Review Questions: Stochastic Growth Model

Econ720. Fall 2020. Prof. Lutz Hendricks

Useful problems from the texts:

- Acemoglu, "Introduction to Modern Economic Growth:" Exercises 17.2.1, 17.5.2, 17.9, 17.11, 17.12, 17.18, 17.19.
- Romer, "Advanced Macro:" Exercises 4.8, 4.9, 4.10, 4.11.

1 Stochastic growth model

[Due to Oksana Leukhina] Consider a neoclassical growth model with uncertainty. Time is discrete: $t = 0, 1, 2, ...$

Demographics: There is a unit mass of identical households.

Preferences: $\mathbb{E}\sum_{t=0}^{\infty}\beta^t \ln c_t$.

Endowments: The household is endowed with one unit of labor time in each period and with initial capital stock k_0.

Technology: Output per capita is given by $\theta_t k_t^{\alpha}$ where k is the capital-labor ratio. The resource constraint is given by $k_{t+1} = \theta_t k_t^{\alpha} - c_t$. θ_t is an i.i.d. random variable that takes on the values θ^1 with probability π and θ^2 with probability $1-\pi$.

a. State the social planner's problem in recursive form.

b. Solve for the value function and the policy function. Start from the guess $V(k,\theta^i) = A + B\ln k + C\ln\theta^i$.

c. Define a Competitive Equilibrium, assuming: (i) θ_t is an aggregate shock; (ii) trading takes place in a sequence of markets; (iii) households trade a complete set of Arrow securities (an Arrow security pays one unit of the good if history $\theta^t = (\theta_1, ..., \theta_t)$ occurs).

1.1 Answer: Stochastic growth model

[I have not checked the details.]

a. The Bellman equation is formulated as

$$V(k,\theta_i) = \max_{c,k'\geq 0}\{\ln(c) + \beta\mathbb{E}V_1(\theta',k')\}$$
$$\text{s.t. } c + k' \leq \theta_i k^{\alpha},$$

where θ is a discrete i.i.d. r.v. that takes on two values θ_1 with probability π and θ_2 with probability $(1-\pi)$.

Find the value function using the method of undertermined coefficients. Use $V(k,\theta_i) = A + B\ln k + C\ln\theta_i$ as a guess.

The Bellman equation is formulated as

$$V(k,\theta_i) = \max_{k'}\{\ln(\theta_i k^{\alpha} - k') + \beta\left[\pi V(k',\theta_h) + (1-\pi)V(k',\theta_h)\right]\}.$$

Plugging the guess into the functional equation gives

$$\begin{aligned} V(k,\theta_i) &= \max_{k'}\{\ln(\theta_i k^{\alpha} - k') + \beta[\pi(A + B\ln k' + C\ln\theta_h) + (1-\pi)(A + B\ln k' + C\ln\theta_l)]\} = \\ &= \max_{k'}\{\ln(\theta_i k^{\alpha} - k') + \beta B\ln k' + \beta A + \beta C[\pi\ln\theta_h + (1-\pi)\ln\theta_l]\}. \end{aligned}$$

Taking first-order conditions, which are necessary and sufficient, gives

$$\frac{-1}{(\theta_i k^\alpha - k')} + \frac{\beta B}{k'} = 0.$$

The optimal policy function is then

$$k' = \frac{\beta B}{1+\beta B}\theta_i k^\alpha,$$

i.e., a constant share of output is saved in each period.

We have

$$\begin{aligned}
A + B\ln k + C\ln\theta_i &= \ln\left(\frac{\theta_i k^\alpha}{1+\beta B}\right) + \beta B \ln\left(\frac{\beta B}{1+\beta B}\theta_i k^\alpha\right) + \beta A + \beta C[\pi \ln\theta_h + (1-\pi)\ln\theta_l] \\
A + B\ln k + C\ln\theta_i &= \ln\left(\frac{\theta_i k^\alpha}{1+\beta B}\right) + \beta B \ln\left(\frac{\beta B}{1+\beta B}\theta_i k^\alpha\right) + \beta A + \beta C[\pi \ln\theta_h + (1-\pi)\ln\theta_l] \\
A + B\ln k + C\ln\theta_i &= \alpha\,(1+\beta B)\ln k + (1+\beta B)\ln\theta_i - (1+\beta B)\ln(1+\beta B) + \beta B\ln\beta B + \beta A + \beta C[\pi\ln\theta_h + (1-\pi)\ln
\end{aligned}$$

We are now able to determine the coefficients:

$$\begin{aligned}
B &= \alpha\,(1+\beta B)\,,\ \text{i.e. } B = \frac{\alpha}{(1-\alpha\beta)} \\
C &= (1+\beta B) \\
A &= ((1+\beta B)\ln\theta_i - (1+\beta B)\ln(1+\beta B) + \beta B\ln\beta B + \beta C[\pi\ln\theta_h + (1-\pi)\ln\theta_l])\,/\,(1-\beta)
\end{aligned}$$