MATH-6600, MOAM: Assignment No. 5, 12-11-15

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1. (a) Consider the singular BVP

$$-y'' + \frac{y'}{x} = f(x), \quad 0 < x < 1$$

$$\lim_{x\to 0} \frac{y(x)}{x} \text{ finite, } y(1) = 0.$$

Use a Green's function to show that the solution may be written

$$y(x) = \int_{0^+}^{1} \frac{f(\xi)}{\xi} G(x,\xi) d\xi$$

where

$$G(x,\xi) = \begin{cases} \frac{1}{2}x^2(1-\xi^2), & x < \xi \\ \frac{1}{2}\xi^2(1-x^2), & x > \xi \end{cases}$$

Solution:

We begin by multiplying through by $\frac{1}{x}$:

$$-\frac{1}{x}y'' + \frac{1}{x^2}y' = \frac{f(x)}{x}$$

We can then rewrite the equation in standard form:

$$(-\frac{1}{x}y')' = \frac{f(x)}{x}$$

Then we have

$$y = \int_{0^+}^{1} \frac{f(\xi)}{\xi} G(x,\xi) d\xi$$

To determine the Green's function we solve the homogeneous problem:

$$(-\frac{1}{x}y')' = 0$$

Integrating twice results in

$$y = -\frac{c}{2}x^2 + d$$

which is a linear combination of the two fundamental solutions $y = 1, y = x^2$. To satisfy the boundary conditions, we let

$$u_1(x) = x^2 \text{ which satisfies only the first boundary condition}$$

$$u_2(x) = 1 - x^2 \text{ which satisfies only the second boundary condition.}$$

We then calculate the Wronskian:

$$W = \begin{vmatrix} x^2 & 1 - x^2 \\ 2x & -2x \end{vmatrix} = -2x$$

Then, since $p(x) = -\frac{1}{x}$ we have $pW = -2$. Now we have the Green's function:

$$G(x, \xi) = \begin{cases} \dfrac{u_1(x)u_2(\xi)}{pW}, & x < \xi \\ \dfrac{u_1(\xi)u_2(x)}{pW}, & x > \xi \end{cases} = \begin{cases} \frac{1}{2}x^2(1 - \xi^2), & x < \xi \\ \frac{1}{2}\xi^2(1 - x^2), & x > \xi \end{cases}$$

(b) Determine the generalized Green's function for the singular problem

$$-(\frac{1}{x}u')' = f(x), \quad 0 < x < 1,$$

$$\lim_{x \to 0^+} \frac{u}{x} \text{ finite}, \quad 2u(1) - u'(1) = 0.$$

Solution:

We note that the solution to the homogeneous problem

$$Lu = -(\frac{1}{x}u')' = 0$$

is the same as above, $u = \frac{c}{2}x^2 + d$. Then the eigenfunction which satisfies both boundary conditions is

$$\varphi(x) = x^2.$$

Using property (iv) and taking $r(x) = \frac{1}{x}$, we get

$$LG^\dagger(x, \xi) = \delta(x - \xi) + cr(x)\phi(x)$$

$$\implies -(x^{-1}G_x^\dagger)' = \delta(x - \xi) + cx$$

Before proceeding we take

$$c = -\frac{\varphi(\xi)}{\int_0^1 r\varphi^2 dx} = -\frac{\xi^2}{1/4} = -4\xi^2,$$

giving us

$$-(x^{-1}G_x^\dagger)' = \delta(x-\xi) - 4\xi^2 x$$

. We then integrate to get

$$-x^{-1}G_x^\dagger = H(x-\xi) - 2\xi^2 x^2 + c_1$$

Then multiplying by $-x$ gives

$$G_x^\dagger = 2\xi^2 x^3 - x(H(x-\xi) + c_1)$$

We integrate one last time to find

$$G^\dagger(x,\xi) = \frac{1}{2}\xi^2 x^4 - \frac{1}{2}c_1 x^2 + \frac{1}{2}(\xi^2 - x^2)H(x-\xi) + c_2$$

We can then write this in the form

$$G^\dagger(x,\xi) = \begin{cases} \frac{1}{2}\xi^2 x^4 - \frac{1}{2}c_1 x^2 + c_2, & x < \xi \\ \frac{1}{2}\xi^2 x^4 - \frac{1}{2}d_1 x^2 + \frac{1}{2}(\xi^2 - x^2) + d_2, & x > \xi \end{cases}$$

Then to determine the constants, we use properties (i),(ii), and (iii) as found in Section 4.5 in Herrron and Foster.

- (i): Boundary conditions:

$$\frac{G^\dagger(0^+,\xi)}{x} = \lim_{x\to 0^+} \frac{1}{2}\xi^2 x^3 + c_1 x + \frac{c_2}{x} = 0 \iff c_2 = 0.$$

$$2G^\dagger(1,\xi) + G_x^\dagger(1,\xi) = 0 \implies 2[\frac{1}{2}\xi^2 - \frac{1}{2}d_1 + \frac{1}{2}(\xi^2 - 1) + d_2] - (2\xi^2 - d_1 - 1) = 0$$
$$\implies d_2 = 0.$$

  Then our Green's function is

$$G^\dagger(x,\xi) = \begin{cases} \frac{1}{2}\xi^2 x^4 - \frac{1}{2}c_1 x^2, & x < \xi \\ \frac{1}{2}\xi^2 x^4 - \frac{1}{2}d_1 x^2 + \frac{1}{2}(\xi^2 - x^2), & x > \xi \end{cases}$$

- (ii) Continuity:

$$G^\dagger(\xi^-,\xi) = G^\dagger(\xi^+,\xi)$$

  This gives:

$$c_1 = d_1$$

  and we have

$$G^\dagger(x,\xi) = \begin{cases} \frac{1}{2}\xi^2 x^4 - \frac{1}{2}c_1 x^2, & x < \xi \\ \frac{1}{2}\xi^2 x^4 - \frac{1}{2}c_1 x^2 + \frac{1}{2}(\xi^2 - x^2), & x > \xi \end{cases}$$

- (iii) Jump Condition:

$$\frac{\partial G^\dagger}{\partial x}|_{x=\xi^-}^{x=\xi^+} = -\xi$$
$$\frac{1}{p(\xi)} = -\xi$$

.

- (i') $\int_{0^+}^1 \phi(x)r(x)G^\dagger(x,\xi)dx = 0$ This becomes

$$\int_{0^+}^1 xG^\dagger(x,\xi)dx = \int_{0^+}^{\xi} \frac{1}{2}\xi^2x^5 - \frac{1}{2}c_1x^3dx + \int_{\xi}^1 \frac{1}{2}\xi^2x^5 - \frac{1}{2}c_1x^3 + \frac{1}{2}x\xi^2 - \frac{1}{2}x^3dx$$

Working this expression will give the result:

$$c_1 = \frac{8}{3}\xi^2 - \xi^4 - 1.$$

We then have the final answer:

$$G^\dagger(x,\xi) = \begin{cases} \frac{1}{6}x^2(3x^2\xi^2 + 3\xi^4 - 8\xi^2 + 3), & x < \xi \\ \frac{1}{6}\xi^2(3x^2\xi^2 + 3x^4 - 8x^2 + 3), & x > \xi \end{cases}.$$

2. (a) Beginning with

$$G(t,\tau) = \begin{cases} U(t)C_1, & t < \tau \\ U(t)C_2, & t > \tau \end{cases},$$

where U is a fundamental matrix and C_1 and C_2 are matrices independent of t and if G satisfies:

$$AG(a,\tau) + BG(b,\tau) = 0,$$

and

$$[G]_{t=\tau^-}^{t=\tau^+} = I,$$

derive

$$G(t,\tau) = \begin{cases} -U(t)D^{-1}BU(b)U^{-1}(\tau), & t < \tau \\ U(t)D^{-1}AU(a)U^{-1}(\tau), & t > \tau \end{cases}.$$

Note that we define,

$$AU(a) + BU(b) \equiv D.$$

Solution:
From the second condition we have

$$U(\tau)C_2 - U(\tau)C_1 = I$$

Then, solving for the difference of the C matrices gives

$$C_2 - C_1 = U^{-1}(\tau).$$

We can use this result and the second condition to solve for C_1 and C_2:

$$AU(a)C_1 + BU(b)C_2 = 0 \implies AU(a)C_1 + BU(b)(C_1 + U^{-1}(\tau)) = 0$$
$$\implies C_1 = -D^{-1}BU(b)U^{-1}(\tau)$$

and again,

$$AU(a)(C_2 - U^{-1}(\tau) + BU(b)C_2 = 0 \implies C_2 = D^{-1}AU(a)U^{-1}(\tau).$$

Thus our Green's function is in the form desired.

(b) Consider the system

$$\frac{d\mathbf{u}}{dt} = \mathbf{Pu} + \mathbf{f}, \quad 0 < t < 2\pi,$$

where

$$\mathbf{u} = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Determine a (2×2) fundamental matrix $\mathbf{U}(t)$ for the system.
Given the boundary conditions

$$u_1(0) - u_1(2\pi) = 0, \quad u_2(0) - u_2(2\pi) = 0,$$

compute the Green's matrix for the differential equation.
Solution:

The fundamental solution will satisfy the homogeneous differential equation:

$$\frac{d\mathbf{u}}{dt} = \mathbf{Pu}, \quad 0 < t < 2\pi.$$

Thus the solution is

$$U = e^{Pt}$$

To compute U we first perform an eigenvalue decomposition of Pt. Fortunately, we already have performed this decomposition algebraically in Problem 5 of Assignment 1. Here, we simply let $\alpha = 1$ and $\beta = 1$. Then we know that the eigenvalue decomposition of P is

$$P = S\Lambda S^{-1} = \frac{1}{2}\begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}\begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}.$$

Then, as computed before, the fundamental matrix U can be written

$$U = e^{Pt} = Se^{\Lambda t}S^{-1} = \begin{bmatrix} e^t \cos t & e^t \sin t \\ -e^t \sin t & e^t \cos t \end{bmatrix}.$$

To then compute the Green's matrix we must define A,B, D, and $U^{-1}(\tau)$. We can easily define A and B through inspection of the boundary conditions. As A and B must satisfy the relationship

$$A\mathbf{u}(0) + B\mathbf{u}(2\pi) = 0,$$

we can clearly see that

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I.$$

Then from the relationship used in the previous part of the question, we can calculate D:

$$D = AU(0) + BU(2\pi) = U(0) - U(2\pi) = I - e^{2\pi}I = (1 - e^{2\pi})I$$

Then we have

$$D^{-1} = \frac{1}{1 - e^{2\pi}}I.$$

Lastly, we can compute $U^{-1}(\tau)$

$$U^{-1}(\tau) = e^{-\tau} \begin{bmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{bmatrix}.$$

Then the Green's function becomes

$$G(t,\tau) = \begin{cases} -e^t \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \dfrac{1}{1 - e^{2\pi}} I(-I)e^{2\pi} I e^{-\tau} \begin{bmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{bmatrix}, & t < \tau \\ e^t \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \dfrac{1}{1 - e^{2\pi}} I(I)(I)e^{-\tau} \begin{bmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{bmatrix}, & t > \tau \end{cases},$$

$$G(t,\tau) = \begin{cases} -\dfrac{e^{t+2\pi-\tau}}{1 - e^{2\pi}} \begin{bmatrix} \cos(t-\tau) & \sin(t-\tau) \\ -\sin(t-\tau) & \cos(t-\tau) \end{bmatrix}, & t < \tau \\ \dfrac{e^{t-\tau}}{1 - e^{2\pi}} \begin{bmatrix} \cos(t-\tau) & \sin(t-\tau) \\ -\sin(t-\tau) & \cos(t-\tau) \end{bmatrix}, & t > \tau \end{cases}.$$

3. (a) Solve the Fredholm integral equation

$$x - 1 = \int_{-1}^{1} (1 + y + 3xy)\phi(y)dy$$

using the component decomposition

$$A_1(x) = 1 \quad B_1(y) = 1 + y$$

$$A_2(x) = 3x \quad B_2(y) = y.$$

Solution:

We begin by showing that $\psi(x) = x - 1$ is a linear combination of A_1 and A_2:

$$x - 1 = \alpha_1 A_1 + \alpha_2 A_2 = \alpha_1 + 3\alpha_2 x$$

$$\implies \alpha_1 = -1, \quad \alpha_2 = \frac{1}{3}.$$

Then we use the fact that

$$\alpha_k = \sum_{j=1}^{k} \beta_j (B_k, B_j)$$

to find the values of β_1, β_2, as these will help us define our solution ϕ. We then find the system of equations dictating the values of the β.

$$\alpha_1 = -1 = \beta_1(B_1, B_1) + \beta_2(B_1, B_2) = \beta_1 \int_{-1}^{1} (1+y)^2 dy + \beta_2 \int_{-1}^{1} (1+y) y dy$$

Which gives

$$-1 = \frac{8}{3}\beta_1 + \frac{2}{3}\beta_2$$

Similarly for α_2:

$$\alpha_2 = \frac{1}{3} = \beta_1(B_2, B_1) + \beta_2(B_2, B_2) = \beta_1 \int_{-1}^{1} (1+y) y dy + \beta_2 \int_{-1}^{1} y^2 dy$$

which results in

$$1 = 2\beta_1 + 2\beta_2.$$

Solving this system of equations will give

$$\beta_1 = -\frac{2}{3}, \quad \beta_2 = \frac{7}{6}.$$

Then we have the solution for ϕ:

$$\phi(x) = \beta_1 B_1 + \beta_2 B_2 = -\frac{2}{3}(1+x) + \frac{7}{6}x = \frac{1}{2}x - \frac{2}{3}.$$

(b) Solve the integral equation

$$u(t) = 1 - \lambda \int_0^1 K(t,s) u(s) ds$$

where

$$K(t,s) = \begin{cases} 0, & s < t \\ 1 & s > t \end{cases}.$$

Solution:

We begin by rewriting the equation using the kernel

$$u(t) = 1 - \lambda \int_t^1 u(s)ds.$$

Differentiating will result in the differential equation

$$u'(t) = \lambda u(t).$$

This equation has solution:

$$u(t) = ke^{\lambda t}.$$

We find the boundary condition $u(1) = 1$ from the original integral equation and solve for k

$$u(1) = ke^{\lambda} = 1 \implies k = e^{-\lambda}$$

Thus we have the solution to the integral equation

$$u(t) = e^{\lambda(t-1)}.$$

4. Use the Fredhold alternative to find all of the values of a and b for which

$$\phi(x) = a \sin x + b \cos x + \frac{8}{\pi} \int_0^{\pi/2} K(x,y)\phi(y)dy$$

can be solved when

$$K(x,y) = \begin{cases} \sin x \sin y, & x < \pi/4 \\ \cos x \sin y, & x > \pi/4 \end{cases}.$$

Note that this is a separable kernel.

Solution:

It is trivial to prove that $\frac{8}{\pi}$ is the characteristic value for this integral equation. Therefore, we can use the Fredholm alternative on the given equation to determine conditions on a and b such that $f(x) = a \sin x + b \cos x$ is orthogonal to $\omega(x)$ - the solution to the homogeneous integral equation. Thus we solve

$$\omega(x) = \frac{8}{\pi} \int_0^{\pi/2} K(x,y)\omega(y)dy.$$

We first list the components of the separable kernel:

$$A = \begin{cases} \sin x, & x < \pi/4 \\ \cos x, & x > \pi/4 \end{cases}, \quad B = \sin x.$$

Then we can use the result derived in class

$$\omega(x) = \alpha A(x)$$

where α is given by

$$\alpha = \lambda\alpha(A, B)$$

however, in this case α cannot be solved for, and is thus arbitrary (further evidence that we are working with an eigenfunction). Since α is arbitrary, for ease of calculation, we choose $\alpha = 1$, thereby giving us $\omega(x) = A(x)$. We then set the inner product of f and $\omega(x)$ to 0, which will give us a condition on a and b.

$$(f, \omega)(x) = \int_0^{\pi/2} (a \sin x + b \cos x) A(x) dx = 0$$

$$\implies \int_0^{\pi/4} a \sin^2 x + b \sin x \cos x dx + \int_{\pi/4}^{\pi/2} a \sin x \cos x + b \cos^2 x dx = 0$$

$$\implies \frac{1}{8}((\pi - 2)a + 2b) + \frac{1}{8}(2a + (\pi - 2)b) = 0$$

which reduces to

$$b = -a.$$

Therefore, the integral equation

$$\phi(x) = a \sin x - a \cos x + \frac{8}{\pi} \int_0^{\pi/2} K(x, y)\phi(y) dy$$

has a solution for any value of a.

5. (a) Use Schur's inequality, to find a bound on λ_1 without solving the problem, based on the kernel $K(x, y) = |x - y|$, $-\frac{1}{2} \leq x, y \leq \frac{1}{2}$.

Solution:

Schur's inequality can be reduced to $\frac{1}{\lambda_1^2} \leq \|K\|^2$. Thus we need only calculate the norm of K:

$$\|K\|^2 = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} |x - y|^2 \, dy dx = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} x^2 - 2xy + y^2 dy dx$$

$$= \int_{-1/2}^{1/2} x^2 + \frac{1}{12} dx = \frac{1}{6}$$

Therefore, we have

$$\frac{1}{\lambda_1^2} \leq \frac{1}{6} \implies \lambda_1 \geq \sqrt{6} \text{ or } \lambda_1 \leq -\sqrt{6}.$$

(b) Determine a full characteristic system for the symmetric kernel

$$K(x,y) = |x-y| \quad -\frac{1}{2} \leq x, y \leq \frac{1}{2}.$$

We begin by rewriting the homogeneous integral equation

$$\phi(x) = \lambda \int_{-1/2}^{1/2} K(x,y)\phi(y)dy$$

as a differential eigenvalue problem on the interval $[-\frac{1}{2}, \frac{1}{2}]$. Differentiating once will give

$$\phi'(x) = \lambda[\int_{-1/2}^{x} \phi(y)dy + \int_{1/2}^{x} \phi(y)dy].$$

Then differentiating again gives the differential eigenvalue problem

$$\phi''(x) = 2\lambda\phi(x).$$

This differential equation has general solution

$$\phi(x) = c_1 e^{\sqrt{2\lambda}x} + c_2 e^{-\sqrt{2\lambda}x}.$$

We can use the fact that the solution $\phi(x)$ and the kernel $K(x,y)$ must satisfy the same boundary conditions to determine the correct boundary conditions for the eigenvalue problem. The values of K and K_x on the boundary are

$$K(-\frac{1}{2}, y) = y + \frac{1}{2}, \quad K(\frac{1}{2}, y) = \frac{1}{2} - y$$

$$K_x(-\frac{1}{2}, y) = -1, \quad K_x(\frac{1}{2}, y) = 1.$$

Then since

$$K(-\frac{1}{2}, y) + K(\frac{1}{2}, y) = 1, \text{ and } K_x(-\frac{1}{2}, y) + K_x(\frac{1}{2}, y) = 0,$$

it would make sense for the boundary conditions on $\phi(x)$ to be

$$\phi(-\frac{1}{2}) + \phi(\frac{1}{2}) = 1, \text{ and } \phi'(-\frac{1}{2}) + \phi'(\frac{1}{2}) = 0.$$

This results in the solution

$$\phi(x) = \frac{\cosh(\sqrt{2\lambda}x)}{2\cosh(\frac{\sqrt{2\lambda}}{2})} + c_2 \sin((2k-1)\pi x).$$

However, since the eigenvalues are $\lambda = -\frac{(2k-1)^2\pi^2}{2}$, the first component of the solution is infinite. Thus we do not have an eigenfunction that satisfies both boundary conditions above. However, if we let the first boundary condition be

$$\phi(-\frac{1}{2}) + \phi(\frac{1}{2}) = 0$$

we can let

$$\phi(x) = c_2 \sin((2k-1)\pi x)$$

where c_2 is an arbitrary constant that can be chosen for normalization, be the eigenfunction, with eigenvalues

$$\lambda = -\frac{(2k-1)^2\pi^2}{2}$$

Interestingly, even though this function does not satisfy the proper boundary conditions as defined earlier, it does satisfy both the differential equation and the integral equation in question.