The Solow Model

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Topics

We write down a basic, but quite general growth model.

The idea: growth is driven by "capital" accumulation.

- but "capital" does not have to be physical capital (machines, structures)
- it could human capital, knowledge capital

Topics

The **Solow model** answers questions such as:

1. How much of cross-country income gaps is due to differences in saving rates?
2. Does **capital** accumulation drive long-run growth?
3. Do country incomes **converge** over time?
4. What happens to growth when production uses **finite resources**?

Objectives

At the end of this section you should be able to

1. Derive properties of the Solow model: steady state, effects of shocks, ...
2. Graph the dynamics of the Solow model.
3. Explain why the contribution of capital (saving) to cross-country output gaps is small.

Note: The Solow model is old (1950s). But it's ideas are durable.

- The basic model structure applies to any factor that is accumulated
- E.g., human capital, knowledge (Romer model)

Models as Measurement Tools

How can we answer **cause - effect questions**?

A key idea

The model as a measurement tool.
To measure the effect of X on Y, we use the implications of a quantitative model

We would like to run **controlled experiments**, but that's not feasible.

We use **models as laboratories** to run those experiments.

Later, we discuss the benefits and drawbacks of this method.

Kaldor Facts

What should a model of growth look like?

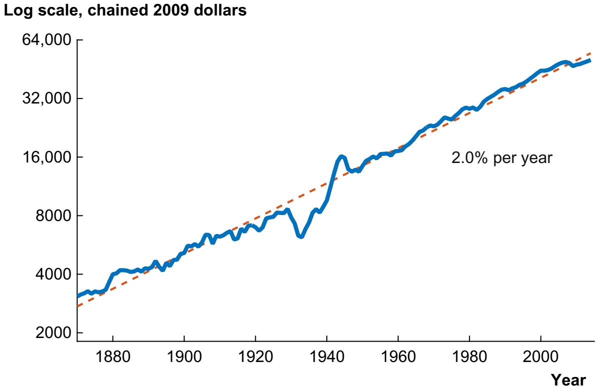
The U.S. growth experience looks a lot like a "**balanced growth path**"

- GDP growth has been essentially constant at 2% per year
- Constant capital output ratio K/Y
- Interest rates have no trend
- The share of labor income in GDP has been constant (2/3)

This is why economists like to write down models with **balanced growth**.

- all growth rates are constant over time.

Constant US Growth

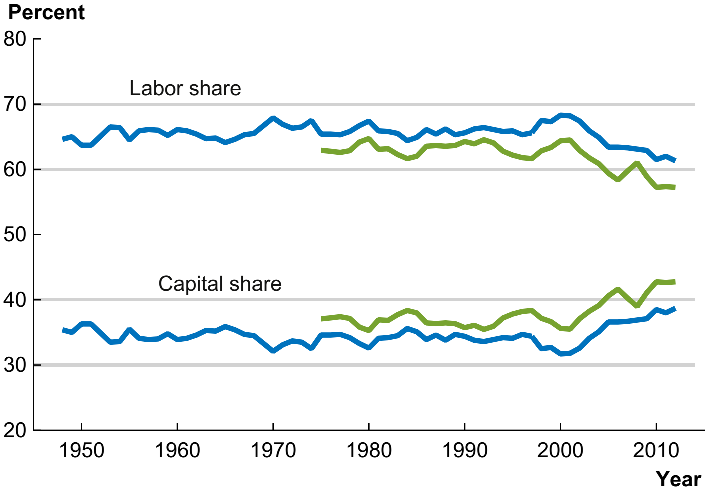


Source: Jones (2016)

US growth has been essentially constant for 140 years.

What does this tell us about determinants of long-run growth?

Constant Labor Share



But labor share has been falling recently.

Model Elements

The world goes on forever.

Time is indexed by the **continuous** variable t.

One good (Y) is produced from two inputs (capital K and labor L).

Productivity (A) grows exogenously

- at a constant rate
- later, we will study where productivity growth comes from

Households save a constant fraction of income.

Is the Model Too Simple?

The model makes strong assumptions:

- only 2 factor inputs (capital and labor)
- an aggregate production function (no firm detail)
- constant saving rate

Why should we take this seriously?

What Makes a Good Model?

A good model starts out as simple as possible.

- A model tells a story in math.
- Simplicity is good (to start with).

But need to check robustness.

The Solow Story

Economic growth is driven by

- physical capital accumulation (investment)
- productivity growth

The key insight:

- investment alone cannot drive growth
- due to diminishing marginal product of capital
- but investment is important for output levels

Romer model:

- investment can drive growth if it is not subject to diminishing returns
- knowledge accumulation

Production Structure

Aggregate production function:

$$Y(t) = F[K(t), L(t), A(t)] \tag{1}$$

There is one output good $Y \rightarrow$ GDP

There are two inputs:

1. Capital K: machines, equipment, structures
2. Labor L: hours worked, education, ...

We can extend the analysis to many capital and labor inputs

- the basic message does not change.

Cobb Douglas

The functional form is **Cobb-Douglas**:

$$Y(t) = K(t)^{\alpha} \; [A(t) \; L(t)]^{1-\alpha} \tag{2}$$

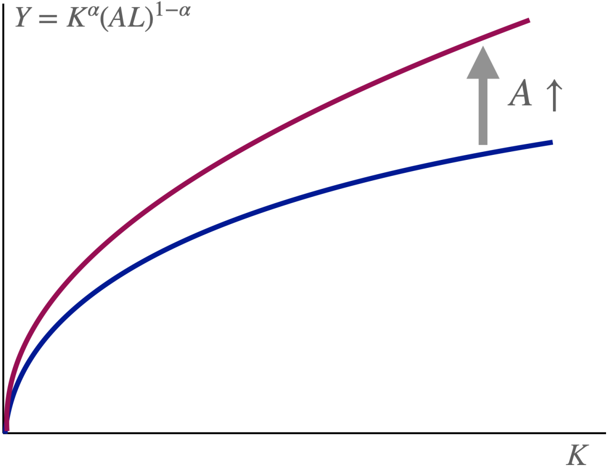
The Cobb-Douglas has properties that fit the data:

- the labor share (labor income / GDP) is constant over time
- the elasticity of substitution between capital and labor is close to 1

α is a parameter between 0 and 1.

- we see later: α is the capital income share

Cobb Douglas



What does higher α do?

Productivity Growth

$A(t)$ is an index of the state of "technology"

- anything that makes people more productive over time

A grows over time for reasons that are not modeled

- a major shortcoming of the model
- the Romer model is all about A growth

The growth rate of A is γ:

$$A(t) = e^{\gamma t} \tag{3}$$

Digression: Growth rates

Rate of change = change per time period = $\frac{A(t+\Delta t)-A(t)}{\Delta t}$

- for some time interval Δt (such as one year)
- e.g. \$1b per year

Growth rate = rate of change / level

$$g(A) = \frac{A(t+\Delta t) - A(t)}{\Delta t} \times \frac{1}{A(t)} \tag{4}$$

- e.g. 3 percent per year

Growth rates

We are in continuous time, so $\Delta t \to 0$

Then

$$\frac{A(t+\Delta t) - A(t)}{\Delta t} \to ? \tag{5}$$

Growth rate

$$g(A) = \frac{dA/dt}{A} \tag{6}$$

Constant growth

Constant growth at rate γ implies

$$A(t) = e^{\gamma t} \tag{7}$$

To check that:

- rate of change

$$dA/dt = \gamma \times e^{\gamma t} = \gamma A(t) \tag{8}$$

- growth rate

$$g(A) = \frac{dA/dt}{A} = \gamma \tag{9}$$

Labor Input

L grows over time at rate n:

$$L(t) = L(0)\, e^{nt}$$

Capital Accumulation

Output is divided between consumption and gross investment:

$$Y(t) = C(t) + I(t) \tag{10}$$

Investment contributes to the capital stock:

$$\dot{K}(t) = I(t) - \delta K(t) \tag{11}$$

$\dot{K}(t) = dK(t)/dt$ is the time derivative of $K(t)$.

- the change in K per "period".

δ is the rate of depreciation.

Capital Accumulation: Discrete Time

To better understand the law of motion for K, we look at a discrete time version.

Enter the period with capital stock $K(t)$.

Lose $\delta K(t)$ to depreciation.

Produce $I(t)$ new machines.

Change in the capital stock: $K(t+1) - K(t) = I(t) - \delta K(t)$.

Capital Accumulation: Discrete Time

Now we look at shorter time periods of length Δt.

$$K(t+\Delta t) - K(t) = [I(t) - \delta K(t)] \times \Delta t \tag{12}$$

or

$$\frac{K(t+\Delta t) - K(t)}{\Delta t} = I(t) - \delta K(t) \tag{13}$$

The change in capital per unit of time is given by investment minus depreciation.

Let $\Delta t \to 0$ then $\frac{K(t+\Delta t)-K(t)}{\Delta t} \to$

Choices

This is a closed economy. Saving equals investment: $S(t) = I(t)$.

Note: All of the above is simply a description of the production technology.

- Nothing has been said about how people behave.

People make two fundamental choices (in macro!):

1. How much to save / consume.
2. How much to work.

Choices

Work: we assume $L(t)$ is exogenous.

Consumption / saving:

- We assume that people save a fixed fraction of income:

$$C(t) = (1-s)\,Y(t) \tag{14}$$

- Equivalently:

$$I(t) = sY(t) \tag{15}$$

Model Summary

We have 3 equations that determine Y, K, I over time.

1. Cobb-Douglas production function

$$Y(t) = K(t)^{\alpha} \; [A(t) \; L(t)]^{1-\alpha} \tag{16}$$

2. Law of motion for capital:

$$\dot{K}(t) = I(t) - \delta K(t) \tag{17}$$

3. Constant saving rate: $I(t) = s \; Y(t)$.

Exogenous driving forces:

1. Constant population growth: $L(t) = L(0) \; e^{nt}$.
2. Constant productivity growth: $A(t) = A(0) \; e^{\gamma t}$.
   For now: $\gamma = 0$.

Model Comments

What have we assumed and why?

1. **Cobb-Douglas** production function
   A harmless simplification
   All we need is diminishing marginal product of K
2. $\dot{K} = I - \delta K$
   This is just accounting
3. **Constant saving rate**
   We currently don't care why people choose some value of s.
   We are looking at long-run growth. Constant s makes sense.
4. **Only K is accumulated**
   No innovation or human capital accumulation.
   But it does not matter exactly what K is, as long as we have diminishing MPK

So the model is really quite general.

The Law of Motion for Capital

Solving the Model

Even this simple model cannot be "solved" algebraically.

- That is, we cannot write the endogenous variables as functions of time.

This is almost never possible in **dynamic** models.

- Dynamic means: there are many time periods.
  All interesting macro models are dynamic.

What we can do is

1. graph the model and trace out qualitatively what happens over time.
2. solve the model for the long-run values of the endogenous variables (e.g. $K(t)$ as $t \to \infty$).

The Solow Diagram

We condense the model into a single equation in K.

- It will be a dynamic equation that tells us how K changes over time as a function of K.
- Then we graph the equation.

The beauty of it all: the **same analysis** applies to any model where some form of "capital" accumulation drives growth.

- Later we will see: a model where growth is due to R&D produces exactly the same graph
  (but with an important wrinkle that changes everything)

The Solow Equation

Start from the law of motion:

$$\dot{K}(t) = I(t) - \delta\, K(t) \tag{18}$$

Impose constant saving:

$$\dot{K}(t) = s\, Y(t) - \delta\, K(t) \tag{19}$$

Impose the production function:

$$\dot{K}(t) = s\underbrace{K(t)^{\alpha}\left[A(t)L(t)\right]^{1-\alpha}}_{Y(t)} - \delta K(t) \tag{20}$$

Per capita growth

We express everything in per capita terms.

- E.g., $y = Y/L$, etc.

Output per capita is derived from

$$Y = K^{\alpha}\ [AL]^{1-\alpha} = \frac{K^{\alpha}}{L^{\alpha}} \times \frac{A^{1-\alpha}L^{1-\alpha}}{L^{1-\alpha}} \tag{21}$$

as

$$\begin{aligned} y &= (K/L)^{\alpha}\ A^{1-\alpha} && (22) \\ &= k^{\alpha}A^{1-\alpha} && (23) \end{aligned}$$

Per capita growth

Let's ignore technical change for now and set A constant.

Start from

$$\dot{K} = sY - \delta K \tag{24}$$

and divide by L:

$$\dot{K}/L = \underbrace{sA^{1-\alpha}(K/L)^{\alpha}}_{Y/L} - \delta K/L \tag{25}$$

or

$$\dot{K}/L = sA^{1-\alpha}k^{\alpha} - \delta k \tag{26}$$

But we want an equation for $\dot{k} = \frac{d(K/L)}{dt}$; not for $\dot{K}/L = \frac{dK}{dt} \times \frac{1}{L}$

The law of motion for capital

Claim: $\dot{k} = \dot{K}/L - nk$.

The law of motion can then be written as

$$\dot{k} = \underbrace{sA^{1-\alpha}k^{\alpha} - \delta k}_{\dot{K}/L} - nk \tag{27}$$

Intuition:

- Suppose you invest nothing $(s = 0)$.
- Then K drops by δ each period due to depreciation.
- K/L declines even more because the number of people increases by n each period.

Proof of the law of motion

- Growth rate rule:
$$\dot{k}/k = \dot{K}/K - n \tag{28}$$
- Multiply by k:
$$\dot{k} = \dot{K}/L \times Lk/K - nk \tag{29}$$
$$= \dot{K}/L - nk \tag{30}$$
- From the law of motion we know $\dot{K}/L$
- Plug that in - done.

Digression: What modern macro would do

- Modern macro would replace the constant saving rate with an **optimizing household**.
- Households **maximize utility** of consumption, summed over all dates.
- They choose time paths of $c(t)$ and $K(t)$.
- The **saving rate** would be endogenous and depend on
  - the interest rate (marginal product of K)
  - productivity
  - population growth
  - **expectations** of all future variables.
- What do we gain from this complication?

Summary

The Solow model captures how economic growth is driven by

- capital accumulation
  - where capital could be more than just machines
  - human capital, knowledge, ...
- productivity growth

The model boils down to one equation:

$$\dot{k} = sA^{1-\alpha}k^{\alpha} - (n+\delta)k \tag{31}$$

Given a value for k at the start, this equation traces out the entire time path of k_t.

Reading

- Jones (2013), ch. 2
- Blanchard (2018), ch. 11

Further Reading:

- Romer (2011), ch. 1
- Barro and Sala-i Martin (1995), ch. 1.2

References I

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