Problem Set 2: OLG Models with Money

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1 Money and Storage

Consider a two-period OLG model with fiat money and a storage technology.

Demographics: In each period $N_t = (1+n)^t$ persons are born. Each lives for 2 periods.

Endowments: The initial old hold capital K_0 and money M_0. Each young person is endowed with a e units of the good.

Preferences: $u(c_t^y) + \beta u(c_{t+1}^o)$.

Technology: Storing k_t units of the good in t yields $f(k_t)$ units in $t+1$. f obeys Inada conditions. The resource constraint is $N_t k_{t+1} = N_t e + N_{t-1} f(k_t) - C_t$ where $C_t = N_t\, c_t^y + N_{t-1}\, c_t^o$.

Government: The government pays a lump-sum transfer of $x_t p_t$ units of money to each old person: $M_{t+1} = M_t + N_{t-1} x_t p_t$. The aggregate money supply grows at the constant rate μ: $M_{t+1} = (1+\mu)\, M_t$.

Markets: In each period, agents buy/sell goods and money in spot markets.

The timing in period t is a follows:

- The old enter period t holding aggregate capital $K_t = N_{t-1} k_t$ and nominal money balances of $M_t = m_t\ N_{t-1}$.
- Each old person produces $f(k_t)$.
- The young buy money (m_{t+1}/p_t) from the old, consume c_t^y and save k_{t+1}.
- The old consume their income.

Questions:

1. State the household's budget constraints when young and old.
2. Derive the household's optimality conditions. Define a solution to the household problem.
3. Define a competitive equilibrium.
4. Does an equilibrium with positive inflation exist? Intuition?
5. Define a steady state as a system of 6 equations in 6 unknowns.
6. Find the money growth rate (μ) that maximizes steady state consumption per young person, $(N_t\, c_t^y + N_{t-1}\, c_t^o)/N_t$.

1.1 Answer: Money and Storage

1. Young: $e_t = c_t^y + k_{t+1} + m_{t+1}/p_t$. Old: $c_{t+1}^o = f(k_{t+1}) + m_{t+1}/p_{t+1} + x_{t+1}$.

2. The household solves

$$\max\ u\left(e_t - k_{t+1} - m_{t+1}/p_t\right) + \beta\, u\left(f(k_{t+1}) + m_{t+1}/p_{t+1} + x_{t+1}\right)$$

   First-order conditions are

$$\begin{aligned} u'(c_t^y) &= \beta\, u'(c_{t+1}^o)\, f'(k_{t+1}) \\ &= \beta\, u'(c_{t+1}^o)\, p_t/p_{t+1} \end{aligned}$$

   A solution is a vector $(c_t^y, c_{t+1}^o, k_{t+1}, m_{t+1})$ that solves the 2 first-order conditions and 2 budget constraints. From the household first-order conditions, money and capital must have the same rate of return: $f'(k_{t+1}) = p_t/p_{t+1}$.

3. A CE consists of sequences $(c_t^y, c_{t+1}^o, k_{t+1}, m_{t+1}, M_t, p_t, x_t)$ that solve 4 household conditions, 2 government conditions, the definition $M_t = m_t\, N_{t-1}$ and goods market clearing (same as resource constraint). In per capita terms: $k_{t+1} = e_t + f\left(k_t\right) / \left(1+n\right) - c_t^y - c_t^o / \left(1+n\right)$. Transfer payments equal new money issues: $N_{t-1}\, p_t\, x_t = \mu\, M_t$.

4. There is no equilibrium with positive inflation. That would imply rate of return dominance, and nobody would hold money.

5. A steady state consists of constants $(c^y, c^o, k, m/p, \pi, x)$ that satisfy:

   (a) Constant m/p requires $(1+\mu) = (1+n)\,(1+\pi)$. This determines π.

   (b) $f'(k) = (1+\pi)^{-1}$ determines k.

   (c) Both consumption levels are determined by the Euler equation and goods market clearing.

   (d) m/p is determined from the young budget constraint.

   (e) $x = \mu\,(m/p)$ from the law of motion for M.

6. Maximizing steady state consumption requires $f'(k) = 1+n$. Therefore, $\mu = 0$. The intuition is that with a constant nominal money supply the per capita money supply shrinks at rate n. For the real money supply to be constant, inflation at rate $-n$ is needed, which satisfies the Golden Rule.

2 Money in the Utility Function in an OLG Model

Demographics: In each period a cohort of constant size N is born. Each person lives for 2 periods.

Endowments: The initial old hold capital K_0 and money M. No new money is ever issued. The young are endowed with one unit of work time.

Preferences: $u(c_t^y) + \beta\, u(c_{t+1}^o) + v(m_t^d/p_t)$. Assume $v' > 0$. Agents derive utility from real money balances as defined below.

Technology: Output is produced with a constant returns to scale production function $F(K_t, L_t)$. The resource constraint is standard. Capital depreciates at rate δ.

Markets: There are spot markets for goods (price p_t), money, labor (wage w_t), and capital rental (price q_t).

Timing:

- The old enter period t holding money M and capital K_t.
- Production takes place.
- The old sell money to the young. m_t^d is the nominal per capita money holding of a young person.
- Consumption takes place.

Questions:

1. Derive a set of 4 equations that characterize optimal household behavior. Show that the household's first-order conditions imply rate of return dominance, i.e., the real return on money is less than the real return on capital (assuming both capital and money are held in equilibrium).
2. Solve the firm's problem.
3. Define a competitive equilibrium.
4. Assume that the utility functions u and v are logarithmic. Solve *in closed form* for the household's money demand function, $m_t^d/p_t = \varphi(w_t, r_{t+1}, \pi_{t+1})$, and for its saving function, $s_{t+1} = \phi(w_t, r_{t+1}, \pi_{t+1})$. $\pi_{t+1} \equiv p_{t+1}/p_t$.

2.1 Answer: Money in the Utility Function in an OLG Model

1. The household solves max $u(c_t^y) + \beta\, u(c_{t+1}^o) + v(m_t^d/p_t)$ subject to the budget constraints $w_t = c_t^y + m_t^d/p_t + s_{t+1}$ and $c_{t+1}^o = s_{t+1}\, R_{t+1} + m_t^d/p_{t+1}$. This is most easily set up using a lifetime budget constraint:

$$w_t - c_t^y - m_t^d/p_t = s_{t+1} = \frac{c_{t+1}^o - m_t^d/p_{t+1}}{R_{t+1}} \tag{1}$$

The first-order conditions are:

$$\begin{aligned} u'(c_t^y) &= \beta\, R_{t+1}\, u'(c_{t+1}^o) \\ v'(m_t^d/p_t) &= \beta\, u'(c_{t+1}^o)\, [R_{t+1} - p_t/p_{t+1}] \end{aligned}$$

Interpretation: If the household saves one unit as money rather than capital, he gains v', but loses some old consumption because the rate of return of money is lower than that of capital. Note that this implies rate of return dominance because positive marginal utility requires $R_{t+1} > p_t/p_{t+1}$. A solution to the household problem is a vector $(c_t^y, c_{t+1}^o, m_t^d, s_{t+1})$ which solves the 2 first-order conditions and the 2 budget constraints.

2. The firm's problem is standard with first order conditions $r_t = F_K(K_t, L_t)$ and $w_t = F_L(K_t, L_t)$.

3. A competitive equilibrium is an allocation $(c_t^y,\, c_{t+1}^o, m_t^d,\, s_{t+1},\, K_t, L_t)$ and a price system $(R_t,\, r_t,\, w_t,\, p_t)$ that satisfies 4 household FOCs, 2 firm FOCs, the identity $R_t = 1 + r_t - \delta$, and market clearing. The capital market clears if $N s_{t+1} = K_{t+1}$. The money market clears if $m_t^d = m_0$. Goods market clearing requires that $F(K_t, L_t) + (1-\delta)\, K_t = N c_t^y + N c_t^o + K_{t+1}$. We have 10 variables and 11 equations, one of which is redundant by Walras' law.

4. With log utility the household first-order conditions become $c_{t+1}^o = \beta\, R_{t+1}\, c_t^y = \beta\, m_t/p_t\, (R_{t+1} - p_t/p_{t+1})$. Define the inflation rate $\pi_{t+1} = p_{t+1}/p_t$. Then the budget constraint 1 together with the first-order condition for c_t^y imply

$$\frac{c_{t+1}^o}{R_{t+1}}\,(1 + 1/\beta) = w_t - \frac{m_t}{p_t}\,(1 - 1/\pi_{t+1} R_{t+1})$$

Substituting out c_{t+1}^o and simplifying yields the *money demand function*

$$w_t = \frac{m_t}{p_t}\,(2+\beta)\,(1 - 1/\pi_{t+1} R_{t+1})\,.$$

This has sensible properties. A higher nominal interest rate reduces money demand. If the household is more patient or richer, more money is held. To solve for the saving function:

$$\begin{aligned} s_{t+1} &= \frac{c_{t+1}^o - m_t/(p_t\,\pi_{t+1})}{R_{t+1}} \\ &= \frac{\beta\, m_t/p_t\,(R_{t+1} - 1/\pi_{t+1}) - m_t/(p_t\,\pi_{t+1})}{R_{t+1}} \\ &= \frac{m_t}{p_t}\,\left(\beta\,(1 - 1/\pi_{t+1} R_{t+1}) - 1/\pi_{t+1} R_{t+1}\right) \end{aligned}$$

Substitute out m_t/p_t using the money demand function.

$$s_{t+1} = w_t \frac{\beta\,(\pi_{t+1} R_{t+1} - 1) - 1}{(2+\beta)\,(\pi_{t+1} R_{t+1} - 1)}$$

From the saving function one could derive a single equation that characterizes the steady state capital stock. In steady state, $\pi = 1$; otherwise real money balances would not be constant. Apply this to the saving function and equate $s = K$. Then note that w and R are functions of K.
