1. On September 4, 1751, Euler writes to his friend Goldbach [196]:

Ich bin neulich auf eine Betrachtung gefallen, welche mir nicht wenig merkwürdig vorkam. Dieselbe betrifft, auf wie vielerley Arten ein gegebenes polygonum durch Diagonallinien in triangula zerchnitten werden könne.	I have recently encountered a question, which appears to me rather noteworthy. It concerns the number of ways in which a given [convex] polygon can be decomposed into triangles by diagonal lines.

Euler then describes the problem (for an n–gon, i.e., $(n-2)$ triangles) and concludes:

Setze ich nun die Anzahl dieser verschiedenen Arten $= x$ *[...]. Hieraus habe ich nun den Schluss gemacht, dass generaliter sey* $$x = \frac{2.6.10.14....(4n-10)}{2.3.4.5....(n-1)}$$ *[...] Ueber die Progression der Zahlen* $1, 2, 5, 14, 42, 132,$ *etc. habe ich auch diese Eigenschaft angemerket, dass* $1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 +$ *etc.* $= \frac{1-2a-\sqrt{1-4a}}{2aa}$.	Let me now denote by x this number of ways [...]. I have then reached the conclusion that in all generality $$x = \frac{2.6.10.14....(4n-10)}{2.3.4.5....(n-1)}$$ [...] Regarding the progression of the numbers $1, 2, 5, 14, 42, 132$, and so on, I have also observed the following property: $1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 +$ etc. $= \frac{1-2a-\sqrt{1-4a}}{2aa}$.

Thus, as early as 1751, Euler knew the solution as well as the associated **generating function**. From his writing, it is however unclear whether he had found complete proofs.

2. In the course of the 1750s, Euler communicated the problem, together with initial elements of the counting sequence, to Segner, who writes in his publication [146] dated 1758: “The great Euler has benevolently communicated these numbers to me; the way in which he found them, and the law of their progression having remained hidden to me” [*“quos numeros mecum beneuolus communicauit summus Eulerus; modo, quo eos reperit, atque progressionis ordine, celatis”*]. Segner develops a recurrence approach to Catalan numbers. By a root decomposition analogous to ours, on p. 35, he proves (in our notation, for decompositions into n triangles)

$$T_n = \sum_{k=0}^{n-1} T_k T_{n-1-k}, \qquad T_0 = 1, \tag{4}$$

a recurrence by which the Catalan numbers can be computed to any desired order. (Segner's work was to be reviewed in [197], anonymously, but most probably, by Euler.)

3. During the 1830s, Liouville circulated the problem and wrote to Lamé, who answered the next day(!) with a proof [399] based on recurrences similar to (4) of the explicit expression:

$$T_n = \frac{1}{n+1}\binom{2n}{n}. \tag{5}$$

Interestingly enough, Lamé's three-page note [399] appeared in the 1838 issue of the *Journal de mathématiques pures et appliquées* (“Journal de Liouville”), immediately followed by a longer study by Catalan [106], who also observed that the T_n intervene in the number of ways of multiplying n numbers (this book, §I. 5.3, p. 73). Catalan would then return to these problems [107, 108], and the numbers $1, 1, 2, 5, 14, 42, \ldots$ eventually became known as the **Catalan numbers**. In [107], Catalan finally *proves* the validity of Euler's generating function:

$$T(z) := \sum_n T_n z^n = \frac{1-\sqrt{1-4z}}{2z}. \tag{6}$$

4. Nowadays, *symbolic methods* directly yield the generating function (6), from which both the recurrence (4) and the explicit form (5) follow easily; see pp. 6 and 35.

Figure I.2. The prehistory of Catalan numbers.