MEK 4300
MANDATORY ASSIGNMENT 1

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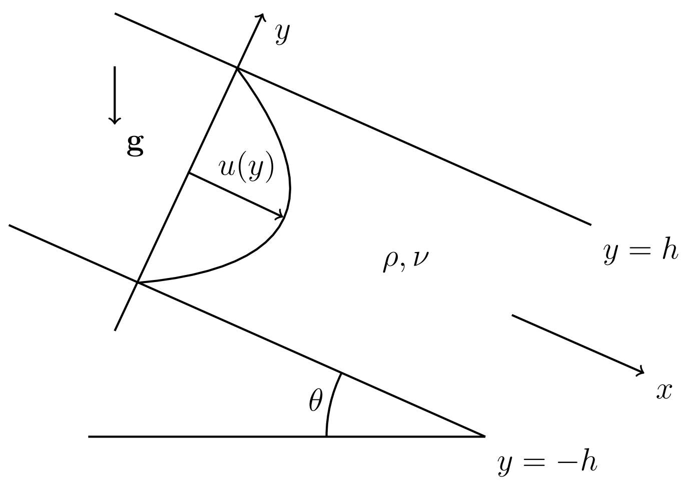
Problem 1 - Poiseuille flow between two planes

In this problem we will consider a variant of steady laminar viscous Poiseuille flow between two paralell planes, as shown in the figure below. The situation consists of two planes which make an angle θ with the horizonal. The flow is driven by gravity, and we will be applying no-slip conditions at the planes where the upper plane is located at $y = h$ and the lower at $y = -h$. We assume the motion is two-dimensional, withthe fluid velocity tangential to the plates, thus only depending on the y-coordinate which is orthogonal to the plates; $u = u(y)$. The fluid has kinematic viscosity coefficient ν, dynamic viscosity coefficient μ and density ρ. The acceleration of gravity is g and we assume the pressure is constant.



a) Initially we need kinematic boundary conditions at $y = \pm h$. Since we are applying no-slip conditions on the upper and lower plane, the kinematic boundary conditions becomes

$$u(h) = 0, \qquad u(-h) = 0$$

b) We now consider the momentum equation, Navier-Stokes equation, in the x-direction. The full equation is given by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + g_x + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

This equation can be simplified for steady flow. First of all there is no fluid flow in the y-direction, hence $v = 0$. Steady flow implies the flow is time independent, thus $\partial u/\partial t = 0$. Since we assume the pressure to be constant, the pressure gradient equals zero, $\partial p/\partial x = 0$. The second derivative term with respect to x becomes zero since the velocity profile is only dependent on the y-direction, $u = u(y)$. With some trigonometry we can express the gravity term in terms of θ, $g_x = g \sin\theta$. Hence the fully simplified momentum equation becomes

$$0 = \rho g \sin\theta + \mu \frac{\partial^2 u}{\partial y^2} \tag{1}$$

c) The velocity profile can be found by moving the first term in equation (1) to the left hand side, dividing by μ and integrating with respect to y. Equivalently,

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\rho g}{\mu} \sin\theta$$

Integration once produces

$$\frac{\partial u}{\partial y} = -\frac{\rho g}{\mu} \sin\theta y + C, \qquad C \in \mathbb{R}$$

A second integration reveals the general solution,

$$u(y) = -\frac{\rho g}{2\mu} \sin\theta y^2 + Cy + D, \qquad C, D \in \mathbb{R}$$

By applying the boundary conditions $u(h) = u(-h) = 0$ we can find the unknown constants C and D,

$$\begin{aligned} u(h) &= -\frac{\rho g}{2\mu} \sin\theta h^2 + Ch + D = 0, \\ u(-h) &= -\frac{\rho g}{2\mu} \sin\theta (-h)^2 - Ch + D = 0 \end{aligned}$$

Adding the two equations lets us express the constant D,

$$-\frac{\rho g}{\mu} \sin\theta h^2 + 2D = 0 \quad \Rightarrow \quad D = \frac{\rho g}{2\mu} \sin\theta h^2$$

This term can now be inserted into one fo the boundary conditions to find C,

$$u(h) = -\frac{\rho g}{2\mu} \sin\theta h^2 + Ch + \frac{\rho g}{2\mu} \sin\theta h^2 = Ch = 0$$

Which implies that $C = 0$. Thus the velocity profile is expressed by

$$u(y) = \frac{\rho g}{2\mu} \sin\theta (h^2 - y^2)$$

To obtain the volume flux we need to compute the following integral,

$$Q = \int_{-h}^{h} u(y)\, dy$$

Inserting the velocity profile into the expression and integrating produces the following volume flux,

$$Q = \int_{-h}^{h} \frac{\rho g}{2\mu} \sin\theta (h^2 - y^2)\, dy = \frac{\rho g}{2\mu} \sin\theta \left[h^2 y - \frac{y^3}{3} \right]_{-h}^{h} = \frac{2}{3} \frac{\rho g}{\mu} \sin\theta h^3$$

d) We define the wall shear stress as the quantity

$$\tau_w = \mu \frac{\partial u}{\partial y} \bigg|_{y=\pm h}$$

At the upper plane this becomes

$$\tau_w = \mu \frac{\partial u}{\partial y} \bigg|_{y=h} = \mu \left(-\frac{\rho g}{\mu} \sin\theta h \right) = -\rho g \sin\theta h$$

Similarly at the lower plane, the wall shear stress becomes

$$\tau_w = \mu \frac{\partial u}{\partial y} \bigg|_{y=-h} = \mu \left(\frac{\rho g}{\mu} \sin\theta h \right) = \rho g \sin\theta h$$

These quantities can be used to find the shear velocity which is defined as

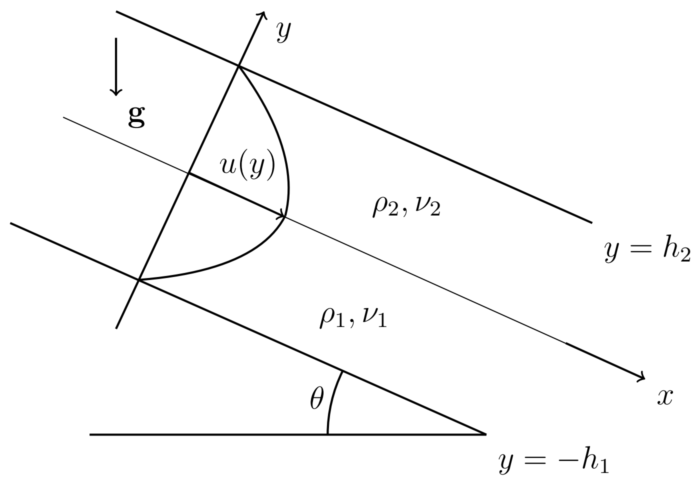
$$u_\tau = \sqrt{\frac{\tau}{\rho}}$$

Consequently the shear velocity becomes

$$u_\tau = \sqrt{\frac{\rho g h \sin\theta}{\rho}} = \sqrt{g h \sin\theta}$$

Problem 2 - Poiseuille flow with two fluids between two planes

We will now consider a variant of Problem 1. The gap between the plates are now occupied with two thin fluid layers, as shown in the figure below. The lower layer has thickness h_1, kinematic viscosity ν_1, dynamic viscosity μ_1 and density ρ_1. The similar quantities of the upper layer are nu_2, μ_2 and ρ_2.



a) Initially we need kinematic boundary conditions at $y = -h_1$ and $y = h_2$. Since we are applying no-slip conditions on the upper and lower plane, the kinematic boundary conditions becomes

$$u_1(-h_1) = 0, \qquad u_2(h_2) = 0$$

b) We also require a kinematic boundary condition at the interface between the layers, at $y = 0$. At this height we expect the fluids to have the same velocity, thus

$$u_1(0) = u_2(0)$$

c) The final boundary condition is also located at the interface between the layers, this being a dynamic boundary condition involving the shear stress. In addition to the velocity being equal, we expect the shear stress at the interface to be the same,

$$\tau_1 = \mu_1 \frac{\partial u_1}{\partial y}\bigg|_{y=0} = \mu_2 \frac{\partial u_2}{\partial y}\bigg|_{y=0} = \tau_2$$

d) We now consider the momentum equation, Navier-Stokes equation, in the x-direction. The full equation is given by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + g_x + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

This equation can be simplified for steady flow. First of all there is no fluid flow in the y-direction, hence $v = 0$. Steady flow implies the flow is time independent, thus $\partial u/\partial t = 0$. Since we assume the pressure to be constant, the pressure gradient equals zero, $\partial p/\partial x = 0$. The second derivative term with respect to x becomes zero since the velocity profile is only dependent on the y-direction, $u = u(y)$. With some trigonometry we can express the gravity term in terms of θ, $g_x = g\sin\theta$. Since this is valid for both fluids, the fully simplified momentum equation becomes

$$0 = \rho_{1,2} g \sin\theta + \mu_{1,2}\frac{\partial^2 u_{1,2}}{\partial y^2} \tag{2}$$

d) The velocity profile can be found by moving the first term in equation (2) to the left hand side, dividing by μ and integrating with respect to y. Equivalently,

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\rho g}{\mu}\sin\theta$$

Integration once produces

$$\frac{\partial u}{\partial y} = -\frac{\rho g}{\mu}\sin\theta y + C, \qquad C \in \mathbb{R}$$

A second integration reveals the general solution for both fluids,

$$u_1(y) = -\frac{\rho_1 g}{2\mu_1}\sin\theta y^2 + C_1 y + D_1, \qquad C_1, D_1 \in \mathbb{R}$$
$$u_2(y) = -\frac{\rho_1 g}{2\mu_2}\sin\theta y^2 + C_2 y + D_2, \qquad C_2, D_2 \in \mathbb{R}$$

By applying the boundary conditions we get the following relations,

$$u_1(-h_1) = -\frac{\rho_1 g}{2\mu_1}\sin\theta h_1^2 - C_1 h_1 + D_1 = 0 \tag{3}$$

$$u_2(h_2) = -\frac{\rho_2 g}{2\mu_2}\sin\theta h_1^2 + C_2 h_2 + D_2 = 0 \tag{4}$$

$$u_1(0) = D_1 = D_2 = u_2(0) \tag{5}$$

$$\tau_1 = \mu_1 C_1 = \mu_2 C_2 = \tau_2 \tag{6}$$

Thus we get the following relations

$$D_1 = D_2, \qquad C_1 = \frac{\mu_2}{\mu_1}C_2$$

Inserting these relations into the first and second equation, we can find expressions for $D_1 = D_2$ and C_2 with the help of some algebra. Combining the first and second equation we get the following expression for C_2 and consequently C_1,

$$C_2\left(h_2+\frac{\mu_2}{\mu_1}h_1\right)=g\sin\theta\left(\frac{\rho_2h_2^2}{2\mu_2}-\frac{\rho_1h_1^2}{2\mu_1}\right)$$
$$\Rightarrow\quad C_2=\frac{\mu_1 g\sin\theta}{\mu_1h_2+\mu_2h_1}\left(\frac{\rho_2h_2^2}{2\mu_2}-\frac{\rho_1h_1^2}{2\mu_1}\right)$$
$$C_1=\frac{\mu_2 g\sin\theta}{\mu_1h_2+\mu_2h_1}\left(\frac{\rho_2h_2^2}{2\mu_2}-\frac{\rho_1h_1^2}{2\mu_1}\right)$$

Inserting C_1 into the first equation produces an expression for $D_1 = D_2$,

$$D_1=D_2=g\sin\theta\left(\frac{\rho_1h_1^2}{2\mu_1}+\frac{\mu_2h_1}{\mu_1h_2+\mu_2h_1}\left(\frac{\rho_2h_2^2}{2\mu_2}-\frac{\rho_1h_1^2}{2\mu_1}\right)\right)$$

Inserting these expressions into the general solutions reveals the two exact solutions of the velocity profile,

$$u_1(y)=g\sin\theta\left(\frac{\rho_1}{2\mu_1}(h_1^2-y^2)+\frac{\mu_2}{\mu_1h_2+\mu_2h_1}\left(\frac{\rho_2h_2^2}{2\mu_2}-\frac{\rho_1h_1^2}{2\mu_1}\right)(h_1+y)\right)$$

$$u_2(y)=g\sin\theta\left(\frac{\rho_1h_1^2}{2\mu_1}-\frac{\rho_2y^2}{2\mu_2}+\frac{1}{\mu_1h_2+\mu_2h_1}\left(\frac{\rho_2h_2^2}{2\mu_2}-\frac{\rho_1h_1^2}{2\mu_1}\right)(\mu_2h_1+\mu_1y)\right)$$

Furthermore, the volume flux can be obtained by computing the following integral,

$$Q=\int u(y)\ dy$$

For the lower layer, the integration limits range from $-h_1$ to 0, while the upper layer is integrated from 0 to h_2. Performing this integration in the lower layer yields the following volume flux,

$$Q_1=\int_{-h_1}^{0}g\sin\theta\left(\frac{\rho_1}{2\mu_1}(h_1^2-y^2)+\frac{\mu_2}{\mu_1h_2+\mu_2h_1}\left(\frac{\rho_2h_2^2}{2\mu_2}-\frac{\rho_1h_1^2}{2\mu_1}\right)(h_1+y)\right)\ dy$$
$$=g\sin\theta\left(\frac{\rho_1h_1^3}{3\mu_1}+\frac{\mu_2}{\mu_1h_2+\mu_2h_1}\left(\frac{\rho_2h_2^2}{2\mu_2}-\frac{\rho_1h_1^2}{2\mu_1}\right)\frac{h_1^2}{2}\right)$$

Similarly, the volume flux in the upper layer becomes

$$Q_1=\int_{0}^{h_2}g\sin\theta\left(\frac{\rho_1h_1^2}{2\mu_1}-\frac{\rho_2y^2}{2\mu_2}+\frac{1}{\mu_1h_2+\mu_2h_1}\left(\frac{\rho_2h_2^2}{2\mu_2}-\frac{\rho_1h_1^2}{2\mu_1}\right)(\mu_2h_1+\mu_1y)\right)\ dy$$
$$=g\sin\theta\left(\frac{\rho_1h_1^2h_2}{2\mu_1}-\frac{\rho_2h_2^3}{6\mu_2}+\frac{1}{\mu_1h_2+\mu_2h_1}\left(\frac{\rho_2h_2^2}{2\mu_2}-\frac{\rho_1h_1^2}{2\mu_1}\right)(\mu_2h_1h_2+\mu_1\frac{h_2^2}{2})\right)$$

f) We define the shear stress as the quantity

$$\tau = \mu \frac{\partial u}{\partial y}$$

At the upper plane this becomes

$$\tau_w = \mu_2 \frac{\partial u_2}{\partial y}\bigg|_{y=h_2} = -g\mu_2 \sin\theta \left(\frac{\rho_2 h_2}{\mu_2} - \frac{\mu_1}{\mu_1 h_2 + \mu_2 h_1} \left(\frac{\rho_2 h_2^2}{2\mu_2} - \frac{\rho_1 h_1^2}{2\mu_1} \right) \right)$$

$$= g \sin\theta \left(\frac{\mu_1 \mu_2}{\mu_1 h_2 + \mu_2 h_1} \left(\frac{\rho_2 h_2^2}{2\mu_2} - \frac{\rho_1 h_1^2}{2\mu_1} \right) - \rho_2 h_2 \right)$$

Similarly at the lower plane, the wall shear stress becomes

$$\tau_w = \mu_1 \frac{\partial u_1}{\partial y}\bigg|_{y=-h_1} = g\mu_1 \sin\theta \left(\frac{\rho_1 h_1}{\mu_1} + \frac{\mu_2}{\mu_1 h_2 + \mu_2 h_1} \left(\frac{\rho_2 h_2^2}{2\mu_2} - \frac{\rho_1 h_1^2}{2\mu_1} \right) \right)$$

$$= g \sin\theta \left(\rho_1 h_1 + \frac{\mu_1 \mu_2}{\mu_1 h_2 + \mu_2 h_1} \left(\frac{\rho_2 h_2^2}{2\mu_2} - \frac{\rho_1 h_1^2}{2\mu_1} \right) \right)$$

We can also calculate the shear stress at the interface, $y = 0$,

$$\tau_1 = \mu_1 \frac{\partial u_1}{\partial y}\bigg|_{y=0} = g\mu_1 \sin\theta \left(\frac{\mu_2}{\mu_1 h_2 + \mu_2 h_1} \left(\frac{\rho_2 h_2^2}{2\mu_2} - \frac{\rho_1 h_1^2}{2\mu_1} \right) \right)$$

$$= g \sin\theta \frac{\mu_1 \mu_2}{\mu_1 h_2 + \mu_2 h_1} \left(\frac{\rho_2 h_2^2}{2\mu_2} - \frac{\rho_1 h_1^2}{2\mu_1} \right) = \mu_2 \frac{\partial u_2}{\partial y}\bigg|_{y=0} = \tau_2$$

These quantities can be used to find the shear velocity which is defined as

$$u_\tau = \sqrt{\frac{\tau}{\rho}}$$

Consequently the shear velocity at $y = h_2$ becomes

$$u_{\tau,2} = \sqrt{\frac{g \sin\theta}{\rho_2} \left(\frac{\mu_1 \mu_2}{\mu_1 h_2 + \mu_2 h_1} \left(\frac{\rho_2 h_2^2}{2\mu_2} - \frac{\rho_1 h_1^2}{2\mu_1} \right) - \rho_2 h_2 \right)}$$

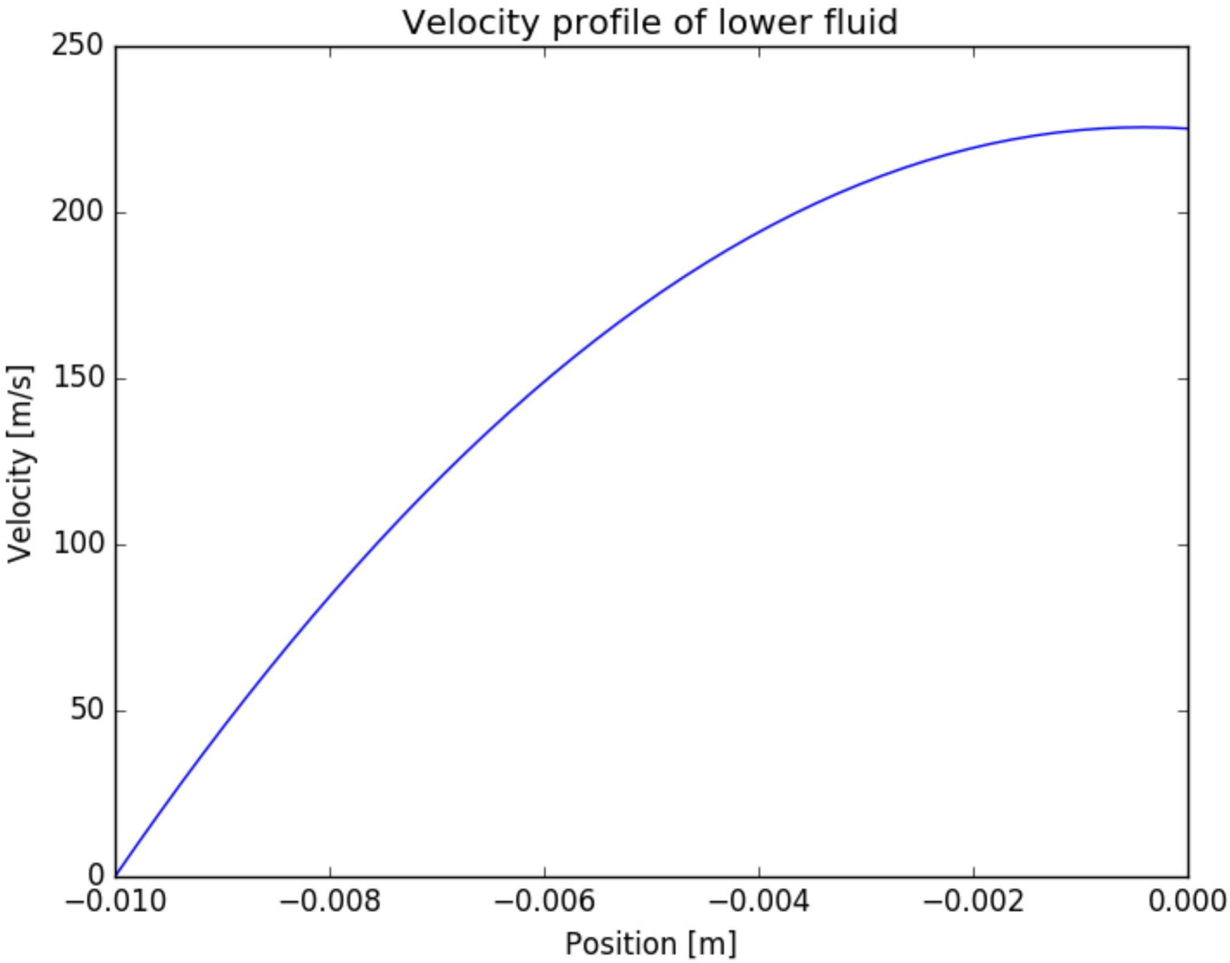
At $y = -h_1$ the shear velocity becomes

$$u_{\tau,1} = \sqrt{\frac{g\sin\theta}{\rho_1}\left(\rho_1 h_1 + \frac{\mu_1\mu_2}{\mu_1 h_2 + \mu_2 h_1}\left(\frac{\rho_2 h_2^2}{2\mu_2} - \frac{\rho_1 h_1^2}{2\mu_1}\right)\right)}$$

g) To visualize the results, we can make plots of the velocity profiles. Running the python script below, using $\rho_2/\rho_1 = 0.01$, $\nu_2/\nu_1 = 10$, $h_1 = h_2 = 1$, $\theta = 30^o$ and $\nu_1 = 10^{-6}\text{m}^2\text{s}^{-1}$, we get the following velocity profiles.

```
import numpy as np
from matplotlib.pyplot import plot, show, xlabel, title, ylabel

# Parameters
g = 9.81
h1 = h2 = 0.01
nu1 = 10.**(-6)
nu2 = 10.*nu1
rho2rho1 = 0.01
nu2nu1 = 10.
theta = np.pi/6

# Velocity profiles
def u_1(y):
    frac = (rho2rho1*nu2nu1) / (h2 + h1*rho2rho1*nu2nu1)
    return g*np.sin(theta)*( (h1**2 - y**2)/(2*nu1) + frac * ( h2**2/(2*
        nu2) - h1**2/(2*nu1))*(h1 + y))

def u_2(y):
    frac1 = (rho2rho1*nu2nu1) / (h2 + h1*rho2rho1*nu2nu1)
    frac2 = 1. / (h2 + h1*rho2rho1*nu2nu1)
    return g*np.sin(theta)*( h1**2/(2*nu1) - y**2/(2*nu2) + (frac1*h1 +
        frac2*y)*( h2**2/(2*nu2) - h1**2/(2*nu1)))

# Visualize
y1 = np.linspace(-h1, 0)
y2 = np.linspace(0, h2)

plot(y1, u_1(y1))
xlabel("Position [m]"); ylabel("Velocity [m/s]")
title("Velocity profile of lower fluid")
show()
plot(y2, u_2(y2))
xlabel("Position [m]"); ylabel("Velocity [m/s]")
title("Velocity profile of upper fluid")
show()
```

Velocity profile of lower fluid:



Velocity profile of upper fluid: