Comparative Dynamics

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Econ720

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Comparative Dynamics

- We use phase diagrams to uncover the dynamic response to shocks.
- We study tax changes in a growth model.

Model

The **household** solves

$$\max \int_0^{\infty} e^{-\rho t} u(c_t)\, dt \tag{1}$$

subject to

$$\dot{k}_t = r_t k_t + w_t - c_t - \tau_t \tag{2}$$

and k_0 given.

Firms produce output using $F(K, L)$.

The **government** uses the tax revenue to finance goverment spending: $G_t = \tau_t$.

Competitive Equilibrium

A competitive equilibrium consists of functions $c(t), k(t), \tau(t), w(t), r(t)$ that satisfy:

1. Household: Budget constraint and

$$g(c) = \frac{r - \rho}{\sigma} \tag{3}$$

2. Firms:

$$\begin{aligned} r &= f'(k) - \delta && (4) \\ w &= f(k) - f'(k)k && (5) \end{aligned}$$

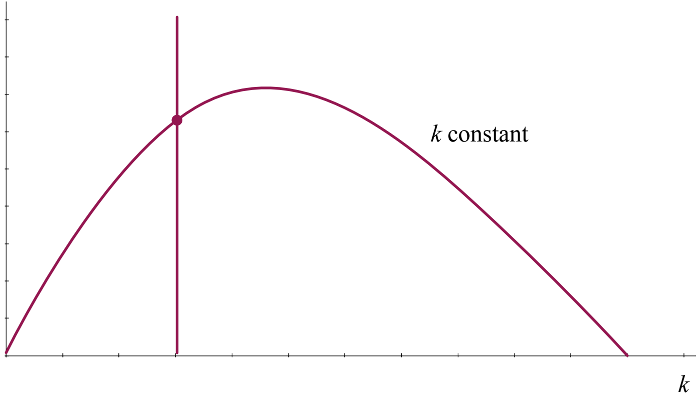
3. Government:

$$\tau = G \tag{6}$$

4. Market clearing:

$$\dot{k} = f(k) - \delta k - c - G \tag{7}$$

Phase Diagram



The only change relative to the model without government:
G shifts the $\dot{k} = 0$ locus down

Permanent Tax Increase

Consider a permanent, unannounced increase in G.

In the phase diagram

- $\dot{k} = 0$ locus shifts down by ΔG.
- k_{ss} remains unchanged because the $\dot{c} = 0$ locus does not shift.

Dynamics: c_{ss} drops to the new saddle path, then moves along it.

- How do I know this is true?

An interesting long-run result: full crowding out of consumption $(\Delta c_{ss} = -\Delta G)$.

Permanent Tax Increase

Key point

The model is almost entirely forward looking.
Only the state variable k links to the past.

Once the shock has occurred, the old phase diagram is forgotten.

Temporary Tax Increase

Consider a *temporary*, unannounced increase in G.

The economy starts on the saddle path for $G_t = G^*$

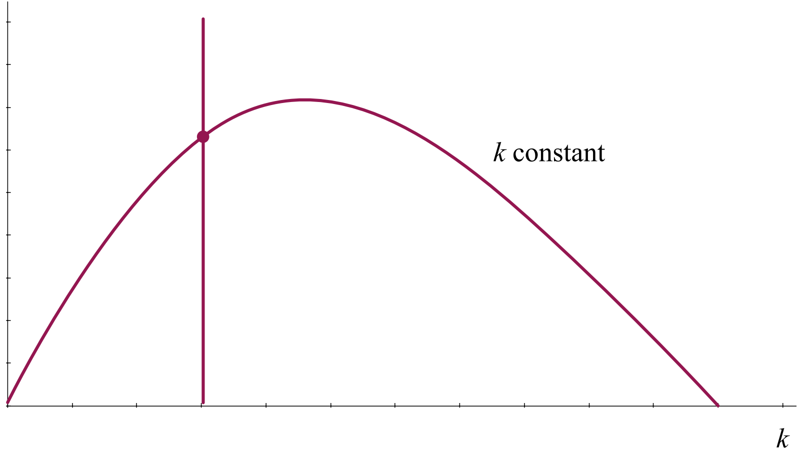
At $t = 0$, agents learn that

- G rises until date T: $G_t = G^* + \Delta G$ for $0 \leq t \leq T$
- then G falls back to $G_t = G^*$ for $t > T$.

To find the dynamics, we work backwards.

- Changes occur at $t = 0$ and $t = T$.

Temporary Tax Increase



Temporary Tax Increase

Step 1: $t = T$.

What does the equilibrium look like?

Temporary Tax Increase

Step 2: $0 < t < T$:

- The phase diagram with taxes applies.
- But the economy is not on the saddle path (why not?).
- What is the right terminal condition for $k(T)$?

Temporary Tax Increase

Step 3: $t = 0$

- The $\dot{k} = 0$ locus shifts down.
- Is c_0 on the saddle path?

Consider $k_0 = k_{ss}$. What paths are feasible?

Temporary Tax Increase

Consider $k_0 < k_{ss}$.

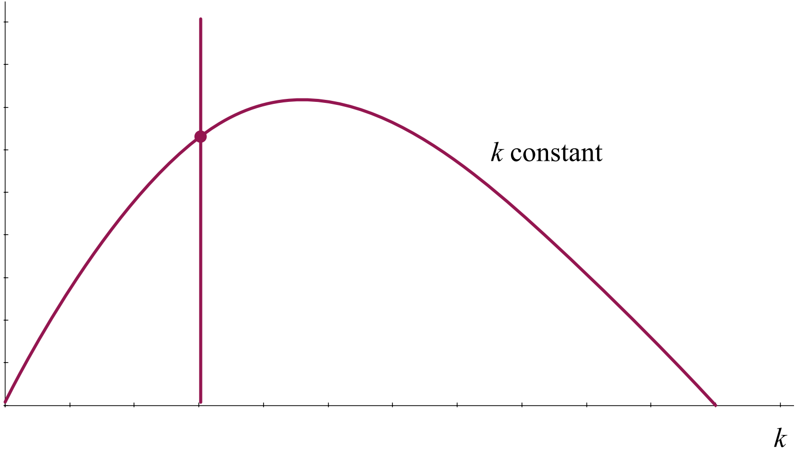
- c_0 drops.
- It cannot drop below the new saddle-path because then it would not reach the old saddle-path at T.

The economy must move north-west and reach the old saddle-path at T.

Announced Tax Cut

- Consider a surprise tax cut that is announced to take place at date T.
- At $t = 0$ the news arrives that taxes remain high until $t = T$, but then fall permanently.
- Again, we work backwards.
- Changes occur at $t = T$ and $t = 0$.

Announced Tax Cut



Summary

To study the dynamic effects of shocks:

1. Find the phase diagram with and without shock.
2. Find the dates at which changes occur:
   2.1 when the shock hits: phase diagram changes
   the control (typically) does not jump
   2.2 when new info arrives: agents reoptimize
   the control jumps
3. Work backwards, starting at the last date at which a change occurs

Phase Diagram for a Simple Human Capital Model

A Human Capital Model

We study the decision of a household how much human capital to accumulate.

This example illustrates two complications:

1. finite horizons
2. binding inequality constraints.

Household problem

The household maximizes

$$\int_0^T e^{-\rho t} u(c(t))\, dt \tag{8}$$

subject to the budget constraint

$$c(t) = w(t)\, h(t)\, [1 - \tau(t)\, v(t)] \tag{9}$$

the human capital technology

$$\dot{h}(t) = v(t) - \delta\, h(t) \tag{10}$$

and $v \geq 0$.

For simplicity, assume that $v \leq 1$ never binds.

Household: Intuition

- Human capital acquired early is more valuable for two reasons:
  1. it lives longer (date T is farther off);
  2. its payoffs are discounted by less.
- We expect the optimal path for $v(t)$ to be falling over time.
- When close to T, we expect $v(t) \geq 0$ to bind.

Hamiltonian

$$H = u(wh[1-\tau v]) + \lambda\ [v - \delta h] \tag{11}$$

First-order conditions

$$u'(c)wh\tau \geq \lambda \tag{12}$$

with equality if $v > 0$ and

$$\dot{\lambda} = \rho\lambda - u'(c)w(1-\tau v) + \lambda\,\delta \tag{13}$$

Summary

The solution to the household problem consists of functions (c, h, v, λ) that solve

1. The first-order conditions

$$
\begin{aligned}
u'(c)wh\tau &\geq \lambda & (14)\\
\dot{\lambda} &= (\rho+\delta)\lambda - u'(c)w(1-\tau v) & (15)
\end{aligned}
$$

   with equality if $v > 0$.

2. The budget constraint

$$c(t) = w(t)\,h(t)\,[1-\tau(t)\,v(t)] \qquad (16)$$

3. The law of motion

$$\dot{h}(t) = v(t) - \delta\, h(t) \qquad (17)$$

4. The boundary conditions: h_0 given and $\lambda_T = 0$.

Log utility

Assume $u(c) = \ln(c)$

Consider two regions of the phase diagram:

1. $v = 0$
2. $v > 0$

Region $v = 0$

$$wh\tau \geq \lambda c \tag{18}$$
$$\dot{\lambda} = (\rho + \delta)\lambda - w/c \tag{19}$$
$$c = wh \tag{20}$$
$$\dot{h} = -\delta h \tag{21}$$

Simplify:

$$\tau \geq \lambda \tag{22}$$
$$\dot{\lambda} = (\rho + \delta)\lambda - 1/h \tag{23}$$
$$\dot{h} = -\delta h \tag{24}$$

Terminal condition: $\lambda_T = 0$

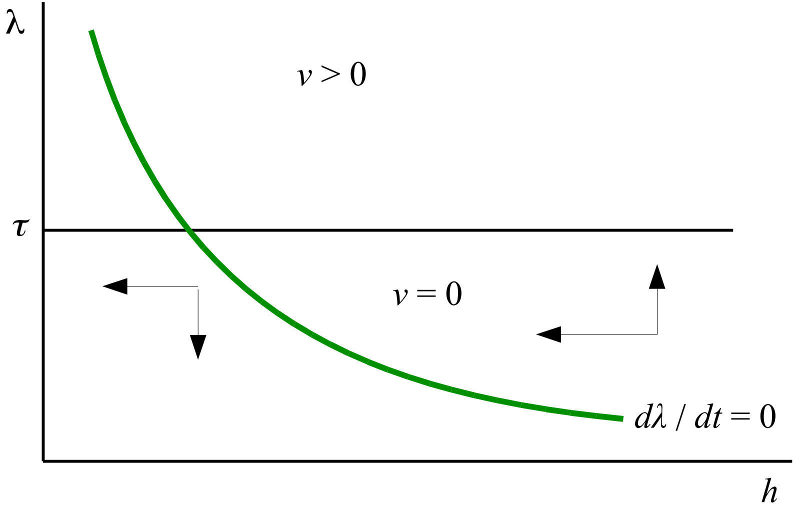
Region v = 0

- The shadow price λ is not large enough to cover the opportunity cost τ.
- The household does not invest in human capital.
- The laws of motion are:

$$\begin{aligned} \dot{\lambda} &= (\rho+\delta)\lambda - 1/h \\ \dot{h} &= -\delta h \end{aligned}$$

- $\lambda \uparrow \Rightarrow \dot{\lambda} \uparrow$.
- $h \uparrow \Rightarrow \dot{\lambda} \uparrow$ and $\dot{h} \downarrow$.
- Hence, $h(t) = h(t_0)\, e^{-\delta\,(t-t_0)}$, where t_0 is any date at which the economy is inside the region.

Phase Diagram: Region v = 0



Region $v > 0$

$$\frac{wh\tau}{c} = \lambda \tag{25}$$

$$\dot{\lambda} = (\rho + \delta)\lambda - \frac{w(1 - \tau v)}{c} \tag{26}$$

$$c = wh(1 - \tau v) \tag{27}$$

$$\dot{h} = v - \delta h \tag{28}$$

Simplify:

$$\begin{aligned}
\lambda &= \frac{\tau}{1 - \tau v} \\
\dot{\lambda} &= (\rho + \delta)\lambda - 1/h \\
\dot{h} &= v - \delta h
\end{aligned}$$

Region $v > 0$

The first-order condition for v holds with equality:

$$\lambda\,(1-\tau v) = \tau$$

or

$$v = 1/\tau - 1/\lambda \tag{29}$$

Substitute v out of the law of motion:

$$\dot{h} = 1/\tau - 1/\lambda - \delta\, h \tag{30}$$

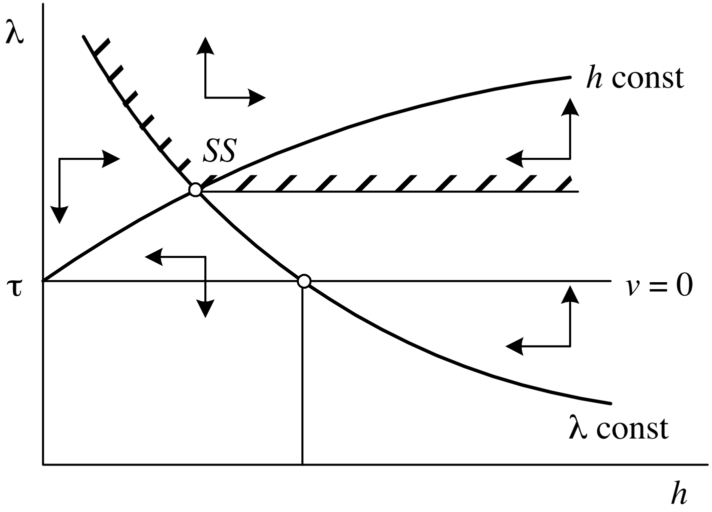
Keep

$$\dot{\lambda} = (\rho+\delta)\lambda - 1/h \tag{31}$$

Region $v > 0$

- In this region, the shadow price of human capital (λ) equals the opportunity cost.
  $\lambda > \tau$
- $\dot{h} = 1/\tau - 1/\lambda - \delta h = 0$
  is upward sloping and starts at $\lambda = \tau$.
- $\dot{\lambda} = (\rho + \delta)\lambda - 1/h = 0$
  is a downward sloping hyperbola (as in region $v = 0$).
- $h \uparrow$ or $\lambda \downarrow \Rightarrow \dot{h} \downarrow$.

Phase Diagram



Steady State

- Assume that w and τ are constant over time and that $T = \infty$.
- Then h and v converge to stationary levels, h_{ss} and v_{ss}.
- We next determine those levels.
- $\dot{\lambda} = 0$ implies

$$(\rho + \delta)\, h\lambda = (\rho + \delta)\, h \frac{\tau}{1 - \tau v} = 1 \tag{32}$$

- $\dot{h} = 0$ implies

$$v = \delta h \tag{33}$$

- Combine both

$$h_{ss} = [\tau(\rho + 2\delta)]^{-1} \tag{34}$$

Steady State

It follows that

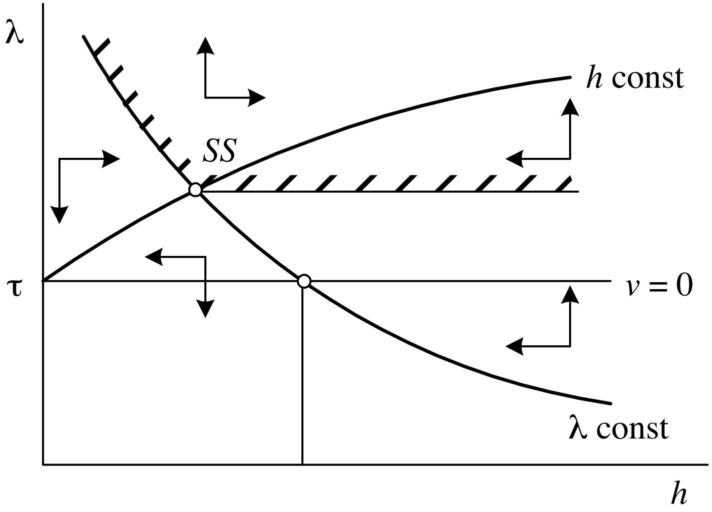
$$\begin{aligned} v_{ss} &= \delta\, h_{ss} = \frac{\delta}{\tau[\rho + 2\,\delta]} \\ c_{ss} &= \frac{(\rho+\delta)\,w}{(\rho+2\,\delta)^2\,\tau} \end{aligned}$$

$$\lambda_{ss} = u'\left(\frac{(\rho+\delta)\,w}{(\rho+2\,\delta)^2\,\tau}\right)\frac{w}{\rho+2\,\delta}$$

Phase Diagram

- The phase diagram has two regions: $v = 0$ and $v > 0$.
- The region boundary occurs when the household just hits the constraint $v \geq 0$: at $\lambda = \tau$.
- For $\lambda > \tau$: $v > 0$.
- For $\lambda \leq \tau$: $v = 0$.

Phase Diagram



Dynamics

- Any path must end with $\lambda_T = 0$ exactly at date T.
- It follows that the shaded region must never be entered.
- What happens as the steady state is approached with $v > 0$?
  - Since all the laws of motion are continuous, $\dot{h} \to 0$ and $\dot{\lambda} \to 0$.
  - The steady state can never be reached.
  - But the economy can spend an arbitrarily long time arbitrarily close to the steady state.

Dynamics

- First consider $h_0 < h_{ss}$.
- λ depends on the horizon T.
- Short T: λ is low.
  - Start in region $v = 0$
  - move south-west until $\lambda_T = 0$.

Dynamics

- To prove this, solve the two differential equations.
- $h(t) = h(t_0)\, e^{-\delta t}$. Substitute this into the law of motion for λ to obtain

$$\dot{\lambda} = (\rho + \delta)\, \lambda - e^{\delta t} / h(t_0) \tag{35}$$

- The solution to this differential equation is

$$\lambda(t) = e^{(\rho+\delta)(t-t_0)} \left[\lambda(t_0) - \frac{\rho}{h(t_0)} \left\{ 1 - e^{-\rho(t-t_0)} \right\} \right]$$

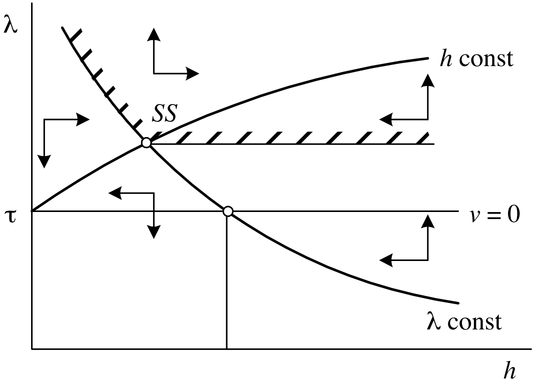
- Imposing the boundary condition $\lambda(T) = 0$ implies $\lambda(t_0)\, h(t_0) = \rho \left\{ 1 - e^{-\rho(T-t_0)} \right\}$.
- For long T: $\lambda(t_0) \to \rho / h(t_0)$ (unless the region $v = 0$ is left).
- But for a short T, $\lambda(t_0) \to 0$.

Dynamics

- Case: $h_0 < h_{ss}$ and long T.
- Initially $v > 0$ and the economy moves south until it crosses into the $v = 0$ region.
- As $T \to \infty$ something bizarre happens:
  - the economy approaches the steady state without ever reaching it.
  - It comes arbitrarily close and stays arbitrarily close for an arbitrarily long time.
  - But when the terminal date comes sufficiently close it leaves the steady state and moves south-west to reach $\lambda_T = 0$.

Dynamics

- Case $h_0 > h_{ss}$.
- Investment is never large enough to increase h.
- The economy may move straight south-west if T is short or it may move towards the steady state, similar to the case where $h_0 < h_{ss}$.



Reading

- Acemoglu, Introduction to modern economic growth, ch. 8.7.
- Hendricks, Lutz (2004). "Taxation and Human Capital Accumulation." *Macroeconomic Dynamics* 8(3): 310-334.
- Sheshinski, Eytan (1968), "On the Individual's Lifetime Allocation Between Education and Work," *Metroeconomica*, 20(1), 42-9.