is

$$E(R) - E(R_f) = -\frac{Cov(MRS, R)}{E(MRS)} \quad (17)$$

- ▶ This defines what **risk** means: **covariance with consumption growth**.
- ▶ Note that risk can be **negative** (insurance).

# The Equity Premium Puzzle

- ▶ Mehra and Prescott (1985): Asset return data pose a puzzle for the theory.
- ▶ The equity premium is "high" (6-7% p.a.)
- ▶ The cov of  $c$  growth and  $R_s$  is low.
  - ▶ The reason: Consumption is very smooth.

# The Equity Premium Puzzle

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TABLE 1  
SUMMARY STATISTICS  
UNITED STATES ANNUAL DATA, 1889–1978

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Sample Means			
$R_t^s$		0.070	
$R_t^b$		0.010	
$C_t/C_{t-1}$		0.018	
Sample Variance-Covariance			
	$R_t^s$	$R_t^b$	$C_t/C_{t-1}$
$R_t^s$	0.0274	0.00104	0.00219
$R_t^b$	0.00104	0.00308	-0.000193
$C_t/C_{t-1}$	0.00219	-0.000193	0.00127

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# The Equity Premium Puzzle

A back-of-the envelope calculation with CRRA utility:

$$EP = - \frac{\text{Cov}(\beta [c_{t+1}/c_t]^{-\sigma}, R_s)}{E\{\beta [c_{t+1}/c_t]^{-\sigma}\}} \quad (18)$$

Take log utility:  $\sigma = 1$ .

- ▶  $\text{Cov}(MRS, R_s) \simeq -0.0022$ .
- ▶  $E(MRS) \simeq 1$ .
- ▶  $EP \simeq 0.2\%$ .
- ▶ Replicating the observed equity premium requires **very high risk aversion** ( $\sigma = 40$ ).

# How severe is the puzzle?

Investors forego very large returns.

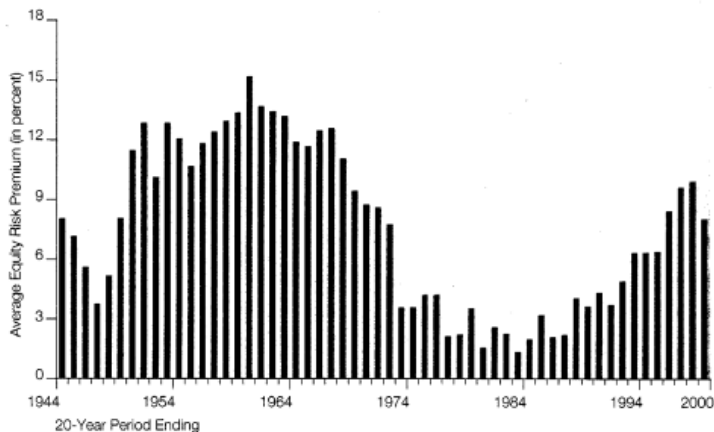
<b>Table 3</b>				
<b>Terminal value of \$1 invested in Stocks and Bonds</b>				
<b>Investment Period</b>	<b>Stocks</b>		<b>T-bills</b>	
	Real	Nominal	Real	Nominal
1802-1997	\$558,945	\$7,470,000	\$276	\$3,679
1926-2000	\$266.47	\$2,586.52	\$1.71	\$16.56

Source: Mehra and Prescott (2003)

## Long holding periods

Over 20 year holding periods: stocks dominate bonds.

Equity Risk Premium Over 20-Year Periods  
1926-2000



Source: Mehra and Prescott (2003)

# Why do we care?

- ▶ The EP puzzle shows that we do not understand
  1. what households view as "risky"
  2. why households place a high value on smooth consumption
- ▶ This has implications for:
  1. The welfare costs of business cycles
    - ▶ They are very low in standard models.
  2. Stock price volatility.
    - ▶ Standard models fail to explain it (see below).

# How to resolve the puzzle

Proposed explanations include:

1. Habit formation:  $u(c_t, c_{t-1}) = \frac{[c_t - \gamma c_{t-1}]^{1-\sigma}}{1-\sigma}$ .
  - ▶ Implies high risk aversion when  $c_t$  is close to  $c_{t-1}$ .
2. Heterogeneous agents
  - ▶ Implicit in the standard model: all idiosyncratic risk is perfectly insured.
3. Borrowing constraints
  - ▶ The young should hold stocks (long horizon), but cannot.
  - ▶ The old receive mostly capital income and find stocks risky.
4. Taxes / regulations (McGrattan and Prescott, 2000)
  - ▶ The runup in stock prices since the 1960s stems from lower dividend taxes & laws permitting institutional investors to hold equity.



# Beta

Now we derive the famous "beta" measure of risk.

Suppose asset *m* (the market) is perfectly correlated with marginal utility:

$$u'(c_{t+1}) = -\gamma R_{m,t+1} \quad (19)$$

The market's expected return is

$$E R_m - R = -\frac{\text{Cov}(MRS, R_m)}{E(MRS)} \quad (20)$$

## Beta

Now we relate the covariance term to marginal utility:

$$\begin{aligned} \text{Cov}(MRS, R_m) &= \text{Cov}\left(\frac{\beta u'(c_{t+1})}{u'(c_t)}, R_{m,t+1}\right) = \beta \frac{\text{Cov}(u'(c_{t+1}), R_{m,t+1})}{u'(c_t)} \\ E(MRS) &= \beta \frac{E(u'(c_{t+1}))}{u'(c_t)} \end{aligned}$$

Therefore:

$$E(R_m) - R = - \frac{\text{Cov}(u'(c_{t+1}), R_{m,t+1})}{E u'(c_{t+1})} = \frac{\gamma \text{Var}(R_{m,t+1})}{E u'(c_{t+1})}$$

## Beta

For any asset  $i$ :

$$E R_i - R = - \frac{\text{Cov}(u'(c_{t+1}), R_i)}{E u'(c_{t+1})} = \frac{\gamma \text{Cov}(R_m, R_i)}{E u'(c_{t+1})}$$

Take the ratio for assets  $i$  and  $m$ :

$$\beta_i = \frac{E R_i - R}{E R_m - R} = \frac{\text{Cov}(R_m, R_i)}{\text{Var}(R_m)} \quad (21)$$

Note:  $\beta_i$  is the coefficient of regressing  $R_i$  on  $R_m$  using OLS.

This is the famous **CAPM** asset pricing equation.

# Beta

- ▶ The risk premium for asset  $i$  depends on:
  - ▶ it's **beta** (essentially the correlation with the market)
  - ▶ the market price of risk:  $E R_m - R$ .
- ▶ A stock's beta can be estimated from data on past returns of the stock ( $R_i$ ) and the market (using a broad stock index).
- ▶ Betas are used to
  - ▶ Measure the risk of an asset.
  - ▶ Calculate the required rate of return for investment projects.
  - ▶ Evaluation of mutual fund managers.

## Securities market line

CAPM prediction:

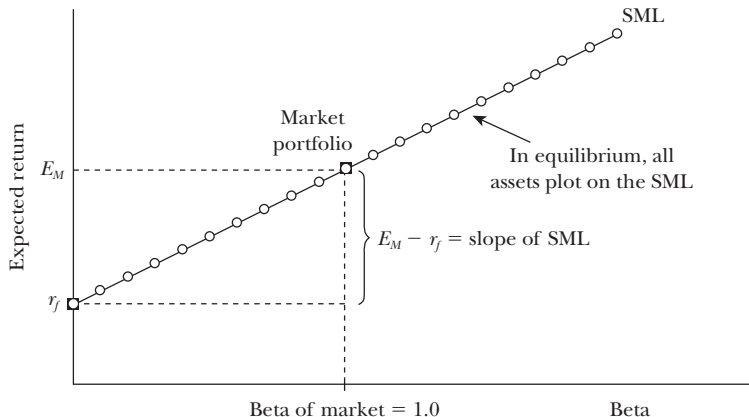
$$\mathbb{E}R_i = (1 - \beta_i)R + \beta_i\mathbb{E}R_m \quad (22)$$

$$= R + \beta_i\mathbb{E}\{R_m - R\} \quad (23)$$

If we plot expected returns against  $\beta$ s, we should get a straight line.  
This is called the **securities market line** (SML)

# Securities market line

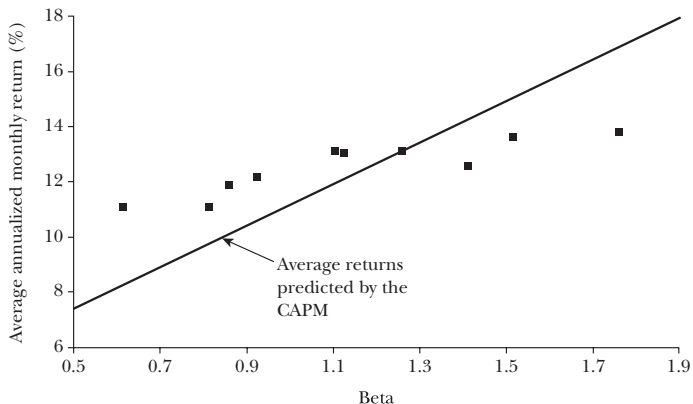
## The Securities Market Line (SML)



Source: Perold (2004)

# Securities market line: Evidence

**Average Annualized Monthly Return versus Beta for Value Weight Portfolios Formed on Prior Beta, 1928–2003**



Source: Fama (2004)

# Implications

Stocks with higher  $\beta$ s have higher expected returns, but the relationship is flatter than predicted.

Again: we don't understand how investors value / measure risk.

- ▶ a fundamental problem.

Oddly,  $\beta$  remains popular, even though it does not work in the data.



## Solving for the asset price

We show that the asset price equals the present discounted value of dividends

$$p_t = \mathbb{E}_t \sum_{j=1}^{\infty} d_{t+j} MRS(t, t+j) \quad (24)$$

The discount factor is the MRS, called the **stochastic discount factor**.

## Solving for the asset price

Start from the Euler equation:

$$u'(c_t) = \beta E_t \left\{ u'(c_{t+1}) \frac{p_{t+1} + d_{t+1}}{p_t} \right\} \quad (25)$$

Solve for the price:

$$p_t = E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} (p_{t+1} + d_{t+1}) \right\} \quad (26)$$

Replace  $p_{t+1}$  with (26) shifted to  $t+1$ :

$$\begin{aligned} p_t = & E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} d_{t+1} \right\} + \\ & E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} E_{t+1} \left[ \frac{\beta u'(c_{t+2})}{u'(c_{t+1})} (p_{t+2} + d_{t+2}) \right] \right\} \quad (27) \end{aligned}$$

## Solving for the asset price

The law of iterated expectations:

$$E_t \{E_{t+1}(x)\} = E_t(x) \quad (28)$$

Eliminate the  $E_{t+1}$ :

$$p_t = E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} d_{t+1} \right\} + E_t \left\{ \frac{\beta^2 u'(c_{t+2})}{u'(c_t)} (p_{t+2} + d_{t+2}) \right\} \quad (29)$$

Iterate forward for  $T$  periods:

$$p_t = E_t \left\{ \sum_{j=1}^T \frac{\beta^j u'(c_{t+j})}{u'(c_t)} d_{t+j} \right\} \quad (30)$$

$$+ E_t \left\{ \frac{\beta^T u'(c_{t+T})}{u'(c_t)} p_{t+T} \right\} \quad (31)$$

## Solving for the asset price

Impose that the last term vanishes in the limit:

$$p_t = E_t \left\{ \sum_{j=1}^{\infty} \frac{\beta^j u'(c_{t+j})}{u'(c_t)} d_{t+j} \right\} \quad (32)$$

- ▶ There is no good reason for this assumption!
- ▶ We will see later: other prices solve the asset pricing equation (bubbles)

The asset price equals the **discounted present value of dividends**.

The stochastic **discount factor** is the marginal rate of substitution.

## Example: Log Utility

In the Lucas model, assume:  $u(c) = \ln(c)$ .  $K = 1$ .

In equilibrium:  $c_t = d_t$ .

$$MRS_{t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)} = \frac{\beta d_t}{d_{t+1}}.$$

The asset pricing equation becomes

$$\begin{aligned} p_t &= E_t \left\{ \sum_{j=1}^{\infty} \frac{\beta^j d_t}{d_{t+j}} d_{t+j} \right\} \\ &= d_t \frac{\beta}{1 - \beta} \end{aligned}$$

## Example: Periodic dividends

In the Lucas model, assume:

- ▶ Utility is  $u(c) = c^{1-\sigma}/(1-\sigma)$ .
- ▶  $d_t$  alternates between  $d^H$  and  $d^L$ .

Asset pricing equation:

$$\begin{aligned} p_t &= \sum \beta^j (d_t/d_{t+j})^\sigma d_{t+j} \\ &= d_t^\sigma \sum \beta^j d_{t+j}^{1-\sigma} \end{aligned} \tag{33}$$

On good days,  $p_t$  is pulled up by low  $u'(c')$ , but is pushed down by low  $d_{t+1}$ .

# The Excess Volatility Puzzle

Consider a stock with dividend process  $d_t$ .

Its price is given by

$$p_t = E_t \left\{ \sum_{j=1}^{\infty} \frac{\beta^j u'(c_{t+j})}{u'(c_t)} d_{t+j} \right\} \quad (34)$$

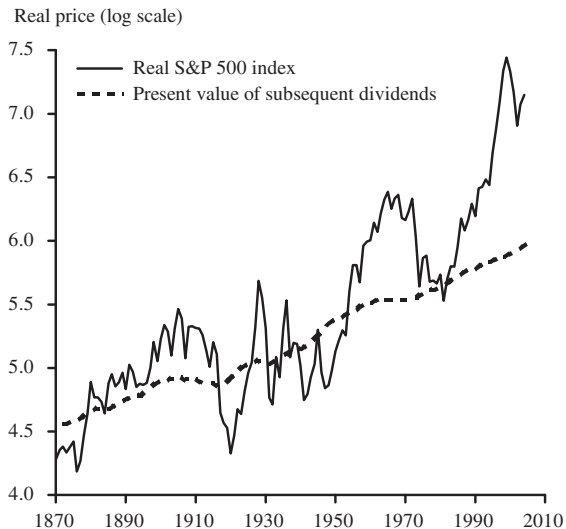
In the data:

- ▶ Dividends are very smooth (a goal of company policy).
- ▶ Stock prices are much more volatile than dividends.

But in the theory: stock prices should be the **average** of future dividends and thus **smoother** than dividends.

This is the flip-side of the Equity Premium Puzzle. See Shiller (1981)

# Excess Volatility



Source: FRBSBF Economic Letter Nov 2007



# Bubbles

- ▶ Recall how the asset pricing formula is derived:
- ▶ We iterate forward on the asset pricing Euler equation

$$p_t = E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} (p_{t+1} + d_{t+1}) \right\} \quad (35)$$

- ▶ We assume that the  $p_{t+1}$  term vanishes in the limit.
- ▶ What if it does not vanish?
- ▶ Then **any** (current) **asset price** can satisfy the asset pricing equation.
- ▶ The deviation between  $p_t$  and the **fundamental price** from (35) is called a **bubble**.
- ▶ It is purely a self-fulfilling expectation.

## Bubbles: Example

- ▶ Consider an asset that pays no dividends.
- ▶ Its **fundamental price** is 0.
- ▶ Assume that the MRS is constant at  $\frac{\beta u'(c_{t+1})}{u'(c_1)} = 1$ .
- ▶ The the asset pricing equation is

$$p_t = E_t p_{t+1} \quad (36)$$

- ▶ One price process that satisfies this:  $p$  doubles with probability 1/2 and drops to 0 otherwise.
- ▶ This satisfies (36) for **any**  $p_t$ .
- ▶ Bubbles are a possible explanation for asset price volatility.
- ▶ Note that the bubble does not offer any excess return opportunities.

# Reading

- ▶ Romer (2011), ch. 7.5
- ▶ Ljungqvist and Sargent (2004), ch. 7.
- ▶ On the equity premium puzzle: Mehra and Prescott (1985, 2003)

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