Asset Pricing: Extensions

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State contingent claims

- Some assets pay out only in particular states of the world
  - e.g. insurance contracts
- Standard asset pricing formulas apply to those assets.
- It just adds notation...

State contingent claims

- We start from the Lucas fruit tree model.
- In addition to stocks and bonds, households can purchase assets that pay out in exactly one state of the world.
  - Arrow securities or state contingent claims
- Their role in theory:
  - given a sufficiently rich set of Arrow securities, we can replicate any asset
  - can set up a model with all possible insurance opportunities (complete markets)

Notation

- quantity purchased of asset that pays out in state d': $y'(d'|d)$.
  - for convenience just write $y'(d')$
- price of that asset: $q(d'|d)$.

Household

States: all assets held, $k, b,$ and all $y(d)$.

- call that s

Choices: $b', k', y'(d')$ for all d'.

Dynamic Program:

$$V(s,d) = \max_{c,k',b',y'(d')} u(c) + \beta EV(s',d')$$

subject to

$$Rb + (p+d)k + y(d) = c + b' + pk' + \sum_{d'} q(d'|d)\, y'(d')$$

Note: only the y matching the realized value of d pays out.

First-order conditions for state contingent claims

Applying the Lucas asset pricing equation:

$$1 = \mathbb{E}\left\{MRS_{t+1}\frac{\mathbb{I}(d')}{q(d',d)}\right\}$$

Special feature of Arrow securities: Only one term in the $\mathbb{E}$ is non-zero:

$$1 = \Pr\left(d'|d\right)\frac{\beta u'(c[s',d'])}{u'(c[s,d])}\frac{1}{q(d'|d)} \tag{1}$$

where the rate of return on the state contingent claim is $1/q$.

The long derivation

FOC:

$$u'(c)\,q\left(d'|d\right) = \beta \,\mathsf{Pr}\left(d'|d\right) V_{y(d')}\left(s', d'\right)$$

Envelope:

$$\begin{aligned} V_{y(d)}(s,d) &= u'(c) && (2) \\ V_{y(d)}\left(s,\hat{d}\right) &= 0, \;\; \hat{d} \neq d && (3) \end{aligned}$$

Note: only the y matching the realized value d has value.

$$u'(c[s,d])\,q\left(d'|d\right) = \beta \,\mathsf{Pr}\left(d'|d\right) u'\left(c[s',d']\right) \qquad (4)$$

Replicating Assets

Note how we can replicate any asset with a combination of state contingent claims:

- risk free bond = hold one claim for every d'
- capital = hold $R'(d')$ claims for every d'

Adding Bonds

Adding Bonds

We add bonds of different maturities to the Lucas model

In each period, maturity n bonds are issued.

- They pay 1 unit of consumption n periods from now.

Bonds issued earlier are traded in secondary markets.

- for maturities $i = 1, ..., n$.
- prices are $p_{t,i}$.

A maturity 1 bond becomes a maturity 0 bond next period.

- It pays 1 unit of consumption and then disappears.

Household Problem

State variables in period t:

- s_t shares
- $b_{t,i}$ bonds for $i = 0, ..., n-1$
- because a maturity $i+1$ bond purchased yesterday is a maturity i bond today.

Controls in period t:

- s_{t+1}: share purchases
- $b_{t+1,i}$ for $i = 0, ..., n-1$: bond purchases
- c_t: consumption

Dynamic Program

$$V(s, b_0, ..., b_{n-1}; d) = \max u(c) + \mathbb{E}\beta V(s', b'_0, ..., b'_{n-1}; d') \quad (5)$$

subject to the budget constraint

$$c + \sum_{i=1}^{n} p_i b'_{i-1} + ps' = (p+d)s + \sum_{i=0}^{n-1} p_i b_i \quad (6)$$

First-order conditions for stocks are standard.

First order conditions: bonds

$$b_i' : u'(c)p_{i+1} = \beta \mathbb{E} V_{b_i}(.') \tag{7}$$

Envelope:

$$V_{b_i} = u'(c)p_i \tag{8}$$

Euler equation:

$$u'(c)p_{i+1} = \beta \mathbb{E} u'(c')p_i' \tag{9}$$

Lucas asset pricing equation

$$1 = \mathbb{E}\left\{\frac{\beta u'(c')}{u'(c)}\frac{p_i'}{p_{i+1}}\right\} \tag{10}$$

The bond return is p_i'/p_{i+1}

- because buying a bond of maturity $i+1$ today gives a bond of maturity i tomorrow

Bond prices

The price sequence of a given bond is:

$$p_{n-1,t}, p_{n-2,t+1}, p_{n-3,t+2}, \ldots, 1 \tag{11}$$

Solve this by backward induction:

$$p_0 = 1 \tag{12}$$

Sub that into the Euler equation and iterate to find

$$p_{t,i} = \beta^i \mathbb{E} \frac{u'(c_{t+i})}{u'(c_t)} \tag{13}$$

with $c_t = d_t$.

Bond prices

These are actually the standard Lucas asset pricing equations

The per period return on the bond is $1+r_{t,i} = (1/p_{t,i})^{1/i}$

Therefore:

$$u'(c_t) = \beta^i \mathbb{E} u'(c_{t+i})(1+r_{t,i})^i \tag{14}$$

$r_{t,i}$ is not stochastic and $Eu'(c_{t+i}) = Eu'(d_{t+i})$ does not depend on the current state d.

Yield curve

- Yield: $1+r_{t,i}=\left[u'(c_t)/Eu'(c_{t+i})\right]^{1/i}/\beta$
- With iid dividends: high consumption implies low yields for all maturities
- When c is above average $(u'(c_t)<\mathbb{E}u'(c_{t+i}))$, the yield curve is downward sloping
- This is consistent with data (the yield curve "predicts" slow growth).

Reading

- Romer (2011), ch. 7.5
- Ljungqvist and Sargent (2004), ch. 7.

References I

Ljungqvist, L. and T. J. Sargent (2004): *Recursive macroeconomic theory*, 2nd ed.

Romer, D. (2011): *Advanced macroeconomics*, McGraw-Hill/Irwin.