The Growth Model in Continuous Time
Competitive Equilibrium

Prof. Lutz Hendricks

Econ720

September 22, 2022

Competitive Equilibrium

- Firms solve the same problem as in the Solow model.
- We add a government that imposes lump-sum taxes to finance government spending.
- The budget constraint is $\tau_t = G_t$.

Households

$$\max \int_{t=0}^{\infty} e^{-(\rho-n)t} u(c_t) dt \tag{1}$$

subject to: k_0 given, the TVC, and the budget constraint

$$\dot{k}_t = w_t + (r_t - \delta - n)k_t - c_t - \tau_t \tag{2}$$

Households

Hamiltonian:

$$H = u(c) + \lambda[w + (r - \delta - n)k - c - \tau] \tag{3}$$

First-order conditions

$$\partial H/\partial c = 0 \Rightarrow u'(c) = \lambda \tag{4}$$

$$\begin{aligned} \dot{\lambda} &= (\rho - n)\lambda - \partial H/\partial k \\ &= \lambda[\rho - n - (r - \delta - n)] \\ &= \lambda(\rho - r + \delta) \end{aligned}$$

Transversality:

$$\lim_{t \to \infty} e^{-(\rho - n)t} \lambda_t k_t = 0 \tag{5}$$

Households

Eliminate λ:

$$u''(c)\dot{c} = \dot{\lambda} \tag{6}$$

Substitute into the law of motion for λ:

$$\dot{c} = u'(c)/u''(c) \cdot [\rho + \delta - r]$$

or

$$g_c = (r - \delta - \rho)/\sigma \tag{7}$$

Solution: Functions c_t, k_t that solve the Euler equation, the budget constraint, and the boundary conditions.

Competitive Equilibrium

Objects: Functions $c_t, k_t, \tau_t, w_t, r_t$.

Equilibrium conditions:

- Household (2)
- Firm (2)
- Government (1)
- Market clearing (1)

Dynamics

Simplify to obtain two differential equations:

$$\dot{c} = u'(c)/u''(c) \cdot [\rho + \delta - f'(k)] \tag{8}$$

$$\dot{k} = f(k) - (n+\delta)k - c - G \tag{9}$$

The planning solution and the CE coincide (with $G = 0$).

Detrending the Model

Detrending a model

Consider the growth model with productivity growth:

$$\max \int_0^\infty e^{-(\rho-n)t} u(c_t) dt \tag{10}$$

$$\dot{k}_t = F(k_t, A_t) - (n+\delta)k_t - c_t \tag{11}$$

with

$$A_t = e^{gt} \tag{12}$$

What does the Planner's solution look like?

The problem: the model has no steady state.

How can we analyze its dynamics?

Approach 1: Solve and detrend

Unchanged: the Planner's optimality conditions in terms of original variables:

$$\dot{c}/c = \frac{\frac{\partial F(k,A)}{\partial k} - n - \delta - (\rho - n)}{\sigma(c)} \tag{13}$$

But we cannot draw the phase diagram without a steady state.

Solution: detrend the variables to make them stationary.

1. Find the balanced growth rate for each variable.
   1.1 Divide each variable by a scale factor that grows at its balanced growth rate.

Balanced growth rates

- The same as in the Solow model with growth:

$$g(c) = g(k) = g \tag{14}$$

- Define the detrended variables:

$$\begin{aligned} \tilde{c}_t &= c_t / A_t & (15) \\ \tilde{k}_t &= k_t / A_t & (16) \end{aligned}$$

- Law of motion:

$$\begin{aligned} g\left(\tilde{k}\right) &= g(k) - g \\ &= \frac{F\left(\tilde{k}, 1\right)A - (n+\delta)\tilde{k}A - \tilde{c}A}{k} - g \\ d\tilde{k}/dt &= f\left(\tilde{k}\right) - (n+\delta+g)\tilde{k} - \tilde{c} & (17) \end{aligned}$$

Detrended first-order conditions

- Optimality conditions in terms of detrended variables:

$$\begin{aligned} \frac{d\tilde{c}/dt}{\tilde{c}} &= \frac{\dot{c}}{c} - g \\ &= \frac{f'\left(\tilde{k}\right) - \delta - \rho}{\sigma\left(c\right)} - g \end{aligned} \tag{18}$$

- This is true because

$$\frac{\partial F\left(k, A\right)}{\partial k} = \frac{\partial F\left(\tilde{k}A, A\right)}{\partial \tilde{k}} \frac{\partial \tilde{k}}{\partial k} = Af'\left(\tilde{k}\right)\frac{1}{A} \tag{19}$$

Detrended first-order conditions

- Assume CRRA preferences:

$$u(c) = c^{1-\sigma}/(1-\sigma) \tag{20}$$

- Then $\sigma(c) = \sigma$ is constant.
- **CRRA is required for balanced growth** - an important result.
  - Otherwise $\sigma(c)$ is not constant.

Model Solution

Functions of time c_t, k_t that satisfy:

1. Euler equation

$$g(\tilde{c}) = \frac{f'\left(\tilde{k}\right) - \delta - \rho}{\sigma(c)} - g \tag{21}$$

2. Resource constraint

$$d\tilde{k}/dt = f\left(\tilde{k}\right) - (n + \delta + g)\tilde{k} - \tilde{c} \tag{22}$$

3. Initial condition: $\tilde{k}_0$ given
4. TVC $\lim_{t\to\infty} e^{-(\rho-n)t} u'(c_t) k_t = 0$.
   With $u'(c_t) = c_t^{-\sigma} = \tilde{c}_t^{-\sigma} e^{-\sigma g t}$ and $k_t = \tilde{k}_t e^{gt}$, this becomes

$$\lim_{t\to\infty} e^{-(\rho-n)t} e^{(1-\sigma)gt} u'(\tilde{c}_t) \tilde{k}_t = 0 \tag{23}$$

Approach 2: Detrend and solve

- Steps:
    1. Find balanced growth rates - as before.
    2. Write the economy in detrended variables.
    3. Take the first-order conditions.
    4. Define the solution.
    5. Convert back into (undetrended) variables.
- This is useful for solution methods that only work on stationary problems (such as DP).
- Exercise: show that this yields the same answer for the growth model.

Detrending the Model

Summary

In the growth model, optimality conditions change only by adding the 2 occurrences of g:

$$g(\tilde{c}) = \frac{f'\left(\tilde{k}\right) - \delta - \rho}{\sigma} - g \tag{24}$$

$$d\tilde{k}/dt = f\left(\tilde{k}\right) - (n + \delta + g)\tilde{k} - \tilde{c} \tag{25}$$

Detrending the Model

Why do we care?

1. The balanced growth $\tilde{k}$ now depends on preferences:

$$g(\tilde{c}) = 0 \Rightarrow f\left(\tilde{k}\right) = \delta + \rho + \sigma g \tag{26}$$

2. We see that preferences must be CRRA for a steady state to exist.
3. Quantitative differences.

Reading

- Acemoglu (2009), ch. 8. Ch. 8.6 covers the detrended model. Ch. 7 covers Optimal Control.
- Barro and Sala-i Martin (1995), ch. 2, explains the Cass-Koopmans/Ramsey model in great detail.
- Blanchard and Fischer (1989), ch. 2
- Romer (2011), ch. 2A
- Phase diagram: Barro and Sala-i Martin (1995), ch. 2.6

References I

Acemoglu, D. (2009): *Introduction to modern economic growth*, MIT Press.

Barro, R. and X. Sala-i Martin (1995): "Economic growth," *Boston, MA*.

Blanchard, O. J. and S. Fischer (1989): *Lectures on macroeconomics*, MIT press.

Romer, D. (2011): *Advanced macroeconomics*, McGraw-Hill/Irwin.