Stochastic Growth Model

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Introduction

We now return to the stochastic growth model.

We study

- the planner's problem
- the competitive equilibrium

Then we introduce heterogeneity and risk sharing.

Planning solution

The history of shocks is θ^t.

Preferences:

$$\sum_{t=0}^{\infty} \beta^t \sum_{\theta^t} \Pr(\theta^t|\theta_0)\, u(c[\theta^t]) \tag{1}$$

Technology:

$$X = F(K, L, \theta) + (1-\delta)K - c \tag{2}$$

$$K' = X \tag{3}$$

Bellman equation

Define $k = K/L$.

$$V(k,\theta) = \max_{k' \in [0, f(k,\theta)+(1-\delta)k]} u\left(f(k,\theta)+(1-\delta)k-k'\right) \tag{4}$$
$$+\beta E\left[V\left(k',\theta'\right)|\theta\right] \tag{5}$$

First-order conditions

- Verify that A1-A5 hold ... Theorems 1-6 apply.
- FOC
$$u'(c) = \beta E V_k(k', \theta')$$
- Envelope
$$V_k(k,\theta) = u'(c)\left[f_k(k,\theta) + 1 - \delta\right]$$
- Euler
$$u'(c) = \beta E\left[u'(c')\left\{f_k(k',\theta') + 1 - \delta\right\}|\theta\right] \tag{6}$$
- Solution: $V(k,\theta)$ and $\pi(k,\theta)$ that "solve" the Bellman equation

Characterization

- Now for the bad news ... there really isn't much one can say about the solution analytically.
- But see Campbell (1994) for a discussion of a log-linear approximation.

Competitive Equilibrium

Competitive equilibrium

The model comes in 2 flavors.

1. Complete markets
   - for every history, there exists an asset that pays in that state of the world
   - the implication is complete risk sharing: all idiosyncratic risks are insured
   - aggregate risks remain
2. Incomplete markets
   - some securities are missing
   - there is no representative agent

Trading arrangements

- With complete markets, date 1 Arrow-Debreu trading is convenient
  - Uncertainty essentially disappears from the model.
- With incomplete markets, it is easiest to specify the set of securities available at each date.
  - Sequential trading.

Complete markets - Arrow Debreu trading

- The environment is standard.
- The history is of shocks is θ^t.
- Trading takes place at date 1.
- The point: This looks like a static model without uncertainty.

Market arrangements

Goods markets: standard

- buy and sell consumption at each node θ^t
- price $p(\theta^t)$

Labor markets: standard

- wage $w(\theta^t)$

Capital rental:

- households can buy goods in θ^t and give them to firms
- firms then pay $R\left(\theta^{t+1}\right)$ tomorrow
- this includes returning the undepreciated capital

Household: budget constraint

Expenditures in state θ^t:

$$x(\theta^t) = p(\theta^t)\left[c(\theta^t) + s(\theta^t)\right] \tag{7}$$

$p(\theta^t)$ is the price of the good in state θ^t.

c is consumption

s is "saving:" buy goods (capital) and rent to firms.

Household: budget constraint

Income in state θ^t:

$$y(\theta^t) = w(\theta^t) + R(\theta^t)\,s\left(\theta^{t-1}\right) \tag{8}$$

$w(\theta^t)$ is the wage.

$R(\theta^t)$ is the payoff from renting a unit of the good to the firm.

Both are state contingent.

Poor notation: keep in mind that θ^t follows θ^{t-1}

Household: budget constraint

Lifetime budget constraint:

$$\sum_{t=0}^{\infty}\sum_{\theta^t}\left[y(\theta^t)-x(\theta^t)\right]+p(\theta_0)s_0=0 \tag{9}$$

s_0 is the initial endowment of goods.

With Arrow-Debreu trading, there is a lifetime budget constraint, even under uncertainty.

- Because there really is no uncertainty any more.
- At each node, the household's spending and income are fully predictable.

Firms

Firms maximize the total value of profits.

- There is no discounting because of Arrow-Debreu trading.

Profits in state θ^t:

$$
\begin{aligned}
& p(\theta^t)[F(K[\theta^t], L[\theta^t], \theta_t) + (1-\delta)K[\theta^t]] \\
& -R(\theta^t)K(\theta^t) - w(\theta^t)L(\theta^t)
\end{aligned}
$$

Value of the firm: sum of profits over all states.

FOCs are standard:

- since the firm does not own anything, it maximizes profits state-by-state.

Competitive Equilibrium

- Allocation: $c(\theta^t), s(\theta^t), K(\theta^t), L(\theta^t)$.
- Price system: $p(\theta^t), w(\theta^t), R(\theta^t)$ for all histories θ^t.
- These satisfy:
  1. Household optimality.
  2. Firm optimality.
  3. Market clearing:
     - $L(\theta^t) = 1$.
     - $K(\theta^t, \theta_{t+1}) = s(\theta^t)$.
     - Goods market.

Competitive Equilibrium

Comments

- This looks like a static model without uncertainty.
  - Each history defines new goods: output, labor, capital rental.
- The setup is far more complicated than the recursive one.

Risk Sharing

- What if agents are heterogeneous?
- With complete markets, risk is perfectly shared.
- The simplest case: An endowment economy with Arrow-Debreu trading.
- The state is θ^t.

Risk Sharing
Households

- There are I types of households, indexed by i.
- Endowments are $y^i(\theta^t)$.
- Preferences are

$$\sum_t \sum_{\theta^t} \beta^t q(\theta^t) u^i\left(c^i[\theta^t]\right)$$

- Budget constraints:

$$\sum_t \sum_{\theta^t} p(\theta^t)\left[c^i(\theta^t) - y^i(\theta^t)\right] = 0 \tag{10}$$

First-order conditions are as usual:

$$q\left(\theta^t\right)\beta^t \frac{\partial u^i\left(c^i\left[\theta^t\right]\right)}{\partial c^i\left[\theta^t\right]} = \lambda_i p\left(\theta^t\right) \tag{11}$$

where λ_i is the Lagrange multiplier.

Risk Sharing

Complete risk sharing: For all θ^t the MRS is equated across households:

$$MRS\left(\theta^t, \hat{\theta}^\tau\right) = -\frac{\beta^t \, \partial u^i \left(c^i \left[\theta^t\right]\right) / \partial c^i \left[\theta^t\right]}{\beta^\tau \, \partial u^i \left(c^i \left[\hat{\theta}^\tau\right]\right) / \partial c^i \left[\hat{\theta}^\tau\right]} = \frac{p\left(\theta^t\right) / q\left(\theta^t\right)}{p\left(\hat{\theta}^\tau\right) / q\left(\hat{\theta}^\tau\right)}$$

Equivalently, the ratio of marginal utilities between 2 agents is the same for all θ^t:

$$\frac{\partial u^i \left(c^i \left[\theta^t\right]\right) / \partial c^i \left[\theta^t\right]}{\partial u^j \left(c^i \left[\theta^t\right]\right) / \partial c^j \left[\theta^t\right]} = \frac{\lambda_i}{\lambda_j} \tag{12}$$

Implications

Individual consumption still fluctuates because the aggregate endowment changes over time.

- aggregate risk cannot be insured

If there is no aggregate uncertainty, then individual consumption is constant.

Proof:

$$\partial u^i / \partial c^i = (\lambda_i / \lambda_1) \, \partial u^1 / \partial c^1 \tag{13}$$

That implies an increasing function $c^i = f_i\left(c^1\right)$ that is the same for all states θ^t.

Market clearing: $\sum_i c^i = \sum_i f_i\left(c^1\right) = y$.

This has a unique solution c^1. $\square$

Sequential Trading

Sequential Trading

- We set up the C.E. with sequential trading.
- If we want complete markets, we need **Arrow securities**.
- Each security, $a\left(\theta^{t+1}\right)$ is indexed by the state of the world in which it pays off: θ^{t+1}.
- The asset is purchased for price $\bar{p}(\theta^t, \theta')$ in state θ^t.
- It pays one unit of consumption if $\theta^{t+1} = [\theta^t, \theta']$.

Household

- Budget constraint:

$$
\begin{aligned}
c(\theta^t) + s(\theta^t) &= w(\theta^t) + a(\theta^t) + R(\theta^t) k(\theta^t) && (14) \\
s(\theta^t) &= \sum_{\theta_{t+1}} \bar{p}(\theta^t, \theta_{t+1}) a(\theta^t, \theta_{t+1}) + x(\theta^t) && (15) \\
k(\theta^t, \theta_{t+1}) &= x(\theta^t) && (16)
\end{aligned}
$$

- Numeraire: consumption at each node θ^t.

Household

- Household problem:

$$\max \sum_{t=0}^{\infty} \beta^t \sum_{\theta^t} \Pr(\theta^t|\theta_0)\, u(c[\theta^t]) \tag{17}$$

s.t. budget constraints for all θ^t.

Recursive household problem

- State: $(\overrightarrow{a}, k, \theta)$.
  - $\overrightarrow{a}$: holdings of all the $a(\theta)$.
- Given prices: w and $\bar{p}(\theta, \theta')$.
- Bellman equation:

$$V(\overrightarrow{a}, k, \theta) = \max_{c, a'(\theta'), k'} u(c) + \beta \sum_{\theta'} q(\theta'|\theta) V(\overrightarrow{a}', k', \theta')$$

s.t. budget constraint

$$\sum_{\theta'} \bar{p}(\theta, \theta') a'(\theta') + k' + c = w + a(\theta) + Rk$$

First order conditions

- For $a'(\theta')$:

$$u'(c)\bar{p}(\theta,\theta') = \beta q(\theta'|\theta)\frac{\partial V(\overrightarrow{a}'[\theta'],k',\theta')}{\partial a(\theta')} \tag{18}$$

- For k':

$$u'(c) = \beta\sum_{\theta'} q(\theta'|\theta)\frac{\partial V(\overrightarrow{a}',k',\theta')}{\partial k'} \tag{19}$$

First order conditions

- Envelope:

$$
\begin{aligned}
\partial V(\overrightarrow{a}, k, \theta) / \partial a(\theta) &= u'(c) && (20) \\
\partial V\left(\overrightarrow{a}, k, \hat{\theta}\right) / \partial a(\theta) &= 0 && (21) \\
\partial V(\overrightarrow{a}, k, \theta) / \partial k &= u'(c) R && (22)
\end{aligned}
$$

- Euler equation holds state by state for state contingent claims:

$$u'(c)\bar{p}(\theta, \theta') = \beta q(\theta'|\theta)\ u'\left(c\left[a'(\theta'), \theta'\right]\right) \quad (23)$$

- Euler equation for capital:

$$
\begin{aligned}
u'(c) &= \beta \sum_{\theta'} q(\theta'|\theta) R(\theta, \theta') u'\left(c\left[a'(\theta'), k', \theta'\right]\right) && (24) \\
&= \beta E\ R'\ u'(c')
\end{aligned}
$$

No arbitrage

- Since capital can be replicated by buying a set of Arrow securities:

$$\sum_{\theta'} \bar{p}\left(\theta, \theta'\right) R\left(\theta, \theta'\right) = 1 \tag{25}$$

- Proof: Solve (23) for $q(\theta'|\theta)$ and substitute into (24).

Equilibrium

- We can write down a sequential equilibrium definition, similar to the Arrow-Debreu.
  - Everything is indexed by θ^t.
- More powerful: Recursive Competitive Equilibrium.
  - Everything is a function of the current state.

Recursive CE

- Define an aggregate state vector: $S = (\theta, K)$.
  - In general: we need to keep track of the distribution of (θ_i, k_i) across households.
  - Here: all households are identical.
- The law of motion for the aggregate state:

$$\begin{aligned} \Pr\left(\theta' | \theta\right) &= q\left(\theta' | \theta\right) \\ K' &= G(\theta, K) \end{aligned}$$

where G is endogenous.

Recursive CE

Household

- Given:
  - aggregate state and its law of motion.
  - price functions: $w(S), R(S)$ and $\bar{p}(S, \theta')$.
- Bellman equation:

$$V(\overrightarrow{a}, k, S) = \max_{c, a'(\theta'), k'} u(c) + \beta \sum_{\theta'} q(\theta'|\theta) V(\overrightarrow{a}'[\theta'], k', S')$$

s.t. budget constraint

$$\sum_{\theta'} \bar{p}(\theta, \theta') a'(\theta') + k' + c = w(S) + a(\theta) + R(S) k$$

and aggregate law of motion

$$S' = G(S)$$

Recursive CE

- First-order conditions: unchanged.
- Solution: $V(a,k,S)$ and policy functions $c(a,k,S)$, $k' = \kappa(a,k,S)$.

Recursive CE

Firm

- Always the same because the firm has a static problem:
- Solution: $R(S), w(S)$.

Recursive CE

- Equilibrium objects:
  1. Household: Value function and policy functions.
  2. Firm: Price functions.
  3. Aggregate law of motion: $K' = G(\theta, K)$.
- Equilibrium conditions:
  1. Household optimality.
  2. Firm optimality.
  3. Market clearing.
  4. Consistency:
  $$G(\theta, K) = \kappa(K, \theta, K) \tag{26}$$
  where the household's policy function is $k' = \kappa(k, \theta, K)$.

Recursive CE

- Note: We could toss out all the Arrow securities without changing anything.
- The model boils down to:
  1. Euler equation for K: $u'(c) = \beta E[R'u'(c')]$
  2. Law of motion for K: $K' = F(K,L) + (1-\delta)K - c$.
  3. FOC: $R = F_K(K,L) + 1 - \delta$.
- This changes when individuals are not identical.

Recursive CE

What do we gain?

- Avoid having to carry around infinite histories.
- Equilibrium contains few objects.
  - Especially when the economy is **stationary**.
- All endogenous objects are functions.
  - Results from functional analysis can be used to determine their properties.
- Recursive CE is easy to compute.

Reading

- Acemoglu (2009) ch. 16-17.
- Krusell (2014) ch. 6
- Stokey et al. (1989) discuss the technical details of stochastic Dynamic Programming.
- Ljungqvist and Sargent (2004), ch. 2 talk about Markov chains. Ch. 7 covers complete market economies (Arrow-Debreu and sequential trading). Ch. 6: Recursive CE.
- Campbell (1994) discusses an analytical solution (approximate)

References I

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