Problem Set 5: Growth Model in Continuous Time

Econ720. Fall 2022. Prof. Lutz Hendricks

1 Continuous Time CIA Model. Cash and Credit Goods.

Demographics: A single representative household who lives forever.

Preferences:

$$\int_0^\infty e^{-\rho t} u(c_t, g_t)\, dt$$

where c and g are two consumption goods.

Technology:

$$f(k) = \dot{k} + c + g \tag{1}$$

Government: The government costlessly produces money M and hands it to households as lump-sum transfers. The money growth rate is constant at $g(M)$. The government budget constraint is $\dot{M} = p\,x = g(M)\,M$ where p is the price level and x is the real lump-sum transfer.

Markets: There are competitive markets for goods and money. Households operate the technology (there are no firms).

CIA constraint: c has to be bought with cash:

$$c_t \leq M_t/p_t$$

g may be bought with credit.

Denote real balances by $m_t = M_t/p_t$.

Questions

1. Write down the household's Hamiltonian. Which are his states and controls? Derive first-order conditions for two cases: either the CIA constraint always binds or it never binds. Hint: the budget constraint is given by
$$\dot{k}_t + c_t + g_t + \dot{M}_t/p_t = f(k_t) + x_t$$
2. Define a competitive equilibrium.
3. Derive a set of equations that characterize the steady state. Show that the nominal interest rate equals zero, if the CIA constraint does not bind.
4. Determine the effects of a higher money growth rate on the steady state allocation. Assume that the utility function takes the form $u(c, g) = U(c) + V(g)$, where U and V are strictly concave functions.

1.1 Answer: Continuous Time CIA Model. Cash and Credit Goods.

1. The state is $a = m + k$. The controls are c, g, m. The Hamiltonian is
$$H = u(c, g) + \lambda\,[f(a - m) + x - c - g - \pi\,m] + \varphi\,[m - c]$$
where $\pi = \dot{p}/p$ is the inflation rate. First-order conditions are:
$$\begin{aligned} u_c &= \lambda + \varphi \\ u_g &= \lambda \\ \lambda\,(f'(a-m) + \pi) &= \varphi \\ \dot{\lambda} &= \lambda\,(\rho - f'(a-m)) \end{aligned}$$

These can be simplified as follows.

$$\begin{aligned} g(\lambda) &= \rho - f'(k) = g(u_g) \qquad (2) \\ u_c &= u_g \; [1 + f'(k) + \pi] \qquad (3) \end{aligned}$$

A solution to the household problem then consists of a set of functions (c_t, g_t, m_t, k_t) that solve the first-order conditions (2) and (3), the budget constraint, and the CIA constraint if it binds, or $u_c = u_g$ if it does not.

2. A competitive equilibrium is a set of functions $(c_t, g_t, m_t, k_t, p_t, M_t, x_t)$ that solve 4 household equations, $m = M/p$, $\dot{M} = p\,x = g(M)\,M$, and goods market clearing $f(k) = c + g + \dot{k}$.

3. The steady state consists of scalars (c, g, m, k, π, x) which are determined near-recursively as follows. A constant real money stock requires $\pi = g(M)$. The Euler equation determines the capital stock: $f'(k) = \rho$. The two consumption flows are determined by $f(k) = c + g$ and $u_c/u_g - 1 = \rho + \pi$.

   If the CIA constraint does not bind, then the last equation is $u_c = u_g$, so that $\pi = -\rho$ and the nominal interest rate is zero. This can only happen, if the money growth rate is set to $g(M) = -\rho$. If the CIA constraint binds, then $m = c$.

4. If the CIA constraint does not bind, then it will bind once the money growth rate is increased. So I only consider the effects when the CIA constraint binds. Clearly, money growth does not affect k. The effect on consumption is determined by $f(k) = c + g$ and $U'(c)/V'(g) = 1 + \rho + g(M)$. It is easy to see that higher money growth increases g and reduces c. Since $m = c$, real balances decline as well. Standard substitution is the intuition.

2 Money in the Utility Function

Demographics: Time is continuous. There is a single representative household who lives forever.

Preferences:

$$\int_{t=0}^{\infty} e^{-\rho t} u(c_t, m_t) dt \qquad (4)$$

where c is consumption and m denotes real money balances.

Endowments: Households work 1 unit of time at each instant. Households are initially endowed with k_0 units of capital and m_0 units of real money.

Technology:

$$f(k_t) - \delta k_t = c_t + \dot{k}_t \qquad (5)$$

Money: nominal money grows at exogenous rate $g(M)$. New money is handed to households as a lump-sum transfer: $\dot{M}_t = p_t x_t$.

Markets: money (numeraire), goods, capital rental (price r), labor (w).

Questions:

1. The household's budget constraint is given by

$$\dot{k}_t + \dot{m}_t = w + r_t k_t + x_t - c_t - \pi_t m_t - \phi(\dot{m}_t) \qquad (6)$$

   where $\phi(\dot{m}_t)$ is the cost of adjusting the money stock. $\phi'(0) = 0$ and $\phi''(\dot{m}_t) > 0$. State the Hamiltonian. If you cannot figure this out, assume $\phi(\dot{m}) = 0$ and proceed (for less than full credit). Hint: Make m and k separate state variables.

2. State the first-order conditions.

3. Define a competitive equilibrium.

4. Characterize the steady state to the extent possible. What is the effect of a permanent change in $g(M)$?

5. What is the optimal rate of inflation? Explain.

2.1 Answer: Money in the Utility Function

1. We have to invent a control $z = \dot{m}$. Then

$$H = u(c, m) + \lambda \left[w + rk + x - c - \pi m - z - g(z) \right] + \mu z \tag{7}$$

2. FOCs

$$\begin{aligned} u_c &= \lambda & (8) \\ \lambda \left[\phi'(z) + 1 \right] &= \mu & (9) \\ \dot{\lambda} &= (\rho - r)\lambda & (10) \\ \dot{\mu} &= \rho\mu - u_m + \lambda\pi & (11) \end{aligned}$$

3. CE: $\{c, k, m, z, \lambda, \mu; w, r, \pi\}$ that satisfy

   - household: 4 focs, 2 constraints, boundary conditions
   - firm: standard focs
   - goods market: feasibility
   - capital and labor markets: implicit
   - money growth: $g(M) = g(m) + \pi$

4. The BGP is recursive: The Euler equation fixes $r = \rho$. The firm's foc fixes k. From k we have w, output, and $c = f(k) - \delta k$. Money is super-neutral.

   Constant real money fixes $\pi = g(M)$. That only leaves the money demand equation $u_m = (\rho - g(M))u_c$ to determine m. Higher $g(M)$ raises inflation and changes money demand, m. Real variables are not affected.

5. Optimal inflation rate: Friedman rule. Saturate the household with money.