2 INTRODUCTION TO FLUID DYNAMICS FOR MICROFLUIDIC FLOWS

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2.1 INTRODUCTION

This book is evidence of the many fields and researchers who are interested in devices for manipulating liquids on small (micro and nano) length scales. This particular chapter has been written from the perspective that the reader will come from an electrical engineering or physics training and so will have only had limited, if any, previous exposure to fluids and their motion. As such, the discussion is necessarily focused on the fundamental concepts most relevant to understanding flows of liquids and gases in small devices. A discussion of current research ideas and trends is given in recent review articles [1, 2] and books [3, 4].

We begin by describing in Section 2.2 a few elementary physical ideas helpful for describing fluid motions in channel-like configurations. In particular, we introduce the fluid viscosity, describe qualitatively the incompressible flow approximation, sketch the most common velocity distributions in channel flows, and introduce the Reynolds number, which is a dimensionless parameter useful for characterizing different possible fluid motions. In Section 2.3 we mention briefly the well-known elementary laws of Ohm and Kirchhoff for electrical circuits and give their fluid analogs, which serve to introduce the notions of flow rate, pressure drop, and viscous resistance that are useful in the most basic characterizations of fluid motions. These introductory connections should hopefully assist the reader in developing physical intuition for simple fluid flows as well as thinking about network-like ideas useful for design considerations in microfluidics. The partial differential equations for describing fluid flows are given and typical estimates are summarized in Section 2.4; these equations are frequently too difficult to solve analytically but there are now available commercial packages in computational mechanics that can numerically solve these equations for many practical configurations. A few elementary solutions relevant to microfluidics are given in Section 2.5.

The use of pipes and channels to convey fluids in an organized manner is essentially as old as the living world, since living systems have veins and arteries to transport water, air, gas, etc. The use of channels for the transport and mixing of gases and liquids is part of the industrial and civil infrastructure of our society and so, not surprisingly, many aspects of the dynamics of flow in channels are well understood. At the larger scales (e.g. length scales and typical speeds) of many common flows, the inertia of the motion is most relevant to the dynamics, and in this case turbulence is the rule: such flows are irregular, stochastic, dominated by fluctuations, and often require statistical ideas, correlations or large-scale numerical computation to quantify (if that is even possible).

The recent explosion of interest in fluid flows, and indeed their active manipulation and control, in micro- and nano-environments has turned attention to dynamics where viscous effects, which can be thought of, as a first approximation, as frictional influences interior to the fluid, are most significant: such flows are regular, reproducible, and generally laminar, which makes detailed control possible at small length scales. In many cases relatively simple quantitative estimates of important flow parameters are possible. Several examples are provided in this chapter.

We are all familiar with the concept of force. In mechanics it is generally important to speak in terms of the *stress* or force/area. The term *fluid* refers to either a liquid or a gas, or more generally any material that flows (the specialist might say “deforms continuously”) in response to tangential stresses. It is best to think first about the flow of a single phase fluid in a channel. The most common way to create such a flow is to apply a pressure difference across the two ends of the channel: the resulting flow speed, or flow rate (volume per time), typically varies linearly with the applied pressure difference, at least at low enough flow speeds or Reynolds numbers, which is a dimensionless parameter introduced below. This flow may be used to transport some chemical species or suspended particles (e.g. cells). It is important to recognize that the velocity varies across the channel, with the highest speed at or near the center of the channel and the lowest speed (zero) at the boundaries. Because different points in the fluid move at different speeds there is a natural *dispersion* of suspended matter: a tracer put in the flow, for example, moves much faster in the middle of a channel than near the walls and so spreading of a tracer takes place at a rate controlled by the flow.

The potential opportunities to use microfluidic "plumbing" to create the lab-on-a-chip concept requires the integration of channels, valves, etc. in a systematic way that allows control. Two examples from the lab of Steve Quake, which make clear the distinct fluid bearing components that have been successfully integrated into microfluidic devices used for mixing and

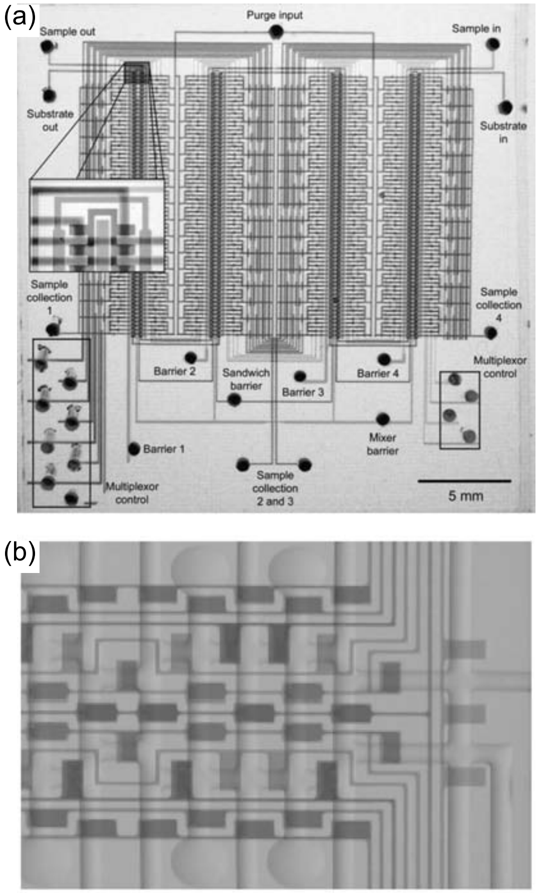

Figure 2.1 Large-scale integration of channels and valves for microfluidic systems. (a) Hundreds of channels and distinct chambers that have been integrated with more than two thousand valves [5]. Food dyes have been used to visualize the different channels and distinct chambers. (b) Example of large-scale integrated microfluidic system for measuring protein interactions [1]. Circular chambers are 250 μm in diameter. Figures courtesy of S. Quake.

reactions, are shown in Fig. 2.1; the different gray levels are dyes labeling distinct aqueous streams.

In other cases, two phases flow in a channel. In one variant of this kind of situation the two fluids are miscible (one fluid can dissolve in the other); see Fig. 2.2. Because the flows are generally laminar, in the absence of significant density variations, the two fluids can flow side-by-side down the channel which means that placement of the streams can be controlled and manipulated. For example, for these two-phase flows, chemical reactions between the two phases can be controlled and in cases such as shown in Fig. 2.2, the interface between the two fluids is the reaction zone [7]. Mixing common to turbulent flows, which is much faster than simple molecular diffusion, then does not take place. Instead, when mixing in a laminar flow in small devices is desired, some strategies need to be implemented [1, 2]. Recent research has made much progress in this area of microfluidic mixing [8].

In other two-phase flows the two fluids are immiscible: interfacial tension acts at the interface to minimize interface deformations. Here it is com-

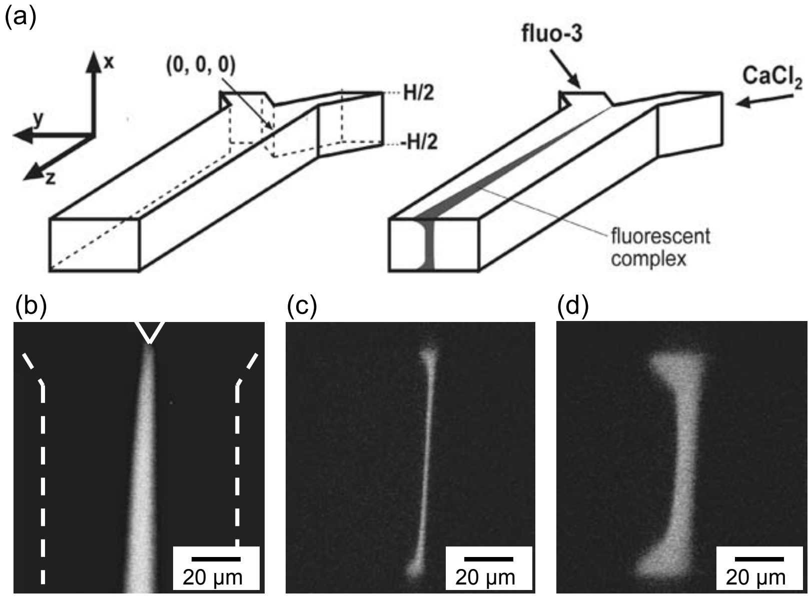

Figure 2.2 Flow of two miscible fluids along a rectangular microchannel [6]. (a) Schematic showing the region of interdiffusion (black). (b) Top view of an experiment using fluorescence to visualize the part of the flow where diffusion mixes the liquids. (c, d) Images taken with a confocal microscope at two different locations downstream, which show the diffusion between the two streams. The liquid near the wall has slower speeds so diffuses further near the wall than in the middle of the channel. These effects can be quantified [6].

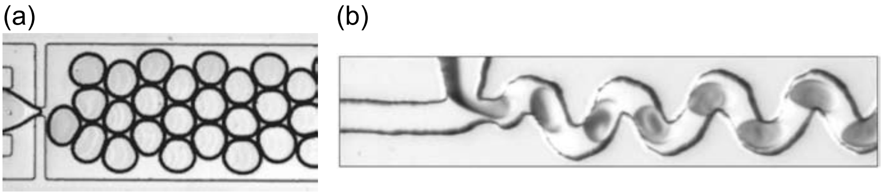

Figure 2.3 Formation of bubbles and drops in microdevices. (a) Formation of disk-shaped gas bubbles in a continuous liquid phase (e.g. [9]); figure courtesy of P. Garstecki. (b) Mixing and reaction inside of droplets, which serve as isolated chemical containers. The mixing is enhanced by the waviness of the channel [10]; figure courtesy of R. Ismagilov.

mon to disperse one phase as droplets in a continuous phase (see Fig. 2.3). The droplets may be used as small chemical reactors, can simply be plugs to separate distinct regions of a fluid column in order to minimize chemical dispersion in the continuous phase, or can be made into solid particles of a variety of size, shapes and compositions [11, 12]. The number of applications of these two-phase flows seems quite large and they are finding many uses in chemistry and biology as well as basic material science.

2.2 CONCEPTS IMPORTANT TO THE DESCRIPTION OF FLUID MOTIONS

2.2.1 Basic Properties in the Physics of Fluids

To introduce concepts needed to describe motion we note that the term fluid generally refers to either liquids or gases. One property needed to characterize a material is the *density* ρ, which measures the mass/volume. A second property important for understanding the flow response of a material is the *viscosity*. Fluids, like solids, can support normal forces without undergoing motion - the equilibrium fluid *pressure* measures the normal force under equilibrium conditions. Recall that the SI unit of pressure is the Pascal, which is defined as 1 N/m. For example, in a container of fluid at rest in a gravitational field, the pressure increases with vertical distance downward simply because of the mass of the column of material vertically above any position. In small devices, because of the small lengths scales involved, such vertical variations of *hydrostatic* pressure are generally not significant relative to other flow-associated stresses.

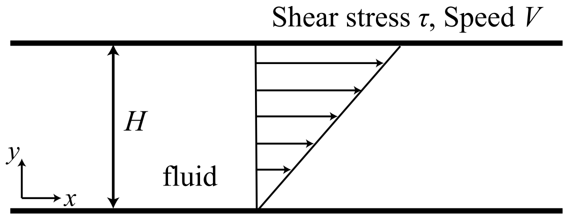

Figure 2.4 Shear flow: relative motion between two planes. A model experiment of this type is how the viscosity of a fluid is defined and measured.

2.2.2 Viscosity and the Velocity Gradient

In contrast, an ordinary (simple) fluid, such as air, water or oil, is set into motion whenever any kind of tangential force, or stress, is applied. The *viscosity* of a fluid measures the resistance to flow or the resistance to the rate of deformation. The simplest way to introduce the physical meaning of the viscosity is to consider the relative motion of two planar surfaces a distance H apart, as sketched in Fig. 2.4. A force/area, or tangential stress τ is needed to translate one surface at speed V relative to the other. In this case, the viscosity μ is the proportionality coefficient between τ and the shear rate V/H: $\tau = \mu(V/H)$. Notice that the dimensions of viscosity are mass/length/time. Water at room temperature has a viscosity $\mu_{\text{water}} \approx 10^{-3}$ Pa·sec. Typical cooking oils have viscosities 10-100 times that of water.

Once this idea is appreciated we note that in the situation considered in Fig. 2.4 it is more precise to introduce x and y coordinates, respectively, parallel and perpendicular to the boundary, and denote the shear stress along the upper boundary (whose normal is in the y-direction) as τ_{yx}. It is a fact of mechanics (unfortunate for the uninitiated) that the detailed description of the state of stress in a material requires two subscripts, one to indicate the direction of the normal to a surface and the second to indicate the direction of a force. In the simplest cases to think about for describing the viscosity (see Fig. 2.4), the velocity is directed parallel to the boundaries, $\mathbf{u} = u_x(y)\mathbf{e}_x$, where $\mathbf{u}$ denotes the velocity field (it is a vector) and $\mathbf{e}_x$ is a unit vector in the x-direction. The velocity is expected to vary linearly between the two surfaces, so $u_x = Vy/H$, and consequently, consistent with the definition in the preceding paragraph, the definition of viscosity can be written

$$\text{shear stress} = \frac{\text{force}}{\text{area}} = \tau_{yx} = \mu \frac{du_x}{dy}. \tag{2.1}$$

The *velocity gradient*, du_x/dy has dimensions 1/time and is frequently referred to as the *shear rate*. Equation (2.1) is a *linear* relation between the shear stress τ_{yx} and the shear rate, du_x/dy; materials that satisfy such linear relations are referred to as *Newtonian fluids*. For practical purposes, all gases and most common small molecule liquids, such as water and other aqueous solutions containing dissolved ions, are Newtonian. On the other hand, small amounts of dissolved macromolecules introduce additional molecular scale stresses – most significantly, these microstructural elements deform in response to the flow, and the description of the motion of the fluid, when viewed on length scales much larger than the macromolecules, generally requires nonlinear relations between the stress and the strain rate. These responses are termed *nonNewtonian*; dilute polymer solutions, biological solutions such as blood, or aqueous solutions containing proteins are in this category.

2.2.3 Compressible Fluids and Incompressible Flows

All materials are compressible to some degree; increasing the pressure usually decreases the volume. Nevertheless, when considering the motion of liquids and gases there are many cases where the density remains close to a constant value, in which case we refer to the "flow as incompressible." Such an incompressible flow is a significant simplification since we then take the density as constant for all calculations and estimates. It then follows that within the incompressibility assumption, the flows of liquids and gases are treated the same. It also follows that the value of the background or reference pressure plays no role in the dynamics other than setting the conditions where the fluid properties of density and viscosity are evaluated. There is only a need when describing fluid flows to distinguish liquids from gases when compressibility of the fluid is important.

A few additional comments may be helpful. It is almost always true under ordinary flow conditions where the pressure changes are modest (e.g. fractions of an atmosphere) that liquids can be treated as *incompressible*, i.e. the density can be taken as a constant. Even though gases are compressible in the sense that according to the ideal gas law their mass density ρ varies linearly with pressure, under common experimental conditions the pressure changes in the gas are sufficiently small that density changes in the gas are usually also small: again, we have the approximation of an incompressible flow. The one case of gas flows where more care is needed is when microchannels are sufficiently long that a gas flow is accompanied by a significant pressure change (say a 20% change in pressure); then the density will also change by approximately this amount. Unless otherwise stated below we will assume that the motions of the fluids occur under conditions where the incompressible flow approximation is valid.

2.2.4 The Reynolds Number

For the simplest qualitative description of a fluid motion we need to recall Newton's second law: the product of mass and acceleration, or more generally the time rate of change of its linear momentum, equals the sum of the forces acting on the body. When we apply this law to fluids it is convenient to consider labeling some set of material points (imagine placing a small amount of dye in the fluid) and following their motion through the system. In addition, we can think about the hydrodynamic pressure as acting to either accelerate the fluid elements (i.e. overcome the inertia) or to overcome friction (viscosity) to maintain the motion. It then follows that the mechanical response to pressure forces that cause flow depends on the relative magnitude of the inertial response to the viscous response; this ratio is known as the *Reynolds number.*

To quantify this idea, consider some body (say a sphere) with radius ℓ translating with speed U through a liquid with viscosity μ and density ρ. The typical acceleration of fluid moving around the object is $U/\Delta t$, where $\Delta t \approx \ell/U$ is the typical time over which changes occur in the fluid when the body moves. Then, we compare (the symbol O simply indicates the order of magnitude)

$$\begin{aligned}\frac{\text{mass}\cdot\text{acceleration}}{\text{viscous forces}} &= \frac{O(\rho\ell^3 U/(\ell/U))}{O(\mu U/\ell^2)} = \frac{\rho U\ell}{\mu} \\ &= \mathcal{R} = \text{Reynolds number}.\end{aligned} \tag{2.2}$$

The reader can verify that the Reynolds number $\mathcal{R}$ is dimensionless, i.e. it has no dimensions.[1]

The Reynolds number is a *dimensionless* parameter that is useful for characterizing flow situations; it is not simply a property of the fluid but rather combines fluid properties (ρ and μ), geometric properties (a length scale ℓ)

[1] The Reynolds number is named after Osborne Reynolds (1842-1912), a professor at Manchester University, who introduced this ratio of variables in 1883 when characterizing the different observed motions for pressure-driven flow in a pipe. The parameter was apparently named the Reynolds number some years later by the German physicist Arnold Sommerfeld. For some historical remarks the reader can refer to [13].

and the typical flow speed. Roughly speaking, high Reynolds number flows tend towards turbulence, they have large domains that tend to be in uniform motion (possibly in a statistical sense) and narrow boundary layers where viscous effects are especially important and viscous stresses are large. Low Reynolds number flows are laminar. In most small devices, because the length scale is measured in tens of microns or less, and the speeds are typically tens of cm/sec, then even for water ($\mu = 10^{-3}$ Pa·sec), the Reynolds number tends to be on the order of unity or smaller. For example, with these numbers $\mathcal{R} = [10^3\ (\text{kg/m}^3) \times 0.1\ (\text{m/sec}) \times 10^{-5}\ (\text{m})]/[10^{-3}\ (\text{Pa·sec})] = 1$.

As a final remark we note that as the Reynolds number is increased from small to large there is a typical progression in "complexity" of a flow. For example, for flow past objects the laminar flow first develops a downstream wake of increasing length, before there is a transition to turbulence, i.e. stochastically fluctuating velocities in at least some region of the flow. For channel flows the laminar flow becomes suddenly turbulent if the Reynolds is large enough. This transition Reynolds number is known to depend on the magnitude of fluctuations in the inlet flow, but for common lab conditions is on the order of 2000 (based on the mean velocity and the diameter) for flow in a circular pipe.

2.2.5 Pressure-driven and Shear-driven Flows in Pipes or Channels

We have already mentioned that a detailed examination of the velocity distribution shows that it varies across a channel. As sketched in Fig. 2.5, the two typical cases are a shear flow, caused by relative movement of two surfaces, and a flow driven by a pressure gradient. In a shear flow the velocity varies linearly across the channel. In a pressure-driven flow in a cylindrical

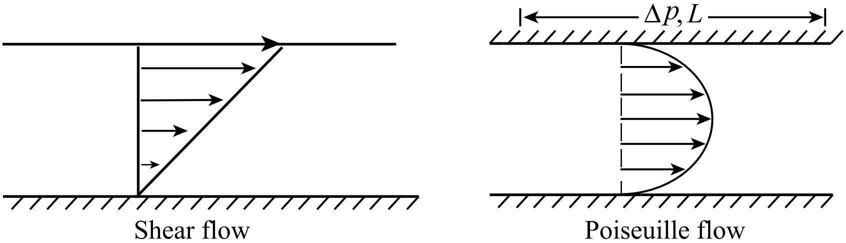

Figure 2.5 (Left) Shear flow driven by a moving boundary. The velocity varies linearly with position perpendicular to the flow direction. (Right) Flow driven by an applied pressure gradient. The velocity varies parabolically with position perpendicular to the flow direction. This situation is commonly called Poiseuille flow.

tube or between two parallel plates the velocity varies parabolically across the channel, with a maximum velocity in the center of the channel (see Subsection 2.5.1). For a rectangular cross section, the velocity distribution is somewhat more complicated but the basic velocity distribution is parabolic across the shortest dimension (see Subsection 2.5.2). Elementary textbooks give the details of the most common laminar flow configurations.

2.3 ELECTRICAL NETWORKS AND THEIR FLUID ANALOGS

2.3.1 Ohm's and Kirchhoff's Laws

The first elementary rule of circuit design is Ohm's law, which relates the change in electrical potential ΔV to the current I in the familiar form $\Delta V = IR$. The resistance R is dependent on the system, the materials, etc. The second rule involves charge conservation, which for steady states gives Kirchhoff's law for a node of a circuit where N currents I_m meet: $\Sigma_{m=1}^{N} I_m = 0$.

The above description by way of algebraic relations for characterizing elementary circuits has analogs when transport of fluid in microchannels is considered (See Table 2.1 for summary). For now we consider single-phase flow. Rather than charge, potential drop, and resistance, for example, as basic

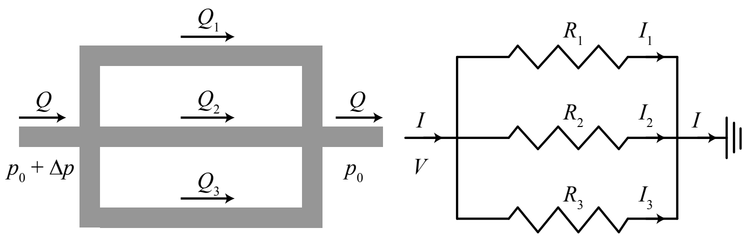

Figure 2.6 Pressure-driven flow in a network of parallel channels. The pressure at the inlet is Δp higher than the pressure at the exit. (Left) Schematic of the flows rates Q_i in each of three channels in parallel. The individual flow rates are related to the total flow rate by $Q = Q_1 + Q_2 + Q_3$ and the pressure drop Δp and Q are related by $\Delta p = QR_H$, where R_H is the equivalent hydrodynamic resistance. In this case, since the same pressure drop acts across each channel, then for each element $\Delta p = Q_i R_{Hi}$. (Right) The equivalent representation as an electrical circuit with resistances in parallel.

parameters, in micro-fluidics we keep track of the rate of mass (or more commonly volume) of fluid transported, a measure of the force needed to move the fluid (e.g. a change of pressure), and the viscosity of the fluid, which is a measure of the resistance of the fluid to motion. In many circumstances the motion of fluids is sufficiently simple that linear relation may be applied betweeen the forcing (e.g. pressure) and the output, e.g. the flow rate of liquid. Hence, given a pressure difference Δp across a fluid-filled channel, there is a flow rate Q (volume/time) with $\Delta p = QR_H$, where the hydrodynamic resistance R_H is a function of the geometry of the channel and the viscosity of the fluid in the channel. At sufficiently high Reynolds number, turbulence is expected and then the resistance R_H is also a function of the average fluid velocity or the Reynolds number.

Table 2.1 Comparing the basic relations for electrical systems.

Electrical relation	Fluid mechanical relation
$\Delta V = IR$	$\Delta p = QR_H$
$\sum_{k=1}^{N} I_k = 0$	$\sum_{k=1}^{N} Q_k = 0$
Power: $I\Delta V = I^2 R$	Power: $Q\Delta p = Q^2 R_H$
$\mathbf{j} = \sigma \mathbf{E}$ (Ohm's law)	$\mathbf{u}_{\text{avg}} = -\frac{k}{\mu}\nabla p$ (Darcy's law)
$R = \dfrac{\sigma^{-1} L}{\pi a^2}$	$R_H = \dfrac{8\mu L}{\pi a^4}$

For electrical systems, ΔV is the potential difference, I is the current and R is the resistance. With single-phase laminar flow, Δp is the pressure difference, Q is the volumetric flow rate, and R_H is the hydrodynamic resistance. The power is used in the usual way as the energy/time that is dissipated, so equals I^2R for an electrical circuit and Q^2R_H for a fluid circuit. In the next to last row, we first give Ohm's law in the form relating the current density $\mathbf{j}$ to the electrical field $\mathbf{E}$ (here σ is the conductivity of the material) while the corresponding flow description relating the average velocity $\mathbf{u}_{\text{avg}}$ to the pressure gradient is known as Darcy's law; it is an approximation valid for low Reynolds number motions. For this description we also introduce the permeability k which, for example, depends on the detailed cross-sectional shape of a channel. In the last row of the table we consider a circular element of radius a, length L, and resistivity σ^{-1}, and a circular pipe of radius a and length L, which is filled with fluid of viscosity μ. The electrical resistance R has a similar though not identical functional form to the hydrodynamic resistance R_H.

2.3.2 Channels in Parallel or in Series

Consider N elements with individual resistances R_m, $m = 1,\ldots, N$. In the two simplest situations the resistances are either in series or in parallel, as is familiar from elementary courses:

- When the elements are placed in series, the effective resistance R_{eff} of the combination is $R_{\mathrm{eff}} = \sum_{m=1}^{N} R_m$.
- In contrast, when the elements are placed in parallel, the effective resistance is $R_{\mathrm{eff}} = 1 \big/ \sum_{m=1}^{N} R_m^{-1}$.

The same results apply to fluid circuits except that the hydrodynamic resistances R_H are used instead of the electrical resistances R.

2.3.3 Resistances in terms of Resistivities, Viscosities and Geometry

The electrical and fluid analogs extend to more microscopic pictures. The "local" version of Ohm's law relates the current density **j** (a vector) to the local electric field $\mathbf{j} = \sigma\mathbf{E}$, where σ is the conductivity and **E** is the local electric field. Thus, in a homogeneous circular wire of radius a and length L, the electrical resistance is $R = \sigma^{-1}L/(\pi a^2)$, where we have written this expression using the resistivity σ^{-1}.

As will be shown below, the corresponding hydrodynamic resistance, $R_H = \Delta p/Q$, which relates the pressure drop to the volumetric flow rate, has a similar form, $R_H = c\mu L/a^4$, where $c = 8/\pi$. The only significant difference in the fluid case is the dependence on the fourth power of the radius rather than the second power of the radius that occurs in the electrical case, which results from the distribution of velocities across the channel (see Fig. 2.5). The variation with the fourth power of the radius obviously will have significant influence when designs of significantly different radius are considered. As stressed in most elementary fluid mechanics treatments, the dependence of R_H on the fourth power of the radius means that a 10% change in radius produces approximately a 40% change in flow for a given pressure drop (see also Subsection 2.5.1).

Rectangular channels are common in microfluidic configurations simply because these are formed naturally by most existing fabrication methods. For a rectangular channel of height h and width w, with $h < w$, then to a good approximation $R_H = c_r\mu L/(h^3 w)$, where $c_r = 12$. In this case, the variation of the hydrodynamic resistance with the third power of the height is to be kept in mind when design is considered.

To be more specific, consider a water-filled rectangular microfluidic channel with height and width 20 μm. Using the above results, if the channel is 5 mm long and the pressure drop is 5 psi (≈ 1/3 atm), the hydrodynamic resistance is $R_H \approx \frac{3}{8}\cdot 10^{15}$ kg/m^4·sec. The corresponding flow rate is about 10^{-10} m^3/sec ≈ 0.1 μl/sec, with a typical velocity 25 cm/sec. The corresponding Reynolds number is $\mathcal{R} = \rho u \ell / \mu = 5$ which is much smaller than the Reynolds number (2000) for transition from laminar to turbulent flow. If the cross-sectional dimensions are reduced by a factor of 2, the hydrodynamic resistance increases by 2^4, the flow rate decreases by 2^4, the average velocity decreases by 2^2, and the Reynolds number decreases by 2^3.

2.4 BASIC FLUID DYNAMICS VIA THE GOVERNING DIFFERENTIAL EQUATIONS

2.4.1 Goals

In this subsection we provide a brief survey of the most important quantitative ideas from fluid dynamics, starting with the basic differential equations, that are needed for describing flows in small devices. Of course, there are many undergraduate and graduate texts available, and even specialized texts for microfluidics. We have selected mathematical material with the view towards a stepwise, yet terse, description of fundamental ideas that recur frequently when characterizing microflows. In particular, we use the familiar continuum description of materials and speak of the velocity, pressure, and stress fields. Our goal is to give the reader familiarity with the concepts and equations with which fluid motions are described, solved for using computational packages, and estimated using order-of-magnitude arguments.

The mathematical description is familiar both at the level of conservation laws for charge and current, and at the scale of transport processes such as thermal conduction or molecular diffusion. Here the basic equations needed for studying the fluid motion are given with emphasis on incompressible flows where the fluid density can be treated as a constant, which is applicable to the overwhelming majority of flows on small scales (even most cases when gases are considered). As is common in engineering and physics, dimensionless parameters play an important role in characterizing problems.

Some elementary flows representative of motions in microdevices are given in Section 2.5.

2.4.2 Continuum Descriptions

Although our subject will be micro- and nanofluidics, our basic starting point for estimates and calculations will be the familiar continuum equations from classical physics. Why should these equations apply at the "small scales" of microdevices? The answer is simply that the equations represent the average of what all the molecules in the fluid are doing, and, if we consider a liquid in a cubic volume with side of length just five times that of a molecule, then there are already more than 100 molecules: averages computed with statistics based on a hundred objects are usually pretty good.

In order to take this idea one step further and to appreciate why the continuum equations are quite reasonable, even at such small scales, imagine computing some average extensive material property f based on N molecules in a small measuring volume. We then expect $f \propto N$. Because of thermal fluctuations there will be variations δN of the number of molecules in the measuring volume with $\delta N \propto N^{1/2}$ (this is a standard result in statistical physics). Hence, the corresponding variation of the average extensive property is $\delta f \propto \delta N$. Next compare the fluctuations with the average value. We see that

$$\frac{\delta f}{f} = \frac{\delta N}{N} = N^{-1/2}. \tag{2.3}$$

Thus, for the fluctuations to be small compared to the mean value, i.e. $\delta f/f \ll 1$, we need the number of molecules to exceed $N > (f/\delta f)^2$. In particular, fluctuations are less than 10% of a measured property, when $\delta f/f \approx 0.1$ or by Eq. (2.3) when $N > 100$. This number of molecules can be found in a cube with only 4-5 molecules on an edge, which is a very small length scale indeed, and justifies the characterization given in the first paragraph of this subsection. For example, a water molecule has a typical dimension of only a few angstroms so five water molecules corresponds to a length smaller than 2 nm. We see that it should not be surprising that continuum calculations often provide reasonable estimates even down towards the molecular scale.

As a final way to introduce the continuum description, one can imagine measuring a fluid property (Fig. 2.7). We distinguish molecular and intermolecular dimensions, ℓ_{mol}, a length scale over which a measuring devices makes a local measurement of the system, ℓ_{avg}, and distances over which

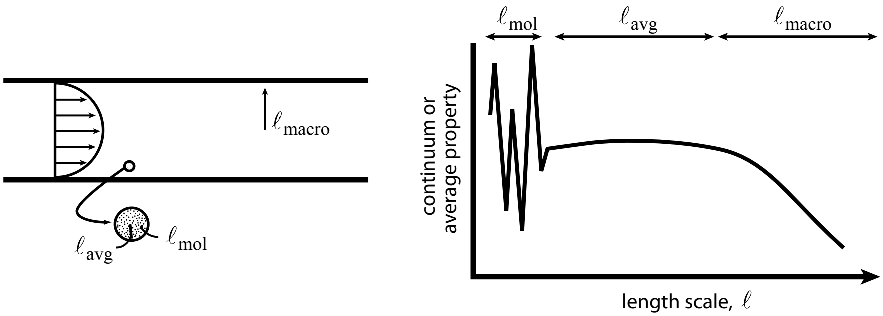

Figure 2.7 Different length scales in a flow when considering the continuum description.

variations are measured, ℓ_{macro}. The continuum variables measured on the scale ℓ_{avg} are such that $\ell_{mol} < \ell_{avg} < \ell_{macro}$.

In such a continuum description all field variables, such as velocity and pressure, are function of position $\mathbf{x} = (x, y, z)$ and time t, i.e. $\mathbf{u}(\mathbf{x}, t)$ and $p(\mathbf{x}, t)$. These variables are related by partial differential equations obtained from mass, momentum and energy considerations in the fluid.

2.4.3 The Continuity and Navier-Stokes Equations

2.4.3.1 Continuity - Local Mass Conservation

We know from courses in basic physics and chemistry that all materials are compressible to some degree (gases much more so than liquids or solids). Thus, for the most general case of motion of a material the velocity ($\mathbf{u}$) and the density (ρ) variations are linked, and this is expressed by the *continuity equation*:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{2.4}$$

where the gradient operator, $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$.

Here we will primarily be concerned with fluid motions that are essentially incompressible, as introduced qualitatively in Subsection 2.2.3. Effectively, this means that the pressure variations that accompany the flow create insignificant density changes ($\delta\rho$), i.e. $\delta\rho/\rho \ll 1$. Hence, we treat the density of the fluid as constant. Justification for this approximation in terms of features of the flow are given in Subsection 2.4.5. From Eq. (2.4) we then see that the velocity field for incompressible motions satisfies

$$\nabla \cdot \mathbf{u} = 0. \tag{2.5}$$

In Cartesian coordinates $\mathbf{u} = (u_x, u_y, u_z)$, the continuity equation is written $\partial u_x/\partial x + \partial u_y/\partial y + \partial u_z/\partial z = 0$, which may be considered a constraint on the allowed form of the velocity variations.

2.4.3.2 The Navier-Stokes Equation - a Linear Momentum Balance

Here we only consider the form of the linear momentum statement for incompressible fluid motions of a Newtonian fluid; a Newtonian fluid is where the stress is linearly related to the rate-of-strain in a generalization of the simplified discussion given in Subsection 2.2.2. In this case, the velocity vector $\mathbf{u}$ and pressure p are related by the *Navier-Stokes equation* (here we are assuming that the viscosity is constant as well)

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mu\nabla^2\mathbf{u} + \mathbf{f}_b, \tag{2.6}$$

where $\mathbf{f}_b$ represents body forces (per unit volume) that act on the fluid. The physical interpretation of this equation is that the left-hand side of Eq. (2.6) refers to the product of mass and acceleration (per unit volume) when following a fluid element (think of a dyed piece of fluid) moving along with the flow. The right-hand side of Eq. (2.6) corresponds to all of the forces acting on the fluid element. For a good physical discussion of the basic equations and a wide range of fluid flow phenomena, the reader is referred to [14].

The most common body force in most fluid systems is the gravitational force, though for most microfluidic (and smaller) flows this body force is generally negligible. Electric fields can also apply forces and we refer the reader to review articles [1, 2]. Note that in writing Eq. (2.6) we have also assumed that the fluid viscosity μ is constant. In some microfluidic flows, temperature variations occur and these can cause significant variations in μ, which can modify the velocity profiles.

Equations (2.5) and (2.6) represent four equations for the four unknowns of the velocity vector and the pressure. These equations are difficult to solve analytically for all but the simplest geometries, some of which fortunately arise in microfluidics (see Section 2.5). For example, when there is incompressible flow along a straight channel with uniform cross-sectional shape, then away from an entrance or an exit the nonlinear term in Eq. (2.6) $\mathbf{u}\cdot\nabla\mathbf{u} = \mathbf{0}$; the elimination of the nonlinear term is a significant simplification and many analytical results are available. Even then, it is worth noting that steady solutions for simple geometries are only realistic if the flow speeds are sufficiently small (as measured by the Reynolds number), since otherwise the steady solutions are unstable to small perturbations and evolve to turbulent states. It is also now feasible to solve numerically these equations (several commercial packages exist), even for complicated three-dimensional geometries, at least in those cases where the Reynolds number is small enough that there is a stable steady flow solution. Cases there the geometry is complicated and the flow is continually evolving in time are generally still subjects of research and code development.

In order to solve the Navier-Stokes equations, it is necessary to impose boundary conditions. Most commonly the velocity is zero on all stationary boundaries, which is referred to as the no-slip boundary condition, and the pressure, velocity distribution or flow rate are prescribed on the two ends of the device.

2.4.4 The Reynolds Number

A useful characterization of any flow is obtained by comparing the ratio of the typical inertia of the fluid motion – the left-hand side of Eq. (2.6) – to the viscous terms – the second term on the right-hand side. This ratio is referred to as the Reynolds number $\mathcal{R}$, which was introduced in Subsection 2.2.4. For flows with a typical speed u, in a geometry with typical channel height h (assumed to be smaller than the width), simple dimensional estimates applied to Eq. (2.6) yield

$$\mathcal{R} = \frac{O(\rho\mathbf{u}\cdot\nabla\mathbf{u})}{O(\mu\nabla^2\mathbf{u})} = \frac{\rho u^2/h}{\mu u/h^2} = \frac{\rho u h}{\mu}. \tag{2.7}$$

In many microfluidic applications the Reynolds number is not too large. For example, consider the flow in a device with $h = 100$ µm, $u = 1$ cm/sec, density comparable to water and a viscosity $\mu = 10\mu_{\text{water}}$. Then, $\mathcal{R} \approx 0.1$ and we should expect inertial influences not to be significant.

The above estimates in fact are rather conservative. In common microfluidic geometries there is a channel length $\ell \gg h$ over which the fluid flows, often changing the velocity at least in part because the channel height or width changes. In these steady flow cases a better estimate of the effective Reynolds number $\mathcal{R}_{\text{eff}}$ of the flow, based on the ratio of the inertial to viscous terms is

$$\mathcal{R}_{\text{eff}} = \frac{O(\rho \mathbf{u} \cdot \nabla \mathbf{u})}{O(\mu \nabla^2 \mathbf{u})} = \frac{\rho u^2/\ell}{\mu u/h^2} = \frac{\rho u h}{\mu}\frac{h}{\ell} = \mathcal{R}\frac{h}{\ell} \tag{2.8}$$

With $h/\ell = 1/50$, such an effective Reynolds number is small even if the usual Reynolds number $\mathcal{R} = 10$. This kind of estimate is useful in so-called "lubrication" configurations, which are flows that are nearly unidirectional and parallel to the bounding surfaces.

In those cases where the Reynolds number is small – formally we consider $\mathcal{R} \ll 1$, but in practice often $\mathcal{R} < 1$ is sufficient – we neglect inertial terms all together. If the forcing of the flow is steady then we simplify Eq. (2.6) to the Stokes equations:

$$0 = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f}_b \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0. \tag{2.9}$$

In this case the only length scales that matter are geometric. Equation (2.9) is the starting point for many detailed calculations of micro- and nanoflows. As a final remark for this viscously dominated flow case, the typical order of magnitude of pressures and stresses are $O(\mu u/\ell)$ where ℓ is the smallest dimension in the flow.

2.4.5 Brief Justification for the Incompressibility Assumption

In any given flow situation we have to ask under what conditions is $\delta\rho/\rho \ll 1$? The thermodynamic state of a fluid relates the density, pressure and temperature. For isothermal conditions, $\rho(p)$ and then $\delta\rho \approx \delta p(d\rho/dp) = dp/c^2$, where c is the (isothermal or adiabatic) sound speed, which is about 10^3 m/sec for small molecule liquids.

In high-speed flows commonly discussed in first-year physics or undergraduate fluid mechanics, the pressure-velocity relation is algebraic and given simply by the Bernoulli relation ($p + \tfrac{1}{2}\cdot\rho\mathbf{u}^2 = \text{constant}$, neglecting gravity and viscous effects), so that pressure changes vary quadratically with the typical flow speed. Hence, the assumption of incompressibility requires

$$\frac{\delta\rho}{\rho} \approx \frac{\delta p}{\rho}\frac{d\rho}{dp} = \frac{u^2}{c^2} = \mathcal{M}^2 \ll 1, \tag{2.10}$$

where the dimensionless ratio $\mathcal{M} = u/c$ is the *Mach number*. Even speeds of liquids about 10 m/sec, which are much faster than those found in typical small devices, have $\mathcal{M} \ll 1$, so density variations can be neglected.

Much more common in small devices are lower Reynolds number flows. In such flows the pressure changes occur owing to viscous effects in the liquid. In this case, in a device with typical dimension ℓ (the small geometric dimension that impacts the flow), $\delta p \approx \mu u/\ell$. Hence, the assumption of incompressibility requires

$$\frac{\delta\rho}{\rho} \approx \frac{\delta p}{\rho}\frac{d\rho}{dp} = \frac{\mu u}{\rho\ell c^2} = \frac{\mathcal{M}^2}{\mathcal{R}} \ll 1, \tag{2.11}$$

where $\mathcal{R} = \rho u \ell/\mu$ is the Reynolds number. Since most microfluidic flows occur with conditions such that $u < 1$ m/sec then $\mathcal{M}^2 \approx 10^{-6}$ which is much smaller than the usual Reynolds numbers of the flows and again density variations can be neglected.

In other words, in almost all cases of interest in micro- and nanofluidics, the fluid flows can be treated as incompressible. The only caveat worth mentioning, as discussed briefly above in Section 2.2, is the case of gas flows in long channels where the pressure change is sufficiently large that the gas density, which varies in proportion to the pressure, needs to be taken into account.

2.5 MODEL FLOWS

There are some simple geometries where the Navier-Stokes equations can be solved exactly. Since some of the solutions for these laminar flows are useful for estimating flow speeds, shear rates, or the effect of a change in geometry, we give a few results, largely without derivation, in the hope that readers will find them helpful.

2.5.1 Pressure-driven Flow in a Circular Tube

Consider steady pressure-driven flow in a circular tube of radius a. Using cylindrical coordinates (r, z), the velocity distribution $u(r)$ is *parabolic* (recall Fig. 2.5) with the form

$$u(r)=\frac{1}{4\mu}\left|\frac{dp}{dz}\right|\left(a^2-r^2\right), \tag{2.12}$$

where *dp*/*dz* is the axial pressure gradient, which is a constant for this flow. The corresponding flow rate Q (volume/time) is then

$$Q=2\pi\int_0^a u(r)r\,\mathrm{d}r=\frac{\pi a^4}{8\mu}\left|\frac{dp}{dz}\right|. \tag{2.13}$$

The dependence of the flow rate on the *fourth* power of the radius has a significant impact on flow in small systems. For the same pressure gradient, a factor of two reduction in radius produce a 16-fold reduction in flow rate. To appreciate further how this small effect has such a big influence, consider flow in small blood vessels: a 10% decrease in the radius (i.e. due to eating too much fatty foods!) produces more than a 40% decrease in the flow rate of blood.

The moral here is that small changes in geometry can have a big impact on flow in micro- and nanoscale devices. In fact, poor comparison of theory and experiment in some studies with small geometries led some authors to conclude that the Navier-Stokes equation did not apply. However, more careful studies have concluded that, in large part because of the significant effect of small changes in scale, very careful experiments are needed and then theory based on the Navier-Stokes equation and experiment are in excellent agreement [15].

It is also convenient to work in terms of an average velocity, $U=\frac{Q}{\pi a^2}=\frac{a^2}{8\mu}\left|\frac{dp}{dz}\right|$. Recall the definition of the Reynolds number: $\mathcal{R}=\frac{\rho U a}{\mu}$. For a fixed pressure gradient, $\mathcal{R}\propto a^3$ and so a factor of two change in radius produces a factor of eight change in Reynolds number.

Since the average velocity is proportional to the pressure gradient it is convenient to write the flow field in vector form as

$$\frac{\mu\mathbf{U}}{k}=-\nabla p \tag{2.14}$$

where k is known as the permeability (k has dimensions length^2). We thus see that the permeability of a channel of radius a is $k=a^2/8$. Equation (2.14) is referred to as Darcy's law (see Table 2.1); the basic steps at the origin of this linear relation between velocity and pressure gradient can be traced to the lack of an effect of inertia in these laminar flows. As a general point, the order-of-magnitude of the permeability of a uniform system is typically the square of the smallest dimension.

In Section 2.3 we discussed the hydrodynamic resistance of an element of a channel. With the above results we see that the hydrodynamic resistance, which is defined as $R_H = Q^{-1}\Delta p$, where $-dp/dz = \Delta p/L$, leads to $R_H = 8\mu L/(\pi a^4)$ (see Table 2.1).

The shear rate (often denoted $\dot{\gamma}$) at the wall is important to estimate. Using Eq. (2.12) we see that

$$\dot{\gamma}_{\text{wall}} = \left|\frac{du}{dr}\right|_{r=a} = \frac{4U}{a}. \tag{2.15}$$

The typical dimensional idea that the shear rate is the maximum centerline velocity divided by the distance from the centerline to the wall should typically be within a factor of two of the actual value for any geometry simply because cross-sectional velocities for most geometries tend to be parabolic.

We make two further remarks that involve physical and dimensional ideas.

- The basic results in this subsection could have been anticipated using only the characteristic scale for velocity established when making the problem statement non-dimensional. For example, consider the dimensions representative of Eq. (2.9). The magnitude of the velocity in laminar pipe flows is $u_c = a^2\Delta p/(\mu L)$ and so the typical volume flow rate is $Q \approx u_c a^2 \approx a^4\Delta p/(\mu L)$. Note that we have arrived at physical conclusions (e.g. $Q \propto a^4\Delta p$) regarding the order of magnitude of quantities without actually having solved the differential equation.
- The fluid motion arises as a balance between the pressure drop driving motion and the frictional (viscous) resistance from the bounding walls. This balance applies independent of the cross-sectional shape of the channel.

2.5.2 Pressure-driven Flow in a Rectangular Channel

Most microfluidic fabrication methods lead to channels with rectangular or nearly rectangular cross sections. Here we summarize the main flow features using the same concepts as described in the previous subsection. First, we consider the simplest case, which is a channel much wider than the height, $h \ll w$. For pressure-driven flow where a pressure drop Δp acts over a length L, the velocity profile across the channel is parabolic:

$$u(y) = \frac{\Delta p}{2\mu L}\left(\left(\frac{h}{2}\right)^2 - y^2\right). \tag{2.16}$$

The corresponding flow rate in a channel of width w is then approximately

$$Q = w\int_0^h u(y)\,\mathrm{d}y = \frac{wh^3\Delta p}{12\mu L}, \tag{2.17}$$

with an average velocity

$$U = \frac{h^2\Delta p}{12\mu L}. \tag{2.18}$$

The corresponding permeability is $k = h^2/12$ and the hydrodynamic resistance is $R_H = 12\mu L/(wh^3)$.

For the general case, the axial velocity in the cross-sectional (x, y) plane is known analytically in terms of a Fourier series:

$$\begin{aligned} u(x,y) &= \frac{\Delta p}{2\mu L}\left\{\left[\left(\frac{h}{2}\right)^2 - y^2\right] - \sum_{n=0}^{\infty} a_n \cos\left(\frac{\lambda_n y}{h/2}\right)\cosh\left(\frac{\lambda_n x}{h/2}\right)\right\} \\ \lambda_n &= \frac{(2n+1)\pi}{2}. \end{aligned} \tag{2.19}$$

The coefficients a_n follow from the no-slip boundary conditions at x = $\pm w/2$, which leads to $a_n = h^2(-1)^n/[(\lambda_n)^3 \cdot \cosh(\lambda_n w/h)]$. The corresponding flow rate is given by

$$\begin{aligned} Q &= 4\int_0^{w/2}\int_0^{h/2} u(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{wh^3\Delta p}{12\mu L}\left[1 - 6\left(\frac{h}{w}\right)\sum_{n=0}^{\infty}\lambda_n^{-5}\tanh\left(\frac{\lambda_n w}{h}\right)\right]. \end{aligned} \tag{2.20}$$

This equation can be evaluated for $0 \le h/w \le 1$ and in fact the result is nearly linear in the aspect ratio h/w. Consequently, a reasonable approximation is

$$\frac{12\mu LQ}{wh^3\Delta p} = 1 - \frac{6(2^5)}{\pi^5}\frac{h}{w} \quad \text{(less than 10\% error for } h/w \le 0.7\text{)}. \tag{2.21}$$

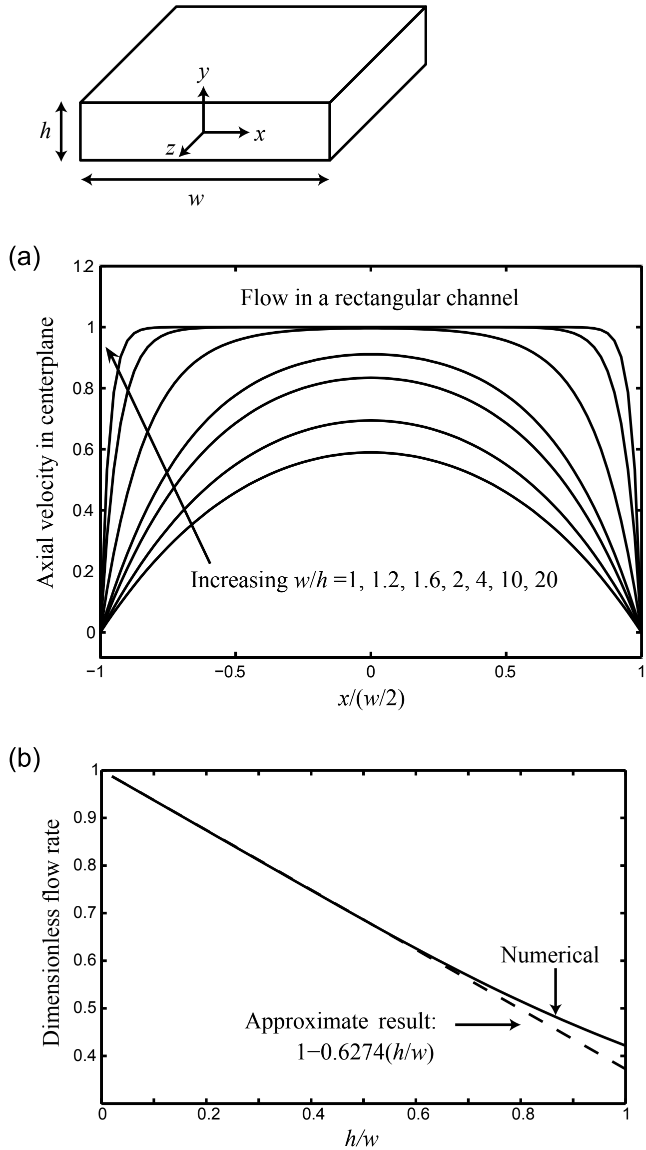

Figure 2.8 (a) Axial velocity distribution evaluated along the center-plane ($y = 0$), when viewed from above, in a channel with rectangular cross-section. The velocity has been scaled proportional to the average velocity. Note that as $w/h > 2$ the velocity distribution in the center of the channel becomes increasingly flat. (b) Dimensionless flow rate $(12\mu LQ)/(wh^3\Delta p)$ as a function of the channel aspect ratio, w/h.

To get a physical feel for the kinds of speeds and flow rates to expect, consider a water-filled rectangular channel of height $h = 20$ μm, width $w = 50$ μm, and length 1 cm. The viscosity of water is $\mu_{\text{water}} \approx 10^{-3}$ kg/m/sec. If a pressure drop of 1 atm (10^5 Pa) is applied then we find $U \approx 33$ cm/sec, $\mathcal{R} \approx 7$, and $Q = 33\times10^{-4}\,\text{cm}^3/\text{sec} = 0.3$ μl/sec.

There is an important feature of the velocity distribution to appreciate when considering rectangular cross sections with $h \ll w$. In the *shortest* dimensions, which is the height, the velocity distribution is parabolic. However, what is the distribution of the average velocity when viewed from above? In this case, across the width of the cross section the velocity distribution is largely unchanged, except for regions $O(h)$ near the walls where the fluid velocity has to change to satisfy the no-slip conditions. The evolution of the velocity distribution as the aspect ratio of a rectangular channel is changed is shown in Fig. 2.8: when the cross section is square the velocity distribution is very nearly parabolic and as the width increases then the velocity distribution across the width gets progressively flatter.

2.6 CONCLUSIONS AND OUTLOOK

The purpose of this chapter was to introduce some of the basic ideas and concepts of fluid dynamics as they frequently appear in applications involving micro- and nanofluidics. The basic equations, including the Reynolds number, were introduced and some elementary but useful order-of-magnitude estimates were given.

The fluid environment is finding many uses in biology, chemistry, and material science. The discussion here focused on pressure-driven flows but it is frequently the case that electrically driven motions are used as the device scale shrinks. In addition to simply transporting material, it is necessary to mix fluids, minimize dispersion, implement combinatorial approaches, sense and detect chemicals, etc. All of these topics are areas of active research and new applications appear regularly. Hopefully, the discussion given in this chapter will serve as a useful introduction for those who wish to go deeper into the physics of fluids in small-scale devices.

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