Binary Search Trees: Splay Trees

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Data Structures
Data Structures and Algorithms

Learning Objectives

- Implement a splay tree.
- Understand the ideas behind the runtime analysis.
- Know some other properties of splay tree runtimes.

Outline

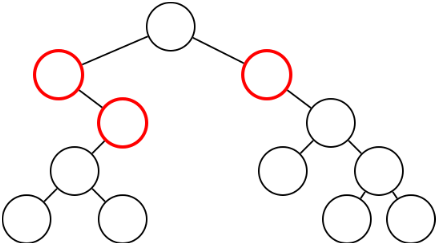
1. Non-Uniform Input Sequences
2. Analysis
3. Operations
4. Other Properties

Non Uniform Inputs

- Search for random elements $O(\log(n))$ best possible.

Non Uniform Inputs

- Search for random elements $O(\log(n))$ best possible.
- If some items more frequent than others, can do better putting frequent queries near root.



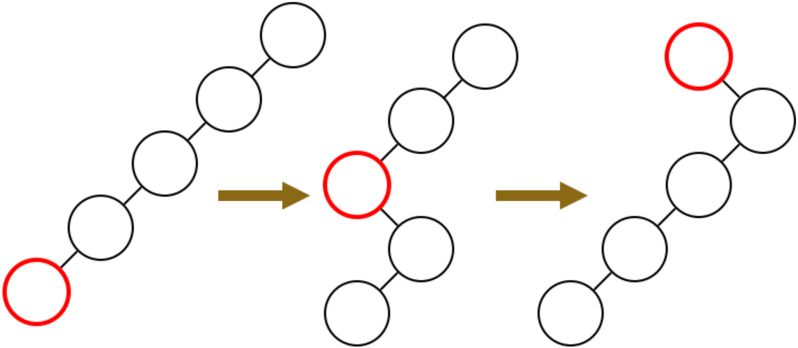
Idea

Bring query node to the root.

Simple Idea

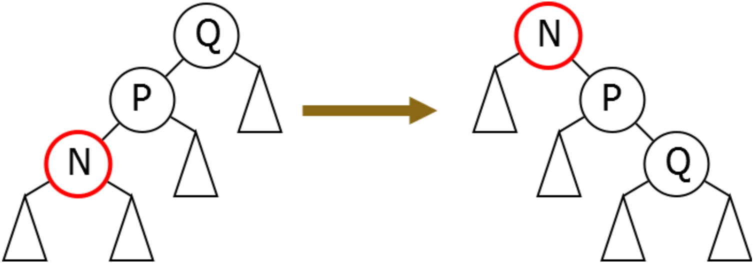
Just rotate to top.

Doesn't work.



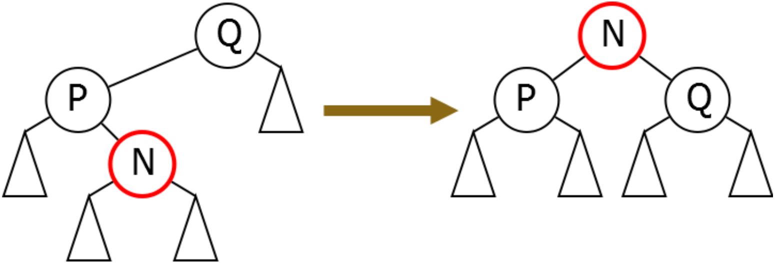
Modification

Zig-Zig



Modification

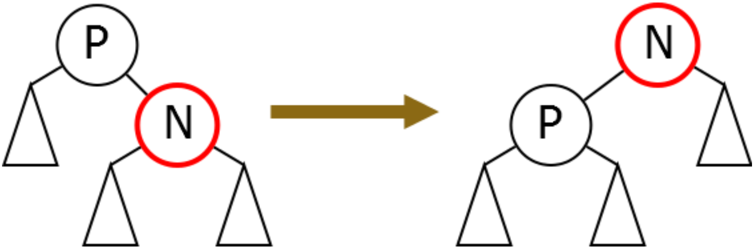
Zig-Zag



Modification

If just below root:

Zig



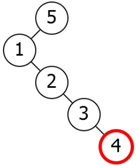
Splay

Splay(N)

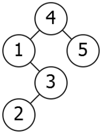
```
Determine proper case
Apply Zig-Zig, Zig-Zag, or Zig as
appropriate
if N.Parent ≠ null:
    Splay(N)
```

Problem

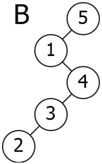
Which of the following is the result of splaying the highlighted node?



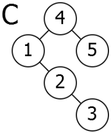
A



B



C



Problem

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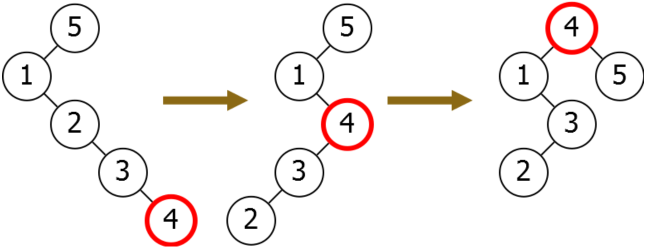

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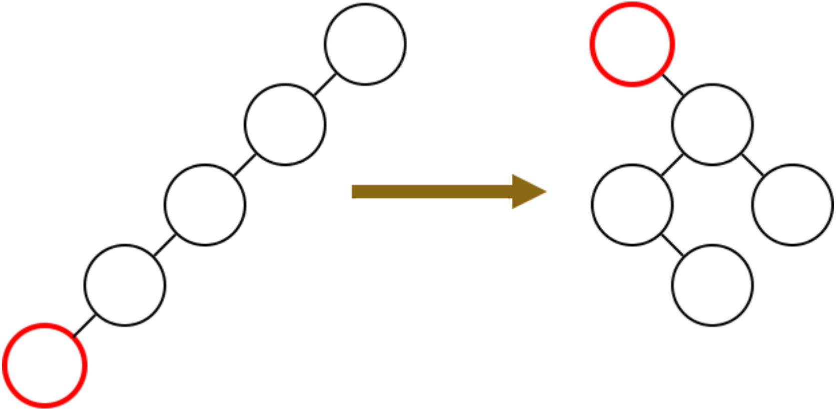
Sometimes Slow

Splay operation is sometimes slow:



Amortized Analysis

Need to amortize. Pick correct potential function.

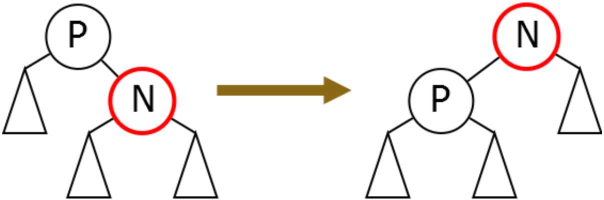
Rank

$R(N) = \log_2(\text{Size of subtree of } N)$.
Potential function

$$\Phi = \sum_N R(N).$$

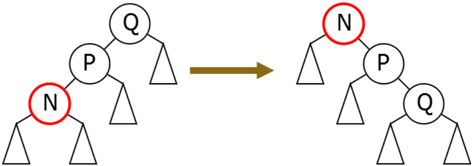
Zig Analysis

$$
\begin{aligned}
\Delta\Phi &= R'(N) + R'(P) - R(N) - R(P) \\
&= R'(P) - R(N) \\
&\le R'(N) - R(N).
\end{aligned}
$$



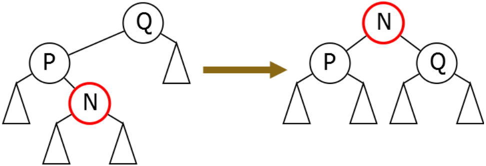
Zig-Zig Analysis

$$
\begin{aligned}
\Delta\Phi &= R'(N) + R'(P) + R'(Q) \\
&\quad - R(N) - R(P) - R(Q) \\
&= (R'(P) - R(P)) + (R'(Q) - R(N)) \\
&\leq 3(R'(N) - R(N)) - 2
\end{aligned}
$$



Zig-Zag Analysis

$$\begin{aligned}\Delta\Phi &= R'(N) + R'(P) + R'(Q) \\ &\quad - R(N) - R(P) - R(Q) \\ &= (R'(P) - R(P)) + (R'(Q) - R(N)) \\ &\leq 2(R'(N) - R(N)) - 2\end{aligned}$$



Total Change

$$\begin{aligned}
\Delta\Phi &\leq 3(R_k(N) - R_{k-1}(N)) - 2 \\
&\qquad + 3(R_{k-1}(N) - R_{k-2}(N)) - 2 + \cdots \\
&= 3(R'(N) - R(N)) - \Omega(\mathrm{Depth}(N)) \\
&= O(\log(n)) - \mathrm{Work}
\end{aligned}$$

Amortized cost of Find+Splay is $O(\log(n))$.

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Find

STFind(k, R)

$N \leftarrow$ `Find`(k, R)
`Splay`(N)
`return` N

Insert

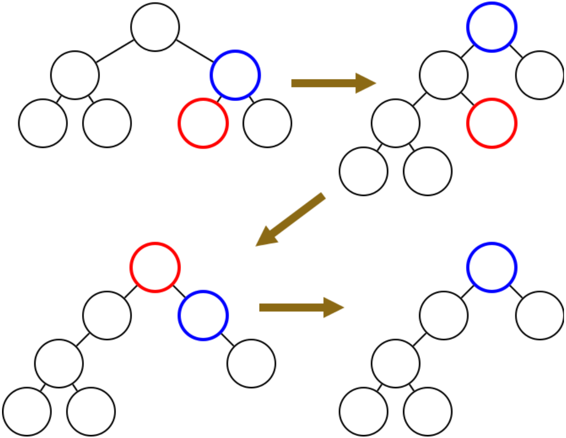
Insert, then splay

`STInsert`(k, R)

`Insert`(k, R)
`STFind`(k, R)

Delete

Bring N and successor to top. Deletes easily.



Delete

STDelete(N)

```
Splay(Next(N))
Splay(N)
Delete(N)
```

Split

$\texttt{STSplit}(R, x)$

$N \leftarrow \texttt{Find}(x, R)$
$\texttt{Splay}(N)$
split off appropriate subtree of N

Merge

$\texttt{STMerge}(R_1, R_2)$

$N \leftarrow \texttt{Find}(\infty, R_1)$
$\texttt{Splay}(N)$
$N.\texttt{Right} \leftarrow R_2$

Summary

Performs all operations in $O(\log(n))$ amortized time.

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Other Bounds

Splay trees have many other wonderful properties.

Weighted Nodes

If you assign weights so that

$$\sum_{N} \mathrm{wt}(N) = 1,$$

accessing N costs $O(\log(1/\mathrm{wt}(N)))$.

Dynamic Finger

Cost of accessing node $O(\log(D + 1))$ where D is distance between last access and current access.

Working Set Bound

Cost of accessing N is $O(\log(t+1))$ where t is time since N was last accessed.

Dynamic Optimality Conjecture

It is conjectured that for any sequence of binary search tree operations that a splay tree does at most a constant factor more work than the best search tree for that sequence.

Conclusion

Splay Trees

- Easy to implement.
- $O(\log(n))$ time per operation.
- Can be much better if queries have structure.