Problem Set 7: Asset Pricing

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1 Lucas Fruit Trees With Crashes

Demographics: There is a single, representative household who lives forever.

Preferences: $U = \mathbb{E}\sum_{t=0}^{\infty} \beta^t u(c_t)$ where $u(c) = c^{1-\sigma}/(1-\sigma)$.

Endowments: The agent is endowed at $t = 0$ with 1 tree. In each period, the tree yields stochastic consumption d_t, which cannot be stored. d_t evolves as follows:

- If $d_t = d_{t-1}$, then $d_{t+1} = d_t$ forever after.
- If $d_t \neq d_{t-1}$, then $d_{t+1} = \gamma d_t$ with probability π and $d_{t+1} = d_t$ with probability $1 - \pi$. $\gamma > 1$.

In words: d grows at rate $\gamma - 1$ until some random event occurs (with probability $1 - \pi$), at which point growth stops forever.

Markets: There are competitive markets for consumption (numeraire) and trees (price p_t). Assume that p_t is *cum dividend*, meaning that d_t accrues to the household who buys the tree in t and holds it into $t + 1$.

Questions:

1. State the household's dynamic program.
2. Derive the Euler equation.
3. Define a recursive competitive equilibrium. Key: what is the state vector?
4. Characterize the stochastic process of p_t. Is p_t a Markov process? Hint: there are 2 phases: before and after dividends have stopped growing. Assume that p/d is constant during the phase with growth.
5. What happens to the stock market when the economy stops growing? Does it crash (does the price decline)? Under what condition?

2 Answer: Lucas Fruit Trees With Crashes[1]

1. The household problem is standard, except for the stock price being *cum dividend*:
$$V(k, S) = \max u(c) + \beta \mathbb{E}\{V(k', S')\} \tag{1}$$
where S is the aggregate state, subject to
$$p(k' - k) + c = dk' \tag{2}$$

[1] Based on a question due to Rodolfo Manuelli

2. Euler:
$$p_t = d_t + \beta \mathbb{E}\left\{\frac{u'(c_{t+1})}{u'(c_t)} p_{t+1}\right\} \tag{3}$$
3. The distribution of d_{t+1} depdends on d_t and d_{t-1}. These form the state vector.
   Objects: $V(k;S)$, $\kappa(k,S)$, $p(S)$
   Equilibrium conditions:
   - V and κ solve the household problem
   - goods market clearing: $c(k,S) = d$
   - asset market clearing: $\kappa(k,S) = 1$
   - law of motion of the aggregate state (exogenous)
4. Stock prices:
   Once growth has stopped: $c = d' = d$, so that $p - d = \beta p \implies p = d/(1-\beta)$.
   While growth continues:
$$p_g = d + \beta\pi\gamma^{-\sigma} p'_g + \beta(1-\pi) d/(1-\beta) \tag{4}$$
   Divide by d:
$$p_g/d = 1 + \beta\pi\gamma^{1-\sigma} p'_g/d' + \beta(1-\pi)/(1-\beta) \tag{5}$$
   or:
$$p_g/d = \frac{1+\beta(1-\pi)/(1-\beta)}{1-\beta\pi\gamma^{1-\sigma}} \tag{6}$$
   The price is Markov in the state $S = (d, d_{-1})$.
5. The growth slowdown leads to a stock market crash, if $p_g/d > p/d$ or
$$\frac{1+\beta(1-\pi)/(1-\beta)}{1-\beta\pi\gamma^{1-\sigma}} > \frac{1}{1-\beta} \tag{7}$$
   or
$$\beta\pi < \beta\pi\gamma^{1-\sigma} \tag{8}$$
   Whether the growth slowdown crashes the stock market depends on the sign of $1-\sigma$ (the curvature of preferences or the relative strengths of income and substitution effects).

3 Two stocks

Demographics: There is a single representative household who lives forever.

Preferences:
$$E\sum_{t=0}^{\infty} \beta^t u(c_t) \tag{9}$$
where u is strictly increasing, strictly concave, and satisfies $u'(0) = \infty$.

Technologies: None (endowment economy).

Endowments: There are two fruit trees. In each period, tree j yields $\theta_{j,t}y$ units of the consumption good. $y > 0$ is a constant. The $\theta_{j,t}$ are i.i.d. random variables.

Markets: There are competitive markets for goods (numeraire) and trees with prices $p_{j,t}$. Shares are traded after the $\theta_{j,t}$ are realized and dividends are paid. Households also trade a riskless bond in zero net supply with endogenous price q_t.

Questions:

1. State the household problem in recursive form.
2. Derive the Euler equations. Assume an interior solution where the household holds both stocks.
3. From now on assume that $\sum_j \theta_j = 1$. Assume that there are no bubbles. Solve for the equilibrium asset prices.
4. Derive the equity premium. Explain your finding.
5. Now assume that y_t is random and drawn from a finite Markov chain. Utility is $u(c) = \ln(c)$. Solve for the equilibrium stock prices and expected returns. What can you say about the connection between expected stock returns and growth, i.e., are stocks expected to do well when there are strong growth prospects? What is the intuition for this?

3.1 Answer: Two stocks[2]

1. Household problem. Let $\Theta = (\theta_1, \theta_2)$. The Bellman equation is given by

$$V(s_1, s_2, b; \Theta) = \max u(c) + \beta E\left[V(s_1', s_2', b'; \Theta') \,|\Theta\right] \tag{10}$$

subject to the budget constraint

$$\sum_j s_j\left[\theta_j y + p_j\right] + b = \sum_j s_j' p_j + qb' + c \tag{11}$$

2. Euler equations. First order conditions yield standard Lucas asset pricing equations:

$$\begin{aligned} u'(c) &= \beta E\left[u'(c')\,\frac{\theta_j' y' + p_j'}{p_j}\right] && (12)\\ u'(c) &= \beta E\left[u'(c')/q\right] && (13)\end{aligned}$$

[2]Based on a problem due to Steve Williamson.

3. Equilibrium prices. c is constant at y. Therefore,

$$p_j = \beta E\left[\theta_j' y + p_j' | \Theta\right] \quad (14)$$

$$q = \beta \quad (15)$$

Since θ is i.i.d., stock prices are constant over time:

$$p_j (1-\beta) = \beta y E[\theta_j] \quad (16)$$

4. Equity premium. Expected gross stock returns equal β^{-1}. The equity premium is zero. The two stocks are jointly riskless.

5. Random endowment. Use the standard asset pricing equation:

$$p_{j,t} = E \sum_{k=1}^{\infty} \beta^k \frac{u'(c_{t+k})}{u'(c_t)} [\theta_{j,t+k} y_{t+k}] \quad (17)$$

Using $c_t = y_t$, this simplifies to

$$p_{j,t} = y_t \; E \sum_{k=1}^{\infty} \beta^k \theta_{j,t+k} \quad (18)$$

$$= y_t E(\theta_j) \frac{\beta}{1-\beta} \quad (19)$$

with expected return $\frac{1}{\beta} E\left[\frac{y'}{y}\right]$. Expected returns are high in states with rising expected y, i.e. when $E(y'/y)$ is high. Intuition: When y' is expected to be high relative to today's y, the same is true for consumption. Marginal utility behaves in the opposite way. A high expected return is needed to entice the household to save in such a state.