Mathematical Model and Analysis of the Effects of Overfishing on Coral Reef Ecosystems

Coral Reefsearchers

Aaron Bumagat, Michelle Luces, Henry Song

University of Guam

July 25, 2021
GECCO 2021 Final Preentation

Table of Contents

1. Background
   - General Background
   - Question
   - Definitions

Background

- Coral Reefs are large underwater structures composed of the skeletons of colonial marine invertebrates known as coral[13].

Background

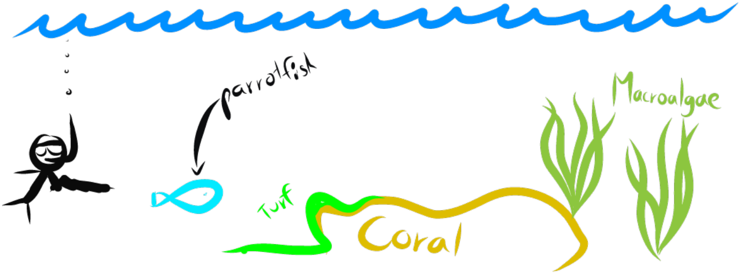
- Coral Reefs are large underwater structures composed of the skeletons of colonial marine invertebrates known as coral[13].
- They play a crucial role in the marine ecosystem's biodiversity along with other functions, such as coastal defense from storms and economic benefits from tourism or local fisheries[8].

Background

- Coral Reefs are large underwater structures composed of the skeletons of colonial marine invertebrates known as coral[13].
- They play a crucial role in the marine ecosystem's biodiversity along with other functions, such as coastal defense from storms and economic benefits from tourism or local fisheries[8].
- Factors that affect coral reefs include climate change, coral reef resilience[12], and exploitative fishing practices[11].

Our Question

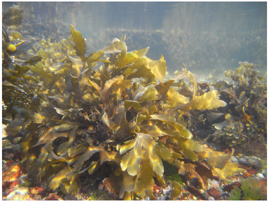
- Specific Question: How will overfishing affect Guam's coral reef ecosystem over time?



Identify Our Terms

(a) Corals[1]

(b) Algal Turfs[2]



(c) Macroalgae[3]

Figure 1: Images of Ecosystem

Identify Our Terms (Cont.)

Figure 2: Parrot Fish[4]

- Parrot fish are common reef fish found in many tropical reefs[5] and are known to feed on algal turfs and macroalgae.
- Their bites on corals have been shown to improve and promote coral growth[9].
- Parrot fish are one of the most overfished reef fish in the Caribbean, and potentially on Guam as well[5].

2 Mathematical Model

- Assumptions
- Compartments
- Parameters
- Compartment Model
- Differential Equations
- Plots

Assumptions

- Ecosystem:
  - is closed (i.e. no migration).
  - consists of only corals (C), algal turfs (T), and macroalgae (M).
  - supports maximum carrying capacity of parrotfish.
- Macroalgae is the only predator of corals.
- Coral recruit to and overgrow algal turfs[8].
- Corals are overgrown by macroalgae[8].
- Macroalgae colonize dead coral by spreading vegetative over algal turfs[8].
- Corals do not naturally die.

Coral Reef Ecosystem Model Compartments

Compartments

The Ecosystem Model consists of 4 compartments:

- C: Corals
- T: Algal Turfs
- M: Macroalgae
- P: Parrotfish

$$\text{where } C + T + M = 1.$$

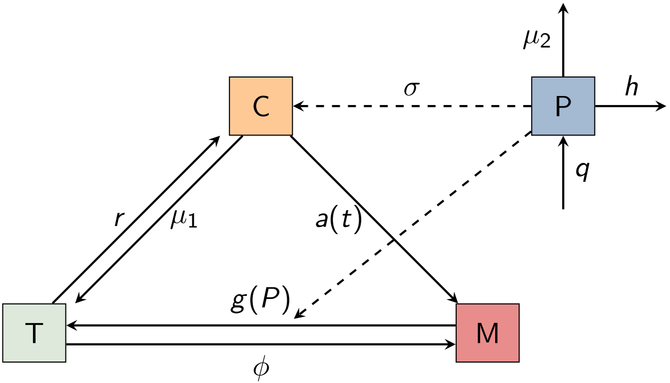
Coral Reef Ecosystem Model Parameters

Parameter	Description	Value	Units[10][8][5]
μ_1	death rate of coral reefs	0.15 [14]	$year^{-1}$
μ_2	natural death rate of parrotfish	0.22 [10]	$year^{-1}$
r	rate that coral recruit to overgrow algal turfs	0.5 [14]	$year^{-1}$
ϕ	rate that macroalgae spread vegetative over algal turfs	0.8[8]	$year^{-1}$
q	intrinsic growth rate for parrotfish	0.47 [10]	$year^{-1}$
h	harvesting rate for parrotfish	0.14 [10]	$year^{-1}$
σ	rate that parrot fish bite coral	0.01*	$bites \cdot year^{-1}$
ω	maximum grazing intensity	1[5]	-
β	carrying capacity of parrotfish	1	-
a_0	control variable to simulate seasonal changes	0.99	-

* = *estimated value*

= *Guam Data*

Coral Reef Ecosystem Model[8]



Differential Equations (Modified from [5])

System of differential equations derived from compartment model:

$$\frac{dC}{dt} = rTC + \sigma PC - (a(t)M + \mu_1)C$$
$$\frac{dT}{dt} = \mu_1 C + \frac{g(P)M}{M+T} - T(rC + \phi M)$$
$$\frac{dM}{dt} = (a(t)C + \phi T)M - \frac{g(P)M}{M+T}$$
$$\frac{dP}{dt} = qP\left(1 - \frac{P}{\beta C}\right) - P\left(h + \mu_2\right)$$

where $g(P) = \dfrac{\omega P}{\beta}$, $\quad a(t) = \left|\dfrac{a_0(9\sin(\pi t) + 1)}{10}\right|$.

Simulating $a(t)$

$$a(t) = \left| \frac{a_0(9\sin(\pi t) + 1)}{10} \right|$$

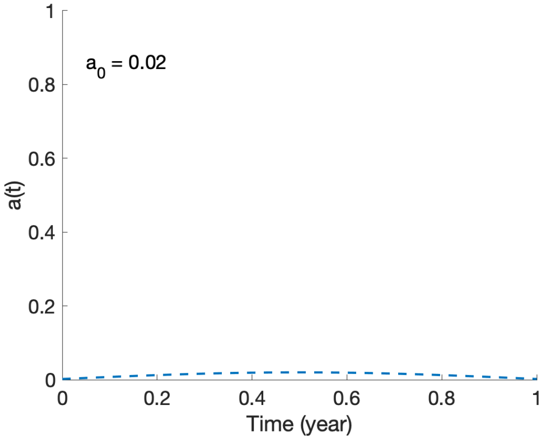


Figure 3: Amplitude of $a(t)$ as a_0 changes

Parameter Changes: a_0

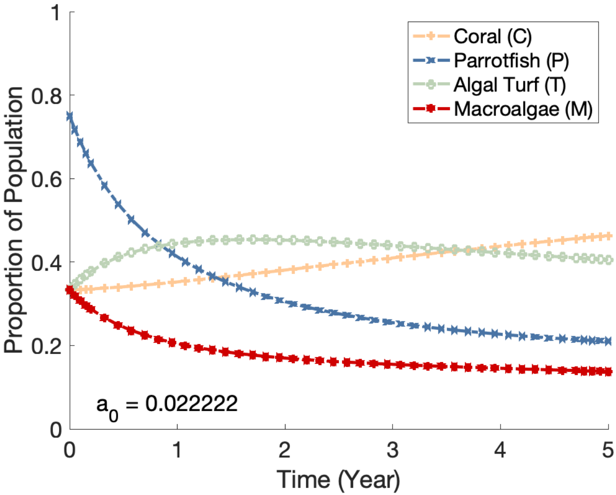


Figure 4: Initial Conditions: $C = T = M = \frac{1}{3}$, and $P = \frac{3}{4}$

Parameter Changes: h

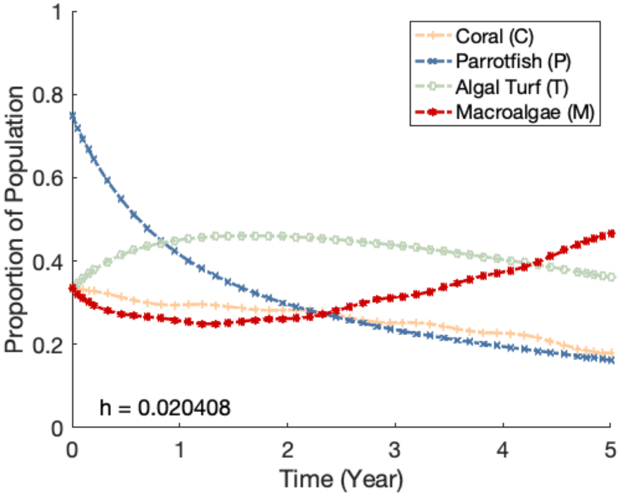


Figure 5: Initial Conditions: $C = T = M = \frac{1}{3}$, and $P = \frac{3}{4}$

Parameter Changes: ϕ

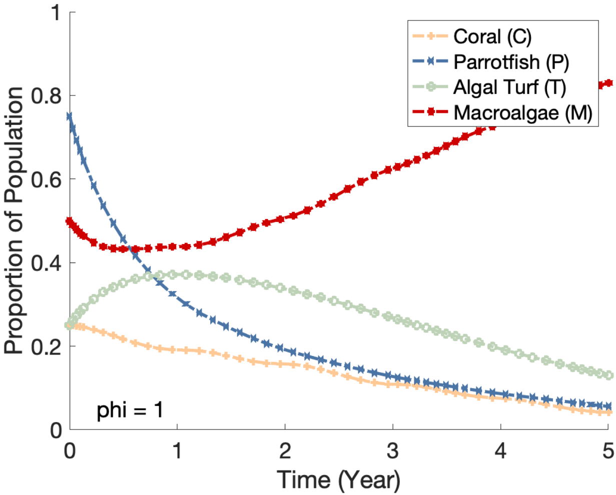


Figure 6: Initial Conditions: $M = \frac{1}{2}$, $C = T = \frac{1}{4}$, and $P = \frac{3}{4}$

3 Equilibria
- Disease Free Equilibrium
- Endemic Equilibrium
- Basic Reproduction Number

Disease Free Equilibrium

Definition

- The disease-free equilibrium is the point at which no disease is present in the system.
- In our model, we classify macroalgae (M) as our "disease" compartment, and since the system is disease free, we set $M^0 = 0$.

Disease Free Equilibrium

$$C^0 = 1 - \frac{\mu_1}{r}$$

$$T^0 = \frac{\mu_1}{r}$$

$$M^0 = 0$$

$$P^0 = -\frac{\beta(1 - \frac{\mu_1}{r})(h - \mu_2 - q)}{q}$$

Endemic Equilibrium

Definition

- The Endemic Equilibrium determines at what point will the disease not spread nor will it be fully eradicate.
- In order to find the endemic equilibrium, we have to set each differential equation equal to 0 and solve for each respective compartment variable.

Endemic Equilibrium

$$C^* = 1 - \left(\frac{\mu_1 + a(t)M^*}{r} + M^* \right)$$

$$T^* = \frac{\mu_1 + a(t)M^*}{r}$$

$$P^* = \beta \left(1 - \left(\frac{\mu_1 + a(t)M^*}{r} + M^* \right) \right) \left(\frac{q - (h + \mu_2)}{q} \right)$$

Endemic Equilibrium

Calculation for M^*:

$$\frac{-k \pm \sqrt{k^2 - 4jn}}{2j}$$

where

$$j = (a(t)q - 2a(t)r + r\phi) - q(r^2 + a(t)^3)$$

$$\begin{aligned} k = {} & a(t)q(r(r - 2\mu_1 + a(t)) - 2\mu_1(a(t) + \phi)) \\ & + r(q(\phi\mu_1 + r\omega) - \omega(hr + a(t)\mu_2 + r\mu_2) \end{aligned}$$

$$\begin{aligned} n = {} & -qr^2\omega + qr\mu_1\omega + hr^2\omega - hr\mu_1\omega + r^2\mu_2\omega \\ & - r\mu_1\mu_2\omega + a(t)qr\mu_1 - a(t)q\mu_1^2 + q\phi\mu_1^2 \end{aligned}$$

Basic Reproduction Number: $\mathscr{R}_0$

Definition

- A metric used to describe the contagiousness or transmissibility of infectious agents[7], i.e. the number of secondary infections.
- $\mathscr{R}_0$ is defined as the expected number of secondary cases produced by a single macroalgae (M) in a completely coral (C) population.

Basic Reproduction Number: Next Generation Matrix

Definition

- i : Set of infectious compartments $\{M\}$
- $\mathscr{F}_i$: Rate of new infections in compartment i
- $\mathscr{V}_i$: Rate of transfer individuals out of compartment i minus rate into compartment i
- F : Jacobian Matrix($\mathscr{F}_i$)
- V : Jacobian Matrix($\mathscr{V}_i$)
- FV^{-1} : Next Generation Matrix

Basic Reproduction Number: $\mathscr{R}_0$

$$\mathscr{F} = \left[a(t)CM + \phi TM\right] \qquad \mathscr{V} = \left[\frac{g(P)M}{M+T}\right]$$

$$\Downarrow \qquad \Downarrow$$

$$F = \left[a(t)C^0 + \phi T^0\right] \qquad V = \left[\frac{g(P)T^0}{(M^0+T^0)^2}\right]$$

$$\mathscr{R}_0 = \rho(\det(FV^{-1} - \lambda I))$$

$$\Downarrow$$

$$\mathscr{R}_0 = \frac{\beta\mu_1 q(a(t)(1 - \frac{\mu_1}{r}) + \frac{\mu_1}{r})}{\omega r(\beta h(\frac{\mu_1}{r} - 1) + \beta\mu_2(\frac{\mu_1}{r} - 1) - \beta q(\frac{\mu_1}{r} - 1))}$$

4 Harvesting Game Theory
- Harvesting Threshold
- Sensitivity Analysis
- Expected Payoff - Harvesting
- Nash Equilibrium

Harvesting Threshold

Definition (Harvesting Threshold)

The proportion of parrotfish that can be harvested in order for macroalgae growth to maintain stability in the ecosystem.

When $\mathscr{R}_0 = 1$,

$$h_{TH} = q - \mu_2 + \frac{\mu_1 q(a(t)\mu_1 - a(t)r - \phi\mu_1)}{\omega r(r - \mu_1)}$$

$h_{pop} < h_{TH}$: Macroalgae growth is stable.

$h_{pop} > h_{TH}$: Macroalgae growth is unstable.

Harvesting Threshold Graph

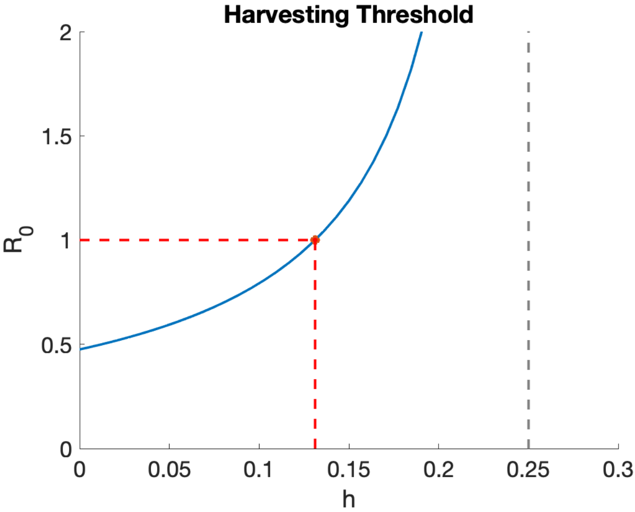


Figure 7: When $\mathscr{R}_0 = 1,\ h = 0.1312$

Sensitivity Analysis for $\mathscr{R}_0$

The Sensitivity Analysis for $\mathscr{R}_0$ determines the most influential parameters affecting $\mathscr{R}_0$. It is defined by

$$S_\lambda = \frac{\frac{\Delta \mathscr{R}_0}{\mathscr{R}_0}}{\frac{\Delta x}{x}} = \frac{\lambda}{\mathscr{R}_0} \cdot \frac{\partial \mathscr{R}_0}{\partial \lambda}$$

where λ is a parameter in the quantity $\mathscr{R}_0$.

λ	S_λ
μ_1	**8.3513**
μ_2	**5.2819**
q	-3.5962
ω	-0.7923
σ	0
r	-2.5054
ϕ	0.4029
β	0
h	**5.2819**
$a(t)$	0.94

Table 1: Sensitivity Analysis

Sensitivity Analysis for $\mathscr{R}_0^{\mu_1}$

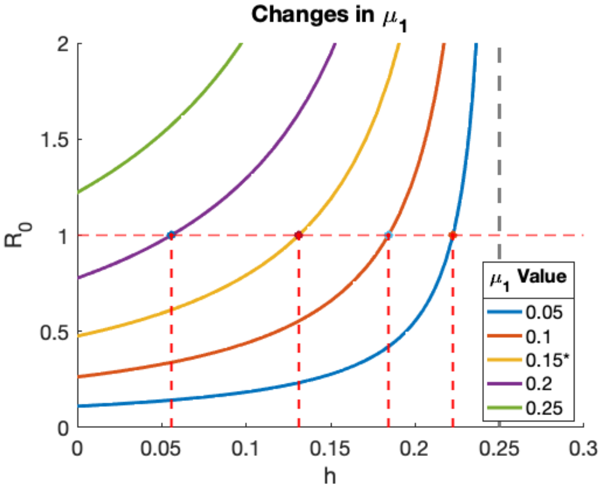


Figure 8: Effects of various μ_1 values on $\mathscr{R}_0$

$*$ = our harvesting threshold(h_{TH})

Sensitivity Analysis for $\mathscr{R}_0^{\mu_1}$

μ_1	h_{TH}
0.05	0.223
0.1	0.1842
0.15	0.1312
0.2	0.0557
0.25	n/a

Table 2: h_{TH} when $\mathscr{R}_0 = 1$ while μ_1 changes

Harvesting Game

Game of Harvesting

- **Player:** Individual
- **Strategy:**
    - Proportion of Individual Harvesting (h) based on the proportion the population is harvesting (h_{pop})

Expected Payoff - Harvesting

Let h $\in$ [0,1] be the proportion at which an individual can harvest parrotfish.

$$E(h, h_{pop}) = -hC_h - \left(\frac{h_{pop}}{h_{pop} + \mu_2} \cdot \frac{g(P^*)(1-h)M^*}{M^* + T^*}\right) C_D$$

Symbol	Definition
$E(h, h_{pop})$	Expected payoff for an individual to harvest based on the harvesting proportion of the population
C_h	Cost of harvesting
C_D	Cost of coral disease (i.e. less protection from storms)

Expected Payoff - Harvesting

$$E(h, h_{pop}) = -hC^h - \frac{h_{pop}}{h_{pop} + \mu_2} \cdot \frac{g(P^*)(1-h)M^*}{M^* + T^*}$$

where $C^h = \frac{C_h}{C_D}$

Expected Payoff - Harvesting

$E(h, h_{pop})$ is a convex function since $\frac{\partial^2 E(h,h_{pop})}{\partial h^2} > 0$

$$\frac{\partial^2 E}{\partial h^2} = \frac{2\omega P\mu 2(M^* + T^*)^2(\mu_1 + 1)}{\beta(h(M^* + T^*) + \mu_2(M^* + T^*))^3} > 0$$

Thus, E achieves a maximum value at $h = 0$ or $h = 1$.

$$\Delta E = E(1, h_{pop}) - E(0, h_{pop})$$
$$\Delta E > 0 : \text{Harvest}$$
$$\Delta E < 0 : \text{Do Not Harvest}$$

What is Nash Equilibrium?

Definition[6]

- The **Nash equilibrium** is a decision-making theorem within game theory that states a player can achieve the desired outcome by not deviating from their initial strategy.
- Each player's strategy is optimal when considering the decisions of other players. Every player wins because everyone gets the outcome they desire.

Nash Equilibrium - Harvesting

To obtain the Nash Equilibrium, we find h_{NE} such that $E(0, h_{pop}) = E(1, h_{pop})$.

$$E(0, h_{pop}) = -\frac{h_{pop}}{h_{pop} + \mu_2} \cdot \frac{g(P^*)M^*}{M^* + T^*}$$
$$E(1, h_{pop}) = -C^h$$

$$\Downarrow$$

$$\frac{h_{pop}}{h_{pop} + \mu_2} \cdot \frac{g(P^*)M^*}{M^* + T^*} = C^h$$

Nash Equilibrium - Graph

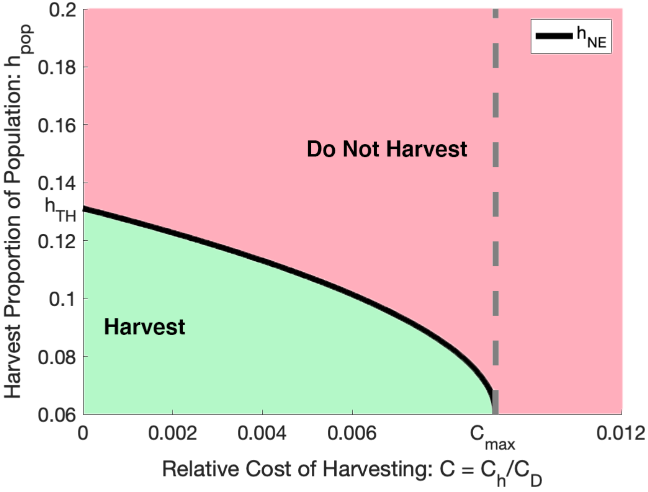


Figure 9: Nash Equilibrium Graph

5 Discussion

Discussion

- Overfishing can have a detrimental impact on coral reef ecosystems.

Discussion

- Overfishing can have a detrimental impact on coral reef ecosystems.
- Based on our mathematical model:
  - A higher a_0 will result in higher macroalgae growth over corals, especially over warmer months of the year.
  - A higher h will result in the increased dominance of macroalgae in the ecosystem.
  - A lower ϕ will result in decreased proportion of macroalgae and increase in algal turfs, and subsequently corals.

Discussion

- Overfishing can have a detrimental impact on coral reef ecosystems.
- Based on our mathematical model:
  - A higher a_0 will result in higher macroalgae growth over corals, especially over warmer months of the year.
  - A higher h will result in the increased dominance of macroalgae in the ecosystem.
  - A lower ϕ will result in decreased proportion of macroalgae and increase in algal turfs, and subsequently corals.
- Our sensitivity analysis indicates that our coral death rate (μ_1), parrotfish natural death rate (μ_2), and harvesting rate (h) have the largest impact on our basic reproduction number.

Discussion

- The maximum sustainable harvest proportion is approximately 0.131157 (13.1157%) of the parrot fish population.

Discussion

- The maximum sustainable harvest proportion is approximately 0.131157 (13.1157%) of the parrot fish population.
- According to our harvesting game theory, as the relative cost of fishing increases, the harvesting Nash equilibrium decreases until it reaches the maximum cost.

Future Research

- Collect data from Guam sources to more accurately model Guam's Reef Ecosystem.
- Application of game theory on ϕ parameter control measures.
- Incorporate more elements of coral reef ecosystems (i.e. other types of plant and aquatic life) to our compartment model.

Acknowledgements

Support for the Young Scholars Research Experience in Mathematics (YSREM) is through the MAA Tensor SUMMA Program. Support for the MAA National Research Experience for Undergraduates Program (NREUP) is provided by the National Science Foundation (Grant Number DMS-1950644). Support for the NSF EPSCoR project, Guam Ecosystems Collaboratorium for Corals and Oceans (GECCO) is provided by the National Science Foundation (Grant Number DMS-1946352).

Special thanks to the UOG Marine Laboratory (Dr. Laurie Raymundo, Dr. Bastian Bentlage & Ms. Grace McDermott), our faculty mentors (Dr. JaeYong Choi, Dr. Hyunju Oh, & Dr. Leslie Aquino), and our Research Assistants (Jaron Bautista & Regina-Mae Dominguez).

MAA MATHEMATICAL ASSOCIATION OF AMERICA NSF GUAM EPSCOR GECCO UNIVERSITY OF GUAM UNIBETSEDÅT GUÅHAN

Questions or Comments?

Bibliography I

[1] URL: https://static.euronews.com/articles/stories/05/00/14/58/1440x810_cmsv2_3031d677-63ce-54f2-baa9-1deb93f016c1-5001458.jpg.

[2] URL: https://i0.wp.com/urbanmarineecology.org/wp-content/uploads/2018/03/Diversity-Algae.jpg?resize=300,225.

[3] URL: https://upload.wikimedia.org/wikipedia/commons/6/67/Fucus_serratus_2015-09-08_ag_M0010140.jpg.

[4] URL: https://seaunseen.com/wp-content/uploads/2020/12/MG_9789.jpg.

[5] J. C. Blackwood, A. Hastings, and P. J. Mumby. "The effect of fishing on hysteresis in Caribbean coral reefs". In: *5* 1 (Nov. 2010), pp. 105–114. DOI: 10.1007/s12080-010-0102-0.

[6] James Chen. *Nash Equilibrium*. May 2021. URL: https://www.investopedia.com/terms/n/nash-equilibrium.asp.

[7] Paul L. Delamater et al. *Complexity of the Basic Reproduction Number(R0) - Emerging Infectious Diseases journal - CDC*. Jan. 2019. URL: https://wwwnc.cdc.gov/eid/article/25/1/17-1901_article.

Bibliography II

[8] Xiong Li et al. "Mathematical analysis of coral reef models". In: *Journal of Mathematical Analysis and Applications* 416.1 (2014), pp. 352–373. ISSN: 0022-247X. DOI: https://doi.org/10.1016/j.jmaa.2014.02.053. URL: https://www.sciencedirect.com/science/article/pii/S0022247X14001917.

[9] Grace McDermott. *Q & A on Coral Ecosystem Model*. July 2021.

[10] Marc O. Nadon. 2019, pp. 1–100.

[11] Sarai Quintero et al. "A Mathematical Model of Coral Reef Response to Destructive Fishing Practices with Predator-Prey Interactions". In: (July 2016).

[12] Bernhard M. Riegl and Samuel J. Purkis. "Model of coral population response to accelerated bleaching and mass mortality in a changing climate". In: *Ecological Modelling* 220.2 (Jan. 2009), pp. 192–208. DOI: 10.1016/j.ecolmodel.2008.09.022.

[13] Rachel Ross. *What Are Coral Reefs?* Sept. 2018. URL: https://www.livescience.com/40276-coral-reefs.html.

Bibliography III

[14] Eric Wolanski, Robert H. Richmond, and Laurence Mccook. "A model of the effects of land-based, human activities on the health of coral reefs in the Great Barrier Reef and in Fouha Bay, Guam, Micronesia". In: *Journal of Marine Systems* 46.1-4 (2004), pp. 133–144. DOI: 10.1016/j.jmarsys.2003.11.018.

Thank you!