# Long-run Growth:
# The Solow Model

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# Topics

We write down a basic, but quite general growth model.

The idea: growth is driven by "capital" accumulation.

- but "capital" does not have to be physical capital (machines, structures)
- it could human capital, knowledge capital
- this is why the model is quite general

# Topics

The **Solow model** answers questions such as:

1. How much of cross-country income gaps is due to differences in saving rates?
2. Does **capital** accumulation drive long-run growth?
3. Do country incomes **converge** over time?
4. What happens to growth when production uses **finite resources**?

# Objectives

At the end of this section you should be able to

1. Derive properties of the Solow model: steady state, effects of shocks, ...
2. Graph the dynamics of the Solow model.

Note: The Solow model is old (1950s). But it's ideas are durable.

- The basic model structure applies to any factor that is accumulated
- E.g., human capital, knowledge (Romer model)

# 2. Long-run Growth: Evidence

## Fact

*Long-run growth rates vary across countries.*
*Poor countries do not grow faster than rich countries.*
*No convergence*

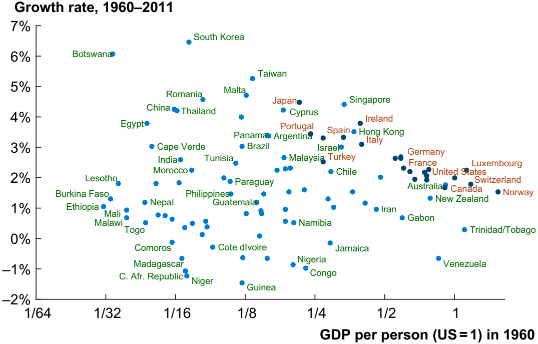



**Fig. 26** The lack of convergence worldwide. Source: *The Penn World Tables 8.0.*

Source: Jones (2016)

# Kaldor Facts

What should a model of growth look like?

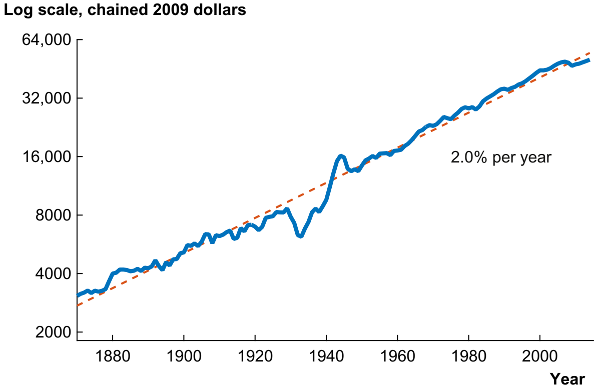
The U.S. growth experience looks a lot like a "**balanced growth path**"

- GDP growth has been essentially constant at 2% per year
- Constant capital output ratio $K/Y$
- Interest rates have no trend
- The share of labor income in GDP has been constant (2/3)

This is why economists like to write down models with **balanced growth**.

- all growth rates are constant over time.

# Constant US Growth

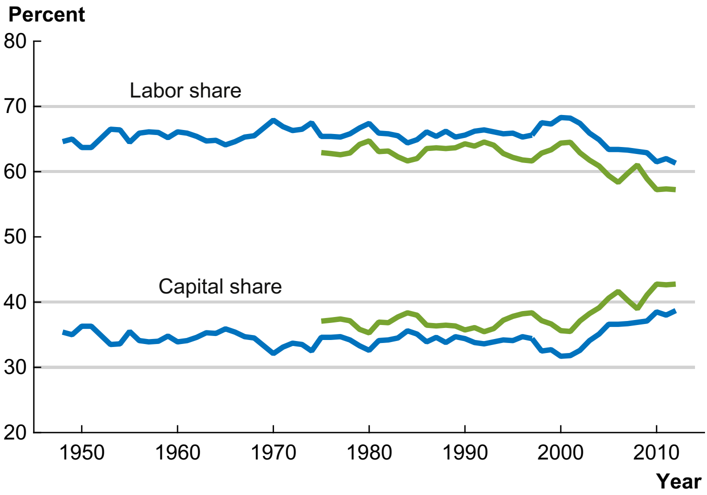



Source: Jones (2016)

US growth has been essentially constant for 140 years.

What does this tell us about determinants of long-run growth?

# Constant Labor Share




But labor share has been falling recently.

# 3. The Solow Model: Structure

# Model Elements

The world goes on forever.

Time is indexed by the **continuous** variable $t$.

One good $(Y)$ is produced from two inputs (capital $K$ and labor $L$).

Productivity $(A)$ grows exogenously

- at a constant rate
- later, we will study where productivity growth comes from

Households save a constant fraction of income.

# Is the Model Too Simple?

The model makes strong assumptions:

- only 2 factor inputs (capital and labor)
- an aggregate production function (no firm detail)
- constant saving rate

Why should we take this seriously?

# What Makes a Good Model?

A good model starts out as simple as possible.

- A model tells a story in math.
- Simplicity is good (to start with).

But need to check robustness.

# The Solow Story

Economic growth is driven by

- physical capital accumulation (investment)
- productivity growth

The key insight:

- investment alone cannot drive growth
- due to diminishing marginal product of capital
- but investment is important for output levels

Romer model:

- investment can drive growth if it is not subject to diminishing returns
- knowledge accumulation (R&D)

# 3.1 Production Structure

Aggregate production function:

$$Y(t) = F[K(t), L(t), A(t)] \tag{1}$$

There is **one output** good $Y \rightarrow$ GDP

There are **two inputs**:

1. Capital $K$: machines, equipment, structures
2. Labor $L$: hours worked, education, ...

We can extend the analysis to many capital and labor inputs

- the basic message does not change.

# Cobb Douglas

The functional form is **Cobb-Douglas**:

$$Y(t) = K(t)^{\alpha} \; [A(t) \; L(t)]^{1-\alpha} \tag{2}$$

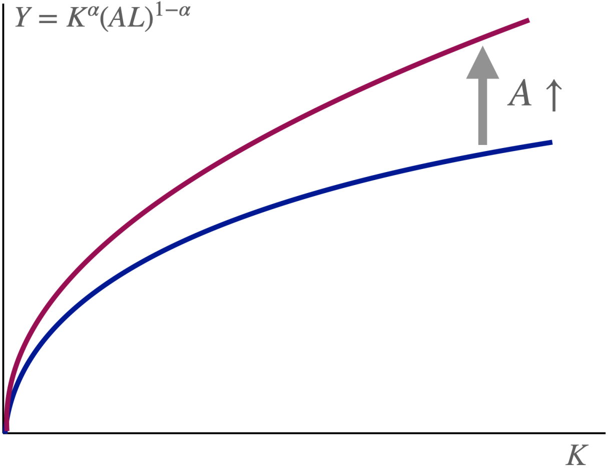
The Cobb-Douglas has properties that fit the data:

- the labor share (labor income / GDP) is constant over time
- the elasticity of substitution between capital and labor is close to 1

$\alpha$ is a parameter between $0$ and $1$.

- we see later: $\alpha$ is the capital income share

# Cobb Douglas




The graph holds $AL$ constant.

What does higher $\alpha$ do?

# Labor Input

$L$ grows over time at constant rate $n$:

$$L(t) = L(0)\, e^{nt}$$

For simplicity ...

Population growth doesn't matter much in the Solow world

- but it does matter when growth is driven by innovation

# Productivity Growth

$A(t)$ is an index of the state of "technology"

- anything that makes people more productive over time

$A$ grows over time for reasons that are not modeled

- a major shortcoming of the model
- the Romer model is all about $A$ growth

The growth rate of $A$ is constant $\gamma$:

$$A(t) = e^{\gamma t} \tag{3}$$

# 3.2 Digression: Growth rates

**Rate of change** = change per time period = $\frac{A(t+\Delta t)-A(t)}{\Delta t}$

- for some time interval $\Delta t$ (such as one year)
- e.g. \$1b per year

**Growth rate** = rate of change / level

$$g(A) = \frac{A(t+\Delta t) - A(t)}{\Delta t} \times \frac{1}{A(t)} \tag{4}$$

- e.g. 3 percent per year

# Growth rates

We are in continuous time, so $\Delta t \to 0$

Then

$$\frac{A(t+\Delta t) - A(t)}{\Delta t} \to ? \tag{5}$$

Growth rate

$$g(A) = \frac{dA/dt}{A} \tag{6}$$

# Constant growth

Constant growth at rate $\gamma$ implies

$$A(t) = e^{\gamma t} \tag{7}$$

To check that:

- rate of change

$$dA/dt = \gamma \times e^{\gamma t} = \gamma A(t) \tag{8}$$

- growth rate

$$g(A) = \frac{dA/dt}{A} = \gamma \tag{9}$$

# 3.3 Output per worker

We are interested in the growth rate of **output per worker** $y$

$$
\begin{aligned}
y_t = \frac{Y_t}{L_t} &= K_t^{\alpha} A_t^{1-\alpha} \frac{L_t^{1-\alpha}}{L_t} \\
&= A_t A_t^{-\alpha} K_t^{\alpha} L_t^{-\alpha} \\
&= A_t \left( \frac{K_t}{A_t L_t} \right)^{\alpha}
\end{aligned}
$$

Recall: $x^{\alpha} \times x^{\beta} = x^{\alpha+\beta}$

Intuition ... why does $\bar{k}_t = \left( \frac{K_t}{A_t L_t} \right)$ matter for output per worker?

# What is $\bar{k}$?

Basic intuition:

- $AL$: labor input in "efficiency units"
- $AL$ is what grows exogenously
- When $K$ grows faster than $AL$, $\bar{k}$ rises

Why does it matter?

We will see that

- long-run growth rate of $K$ equals $g(AL)$
- when $\bar{k}$ rises, the economy runs into diminishing returns
- that slows down $K$ growth

# Output growth

Output level:

$$y_t = A_t \left( \frac{K_t}{A_t L_t} \right)^{\alpha} = A_t \bar{k}_t^{\alpha}$$

What is the growth rate of $y$?

Growth rate rule: $g(x \times y) = g(x) + g(y)$
Therefore: $g(y) = g(A) + g\left(\bar{k}^{\alpha}\right)$

# Output growth

Growth rate rule: $g\left(x^{\alpha}\right) = \alpha g\left(x\right)$

Therefore:

$$\begin{aligned} g\left(\bar{k}^{\alpha}\right) &= \alpha g\left(\bar{k}\right) \\ &= \alpha \left[g\left(K/L\right) - g\left(A\right)\right] \end{aligned}$$

# Output growth

Combine:

- $g(y) = g(A) + g\left(\bar{k}^{\alpha}\right)$
- $g\left(\bar{k}^{\alpha}\right) = \alpha\left[g(K/L) - g(A)\right]$

$$g(y) = (1-\alpha)\,g(A) + \alpha g(K/L)$$

Two reasons why output per worker grows:

1. Exogenous technical change $g(A)$
   Production function shifts up
2. Capital deepening ($K/L$ grows)
   Move along production function

## 3.4 Capital Accumulation

We need $g(K/L)$

Output is divided between consumption and gross investment:

$$Y(t) = C(t) + I(t) \tag{10}$$

Investment contributes to the capital stock:

$$dK(t) = [I(t) - \delta K(t)]\, dt \tag{11}$$

$\delta$ is the rate of depreciation.

Growth rate of capital:

$$g(K) = \frac{dK/dt}{K} = I/K - \delta$$

In words: $K$ grows when investment outpaces depreciation.

# Capital Accumulation

We assume that people save a fixed fraction of income:

$$C(t) = (1 - s)\,Y(t) \tag{12}$$

Equivalently:

$$I(t) = sY(t) \tag{13}$$

Then

$$\frac{I}{K} = s\frac{Y}{K}$$

# Capital Accumulation

$$\frac{Y}{K} = \frac{y}{K/L} = \frac{A_t}{K/L}\left(\frac{K_t}{A_tL_t}\right)^{\alpha} = \bar{k}^{\alpha-1}$$

$$g(K) = s\bar{k}^{\alpha-1} - \delta$$

$$g(k) = g(K) - g(L) = s\bar{k}^{\alpha-1} - (n+\delta)$$

# Model Summary

We have 3 equations that determine $Y, K, I$ over time.

1. Cobb-Douglas production function

$$Y(t) = K(t)^{\alpha} \; [A(t) \; L(t)]^{1-\alpha} \tag{14}$$

2. Law of motion for capital:

$$g(K) = I(t)/K - \delta \tag{15}$$

3. Constant saving rate: $I(t) = s \; Y(t)$.

Exogenous driving forces:

1. Constant population growth: $L(t) = L(0) \; e^{nt}$.
2. Constant productivity growth: $A(t) = A(0) \; e^{\gamma t}$.

# Model Summary

The key equation that tells us how $k$ grows over time:

$$g(k) = s\bar{k}^{\alpha-1} - (n+\delta)$$

where $\bar{k} = \frac{K}{AL}$

Once we have $k(t)$ we can get the rest:

- $y = A\bar{k}^{\alpha}$
- $g(y) = (1-\alpha)g(A) + \alpha g(K/L)$

# Model Comments

What have we assumed and why?

1. **Cobb-Douglas** production function
   A harmless simplification
   All we need is diminishing marginal product of $K$
2. $g(K) = I/K - \delta$
   This is just accounting
3. **Constant saving rate**
   We currently don't care why people choose some value of $s$.
   We are looking at long-run growth. Constant $s$ makes sense.
4. **Only $K$ is accumulated**
   No innovation or human capital accumulation.
   But it does not matter exactly what $K$ is, as long as we have diminishing $MPK$

So the model is really quite general.

# Solving the Model

Even this simple model cannot be "solved" in closed form for $k(t)$

- That is, we cannot write the endogenous variables as functions of time.

What we can do is

1. graph the model and trace out qualitatively what happens over time.
2. solve the model for the long-run values of the endogenous variables (e.g. $K(t)$ as $t \to \infty$).

# The Solow Diagram

We develop a graph that summarizes how $k$ grows over time.

We simply plot

$$g(k) = s\bar{k}^{\alpha-1} - (n+\delta)$$

The beauty of it all:

The **same analysis** applies to any model where some form of "capital" accumulation drives growth.

- Later we will see: a model where growth is due to R&D produces exactly the same graph
  (but with an important wrinkle that changes everything)

# Summary

The Solow model captures how economic growth is driven by

- capital accumulation
  - where capital could be more than just machines
  - human capital, knowledge, ...
- productivity growth

The model boils down to one equation:

$$g(k) = s\bar{k}^{\alpha-1} - (n+\delta)$$

Given a value for $k$ at the start, this equation traces out the entire time path of $k(t)$.

# Reading

- Jones / Vollrath, Introduction to Economic Growth, 3rd or 4th ed., ch. 2
- Blanchard (2018), ch. 11

Further Reading:

- Romer (2011), ch. 1
- Barro and Sala-i Martin (1995), ch. 1.2

# References I


Barro, R. and X. Sala-i Martin (1995): "Economic growth," *Boston, MA*.

Blanchard, O. (2018): *Macroeconomics*, Boston: Pearson, 8th ed.

Jones, C. I. (2016): "The Facts of Economic Growth," in *Handbook of Macroeconomics*, ed. by J. B. Taylor and H. Uhlig, Elsevier, vol. 2, chap. 1, 3–69.

Romer, D. (2011): *Advanced macroeconomics*, McGraw-Hill/Irwin.