Lecture on: Acoustic (secondary) streaming. MEK4300 Viscous flow and turbulence, Univ. of Oslo.

by John Grue, September 12, 2017

The non-stationary boundary layer equations read:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2}$$

with boundary conditions $u = v = 0$ at $y = 0$ and $u = U$ for $y \to \infty$. We assume that the motion outside the boundary layer is represented by the velocity field $U(x,t)$. The pressure gradient within the boundary layer is then given by

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}. \tag{3}$$

Combining (3) with (1) gives

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}. \tag{4}$$

The method of successive approximations is used to solve the boundary layer equations. We assume that the velocity vector (u,v) may be obtained by

$$u = u_0 + u_1 + u_2 + ...+, \tag{5}$$

$$v = v_0 + v_1 + v_2 + ...+, \tag{6}$$

where $u_0 >> u_1$, $u_1 >> u_2$, ..., $v_0 >> v_1$, $v_1 >> v_2$. The first approximation to the equation of motion reads

$$\frac{\partial u_0}{\partial t} - \nu\frac{\partial^2 u_0}{\partial^2 y} = \frac{\partial U}{\partial t}, \tag{7}$$

which together with the continuity equation determines the leading approxiation of the velocity field. The boundary conditions read: $u_0 = v_0 = 0$ at $y = 0$, and $u_0 = U$ for $y \to \infty$.

The next approximation gives

$$\frac{\partial u_1}{\partial t} - \nu\frac{\partial^2 u_1}{\partial y^2} = U\frac{\partial U}{\partial x} - u_0\frac{\partial u_0}{\partial x} - v_0\frac{\partial u_0}{\partial y}, \tag{8}$$

which together with the continuity equation determines u_1 and v_1. The boundary conditions read: $u_1 = v_1 = 0$ at $y = 0$, and $u_1 \to 0$ for $y \to \infty$.

Periodic boundary layer

We assume that $U(x,t) = U_0(x)\cos\omega t$. Applying a pertubation method the boundary layer equations read:

$$\frac{\partial u_0}{\partial t} - \nu\frac{\partial^2 u_0}{\partial y^2} = \frac{\partial U}{\partial t} \tag{9}$$

$$\frac{\partial u_1}{\partial t} - \nu\frac{\partial^2 u_1}{\partial y^2} = U\frac{\partial U}{\partial x} - u_0\frac{\partial u_0}{\partial x} - v_0\frac{\partial u_0}{\partial y}. \tag{10}$$

The perturbation is valid provided that

$$\frac{\partial U}{\partial t} >> U\frac{\partial U}{\partial x}. \tag{11}$$

We introduce the dimensionless coordinate $\eta = y/\delta$ where $\delta = \sqrt{2\nu/\omega}$ and assume that

$$u_0(x,y,t) = Re(U_0(x)\zeta_0'(\eta)e^{\mathrm{i}\omega t}. \tag{12}$$

We further introduce the stream function such that $u_0 = \partial\psi_0/\partial y$ and $v_0 = -\partial\psi_0/\partial x$. We obtain

$$\psi_0 = Re(\delta U_0(x)\zeta_0(\eta)e^{\mathrm{i}\omega t}) \tag{13}$$

$$v_0 = Re\Big(-\delta\frac{dU_0}{dx}\zeta_0(\eta)e^{\mathrm{i}\omega t}\Big). \tag{14}$$

The function ζ_0 is determined by the following equation

$$\zeta_0^{'''} - 2\mathrm{i}\zeta' = -2\mathrm{i}. \tag{15}$$

The solution of this equation, satisfying the boundary conditions $\zeta_0' = 0$ for $\eta = 0$ and $\zeta_1' = 1$ for $\eta \to \infty$, reads

$$\zeta_0' = 1 - e^{-(1+\mathrm{i})\eta}, \tag{16}$$

giving

$$u_0(x,y,t) = U_0(x)\Big(\cos\omega t - e^{-\eta}\cos(\omega t - \eta)\Big). \tag{17}$$

Large Acoustic streaming. Steady streaming.

An important effect of the oscillatory boundary layer is a steady streaming that becomes introduced outside the boundary layer. The streaming is a consequence of the boundary layer at the wall. The streaming is obtained in the following way. The next term u_1 of the velocity expansion is evaluated. We assume that this term is expressed on the following form

$$u_1(x,y,t) = Re\Big(\frac{U_0}{\omega}\frac{dU_0}{dx}(\zeta_{1b}' + \zeta_{1a}'e^{2\mathrm{i}\omega t})\Big). \tag{18}$$

We shall be interested in the term ζ_{1b}' only. Putting the expression of u_1 into the boundary layer equation for u_1 we obtain

$$-\tfrac{1}{2}\zeta_{1b}''' = \tfrac{1}{2} - \zeta_0'\zeta_0'^* + \tfrac{1}{4}(\zeta_0\zeta_0^*)''. \tag{19}$$

Introduce $\kappa = 1 + \mathrm{i}$ such that

$$\zeta_0' = 1 - e^{-\kappa\eta}, \tag{20}$$

$$\zeta_0 = \eta - \zeta_0'/\kappa \tag{21}$$

$$\zeta_0'\zeta_0'^* = 1 - e^{-\kappa\eta} - e^{-\kappa^*\eta} + e^{-2\eta}. \tag{22}$$

Integration of the equation (19) for ζ_{1b}''', using (22), gives

$$\zeta_{1b}'' = a + \eta + \frac{2}{\kappa}e^{-\kappa\eta} + \frac{2}{\kappa^*}e^{-\kappa^*\eta} - e^{-2\eta} - \tfrac{1}{2}(\zeta_0\zeta^*)', \tag{23}$$

where a is a constant of integration. Integrating once more gives

$$\zeta_{1b}' = b + a\eta + \tfrac{1}{2}\eta^2 - \tfrac{2}{\kappa^2}e^{-\kappa\eta} - \tfrac{2}{\kappa^{*2}}e^{-\kappa^*\eta} + \tfrac{1}{2}e^{-2\eta} - \tfrac{1}{2}(\zeta_0\zeta^*). \tag{24}$$

The boundary condition $\zeta_{1b}'(0) = 0$ gives that $b = -\frac{1}{2}$. At large η we obtain

$$\zeta_0\zeta_0^* \sim \eta^2 - \eta + \tfrac{1}{2}, \quad \eta \to \infty, \tag{25}$$

giving

$$\zeta_{1b}' = -\frac{3}{4} + \left(a + \tfrac{1}{2}\right)\eta, \quad \eta \to \infty. \tag{26}$$

A mathematical solution giving an unbounded velocity in the fluid is unphysical, giving that $a + 1/2 = 0$. This gives that

$$\zeta_{1b}' = -\frac{3}{4}, \quad \eta \to \infty, \tag{27}$$

and that

$$u_1(x,\infty) = -\frac{3}{4}\frac{U_0}{\omega}\frac{dU_0}{dx}. \tag{28}$$

Let $U_0(x) = 2U_\infty \sin(x/R) = 2U_\infty \sin\theta$ for the flow at a cylinder, giving

$$\frac{u_1(x,\infty)}{U_\infty^2/\omega R} = -\frac{3}{2}\sin 2\theta. \tag{29}$$

The velocity is illustrated in the figure 1.

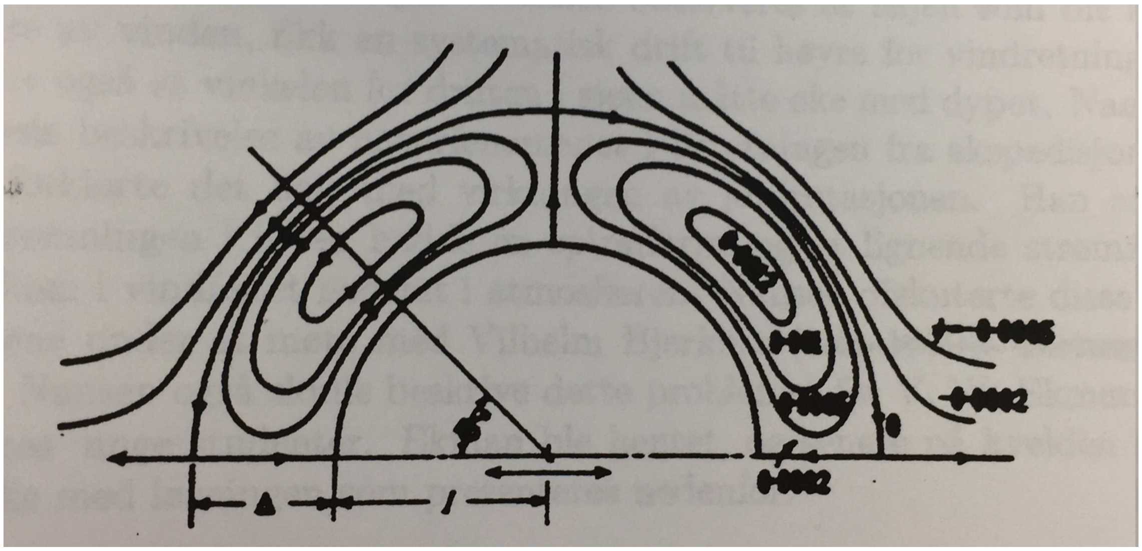

Figure 1: Oscillating cylinder. Convective cells of secondary streaming.

Consider the boundary layer at a sea bed below a standing wave. The horizontal velocity at the bottom is given by

$$U(x,t) = U_0(x)\cos\omega t, \quad \text{where} \quad U_0 = U_\infty \sin(x/R). \tag{30}$$

This gives

$$\frac{u_1(x,\infty)}{U_\infty^2/\omega R} = -\frac{3}{2}\sin\frac{2x}{R}. \tag{31}$$

The flow pattern is illustrated in figure 2 where particles at the sea bed are collected at $2x/R = \pm\pi/2, \pm 3\pi/2, ...$

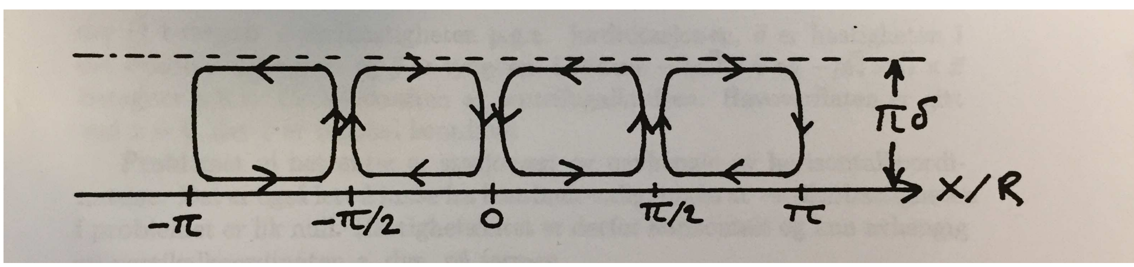

Figure 2: Convective cells of secondary streaming at the sea bed below a standing wave.