The Growth Model: Discrete Time Dynamic Programming

Prof. Lutz Hendricks

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The Issue

We solved the growth model in **sequence language**.

- the solution is a sequence of objects that satisfies a bunch of difference equations

An alternative: **recursive** formulation.

- dynamic programming
- the solution is a set of functions

Dynamic Programming: An Informal Introduction

The basic idea of DP is to transform a many period optimization problem into a static problem.

To do so, we summarize the entire future by a **value function**.

The value function

- tells us the maximum utility obtainable from tomorrow onwards for any value of the state variables.
- an indirect utility function

Simplest example

A household lives for two periods.

$$\max u(c) + \beta v\left(c_1', c_2'\right) \tag{1}$$

subject to

$$s' = e + Rs - c \tag{2}$$

$$c_1' + pc_2' = Rs' \tag{3}$$

Notational convention: prime means tomorrow.

Lagrangian

$$\mathscr{L} = u\left(e + Rs - s'\right) + \beta v\left(Rs' - pc_2', c_2'\right) \tag{4}$$

FOC:

$$u'\left(c\right) = \beta v_1\left(.'\right) R \tag{5}$$

$$v_1\left(.'\right) p = v_2\left(.'\right) \tag{6}$$

Solution: c, c_1', c_2', s' that solve

- 2 FOCs
- 2 budget constraints

Dynamic Programming

The idea:

- solve one period at a time
- summarize the entire future with a "value function"
  (an indirect utility function)

If I know the value of saving, I can solve the first period problem.

Backward Induction

We solve this **backwards**, starting from the last period ("backwards induction").

The last period problem is a simple static one:

$$\max v(Rs - pc_2, c_2) \tag{7}$$

- All we need to know about the past is saving s (assets at the start of this period)

FOC:

- The same as the static condition from the Lagrangian

$$v_1(c_1, c_2)p = v_2(c_1, c_2) \tag{8}$$

Decision Rules

The FOC implicitly defines two decision rules of the form $c_j(s)$
Indirect utility is then also just a function of s:

$$\begin{aligned} W(s) &= v(c_1(s), c_2(s)) && (9) \\ &= \max_{c_2} v(Rs - pc_2, c_2) && (10) \end{aligned}$$

We call s the **state variable** and W the **value function.**

Log utility

Assume log utility tomorrow:

$$v(c_1, c_2) = \alpha \ln c_1 + (1-\alpha) \ln(c_2) \tag{11}$$

Then the static condition becomes

$$p\alpha / c_1 = (1-\alpha)/c_2 \tag{12}$$

or

$$pc_2 = \frac{1-\alpha}{\alpha} c_1 \tag{13}$$

With log utility, expenditure shares are constant (α for c_1 and $1-\alpha$ for c_2).

Consumption levels:

$$c_1 = \alpha Rs \tag{14}$$

$$c_2 = (1-\alpha) Rs/p \tag{15}$$

Value function

Then we can compute tomorrow's value function:

$$W(s) = \alpha \ln(\alpha R s) + (1-\alpha) \ln((1-\alpha) R s/p) \tag{16}$$

with marginal utility of wealth

$$W'(s) = \alpha/s + (1-\alpha)/s = 1/s \tag{17}$$

Solution:

- value function $W(s)$ and policy functions $c_j = f_j(s)$
- policy functions maximize $W(s)$ point by point
- W is the max of the RHS

Today's problem

We can now write today's problem as

$$V(s) = \max_{s'} u(s' - e) + \beta \underbrace{\left[\alpha \ln(\alpha R s') + (1-\alpha) \ln((1-\alpha) R s'/p)\right]}_{W(s')} \tag{18}$$

with FOC

$$u'(c) = \beta W'(s') = \beta / s' \tag{19}$$

Solution:

- value function $V(s)$ and policy function $s' = g(s)$
- g maximizes Bellman equation given W
- V is the max of the RHS

Cross check

Check against the Euler equation:

$$u'(c) = \beta v_1 \left(.'\right) R \tag{20}$$

$$= \beta R \frac{\alpha}{\alpha R s'} = \beta / s' \tag{21}$$

Both solutions given (of course) the same result.

Backward Induction I

Now consider a more general finite horizon problem

$$\max \sum_{t=1}^{T} u(c_t) \tag{22}$$

subject to $k_{t+1} = Rk_t - c_t$ and $k_{T+1} \geq 0$.

We again solve it backwards, starting from the last period.

Last Period

Consider the last date $t = T$.

The household cannot save: $k_{T+1} = 0$

Continuation value: $V(k, T+1) = 0$

- The problem is static: just eat all income

Terminal value:

$$V(k,T) = u(Rk) \tag{23}$$

Backward Induction

Now step back to $t = T - 1$

$$V(k, T-1) = \max u(Rk - k') + \beta \underbrace{u(Rk')}_{V(k',T)} \tag{24}$$

We can (in principle) solve for $V(k, T-1)$

Now step back to $t = T - 2$, etc.

Bellman equation for any period:

$$V(k, t) = \max u(Rk - k') + \beta V(k', t+1) \tag{25}$$

Backward Induction

This is mainly useful for numerically solving the problem.

Sometimes, one can solve finite horizon problems analytically (see Huggett et al. (2006) for an example).

Infinite Horizon

Conceptually, we do exactly the same thing.

But now we don't have a last period that would solve for $W(s')$ on the RHS of the Bellman equation.

Instead, we impose that tomorrow's value function is the same as today's:

$$V(s) = W(s) \tag{26}$$

Infinite Horizon Problem

Suppose we solve the planner's problem with starting date t^*:

$$V = \max_{\{c_t, k_t\}} \sum_{t=t^*}^{\infty} \beta^{t-t^*} u(c_t) \tag{27}$$

subject to $k_{t+1} = f(k_t) - c_t \ \forall t$.

Call the optimal solution c_t^*, k_t^* and the implied lifetime utility V.

Value function

Claim: The only fact that we need to know to figure out V is k_{t^*}

- other past choices do not show up in preferences or constraints
- k_{t^*} is the **state variable** of the problem.

Therefore, we can define the **value function** (indirect utility function)

$$V(k_{t^*}) = \sum_{t=t^*}^{\infty} \beta^{t-t^*} u(c_{t^*}) \tag{28}$$

$$= \max \sum_{t=t^*}^{\infty} \beta^{t-t^*} u(f(k_t) - k_{t+1}) \tag{29}$$

Stationarity

Claim: $V(k_{t*})$ does not depend on t^*.

Compare the value functions obtained from the problems starting at t^* and at $t^* + 1$.

They are the same functions.

That is, solving the problem yields the same value function regardless of the starting date.

Such a problem is called **stationary**.

Not all optimization problems have this property.

For example, if the world ended at some finite date, then the problem at $t * + 1$ looks different from the problem at $t*$.

Time consistency

- What if we start the problem at $t^* + 1$?
- Would the planner want to change his optimal choices of k_{t^*+2}, k_{t^*+3}, and so on?
- The answer is obviously "no," ... although I won't prove this just yet.
- A problem with this property is known as **time consistent**:
  - Give the decision maker a choice to change his mind at a later date and he will choose the same actions again.
- Not all optimization problems have this property.
  - For example, changing the specification of discounting easily destroys time consistency (self-control problems arise).

Recursive structure

Now comes the key insight:

$$V(k_{t^*}) = u(c_t^*) + \beta \left[\sum_{t=t^*+1}^{\infty} \beta^{t-(t^*+1)} u(c_{t^*}) \right] \tag{30}$$

$$= u(c_t^*) + \beta V(k_{t^*+1}) \tag{31}$$

$$= \max_{k_{t^*+1}} u(f(k_{t^*}) - k_{t^*+1}) + \beta V(k_{t^*+1}) \tag{32}$$

We have

- one term reflecting current period utility
- a second term summarizing everything that happens in the future, given optimal behavior, as a function of k_{t^*+1}.

Recursive structure

Since this equation holds for any arbitrary start date, we may drop date subscripts.

Unfortunate convention in macro:

- no subscript = today
- prime = tomorrow: $k' = k_{t+1}$

This yields a **Bellman equation**:

$$\boxed{V(k) = \max_{k'} u\left(f(k) - k'\right) + \beta V\left(k'\right)}$$

Claim: **Solving the DP is equivalent to solving the original problem** (the Lagrangian).

- We will see conditions when this is true later.

Recursive structure

The convenient part of this is: we have transformed a multiperiod optimization problem into a two period (almost static) one.

If we knew the value function, solving this problem would be trivial.

The bad news is that we have transformed an algebraic equation into a **functional equation**.

The solution of the problem is a value function V and an optimal policy function

$$c = \phi(k)$$

Note that c cannot depend on anything other than k, in particular not on k's at other dates, because these don't appear in the Bellman equation.

Solution

A solution to the planner's problem is now a pair of functions

- value function $V(k)$
- policy function $\phi(k)$

These solve the Bellman equation in the following sense.

1. Given $V(k)$, setting $c = \phi(k)$ solves the max part of the Bellman equation.
2. Given that $c = \phi(k)$, the value function solves

$$V(k) = u(\phi(k)) + \beta V(f(k) - \phi(k))$$

Solution: Intuition

Given $V(k)$, setting $c = \phi(k)$ solves the max part of the Bellman equation.

This means:

Point by point, for each k:

$$\phi(k) = \arg\max_c u(c) + \beta V(f(k) - c) \tag{33}$$

$\phi(k)$ simply collects all the optimal c's – one for each k.

Solution: Intuition

$$V(k) = u(\phi(k)) + \beta V(f(k) - \phi(k))$$

Note that this uses the optimal policy function for c.

Think of the Bellman equation as a mapping in a function space:

$$V^{n+1} = T(V^n) = \max u(c) + \beta V^n (f(k) - c)$$

Given an input argument V^n the mapping produces an output arguments V^{n+1}.

The solution to the Bellman equation is the V that satisfies $V = T(V)$.

- a **fixed point**.

The Planner's Problem with DP

The Planner's Bellman equation is

$$V(k) = \max_c u(c) + \beta V(f(k) - c)$$

with state k and control c.

The FOC for c is

$$u'(c) = \beta \; V'(k')$$

Problem: we do not know V'.

The Planner's Problem with DP

Differentiate the Bellman equation to obtain the

envelope condition

(aka Benveniste-Scheinkman equation):

$$V'(k) = \beta V'(k')f'(k) + \frac{\partial c}{\partial k}\underbrace{\left[u'(c) - \beta V'(k')\right]}_{=0}$$

Key point:

in the envelope condition, we can always ignore that changing the state (k) affects the controls (c).

The Planner's Problem with DP

Combine the FOC and the envelope condition to sub out all terms involving V':

$$\beta V'\left(k'\right) = \beta\beta V'\left(k''\right) f'\left(k'\right) \tag{34}$$

$$u'\left(c\right) = \beta u'\left(c'\right) f'\left(k'\right) \tag{35}$$

We obtain the same Euler equation as from the Lagrangian approach (of course).

DP also tells us that the optimal c is a function only of k.

Therefore k' also depends only on k:

$$\begin{aligned} k' &= f(k) - \phi(k) \\ &= h(k) \end{aligned}$$

Capital as control variable

There are other ways of setting up the Bellman equation.

With capital as the control:

$$V(k) = \max_{k'} u(f(k) - k') + \beta V(k')$$

FOC:

$$u'(c) = \beta V'(k')$$

Envelope condition

$$V'(k) = u'(c) f'(k)$$

The general point: We cannot choose the state variables, but we can choose the control variables.

Characterizing the Planner's Solution

It is here where DP has serious advantages over the Lagrangean: one can use results from **functional analysis** to establish properties of the value function and the policy function.

In our example, it can be shown that the economy converges monotonically from any k_0 to the steady state [Sargent (2009), p. 25, fn. 2]:

Note the difference relative to the OLG economy where much stronger assumptions are needed for this result.

Nonstationary Dynamic Programming

What if time matters?

Case 1: Time matters because of a **time-varying state variable**.

- Example: $f(k_t, A_t)$ where $A_{t+1} = G(A_t)$.
- Solution: Add A_t as a state variable to the value function.

Case 2: **Finite horizon** problems.

- Example: the household lives until date T.
- Solution: Add t as a state variable to the value function.
- We have one value function per date (see below).

Additional Constraints

Constraints are treated as in any optimization problem.

Example:

$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$ subject to

- $k' = f(k) - c$
- $k' \geq 0$

Bellman equation:

$$V(k) = \max_{c,k'} u(c) + \beta V\left(k'\right) + \lambda\left(f(k) - c - k'\right) + \mu k' \tag{36}$$

First-order conditions: Kuhn Tucker for k'.

Example: Non-separable Utility

Example: Non-separable Utility

Consider the following growth economy, modified to include **habit persistence** in consumption.

The social planner solves

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, c_{t-1})$$

subject to the feasibility constraints

$$k_{t+1} + c_t = f(k_t) \tag{37}$$

f satisfies Inada conditions.

Compute and interpret the first-order necessary conditions for the planner's problem.

Sequential Solution

This problem does not fit the DP approach without some modification.

We first solve it using a Lagrangian:

$$\Gamma = \sum_{t=1}^{\infty} \beta^t u\left(f(k_t) - k_{t+1}, f(k_{t-1}) - k_t\right) \tag{38}$$

First order conditions:

$$\beta^t u_1(t, t-1) f'(k_t) - \beta^{t-1} u_1(t-1, t-2)$$
$$+\beta^{t+1} u_2(t+1, t) f'(k_t) - \beta^t u_2(t, t-1) = 0$$

Sequential Solution

Define the total marginal utility of consumption as

$$U'(c_{t-1}) = u_1(t-1, t-2) + \beta u_2(t, t-1)$$

The Euler Equation then becomes:

$$U'(c_{t-1}) = \beta U'(c_t) f'(k_t) \tag{39}$$

Interpretation

$$U'(c_{t-1}) = \beta U'(c_t) f'(k_t) \tag{40}$$

- Give up one unit of c_{t-1}. This costs $U'(c_{t-1})$.
- We can increase x_{t-1} by 1 and raise k_t by 1.
- We eat the results next period at marginal utility $U'(c_t)$.
- We can eat
  - the additional output $f'(k_t)$;
  - the undepreciated capital $1-\delta$; (zero, in this case)

Sequential Solution

A **solution** of the hh problem is:

Sequences $\{c_t, k_t\}$ that satisfy

1. the EE
2. the flow budget constraint.
3. The boundary conditions k_1 given and a TVC:

$$\lim_{t \to \infty} U'(c_t) k_t = 0$$

DP Solution

For DP to work, it must be possible to write the problem as

$$V(s) = \max_{c} \ u(s, c) + \beta \ V(s')$$

subject to $s' = g(s, c)$

where s is the state and c is the control.

The current problem does not fit that pattern:

$$V(k) = \max \ u(c, c_{-1}) + \beta \ V(k')$$

subject to the law of motion

$$k' = f(k) - c$$

Nonseparable utility is the problem.

Adding a State Variable

The solution is to define an additional state variable

$$z = c_{-1}$$

or

$$z' = c = f\left(k\right) - k' \tag{41}$$

Then the Bellman equation is

$$V(k,z) \quad = \quad \max_{k'} u\left(f(k) - k', z\right) + \beta V\left(k', f(k) - k'\right)$$

Note that this looks "wrong" b/c z appears only once on the RHS, but everything is fine...

FOC

$$u_1(c,z) = \beta V_k(k',z') - \beta V_z(k',z')$$

Adding a state variable

The envelope conditions are

$$\begin{aligned} V_z &= u_2(c,z) \\ V_k &= u_1(c,z)f'(k) + \beta V_z(.')f'(k) \end{aligned}$$

Now define

$$U'(c) = u_1(c,z) + \beta u_2(c',z')$$

Then substitute out the V_z terms:

$$\begin{aligned} U'(c) &= \beta V_k(.') \\ V_k &= U'(c)f'(k) \end{aligned}$$

Substitute out the V_k terms and we get the same EE as with the Lagrangean.

The key point

If lagged variables occur in the problem, simply define new variables for date t: $z_t = c_{t-1}$.

Guess and Verify

Guess and Verify

In very special cases it is possible to solve for the value function in closed form.

A common case is

- log utility, $u(c) = \ln(c)$, and
- Cobb-Douglas technology with full depreciation: $f(k) = Ak^{\theta}$.

Then we can use the "guess and verify" method.

Guess and Verify

The general approach is:

1. Guess a functional form for V. Stick this into the right-hand-side of the Bellman equation.
2. Solve the max problem given the guess for V. The result is on the left hand side a new value function, V^1.
3. If $V = V^1$ the guess was correct.

This nicely illustrates what defines a solution to the Bellman equation.

Guess and Verify: Example

Consider the growth model with log utility and Cobb-Douglas production / full depreciation.

The planner solves:

$$\max \sum_{t=0}^{\infty} \beta^t \ \ln(c_t)$$

$$s.t. \ \ k_{t+1} = A \ k_t^{\theta} - c_t$$

Guess

Guess

$$V(k) = E + F\ln(k)$$

This is inspired by the hope that V should inherit the form of u. Having capital stock k amounts to having output Ak^{θ}, which would suggest

$$\begin{aligned} V(k) &\cong \ln(Ak^{\theta}) \\ &= \ln(A) + \theta\ln(k) \end{aligned}$$

Note that the guess for V contains some unknown constants (E, F) which we determine as we go along.

First-order Conditions

FOC:

$$u'(c) = \beta V'(k')$$

or

$$1/c = \beta F/k' \tag{42}$$

Envelope condition

$$V'(k) = u'(c)f'(k)$$

or

$$F/k = f'(k)/c \tag{43}$$

Policy Function

We can use the FOC to obtain the policy function in terms of the unknown parameters.

$$Fc = k'/\beta = f'(k)k \tag{44}$$

Note that

$$f'(k)k = \theta f(k) \tag{45}$$

Here, we are lucky and the F drops out

$$k' = h(k) = \beta\theta f(k) \tag{46}$$
$$c = (1-\beta\theta)f(k) \tag{47}$$

Result (as expected): the saving rate is constant.

Recover F

Now we need to recover E and F (and make sure they are indeed constants)

We know:

$$Fc = k'/\beta \tag{48}$$

$$c/k' = F/\beta \tag{49}$$

From the policy rules:

$$c/k' = (1 - \beta\theta)/\theta \tag{50}$$

Therefore

$$F = \frac{\theta}{1 - \theta\beta} \tag{51}$$

Recover E

Substitute everything we know into the Bellman equation:

$$E + F\ln(k) = \ln((1-\beta\theta)f(k)) + \beta\{E + F\ln(\beta\theta f(k))\} \tag{52}$$

Note that $\ln(f(k)) = \ln(A) + \theta\ln(k)$.

Collect all the constant terms to solve for E

$$E = \ln(1-\theta\beta) + \ln(A) + \beta E + \beta F\ln(\theta\beta) + \beta FA \tag{53}$$

Summary: Guess and Verify

1. Guess a value function (including unknown parameters).
2. Write first-order and Envelope conditions using the guess.
3. Solve for policy function.
4. Substitute policy function into Bellman equation to recover unknown parameters (and check the guess).

Applications

Examples where guess + verify is used:

Huggett et al. (2006), Huggett et al. (2011), Manuelli and Seshadri (2014)

(all models of human capital accumulation)

DP vs Lagrangian

What does DP buy us compared with a Lagrangian?

- With **uncertainty**, DP tends to be more convenient than a Lagrangian.
- Results from functional analysis can often be used to find **properties** of the optimal policy function such as monotonicity, continuity, and existence.
- DP can have **computational** advantages. There are methods for numerically approximating policy functions.

Reading

- Acemoglu (2009), ch. 6. Also ch. 5 for background material we will discuss in detail later on.
- Ljungqvist and Sargent (2004), ch. 3 (Dynamic Programming), ch. 7 (Recursive CE).
- Stokey et al. (1989), ch. 1 is a nice introduction.

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