Problem Set 3: Growth Model in Discrete Time

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1 Land Prices with Capital Accumulation

Consider the following economy with land and capital.

Demographics: There is a representative household of unit mass who lives forever.

Preferences: $\sum_{t=0}^{\infty} \beta^t u(c_t)$

Endowments: At $t = 0$ the household is endowed with capital K_0 and land L. The aggregate endowment of land is fixed.

Technologies:

$$K_{t+1} = A\,F(K_t, L_t) + (1-\delta)\,K_t - c_t \tag{1}$$

where A is an exogenous productivity factor, δ is the depreciation rate of capital, and c is consumption. The production function has constant returns to scale.

Markets: Production takes place in a representative firm which rents capital and land from households. There are competitive markets for goods (price 1), land (p_t), capital rental (r_t), and land rental (q_t).

Questions:

1. Set up the household's Bellman equation. Define a solution to the household problem.
2. Define a competitive equilibrium.
3. Determine the effects of the following changes on steady state prices and quantities. A qualitative characterization is sufficient (which variables increase/decrease?): L increases, A increases.

1.1 Answer: Land Prices with Capital Accumulation

(a) Since the household's portfolio composition will be indeterminate, it can be set up with a single asset a with gross return R. This is standard and leads to an Euler equation $u'(c) = \beta\,R'\,u'(c')$. The budget constraint is $a' = R\,a - c$.

An alternative is to set up a problem with two assets. The budget constraint is then $k' + p\,l' + c = (r+1-\delta)\,k + (p+q)\,l$. The Bellman equation is

$$V(k,l) = \max\, u\left((r+1-\delta)\,k + (p+q)\,l - k' - p\,l'\right) + \beta\,V(k',l') \tag{2}$$

The first-order conditions are

$$\begin{aligned} u'(c) &= \beta\,V_k(k',l') \\ u'(c)\,p &= \beta\,V_l(k',l') \end{aligned} \tag{3}$$

The envelope conditions are

$$\begin{aligned} V_k(k,l) &= u'(c)\,(r+1-\delta) \\ V_l(k,l) &= u'(c)\,(p+q) \end{aligned}$$

Combining those yields the Euler equation and the arbitrage condition $r' + 1 - \delta = (p' + q')/p$ which says that both assets must yield the same rate of return.

(b) A competitive equilibrium is a set of sequences $(c_t, a_t, K_t, R_t, r_t, q_t, p_t)$ which satisfy

- Household: Euler equation and budget constraint.
- Firms: $r_t = A\,f'(k_t^F)$ and $q_t = A\,[f(k_t^F) - f'(k_t^F)\,k_t^F]$ where $k_t^F = K_t/L_t$.
- Identitites: $a_t = K_t + p_t\,L_t$ and $R_t = 1 - \delta + r_t$.
- Goods market clearing (1).
- Asset market clearing: $R_t = (p_t + q_t)/p_{t-1}$.

(c) The steady state is characterized by a recursive system. $R = 1/\beta$. $k^F = K/L$ is determined from $r = R - 1 + \delta = f'(k^F)$. Then q follows from the firm's first-order condition. Market clearing implies $c = L\,[A\,f(k^F) - \delta k^F]$. Finally, the price of land equals $p = q/(R-1)$.

An increase in L has no effect on R, k^F, q, p. c and K rise in proportion to L. The intuition is that the economy has constant returns to scale. Increasing the fixed factor simply raises all real variables in proportion, but leaves prices unaffected.

An increase in A has no effect on R. Hence, $f'(k^F)$ must fall and k^F must rise. This in turn implies a higher q. Therefore p rises. From market clearing, consumption increases.

2 Education Costs

Consider the following version of a standard growth model with human capital. The planner solves

$$\max \sum_{t=1}^{\infty} \beta^t u\,(c_t) \tag{4}$$

s.t.

$$k_{t+1} = (1-\delta)\,k_t + x_{kt} \tag{5}$$

$$h_{t+1} = (1-\delta)\,h_t + x_{ht} \tag{6}$$

$$c_t + x_{kt} + \eta x_{ht} = f\,(k_t, h_t) \tag{7}$$

with k_1 and h_1 given. Here c is consumption, k is physical capital, h is human capital, and η is a constant representing education costs. Assume that the production function is Cobb-Douglas:

$$f\,(k,h) = z k^{\alpha} h^{\varepsilon} \tag{8}$$

where z is a constant technology parameter and $\alpha + \varepsilon < 1$.

Questions:

1. Derive the first-order condition for the planner's problem using Dynamic Programming. Define a solution in sequence language and in functional language.
2. Solve for the steady state levels of k/h and k.
3. Characterize the impact of cross-country differences in education costs (η) on output per worker in steady state. In particular, calculate the ratio of outputs per worker for two countries that only differ in their η's.

2.1 Answer: Education Costs

(a) The planner's Bellman equation is

$$\begin{aligned} V(k,h) &= \max u(c) + \beta V((1-\delta)k + x_k, (1-\delta)h + x_h) \\ &\quad + \lambda[f(k,h) - c - x_k - \eta x_h - g] \end{aligned}$$

First-order conditions:

$$\begin{aligned} u'(c) &= \lambda \\ \beta V_k(.') &= \lambda \\ \beta V_h(.') &= \eta\lambda \end{aligned}$$

Envelope conditions:

$$\begin{aligned} V_k(k,h) &= \beta V_k(k',h')(1-\delta) + \lambda f_k(k,h) \\ V_h(k,h) &= \beta V_h(k',h')(1-\delta) + \lambda f_h(k,h) \end{aligned}$$

Simplify to obtain an Euler equation, which is perfectly standard:

$$u'(c) = \beta u'(c')[1-\delta+f_k(k',h')]$$

In addition, there is a second Euler equation

$$u'(c) = \beta u'(c')[1-\delta+f_h(k',h')/\eta]$$

which can be made into a static condition

$$1-\delta+f_k(k',h') = 1-\delta+f_h(k',h')/\eta$$

A solution consists of sequences c, k, h, x_k, x_h that solve 2 laws of motion, 1 feasibility condition, 2 first-order conditions.

(b) Imposing functional forms: $k/h = \eta\alpha/\varepsilon$. The steady state capital stock is determined by

$$1/\beta = z\alpha k^{\alpha-1+\varepsilon}[\varepsilon/(\alpha\eta)]^{\varepsilon} + 1 - \delta$$

Steady state output is

$$f(k_{ss}, h_{ss}) = z k_{ss}^{\alpha+\varepsilon}[\varepsilon/(\alpha\eta)]^{\varepsilon}$$

(c) An increase in η reduces both k and h in steady state. How much do education costs affect output per worker? The output ratio of two countries is

$$\frac{f^A}{f^B} = \left(\frac{k_{ss}^A}{k_{ss}^B}\right)^{\alpha+\varepsilon} \left(\frac{\eta_B}{\eta_A}\right)^{\varepsilon}$$

The ratio of capital stocks can be derived from the steady state k equation:

$$k_{ss}^A / k_{ss}^B = (\eta_A/\eta_B)^{\varepsilon/(\alpha+\varepsilon-1)}$$

Finally,

$$f^A/f^B = (\eta_A/\eta_B)^{\varepsilon/(1-\alpha-\varepsilon)}$$