The Growth Model: Discrete Time

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The standard growth model

- The neoclassical growth model, aka the standard growth model, is the most important model in macro.
- It underlies entire branches of the literature (parts of growth theory and business cycle theory, for example).
- Here, we study this model in discrete time.
- Dynamic programming is the solution method.

Model structure

There are many versions of the growth model. This is a basic version.

1. Households are identical and live forever.
2. Firms produce a single good using capital and labor.
3. All agents are price takers.
4. All prices are perfectly flexible. All markets clear at all times.

Infinite horizons

So far we have assumed that agents are finitely lived.

Analytically more convenient: infinite lifetimes.

How to justify this assumption?

- need to show that it is harmless for a given application

Demographics

There is a continuum of households (uncountably infinite number).

All households are identical.

- This is stronger than needed (see notes on aggregation later on).

We can think of a single, price-taking household.

The measure of households is 1.

Therefore, per capita and aggregate variables are the same.

Exercise: Redo everything when the number of households is $N_t = (1+n)^t$.

Preferences

The household values discounted utility from consumption:

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1 \tag{1}$$

Utility is time separable (for tractability).

Discounting is exponential (to avoid time consistency problems).

Time consistency means:

- If $\{c_t\}_{t=0}^{\infty}$ solves the problem with start date 0, then $\{c_t\}_{t=\tau}^{\infty}$ solves the problem with start date τ.
- The household does not want to change past plans.

Endowments

The household has

- k_0 units of the good at $t = 0$
- 1 unit of time in each period

Technology

Resource constraint:

$$k_{t+1} = f(k_t) - c_t \tag{2}$$

$f(k) = g(k) + (1 - \delta)k$

We assume Inada conditions for g.

Capital cannot be negative: $k_t \geq 0$.

Markets

Goods: numeraire.

Labor: w_t

Capital rental: q_t

All markets are competitive.

Planning Problem

The planner maximizes discounted utility of the representative household

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1$$

Constraints:

$$\begin{aligned} k_{t+1} &= f(k_t) - c_t \\ k_{t+1} &\geq 0 \\ & k_0 \text{ given} \end{aligned}$$

Lagrangian

$$\begin{aligned} \Gamma &= \sum_{t=0}^{\infty} \beta^t u(c_t) \\ &\quad + \sum_{t=0}^{\infty} \lambda_t [f(k_t) - c_t - k_{t+1}] \end{aligned}$$

FOCs for an interior solution:

$$\begin{aligned} \beta^t u'(c_t) &= \lambda_t \\ \lambda_{t+1} f'(k_{t+1}) &= \lambda_t \end{aligned}$$

In words...

Euler equation

$$u'(c_t) = \beta u'(c_{t+1}) f'(k_{t+1}) \tag{3}$$

This is exactly the same Euler equation we saw many times before.

The Euler equation implicitly defines a law of motion for the capital stock:

$$u'(f(k_t) - k_{t+1}) = \beta u'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}) \tag{4}$$

This is a second order difference equation.

- but later we will see that it isn't...

Planner: Solution

A solution is a sequence $\{c_t, k_{t+1}\}_{t=0}^{\infty}$.

These satisfy the **necessary** conditions:

1. Euler equation $\forall t$
2. Resource constraint $\forall t$
3. k_0 given

We have two difference equations, but only one **boundary condition.**

Uniqueness requires an additional restriction: transversality

Transversality Condition

$$\lim_{t\to\infty} \beta^t \; u'(c_t) \; k_{t+1} = 0 \tag{5}$$

This rules out paths that satisfy the Euler equation, but are not optimal.

In this case, possible "pathological" paths feature

- $c_t \to 0$ and $k_t \to k_{GR}$

Such paths could violate TVC

- if $\beta u'(c_t) \to 0$ as $k_t \to k_{GR}$

Digression: Transversality Conditions

Digression: Transversality Conditions

Consider the following example:

$$\begin{aligned} & \max \sum_{t=0}^{T} \beta^t u(c_t) \\ s.t. \;\; k_{t+1} &= e_t + (1+r_t)k_t - c_t \end{aligned}$$

Note the finite horizon: $T < \infty$.

As stated, this problem does not have a solution (why not?).

Digression: Transversality Conditions

Let's proceed to solve the problem as stated.

Lagrangian

$$\begin{aligned}\Gamma &= \sum_{t=0}^{T} \beta^t u(c_t) \\ &\quad + \sum_{t=0}^{T} \lambda_t \{e_t + (1+r_t) k_t - c_t - k_{t+1}\}\end{aligned}$$

FOCs (necessary):

$$u'(c_t) = \beta \; u'(c_{t+1}) \; (1 + r_{t+1})$$

Solution

Sequences $\{c_t, k_{t+1}\}$ that satisfy:

- Euler equation
- budget constraint
- k_0 given

Problem 1

We allowed the household to choose $c_t \to \infty$ and $k_{t+1} \to -\infty$.

- The household problem has no solution.

Solution:

- We need a constraint that bounds consumption
- A natural candidate in this example: $k_{T+1} \geq 0$.
- The household cannot die in debt.
- This is part of the economic model.

Problem 2

We have 2 difference equations, but only one boundary condition.

- The solution is not uniquely determined by those.

We need one more boundary condition to ensure that utility is finite.

In this model, the right boundary condition is obvious: $k_{T+1} = 0$.

- Leaving food on the table cannot be optimal.

Infinite horizon case

What if $T \to \infty$?

We could impose $\lim_{T\to\infty} k_{T+1} = 0$, but it does not make economic sense.

- This would prevent the household from perpetually growing its capital stock.

Economically, the problem is that the household could choose $c_t \to 0$ and $k_t \to k_{\max}$ (the max sustainable capital stock).

Infinite horizon case

The correct boundary condition to rule out such paths:

$$\lim_{T \to \infty} k_{T+1}/R_{T+1} = 0 \tag{6}$$

Intuition:

- If the interest rate is positive, it makes sense for the household to maintain a high capital stock forever.
- But at least some of the interest that the capital generates will be eaten.

Infinite horizon intuition

Write the present value budget constraint as

$$\sum_{t=0}^{T} \frac{c_t}{R_t} = \sum_{t=0}^{T} \frac{e_t}{R_t} + k_0 - \frac{k_{T+1}}{R_{T+1}}$$

where $R_t = (1+r_1) \times ... \times (1+r_t)$ is a cumulative discount factor.

- **Exercise**: show that this is the present value budget constraint.

$$\lim_{T\to\infty} k_{T+1}/R_{T+1} = 0 \tag{7}$$

ensures finite consumption.

Infinite horizon case

An equivalent solution:

Impose

$$\lim_{T\to\infty} \beta^T \; u'(c_T) \; k_{T+1} = 0$$

This is the same as

$$\lim_{T\to\infty} k_{T+1}/R_{T+1} = 0 \tag{8}$$

because, by the Euler equation:

$$\beta^T u'(c_T) R_T = u'(c_0)$$

The general point

In dynamic optimization problems, the flow budget constraint is usually not enough to ensure finite utility.

- The economics of the model provides the condition that fixes this problem.

Even then, the solution is not unique.

With finite T, the correct boundary condition is usually obvious (don't leave assets on the table).

With infinite T, the correct boundary condition is not obvious.

Reading

- Acemoglu (2009), ch. 6. Also ch. 5 for background material we will discuss in detail later on.
- Stokey et al. (1989), ch. 1 is a nice introduction.
- Blanchard and Fischer (1989) is a good introduction to the standard growth model.
- Krusell (2014) ch. 2 discusses why the assumptions made in the growth model are popular.

References I

Acemoglu, D. (2009): *Introduction to modern economic growth*, MIT Press.

Blanchard, O. J. and S. Fischer (1989): *Lectures on macroeconomics*, MIT press.

Krusell, P. (2014): "Real Macroeconomic Theory," Unpublished.

Stokey, N., R. Lucas, and E. C. Prescott (1989): "Recursive Methods in Economic Dynamics," .