Last week we dealt with Bessel's equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \tag{1}$$

and the functions that arise from it. It is a special case of

$$x^2(1 + r_m x^m)y'' + x(p_0 + p_m x^m)y' + (q_0 + q_m x^m)y = 0. \tag{2}$$

Modified Bessel functions.
An important related equation related to (1) is

$$x^2y'' + xy' - (x^2 + \nu^2)y = 0. \tag{3}$$

Its solutions are called **modified Bessel functions**. A canonical pair of solutions are given by $I_\nu(x)$ and $K_\nu(x)$,

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{(\frac{x}{2})^{2k+\nu}}{\Gamma(k+\nu+1)k!}, \tag{4}$$

$$K_\nu(x) = \frac{\pi}{2}\frac{(I_{-\nu}(x) - I_\nu(x))}{\sin(\nu\pi)},\ \nu \text{ not an integer.} \tag{5}$$

When ν is an integer, then $K_\nu(x)$ is defined in a limiting sense, and terms involving $\log(x)$ will arise. (Bessel Functions, Lozier, D., Olver, F., Clark, C. and Boisvert, R., eds., *Digital Library of Mathematical Functions*, NIST, 2005, http://dlmf.nist.gov/).
These functions do not have the oscillatory properties of the standard Bessel functions. Notice $(-1)^k$ is absent from the summations, in much the same way as hyperbolic functions differ from sines and cosines. Example 3.11 in the text shows one use in the series solution to a pde.

Hypergeometric Functions. The hypergeometric differential equation is defined as

$$x(1-x)y'' + (\gamma - (\alpha + \beta + 1)x)y' - \alpha\beta y = 0. \tag{6}$$

It too is of the form (2) with $m = 1$, $p_0 = \gamma,\ p_1 = \alpha + \beta + 1,\ q_0 = 0,\ q_1 = -\alpha\beta,\ r_1 = -1.$
Assume

$$y = \sum_{k=0}^{\infty} A_k x^{k+s},$$

find $s^2 + (\gamma - 1)\,s = 0 \Rightarrow s = 0, 1-\gamma$
and the series solution has the recurrence relation

$$A_k = A_{k-1}\frac{(\alpha + k - 1)\,(\beta + k - 1)}{(\gamma + k - 1)\,k},\ k \geq 1$$

$$\left|\frac{A_{k+1}}{A_k}\right| = \left|\frac{(\alpha+k)\,(\beta+k)}{(\gamma+k)\,(k+1)}\right|, \quad k \geq 0$$
$$\to 1 \quad \text{as } k \to \infty.$$

Thus the series converges for $|x| < 1$.
It may happen that the terms of the series are complex, the $|z|$ means the modulus of z.
A useful criterion of Weierstrass applies to a series with a unit radius of convergence, i.e. the unit circle.
In the series $\sum c_k z^k$, let

$$\frac{c_{k+1}}{c_k} = 1 - \frac{b}{k} + \frac{B(k)}{k^\lambda},$$

where b is a complex constant, $\lambda > 1$ is a real constant, and $B(k)$ is some bounded function as $k \to \infty$.
Then, for points on the circle of convergence $|z| = 1$, the series either

1. will converge absolutely if $\mathrm{Re}(b) > 1$,
2. will diverge if $\mathrm{Re}(b) \leq 0$,
3. will converge (but not absolutely) if $0 < \mathrm{Re}(b) \leq 1$, except at $z = 1$, where it will diverge.
   For a proof see Konrad Knopp, "Theory and Application of Infinite Series", p.401.

As an example the series

$$\sum \frac{z^k}{k}$$

converges everywhere on the unit circle, except at $z = 1$. Notice that $c_k = 1/k$,

$$\frac{c_{k+1}}{c_k} = \frac{k}{k+1} = 1 - \frac{1}{k} + \frac{1}{k(k+1)}.$$

Likewise, the hypergeometric series can be analyzed. The convergence at the end-points of the interval or on the unit circle in the complex plane depends on the coefficients, α, β, and γ.

$$\begin{aligned}
\frac{c_{k+1}}{c_k} &= \frac{k^2 + (\alpha+\beta)k + \alpha\beta}{k^2 + (\gamma-1)k + \gamma} \\
&= 1 + \frac{(\alpha+\beta-\gamma-1)k + \alpha\beta - \gamma}{k^2 + (\gamma-1)k + \gamma} \\
&= 1 + \frac{(\alpha+\beta-\gamma-1)}{k} + \frac{B(k)}{k^2}
\end{aligned}$$

where $B(k)$ is bounded. So, either

1. Absolute convergence on $|x| = 1$ is assured if $\Re\text{eal}(\gamma - \alpha - \beta) > 0$,
2. Diverges if $\Re\text{eal}(\gamma - \alpha - \beta) \leq -1$,
3. Converges conditionally if

   $$0 < \Re\text{eal}(\gamma - \alpha - \beta + 1) \leq 1$$

   except at $x = 1$ where it diverges.
   As long as γ is neither zero nor a negative integer, the power series solution to (6) leads to

   $$y = 1 + \frac{\alpha\beta}{1 \cdot \gamma} x + \frac{\alpha\,(\alpha+1)\,\beta\,(\beta+1)}{2!\gamma\,(\gamma+1)} x^2 + \cdots .$$

   This solution is defined as

   $${}_2F_1(\alpha; \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{k!\Gamma(\gamma+k)} x^k. \tag{7}$$

   The series converges absolutely for $|x| < 1$.

The hypergeometric series terminates for certain values of α and β. Rather than examine that situation in general, we note that using the transformation

$$x \to 1 - 2x, \alpha = -n, \beta = n + 1, \gamma = 1$$

in (6), results in

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0. \tag{8}$$

which is Legendre's equation, so it is possible to make the connection that

$${}_2F_1(-n; n+1; 1; \frac{1}{2}(1-x)) = P_n(x),$$

which are the *Legendre polynomials*.

Problem 2.18 It turns out that the second solution to Legendre's equation is more valuable when $|x| > 1$. Make the change of independent variable $z = x^{-1}$ in (2.16). Show that the resulting differential equation for $w(z) = y(x)$ is

$$(1 - z^2)w'' - 2zw' - n(n+1)z^{-2}w = 0. \tag{9}$$

Hence show that when $n = 0$ there is a solution which is

$$Q_0(z) = \frac{1}{2} \log\left(\frac{1+z}{1-z}\right), \ |z| < 1,$$

the *Legendre function of the second kind* of order zero. The succeeding functions of higher orders may be found from the same recurrence formulas as for Legendre polynomials [cite, Abramowitz& Stegun]. However, it may be necessary to express them in closed form. Show that the substitution

$$w(z) = z^{n+1}u(\xi),\ \xi = z^2,$$

in (9) leads to a hypergeometric differential equation, and that a solution to this equation is

$$u(\xi) = \ {}_2F_1\left(\frac{n+2}{2}, \frac{n+1}{2}; n+\frac{3}{2}; \xi\right), \quad |\xi| < 1.$$

The $Q-$functions are thereby defined for $|x| > 1$ as

$$Q_n(x) = \frac{\sqrt{\pi}n!}{\Gamma(n+\frac{3}{2})\,(2x)^{n+1}}\ {}_2F_1\left(\frac{n+2}{2}, \frac{n+1}{2}; n+\frac{3}{2}; \frac{1}{x^2}\right).$$

Confluent Hypergeometric Functions
In the hypergeometric equation there are three adjustable parameters, which though of great generality leads to a function of four variables including x. Of less generality is the equation obtained when $\beta \to \infty$.
This is done by setting $x = \xi/\beta$, $u(\xi) = y(x)$ and then taking the limit as $\beta \to \infty$. There is a price to paid however. The singularity at infinity is a confluence of two regular singularities, rendering it irregular. The resulting DE is

$$\xi u'' + (\gamma - \xi)u' - \alpha u = 0.$$

Now return to the standard variables x, y. Write

$$xy'' + (\gamma - x)y' - \alpha y = 0.$$

It too is the form of (2) with $m = 1$, $r_1 = q_0 = 0$, $p_0 = \gamma$, $p_1 = -1$, $q_1 = -\alpha$. The series solutions are based on *Kummer's function*

$$\begin{aligned} M(a, \gamma; x) &= 1 + \frac{\alpha x}{\gamma} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)2!} + \cdots \\ &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k)}{\Gamma(\gamma+k)}\frac{x^k}{k!} \end{aligned} \tag{10}$$

This series converges for all $|x| < \infty$. It terminates if α is a negative integer.
An alternative notation for Kummer's function is ${}_1F_1(\alpha; \gamma; x)$.

Chebyshev's Equation.

Though not usually arising from the method of separation of variables as some other special functions, Chebyshev functions are ideally suited to the numerical solution of many problems. They satisfy

$$(1 - x^2)y'' - xy' + p^2 y = 0, \quad -1 < x < 1. \tag{11}$$

This is an equation of the form (2) with $m = 2$, $r_2 = -1$, $p_0 = 0$, $p_2 = -1$, $q_0 = 0$, $q_2 = p^2$. It is possible then to look for a series solution of the form $y(x) = \sum_{k=0}^{\infty} c_k x^k$. the recurrence formula is given by

$$c_{k+2} = \frac{k^2 - p^2}{(k+2)(k+1)} c_k, \quad k = 0, 1, 2, \ldots.$$

So, if p is an integer, one of the series terminates.
See the reference on the course website to an article by I. H. Herron, Am. Math. Monthly. These are called the *Chebyshev polynomials* $T_p(x)$. They are

$$T_0(x) = 1, T_1(x) = x,$$
$$T_2(x) = 2x^2 - 1, T_3(x) = 4x^3 - 3x, \ldots$$

normalized by the condition $T_p(1) = 1$. A general formula for $T_p(x)$ may be given as a series

$$T_p(x) = \frac{p}{2} \sum_{k=0}^{\left[\frac{p}{2}\right]} (-1)^k \frac{(p-k-1)!}{k!(p-2k)!} (2x)^{p-2k},$$

where $\left[\frac{p}{2}\right]$ is the greatest integer less than or equal to $\frac{p}{2}$.

Airy Functions

Another differential equation which defines important special functions is

$$y'' - xy = 0. \tag{12}$$

Though $x = 0$ is an ordinary point for this equation, it is also of the form (2) with $m = 3$, with $r_3 = p_0 = p_3 = q_0 = 0$, $q_3 = -1$. If a solution is sought of the form

$$y(x) = \sum_{k=0}^{\infty} c_k x^k,$$

the result is the recurrence formula

$$c_{k+2} = \frac{c_k}{(k+2)(k+1)}, \quad k = 0, 1, \ldots,$$

with $c_2 = 0$. The general solution may be written as

$$\begin{aligned} y &= c_0 \left(1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{(2 \cdot 3)(5 \cdot 6)} + \cdots\right) \\ &\quad + c_1 \left(x + \frac{x^4}{3 \cdot 4} + \frac{x^7}{(3 \cdot 4)(6 \cdot 7)} + \cdots\right). \end{aligned} \tag{13}$$

The values of coefficients c_0 and c_1 determine the Airy functions of the first and second kinds, $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ respectively. By means of the transformations for *Other forms of Bessel's equation* we can show that a general solution is

$$y(x) = x^{1/2}\left[c_1 I_{1/3}(\tfrac{2}{3}x^{3/2}) + c_2 I_{-1/3}(\tfrac{2}{3}x^{3/2})\right].$$

In fact with $\zeta = \frac{2}{3}x^{3/2}$

$$\begin{aligned}\mathrm{Ai}(x) &:= \frac{1}{3}x^{1/2}\left[I_{-1/3}(\zeta) - I_{1/3}(\zeta)\right]\\ &= \pi^{-1}\sqrt{x/3}K_{1/3}(\zeta),\\ \mathrm{Bi}(x) &:= \sqrt{x/3}\left[I_{1/3}(\zeta) + I_{-1/3}(\zeta)\right]\end{aligned}$$