# The Growth Model in Continuous Time (Ramsey Model)

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# The Growth Model in Continuous Time

We add optimizing households to the Solow model.

We first study the planner's problem, then the CE.

# 2. Planning Problem

# Planning Problem

The social planner maximizes

$$\int_{t=0}^{\infty} e^{-(\rho-n)t} u(c_t) dt \tag{1}$$

subject to the resource constraint

$$\dot{k}_t = f(k_t) - (n+\delta)k_t - c_t \tag{2}$$

$$k_0 \text{ given} \tag{3}$$

$$k_t \geq 0 \tag{4}$$

# Planning Problem

The current value Hamiltonian is

The state is $k$ and the control is $c$.

The optimality conditions are

## Planner: TVC

The TVC is:
$$\lim_{t \to \infty} e^{-(\rho-n)t} \mu(t) k(t) = 0 \tag{5}$$

To check this:

▶ we need $u$ and $g(k,c)$ to be monotone

▶ $u$ is obvious.

▶ $g(k,c) = f(k) - c - (n+\delta)k$ is monotone in $c$ but not $k$.

▶ However, we "know" that $k$ never rises above the golden rule point where $f'(k) = \delta$ - unless $k(0)$ is too high.

▶ Then $g$ is increasing in $k$.

# Sufficiency

This is an example where the easiest (1st) set of sufficiency conditions applies:

- ▶ $u$ is strictly concave in $c$ (only).
- ▶ $g(k,c)$ is jointly concave in $k$ and $c$.

First order conditions are sufficient.

# Planner: Solution

A solution consists of functions of time

$$c_t, k_t, \mu_t$$

that satisfy:

1. The first-order conditions (2)
2. The resource constraint
3. The boundary conditions $k_0$ given and the TVC

$$\lim e^{-(\rho-n)t} \mu_t k_t = 0 \tag{6}$$

# Planner: Euler Equation

We eliminate the multiplier, starting from

$$g(\mu) = -\left[f'(k) - \delta - \rho\right] \tag{7}$$
$$= g\left(u'(c)\right) \tag{8}$$
$$= -\sigma g(c) \tag{9}$$

where $\sigma$ is the elasticity of $u'$ w.r.to $c$:

$$\sigma = -u''_c c / u' \tag{10}$$
$$= -\frac{du'(c)}{dc}\frac{c}{u'(c)} \tag{11}$$

Therfore:

$$\boxed{g(c) = [f'(k) - \delta - \rho]/\sigma} \tag{12}$$

# More direct derivation

Differentiating the FOC yields

$$\dot{\mu} = u''(c)\dot{c} \tag{13}$$

and therefore

$$\dot{\mu}/\mu = u''(c)\dot{c}/u'(c) \tag{14}$$
$$= -\left[f'(k) - \delta - \rho\right] \tag{15}$$

# Planner: Euler Equation

$$g(c) = [f'(k) - \delta - \rho]/\sigma \qquad (16)$$

Recall the discrete time version:

$$\frac{c_{t+1}}{c_t} = [\beta R]^{1/\sigma} \qquad (17)$$

The same idea:

▶ consumption growth rises with the interest rate

▶ declines with the discount rate

▶ $\sigma$ governs how responsive consumption growth is

# Note on $\sigma$

$\sigma$ is a key parameter for asset pricing, business cycle volatilities, ...

$$
\begin{align}
\sigma &= -u_c''c/u' \tag{18} \\
&= -\frac{du'(c)}{dc}\frac{c}{u'(c)} \tag{19}
\end{align}
$$

$\sigma$ is

1. the elasticity of marginal utility w.r.to $c$
2. the inverse intertemporal elasticity of substitution
3. the coefficient of relative risk aversion

$u(c) = c^{1-\phi}/1 - \phi$ implies $\sigma = \phi$.

# Planner: Summary

▶ The planner's problem solves for functions of time $c(t)$ and $k(t)$.

▶ These satisfy two differential equations

$$g(c) = \frac{f'(k) - \delta - \rho}{\sigma} \qquad (20)$$

$$\dot{k} = f(k) - (n+\delta)k - c \qquad (21)$$

and two boundary conditions

$$k_0 \text{ given}$$

$$\lim_{t\to\infty} \beta^t u'(c(t))\, k(t) = 0$$

▶ How can we analyze the dynamics of this system?

# 3. Phase Diagram Example

# 3.1. Phase Diagram Example

Phase diagrams can be used to analyze the dynamics of systems of 2 differential equations.

Consider the example

$$
\begin{aligned}
\dot{x} &= A - ax + by \\
\dot{y} &= B + cx - dy
\end{aligned}
$$

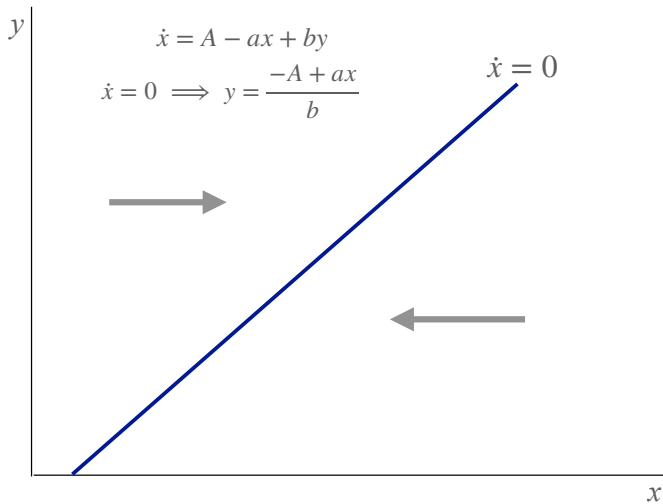Boundary conditions: $x_0, y_0$ given.

Assume $a, b, c, d > 0$.

# Basic Idea

$$\dot{x} = A - ax + by \qquad (22)$$

This divides the $(x, y)$ plane into two regions:

- one where $x$ rises (moving east over time)
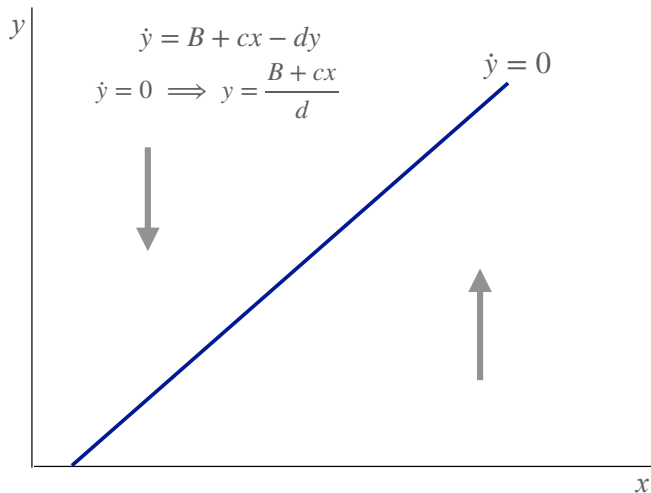- one where $x$ falls (moving west over time)

# Basic idea



$$\dot{x} = A - ax + by$$

$$\dot{x} = 0 \implies y = \frac{-A + ax}{b}$$

$\dot{x} = 0$

# Basic idea

$$\dot{y} = B + cx - dy \tag{23}$$

also divides the plane into two regions

► one where $y$ rises (moving north over time)
► one where $y$ falls (moving south over time).

# Basic idea



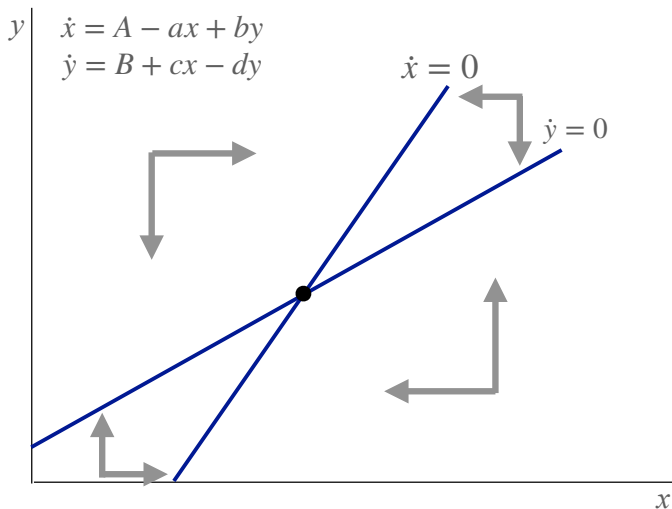$$\dot{y} = B + cx - dy$$

$$\dot{y} = 0 \implies y = \frac{B + cx}{d}$$

$\dot{y} = 0$

$y$

$x$

# The Diagram

So we end up with a diagram that looks something like this:



$\dot{x} = A - ax + by$
$\dot{y} = B + cx - dy$
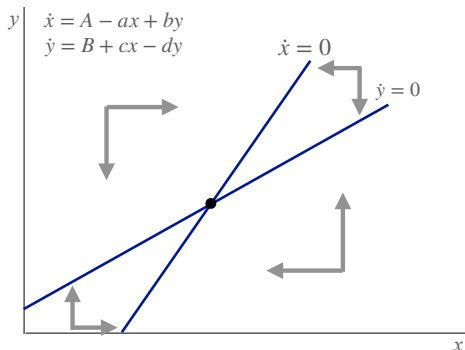
All points in a given quadrant move (qualitatively) in the same direction.

Now we can use logic to figure out possible paths.

# Phase Diagram



What happens for points on the region boundaries?

# Think about possible paths

For each quadrant, determine whether the path can leave the quadrant.

In this example: the answer is

- ▶ "no" for two quadrants.
- ▶ "yes" for two quadrants.

Now think about possible paths...

Conclude that the steady state is globally stable.

# Summary

Given:

- a model characterized by two differential equations $(\dot{x}, \dot{y})$.
- boundary conditions (here $x_0, y_0$)

Steps:

1. Find region boundaries where $\dot{x} = 0$ and $\dot{y} = 0$.
2. Find direction of movement in each of the four regions.
3. Determine which regions can be exited.
4. Think about possible paths.

# Applications

Unified growth theory:

- ▶ Galor (2000) studies transition from Malthusian stagnation to industrialization using a sequence of phase diagrams
- ▶ Galor (2005)

Models of human capital accumulation over the life-cycle:

- ▶ Heckman (1976)

# 3.2. Phase Diagram: Growth Model

Movement of $c$:
$$g(c) = \frac{f'(k) - \delta - \rho}{\sigma} \qquad (24)$$
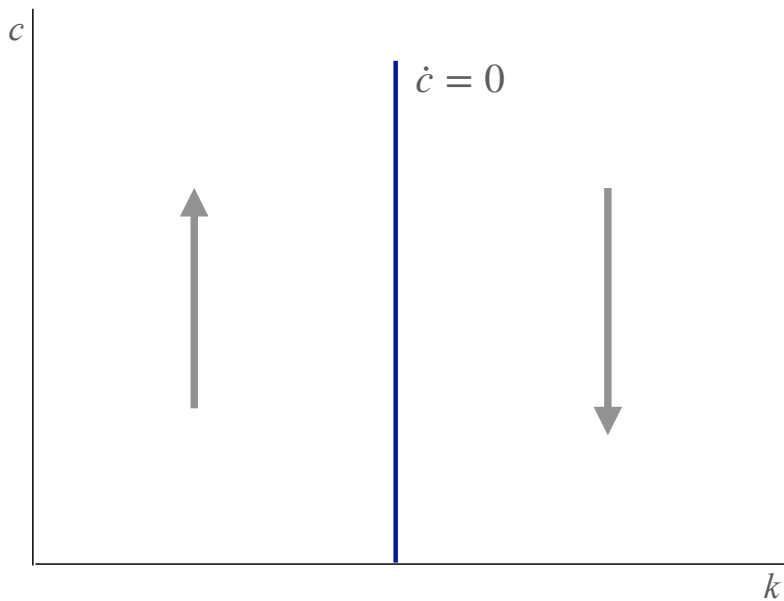
The $\dot{c} = 0$ locus is characterized by
$$f'(k^*) = \rho + \delta \qquad (25)$$

A vertical line in the $(k, c)$ plane.

Higher $k$ implies lower $\dot{c}$

$c \downarrow$ to the right of the $\dot{c} = 0$ locus.

# Movement of $c$

# Movement of $k$

$$\dot{k} = f(k) - (n + \delta)k - c \qquad (26)$$

The $\dot{k} = 0$ locus is hump-shaped:
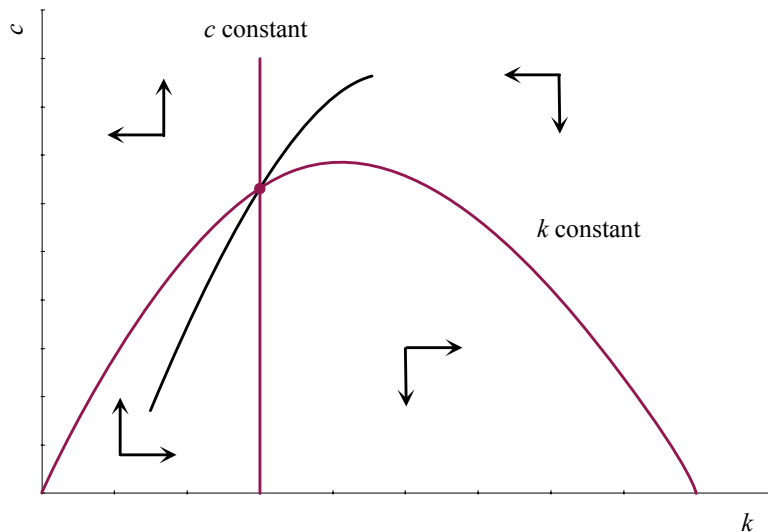
$$c = f(k) - (n + \delta)k \qquad (27)$$

with a maximum at

$$f'(k^*) = n + \delta \qquad (28)$$
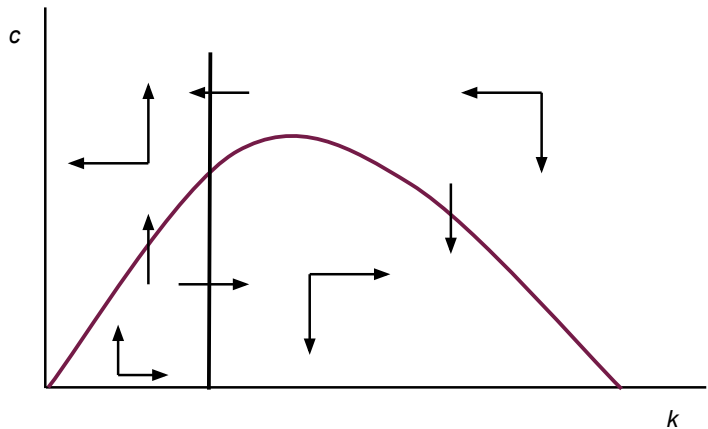
Higher $c$ implies lower $\dot{k}$.

$k$ falls over time above the $\dot{k} = 0$ locus.

# Phase Diagram



Since $\rho - n > 0$, the $\dot{c} = 0$ locus lies to the left of the peak of the $\dot{k} = 0$ locus.

# Dynamics: Possible Paths

# Ruling out the "north-west" path

$$g(c) = \frac{f'(k) - \delta - \rho}{\sigma} \qquad (29)$$

$g(c)$ rises over time as $k \to 0$.

Eventually, this violates feasibility.

# Ruling out the "south-east" path

Properties of that path:

$$c \to 0 \implies k \to k_{max} > k_{GR} \tag{30}$$

and therefore $f'(k_{max}) - \delta < n$.

This is exactly the kind of path that the TVC rules out.

Note:

- Even though $g(c)$ is strictly negative, $\dot{c} \to 0$. Therefore $c$ does not turn negative.
- Any such path asymptotes towards $c = 0$ and $k = k_{max}$.

# Ruling out the "south-east" path

Transversality

$$\lim_{t\to\infty} e^{-(\rho-n)t} u'(c_t) k_t = 0 \tag{31}$$

requires

$$g\left(e^{-(\rho-n)t} u'(c)\right) = g\left(u'\right) - (\rho-n) < 0 \tag{32}$$

as $t \to \infty$.

Euler:

$$g\left(u'\right) = \rho + \delta - f'(k) \tag{33}$$

Higher $k$ implies lower $f'$ and higher $g\left(u'\right)$.

Any $k > k_{GR}$ violates TVC because

$$g\left(u'\right) - (\rho-n) = -\left[f'(k) - \delta - n\right] > 0 \tag{34}$$

# Dynamics: Saddle-path Stability

Only one value of $c$ avoids moving into "forbidden" regions for given $k$.

For this $c$, the economy converges to the steady state.

Such a system is called "**saddle-path stable**."

▸ Details

A great way of exploring the phase diagram can be found at EconGraphs

# Summary

The growth model in continuous time behaves like the one in discrete time (no surprise here):

1. The allocation is determined by Euler equation and resource constraint.
   Boundary conditions also look like in discrete time.

2. There is a unique, globally stable steady state.
   It satisfies "interest rate" = discount rate.

3. In continuous time, we can use a phase diagram to study the entire path.

# Reading

- Acemoglu (2009), ch. 8. Ch. 8.6 covers the detrended model. Ch. 7 covers Optimal Control.
- Barro and Sala-i Martin (1995), ch. 2, explains the Cass-Koopmans/Ramsey model in great detail.
- Blanchard and Fischer (1989), ch. 2
- Romer (2011), ch. 2A
- Phase diagram: Barro and Sala-i Martin (1995), ch. 2.6

# Technical notes: Unique saddle path

## Theorem

*Take as given*
$\dot{x}(t) = G[x(t)]$ *with initial value* $x(0)$ *given, where* $G$ *is continuously differentiable.*
*The steady state is* $G(x^*) = 0$. *Define* $A = DG(x^*)$.
*Suppose that* $m$ *eigenvalues of* $A$ *have negative real parts while* $n - m$ *have positive real parts.*
*Then there exists an* $m$ *dimensional manifold in the neighborhood around the steady state such that starting from any* $x(0)$ *in that manifold a unique* $x(t) \to x^*$.

See Acemoglu (2009), Theorem 7.15.

# What this says in words

Suppose we have a system of $n = 2$ differential equations (in $c$ and $k$).

The local dynamics around the steady state can be approximated by a linear differential equation with matrix $A$.

$$\begin{bmatrix} \dot{k} \\ \dot{c} \end{bmatrix} = A \begin{bmatrix} k \\ c \end{bmatrix} \tag{35}$$

If that matrix has $m = 1$ negative eigenvalues, then **locally** around the steady state there is a line (dimension $m = 1$) of points $(c, k)$ that converge to the steady state.

▶ This is the saddle path.

Other points could, in principle, converge as well, but we can rule that out as above.

# Application to the growth model

First, establish that the saddle path is locally unique.

Start from a linear approximation to the two differential equations:

$$\begin{bmatrix} \dot{k} \\ \dot{c} \end{bmatrix} = \begin{bmatrix} f'(k^*) - n - \delta & -1 \\ c^* f''(k^*) \sigma & 0 \end{bmatrix} \begin{bmatrix} k - k^* \\ c - c^* \end{bmatrix} \tag{36}$$

Details:

$$\dot{k} = f(k) - (n + \delta)k - c \tag{37}$$
$$= f'(k - k^*) - (n + \delta) - (c - c^*) \tag{38}$$

and

$$\dot{c} = c[f(k) - \delta - \rho]/\sigma \tag{39}$$

$$= (c - c^*)\underbrace{[f(k^*) - \delta - \rho]/\sigma}_{=0} + c^* \frac{f''(k^*)}{\sigma}(k - k^*) \tag{40}$$

## Eigenvalues I

The eigenvalues $\lambda$ solve $Ax = \lambda x$. In this case:

$$\begin{bmatrix} f'(k^*) - n - \delta & -1 \\ c^* f''(k^*)/\sigma & 0 \end{bmatrix} x = \lambda x \tag{41}$$

Also $\det(A - I\lambda)x = 0$. In this case:

$$\det \begin{bmatrix} f'(k^*) - n - \delta - \lambda & -1 \\ c^* f''(k^*)/\sigma & 0 - \lambda \end{bmatrix} = 0 \tag{42}$$

$$\det(A - I\lambda) = -\left[ f'(k^*) - \delta - n - \lambda \right]\lambda + c^* f''(k^*)/\sigma \tag{43}$$

$$\lambda^2 - \left( f' - \delta - n \right)\lambda + c^* f''/\sigma = 0 \tag{44}$$

# Eigenvalues II

Apply

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (45)$$

and obtain

$$\lambda = \left\{ f'(k^*) - n - \delta \pm \sqrt{(f' - n - \delta)^2 - 4c^*f''/\sigma} \right\} / 2 \qquad (46)$$

Since $\sqrt{(f' - n - \delta)^2 - 4c^*f''/\sigma} > f' - n - \delta$, there is exactly one negative eigenvalue.

Therefore: in a neighborhood of the steady state, the saddle path is unique.

## Application to the growth model

How do we know that the saddle is globally unique?

Define one saddle path that converges.

Take a point not on it. We know:

1. The path cannot reach or cross the saddle path in finite time.
2. The path cannot asymptote to the saddle path because that would get into a neighborhood of the steady state where the saddle is unique.
3. Therefore, the path cannot converge to the steady state.

## References I

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