

## Problem Set 5: Growth Model in Continuous Time

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### 1 Continuous Time CIA Model. Cash and Credit Goods.

Demographics: A single representative household who lives forever.

Endowments: At the beginning of time, households hold  $k_0$  units of capital and  $M_0$  units of money.

Preferences:

$$\int_0^\infty e^{-\rho t} u(c_t, d_t) dt$$

where  $c$  and  $d$  are two consumption goods.

Technology:

$$\dot{f}(k) = \dot{k} + c + d \tag{1}$$

Government: The government costlessly produces money  $M$  and hands it to households as lump-sum transfers. The money growth rate is constant at  $\dot{g}(M) = \gamma$ . The government budget constraint is  $\dot{M} = p x = g(M) M$  where  $p$  is the price level and  $x$  is the real lump-sum transfer.

Markets: There are competitive markets for goods and money. Households operate the technology (there are no firms).

CIA constraint:  $c$  has to be bought with cash:

$$c_t \leq M_t/p_t$$

$d$  may be bought with credit.

Denote real balances by  $m_t = M_t/p_t$ .

#### Questions

1. Write down the household's Hamiltonian. Which are his states and controls? Derive first-order conditions for two cases: either the CIA constraint always binds or it never binds. Hint: the budget constraint is given by

$$\dot{k}_t + c_t + d_t + \dot{M}_t/p_t = f(k_t) + x_t$$

2. Define a competitive equilibrium.
3. Derive a set of equations that characterize the steady state. Show that the nominal interest rate equals zero, if the CIA constraint does not bind.
4. Determine the effects of a higher money growth rate on the steady state allocation. Assume that the utility function takes the form  $u(c, d) = U(c) + V(d)$ , where  $U$  and  $V$  are strictly concave functions.

## 2 Money in the Utility Function

Demographics: Time is continuous. There is a single representative household who lives forever.

Preferences:

$$\int_{t=0}^{\infty} e^{-\rho t} u(c_t, m_t) dt \quad (2)$$

where  $c$  is consumption and  $m$  denotes real money balances.

Endowments: Households work 1 unit of time at each instant. Households are initially endowed with  $k_0$  units of capital and  $m_0$  units of real money.

Technology:

$$f(k_t) - \delta k_t = c_t + \dot{k}_t + \phi(\dot{m}) \quad (3)$$

where  $\phi(\dot{m})$  is the cost of adjusting household money holdings.  $\phi'(0) = 0$  and  $\phi''(\dot{m}_t) > 0$ .

Money: nominal money grows at exogenous rate  $g(M)$ . New money is handed to households as a lump-sum transfer:  $\dot{M}_t = p_t x_t$ .

Markets: money (numeraire), goods, capital rental (price  $q$ ), labor ( $w$ ).

### Questions:

1. The household's budget constraint is given by

$$\dot{k}_t + \dot{m}_t = w + r_t k_t + x_t - c_t - \pi_t m_t - \phi(\dot{m}_t) \quad (4)$$

where  $r = q - \delta$ . State the Hamiltonian. If you cannot figure this out, assume  $\phi(\dot{m}) = 0$  and proceed (for less than full credit). Hint: Make  $m$  and  $k$  separate state variables.

2. State the first-order conditions.
3. Define a competitive equilibrium.
4. Characterize the steady state to the extent possible. What is the effect of a permanent change in  $g(M)$ ?
5. What is the optimal rate of inflation? Explain.