1. Improving the accuracy of the spherical spline in tension

Paul Wessel, June 2013

From Parker's January 2013 notes (his eq. 5) we now have a Legendre series solution for the desired Green's function, given as

$$g_p = -\ln 2 + \frac{p^2-1}{p^2} + \sum_{l=1}^{\infty} \frac{(2l+1)p^2}{l(l+1)(l^2+l+p^2)} P_l(\cos\theta). \tag{1}$$

The main reason for replacing the original implementation with (1) is that the old solution required us to compute the difference between two near-singular expressions, one of which ($P_\nu(x)$) is not very stable while the other ($\log(1-x)$) is. For large angles (maybe in the 90–180 range) and for small tensions (maybe $p < 0.1$) the resulting expressions became unstable. The new series solution is much more stable and can handle the range of tensions and angles. Given (1) we also need a series solution for the gradient of g_p. The gradient is given by

$$\nabla g_p = \frac{dg_p(x)}{dx}\frac{dx}{d\theta}\hat{n} = -\frac{dg_p(x)}{dx}\sin\theta\hat{n}, \tag{2}$$

where $x = \cos\theta$. We can now use the relation

$$\frac{x^2-1}{l}\frac{d}{dx}P_l(x) = xP_l(x) - P_{l-1}(x) \tag{3}$$

and by combining (1) with (5) we arrive at

$$\nabla g_p = \frac{1}{\sin\theta}\sum_{l=1}^{\infty}\frac{(2l+1)p^2}{(l+1)(l^2+l+p^2)}(xP_l(x) - P_{l-1}(x)). \tag{4}$$

As of June, 2013, both (1) and (4) have been implemented in greenspline.c for GMT5.

2. Speeding up the Green's function series solution

There are some further improvements we can consider when having more time. Since $P_l(\cos\theta)$ will decay as $l^{-3/2}$ then g_p will decay as $l^{-7/2}$. To speed up its evaluation we could implement a "remove-restore" approach in which we remove and add the series expansion for the dilogarithm. We have a fast implementation of the dilogarithm, but there is also a suitable series solution that will allow us to cancel some terms and improve the convergence of (1). A comparable series solution for the dilogarithm can be obtained from the solution to the spherical biharmonic equation for minimum curvature spline [Parker, 1994], which at an intermediate step yields

$$Li_2 = \frac{\pi^2}{6} - 1 + \sum_{l=1}^{\infty} \frac{(2l+1)}{l^2(l+1)^2} P_l(\cos\theta). \tag{5}$$

Actually, Parker says to remove an expansion whose coefficients have the asymptotic behavior

$$a_l \sim 2p^2 l^{-3} \tag{6}$$

so we need to subtract $p^2 Li_2$ instead. This would then give

$$g'_p = -\ln 2 + \frac{p^2-1}{p^2} - p^2\frac{\pi^2}{6} + p^2 + \sum_{l=1}^{\infty} \frac{-(2l+1)p^4}{l^2(l+1)^2(l^2+l+p^2)} P_l(\cos\theta). \tag{7}$$

Now, g'_p will decay as $l^{-9/2}$ which is l^{-1} faster than (1). This will result in some speed-up during the evaluation.

To determine when to terminate the infinite series we would follow Parker's notes and assume we evaluate (7) up to $l = L-1$ and thus wish to estimate the magnitude of

$$R_L = \sum_{l=L}^{\infty} \frac{-(2l+1)p^4}{l^2(l+1)^2(l^2+l+p^2)} P_l(\cos\theta). \tag{8}$$

The plan then is to revise the derivation in Parker's notes to determine a new (and lower) limit on L.