

# The Growth Model: Discrete Time

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# The standard growth model

- ▶ The neoclassical growth model, aka the standard growth model, is the most important model in macro.
- ▶ It underlies entire branches of the literature (parts of growth theory and business cycle theory, for example).
- ▶ Here, we study this model in discrete time.
- ▶ The **main issues** of this section are:
  - ▶ Tools: Dynamic programming
  - ▶ The neoclassical growth model

# Model structure

There are many versions of the growth model. This is a basic version.

1. Households are identical and live forever.
2. Firms produce a single good using capital and labor.
3. All agents are price takers.
4. All prices are perfectly flexible. All markets clear at all times.

# Infinite horizons

- ▶ So far we have assumed that agents are finitely lived.
- ▶ Analytically more convenient: infinite lifetimes.
- ▶ How to justify this?
  - ▶ Reduced form of an OLG model with **altruism**.
  - ▶ Stochastic deaths (perpetual youth models).
  - ▶ But really: convenience + show it does not matter.

# Demographics

There is a continuum of households (uncountably infinite number).  
All households are identical.

- ▶ This is stronger than needed (see notes on aggregation later on).

We can think of a single, price-taking household.

The measure of households is 1.

Therefore, per capita and aggregate variables are the same.

**Exercise:** Redo everything when the number of households is  
 $N_t = (1 + n)^t$ .

# Preferences

The household values discounted utility from consumption:

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1 \quad (1)$$

Utility is time separable (for tractability).

Discounting is exponential (to avoid time consistency problems).

Time consistency means:

- ▶ If  $\{c_t\}_{t=0}^{\infty}$  solves the problem with start date 0, then  $\{c_t\}_{t=\tau}^{\infty}$  solves the problem with start date  $\tau$ .
- ▶ The household does not want to change past plans.

# Endowments

The household has

- ▶  $k_0$  units of the good at  $t = 0$
- ▶ 1 unit of time in each period

# Technology

- ▶ Resource constraint:

$$k_{t+1} = f(k_t) - c_t \quad (2)$$

- ▶ We assume Inada conditions for  $f$ .
- ▶ Capital cannot be negative:  $k_t \geq 0$ .



# Markets

Goods: numeraire.

Labor:  $w_t$

Capital rental:  $q_t$

All markets are competitive.

# Planning Problem

The planner maximizes discounted utility of the representative household

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1$$

Constraints:

$$k_{t+1} = f(k_t) - c_t$$

$$k_{t+1} \geq 0$$

$k_0$  given

# Lagrangian

$$\begin{aligned}\Gamma = & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ & + \sum_{t=0}^{\infty} \lambda_t [f(k_t) - c_t - k_{t+1}]\end{aligned}$$

FOCs for an interior solution:

$$\begin{aligned}\beta^t u'(c_t) &= \lambda_t \\ \lambda_{t+1} f'(k_{t+1}) &= \lambda_t\end{aligned}$$

## Euler equation

$$u'(c_t) = \beta u'(c_{t+1}) f'(k_{t+1}) \quad (3)$$

This is exactly the same Euler equation we saw many times before.  
The Euler equation implicitly defines a law of motion for the capital stock:

$$u'(f(k_t) - k_{t+1}) = \beta u'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}) \quad (4)$$

This is a second order difference equation.

► but later we will see that it isn't...

# Planner: Solution

A solution is a sequence  $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ .

These satisfy the **necessary** conditions:

1. Euler equation
2. Resource constraint
3.  $k_0$  given

We have two difference equations, but only one **boundary condition**.

Uniqueness requires an additional restriction:

**Transversality:**

$$\lim_{t \rightarrow \infty} \beta^t u'(c_t) k_{t+1} = 0 \quad (5)$$

Digression: Transversality Conditions

## Digression: Transversality Conditions

Consider the following example:

$$\begin{aligned} \max \sum_{t=0}^T \beta^t u(c_t) \\ \text{s.t. } k_{t+1} &= e_t + (1 + r_t)k_t - c_t \end{aligned}$$

Note the finite horizon:  $T < \infty$ .

As stated, this problem does not have a solution (why not?).

## Digression: Transversality Conditions

Let's proceed to solve the problem as stated.

Lagrangian

$$\begin{aligned}\Gamma = & \sum_{t=0}^T \beta^t u(c_t) \\ & + \sum_{t=0}^T \lambda_t \{e_t + (1 + r_t)k_t - c_t - k_{t+1}\}\end{aligned}$$

FOCs (necessary):

$$u'(c_t) = \beta u'(c_{t+1}) (1 + r_{t+1})$$



# Solution

Sequences  $\{c_t, k_{t+1}\}$  that satisfy:

- ▶ Euler equation
- ▶ budget constraint
- ▶  $k_0$  given

# Problem 1

We allowed the household to choose  $c_t \rightarrow \infty$  and  $k_{t+1} \rightarrow -\infty$ .

- ▶ The household problem has no solution.

Solution:

- ▶ We need a constraint that bounds consumption
- ▶ A natural candidate in this example:  $k_{T+1} \geq 0$ .
- ▶ The household cannot die in debt.
- ▶ This is part of the economic model.

## Problem 2

We have 2 difference equations, but only one boundary condition.

- ▶ The solution is not uniquely determined by those.

We need one more boundary condition to ensure that utility is finite.

In this model, the right boundary condition is obvious:  $k_{T+1} = 0$ .

- ▶ Leaving food on the table cannot be optimal.

## Infinite horizon case

What if  $T \rightarrow \infty$ ?

We could impose  $\lim_{T \rightarrow \infty} k_{T+1} = 0$ , but it does not make economic sense.

- ▶ This would prevent the household from perpetually growing its capital stock.

Economically, the problem is that the household could choose  $c_t \rightarrow 0$  and  $k_t \rightarrow k_{\max}$  (the max sustainable capital stock).

## Infinite horizon case

The correct boundary condition to rule out such paths:

$$\lim_{T \rightarrow \infty} k_{T+1}/R_{T+1} = 0 \quad (6)$$

Intuition:

- ▶ If the interest rate is positive, it makes sense for the household to maintain a high capital stock forever.
- ▶ But at least some of the interest that the capital generates will be eaten.

## Infinite horizon intuition

Write the present value budget constraint as

$$\sum_{t=0}^T \frac{c_t}{R_t} = \sum_{t=0}^T \frac{e_t}{R_t} + k_0 - \frac{k_{T+1}}{R_{T+1}}$$

where  $R_t = (1 + r_1) \times \dots \times (1 + r_t)$  is a cumulative discount factor.

- **Exercise:** show that this is the present value budget constraint.

$$\lim_{T \rightarrow \infty} k_{T+1}/R_{T+1} = 0 \tag{7}$$

ensures finite consumption.

## Infinite horizon case

An equivalent solution:

Impose

$$\lim_{T \rightarrow \infty} \beta^T u'(c_T) k_{T+1} = 0$$

This is the same as

$$\lim_{T \rightarrow \infty} k_{T+1}/R_{T+1} = 0 \quad (8)$$

because, by the Euler equation:

$$\beta^T u'(c_T) R_T = u'(c_0)$$

# The general point

In dynamic optimization problems, the flow budget constraint is usually not enough to ensure finite utility.

- ▶ The economics of the model provides the condition that fixes this problem.

Even then, the solution is not unique.

With finite  $T$ , the correct boundary condition is usually obvious (don't leave assets on the table).

With infinite  $T$ , the correct boundary condition is not obvious.



# Reading

- ▶ Acemoglu (2009), ch. 6. Also ch. 5 for background material we will discuss in detail later on.
- ▶ Stokey et al. (1989), ch. 1 is a nice introduction.
- ▶ Blanchard and Fischer (1989) is a good introduction to the standard growth model.
- ▶ Krusell (2014) ch. 2 discusses why the assumptions made in the growth model are popular.

# References I

Acemoglu, D. (2009): *Introduction to modern economic growth*, MIT Press.

Blanchard, O. J. and S. Fischer (1989): *Lectures on macroeconomics*, MIT press.

Krusell, P. (2014): "Real Macroeconomic Theory," Unpublished.

Stokey, N., R. Lucas, and E. C. Prescott (1989): "Recursive Methods in Economic Dynamics," .