

# Stochastic Optimization

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# Stochastic Optimization

- ▶ We add shocks to the growth model.
- ▶ Recursive methods are needed.
- ▶ The resulting model is used to study
  - ▶ business cycles
  - ▶ asset pricing
  - ▶ wealth distribution

# Model

Demographics:

- ▶ 1 representative household lives forever

Preferences: expected utility

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \quad (1)$$

- ▶  $\mathbb{E}_0$  denotes the expectation given information at date 0.
- ▶ The household needs to know the probability distribution of  $c_t$  implied by his choices

Endowments:  $k_0$  at  $t = 0$

# Technology

$$k_{t+1} = f(k_t, \theta_t) - c_t \quad (2)$$

$\theta_t$ : productivity shock

a Markov process with  $N$  (finite) discrete values

$$\theta_t \in \{\theta^1, \dots, \theta^N\} \quad (3)$$

and transition matrix

$$\Pr(\theta_{t+1} = \theta^j \mid \theta_t = \theta^i) = \Omega_{ij}$$

# Planner's Problem

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \quad (4)$$

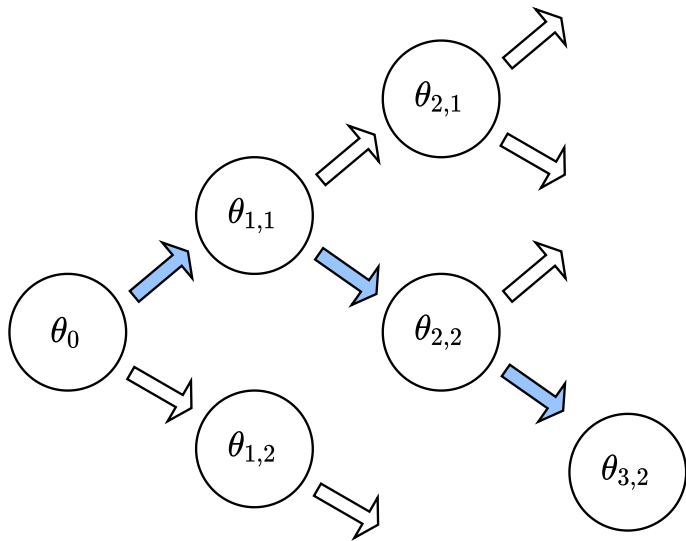
subject to

$$k_{t+1} = f(k_t, \theta_t) - c_t \quad (5)$$

Key problem: **what does the planner choose?**

- ▶ He cannot choose *sequences*  $\{c_t, k_t\}$  because we don't know the realizations of  $\theta_t$ .
- ▶ He must choose a sequence  $\{c_t, k_t\}$  for *every possible history* of shocks  $(\theta_0, \theta_1, \dots)$ .
- ▶ *state contingent plans*.

## State Contingent Plans



# Notation

Define a history up to date  $t$  as

$$s^t = (\theta_0, \dots, \theta_t) \quad (6)$$

A state contingent plan is a mapping from histories to choices:

$$c_t = c(s^t) \quad (7)$$

and

$$k_{t+1} = \kappa(s^t) \quad (8)$$

Two period example



## Two period example

The problem is

$$\max \mathbb{E}_1 \sum_{t=1}^2 \beta^{t-1} u(c_t)$$

subject to

$$k_2 = f(k_1, \theta_1) - c_1$$

$$c_2 = f(k_2, \theta_2)$$

$$\theta_1 \text{ known}$$

## Two period example

Write out the expectation explicitly:

$$\begin{aligned} \max & u(c_1) + \sum_{j=1}^N \Pr(\theta_2 = \theta^j) \beta u(c_2(\theta^j)) \\ & + \lambda_1 [f(k_1, \theta_1) - c_1 - k_2] \\ & + \sum_{j=1}^N \lambda_2(\theta^j) [f(k_2, \theta^j) - c_2(\theta^j)] \end{aligned}$$

The household chooses  $c_1, k_2$  and  $c_2(\theta^j)$ .

The budget constraint must hold in every history.

## First-order conditions

$$\begin{aligned}c_1 &: u'(c_1) = \lambda_1 \\k_2 &: \lambda_1 = \sum \lambda_2(\theta^j) f_k(k_2, \theta^j) \\c_2(\theta^j) &: \Pr(\theta_2 = \theta^j) \beta u'(c_2(\theta^j)) = \lambda_2(\theta^j)\end{aligned}$$

Euler equation:

$$\begin{aligned}u'(c_1) &= \beta \sum \Pr(\theta_2 = \theta^j) u'(c_2(\theta^j)) f_k(k_2, \theta^j) \\&= \beta E \{ u'(c_2) f_k(k_2, \theta_2) | \theta_1 \}\end{aligned}$$

Interpretation ...

# Notes

1.  $\mathbb{E}\{u'(c_2(\theta_2))f_k(k_2, \theta_2)\} \neq \mathbb{E}u'(c_2(\theta_2)) \times \mathbb{E}f_k(k_2, \theta_2)$
2. Naively maximizing

$$u(c_1) + \mathbb{E}\beta u(f(k_1; \theta_1) - c_1) \quad (9)$$

treating  $\mathbb{E}$  as a number (!)  
would have produced the right answer.

Many periods

## Many periods

A history of length  $t$  is  $s^t$ .

► e.g.,  $(\theta^2, \theta^6, \dots, \theta_t)$

The household chooses  $c(s^t)$  and  $k(s^t)$  to maximize

$$\sum_t \sum_{s^t} p(s^t) \beta^t u(c(s^t))$$

subject to

$$\begin{aligned} x(s^t) + c(s^t) &= f(k(s^t), \theta(s^t)), \quad \forall s^t \\ k(s_{t+1}, s^t) &= x(s^t), \quad \forall s^t, s_{t+1} \end{aligned}$$

## Important point

You need the constraint

$$k(s_{t+1}, s^t) = x(s^t), \quad \forall s^t, s_{t+1} \quad (10)$$

that ensures  $k(s^{t+1})$  is the same for all  $s^t$ !

The following would be **wrong**:

$$k(s^{t+1}) + c(s^t) = f(k(s^t), \theta(s^t)) \quad (11)$$

Try to write down a Lagrangian and take FOCs - it does not work.

# Lagrangian

$$\sum_t \sum_{s^t} p(s^t) \beta^t u(f(k(s^t), \theta(s^t)) - x(s^t)) \\ + \sum_t \sum_{s^t} \sum_{s_{t+1}} \varphi(s_{t+1}, s^t) [k(s_{t+1}, s^t) - x(s^t)]$$

FOC:

$$x(s^t) : \beta^t p(s^t) u'(s^t) = \sum_{s_{t+1}} \varphi(s_{t+1}, s^t)$$

$$k(s^t) : \beta^t p(s^t) u'(s^t) f_k(s^t) = \varphi(s^t)$$

What do these say in words?



## Euler equation

$$\beta^t p(s^t) u'(s^t) = \sum_{s_{t+1}} f_k(s_{t+1}, s^t) \beta^{t+1} p(s_{t+1}, s^t) u'(s_{t+1}, s^t) \quad (12)$$

Divide by  $\beta^t p(s^t)$  and note that

$$p(s_{t+1}, s^t) / p(s^t) = \Pr(s_{t+1} | s^t) \quad (13)$$

This yields the usual Euler equation:

$$u'(s^t) = \beta \sum_{s^{t+1}} \Pr(s_{t+1} | s^t) f_k(s_{t+1}, s^t) u'(s_{t+1}, s^t) \quad (14)$$

$$= \beta \mathbb{E} \{ f_k(t+1) u'(c(t+1)) \mid s^t \} \quad (15)$$

# Euler equation

- ▶ Be careful with notation.
- ▶ It would be wrong to write

$$\beta^t p(s^t) u'(s^t) = \sum_{s^{t+1}} f_k(s^{t+1}) \beta^{t+1} p(s^{t+1}) u'(s^{t+1}) \quad (16)$$

- ▶ Why is this wrong?

# The point

- ▶ With uncertainty, the sequence approach is a mess.
- ▶ Two solutions:
  1. Recursive methods.
  2. A shortcut: Maximize as if one could choose sequences.

## A shortcut

Let's proceed mechanically as if we were choosing sequences  $\{c_t, k_t\}$ :

$$\begin{aligned}\Gamma &= E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \\ &\quad + E_0 \sum_{t=0}^{\infty} \lambda_t [f(k_t, \theta_t) - c_t - k_{t+1}]\end{aligned}$$

This is not quite right: the budget constraint should bind state by state, not just in expectation.

- See the 2 period example.

## A shortcut

- First-order conditions

$$\begin{aligned}E_0 \beta^t u'(c_t) &= E_0 \lambda_t \\E_0 \lambda_{t-1} &= E_0 \lambda_t f_k(k_t, \theta_t)\end{aligned}$$

- Euler equation:

$$E_0 u'(c_t) = \beta E_0 u'(c_{t+1}) f_k(k_{t+1}, \theta_{t+1})$$

- When date  $t$  arrives:

$$u'(c_t) = \beta E_t u'(c_{t+1}) f_k(k_{t+1}, \theta_{t+1})$$

- The point: Treating the  $E$  as a constant in the optimization problem actually yields the right result!

## A shortcut

This approach is advocated by Chow (1997)

Why does the shortcut work, even though it is entirely wrong?

- ▶ One reason:  $\mathbb{E}$  is linear:  $\mathbb{E}(x) = \sum p(x_i) x_i$ .
- ▶ The recursive approach makes this clearer...

## Recursive Approach

# Recursive Approach

- ▶ We generalize the DP approach introduced for deterministic problems.
- ▶ Nothing of substance changes, except there is an  $E$  in front of each equation.
- ▶ Why does nothing change?
  - ▶ Because  $E$  is a linear operator - just the sum of  $\Pr(s'|s) \times \text{outcome}(s')$ .
- ▶ We start by assuming that stochastic DP works as expected.
- ▶ Then we state the conditions under which it works.



# Recursive Approach to the Growth Model

State vector:  $s_t = (k_t, \theta_t)$ .

Bellman equation:

$$\begin{aligned} V(k, \theta) &= \max u(c) + \beta E V(f(k, \theta) - c, \theta') \\ &= \max u(c) + \beta \sum_{\theta'} \Pr(\theta' | \theta) V(f(k, \theta) - c, \theta') \end{aligned}$$

First-order conditions:

$$u'(c) = \beta E V_k(k', \theta')$$

Envelope condition:

$$V_k(k, \theta) = \beta E V_k(k', \theta') f_k(k, \theta)$$

Now we can see why the shortcut works.

## Euler Equation

$$u'(c) = \beta \sum \Pr(\theta'|\theta) V_k(k', \theta') \quad (17)$$

$$= \beta \sum \Pr(\theta'|\theta) f_k(k', \theta') \underbrace{\left[ \beta \sum \Pr(\theta''|\theta') V_k(k'', \theta'') \right]}_{u'(c')} \quad (18)$$

Or

$$u'(c) = \beta E \{ u'(c') f_k(k', \theta') \} \quad (19)$$

# Recursive Solution

Solution:  $V(k, \theta), c(k, \theta)$  that satisfy:

1. Given  $V$ ,  $c(k, \theta)$  maximizes the right-hand-side of the Bellman equation.
2.  $V$  is a fixed point of the Bellman operator:

$$V(k, \theta) = u(c[k, \theta]) + \beta E V(f[k, \theta] - c[k, \theta], \theta') \quad (20)$$

# Continuous state Markov chains

- ▶ What if the random vector takes on a continuum of values?
- ▶ Simply replace sums with integrals when calculating expectations.

# Continuous state Markov chains

- ▶ Assume that  $\theta \in \Theta = [\underline{\theta}, \bar{\theta}]$ .
- ▶ The evolution of  $\theta$  is governed by a **transition function**:

$$\Pr(\theta' \leq x | \theta) = \Pi(x, \theta) \quad (21)$$

- ▶ This is really a cdf conditional on  $\theta$ .
- ▶ The transition density for this CDF is  $\pi$  with

$$\int_{\underline{\theta}}^x \pi(\theta', \theta) d\theta' = \Pi(x, \theta) \quad (22)$$

- ▶ This is the analogue to the transition matrix  $\Pr(\theta' | \theta)$  in the discrete case.

# Continuous state Markov chains

- ▶ The conditional expectation of  $f$  is then

$$\begin{aligned} E[f(\theta') | \theta] &= \int_{\underline{\theta}}^{\bar{\theta}} f(\theta') \pi(\theta', \theta) d\theta' \\ &= \int_{\underline{\theta}}^{\bar{\theta}} f(\theta') \Pi(d\theta', \theta) \end{aligned}$$

- ▶ The point: In the continuous case, simply replace all the  $\sum_{\theta'} \Pr(\theta' | \theta)$  with  $\int \pi(\theta', \theta) d\theta'$ .

# Reading

- ▶ Acemoglu (2009), ch. 16.1-16.2.
- ▶ Krusell (2014), ch. 6.
- ▶ Stokey et al. (1989) discuss the technical details of stochastic Dynamic Programming.
- ▶ Ljungqvist and Sargent (2004), ch. 2 talk about Markov chains.

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- Acemoglu, D. (2009): *Introduction to modern economic growth*, MIT Press.
- Chow, G. C. (1997): *Dynamic Economics: Optimization by the Lagrange Method*, Oxford University Press, USA.
- Krusell, P. (2014): “Real Macroeconomic Theory,” Unpublished.
- Ljungqvist, L. and T. J. Sargent (2004): *Recursive macroeconomic theory*, 2nd ed.
- Stokey, N., R. Lucas, and E. C. Prescott (1989): “Recursive Methods in Economic Dynamics,” .