Optimal Control

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Topics

Optimal control is a method for solving dynamic optimization problems in continuous time.

Example: Growth Model

A household chooses optimal consumption to

$$\max \int_0^T e^{-\rho t} u[c(t)] dt \tag{1}$$

subject to

$$\dot{k}(t) = rk(t) - c(t) \tag{2}$$

$$c(t) \in [0, \bar{c}] \tag{3}$$

$$k(0) = k_0, \text{given} \tag{4}$$

$$k(T) \geq 0 \tag{5}$$

Generic Optimal control problem

Choose functions of time $c(t)$ and $k(t)$ so as to

$$\max \int_0^T v[k(t), c(t), t]dt \tag{6}$$

Constraints:

1. Law of motion of the **state** variable $k(t)$:
$$\dot{k}(t) = g[k(t), c(t), t] \tag{7}$$
2. Feasible set for **control** variable $c(t)$:
$$c(t) \in Y(t) \tag{8}$$
3. Boundary conditions, such as:
$$k(0) = k_0, \text{given} \tag{9}$$
$$k(T) \geq k_T \tag{10}$$

Generic Optimal control problem

- c and k can be vectors.
- $Y(t)$ is a compact, nonempty set.
- T could be infinite.
  - Then the boundary conditions change
- Important: the state cannot jump; the control can.
- Note that this looks exactly like the kind of problem that could be solved with Dynamic Programming in discrete time.

2. A Recipe for Solving Optimal Control Problems

Step 1: Hamiltonian

$$H(t) = v(k,c,t) + \mu(t)\underbrace{g(k,c,t)}_{\dot{k}(t)} \tag{11}$$

μ is essentially a Lagrange multiplier (called a **co-state**).

Intuition:

- similar to the dynamic program: current utility + continuation value (but not quite)
- $v(k,c,t)$: current utility
- $\mu(t)$: the marginal value of increasing k for the future
- $g(k,c,t)$: captures how current actions affect future k

Step 2: First-order conditions

Derive the **first order conditions** which are **necessary** for an optimum:

$$\begin{aligned} \partial H/\partial c &= 0 \quad (12) \\ \partial H/\partial k &= -\dot{\mu} \quad (13) \end{aligned}$$

Intuition below ...

Step 3: TVC

Impose the **transversality** condition:

- for finite horizon:

$$\mu(T) = 0 \tag{14}$$

- for infinite horizon:

$$\lim_{t \to \infty} H(t) = 0 \tag{15}$$

This depends on the terminal condition (see below).

Step 4: Solution

A **solution** is the a set of functions $[c(t), k(t), \mu(t)]$ which satisfy

- the FOCs
- the law of motion for the state
- the boundary / transversality conditions

2.1. Intuition $\partial H/\partial c = 0$

Maximize Hamiltonian w.r.to control.

Implies

$$v_c + \mu g_c = 0 \tag{16}$$

$v_c(k,c,t)$ picks up current marginal utility of c

$\mu(t)$ is marginal value of additional "future" k.

$\mu(t) g_c(k,c,t)$ picks up change in continuation value

(change in $\dot{k}$ times marginal value of future k)

Intuition: $\partial H / \partial k = -\dot{\mu}$

Implies

$$v_k(k, c, t) + \mu g_k(k, c, t) = -\dot{\mu} \tag{17}$$

Think of this as

$$[\partial H / \partial k] / \mu = -\dot{\mu} / \mu \tag{18}$$

- $\dot{\mu} / \mu$ is the growth rate of marginal utility
- $[\partial H / \partial k] / \mu$ is like a rate of return (marginal value of k now versus the future)
- if the rate of return is high, it is optimal to postpone consumption and let it grow
- then marginal utility declines over time

2.2. Example: Growth Model

$$\max \int_0^\infty v(k,c,t)\,dt \to \max \int_0^\infty e^{-\rho t} u(c(t))\,dt \tag{19}$$

subject to

$$\begin{aligned}
\dot{k}(t) &= g(k,c,t) \equiv f(k(t)) - c(t) - \delta k(t) && (20)\\
c(t) &\in Y(t) \equiv [0, f(k_{max}) - \delta k_{max}] && (21)\\
& k(0) \text{ given} && (22)
\end{aligned}$$

For this to work, we need to bound $k \leq k_{max}$.

Growth Model: Hamiltonian

$$H(k,c,\mu) = \underbrace{e^{-\rho t}u(c(t))}_{v(k,c,t)} + \mu(t)\underbrace{[f(k(t)) - c(t) - \delta k(t)]}_{\dot{k}} \tag{23}$$

Necessary conditions:

$$\begin{aligned} H_c &= e^{-\rho t}u'(c) - \mu = 0 \\ H_k &= \mu\left[f'(k) - \delta\right] = -\dot{\mu} \end{aligned}$$

Interpretation

$$\mu = e^{-\rho t} u'(c) \tag{24}$$

μ is indeed the marginal value of capital

- the same as the marginal value of consumption

Note: μ is discounted to date 0

- it falls over time, even in steady state

$$-g(\mu) = f'(k) - \delta \tag{25}$$

When the rate of return is high, marginal utility falls over time

- getting paid to postpone consumption

Substitute out the co-state

FOC imply two expressions for $g(\mu)$:

$$
\begin{aligned}
g(\mu) &= \delta - f'(k) && (26)\\
&= g\left(e^{-\rho t} u'(c_t)\right) && (27)
\end{aligned}
$$

The growth rate of marginal utility (MRS) equals the "interest rate" (relative price).

Using the growth rate rule:

$$
\begin{aligned}
g\left(e^{-\rho t} u'(c)\right) &= -\rho + g\left(u'(c)\right) && (28)\\
&= -\rho - \sigma(c)\, g(c) && (29)
\end{aligned}
$$

where $\sigma(c) = -u''/u' \times c$ is the elasticity of marginal utility w.r.to c

Substitute out the co-state

Direct derivation:

$$
\begin{aligned}
g\left(e^{-\rho t}u'(c_t)\right) &= \frac{d \ln\left(e^{-\rho t}u'(c_t)\right)}{dt} && (30)\\
&= \frac{d}{dt}\left[-\rho t + \ln u'(c_t)\right] && (31)\\
&= -\rho + \frac{u''(c_t)\,\dot{c}_t}{u'(c_t)} && (32)
\end{aligned}
$$

Euler Equation

$$-g(\mu) = f'(k) - \delta = \rho + \sigma(c)\, g(c) \tag{33}$$

$$g(c) = \frac{f'(k) - \delta - \rho}{\sigma(c)}$$

Analogous to the discrete time version:

$$\frac{c_{t+1}}{c_t} = \left(\beta R'\right)^{1/\sigma(c)} \tag{34}$$

Solution: c_t, k_t that solve Euler equation and resource constraint, plus boundary conditions.

2.3. Details

First order conditions are **necessary**, not sufficient.

They are necessary only if we **assume** that

1. a continuous, interior solution exists;
2. the objective function v and the constraint function g are continuously differentiable.

Acemoglu (2009), ch. 7, offers some insight into why the FOCs are necessary.

- Also discusses sufficient conditions.

Details

If there are multiple states and controls, simply write down one FOC for each separately:

$$\begin{aligned} \delta H / \delta c_i &= 0 \\ \partial H / \partial k_j &= -\dot{\mu}_j \end{aligned}$$

There is a large variety of cases depending on the length of the horizon (finite or infinite) and the kinds of boundary conditions.

- Each has its transversality condition (see Leonard and Van Long 1992).

Equality constraints

Equality constraints of the form

$$h[c(t), k(t), t] = 0 \tag{35}$$

are simply added to the Hamiltonian as in a Lagrangian problem:

$$H(t) = v(k, c, t) + \mu(t) g(k, c, t) + \lambda(t) h(k, c, t) \tag{36}$$

FOCs are unchanged:

$$\begin{aligned} \partial H / \partial c &= 0 \\ \partial H / \partial k &= -\dot{\mu} \end{aligned}$$

For inequality constraints:

$$h(c, k, t) \geq 0;\ \lambda h = 0 \tag{37}$$

3. Recipe with Discounting

Discounting: Current value Hamiltonian

Problems with discounting:

- Current utility depends on time only through an exponential discounting term $e^{-\rho t}$.

The generic discounted problem is

$$\max \int_0^T e^{-\rho t} v[k(t), c(t)] dt \tag{38}$$

subject to the same constraints as above.

Shortcut

Discounted Hamiltonian (drop the discounting term):

$$H = v(k,c) + \mu g(k,c) \tag{39}$$

FOCs:

$$\partial H / \partial c = 0 \tag{40}$$

$$\partial H / \partial k = \underbrace{\mu(t)\rho}_{\text{added}} - \dot{\mu}(t) \tag{41}$$

and the TVC

$$\lim_{T \to \infty} e^{-\rho T} \mu(T) k(T) = 0 \tag{42}$$

Deriving the Shortcut

Start from the standard recipe:

$$H(t) = e^{-\rho t} v(k,c) + \hat{\mu} g(k,c) \tag{43}$$

$$\frac{\partial H}{\partial c_t} = 0 \implies e^{-\rho t} v_c(k_t, c_t) = -\hat{\mu}_t g_c(k_t, c_t) \tag{44}$$

$$\frac{\partial H}{\partial k_t} = e^{-\rho t} v_k(k_t, c_t) + \hat{\mu}_t g_k(k_t, c_t) = -\dot{\hat{\mu}}_t \tag{45}$$

$\hat{\mu}$ is the **discounted** marginal value of k.

Deriving the Shortcut

Let

$$\mu_t = e^{\rho t}\hat{\mu}_t \tag{46}$$

and multiply through by $e^{\rho t}$:

$$\frac{\partial H}{\partial c_t} = 0 \implies \underbrace{e^{\rho t} e^{-\rho t}}_{1} v_c(k_t, c_t) = -\underbrace{e^{\rho t}\hat{\mu}_t}_{\mu_t} g_c(k_t, c_t) \tag{47}$$

$$v_c(t) = -\mu_t g_c(t)$$

This is the standard FOC, but with μ instead of $\hat{\mu}$.

μ is the **current** marginal value of k.

Deriving the Shortcut

$$v_k(t) + e^{\rho t}\hat{\mu}_t g_k(t) = -e^{\rho t}\dot{\hat{\mu}}_t \tag{48}$$

Substitute out $\dot{\hat{\mu}}_t$ using

$$\dot{\mu}_t = \frac{de^{\rho t}\hat{\mu}_t}{dt} = \rho\mu_t + e^{\rho t}\dot{\hat{\mu}}_t$$

we have

$$v_k(t) + \mu_t g_k(t) = -\dot{\mu}_t + \rho\mu_t$$

This is the standard condition with an additional $\rho\mu$ term.

Example: Growth Model

$$\max \int_0^\infty e^{-\rho t} u(c(t))\, dt \tag{49}$$

subject to

$$\begin{aligned} \dot{k}(t) &= f(k(t)) - c(t) - \delta k(t) && (50) \\ c(t) &\in [0, f(k_{max}) - \delta k_{max}] && (51) \\ & k(0) \text{ given} && (52) \end{aligned}$$

Discounted Hamiltonian

$$H(k,c,\mu) = u(c(t)) + \mu(t)\underbrace{[f(k(t)) - c(t) - \delta k(t)]}_{\dot{k}} \tag{53}$$

Necessary conditions:

$$\begin{aligned} H_c &= u'(c) - \mu = 0 \\ H_k &= \mu\left[f'(k) - \delta\right] = \rho\mu - \dot{\mu} \end{aligned}$$

Euler Equation

$$
\begin{aligned}
-g(\mu) &= f'(k) - \delta - \rho && (54)\\
&= -g\left(u'(c)\right) && (55)\\
&= \sigma(c)\, g(c) && (56)
\end{aligned}
$$

or

$$g(c) = \frac{f'(k) - \delta - \rho}{\sigma(c)} \tag{57}$$

4. Transversality Conditions

4.1. Finite horizon: Scrap value problems

The horizon is T.

The objective function assigns a scrap value to the terminal state variable: $e^{-\rho T}\phi(k(T))$:

$$\max \int_0^T e^{-\rho t} v[k(t), c(t), t] dt + e^{-\rho T}\phi(k(T)) \tag{58}$$

Hamiltonian and FOCs: unchanged.

The TVC is

$$\mu(T) = \phi'(k(T)) \tag{59}$$

Intuition:

- μ is the marginal value of the state k.
- Recall that μ is the **current** value of k.

Scrap value examples

Household with bequest motive

$$U = \int_0^T e^{-\rho t} u(c(t)) + e^{-\rho t} V(k_T) \tag{60}$$

with $\dot{k} = w + rk - c$.

TVC:

$$\mu(T) = u'(c(T)) = V'(k_T) \tag{61}$$

Scrap value examples

Maximizing the present value of earnings

$$Y = \int_0^T e^{-rt} w h(t) [1 - l(t)] \tag{62}$$

subject to $\dot{h}(t) = A h(t)^{\alpha} l(t)^{\beta} - \delta h(t)$

Scrap value is 0.

TVC: $\mu(T) = 0$.

4.2. Infinite horizon TVC

The finite horizon TVC with the boundary condition $k(T) \geq k_T$ is $\mu(T) = 0$.

- Intuition: capital has no value at the end of time.

But the infinite horizon boundary condition is NOT $\lim_{t \to \infty} \mu(t) = 0$.

The next example illustrates why.

Infinite horizon TVC: Example

$$
\begin{aligned}
& \max \int_0^\infty \left[\ln\left(c\left(t\right)\right) - \ln\left(c^*\right)\right] dt \\
& \textit{subject to} \\
\dot{k}\left(t\right) &= k\left(t\right)^\alpha - c\left(t\right) - \delta k\left(t\right) \\
k\left(0\right) &= 1 \\
\lim_{t\to\infty} k\left(t\right) &\geq 0
\end{aligned}
$$

c^* is the max steady state (golden rule) consumption.

No discounting - subtracting c^* makes utility finite.

Infinite horizon TVC

Hamiltonian

$$H(k, c, \lambda) = \ln c - \ln c^* + \lambda \left[k^\alpha - c - \delta k\right] \tag{63}$$

Necessary FOCs

$$\begin{aligned} H_c &= 1/c - \lambda = 0 & (64) \\ H_k &= \lambda \left[\alpha k^{\alpha-1} - \delta\right] = -\dot{\lambda} & (65) \end{aligned}$$

Infinite horizon TVC

We show: $\lim_{t\to\infty} c(t) = c^*$ [why?]

Limiting steady state solves

$$\begin{aligned}\dot{\lambda}/\lambda &= \alpha k^{\alpha-1} - \delta = 0 \\ \dot{k} &= k^{\alpha} - 1/\lambda - \delta k = 0\end{aligned}$$

Solution is the golden rule:

$$k^* = (\alpha/\delta)^{1/(1-\alpha)} \tag{66}$$

Verify that this max's steady state consumption.

Infinite horizon TVC

Implications for the TVC...

$\lambda(t) = 1/c(t)$ implies $\lim_{t\to\infty} \lambda(t) = 1/c^*$.

Therefore, neither $\lambda(t)$ nor $\lambda(t)k(t)$ converge to 0.

The **generically correct TVC**:

$$\lim_{t\to\infty} H(t) = 0 \tag{67}$$

The only reason why the standard TVC does not work:

- there is **no discounting** in the example.

Infinite horizon TVC: Discounting

With discounting, the TVC is easier to check.

Assume:

- the objective function is $e^{-\rho t} v\left[k(t), c(t)\right]$
- it only depends on t through the discount factor
- v and g are weakly monotone

Then the TVC becomes

$$\lim_{t \to \infty} e^{-\rho t} \mu(t) k(t) = 0 \tag{68}$$

where μ is the costate of the current value Hamiltonian.

This is exactly analogous to the discrete time version

$$\lim_{t \to \infty} \beta^t u'(c_t) k_t = 0 \tag{69}$$

5. Example: renewable resource

Setup

$$\max \int_0^\infty e^{-\rho t} u(y(t))\, dt \tag{70}$$

$$\textit{subject to} \tag{71}$$

$$\dot{x}(t) = -y(t) \tag{72}$$

$$x(0) = 1 \tag{73}$$

$$x(t) \geq 0 \tag{74}$$

Hamiltonian

Current value Hamiltonian

Necessary FOCs

Solution

Therefore:

$$\mu(t) = \mu(0)\, e^{\rho t} \tag{75}$$

$$y(t) = u'^{-1}\left[\mu(0)\, e^{\rho t}\right] \tag{76}$$

The optimal path has $\lim x(t) = 0$ or

$$\int_0^\infty y(t)\, dt = \int_0^\infty u'^{-1}\left[\mu(0)\, e^{\rho t}\right] dt = 1 \tag{77}$$

This solves for $\mu(0)$.

TVC

TVC for infinite horizon case:

$$\lim e^{-\rho t}\mu(0)e^{\rho t}x(t)=0 \tag{78}$$

Equivalent to

$$\lim x(t)=0 \tag{79}$$

Reading

- Acemoglu (2009), ch. 7. Proves the Theorems of Optimal Control.
- Barro and Sala-i Martin (1995), appendix.
- Leonard and Van Long (1992): A fairly comprehensive treatment. Contains many variations on boundary conditions.

References I

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