

The Growth Model in Continuous Time (Ramsey Model)

Prof. Lutz Hendricks

Econ720

October 13, 2023

The Growth Model in Continuous Time

We add optimizing households to the Solow model.

We first study the planner's problem, then the CE.

Planning Problem

Planning Problem

The social planner maximizes

$$\int_{t=0}^{\infty} e^{-(\rho-n)t} u(c_t) dt \quad (1)$$

subject to the resource constraint

$$\dot{k}_t = f(k_t) - (n + \delta)k_t - c_t \quad (2)$$

$$k_0 \text{ given} \quad (3)$$

$$k_t \geq 0 \quad (4)$$

Planning Problem

The current value Hamiltonian is

The state is k and the control is c .

The optimality conditions are

Planner: TVC

The TVC is:

$$\lim_{t \rightarrow \infty} e^{-(\rho-n)t} \mu(t) k(t) = 0 \quad (5)$$

To check this:

- ▶ we need u and $g(k, c)$ to be monotone
- ▶ u is obvious.
- ▶ $g(k, c) = f(k) - c - (n + \delta)k$ is monotone in c but not k .
- ▶ However, we "know" that k never rises above the golden rule point where $f'(k) = \delta$ - unless $k(0)$ is too high.
- ▶ Then g is increasing in k .

Sufficiency

This is an example where the easiest (1st) set of sufficiency conditions applies:

- ▶ u is strictly concave in c (only).
- ▶ $g(k, c)$ is jointly concave in k and c .

First order conditions are sufficient.

Planner: Solution

A solution consists of functions of time

$$c_t, k_t, \mu_t$$

that satisfy:

1. The first-order conditions (2)
2. The resource constraint
3. The boundary conditions k_0 given and the TVC

$$\lim e^{-(\rho-n)t} \mu_t k_t = 0 \quad (6)$$

Planner: Euler Equation

We eliminate the multiplier, starting from

$$g(\mu) = -[f'(k) - \delta - \rho] \quad (7)$$

$$= g(u'(c)) \quad (8)$$

$$= -\sigma g(c) \quad (9)$$

where σ is the elasticity of u' w.r.to c :

$$\sigma = -u''_c c / u' \quad (10)$$

$$= -\frac{du'(c)}{dc} \frac{c}{u'(c)} \quad (11)$$

Therefore:

$$\boxed{g(c) = [f'(k) - \delta - \rho] / \sigma} \quad (12)$$

More direct derivation

Differentiating the FOC yields

$$\dot{\mu} = u''(c)\dot{c} \quad (13)$$

and therefore

$$\dot{\mu}/\mu = u''(c)\dot{c}/u'(c) \quad (14)$$

$$= -[f'(k) - \delta - \rho] \quad (15)$$

Planner: Euler Equation

$$g(c) = [f'(k) - \delta - \rho] / \sigma \quad (16)$$

Recall the discrete time version:

$$\frac{c_{t+1}}{c_t} = [\beta R]^{1/\sigma} \quad (17)$$

The same idea:

- ▶ consumption growth rises with the interest rate
- ▶ declines with the discount rate
- ▶ σ governs how responsive consumption growth is

Note on σ

σ is a key parameter for asset pricing, business cycle volatilities, ...

$$\sigma = -u''_c c / u' \quad (18)$$

$$= -\frac{du'(c)}{dc} \frac{c}{u'(c)} \quad (19)$$

σ is

1. the elasticity of marginal utility w.r.to c
2. the inverse intertemporal elasticity of substitution
3. the coefficient of relative risk aversion

$u(c) = c^{1-\phi} / 1 - \phi$ implies $\sigma = \phi$.

Planner: Summary

- ▶ The planner's problem solves for functions of time $c(t)$ and $k(t)$.
- ▶ These satisfy two differential equations

$$g(c) = \frac{f'(k) - \delta - \rho}{\sigma} \quad (20)$$

$$\dot{k} = f(k) - (n + \delta)k - c \quad (21)$$

and two boundary conditions

$$\lim_{t \rightarrow \infty} \beta^t u'(c(t)) k(t) = 0 \quad k_0 \text{ given}$$

- ▶ How can we analyze the dynamics of this system?

Phase Diagram

Phase Diagram

Phase diagrams can be used to analyze the dynamics of systems of 2 differential equations.

Consider the example

$$\dot{x} = A - ax + by$$

$$\dot{y} = B + cx - dy$$

Boundary conditions: x_0, y_0 given.

Assume $a, b, c, d > 0$.

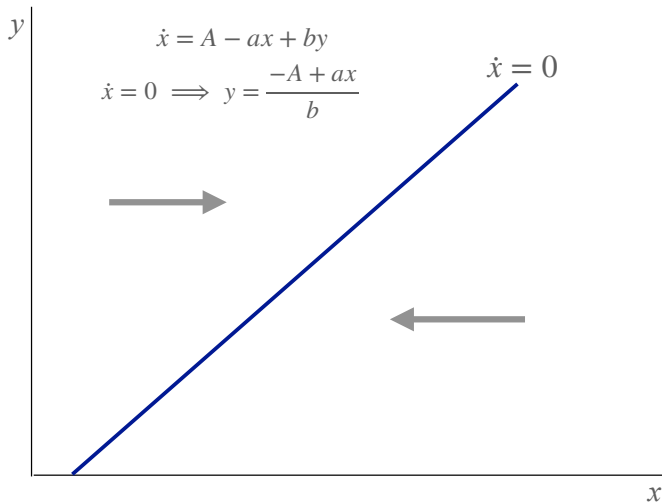
Basic Idea

$$\dot{x} = A - ax + by \quad (22)$$

This divides the (x,y) plane into two regions:

- ▶ one where x rises (moving east over time)
- ▶ one where x falls (moving west over time)

Basic idea



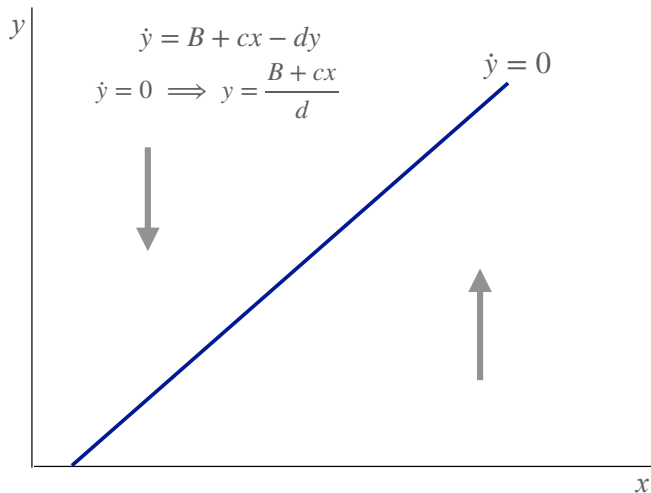
Basic idea

$$\dot{y} = B + cx - dy \quad (23)$$

also divides the plane into two regions

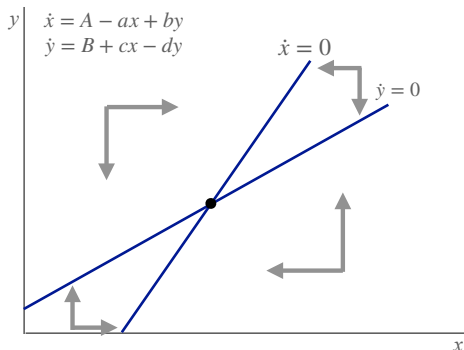
- ▶ one where y rises (moving north over time)
- ▶ one where y falls (moving south over time).

Basic idea



Basic Idea

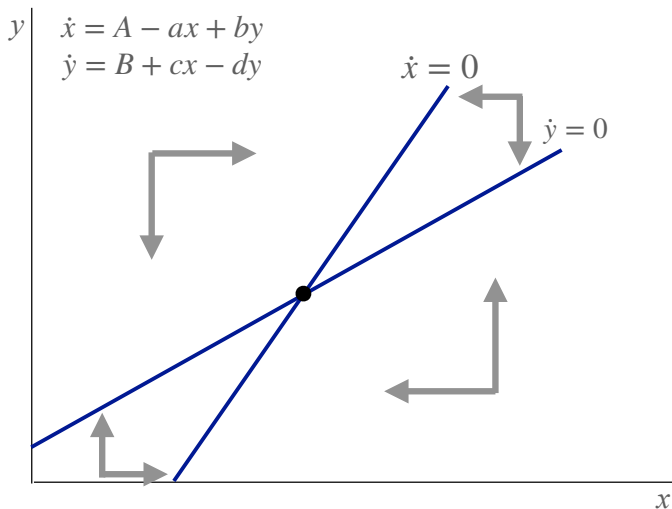
So we end up with a diagram that looks something like this:



All points in a given quadrant move (qualitatively) in the same direction.

Now we can use logic to figure out possible paths.

Phase Diagram



What happens for points on the region boundaries?

Think about possible paths

For each quadrant, determine whether the path can leave the quadrant.

In this example: the answer is

- ▶ “no” for two quadrants.
- ▶ “yes” for two quadrants.

Now think about possible paths...

Conclude that the steady state is globally stable.

Summary

Given:

- ▶ a model characterized by two differential equations (\dot{x}, \dot{y}) .
- ▶ boundary conditions (here x_0, y_0)

Steps:

1. Find region boundaries where $\dot{x} = 0$ and $\dot{y} = 0$.
2. Find direction of movement in each of the four regions.
3. Determine which regions can be exited.
4. Think about possible paths.

Applications

Unified growth theory:

- ▶ Galor (2000) studies transition from Malthusian stagnation to industrialization using a sequence of phase diagrams
- ▶ Galor (2005)

Models of human capital accumulation over the life-cycle:

- ▶ Heckman (1976)

Phase Diagram: Growth Model

Movement of c :

$$g(c) = \frac{f'(k) - \delta - \rho}{\sigma} \quad (24)$$

The $\dot{c} = 0$ locus is characterized by

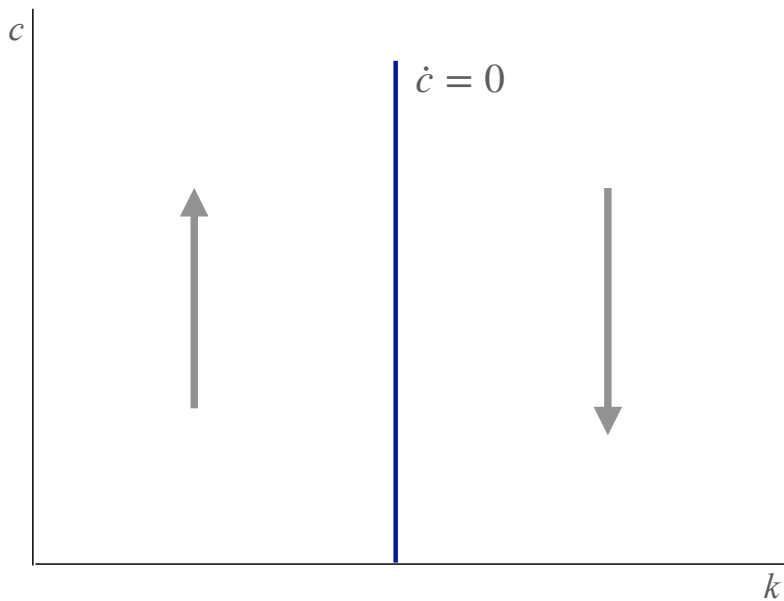
$$f'(k^*) = \rho + \delta \quad (25)$$

A vertical line in the (k, c) plane.

Higher k implies lower \dot{c}

$c \downarrow$ to the right of the $\dot{c} = 0$ locus.

Movement of c



Movement of k

$$\dot{k} = f(k) - (n + \delta)k - c \quad (26)$$

The $\dot{k} = 0$ locus is hump-shaped:

$$c = f(k) - (n + \delta)k \quad (27)$$

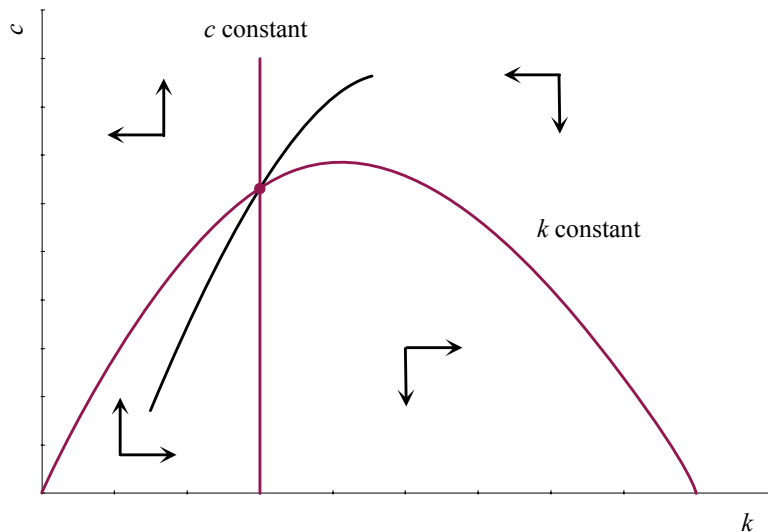
with a maximum at

$$f'(k^*) = n + \delta \quad (28)$$

Higher c implies lower \dot{k} .

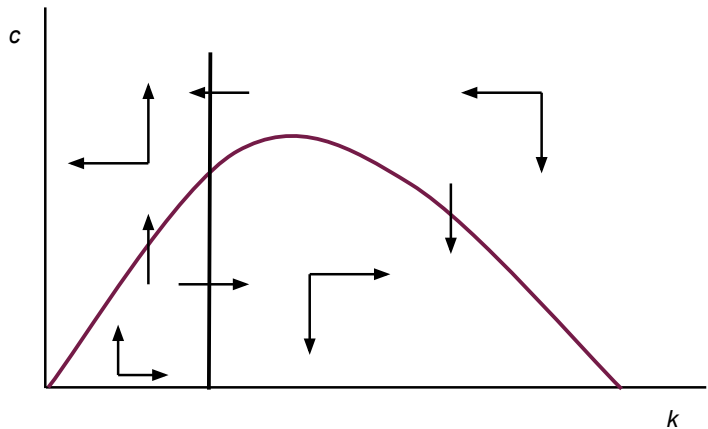
k falls over time above the $\dot{k} = 0$ locus.

Phase Diagram



Since $\rho - n > 0$, the $\dot{c} = 0$ locus lies to the left of the peak of the $\dot{k} = 0$ locus.

Dynamics: Possible Paths



Ruling out the “north-west” path

$$g(c) = \frac{f'(k) - \delta - \rho}{\sigma} \quad (29)$$

$g(c)$ rises over time as $k \rightarrow 0$.

Eventually, this violates feasibility.

Ruling out the “south-east” path

Properties of that path:

$$c \rightarrow 0 \implies k \rightarrow k_{max} > k_{GR} \quad (30)$$

and therefore $f'(k_{max}) - \delta < n$.

This is exactly the kind of path that the TVC rules out.

Note:

- ▶ Even though $g(c)$ is strictly negative, $\dot{c} \rightarrow 0$. Therefore c does not turn negative.
- ▶ Any such path asymptotes towards $c = 0$ and $k = k_{max}$.

Ruling out the “south-east” path

Transversality

$$\lim_{t \rightarrow \infty} e^{-(\rho-n)t} u'(c_t) k_t = 0 \quad (31)$$

requires

$$g\left(e^{-(\rho-n)t} u'(c)\right) = g(u') - (\rho - n) < 0 \quad (32)$$

as $t \rightarrow \infty$.

Euler:

$$g(u') = \rho + \delta - f'(k) \quad (33)$$

Higher k implies lower f' and higher $g(u')$.

Any $k > k_{GR}$ violates TVC because

$$g(u') - (\rho - n) = -[f'(k) - \delta - n] > 0 \quad (34)$$

Dynamics: Saddle-path Stability

Only one value of c avoids moving into “forbidden” regions for given k .

For this c , the economy converges to the steady state.

Such a system is called "**saddle-path stable**."

► Details

A great way of exploring the phase diagram can be found at EconGraphs

Summary

The growth model in continuous time behaves like the one in discrete time (no surprise here):

1. The allocation is determined by Euler equation and resource constraint.
Boundary conditions also look like in discrete time.
2. There is a unique, globally stable steady state.
It satisfies “interest rate” = discount rate.
3. In continuous time, we can use a phase diagram to study the entire path.

Reading

- ▶ Acemoglu (2009), ch. 8. Ch. 8.6 covers the detrended model. Ch. 7 covers Optimal Control.
- ▶ Barro and Sala-i Martin (1995), ch. 2, explains the Cass-Koopmans/Ramsey model in great detail.
- ▶ Blanchard and Fischer (1989), ch. 2
- ▶ Romer (2011), ch. 2A
- ▶ Phase diagram: Barro and Sala-i Martin (1995), ch. 2.6

Technical notes: Unique saddle path

Theorem

Take as given

$\dot{x}(t) = G[x(t)]$ with initial value $x(0)$ given, where G is continuously differentiable.

The steady state is $G(x^) = 0$. Define $A = DG(x^*)$.*

Suppose that m eigenvalues of A have negative real parts while $n - m$ have positive real parts.

Then there exists an m dimensional manifold in the neighborhood around the steady state such that starting from any $x(0)$ in that manifold a unique $x(t) \rightarrow x^$.*

See Acemoglu (2009), Theorem 7.15.

What this says in words

Suppose we have a system of $n = 2$ differential equations (in c and k).

The local dynamics around the steady state can be approximated by a linear differential equation with matrix A .

$$\begin{bmatrix} \dot{k} \\ \dot{c} \end{bmatrix} = A \begin{bmatrix} k \\ c \end{bmatrix} \quad (35)$$

If that matrix has $m = 1$ negative eigenvalues, then **locally** around the steady state there is a line (dimension $m = 1$) of points (c, k) that converge to the steady state.

► This is the saddle path.

Other points could, in principle, converge as well, but we can rule that out as above.

Application to the growth model

First, establish that the saddle path is locally unique.

Start from a linear approximation to the two differential equations:

$$\begin{bmatrix} \dot{k} \\ \dot{c} \end{bmatrix} = \begin{bmatrix} f'(k^*) - n - \delta & -1 \\ c^* f''(k^*) \sigma & 0 \end{bmatrix} \begin{bmatrix} k - k^* \\ c - c^* \end{bmatrix} \quad (36)$$

Details:

$$\dot{k} = f(k) - (n + \delta)k - c \quad (37)$$

$$= f'(k - k^*) - (n + \delta) - (c - c^*) \quad (38)$$

and

$$\dot{c} = c[f(k) - \delta - \rho] / \sigma \quad (39)$$

$$= (c - c^*) \underbrace{[f(k^*) - \delta - \rho]}_{=0} / \sigma + c^* \frac{f''(k^*)}{\sigma} (k - k^*) \quad (40)$$

Eigenvalues I

The eigenvalues λ solve $Ax = \lambda x$. In this case:

$$\begin{bmatrix} f'(k^*) - n - \delta & -1 \\ c^* f''(k^*) / \sigma & 0 \end{bmatrix} x = \lambda x \quad (41)$$

Also $\det(A - I\lambda)x = 0$. In this case:

$$\det \begin{bmatrix} f'(k^*) - n - \delta - \lambda & -1 \\ c^* f''(k^*) / \sigma & 0 - \lambda \end{bmatrix} = 0 \quad (42)$$

$$\det(A - I\lambda) = -[f'(k^*) - \delta - n - \lambda] \lambda + c^* f''(k^*) / \sigma \quad (43)$$

$$\lambda^2 - (f' - \delta - n) \lambda + c^* f'' / \sigma = 0 \quad (44)$$

Eigenvalues II

Apply

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (45)$$

and obtain

$$\lambda = \left\{ f'(k^*) - n - \delta \pm \sqrt{(f' - n - \delta)^2 - 4c^*f''/\sigma} \right\} / 2 \quad (46)$$

Since $\sqrt{(f' - n - \delta)^2 - 4c^*f''/\sigma} > f' - n - \delta$, there is exactly one negative eigenvalue.

Therefore: in a neighborhood of the steady state, the saddle path is unique.

Application to the growth model

How do we know that the saddle is globally unique?

Define one saddle path that converges.

Take a point not on it. We know:

1. The path cannot reach or cross the saddle path in finite time.
2. The path cannot asymptote to the saddle path because that would get into a neighborhood of the steady state where the saddle is unique.
3. Therefore, the path cannot converge to the steady state.

References I

- Acemoglu, D. (2009): *Introduction to modern economic growth*, MIT Press.
- Barro, R. and X. Sala-i Martin (1995): "Economic growth," *Boston, MA*.
- Blanchard, O. J. and S. Fischer (1989): *Lectures on macroeconomics*, MIT press.
- Galor, O. (2000): "Ability Biased Technological Transition, Wage Inequality, and Economic Growth," *Quarterly Journal of Economics*, 115, 469–498.
- (2005): "From Stagnation to Growth: Unified Growth Theory," in *Handbook of Economic Growth*, ed. by P. Aghion and S. N. Durlauf, Elsevier, vol. 1A, 171–293.
- Heckman, J. J. (1976): "A Life-Cycle Model of Earnings, Learning, and Consumption," *Journal of Political Economy*, 84, pp. S11–S44.
- Romer, D. (2011): *Advanced macroeconomics*, McGraw-Hill/Irwin.