Perpetual Youth Model

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Perpetual youth

The standard growth model is very tractable.

But it has an important limitation: all households are identical.

For some questions, it is important to have households of **different ages**:

- fiscal policies that redistribute across ages
- models with life-cycle features: job search, matching, ...

An analytically tractable version of the OLG model is the **Blanchard-Yaari** model of **perpetual youth**.

The key analytical trick:

- all stochastic events are i.i.d.
- in continuous time: they are drawn from a Poisson process

Poisson Process

The Poisson process is the continuous time analog of i.i.d.

Mental image:

- randomly distribute points on a real line
- on average, there are ν points per unit length
- as time passes, move along the line and count the points

Poisson Process

Let $N(t)$ denote the (random) number of events that occur during an interval of length t.

The parameter $\nu > 0$ is the **arrival rate**:

$$\mathbb{E}\{N(t)\} = \nu t \tag{1}$$

For a short interval t,

- the probability of more than one event is 0.
- the probability of one event: νt

Poisson Process

The PDF for $N(t)$ is the Poisson PDF:

$$\Pr(N(t) = n) = \frac{(\nu t)^n}{n!} e^{-\nu t} \tag{2}$$

The probability of **no event** over a period of length τ is $\exp(-\nu\tau)$.

- the continuous time analogue of $(1-p)^t$

Example

If the instantaneous probability of retirement is ν, then the probability of working more than τ "periods" is $\exp(-\nu t)$

Model Setup

Time t is continuous and goes on forever.

At each t, persons from all birth cohorts $\tau \geq 0$ are alive (but not all of them).

Agents die with constant probability v (perpetual youth).

Otherwise, it's a standard growth model.

Demographics

$L(t)$ is the population size.

At $t = 0$, $L(0) = 1$ identical persons are born.

Each person dies at each instant with **Poisson** probability ν.

- There are $\nu L(t)$ deaths at t.

At each instant, $nL(t)$ identical persons are born.

Therefore: $\dot{L}(t) = (n - \nu)L(t)$

The population growth rate is $n - \nu > 0$:

$$L(t) = \exp([n - \nu]t) \tag{3}$$

Demographics

Probability of living to at least age $t-\tau$: $e^{-\nu(t-\tau)}$

At time t, the mass of persons aged $t-\tau$ is

$$L(t|\tau) \quad = \quad \underbrace{\exp\left(-\nu\left(t-\tau\right)\right)}_{\text{survival rate}} \times \underbrace{n\exp\left(\left(n-\nu\right)\tau\right)}_{nL(\tau)}$$

Notation: $x(t|\tau)$ means x at t for those born at τ.

Preferences

Conditional on surviving, households utility at date t is $e^{-\rho(t-\tau)} \ln(c(t|\tau))$.

Expected utility for date t is

$$\underbrace{e^{-\nu(t-\tau)}}_{\text{survival prob.}} \underbrace{e^{-\rho(t-\tau)} \ln(c(t|\tau))}_{\text{utility if alive}} \tag{4}$$

Expected lifetime utility is

$$\int_{\tau}^{\infty} e^{-(\rho+\nu)(t-\tau)} \ln(c(t|\tau))\, dt \tag{5}$$

Interesting: mortality simply increases the discount factor: $\rho + \nu$.

Endowments

Households work 1 unit of time.

Newborn households do not own any assets.

This is how age matters: older households are richer.

Technology

- The resource constraint is

$$\dot{K} + C = F(K, L) - \delta K$$

- In per capita terms

$$\dot{k} = f(k) - c - (n - \nu + \delta) k \tag{6}$$

- $k = K/L$ is capital per capita and capital per worker.

Markets

Competitive markets for

- goods (numeraire)
- labor rental: w
- capital rental: q
- annuities...

Annuities

The problem: what to do with the wealth of households who die?

- "accidental bequests"

Assumption: households buy fair **annuities**.

Each cohort τ household gives $a(t|\tau)$ to the insurance company.

They get paid:

1. interest $r(t)a(t|\tau)$
2. an equal share of accidental bequests of his own cohort:

$$z(a(t|\tau)|t,\tau) = \nu a(t|\tau) \tag{7}$$

Effectively, the interest rate, conditional on survival, is $r(t)+\nu$.

Firms

- A representative firm solves the standard problem.
- Factor prices are

$$
\begin{aligned}
q &= f'(k) \\
w &= f(k) - f'(k)k
\end{aligned}
$$

Equilibrium

Definition

A CE is an allocation

$$[K(t), L(t), C(t), c(t|\tau), a(t|\tau)]_{t=0,\tau\leq t}^{\infty} \tag{8}$$

and a price system

$$[w(t), q(t), r(t)] \tag{9}$$

such that:

1. $c(t|\tau)$ and $a(t|\tau)$ solve the household's problem for cohort $t-\tau$.
2. $w(t)$ and $q(t)$ solve the firm's problem.
3. markets clear (below).
4. identities: $L(t), C(t)$, $r(t) = q(t) - \delta$

Important: we have to keep track of assets and consumption by cohort and age.

Equilibrium

Aggregation function: $\mathscr{A}(x) \equiv \int_0^t L(t|\tau)\, x(t|\tau)\, d\tau$

Market clearing:

- labor: implicit
- capital: $K(t) = \mathscr{A}(a) = \int_0^t L(t|\tau)\, a(t|\tau)\, d\tau$.
- goods: same as resource constraint.

Identities:

- $C(t) = \mathscr{A}(c)$ etc

Math Digression: Leibniz's Rule

We want to differentiate an integral

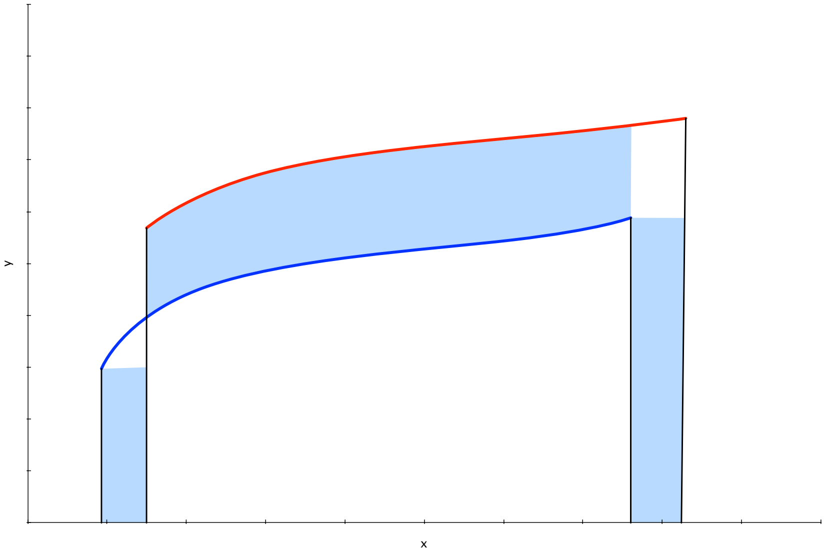
Given

$$F(\theta) = \int_{a(\theta)}^{b(\theta)} f(x,\theta)\, dx \tag{10}$$

We have

$$\frac{\partial F}{\partial \theta} = \underbrace{f(b(\theta),\theta)\, b'(\theta)}_{\text{right bound}} - \underbrace{f(a(\theta),\theta)\, a'(\theta)}_{\text{left bound}} + \underbrace{\int_{a(\theta)}^{b(\theta)} f_\theta(x,\theta)\, dx}_{\text{shift } f} \tag{11}$$

Leibniz's Rule



Households

The representative member of cohort τ solves

$$\max \int_{\tau}^{\infty} e^{-(\rho+\nu)(t-\tau)} \ln(c(t|\tau))\, dt$$

subject to

$$\dot{a}(t|\tau) = [r(t)+\nu]a(t|\tau) - c(t|\tau) + w(t) \tag{12}$$

The standard growth model problem, except:

- the discount rate is $\rho+\nu$;
- the interest rate is $r+\nu$

Household solution

This is a standard problem with Euler equation

$$\frac{\dot{c}(t|\tau)}{c(t|\tau)} = [r(t)+v] - [\rho + v] = r(t) - \rho \tag{13}$$

budget constraint and TVC

$$\lim_{t\to\infty} D_{t,\tau} a(t|\tau) = 0 \tag{14}$$

where

$$D_{t,\tau} = \exp\left(-\int_{\tau}^{t}[r(z)+v]dz\right) \tag{15}$$

Notation

- $D_{t,\tau}$ discounts a date t payment to τ.
- $D_{\tau,t} = 1/D_{t,\tau}$ discounts a date τ payment to t.
- $PV(x,t) = \int_{s=t}^{\infty} D_{s,t} x(s)\, ds$ is the present value of x.
- Useful later on:

$$\frac{\partial D_{t,\tau}}{\partial t} = (r(t) + v) D_{t,\tau} \tag{16}$$

  This is from $\frac{\partial}{\partial t} \exp(f(t)) = f'(t) \exp(f(t))$ together with Leibniz' Rule

$$\frac{\partial}{\partial t} \int_{\tau}^{t} \mathscr{F}(z)\, dz = \mathscr{F}(t) \tag{17}$$

Household: PIH

Claim: the household consumes a constant fraction of wealth:

$$c(t|\tau) = (\rho + \nu)[a(t|\tau) + \omega(t)] \tag{18}$$

Human wealth is the present value of lifetime earnings

$$\omega(t) = PV(w,t) = \int_t^\infty D_{s,t} w(s)\, ds \tag{19}$$

Note: all persons alive at t have the same ω.

Intuition...

Proof: PIH

Claim 1: We have a standard present value budget constraint:

$$PV(c(.|\tau), \tau) = a(\tau|\tau) + \omega(\tau) \tag{20}$$

In words: present value of c = present value of earnings + initial assets.

Claim 2:

$$PV(c(.|\tau), \tau) = \frac{c(\tau|\tau)}{\rho + v} \tag{21}$$

Together, these imply $c(\tau|\tau) = (\rho + \nu)[a(\tau|\tau) + \omega(\tau)]$.

From the derivation, we see that this holds for any age, not just for $t = \tau$.

Proof Claim 2 I

Integrate the Euler equation to get consumption:

$$c(t|\tau) = c(\tau|\tau)\exp\left(\int_\tau^t [r(z) - \rho]\,dz\right) \tag{22}$$

Verify by differentiating and comparing with Euler.

Multiply both sides by $D_{t,\tau}$:

$$\underbrace{D_{t,\tau}c(t|\tau)}_{\text{pres.value}} = c(\tau|\tau)\exp\left(\int_\tau^t [\underbrace{r(z) - \rho}_{c\text{ growth}} - \underbrace{(r(z) + v)}_{\text{discounting}}]dz\right) \tag{23}$$

$$= c(\tau|\tau)\exp\left(-[\rho + v][t - \tau]\right) \tag{24}$$

Proof Claim 2 II

In words: The present value of $c(t|\tau)$ grows at a rate the equals the difference between the consumption growth rate and the interest rate.

Present value of consumption

$$\int_{\tau}^{\infty} D_{t,\tau} c(t|\tau) dt = c(\tau|\tau) \int_{\tau}^{\infty} e^{-(\rho+\nu)t} dt = \frac{c(\tau|\tau)}{\rho+\nu} \qquad (25)$$

which is (21).

Proof: Claim 1

Claim:

$$D_{t,\tau}a(t|\tau) = a(\tau,\tau) + \int_{\tau}^{t} D_{z,\tau}[w(z) - c(z|\tau)]\,dz \tag{26}$$

In words: The present value of "terminal" assets $a(t|\tau)$ equals initial assets + the present value of savings.

Take $\lim_{t\to\infty}$and the LHS goes to 0 due to TVC.

That gives the lifetime budget constraint

$$PV(c) = \omega + a(\tau,\tau) \tag{27}$$

b/c the RHS is $\omega - PV(c)$.

Lifetime budget constraint

To show that the claim (26) implies the flow budget constraint:

Multiply by $D_{\tau,t}$:

$$a(t|\tau) = a(\tau|\tau) D_{\tau,t} + \int_{\tau}^{t} D_{z,t} [w(z) - c(z|\tau)] \, dz \tag{28}$$

because

$$D_{z,\tau} \times D_{\tau,t} = D_{z,t} \tag{29}$$

In words: discounting from z to t $(D_{z,t})$ is the same as

- first discounting from z to τ $(D_{z,\tau})$
- then discounting from τ to t $(D_{\tau,t})$

Next: Differentiate with respect to t and check that the flow budget constraint

$$\dot{a}(t|\tau) = [r(t) + \nu] a(t|\tau) - c(t|\tau) + w(t) \tag{30}$$

emerges.

Lifetime Budget Constraint

Differentiate (28) w.r.to t (Leibniz Rule):

$$\dot{a}(t|\tau) = a(\tau|\tau)\frac{\partial D_{\tau,t}}{\partial t} + D_{t,t}[w(t) - c(t|\tau)] + \int_{\tau}^{t} \frac{\partial D_{z,t}[w(z) - c(z|\tau)]}{\partial t} dz$$

and note that

1. $\frac{\partial D_{\tau,t}}{\partial t} = D_{\tau,t}[r(t) - v]$, so that the first term becomes $(r(t) + v)\, a(\tau|\tau)\, D_{\tau,t}$
2. $D_{t,t} = \exp(0) = 1$, so that the second term becomes $w(t) - c(t|\tau)$
3. from (28) the 3rd term is

$$[r(t) + v] \int_{\tau}^{t} D_{z,t}[w(z) - c(z|\tau)]\, dz = [r(t) + v][a(t,\tau) - a(\tau|\tau) D_{\tau,t}]$$

Add all that up and the flow budget constraint emerges.

Summary

We now have a solution for the individual consumption function:

$$c(t|\tau) = (\rho + \nu)\left[a(t|\tau) + \omega(t)\right] \tag{31}$$

To characterize equilibrium, we need the aggregate consumption function:

$$c(t) = \int_0^t L(t,\tau)c(t|\tau)d\tau / L(t) \tag{32}$$

A nice feature of this model: we can aggregate with paper and pencil.

Aggregation

$$\begin{aligned} c(t) &= \int_0^t L(t|\tau)c(t|\tau)d\tau/L(t) && (33) \\ &= \int_0^t [(\rho+\nu)(a(t|\tau)+\omega(t))]L(t|\tau)/L(t)d\tau && (34) \\ &= (\rho+\nu)[a(t)+\omega(t)] && (35) \end{aligned}$$

where

- $\int L(t|\tau)/L(t)d\tau = 1$
- $\int a(t|\tau)L(t|\tau)/L(t)d\tau \equiv a(t)$

Aggregation

This is a strong form of **aggregation**:

- Aggregate consumption behaves like individual consumption.
- As if a single individual made the choice.

The budget constraint aggregates in the same way.

How general is this?

Equilibrium Dynamics

It would be tempting to say:

- Euler is unchanged relative to growth model
- Resource constraint is unchanged
- Everything behaves like the growth model

But this would be wrong:

- each person has an Euler equation that looks "standard"
- that does not mean that aggregate consumption also behaves that way

Equilibrium Dynamics

We have a system in c, a, ω.

Consumption function

$$c(t) = (\rho + \nu)[a(t) + \omega(t)] \tag{36}$$

Budget constraint

$$\dot{a}(t) = (r(t) - (n - \nu))a(t) + w(t) - c(t) \tag{37}$$

Definition of human wealth

$$\omega(t) = \int_t^\infty \exp\left(-\int_t^s [r(\iota) + \nu]\, d\iota\right) w(s)\, ds \tag{38}$$

Note: The equation for $\dot{a}(t)$ follows directly from integrating the individual budget constraints.

Equilibrium Dynamics I

The strategy:

Derive an Euler equation for aggregate consumption by differentiating the $c(t)$ equation

$$c(t) = (\rho + \nu)[a(t) + \omega(t)] \tag{39}$$

Differentiating gives

$$\dot{c} = (\rho + \nu)[\dot{a} + \dot{\omega}] \tag{40}$$

Sub in budget constraint for $\dot{a}$.

Differentiate def of ω (Leibniz's rule - next slide):

$$\dot{\omega}(t) = (r(t) + \nu)\,\omega(t) - w(t) \tag{41}$$

Equilibrium Dynamics II

Sub that into $\dot{c}$ and collect terms:

$$\dot{c}(t) = [r(t) - \rho] c(t) - (\rho + \nu) n a(t) \tag{42}$$

Sub in $k(t) = a(t)$ and the firm foc for $r(t)$:

$$\frac{\dot{c}(t)}{c(t)} = \underbrace{f'(k(t)) - \delta - \rho}_{\text{standard growth}} - \underbrace{(\rho + \nu) n \frac{k(t)}{c(t)}}_{\text{new}} \tag{43}$$

Intuition for $\dot{\omega}(t)$

Think of human wealth as an asset with price $\omega(t)$.

Its instantaneous payoff consists of:

1. "dividend" $w(t)$
2. capital gain $\dot{\omega}(t)$

The asset price equals [required rate of return] $\times$ [dividend + capital gain]

Required rate of return is $r(t)+\nu$.

$$[r(t)+v]\,\omega(t) = w(t) + \dot{\omega}(t) \tag{44}$$

This is a general asset pricing equation that we will use more in the future.

Note: Deriving $\dot{\omega}(t)$

$$\omega(t) = \int_t^\infty \exp\left(-\int_t^s [r(\iota)+v]\,d\iota\right)\; w(s)\,ds \tag{45}$$

Using Leibniz's Rule, $\dot{\omega}(t)$ has 2 pieces:

1. Effect of changing lower bound of integral
   - integrand evaluated at $s=t$: $w(t)$.
2. Derivative of integrand w.r.to t:

$$\int_t^\infty w(s)\,\frac{d}{dt}\exp\left(-\int_t^s [r(\iota)+v]\,d\iota\right) ds = -[r(t)+v]\,\omega(t) \tag{46}$$

Putting both pieces together gives

$$[r(t)+v]\,\omega(t) = w(t) + \dot{\omega}(t) \tag{47}$$

Note: Deriving $\dot{\omega}(t)$ I

The second step in detail...

By the chain rule

$$\frac{d}{dt}\exp(f(t)) = f'(t)\exp(f(t)) \tag{48}$$

Leibniz's Rule:

$$\frac{d}{dt}\int_t^s [r(\iota)+\nu]\,d\iota = r(t)+\nu \tag{49}$$

Putting it all together:

$$\frac{d}{dt}\exp\left(-\int_t^s [r(\iota)+\nu]\,d\iota\right) = \exp\left(-\int_t^s [r(\iota)+\nu]\,d\iota\right) \times [-(r(t)+\nu)] \tag{50}$$

Note: Deriving $\dot{\omega}(t)$ II

And therefore

$$\int_t^{\infty} w(s) \frac{d}{dt} \exp\left(-\int_t^{s} [r(\iota)+\nu] d\iota\right) ds = -[r(t)+\nu]\,\omega(t) \quad (51)$$

is the second term in the $\dot{\omega}$ equation.

Phase diagram

$$\frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \delta - \rho - (\rho + \nu) n \frac{k(t)}{c(t)} \tag{52}$$

$$\dot{k} = f(k) - c - (n - \delta - \nu) k \tag{53}$$

with boundary conditions $k(0)$ given and TVC (which is not so obvious...)

This looks a lot like a standard growth model...

Steady state

$\dot{c} = 0 \Longrightarrow$

$$c = \frac{(\rho + \nu)n}{f'(k) - \delta - \rho}k \tag{54}$$

Properties:

1. $k \longrightarrow 0 \Longrightarrow c \longrightarrow 0$ [as $f' \longrightarrow \infty$]
2. $k \longrightarrow k^{MGR}$ where $f'\left(k^{MGR}\right) = \delta + \rho \Longrightarrow c \longrightarrow \infty$
3. $c''(k) > 0$ [verify]

Steady state

$\dot{k} = 0 \Longrightarrow$

$$c = f(k) - (n + \delta - \nu)k \tag{55}$$

Properties: as the standard growth model.

Steady state

Solution for steady state k^*

$$\frac{f(k^*)}{k^*} - (n - \nu + \delta) - \frac{(\rho + \nu)n}{f'(k^*) - \delta - \rho} = 0 \tag{56}$$

Unique steady state k^*: $f(k)/k \searrow$ in k. $-1/f'(k) \searrow$ in k.

Dynamic efficiency

Golden Rule maximizes

$$c^* = f(k^*) - (n + \delta - \nu) k^* \tag{57}$$

$$f'(k_{GR}) - \delta = n - \nu \tag{58}$$

Steady state:

$$f'(k^*) - \delta > \rho \tag{59}$$

[otherwise $c/k < 0$]

There can be overaccumulation relative to the Golden Rule.

This happens when households are sufficiently impatient (high ρ).

Similar to the finite lifetime OLG model.

Dynamic efficiency

Modified Golden Rule for planner with discount factor ρ [effects of mortality and "annuities" cancel]:

$$f'(k_{MGR}) - \delta = \rho \tag{60}$$

Equilibrium avoids overaccumulation relative to MGR.

This is not a robust feature of the model.

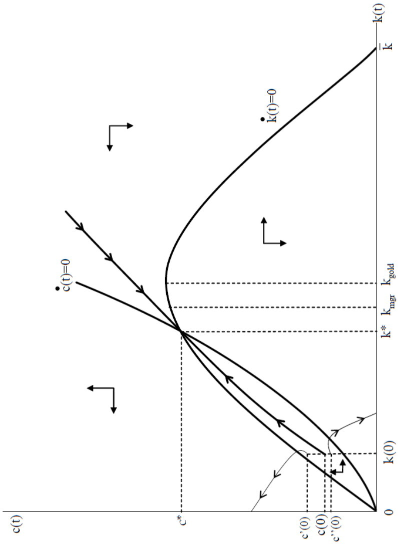
Giving households a stronger motive to save for "old age" can lead to overaccumulation.

Example: labor efficiency declines with age.

Dynamic efficiency

- Finite lifetimes are not necessary to generate overaccumulation.
- In this model, it is the presence of overlapping generations that destroys the welfare theorems.

Phase diagram



Phase diagram

- The dynamics closely resemble the growth model.
- A unique, globally saddle path stable steady state exists.
- Convergence is monotone.
- An analytically tractable model with OLG.

Where Is This Used?

Models of human capital

- combine the convenience of an infinitely lived decision maker
- capture that only young invest in education
- Akyol and Athreya (2005)

Models of income / wealth distribution

- a version of perpetual youth: agents age stochastically
- Castaneda et al. (2003)

Reading

- Acemoglu (2009), ch. 9.7-9.8.
- Blanchard and Fischer (1989), ch. 3.3

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