Macroeconomics Qualifying Examination

June 2022

Department of Economics

UNC Chapel Hill

Instructions:

- This examination consists of **3** questions. Answer all questions.
- The total time is 3 hours. The total number of points is 180.
- If you believe a question is ambiguously stated, ask for clarification. Unnecessary simplifications will be penalized.
- **Write legibly**.
- Number your answers.
- Explain your answers.
- Budget your time wisely. Don't get hung up on one question.
- Good luck!

1 Seignorage Financed Public Capital

Demographics: In each period, $N_t = (1+n)^t$ young households are born. Each household lives for two periods.

Preferences: Households born in t value consumption when old (c_{t+1}) and work hours when young l_t according to $\mathcal{U}(c_{t+1}) - \mathcal{V}(l_t)$.

Technology: A single good is produced from public capital K and time L according to $Y_t = A(K_t/N_t) L_t$. The function $A(K/N)$ is strictly increasing. The resource constraint is

$$Y_t + (1-\delta) K_t = N_{t-1} c_t + K_{t+1} \tag{1}$$

Endowments: At the beginning of time ($t = 0$), the government is endowed with K_0 units of capital.

Government: The government issues fiat money to purchase net investment. If M_t is the stock of money at the start of t, then the government budget constraint is

$$M_{t+1} - M_t = p_t (K_{t+1} - (1-\delta) K_t) \tag{2}$$

The government chooses the sequence of M_t to fix the per capita capital stock at $K_t/N_t = \bar{k}$.

Markets: There are competitive markets for goods (price p_t), money (numeraire), and labor (real wage w_t). Private agents take public capital K as given.

Questions:

1. [10 points] Write out the problem solved by generation t and define a solution.

   Answer

   Budget constraint: $w_t l_t = s_{t+1}$. $c_{t+1} = R_{t+1} s_{t+1} = R_{t+1} w_t l_t$.[1]

   Problem: $\max_{l_t} \mathcal{U}(R_{t+1} w_t l_t) - \mathcal{V}(l_t)$.

   FOC: $\mathcal{U}'(R_{t+1} w_t l_t) R_{t+1} w_t = \mathcal{V}'(l_t)$.

   Solution: (c_{t+1}, l_t) that solve the first-order condition and budget constraint.

   In equilibrium, the gross interest rate R will be negative inflation, but we don't have to worry about this yet.

2. [10 points] Write out the government's budget constraint and the market clearing conditions. All variables should be in per capita real form.

   Answer

[1] Based on Penn State qual in 2016.

Government budget constraint: Let $m_t \equiv M_t / (N_t P_t)$. Then

$$m_{t+1}(1+n)(1+\pi_{t+1}) - m_t = k_{t+1}(1+n) - (1-\delta) k_t \quad (3)$$
$$= \bar{k}(n+\delta) \quad (4)$$

where $1+\pi_{t+1} \equiv p_{t+1}/p_t$.

Firm: $\max A(\bar{k}) L - wL$ implies $w = A(\bar{k})$.

Market clearing:

- Goods: resource constraint: $A(\bar{k}) + (1-\delta) k = \frac{c}{1+n} + k'(1+n)$.
- Labor: $L_t/N_t = l_t$.
- Money: Household saving in nominal terms is $s_{t+1} N_t p_t = M_{t+1} = m_{t+1} N_{t+1} p_{t+1}$ or

$$s_{t+1} = m_{t+1}(1+n)(1+\pi_{t+1}) \quad (5)$$

---

3. [10 points] Define a competitive equilibrium.

Answer

Objects: $\{c_t, l_t, s_t, k_t, m_t, w_t, R_t, \pi_t\}$

Equations:

- household: 3
- government budget constraint and $k_t = \bar{k}$
- firm: 1
- market clearing: 2 (labor is implicit)
- identity: $R_{t+1} = 1/(1+\pi_{t+1})$ (money pays no interest).

---

4. [15 points] Go as far as you can towards characterizing the steady state. Is it unique? What is the intuition?

Answer

The wage is constant at $\bar{w} = A(\bar{k})$.

We now have 3 equations in 3 unknowns:

- household budget constraint: $wl = m(1+n)(1+\pi)$.
- government budget constraint:$\bar{w}l = \bar{k}(n+\delta) + m$
- first order condition, which can be written as $\mathcal{U}'\left(\frac{wl}{1+\pi}\right)\frac{wl}{1+\pi} = \mathcal{V}'(l)\, l$

In addition, consumption follows from goods market clearing, which, in steady state, implies $k(n+\delta) = A(k)\,l - c$ or $c = m(1+n)$.

The steady state is generically not unique. One way of seeing this is consider the log utility case where $\mathcal{U} = \mathcal{V} = \ln$. Then the first-order condition is always satisfied (income and substitution effects cancel). We lose one equation and therefore only have 2 equations for 3 unknowns (l, m, π).

We may simplify the two budget constraints as

$$m(1+n)(1+\pi) - m = \bar{k}(n+\delta) \tag{6}$$

Any combination of m and π that satisfies this condition is a balanced growth path. Increasing m allows the government to lower the "inflation tax" and still pay for its expenditures. Households must work more to support the increased saving (cf the government budget constraint). As long as preferences are such that higher rates of return are consistent with longer work hours, we get another steady state.

---

5. [15 points] State the social planner's problem, derive the first-order conditions, and define a solution.

Answer

This question was, regrettably, ambiguous. What I had in mind was a planner who chooses $\{k_t\}$, but I did not penalize those who assumed that $\bar{k}$ was still fixed (students should have asked!).

The planner solves:

$$V(k) = \max_{l,k'} \omega^{-1}\mathcal{U}(c) - \mathcal{V}(l) + \omega V\left(\frac{A(k)\,l + (1-\delta)\,k - c}{1+n}\right) \tag{7}$$

FOC:

$$\omega^{-1}\mathcal{U}'(c) = \omega V'(k') / (1+n) \tag{8}$$

$$\mathcal{V}'(l) = \omega V'(k') \frac{A(k)}{1+n} \tag{9}$$

Envelope:

$$V'(k) = \omega V'(k') \frac{A'(k)\,l + 1 - \delta}{1+n} \tag{10}$$

This implies a static condition:

$$\omega^{-1}\mathcal{U}'(c)\,A(k) = \mathcal{V}'(l) \tag{11}$$

In words: Working another hour produces $A(k)$ units of consumption today.

Euler equation:

$$\omega^{-2}\mathcal{U}'(c)(1+n) = V'(k') \tag{12}$$

$$= \omega V'(k'') \frac{A'(k')l' + 1 - \delta}{1+n} \tag{13}$$

$$= \omega\omega^{-2}(1+n)\mathcal{U}'(c') \frac{A'(k')l' + 1 - \delta}{1+n} \tag{14}$$

or

$$\mathcal{U}'(c) = \omega\mathcal{U}'(c') \frac{A'(k')l' + 1 - \delta}{1+n} \tag{15}$$

This is actually quite standard. Giving up a unit of consumption today yields $(1+n)^{-1}$ units of capital tomorrow. The next generation may then eat its marginal product $A'(k')l'$

Solution: $\{c_t, l_t, k_t\}$ that solve the static condition, Euler equation, and resource constraint.

Note on the planner's steady state:

- We get the modified golden rule

$$\omega A'(k)l = 1+n \tag{16}$$

- and a "static" condition (which spans generations now) of

$$\mathcal{U}'(c)A(k) = \omega\mathcal{V}'(l) \tag{17}$$

---

2 Government Spending Shocks and Business Cycles

Consider a real business cycle model with government spending. The representative household has preferences

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t)$$

where U is increasing in consumption C_t and decreasing in labor N_t. The representative firm has the technology

$$Y_t = z_t F(K_t, N_t),$$

where F has standard properties. Capital K_t depreciates at rate δ. The productivity shock z_t follows a Markov process.

Every period, a given amount G_t must be allocated to government spending, where G_t follows a Markov process. The government as well as private agents take G_t as given. Government spending is completely wasteful and is just thrown away.

Questions:

1. (5 points) Write down the social planner's problem in recursive form.

   Answer

   $$V(z, G, K) = \max_{C,N,K'} + \beta \mathbb{E} V(z', G', K') \tag{18}$$

   subject to

   $$C + K' + G = (1 - \delta)K + zF(K, N) \tag{19}$$

2. (15 points) Suppose that the utility function U takes the form $U(C, N) = \ln(C - h(N))$, where h is some function satisfying $h' > 0$, $h'' > 0$. Determine how a shock to G_t would affect period-t hours worked, and how the effect is different from a shock to z_t. Prove your answer mathematically, and provide some intuition.

   Answer

   Suppose the utility function is of the form

   $$U(C, N) = \ln(C - h(N)) \tag{20}$$

   for some functions u and h. In this case, the first-order conditions for C and N are

   $$\frac{1}{C - h(N)} = \lambda \tag{21}$$

and

$$\frac{1}{C - h(N)} h'(N) = \lambda z F_N(K, N) \tag{22}$$

These can be combined to get

$$h'(N) = z F_N(K, N), \tag{23}$$

which clearly pins down the optimal N independently of G, though it still depends on z. The above functional form rules out income effects on labor supply, which are the only channel through which G would affect N. On the other hand, the same expression also tells us that z unambiguously increases N, since $h'(N)$ is increasing and $F_N(K, N)$ is decreasing in N. z affects N through a substitution effect.

---

3. (15 points) Now, assume that $U(C, N)$ is of the separable form, $U(C, N) = u(C) + v(1 - N)$, where $u' > 0$, $u'' < 0$, and $v' > 0$, $v'' < 0$. Prove mathematically that a shock to G_t *cannot* simultaneously raise current consumption and hours worked.

Answer

In this case, the first-order conditions for C and N in the planner's problem can be combined to obtain

$$\frac{v'(1 - N)}{z F_N(K, N)} = u'(C) \tag{24}$$

The left-hand side is increasing in N, while the right-hand side is decreasing in C. From this, it is immediate that, for a given z and K, consumption and hours worked must move in opposite directions in response to a government spending shock.

---

4. (10 points) Continue to assume that utility is of the form $U(C, N) = u(C) + v(1 - N)$. Describe in words how you would expect a shock to G_t to affect N_t and Y_t/N_t, and how this is different from a shock to z_t. Describe in words how the model-implied co-movement between N_t and Y_t/N_t would be different from a model with z_t shocks only.

Answer

An increase in G generates an income effect, lowering C and increasing N. In turn, this increase in N lowers $Y/N = zF(K, N)/N$, by the concavity assumption on F. So, a shock to G drives N and Y/N in opposite directions.

On the other hand, an increase in z increases N (through the inter-temporal substitution channel) but also increases Y/N. This is because, unlike G, z directly affects $Y/N = zF(K, N)/N$. So, shocks to z would drive N and Y/N in the same direction.

Because of the above, a model with both G_t and z_t shocks would produce a *lower* correlation between N_t and Y_t/N_t than a model with z_t shocks only.

Now, consider a competitive equilibrium. Assume that households own the capital and rent it out to firms. The government finances its expenditures G_t with a proportional tax on households' period-t capital income.

5. (5 points) Write down the government's budget constraint in a recursive competitive equilibrium.

Answer

Letting $r(z, G, K)$ be the rental rate on capital and $\tau(z, G, K)$ the tax rate on capital income, the government budget constraint reads

$$G = \tau(z, G, K)r(z, G, K)K \tag{25}$$

---

6. (15 points) Prove that the competitive equilibrium allocation does not coincide with the solution to the social planner's problem.

Answer

The Euler equation for the social planner reads

$$u_C(C_t, N_t) = \beta\mathbb{E}(1 - \delta + z_{t+1}F_K(K_{t+1}, N_{t+1}))U_C(C_{t+1}, N_{t+1}) \tag{26}$$

On the other hand, it can be shown that the Euler equation for a household in a competitive equilibrium is

$$u_C(C_t, N_t) = \beta\mathbb{E}(1 - \delta + (1 - \tau_{t+1})z_{t+1}F_K(K_{t+1}, N_{t+1}))U_C(C_{t+1}, N_{t+1}) \tag{27}$$

Clearly, the optimality conditions do not coincide, since $\tau_{t+1} > 0$.

---

3 Government Spending With Heterogeneous Agents

An economy consists of a large measure of infinitely-lived households, who are subject to idiosyncratic income shocks. Each household has utility

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

with $u' > 0$ and $u'' < 0$. Each household's **idiosyncratic** income endowment y_t is i.i.d. across households and over time, and can take on only two values: y^h with probability λ, and y^ℓ with probability $1-\lambda$, where $0 < y^\ell < y^h$. Household can save in a non-contingent bond, but *cannot* borrow.

The government has an exogenously given amount of expenditures, G, every period. It finances this government expenditure through a combination of taxes and bonds: every period, it issues an exogenously fixed amount B in bonds and collects a lump-sum tax T, which is endogenously determined to satisfy its budget constraint. Both B and G are fixed over time. Assume that $G < y^\ell$.

We will consider a stationary competitive equilibrium.

Questions:

1. (10 points) Write down the household's problem in recursive form, using as few state variables as possible. Clearly explain what you are doing. What assumptions simplify this problem?

 Answer

 In general, the household's individual state would be the current income shock y, and its asset holdings, a. Because y is i.i.d., we can combine the two into one individual state, cash-on-hand, defined as

 $$x = (1+r)a + y \tag{28}$$

 Furthermore, because we are considering a stationary competitive equilibrium, we can take the interest rate r and the lump-sum tax T to be constant over time (in particular, we do not have an aggregate state variable here).

 The household's problem is then

 $$V(x) = \max_{c,a'} u(c) + \beta\left[\lambda V((1+r)a' + y^h) + (1-\lambda)V((1+r)a' + y^\ell)\right] \tag{29}$$

 subject to the budget constraint

 $$c + a' = x - T \tag{30}$$

 and the no-borrowing constraint

 $$a' \geq 0 \tag{31}$$

 ---

2. (10 points) Derive the household's inter-temporal optimality condition. Clearly explain how it differs depending on whether the no-borrowing constraint binds.

Answer

The first-order conditions for c and a' respectively are

$$u'(c) = \lambda \tag{32}$$

and

$$\lambda = \beta(1+r)\mathbb{E}V'((1+r)a' + y') \tag{33}$$

The envelope condition is

$$V'(x) = \lambda \tag{34}$$

The first-order and envelope conditions for the household can be combined to get

$$u'(c_t) \geq \beta(1+r)\mathbb{E}u'(c_{t+1}) \tag{35}$$

with equality if the no-borrowing constraint binds, and strict inequality if it does.

---

3. (5 points) Write down the government budget constraint.

Answer

We have

$$G + (1+r)B = T + B \tag{36}$$

which simplifies to

$$G + rB = T \tag{37}$$

---

4. (5 points) Write down the asset market clearing condition.

Answer

Letting $\Phi(a, y)$ be the stationary distribution of assets and income, we must have

$$\int_{A\times\{y^h, y^\ell\}} a'(a, y)\Phi(da, dy) = B \tag{38}$$

Asset demand equals asset supply. In other words, the total saving of the households must equal total borrowing by the government.

---

5. (20 points) Suppose that $B = 0$. Describe the equilibrium. What can you say about the wealth distribution? What can you say about the interest rate r in equilibrium? Prove that the interest rate satisfies $\beta(1+r) < 1$.

Answer

First, $B = 0$ implies that asset demand a' must add up to zero. But, since $a' \geq 0$ for all households, this is only possible if $a' = 0$ for all households. In other words, if net asset demand is zero and no one is borrowing, market clearing implies that no one is lending. So, in equilibrium, all households consume their after-tax income. Since $B = 0$, $T = G$ by the government budget constraint, and so a household with income y consumes $c = y - G$.

The second step is to use this information to characterize the equilibrium interest rate. The Euler equations for the high and low-income households are

$$u'(y^h - G) \geq \beta(1+r)\left[\lambda u'(y^h - G) + (1-\lambda)u'(y^\ell - G)\right] \tag{39}$$

and

$$u'(y^\ell - G) \geq \beta(1+r)\left[\lambda u'(y^h - G) + (1-\lambda)u'(y^\ell - G)\right] \tag{40}$$

Since $u'(y^h - G) < u'(y^\ell - G)$, (40) must be strict, which means the no-borrowing constraint definitely binds for the low-income households. The highest-possible interest rate in equilibrium is the one where (39) holds with equality, so that the no-borrowing constraint does not bind for the high-income households. Or in other words, the interest rate has to be such that the high-income household chooses not to save. From this, it is clear that r must satisfy

$$\beta(1+r) \leq \frac{u'(y^h - G)}{\lambda u'(y^h - G) + (1-\lambda)u'(y^\ell - G)} \tag{41}$$

and therefore $\beta(1+r) < 1$ since $u'(y^h - G) < \lambda u'(y^h - G) + (1-\lambda)u'(y^\ell - G)$.

---

6. (5 points) Suppose that $B > 0$ (for the same G). How is the equilibrium different? What can you say about the wealth distribution? You are *not* asked to define a stationary wealth distribution or compute the interest rate. Just explain carefully what is different and why.

Answer

Now, there will be saving in equilibrium, and therefore there will be a non-trivial wealth distribution. The interest would now have to be determined by the market clearing condition as described above.

End of exam.