

# The Solow Diagram

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# Analyzing the Solow Model

What are the properties of the Solow model?

- ▶ Why do economies grow over time?
- ▶ Does the economy settle down in the long-run?
- ▶ What are the long-run and short-run effects of changes in behavior?

To answer that:

1. Study the steady state (where everything is constant over time).
2. Plot the law of motion for  $k$ .

# The steady state

## Definition

A **steady state** is a situation where all variables are constant over time (in per capita terms).

In the Solow model:

- ▶ Capital per worker is constant:  $\dot{k} = 0$ .

Law of motion:

$$\dot{k}(t) = s \underbrace{k(t)^\alpha A^{1-\alpha}}_{f(k)} - (n + \delta) k(t)$$

The steady state capital stock solves:

$$sf(k^*) = (n + \delta)k^* \quad (1)$$

Intuition?

# The Steady State

With the Cobb-Douglas production function

$$sA^{1-\alpha}k^\alpha = (n + \delta)k \quad (2)$$

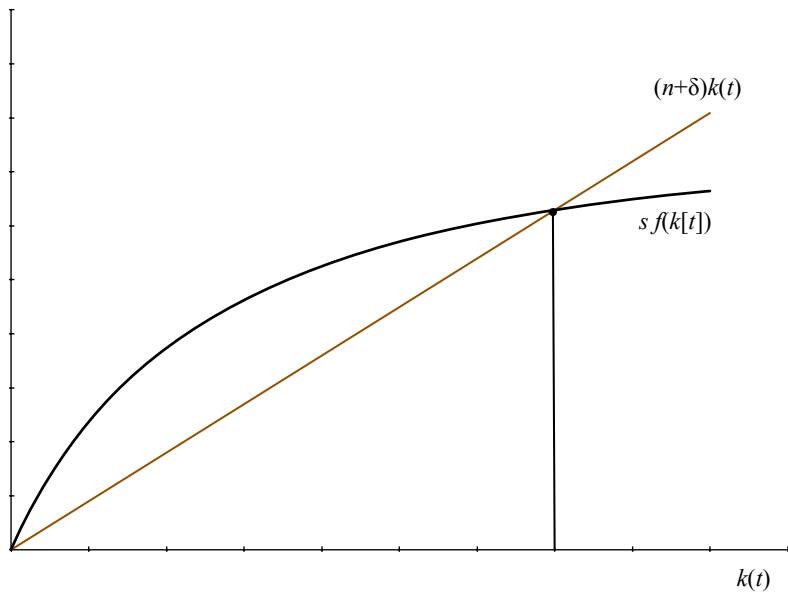
or

$$k^{1-\alpha} = \frac{sA^{1-\alpha}}{n + \delta} \quad (3)$$

Steady state **output** per worker

$$y = A^{1-\alpha}k^\alpha = A \left( \frac{s}{n + \delta} \right)^{\alpha/(1-\alpha)}$$

## Steady state graph

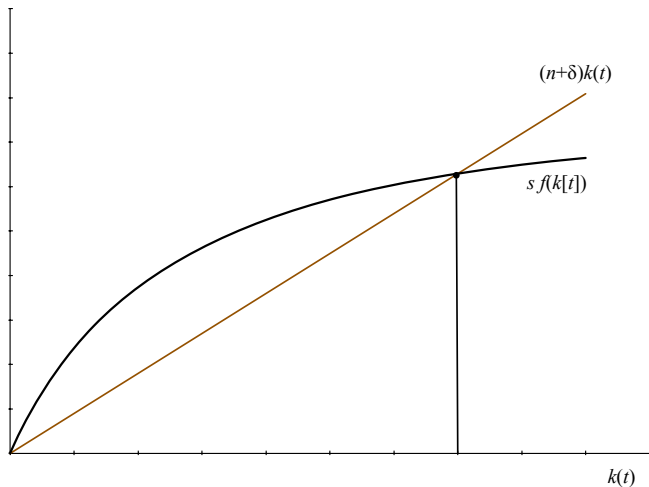


# Properties of the Steady State

1. Unique
2. Higher saving or productivity increase  $k$  and  $y$
3. Higher depreciation or population growth reduce  $k$  and  $y$

How big these effects are is governed by  $\alpha$  (again).

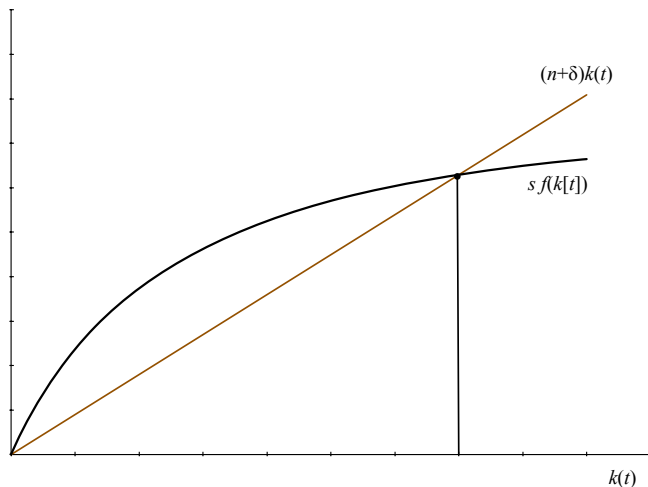
# Dynamics



What can we say about the dynamics?

# Comparative statics (or dynamics)

What happens if households save more?



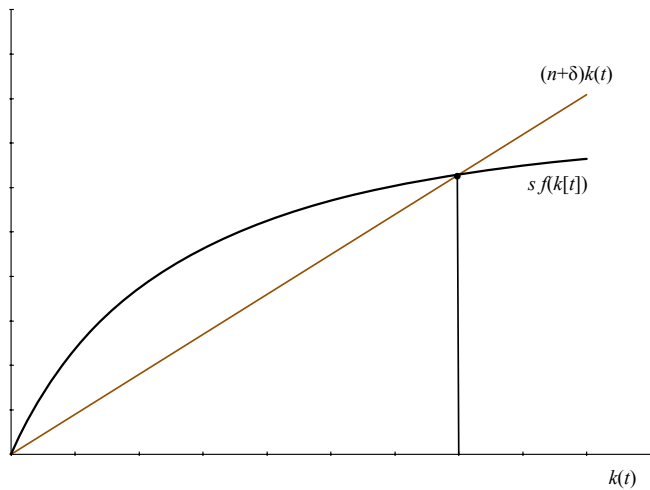
Plot the time paths of output and interest rates.



# Reality Check

- ▶ The model says: more investment (or **lower consumption**) generates a period of **faster** growth.
- ▶ Isn't everybody saying: the U.S. is in a recession (slow growth) because consumption is too low?
- ▶ How does the contradiction get resolved?
- ▶ Where is the effect of lower consumption demand in the Solow model?
- ▶ Where is the demand side anyway?

What happens if there is a baby boom?

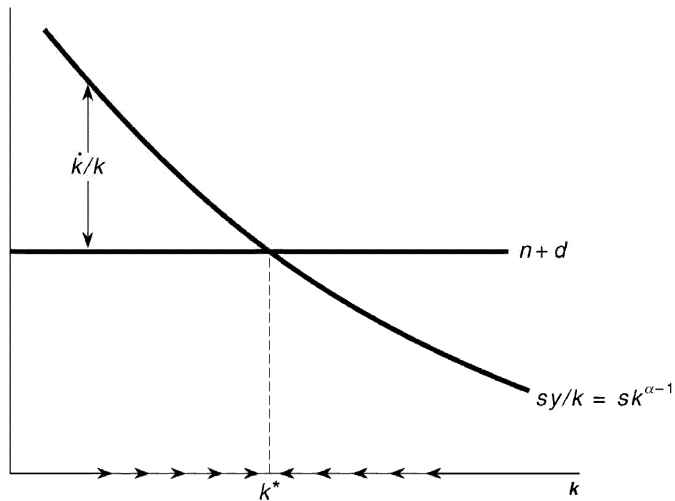


# Economic Growth

- ▶ Why do countries grow?
- ▶ In the Solow model:
  - ▶ Growth can only occur along a **transition path**.
  - ▶ There is **no long-run** growth in GDP per worker ( $y = Y/L$ ).
- ▶ But growth slows as the economy approaches the steady state.
- ▶ To see this, write the law of motion for  $k$  as

$$\dot{k}/k = g(k) = sy/k - (n + \delta)$$

# Economic Growth



# The Principle of Transition Dynamics

## Fact

*In the Solow model, the farther away the economy is from its steady state, the faster it grows (or shrinks)*

*What is the intuition?*

# Why does investment not sustain growth?

- ▶ The problem is the diminishing  $MP_K$ .
- ▶ Giving up one unit of  $C$  today yields  $MP_{K'} - \delta$  in additional output tomorrow.
- ▶ As  $k$  grows,  $MP_K$  eventually falls below  $\delta$ :
  - ▶ Additional investment no longer even pays for its own depreciation.
- ▶ Sustained growth through capital accumulation requires that  $MP_K$  stays above  $\delta$ , even as  $k$  grows without bounds.

Technical Change

# Technical change

- ▶ To sustain long-run growth of  $y$  the Solow model requires **technical change**.
  - ▶ Technical change is modeled as shifting the production function up.
  - ▶ Productivity grows:  $g(A) > 0$ .
- ▶ Later, we treat  $A$  as the product of innovation.
- ▶ Here:  $A$  is exogenous.
- ▶ Assume that technical change is **labor augmenting**:  
 $Y = F[K, AL]$ .
  - ▶ Otherwise, the model is not consistent with the data
  - ▶ “Kaldor facts” (not obvious, but true).



## Steady state?

Law of motion (unchanged):

$$\dot{k}(t) = sA(t)^{1-\alpha} k(t)^\alpha - (n + \delta) k(t) \quad (4)$$

But now  $A$  grows over time:

$$A(t) = A(0) e^{\gamma t} \quad (5)$$

Can we have a steady state?

It would imply

$$sA(t)^{1-\alpha} k^\alpha = (n + \delta) k \quad (6)$$

That can only work with constant  $A$ .

Growing  $A$  implies that  $k$  will grow forever.

# Balanced growth path

We don't have a steady state, so we look for the next best thing.

## Definition

A balanced growth path is an equilibrium where all variables grow at rates that are constant over time.

## What are the balanced growth rates?

Write the law of motion as

$$g(k) = sy/k - (n + \delta)$$

Constant  $g(k)$  requires constant  $y/k$ .

But

$$y/k = sA(t)^{1-\alpha} k(t)^{\alpha-1} \quad (7)$$

So we need constant  $\bar{k} = k/A$ .

Therefore: On the balanced growth path,

$$g(k) = g(y) = g(A) = \gamma \quad (8)$$

## Law of motion

To analyze the dynamics: construct variables that are constant on the BGP

►  $\bar{k} = k/A, \bar{y} = y/A$

We derive a law of motion for  $\bar{k}$ .

By the growth rate rule:

$$\begin{aligned}g(\bar{k}) &= g(k) - g(A) \\ &= sy/k - (n + \delta) - g(A)\end{aligned}$$

Note that  $y/k = \bar{y}/\bar{k}$ .

Law of motion:

$$d\bar{k}/dt = s\bar{y} - (n + \delta + g(A))\bar{k} \quad (9)$$

with

$$\bar{y} = y/A = \frac{k^\alpha A^{1-\alpha}}{A} = \bar{k}^\alpha \quad (10)$$

## What has changed?

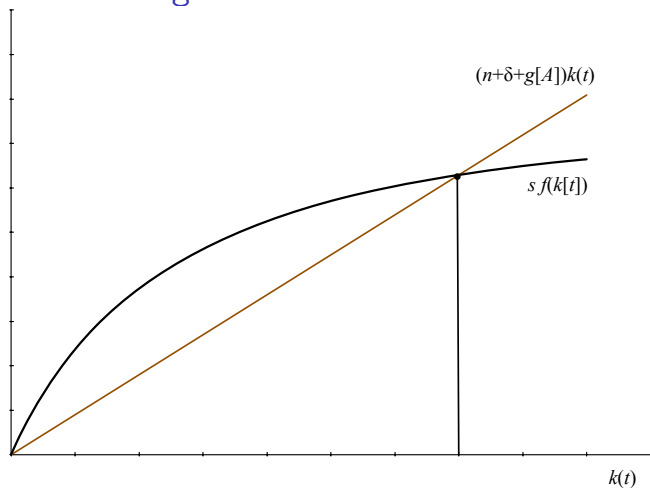
The model with technical change looks exactly like the previous model, except:

1. All variables are "detrended" (divided) by  $AL$ .
2. The steady state has per capita variables growing at rate  $g(A)$ .
3. The law of motion contains an additional  $g(A)$  term.

The model has a steady state in the "detrended" variables  $(\bar{k}, \bar{y})$ .

It has a balanced growth path in per capita variables  $(k, y)$ .

## The Solow diagram

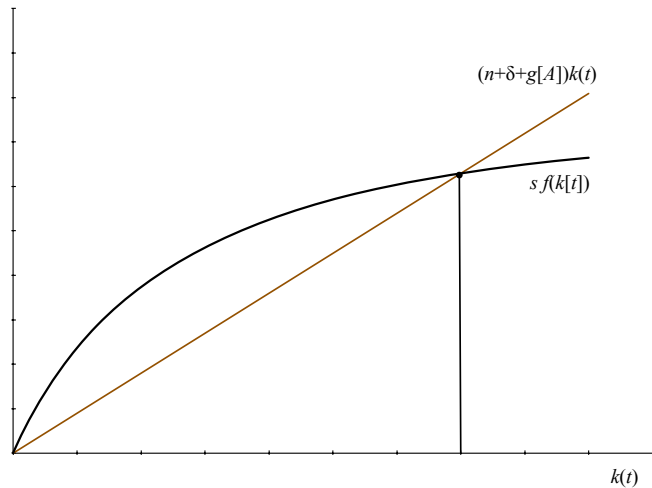


This is essentially the same diagram as without technical change, except:

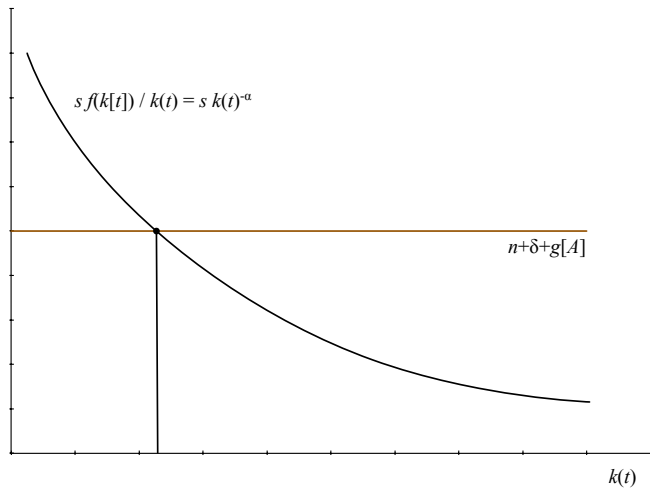
- ▶ variables are detrended.
- ▶ an additional  $g$  term appears in the straight line.

## Comparative statics: higher saving rate

The Solow diagram is familiar:



# Transitional dynamics

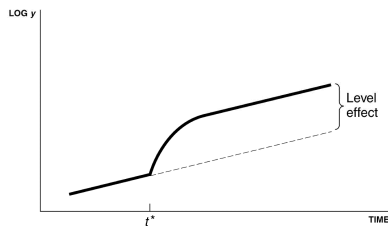




# Policies have level effects

A key implication of the Solow model: Policies, such as taxes, do not affect the long-run growth rate.

The growth rate rises on the transition to the new steady state, then levels off to  $g(A)$ .



# Important Points

- ▶ The Solow model reveals how choices (saving, fertility) affect capital and output (levels and growth).
- ▶ Capital cannot sustain long-run growth.
  - ▶ the reason: diminishing returns
- ▶ Therefore policies have level effects.
- ▶ In the short run: countries grow fast when they are far below their steady states.
- ▶ In the long run: growth is determined by productivity improvements.

## Final Example

Modify the Solow model by assuming that the production function is given by

$$Y_t = AK_t^\alpha L_t^{1-\alpha} - L_t X$$

where  $X > 0$  is a constant.

1. How does the Solow diagram change?
2. How many steady states are there?
3. Which ones are stable?

# Reading

- ▶ Jones (2013), ch. 2
- ▶ Blanchard and Johnson (2013), ch. 11

## Further Reading:

- ▶ Romer (2011), ch. 1
- ▶ Barro and Martin (1995), ch. 1.2

## References I

- Barro, R. and S.-i. Martin (1995): "X., 1995. Economic growth," *Boston, MA*.
- Blanchard, O. and D. Johnson (2013): *Macroeconomics*, Boston: Pearson, 6th ed.
- Jones, Charles; Vollrath, D. (2013): *Introduction To Economic Growth*, W W Norton, 3rd ed.
- Romer, D. (2011): *Advanced macroeconomics*, McGraw-Hill/Irwin.