

# Endogenous Growth: AK Model

Prof. Lutz Hendricks

Econ720

October 14, 2020

# Endogenous Growth

- ▶ Why do countries grow?
  - ▶ A question with large welfare consequences.
- ▶ We need models where growth is endogenous.
- ▶ The simplest model is a variation of the Ramsey model.
  - ▶ Growth can be sustained if the MPK is bounded below.
  - ▶ AK model

# Necessary Conditions for Sustained Growth

- ▶ How can growth be sustained without exogenous productivity growth?
- ▶ A necessary condition: **constant returns to the reproducible factors**.
  - ▶ The production functions for inputs that can be accumulated must be linear in those inputs.
  - ▶ Example: In the growth model,  $K$  would have to be produced with a technology that is linear in  $K$
- ▶ This motivates a simple class of models in which
  1. only  $K$  can be produced and
  2. the production function is  $AK$ .
- ▶ This can be thought of as a reduced form for more complex models (we'll see examples).

## Solow AK model

To see what is required for endogenous growth, consider the Solow model:

$$g(k) = sf(k)/k - (n + \delta) \quad (1)$$

Positive long-run growth requires: As  $k \rightarrow \infty$  it is the case that

$$f(k)/k > n + \delta \quad (2)$$

L'Hopital's rule implies (if  $f'$  has a limit):

$$\lim f(k)/k = \lim f'(k) \quad (3)$$

Sustained growth therefore requires:

$$\lim_{k \rightarrow \infty} f'(k) > n + \delta \quad (4)$$

# Necessary Conditions for Sustained Growth

- ▶ This argument is more general than the Solow model.
  - ▶ It does not matter how  $s$  is determined.
- ▶ If  $\lim_{k \rightarrow \infty} f'(k)$  exists, the production function has **asymptotic constant returns to scale**.

$$f(k) \rightarrow Ak + B \quad (5)$$

- ▶ It is fine to have diminishing returns for finite  $k$ .

## Examples

$$f(k) = Ak + Bk^\alpha \quad (6)$$

- ▶  $0 < \alpha < 1$
- ▶  $f(k)/k \rightarrow A$  as  $k \rightarrow \infty$

CES production function with high elasticity of substitution:

$$F(K, L) = \left[ \mu K^\theta + (1 - \mu) L^\theta \right]^{1/\theta} \quad (7)$$

- ▶  $f(k) = [\mu k^\theta + 1 - \mu]^{1/\theta}$
- ▶ Elasticity of substitution:  $\varepsilon = (1 - \theta)^{-1}$ .
- ▶ If  $\theta > 0$  [ $\varepsilon > 1$ ],  $f(k)/k \rightarrow \mu^{1/\theta}$ .

# AK Solow Model

- ▶ In the Solow model, assume  $f(k) = Ak$ .
- ▶ Law of motion:

$$g(k) = sA - n - \delta \quad (8)$$

- ▶ Changes in parameters alter the growth rate of  $k$ .
- ▶ The model does not have any transitional dynamics:  $k$  always grows at rate  $sA - n - \delta$ .

# AK Solow Model

- ▶ It is not necessary to have constant returns in all sectors of the economy.
- ▶ Imagine that  $c$  is produced from  $k$  with diminishing returns to scale:  $c = [(1-s)Ak]^\varphi$  with  $\varphi < 1$ .
- ▶ The law of motion for  $k$  is unchanged (so is the balanced growth rate of  $k$ ).
- ▶ This model still has a balanced growth path with a strictly positive growth rate, but not  $c$  and  $k$  grow at different constant rates:

$$g(c) = \varphi g(k) \tag{9}$$



# AK Neoclassical Growth Model

# AK neoclassical growth model

This model adds optimizing consumers to the  $Ak$  model.

Households maximize

$$\int_{t=0}^{\infty} e^{-(\rho-n)t} u(c_t) dt \quad (10)$$

subject to the flow budget constraint

$$\dot{k} = (r-n)k - c \quad (11)$$

There is no labor income because in the  $Ak$  world all income goes to capital.

## AK neoclassical growth model

For balanced growth we need

$$u(c) = c^{1-\sigma}/(1-\sigma) \quad (12)$$

The optimality conditions are the same as in the Cass-Koopmans model:

$$g(c) = (r - \rho)/\sigma$$

and the transversality condition (assuming constant  $r$ )

$$\lim_{t \rightarrow \infty} k_t e^{-(r-n)t} = 0 \quad (13)$$

# Firms

Firms maximize period profits.

The first-order condition is  $r = A - \delta$ .

# Equilibrium

An allocation:  $c(t), k(t)$ .

A price system:  $r(t)$ .

These satisfy:

1. Household: Euler, budget constraint (TVC).
2. Firm: 1 foc.
3. Market clearing:

$$\dot{k} = Ak - (n + \delta)k - c \quad (14)$$

## Summary

Simplify into a pair of differential equations:

$$\dot{k} = (A - \delta - n)k - c \quad (15)$$

$$g(c) = (A - \delta - \rho)/\sigma \quad (16)$$

Boundary conditions:  $k_0$  given and the TVC.

Of course, we could have simply taken the equilibrium of the standard growth model and replaced  $f'(k) = A$  and  $f(k) = Ak$ .

## Bounded utility

We need restrictions on the parameters that ensure bounded utility.  
Lifetime utility is

$$\int_0^{\infty} e^{-(\rho-n)t} [c_0 e^{g(c)t}]^{1-\sigma} dt / (1-\sigma) \quad (17)$$

Boundedness then requires that  $n - \rho + (1 - \sigma)g(c) < 0$ .

Instantaneous utility cannot grow faster than the discount factor  $(\rho - n)$ .

# Transitional dynamics

This model has no transitional dynamics.

Consumption growth is obviously constant over time.

To show that  $g(k)$  is constant: we need to solve for  $k(t)$  in closed form.

► Details



# Summary

The AK model has a very simple equilibrium.

1. The saving rate is constant.
2. All growth rates are constant.

This is very convenient, but also very limiting in many applications.

## How to think about AK models?

In the data, there is at least one non-reproducible factor: labor.  
Do models with constant returns to reproducible factors make sense?

The best way of thinking about AK models:

- ▶ a **reduced form** for a model with multiple factors
- ▶ there may be transition dynamics, but it does not matter if you are interested in long-run issues
- ▶ there may be fixed factors, but it does not matter if there are constant returns to reproducible factors.

## Examples: AK as reduced form

1. Human capital:  $F(K, hL)$  with  $K$  and  $h$  reproducible.
2. Externalities:
  - 2.1 Romer (1986). For the firm  $F(k_i, l_i K) = K^{1-\alpha} k_i^\alpha l_i^\theta$
  - 2.2 Firms take  $K$  as given - diminishing returns to  $k_i$ .
  - 2.3 In equilibrium:  $K = \sum k_i$  - constant returns to scale to  $K$ .
3. Increasing returns to scale at the firm level:  $y = Ak^\alpha l^{1-\alpha}$ 
  - 3.1  $A$  can be produced somehow - R&D.
  - 3.2 Need imperfect competition.

Example: Lucas (1988)

## Example: Lucas (1988)

A classic endogenous growth paper.

Growth is due to human capital accumulation.

The model has an AK reduced form.

## Model: Lucas (1988)

Demographics:

- ▶ A representative, infinitely lived household.

Preferences:

$$\int_0^{\infty} e^{-\rho t} u(c_t) dt \quad (18)$$

$$u(c) = c^{1-\sigma} / (1-\sigma) \quad (19)$$

Technology:

$$\dot{k} + c = f(k, h, l) - \delta k \quad (20)$$

$$\dot{h} = e(k, h, l) - \delta h \quad (21)$$

where  $f(k, h, l) = k^{\alpha}(lh)^{1-\alpha}$  and  $e(k, h, l) = B(1-l)h$

## Lucas (1988): Balanced growth rates

Law of motion for  $h$ :

$$g(h) = B(1 - l) - \delta \quad (22)$$

Law of motion for  $k$ :

$$g(k) + c/k = (lh/k)^{1-\alpha} - \delta \quad (23)$$

Therefore:

$$g(c) = g(k) = g(h) \quad (24)$$

## Lucas (1988): Optimality

Current value Hamiltonian:

$$H = u(c) + \lambda [e(k, h, l) - \delta h] + \mu [f(k, h, l) - \delta k - c] \quad (25)$$

FOCs:

$$\partial H / \partial c = u'(c) - \mu = 0 \quad (26)$$

$$\partial H / \partial u = \lambda e_l + \mu f_l = 0 \quad (27)$$

$$\rho \lambda - \dot{\lambda} = \lambda [e_h - \delta] + \mu f_h \quad (28)$$

$$\rho \mu - \dot{\mu} = \mu [f_k - \delta] + \lambda e_k \quad (29)$$

Major simplification from  $e_k = 0$ .



## Optimality

Euler equation (using  $e_k = 0$ )

$$g(c) = \frac{f_k - \delta - \rho}{\sigma} \quad (30)$$

From FOC for  $h$ :

$$-g(\lambda) = e_h - \delta - \rho + \mu/\lambda f_h \quad (31)$$

FOC for  $u$ :

$$\frac{\mu}{\lambda} f_h = -\frac{e_l f_h}{f_u} = (Bh) \frac{l}{h} = Bl \quad (32)$$

Substitute into FOC for  $h$ :

$$-g(\lambda) = B - \delta - \rho \quad (33)$$

This is an exogenous constant!

## Balanced growth

Constant  $f_k$  requires constant  $k/h$ .

Then

$$\frac{\mu}{\lambda} = \frac{Bh}{f_l} = \frac{Bh}{(1-\alpha)(k/h)^\alpha l^{1-\alpha}} \quad (34)$$

requires constant  $\mu/\lambda$ .

Then  $g(u_c) = -g(\mu) = -g(\lambda) = B - \delta - \rho$ .

This determines the interest rate:

$$r = f_k - \delta = B - \delta \quad (35)$$

The balanced growth rate is determined by the linear human capital technology:

$$g(c) = \frac{B - \delta - \rho}{\sigma} \quad (36)$$

# Intuition

- ▶ The household has 2 assets:  $k$  and  $h$ .
- ▶ One asset has a constant rate of return:
  - ▶ give up 1 unit of time to gain a fixed increment of future income
  - ▶ regardless of current values of  $k$  and  $h$ .
- ▶ This pins down the interest rate on the other asset by no arbitrage.
- ▶ All of this has implicitly assumed an interior solution!

# Summary

Sustained growth requires that inputs are produced with constant returns to reproducible inputs.

Then the model is (at least asymptotically) of the AK form:

$$\dot{K} = AK.$$

The AK model is a reduced form of something more interesting.

# Reading

- ▶ Acemoglu (2009), ch. 11.
- ▶ Krueger, "Macroeconomic Theory," ch. 9.
- ▶ Krusell (2014), ch. 8.
- ▶ Barro and Martin (1995), ch. 1.3, 4.1, 4.2, 4.5.
- ▶ Jones and Manuelli (1990)
- ▶ Lucas (1988).

## Digression: Solving for $k(t)$ I

- Law of motion:

$$\dot{k}_t = (A - \delta - n)k_t - c_0 \exp\left(\frac{A - \delta - \rho}{\sigma}t\right) \quad (37)$$

- Solution to  $\dot{x} = ax - b(t)$  is

$$x_t = x_0 e^{at} - e^{at} \int_0^t e^{-as} b(s) ds \quad (38)$$

- To verify:

$$\dot{x}_t = ax_0 e^{at} - a e^{at} \int_0^t e^{-as} b(s) ds - e^{at} e^{-at} b(t) \quad (39)$$

$$= ax_t - b(t) \quad (40)$$

## Digression: Solving for $k(t)$ II

- Define

$$a = r - n = A - \delta - n > 0 \quad (41)$$

$$b = g_c = \frac{A - \delta - \rho}{\sigma} > 0 \quad (42)$$

- Then

$$k_t = k_0 \exp(at) - \exp(at) \int_0^t c_0 \exp([-a + b]s) ds \quad (43)$$

- Note:

$$\int_0^t e^{zs} ds = \frac{e^{zt} - 1}{z} \quad (44)$$

## Digression: Solving for $k(t)$ III

- Therefore:

$$k_t = k_0 e^{at} - \frac{c_0}{b-a} e^{at} \left[ e^{(b-a)t} - 1 \right] \quad (45)$$

$$= \left[ k_0 + \frac{c_0}{b-a} \right] e^{at} - \frac{c_0}{b-a} e^{bt} \quad (46)$$

- Now we show that  $g(k)$  is constant:  $k_t = k_0 e^{bt}$ .
- Transversality:

$$\lim_{t \rightarrow \infty} e^{(r-n)t} k_t = 0 \quad (47)$$

- Note that  $a = r - n = A - \rho - n$ .
- If  $b > a$ :  $g(k) \rightarrow b > a$  and TVC is violated.
- So we need  $b < a$ .



## Digression: Solving for $k(t)$ IV

- ▶ With  $b < a$  capital grows at rate  $a$ , unless the term in brackets is 0:

$$k_0 + \frac{c_0}{b-a} = 0 \quad (48)$$

- ▶ If  $g(k) = a$ , then  $g(e^{-(r-n)t}k_t) = 0$  - because  $a = r - n$ .
  - ▶ That violates TVC.
- ▶ The only value of  $c_0$  consistent with TVC is the one that sets the term in brackets to 0.
- ▶ It implies that  $k$  always grows at rate  $b$ .

## Saving rate

- ▶ We can solve for  $c/k$  and the saving rate.

$$\begin{aligned}g(k) - g(c) &= [A - \delta - n - c/k] - (A - \delta - \rho)/\sigma = 0 \\c/k &= A - \delta - n - (A - \delta - \rho)/\sigma\end{aligned}$$

- ▶ And the gross savings rate is

$$\begin{aligned}s &= (\dot{K} + \delta K)/AK \\&= [g(K) + \delta]/A \\&= [g(c) + n + \delta]/A \\&= [(A - \delta - \rho)/\sigma + n + \delta]/A\end{aligned}$$

- ▶ The savings rate is high, if ( $\sigma, \rho$  or  $A$ ) are low, or if  $n$  is high.

## References I

- Acemoglu, D. (2009): *Introduction to modern economic growth*, MIT Press.
- Barro, R. and S.-i. Martin (1995): "X., 1995. Economic growth," *Boston, MA*.
- Jones, L. E. and R. Manuelli (1990): "A Convex Model of Equilibrium Growth: Theory and Policy Implications," *Journal of Political Economy*, 1008–1038.
- Krusell, P. (2014): "Real Macroeconomic Theory," Unpublished.
- Lucas, R. E. (1988): "On the mechanics of economic development," *Journal of monetary economics*, 22, 3–42.
- Romer, P. M. (1986): "Increasing returns and long-run growth," *The journal of political economy*, 1002–1037.