Stages of Understanding in Mathematics

1   Stages

	Meaning	How to Study
Facts	Basic terminology, facts, principles. You must be able to remember.	Make a list of formula and words that are critical to a topic together with their definition.
Methods	Processes and algorithms. You must be able to do.	Make a list of all the processes that are needed and in addition a list of basic problems that use these methods.
Understanding	Understand meaning. You must be able to explain.	Explain the above concepts to other people. The explanation happens in your mind but you must have a actual person in mind when you do this.
Application	Problem solving You must be able to apply.	Do not give up on a problem but use whatever resources you have to solve it. When you have solved it document your solution.
Analysis	Reflection and making inferences. You must be able to learn from mistakes.	You must discover the reason for not being able to complete a problem or the reason why you had difficulty with a problem and you must learn from this.
Problem Solving	Solve non-routine problems. You must be able to solve problems never seen before.	You must consider more complex problems. You must consider possible strategies and evaluate them. This must happen as a dialogue with yourself.

2   Facts

You can test yourself by covering up some columns and rewriting.

2.1   Example

Word	Topics	Formula	Meaning
Different of Squares	Algebra	$a^2 - b^2 = (a+b)(a-b)$	
Perfect Square	Algebra	$(a+b)^2 = a^2 + 2ab + b^2$	
Surds	Algebra	$\sqrt{72}$	The square root. Inverse of squaring, what do you square to make 72.
	Algebra	$\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$	
	Algebra	$\sqrt{a^2} = (\sqrt{a})^2 = a$	
Quadratic Equation	Algebra	$ax^2 + bx + c$	Shape of a parabola. Has two roots.
	Algebra	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	To solve quadratic equations.
Discriminant	Algebra	$\Delta = b^2 - 4ac$	$\Delta < 0$ no roots, $\Delta = 0$ one repeated root, $\Delta > 0$ two different roots.
Exponents	Algebra	x^a	a is the exponent.
	Algebra	$x^a \times x^b = x^{a+b}$	Rules for Exponents.
	Algebra	$x^a \div x^b = x^{a-b}$	
	Algebra	$(x^a)^b = x^{ab}$	
	Algebra	$(xy)^a = x^a y^b$	
	Algebra	$x^0 = 1$	
	Algebra	$x^{-a} = 1 \div x^a$	
	Algebra	$x^{1/a} = \sqrt[a]{x}$	
Logarithms	Algebra	$\log_a x = b : a^b = x$	Rules for logs
	Algebra	$\log(a \times b) = \log a + \log b$	
	Algebra	$\log(a \div b) = \log a - \log b$	
	Algebra	$\log x^n = n \log x$	
	Algebra	$\log 1 = 0$	

3 Methods

3.1 Example

Method	Topics	Example	Meaning
Linear Equation	Algebra	$\frac{2}{3}x + 13 = \frac{1}{2}(3x - 4)$	Multiply through by the common denominator $4x + 78 = 3(3x - 4)$,
			Expand brackets $4x + 78 = 9x - 12$,
			Collect like terms $90 = 5x$
			Divide by x-coefficient $x = 18$.
Factorize	Algebra	$x^2 + 13x + 36$	Make a list of all factors of 36. $\{36 \times 1, 18 \times 2, 12 \times 3, 9 \times 4, 6 \times 6\}$
			Find two that add to make 13.
			$(x + 9)(x + 4)$
Factorize	Algebra	$x^2 + 16x - 36$	Make a list of all factors of 36. $\{36 \times 1, 18 \times 2, 12 \times 3, 9 \times 4, 6 \times 6\}$
			Find two that subtract to make 16.
			$(x + 18)(x - 2)$
Factorize	Algebra	$6x^2 + 17x + 12$	Multiply the first and last number together $6 \times 12 = 72$
			List of all factors of 72.
			Find two that add to make 17. $9 + 8 = 17$
			Expand the 17 out in this way, $6x^2 + 17x + 12 = 6x^2 + 8x + 9x + 12$
			Factorize each part, $2x(3x + 4) + 3(3x + 4) = (2x + 3)(3x + 4)$
Solve	Algebra	$3^x = 5$	Take logs of both sides, $\log 3^x = x \log 3 = \log 5$, $x = \log 5 \div \log 3$
Surd Equations	Algebra	$\sqrt{2x - 1} + 8 = x$	Get the square-root on one side $\sqrt{2x - 1} = x - 8$
			Square both sides $2x - 1 = (x - 8)^2 = x^2 - 16x + 64$
			Collect terms and factorize $x^2 - 18x + 65 = (x - 5)(x - 13)$

4   Understanding

What does *understanding* mean. It means that you can see the connections between the facts and methods that you have remembered.

To prove that you understand something then you must be able to *demonstrate* your understanding (next section).

IB wants you to demonstrate your understanding by using Mathematics to solve problems.

To prove that you understand something, you must be able to explain it to someone else.

Note that when someone explains something to you, it only enables you to use that information.

Example - Discriminant

Writing down the formula and symbol for discriminant:- is a fact.
Using the discriminant to state the number of roots of a quadratic equation:- is a method
Explaining why the discriminant tells you the number of roots of a quadratic equation:- is understanding.

Example - Equations

Solve the equation $\sqrt{x + 2} = x - 4$.
To do this we square both sides (method).
We use $(\sqrt{a})^2 = a$ (fact) and $(a - b)^2 = a^2 - 2ab + b^2$ (fact) and get $x + 2 = x^2 - 8x + 16$
Then we collect like terms (method) to get $0 = x^2 - 9x + 14$
Then we factorize (method) and get $0 = (x - 7)(x - 2)$
From this we get two answers $x = 2$ and $x = 7$
EXPLAIN why we get two answers where the equation only has one.

5 Application

Here we are applying the Mathematics we know to straight-forward problems.

One important thing that must be learnt at this stage is to recognize the nature of the problem. Are you going to solve an equation, expand brackets, factorize, differentiate, integrate, use a formula, use your calculator, etc.

Your notes should contain everything. Therefore if you come across a new concept, or a new way of doing something then you must add that to your notes. Your notes must be organized so that each part references other parts. When you add something to one part of your notes then you may have to add it, or references to it, to other parts as well.

Example - Discriminant

Consider the problem:- *find the value of m for which the line $y = mx - 9$ is tangent to $y = x^2$.*

You must recognizing that we need to solve these equations simultaneously.
You must realize that solving these two equations will give a quadratic.
You must understand that in this situation there is only one solution.
You must recognize that the discriminant of a quadratic equation tells us the number of roots.

Consider the problem:- *find the value of k so that $x^2 - kx + 6$ is always positive.* You must realize that the equation is a parabola.
You must be able to visualize the shape of a parabola.
You must realize that in this situation the parabola does not go beneath the x-axis.
You must realize that this means that it has no roots.
You must realize that the discriminant tells the number of roots.

6 Analysis

You must be able to learn from mistakes. The best way of doing this is to take an examination or test and analyze your performance. What you do under time pressure is different from how you normally behave.

Here is a series of possible problems areas that could have occurred.

1	Did not understand the question.	You should have looked for command words in the instructions. These words are asking you to do something. Work backwards from them
2	Did not know where to begin.	You did not think of every possible way of solving the problem. See next section on Problem Solving that gives you possible things to think about.
3	Began but the method used was not appropriate to the problem. Wasted a lot of time doing this and not realizing that it was not working.	You should have realized this early and gone onto something else. See section 8 on Monitoring Execution.
4	Started correctly but got stuck and could not proceed. Did not know what to do next.	You rushed into the problem without fully thinking out the solution. You should have evaluated your strategy before you started.
5	Thought that I had completed the problem but did not answer all the question.	You did not re-reading the question again to ensure that you had correctly answered.
6	Did the complete problem but made errors in your working.	You did not maintain an awareness of all places that you could have made mistakes. See section 8 on Monitoring Accuracy.

You must realize where your problem is before you can make improvements.

7   Problem Solving

Solve non-routine problems. You must be able to solve problems never seen before. Consider more complex problems - write down possible strategies without actually using them. Evaluate these strategies.

The 4 stages of problem solving are:

1. Understand the problem

2. Come up with a strategy

3. Implement the strategy

4. Reflect on your method

Problem Solving Strategies

Analogy	Have I ever solved a problem similar to this one?
Working backward	Can I start with the goal and work backwards to something I already know?
Model	Can I write an equation for this problem and solve that?
Stages	Can I break the problem into a sequence of steps?
Draw a Figure	Can I draw a diagram of the problem?
Formula	Is there a formula that will help me in my solution?
Symmetry	Is it possible that symmetry will help me in my solution?
Divide and Conquer	Can I break the problem into special cases and solve them
Information	Have I used every piece of information given in the problem?
Listing	Can I list all the possibilities?
Variation	Can I change the problem into a different one that I can solve?

8 Metacognition

This is a fancy word that means *Thinking about Thinking*. It is not sufficient to be able to solve problems because of mistakes that could creep into your work. As part of your mind is engaged in solving the problem and writing down the answer, another part of your mind must be aware of the process. Sometimes students get too engaged in solving the problem that they are not aware of anything else.

So there are three areas of concern:

Clarity You must keep it in mind constantly that your work is going to be read and assessed by another person.

- Always work down the page.
- Cross out and replace - do not overwrite
- Start each question on a new page
- Write big
- Leave spaces in your working.
- Recognize when some of your numerals and letters look the same. For example for some students a 1 looks like a 7.

Accuracy It is very easy to make simple arithmetic errors and you must keep vigilant to avoid this.

- Does this answer make sense?
- Do not re-read the line, recalculate on separate paper and compare.

Execution This is really important because many students waste time by doing some working that does not help the solution

- Is the next stage obvious?
- Are the numbers beginning to become complicated?
- Is this method taking more time than it should (1 min = 1 mark)?

9 Study Progress

There are the stages that a good student should work through in their study process. Some students will not go far in the process. It would be wrong and bad for you if you started too high up the list. For example practicing exam questions is a waste of time if you have no knowledge of basic facts.

1. Start with your notes. Isolate the facts and methods.

2. Rewrite your notes in concise, annotated form for ease of understanding. Put everything that you need for one topic on one page.

3. Practice basic skills. At this stage you are focusing on one topic.

4. Review your notes.

5. Practice exam questions. Any question that you can't do, goes into a *Can't Do List*. Then add to your notes and practice other similar questions.

6. Review your notes.

7. Do exam questions under exam conditions. Suggest that your do half a paper (1 hour for this). You should reserve the last two available IB exams for this.

8. Review your notes.

9. Evaluate progress. What gaps were revealed? Now you may have to go back to stage 2 and add extra information to your notes, then stage 3.

10. Review your notes.

11. Make up one page containing all the information that is in the course.