The Growth Model: Discrete Time Dynamic Programming

Prof. Lutz Hendricks

Econ720

September 11, 2023

The Issue

We solved the growth model in **sequence language**.

- the solution is a sequence of objects that satisfies a bunch of difference equations

An alternative: **recursive** formulation.

- dynamic programming
- the solution is a set of functions

Dynamic Programming: An Informal Introduction

The basic idea of DP is to transform a many period optimization problem into a static problem.

To do so, we summarize the entire future by a **value function**.

The value function

- tells us the maximum utility obtainable from tomorrow onwards for any value of the state variables.
- an indirect utility function

Simplest example

A household lives for two periods.

$$\max u(c) + \beta v\left(c_1', c_2'\right) \tag{1}$$

subject to

$$s' = e + Rs - c \tag{2}$$

$$c_1' + pc_2' = Rs' \tag{3}$$

Notational convention: prime means tomorrow.

Lagrangian

$$\mathscr{L} = u\left(e + Rs - s'\right) + \beta v\left(Rs' - pc_2', c_2'\right) \tag{4}$$

FOC:

$$u'(c) = \beta v_1\left(.'\right) R \tag{5}$$
$$v_1\left(.'\right) p = v_2\left(.'\right) \tag{6}$$

Solution: c, c_1', c_2', s' that solve

- 2 FOCs
- 2 budget constraints

Dynamic Programming

The idea:

- solve one period at a time
- summarize the entire future with a "value function"
  (an indirect utility function)

If I know the value of saving, I can solve the first period problem.

Backward Induction

We solve this **backwards**, starting from the last period ("backwards induction").

The last period problem is a simple static one:

$$\max v(Rs - pc_2, c_2) \tag{7}$$

- All we need to know about the past is saving s (assets at the start of this period)

FOC:

- The same as the static condition from the Lagrangian

$$v_1(c_1, c_2)p = v_2(c_1, c_2) \tag{8}$$

Decision Rules

The FOC implicitly defines two decision rules of the form $c_j(s)$

Indirect utility is then also just a function of s:

$$
\begin{aligned}
W(s) &= v(c_1(s), c_2(s)) && (9) \\
&= \max_{c_2} v(Rs - pc_2, c_2) && (10)
\end{aligned}
$$

We call s the **state variable** and W the **value function.**

Log utility

Assume log utility tomorrow:

$$v(c_1, c_2) = \alpha \ln c_1 + (1-\alpha) \ln (c_2) \tag{11}$$

Then the static condition becomes

$$p\alpha / c_1 = (1-\alpha) / c_2 \tag{12}$$

or

$$pc_2 = \frac{1-\alpha}{\alpha} c_1 \tag{13}$$

With log utility, expenditure shares are constant (α for c_1 and $1-\alpha$ for c_2).

Consumption levels:

$$c_1 = \alpha Rs \tag{14}$$

$$c_2 = (1-\alpha) Rs / p \tag{15}$$

Value function

Then we can compute tomorrow's value function:

$$W(s) = \alpha \ln(\alpha Rs) + (1-\alpha)\ln((1-\alpha)Rs/p) \tag{16}$$

with marginal utility of wealth

$$W'(s) = \alpha/s + (1-\alpha)/s = 1/s \tag{17}$$

Solution:

- value function $W(s)$ and policy functions $c_j = f_j(s)$
- policy functions maximize $W(s)$ point by point
- W is the max of the RHS

Today's problem

We can now write today's problem as

$$V(s) = \max_{s'} u(s' - e) + \beta \underbrace{\left[\alpha \ln(\alpha R s') + (1-\alpha) \ln((1-\alpha) R s'/p)\right]}_{W(s')} \tag{18}$$

with FOC

$$u'(c) = \beta W'(s') = \beta / s' \tag{19}$$

Solution:

- value function $V(s)$ and policy function $s' = g(s)$
- g maximizes Bellman equation given W
- V is the max of the RHS

Cross check

Check against the Euler equation:

$$
\begin{aligned}
u'(c) &= \beta v_1\left(.'\right)R && (20)\\
&= \beta R \frac{\alpha}{\alpha R s'} = \beta / s' && (21)
\end{aligned}
$$

Both solutions given (of course) the same result.

Backward Induction I

Now consider a more general finite horizon problem

$$\max \sum_{t=1}^{T} u(c_t) \tag{22}$$

subject to $k_{t+1} = Rk_t - c_t$ and $k_{T+1} \geq 0$.

We again solve it backwards, starting from the last period.

Last Period

Consider the last date $t = T$.

The household cannot save: $k_{T+1} = 0$

Continuation value: $V(k, T+1) = 0$

- The problem is static: just eat all income

Terminal value:

$$V(k, T) = u(Rk) \tag{23}$$

Backward Induction

Now step back to $t = T - 1$

$$V(k, T-1) = \max u\left(Rk - k'\right) + \beta \underbrace{u\left(Rk'\right)}_{V(k',T)} \tag{24}$$

We can (in principle) solve for $V(k, T-1)$

Now step back to $t = T - 2$, etc.

Bellman equation for any period:

$$V(k, t) = \max u\left(Rk - k'\right) + \beta V\left(k', t+1\right) \tag{25}$$

Backward Induction

This is mainly useful for numerically solving the problem.

Sometimes, one can solve finite horizon problems analytically (see Huggett et al. (2006) for an example).

Infinite Horizon

Conceptually, we do exactly the same thing.

But now we don't have a last period that would solve for $W(s')$ on the RHS of the Bellman equation.

Instead, we impose that tomorrow's value function is the same as today's:

$$V(s) = W(s) \tag{26}$$

This works because the problem is **stationary**

- every period is the same, except for the value of the state variable s

The Growth Model: Planner's Problem

Infinite Horizon Problem

Suppose we solve the planner's problem with starting date t^*:

$$V = \max_{\{c_t, k_t\}} \sum_{t=t^*}^{\infty} \beta^{t-t^*} u(c_t) \tag{27}$$

subject to $k_{t+1} = f(k_t) - c_t \ \forall t$.

Call the optimal solution c_t^*, k_t^* and the implied lifetime utility V.

Value function

Claim: The only fact that we need to know to figure out V is k_{t^*}

- other past choices do not show up in preferences or constraints
- k_{t^*} is the **state variable** of the problem.

Therefore, we can define the **value function** (indirect utility function)

$$V(k_{t^*}) = \sum_{t=t^*}^{\infty} \beta^{t-t^*} u(c_{t^*}) \tag{28}$$

$$= \max \sum_{t=t^*}^{\infty} \beta^{t-t^*} u(f(k_t) - k_{t+1}) \tag{29}$$

Stationarity

Claim: $V(k_{t*})$ does not depend on t^*.

- Compare the value functions obtained from the problems starting at t^* and at $t^* + 1$.
- They are the same functions.
- That is, solving the problem yields the same value function regardless of the starting date.

Such a problem is called **stationary**.

- Not all optimization problems have this property.
- For example, if the world ends at some finite date, then the problem at $t*+1$ looks different from the problem at $t*$.

Time consistency

- What if we start the problem at $t^* + 1$?
- Would the planner want to change his optimal choices of k_{t^*+2}, k_{t^*+3}, and so on?
- The answer is obviously "no," ... although I won't prove this just yet.
- A problem with this property is known as **time consistent**:
  - Give the decision maker a choice to change his mind at a later date and he will choose the same actions again.
- Not all optimization problems have this property.
  - For example, changing the specification of discounting easily destroys time consistency (self-control problems arise).

Recursive structure

Now comes the key insight:

$$
\begin{aligned}
V(k_{t^*}) &= u(c_t^*) + \beta \left[\sum_{t=t^*+1}^{\infty} \beta^{t-(t^*+1)} u(c_{t^*}) \right] && (30) \\
&= u(c_t^*) + \beta V(k_{t^*+1}) && (31) \\
&= \max_{k_{t^*+1}} u(f(k_{t^*}) - k_{t^*+1}) + \beta V(k_{t^*+1}) && (32)
\end{aligned}
$$

We have

- one term reflecting current period utility
- a second term summarizing everything that happens in the future, given optimal behavior, as a function of k_{t^*+1}.

Recursive structure

Since this equation holds for any arbitrary start date, we may drop date subscripts.

Unfortunate convention in macro:

- no subscript = today
- prime = tomorrow: $k' = k_{t+1}$

This yields a **Bellman equation**:

$$\boxed{V(k) = \max_{k'} u\left(f(k) - k'\right) + \beta V\left(k'\right)}$$

Claim: **Solving the DP is equivalent to solving the original problem** (the Lagrangian).

- We will see conditions when this is true later.

Recursive structure

The convenient part of this is: we have transformed a multiperiod optimization problem into a two period (almost static) one.

If we knew the value function, solving this problem would be trivial.

The bad news is that we have transformed an algebraic equation into a **functional equation**.

The solution of the problem is a value function V and an optimal policy function

$$c = \phi(k)$$

Note that c cannot depend on anything other than k, in particular not on k's at other dates, because these don't appear in the Bellman equation.

Solution

A solution to the planner's problem is now a pair of functions

- value function $V(k)$
- policy function $\phi(k)$

These solve the Bellman equation in the following sense.

1. Given $V(k)$, setting $c = \phi(k)$ solves the max part of the Bellman equation.
2. Given that $c = \phi(k)$, the value function solves

$$V(k) = u(\phi(k)) + \beta V(f(k) - \phi(k))$$

Solution: Intuition

Given $V(k)$, setting $c = \phi(k)$ solves the max part of the Bellman equation.

This means:

Point by point, for each k:

$$\phi(k) = \arg\max_{c} u(c) + \beta V(f(k) - c) \tag{33}$$

$\phi(k)$ simply collects all the optimal c's – one for each k.

Solution: Intuition

$$V(k) = u(\phi(k)) + \beta V(f(k) - \phi(k))$$

Note that this uses the optimal policy function for c.

Think of the Bellman equation as a mapping in a function space:

$$V^{n+1} = T(V^n) = \max u(c) + \beta V^n(f(k) - c)$$

Given an input argument V^n the mapping produces an output arguments V^{n+1}.

The solution to the Bellman equation is the V that satisfies $V = T(V)$.

- a **fixed point**.

Reading

- Acemoglu (2009), ch. 6. Also ch. 5 for background material we will discuss in detail later on.
- Ljungqvist and Sargent (2004), ch. 3 (Dynamic Programming), ch. 7 (Recursive CE).
- Stokey et al. (1989), ch. 1 is a nice introduction.

References I

Acemoglu, D. (2009): *Introduction to modern economic growth*, MIT Press.

Huggett, M., G. Ventura, and A. Yaron (2006): "Human Capital and Earnings Distribution Dynamics," *Journal of Monetary Economics*, 53, 265–290.

Ljungqvist, L. and T. J. Sargent (2004): *Recursive macroeconomic theory*, 2nd ed.

Stokey, N., R. Lucas, and E. C. Prescott (1989): "Recursive Methods in Economic Dynamics," .