

# The Growth Model: Discrete Time Competitive Equilibrium

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# Competitive Equilibrium

We show that the CE allocation coincides with the planner's solution.

Model setup:

- ▶ Preferences, endowments, and technology are the same as before.
- ▶ Markets: goods, capital rental, labor rental

# Households

A single representative household owns the capital and rents it to firms at rental rate  $q$ .

It supplies one unit of labor to the firm at wage rate  $w$ .

Preferences are

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

The budget constraint is:

$$k_{t+1} = (1 - \delta)k_t + w_t + q_t k_t - c_t$$

# Households: DP Representation

State variable:  $k$ .

Control:  $k'$ .

Bellman equation:

FOC

Envelope:

Euler equation:

$$u'(c) = \beta(1 + q' - \delta)u'(c')$$

# Household: Solution

A pair of policy functions  $c = \phi(k)$  and  $k' = h(k)$  and a value function such that:

1. the policy functions solve the “max” part of the Bellman equation, given  $V$ ;
2. the value function satisfies

In terms of **sequences**:  $\{c_t, k_{t+1}\}$  that solve the Euler equation and the budget constraint.

The boundary conditions are  $k_0$  given and the transversality condition (TVC)

$$\lim_{t \rightarrow \infty} \beta^t u'(c_t) k_t = 0$$

# Firms

- ▶ Firms rent capital and labor services from households, taking prices  $(q, w)$  as given.
- ▶ They maximize current period profits:

$$\max F(K, L) - wL - qK$$

- ▶ FOC

$$F_K(K, L) = q$$

$$F_L(K, L) = w$$

# Firms

- ▶ Assume constant returns to scale. Define

$$F(k^F)L = F(K,L)$$

- ▶ FOC's become

$$\begin{aligned}f'(k^F) &= q \\ f(k^F) - f'(k^F)k^F &= w\end{aligned}$$

- ▶ A **solution** is a pair  $(K,L)$  that satisfies the 2 FOC.

# Equilibrium

An equilibrium is a sequence of  
that satisfy:



## Comparison with the Planner's Solution

One way of showing that the Planner's solution coincides with the CE is to appeal to the First and Second Welfare Theorems.

A more direct way is to show that the equations that characterize CE and the planner's solution are the same.

CE	Planner
$u'(c) = (1 + q' - \delta) \beta u'(c')$	$u'(c) = (f'(k') + 1 - \delta) \beta u'(c')$
$k' + c = f(k) + (1 - \delta) k$	$k' + c = f(k) + (1 - \delta) k$
$k' = (1 - \delta) k + w + qk - c$	
$q = f'(k)$	
$w = f(k) - f'(k)k$	

# Recursive Competitive Equilibrium

# Recursive competitive equilibrium

Recursive CE is an alternative way of representing a CE that is more fully consistent with the DP approach.

- ▶ Everything is written as functions of the state variables.
- ▶ There are no sequences.

This is useful especially in models with

- ▶ heterogeneous agents where the distribution of households is a state variable;
- ▶ uncertainty, where we cannot assume that agents take future prices as given.

# Recursive competitive equilibrium

## Key feature of RCE

Everything in the economy is a function of the aggregate state  $S$ .

Agents form expectations using the **law of motion** for  $S$ :  $S' = G(S)$

- ▶ E.g., to form expectations over future interest rate, use the law of motion for  $k$  and the price function  $q = f'(k)$ .

A fixed point problem:

- ▶ Agents' policy functions depends on the laws of motion.
- ▶ The laws of motion depend on agents' policy functions.

## RCE in the example

The economy's *state variable* is aggregate  $K$ .

- ▶ Call its law of motion  $K' = G(K)$ .
- ▶ This is part of the equilibrium.

We solve the household problem for a saving function  $k' = h(k, K)$ .

- ▶ It depends on the private state  $k$  and the aggregate state  $K$ .

We solve the firm's problem for price functions  $q(K), w(K)$ .

# Household

The household solves

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$k_{t+1} = (1 - \delta)k_t + w(K_t) + q(K_t)k_t - c_t$$

The household's problem has an individual state  $k$  and an aggregate state  $K$ .

# Household

Bellman's equation is

$$\begin{aligned} V(k, K) &= \max u([1 - \delta]k + w(K) + q(K)k - k') + \beta V(k', K') \\ K' &= G(K) \end{aligned}$$

Solution:  $k' = h(k, K)$ .

# Firm

Nothing changes in the firm's problem.

Solution:

$$q(K) = f'(K)$$

$$w(K) = f(K) - f'(K)K$$



# RCE

Objects:

- ▶ household: a policy function  $k' = h(k, K)$  and a value function  $V(k, K)$ .
- ▶ firm: price functions  $q(K), w(K)$ ,
- ▶ law of motion for the aggregate state:  $K' = G(K)$ ,

Equilibrium conditions:

- ▶ household: Given  $G(K), q(K), w(K)$ : the policy function solves the household's DP.
- ▶ firm: The price functions satisfy firm FOCs.
- ▶ Markets clear (same as before, except for notation).
- ▶ Household expectations are consistent with household behavior:

$$h(K, K) = G(K)$$

# Consistency

$$h(K, K) = G(K)$$

Basic idea: expectations (governed by  $G$ ) are consistent with actions.

In equilibrium, the household holds  $k = K$  and chooses  $k' = h(K, K)$ .

He expects  $K' = G(K)$ .

Correct expectations requires  $k' = K'$

# Recursive CE: Example

## Households

There are  $N_j$  households of type  $j$ .

The representative type  $j$  household solves:

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \\ \text{s.t.} \quad & k_{t+1} = R_t k_t + w_t l_t - c_t \end{aligned}$$

# Aggregate State

The aggregate state vector is the distribution of wealth:

$$\kappa = (\kappa_1, \dots, \kappa_N) \quad (1)$$

$\kappa_j$  is wealth of household  $j$  in equilibrium.

The household knows the law of motion

$$\kappa' = G(\kappa) \quad (2)$$

with  $j$ th element

$$\kappa'_j = G_j(\kappa) \quad (3)$$

Why not just  $S = K$ ?

# Household Dynamic Program

$$\begin{aligned} V_j(k_j, \kappa) &= \max u(c_j, l_j) + \beta V_j(k'_j, G(\kappa)) \\ k'_j &= R(\kappa)k_j + w(\kappa)l_j - c_j \end{aligned}$$

First-order conditions:

$$u_c(c_j, l_j) = \beta V_{j,1}(k'_j, G(\kappa)) \quad (4)$$

$$u_l(c_j, l_j) = \beta V_{j,1}(k'_j, G(\kappa)) w(\kappa) \quad (5)$$

Envelope:

$$V_{j,1}(k_j, \kappa) = u_c(c_j, l_j) R(\kappa) \quad (6)$$

## Household solution

A solution to the type  $j$  household problem consists of

- ▶ a value function  $V_j$
- ▶ policy functions  $k'_j = h_j(k_j, \kappa)$ ,  $l_j = \ell_j(k_j, \kappa)$ , and  $c_j = g_j(k_j, \kappa)$ .

These satisfy:

1.  $V_j$  is a fixed point of the Bellman equation, given  $h, \ell$  and  $g$ .
2.  $h, \ell$  and  $g$  "max" the Bellman equation.

Implicit: the household takes  $S' = G(S)$  as given.

# Firm

This is standard:

$$\max_{K,L} F(K,L) - w(\kappa)L - q(\kappa)K$$

FOC: Factor prices equal marginal products.

Solution:  $K(\kappa)$  and  $L(\kappa)$ .

# Market clearing

Goods:

$$F(K(\kappa), L(\kappa)) + (1 - \delta)K(\kappa) = \sum_j N_j [g_j(\kappa_j, \kappa) + h_j(\kappa_j, \kappa)] \quad (7)$$

Labor:

$$L(\kappa) = \sum_j N_j \ell(\kappa_j, \kappa) \quad (8)$$

Capital:

$$K(\kappa) = \sum_j N_j \kappa_j \quad (9)$$

## Note

Everything is either exogenous or a function of the state variables.



# Recursive CE

Objects:

- ▶ household: functions  $V_j, h_j, \ell_j, g_j$
- ▶ firm:  $K(\kappa), L(\kappa)$
- ▶ price functions:  $w(\kappa), q(\kappa), R(\kappa)$
- ▶ law of motion:  $G$ .

These satisfy:

1. Household solution (4)
2. Firm first order conditions (2)
3. Market clearing (3 - 1 redundant)
4. Identity:  $R(\kappa) = q(\kappa) + 1 - \delta$ .
5. Consistency:

$$\kappa'_j = G_j(\kappa) = h_j(\kappa_j, \kappa) \quad \forall j \quad (10)$$

# Notes on RCE

All the objects to be found are functions, not sequences.

This helps when there are shocks:

- ▶ We cannot find the sequence of prices without knowing the realizations of the shocks.
- ▶ But we can find how prices evolve for each possible sequence of shocks.
- ▶ The price functions describe this together with the laws of motion for the states.

# Notes on RCE

**Functional analysis** helps determine the properties of the policy functions and the laws of motion.

- ▶ E.g., we strictly concave utility we know that savings are increasing in  $k$ , continuous, differentiable, etc.

RCE helps compute equilibria.

- ▶ Find the household's optimal choices for every possible set of states.
- ▶ Then simulate household histories to find the laws of motion.

## Example: Firms accumulate capital

The physical environment is unchanged.

Markets are now:

1. goods (numeraire)
2. labor rental ( $w$ )
3. shares of firms ( $q$ )  
supply of shares = 1
4. bonds ( $R$ )

Aggregate state:  $K$  with law of motion  $K' = G(K)$

# Household

Budget constraint:

$$c + q(K)a' + b' = w(K) + [q(K) + \pi(K)]a + R(K)b \quad (11)$$

The household also gets a share of the profits  $\pi$ .

$$V(a, b, K) = \max_{c, a', b'} u(c) + \beta V(a', b', G(K)) \quad (12)$$

subject to the budget constraint.

Decision rule  $a' = g(a, b, K)$  and  $b' = h(a, b, K)$

## Firm

Period profits:

$$\pi = F(k, l) + (1 - \delta)k - w(K)l - k' \quad (13)$$

Firms maximize the discounted present value of profits

$$W_0 = \max_{\{k_{t+1}, l_t\}} \sum_{t=0}^{\infty} \frac{\pi_t}{R_1 \times R_2 \times \dots \times R_t} \quad (14)$$

Bellman equation:

$$W(k, K) = \max_{k', l} \pi(k, l, k', K) + \frac{W(k', G(K))}{R(G(K))} \quad (15)$$

Decision rules:  $k'(k, K), l(k, K)$

# Recursive Competitive Equilibrium

Objects:

1. Household:  $V(a, b, K)$ ,  $g(a, b, K)$ ,  $h(a, b, K)$
2. Firm:  $W(k, K)$ ,  $k'(k, K)$ ,  $l(k, K)$ ,  $\pi(k, k', K)$
3. Price functions  $w(K)$ ,  $R(K)$ ,  $q(K)$
4. Aggregate law of motion  $G(K)$

## Conditions:

1. Household optimization
2. Firm optimization
3. Market clearing
  - 3.1 bonds:  $h(1, 0, K) = 0$
  - 3.2 shares:  $g(1, 0, K) = 1$
  - 3.3 goods: RC
4. Consistency:
  - 4.1  $k'(K, K) = G(K)$
  - 4.2  $q(K) = W(K, K)$ : the share price is the present value of profits



Example: Heterogeneous Preferences

# Model

Demographics:

- ▶ There are  $j = 1, \dots, J$  types of households.
- ▶ The mass of type  $j$  households is  $\mu_j$ .

Preferences:

- ▶  $\max \sum_{t=0}^{\infty} \beta^t u_j(c_{jt})$ .
- ▶  $u_j$  is increasing and strictly concave and obeys Inada conditions.

# Model

Technology:  $F(K_t, L_t) + (1 - \delta)L_t = C_t + K_{t+1}$

Endowments:

- ▶ Each household is endowed with one unit of labor in each period.
- ▶ At  $t = 0$  household  $j$  is endowed with  $k_{j0}$  units of capital and with  $b_{j0} = 0$  units of one period bonds.

Market arrangements are standard.

# Household Problem

Nothing new here, except everything is indexed by  $j$ .

Define wealth as  $a_{jt} = k_{jt} + b_{jt}$ .

Impose no-arbitrage:  $R = q + 1 - \delta$

Bellman equation:

Euler Equation:

$$u'_j(c) = \beta R' u'_j(c') \quad (16)$$

Solution (sequence language):  $\{c_{jt}, a_{jt}\}$  that solve the Euler equation and budget constraint.

Boundary conditions:  $a_{j0}$  given and TVC  $\lim_{t \rightarrow \infty} \beta^t u'(c_{jt}) a_{jt} = 0$ .

# Competitive Equilibrium

A CE consists of sequences which satisfy:

- ▶ 2 household conditions
- ▶ 2 firm first-order conditions (standard)  
 $q_t = f'(k_t/n_t) + 1 - \delta$  and  $w_t = f(k_t/n_t) - f'(k_t/n_t)k_t/n_t$
- ▶ Market clearing:

We need to distinguish  $k_{jt}$  from  $k_t = K_t / \sum_j \mu_j$  in the equilibrium definition.

# Steady State

- ▶ Similar to CE without time subscripts.
- ▶ Euler equation becomes:

$$\beta R = 1$$

- ▶ Interesting: we can find  $R$  without knowing preferences or wealth distribution.

## Are there steady states with persistent inequality?

- ▶ Let's solve for steady state  $c_j$  as a function of prices and endowments  $(k_{j0}, b_{j0})$ .
- ▶ With constant prices, the household's present value budget constraint implies
- ▶ Endowing households with any  $k_{j0}$ 's that sum to the steady state  $k$  yields a steady state with persistent inequality.
- ▶ It would be harder to show that persistent inequality follows from *any* initial asset distribution which features capital inequality.

## Redistribution

How does the steady state allocation change when a unit of capital is taken from household  $j$  and given to household  $j'$ ?



## Lump-sum Taxes

Impose a lump-sum tax  $\tau$  on type  $j$  households. The revenues are given to type  $j'$  households.

How does the steady state change?

# Lump-sum Taxes

What if revenues are thrown into the ocean instead?

## Differences in $\beta$

- ▶ Now imagine households differ in their  $\beta$ 's, but not in their  $u$  functions.
- ▶ For simplicity, assume that  $u(c) = c^{1-\sigma}/(1-\sigma)$ .
- ▶ What would the asset distribution look like in the limit as  $t \rightarrow \infty$ ?

# Reading

- ▶ Acemoglu (2009), ch. 6. Also ch. 5 for background material we will discuss in detail later on.
- ▶ Krusell (2014), ch. 5 on Recursive Competitive Equilibrium.
- ▶ Ljungqvist and Sargent (2004), ch. 3 (Dynamic Programming), ch. 7 (Recursive CE).
- ▶ Stokey et al. (1989), ch. 1 is a nice introduction.

# References I

- Acemoglu, D. (2009): *Introduction to modern economic growth*, MIT Press.
- Krusell, P. (2014): "Real Macroeconomic Theory," Unpublished.
- Ljungqvist, L. and T. J. Sargent (2004): *Recursive macroeconomic theory*, 2nd ed.
- Stokey, N., R. Lucas, and E. C. Prescott (1989): "Recursive Methods in Economic Dynamics," .