

# Perpetual Youth Model

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# Perpetual youth

The standard growth model is very tractable.

But it has an important limitation: all households are identical.

For some questions, it is important to have households of **different ages**:

- ▶ fiscal policies that redistribute across ages
- ▶ models with life-cycle features: job search, matching, ...

An analytically tractable version of the OLG model is the **Blanchard-Yaari** model of **perpetual youth**.

The key analytical trick:

- ▶ all stochastic events are i.i.d.
- ▶ in continuous time: they are drawn from a Poisson process

# Poisson Process

The Poisson process is the continuous time analog of i.i.d.

Mental image:

- ▶ randomly distribute points on a real line
- ▶ on average, there are  $\lambda$  points per unit length
- ▶ as time passes, move along the line and count the points

# Poisson Process

Let  $N(t)$  denote the (random) number of events that occur during an interval of length  $t$ .

The parameter  $\nu > 0$  is the **arrival rate**:

$$\mathbb{E}\{N(t)\} = \nu t \quad (1)$$

For a short interval  $t$ ,

- ▶ the probability of more than one event is 0.
- ▶ the probability of one event:  $\nu t$

# Poisson Process

The PDF for  $N(t)$  is the Poisson PDF:

$$\Pr(N(t) = n) = \frac{(vt)^n}{n!} e^{-vt} \quad (2)$$

The probability of **no event** over a period of length  $\tau$  is  $\exp(-v\tau)$ .

- ▶ the continuous time analogue of  $(1-p)^t$

## Example

If the instantaneous probability of retirement is  $v$ , then the probability of working more than  $\tau$  “periods” is  $\exp(-v\tau)$

# Model Setup

Time  $t$  is continuous and goes on forever.

At each  $t$ , persons from all birth cohorts  $\tau \geq 0$  are alive (but not all of them).

Agents die with constant probability  $\nu$  (perpetual youth).

Otherwise, it's a standard growth model.

# Demographics

$L(t)$  is the population size.

At  $t = 0$ ,  $L(0) = 1$  identical persons are born.

Each person dies at each instant with **Poisson** probability  $v$ .

► There are  $vL(t)$  deaths at  $t$ .

At each instant,  $nL(t)$  identical persons are born.

Therefore:  $\dot{L}(t) = (n - v)L(t)$

The population growth rate is  $n - v > 0$ :

$$L(t) = \exp([n - v]t) \quad (3)$$

# Demographics

Probability of living to at least age  $t - \tau$ :  $e^{-v(t-\tau)}$

At time  $t$ , the mass of persons aged  $t - \tau$  is

$$L(t|\tau) = \underbrace{\exp(-v(t-\tau))}_{\text{survival rate}} \times \underbrace{n \exp((n-v)\tau)}_{nL(\tau)}$$

Notation:  $x(t|\tau)$  means  $x$  at  $t$  for those born at  $\tau$ .



# Preferences

Conditional on surviving, households utility at date  $t$  is  $e^{-\rho(t-\tau)} \ln(c(t|\tau))$ .

Expected utility for date  $t$  is

$$\underbrace{e^{-\nu(t-\tau)}}_{\text{survival prob.}} \underbrace{e^{-\rho(t-\tau)} \ln(c(t|\tau))}_{\text{utility if alive}} \quad (4)$$

Expected lifetime utility is

$$\int_{\tau}^{\infty} e^{-(\rho+\nu)(t-\tau)} \ln(c(t|\tau)) dt \quad (5)$$

Interesting: mortality simply increases the discount factor:  $\rho + \nu$ .

# Endowments

Households work 1 unit of time.

Newborn households do not own any assets.

This is how age matters: older households are richer.

# Technology

- ▶ The resource constraint is

$$\dot{K} + C = F(K, L) - \delta K$$

- ▶ In per capita terms

$$\dot{k} = f(k) - c - (n - v + \delta)k \quad (6)$$

- ▶  $k = K/L$  is capital per capita and capital per worker.

# Markets

Competitive markets for

- ▶ goods (numeraire)
- ▶ labor rental:  $w$
- ▶ capital rental:  $q$
- ▶ annuities...

# Annuities

The problem: what to do with the wealth of households who die?

- ▶ “accidental bequests”

Assumption: households buy fair **annuities**.

Each cohort  $\tau$  household gives  $a(t|\tau)$  to the insurance company.

They get paid:

1. interest  $r(t)a(t|\tau)$
2. an equal share of accidental bequests of his own cohort:

$$z(a(t|\tau)|t, \tau) = v a(t|\tau) \quad (7)$$

Effectively, the interest rate, conditional on survival, is  $r(t) + v$ .

# Firms

- ▶ A representative firm solves the standard problem.
- ▶ Factor prices are

$$q = f'(k)$$

$$w = f(k) - f'(k)k$$

# Equilibrium

## Definition

A CE is an allocation

$$[K(t), L(t), C(t), c(t|\tau), a(t|\tau)]_{t=0, \tau \leq t}^{\infty} \quad (8)$$

and a price system

$$[w(t), q(t), r(t)] \quad (9)$$

such that:

1.  $c(t|\tau)$  and  $a(t|\tau)$  solve the household's problem for cohort  $t - \tau$ .
2.  $w(t)$  and  $q(t)$  solve the firm's problem.
3. markets clear (below).
4. identities:  $L(t), C(t), r(t) = q(t) - \delta$

Important: we have to keep track of assets and consumption by cohort and age.

# Equilibrium

Aggregation function:

$$\mathcal{A}(x) \equiv \int_0^t L(t|\tau) x(t|\tau) d\tau \quad (10)$$

Market clearing:

- ▶ labor: implicit
- ▶ capital:  $K(t) = \mathcal{A}(a) = \int_0^t L(t|\tau) a(t|\tau) d\tau$ .
- ▶ goods: same as resource constraint.

Identities:

- ▶  $C(t) = \mathcal{A}(c)$  etc



## Math Digression: Leibniz's Rule

We want to differentiate an integral

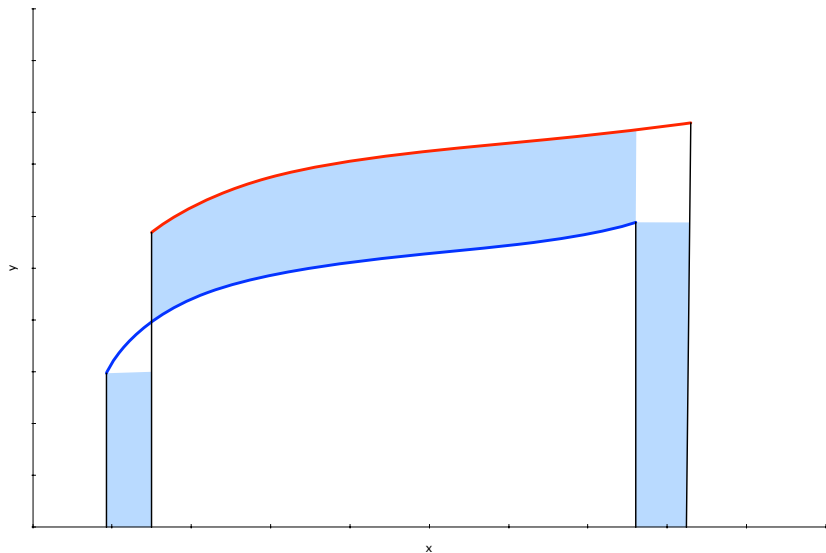
Given

$$F(\theta) = \int_{a(\theta)}^{b(\theta)} f(x, \theta) dx \quad (11)$$

We have

$$\frac{\partial F}{\partial \theta} = \underbrace{f(b(\theta), \theta) b'(\theta)}_{\text{right bound}} - \underbrace{f(a(\theta), \theta) a'(\theta)}_{\text{left bound}} + \underbrace{\int_{a(\theta)}^{b(\theta)} f_{\theta}(x, \theta) dx}_{\text{shift } f} \quad (12)$$

# Leibniz's Rule



# Households

The representative member of cohort  $\tau$  solves

$$\max_{\tau} \int_{\tau}^{\infty} e^{-(\rho+\nu)(t-\tau)} \ln(c(t|\tau)) dt$$

subject to

$$\dot{a}(t|\tau) = [r(t) + \nu]a(t|\tau) - c(t|\tau) + w(t) \quad (13)$$

The standard growth model problem, except:

- ▶ the discount rate is  $\rho + \nu$ ;
- ▶ the interest rate is  $r + \nu$

## Household solution

This is a standard problem with Euler equation

$$\frac{\dot{c}(t|\tau)}{c(t|\tau)} = [r(t) + v] - [\rho + v] = r(t) - \rho \quad (14)$$

budget constraint and TVC

$$\lim_{t \rightarrow \infty} D_{t,\tau} a(t|\tau) = 0 \quad (15)$$

where

$$D_{t,\tau} = \exp \left( - \int_{\tau}^t [r(z) + v] dz \right) \quad (16)$$

# Notation

- ▶  $D_{t,\tau}$  discounts a date  $t$  payment to  $\tau$ .
- ▶  $D_{\tau,t} = 1/D_{t,\tau}$  discounts a date  $\tau$  payment to  $t$ .
- ▶  $PV(x, t) = \int_{s=t}^{\infty} D_{s,t} x(s) ds$  is the present value of  $x$ .
- ▶ Useful later on:

$$\frac{\partial D_{t,\tau}}{\partial t} = (r(t) + v) D_{t,\tau} \quad (17)$$

This is from  $\frac{\partial}{\partial t} \exp(f(t)) = f'(t) \exp(f(t))$  together with Leibniz' Rule

$$\frac{\partial}{\partial t} \int_{\tau}^t \mathcal{F}(z) dz = \mathcal{F}(t) \quad (18)$$

## Household: PIH

Claim: the household consumes a constant fraction of wealth:

$$c(t|\tau) = (\rho + \nu)[a(t|\tau) + \omega(t)] \quad (19)$$

Human wealth is the present value of lifetime earnings

$$\omega(t) = PV(w, t) = \int_t^{\infty} D_{s,t} w(s) ds \quad (20)$$

Note: all persons alive at  $t$  have the same  $\omega$ .

Intuition...

## Proof: PIH

Claim 1: We have a standard present value budget constraint:

$$PV(c(.|\tau), \tau) = a(\tau|\tau) + \omega(\tau) \quad (21)$$

In words: present value of  $c$  = present value of earnings + initial assets.

Claim 2:

$$PV(c(.|\tau), \tau) = \frac{c(\tau|\tau)}{\rho + \nu} \quad (22)$$

Together, these imply  $c(\tau|\tau) = (\rho + \nu)[a(\tau|\tau) + \omega(\tau)]$ .

From the derivation, we see that this holds for any age, not just for  $t = \tau$ .

## Proof Claim 2 I

Integrate the Euler equation to get consumption:

$$c(t|\tau) = c(\tau|\tau) \exp \left( \int_{\tau}^t [r(z) - \rho] dz \right) \quad (23)$$

Verify by differentiating and comparing with Euler.

Multiply both sides by  $D_{t,\tau}$ :

$$\underbrace{D_{t,\tau} c(t|\tau)}_{\text{pres. value}} = c(\tau|\tau) \exp \left( \int_{\tau}^t \underbrace{[r(z) - \rho]}_{c \text{ growth}} - \underbrace{(r(z) + v)}_{\text{discounting}} dz \right) \quad (24)$$

$$= c(\tau|\tau) \exp(-[\rho + v][t - \tau]) \quad (25)$$



## Proof Claim 2 II

In words: The present value of  $c(t|\tau)$  grows at a rate that equals the difference between the consumption growth rate and the interest rate.

Present value of consumption

$$\int_{\tau}^{\infty} D_{t,\tau} c(t|\tau) dt = c(\tau|\tau) \int_{\tau}^{\infty} e^{-(\rho+\nu)t} dt = \frac{c(\tau|\tau)}{\rho + \nu} \quad (26)$$

which is (22).

## Proof: Claim 1

Claim:

$$D_{t,\tau}a(t|\tau) = a(\tau, \tau) + \int_{\tau}^t D_{z,\tau}[w(z) - c(z|\tau)] dz \quad (27)$$

In words: The present value of “terminal” assets  $a(t|\tau)$  equals initial assets + the present value of savings.

Take  $\lim_{t \rightarrow \infty}$  and the LHS goes to 0 due to TVC.

That gives the lifetime budget constraint

$$PV(c) = \omega + a(\tau, \tau) \quad (28)$$

b/c the RHS is  $\omega - PV(c)$ .

## Lifetime budget constraint

To show that the claim (27) implies the flow budget constraint:

Multiply by  $D_{\tau,t}$ :

$$a(t|\tau) = a(\tau|\tau) D_{\tau,t} + \int_{\tau}^t D_{z,t} [w(z) - c(z|\tau)] dz \quad (29)$$

because

$$D_{z,\tau} \times D_{\tau,t} = D_{z,t} \quad (30)$$

In words: discounting from  $z$  to  $t$  ( $D_{z,t}$ ) is the same as

- ▶ first discounting from  $z$  to  $\tau$  ( $D_{z,\tau}$ )
- ▶ then discounting from  $\tau$  to  $t$  ( $D_{\tau,t}$ )

Next: Differentiate with respect to  $t$  and check that the flow budget constraint

$$\dot{a}(t|\tau) = [r(t) + v]a(t|\tau) - c(t|\tau) + w(t) \quad (31)$$

emerges.

## Lifetime Budget Constraint

Differentiate (29) w.r.to  $t$  (Leibniz Rule):

$$\dot{a}(t|\tau) = a(\tau|\tau) \frac{\partial D_{\tau,t}}{\partial t} + D_{t,t} [w(t) - c(t|\tau)] + \int_{\tau}^t \frac{\partial D_{z,t} [w(z) - c(z|\tau)]}{\partial t} dz$$

and note that

1.  $\frac{\partial D_{\tau,t}}{\partial t} = D_{\tau,t} [r(t) - v]$ , so that the first term becomes  $(r(t) + v) a(\tau|\tau) D_{\tau,t}$
2.  $D_{t,t} = \exp(0) = 1$ , so that the second term becomes  $w(t) - c(t|\tau)$
3. from (29) the 3rd term is

$$[r(t) + v] \int_{\tau}^t D_{z,t} [w(z) - c(z|\tau)] dz = [r(t) + v] [a(t, \tau) - a(\tau|\tau) D_{\tau,t}]$$

Add all that up and the flow budget constraint emerges.

## Summary

We now have a solution for the individual consumption function:

$$c(t|\tau) = (\rho + \nu) [a(t|\tau) + \omega(t)] \quad (32)$$

To characterize equilibrium, we need the aggregate consumption function:

$$c(t) = \int_0^t L(t, \tau) c(t|\tau) d\tau / L(t) \quad (33)$$

A nice feature of this model: we can aggregate with paper and pencil.

# Aggregation

$$c(t) = \int_0^t L(t|\tau) c(t|\tau) d\tau / L(t) \quad (34)$$

$$= \int_0^t [(\rho + \nu)(a(t|\tau) + \omega(t))] L(t|\tau) / L(t) d\tau \quad (35)$$

$$= (\rho + \nu)[a(t) + \omega(t)] \quad (36)$$

where

- ▶  $\int L(t|\tau) / L(t) d\tau = 1$
- ▶  $\int a(t|\tau) L(t|\tau) / L(t) d\tau \equiv a(t)$

# Aggregation

This is a strong form of **aggregation**:

- ▶ Aggregate consumption behaves like individual consumption.
- ▶ As if a single individual made the choice.

The budget constraint aggregates in the same way.

How general is this?

# Equilibrium Dynamics

It would be tempting to say:

- ▶ Euler is unchanged relative to growth model
- ▶ Resource constraint is unchanged
- ▶ Everything behaves like the growth model

But this would be wrong:

- ▶ each person has an Euler equation that looks “standard”
- ▶ that does not mean that aggregate consumption also behaves that way



## Equilibrium Dynamics

We have a system in  $c, a, \omega$ .

Consumption function

$$c(t) = (\rho + \nu) [a(t) + \omega(t)] \quad (37)$$

Budget constraint

$$\dot{a}(t) = (r(t) - (n - \nu)) a(t) + w(t) - c(t) \quad (38)$$

Definition of human wealth

$$\omega(t) = \int_t^\infty \exp\left(-\int_t^s [r(l) + \nu] dl\right) w(s) ds \quad (39)$$

Note: The equation for  $\dot{a}(t)$  follows directly from integrating the individual budget constraints.

# Equilibrium Dynamics I

The strategy:

Derive an Euler equation for aggregate consumption by differentiating the  $c(t)$  equation

$$c(t) = (\rho + \nu) [a(t) + \omega(t)] \quad (40)$$

Differentiating gives

$$\dot{c} = (\rho + \nu) [\dot{a} + \dot{\omega}] \quad (41)$$

Sub in budget constraint for  $\dot{a}$ .

Differentiate def of  $\omega$  (Leibniz's rule - next slide):

$$\dot{\omega}(t) = (r(t) + \nu) \omega(t) - w(t) \quad (42)$$

## Equilibrium Dynamics II

Sub that into  $\dot{c}$  and collect terms:

$$\dot{c}(t) = [r(t) - \rho]c(t) - (\rho + v)na(t) \quad (43)$$

Sub in  $k(t) = a(t)$  and the firm foc for  $r(t)$ :

$$\frac{\dot{c}(t)}{c(t)} = \underbrace{f'(k(t)) - \delta - \rho}_{\text{standard growth}} - \underbrace{(\rho + v)n \frac{k(t)}{c(t)}}_{\text{new}} \quad (44)$$

## Intuition for $\dot{\omega}(t)$

Think of human wealth as an asset with price  $\omega(t)$ .

Its instantaneous payoff consists of:

1. "dividend"  $w(t)$
2. capital gain  $\dot{\omega}(t)$

The asset price equals [required rate of return]  $\times$  [dividend + capital gain]

Required rate of return is  $r(t) + v$ .

$$[r(t) + v] \omega(t) = w(t) + \dot{\omega}(t) \quad (45)$$

This is a general asset pricing equation that we will use more in the future.

Note: Deriving  $\dot{\omega}(t)$

$$\omega(t) = \int_t^\infty \exp\left(-\int_t^s [r(l) + v] dl\right) w(s) ds \quad (46)$$

Using Leibniz's Rule,  $\dot{\omega}(t)$  has 2 pieces:

1. Effect of changing lower bound of integral
  - integrand evaluated at  $s = t$ :  $w(t)$ .
2. Derivative of integrand w.r.to  $t$ :

$$\int_t^\infty w(s) \frac{d}{dt} \exp\left(-\int_t^s [r(l) + v] dl\right) ds = -[r(t) + v] \omega(t) \quad (47)$$

Putting both pieces together gives

$$[r(t) + v] \omega(t) = w(t) + \dot{\omega}(t) \quad (48)$$

## Note: Deriving $\dot{\omega}(t)$ I

The second step in detail...

By the chain rule

$$\frac{d}{dt} \exp(f(t)) = f'(t) \exp(f(t)) \quad (49)$$

Leibniz's Rule:

$$\frac{d}{dt} \int_t^s [r(\iota) + v] d\iota = r(t) + v \quad (50)$$

Putting it all together:

$$\frac{d}{dt} \exp \left( - \int_t^s [r(\iota) + v] d\iota \right) = \exp \left( - \int_t^s [r(\iota) + v] d\iota \right) \times [-(r(t) + v)] \quad (51)$$

## Note: Deriving $\dot{\omega}(t)$ II

And therefore

$$\int_t^\infty w(s) \frac{d}{ds} \exp\left(-\int_t^s [r(\iota) + v] d\iota\right) ds = -[r(t) + v] \omega(t) \quad (52)$$

is the second term in the  $\dot{\omega}$  equation.

## Phase diagram

$$\frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \delta - \rho - (\rho + \nu)n \frac{k(t)}{c(t)} \quad (53)$$

$$\dot{k} = f(k) - c - (n - \delta - \nu)k \quad (54)$$

with boundary conditions  $k(0)$  given and TVC (which is not so obvious...)

This looks a lot like a standard growth model...



## Steady state

$$\dot{c} = 0 \implies$$

$$c = \frac{(\rho + v)n}{f'(k) - \delta - \rho} k \quad (55)$$

Properties:

1.  $k \longrightarrow 0 \implies c \longrightarrow 0$  [as  $f' \longrightarrow \infty$ ]
2.  $k \longrightarrow k^{MGR}$  where  $f'(k^{MGR}) = \delta + \rho \implies c \longrightarrow \infty$
3.  $c''(k) > 0$  [verify]

## Steady state

$$\dot{k} = 0 \implies c = f(k) - (n + \delta - \nu)k \quad (56)$$

Properties: as the standard growth model.

## Steady state

Solution for steady state  $k^*$

$$\frac{f(k^*)}{k^*} - (n - v + \delta) - \frac{(\rho + v)n}{f'(k^*) - \delta - \rho} = 0 \quad (57)$$

Unique steady state  $k^*$ :  $f(k)/k \searrow$  in  $k$ .  $-1/f'(k) \searrow$  in  $k$ .

# Dynamic efficiency

**Golden Rule** maximizes

$$c^* = f(k^*) - (n + \delta - v)k^* \quad (58)$$

$$f'(k_{GR}) - \delta = n - v \quad (59)$$

Steady state:

$$f'(k^*) - \delta > \rho \quad (60)$$

[otherwise  $c/k < 0$ ]

There can be overaccumulation relative to the Golden Rule.

This happens when households are sufficiently impatient (high  $\rho$ ).

Similar to the finite lifetime OLG model.

## Dynamic efficiency

**Modified Golden Rule** for planner with discount factor  $\rho$  [effects of mortality and "annuities" cancel]:

$$f'(k_{MGR}) - \delta = \rho \quad (61)$$

Equilibrium avoids overaccumulation relative to MGR.

This is not a robust feature of the model.

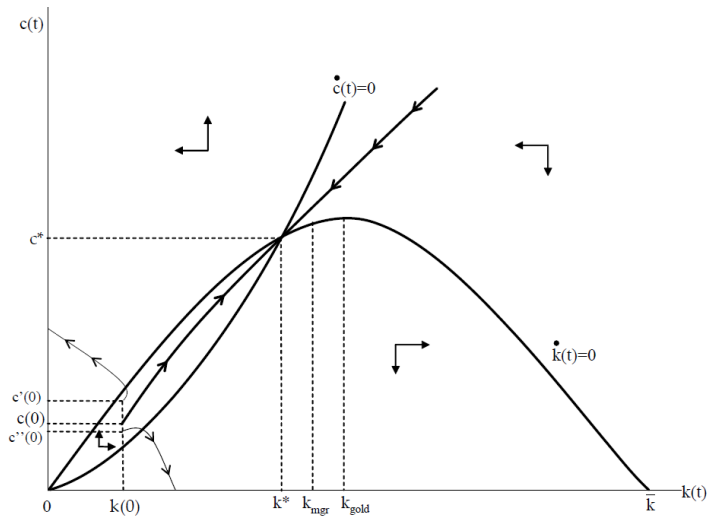
Giving households a stronger motive to save for "old age" can lead to overaccumulation.

Example: labor efficiency declines with age.

# Dynamic efficiency

- ▶ Finite lifetimes are not necessary to generate overaccumulation.
- ▶ In this model, it is the presence of overlapping generations that destroys the welfare theorems.

# Phase diagram



## Phase diagram

- ▶ The dynamics closely resemble the growth model.
- ▶ A unique, globally saddle path stable steady state exists.
- ▶ Convergence is monotone.
- ▶ An analytically tractable model with OLG.



# Where Is This Used?

## Models of human capital

- ▶ combine the convenience of an infinitely lived decision maker
- ▶ capture that only young invest in education
- ▶ Akyol and Athreya (2005)

## Models of income / wealth distribution

- ▶ a version of perpetual youth: agents age stochastically
- ▶ Castaneda et al. (2003)

# Reading

- ▶ Acemoglu (2009), ch. 9.7-9.8.
- ▶ Blanchard and Fischer (1989), ch. 3.3

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