

Production Systems

Henri LEFEBVRE

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Introduction

Any system can be regarded as some sort of "black box" or "machine" that turns a certain number of inputs into one or multiple output. Production systems do not escape that definition and can be viewed as a system transforming raw materials into final products. Actually, the inputs are more general and complex than just raw materials, for example, one can consider energy, workforce, decisions and many others as inputs of our system. As well, the outputs are generally not only a final product but also scraps or waste. Scrap corresponds to the portion of raw material that we loose with respect to the technology we utilize for transforming the raw materials into final products. Wastes, on the other hand, are immaterial. The figure (1) sums up what have been said.

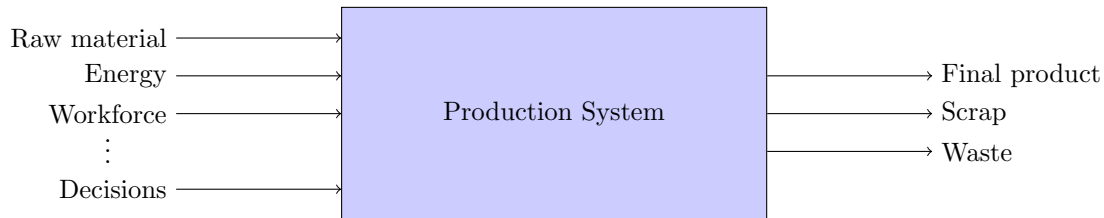


Figure 1: General production system

The objective of this course is to analyse how we can make our production system more efficient with respect to some criteria (time, money, ...). Of course, our production depends on the methods and on the technologies we use to manufacture our goods, but these are rather concerns for mechanical engineers or more specific fields (chemistry, process engineering, ...). Actually, this course focuses on the only input we can, at our level, act upon, which is : decisions.

To understand how decisions can improve (or worsen) our productivity, let's take a very simple example. Consider the following task : "Drilling four holes in a piece of wood". And let X be the rate of production. Consider as well that the "drilling" operation needs some set-up time in order to prepare the right borer and to turn on the machine. Let's denote by T_S this set-up time and by T_H the operation time which is necessary to actually drill the wooden piece. There are plenty of ways to perform such task, one idea is to drill one hole at a time, stopping the machine after each drill. But this wouldn't be a very clever way since we would have to wait for the set-up time for each hole we want to make. Another idea is to perform a batch of holes in a row so that we only account for the set-up time once for all the batch. The two situations are presented in the GANT diagram in figure (2).

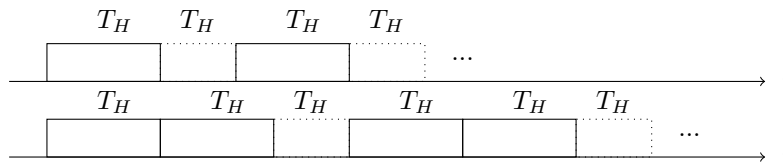


Figure 2: Two GANT diagrams showing how decisions can influence productivity

In the first situation, the rate of production can be expressed as

$$X_1 = \frac{1}{T_H + T_S}$$

meaning that we produce one item every $T_H + T_S$ unit of times. For the second situation however, the rate can be computed as

$$X_2 = \frac{2}{2T_H + T_S} = \frac{1}{T_H + \frac{T_S}{2}}$$

which means that we produce one item every $T_H + \frac{T_S}{2}$ unit of time, which is smaller. In fact, as long as the batch size increases, the set-up time becomes less and less influent on the production rate since

$$X_k = \frac{k}{kT_H + T_S} \xrightarrow{k \rightarrow \infty} \frac{1}{T_H}$$

which clearly demonstrates that we can influence our production by making some decisions rather than others.

The first part of this course will deal with the inventory management in which we'll try to answer to questions like "How and when should we order goods for our production system ?". In fact, we will deal with the different stakeholders to establish good strategies to make our orders. These stakeholders are represented in figure (3). Firstly, the suppliers provide us with some raw materials which can then be stored in a warehouse. When needed, the production system will take some of these materials to transform them into final products. Once done, they will then leave them in another warehouse which will be used to fulfill the demand of such final goods.

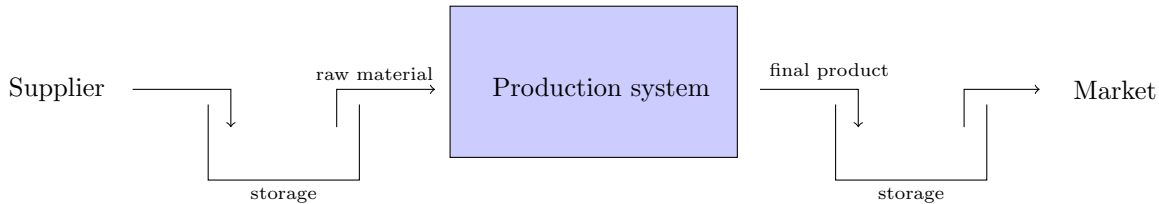


Figure 3: Production system agents

That being said, it is clear now that any production system aims to fulfill a certain demand taking into account the supply constraints it may encounter. This will be largely discussed further on.

The figure (4) depicts the different production system types we can encounter :

- Job shops : very specialized entities where, often, solely one product can be produced at the same time. Car repairers are a good example of such production systems. This kind of system is not easily formalized because they are too specific. We will not discuss them in this course.
- Mass production : produces an average number of varieties of goods at average to high rate.
- Continuous flow process : produces continuously a given product at very high rate but with very small variety of products.

We will focus on both mass production and Continuous flow processes.

The second part of this course will take interest in the bill of materials (which will be further defined) in which we'll look at how to optimize production with respect to the production rate taking into account bottleneck machines in a production site.

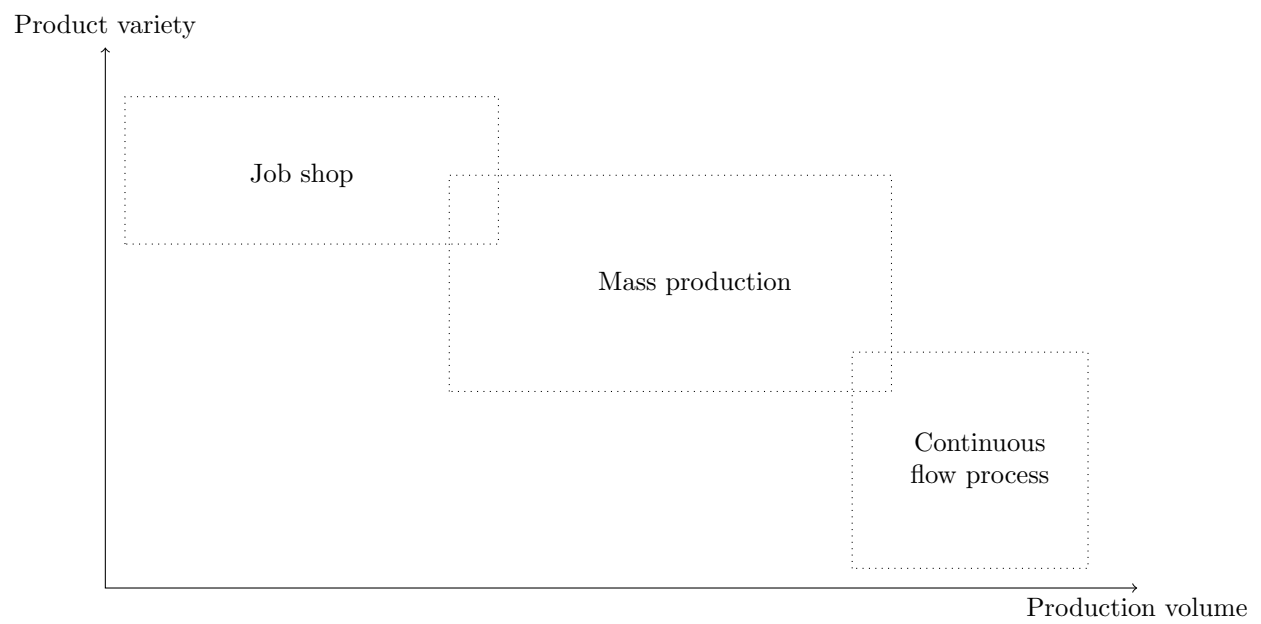


Figure 4: Production system types

Part I

Inventory management

Chapter 1

Continuous review

Inventory management, or storage management, can be qualified as the art of storing and ordering goods in a clever way with respect to some criteria (most often the cost). It can be applied at both the beginning and the end of the production system (even in between actually). At the beginning of this chain, raw materials are delivered by the suppliers and are stored in a warehouse. The storage is emptied when the production site requires some raw materials to be used for production. On the other side of the chain, at the end, the storage is filled with final products and is emptied by the market demand. Though the two situation may appear different in some kind, we can jointly study them considering some sort of "abstract demand" (either production demand or market demand) and some sort of "abstract supplier" (either a real supplier which provides raw material or the production site itself). See figure (1.1) for graphical representation.

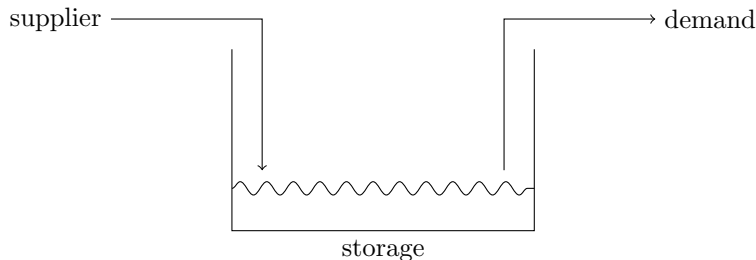


Figure 1.1: Storage management parameters

To study the inventory management, we introduce models that tend to represent real life cases. Making different assumptions on the demand and supplier, we end with different models. The simplest model is the so called "deterministic model" where all the input data for our problem are supposed to be perfectly known in advance (both the supply and the demand). We will start by studying this model before focusing on a "stochastic model" where the supplier and the demand are modeled by random variables.

In this chapter, we will study what is called "continuous inventory management". What we mean by this is that the time is modeled by a continuous variable ($t \in \mathbb{R}$). In the next chapter, we'll deep into "periodic inventory management" where the time is modeled by a discrete variable ($t \in \mathbb{N}$).

1.1 Deterministic model

The deterministic model supposes that we exactly know both the demand and the characteristics of our supplier. We denote by λ the rate at which the storage is emptied. In the case where our warehouse is at the end of our chain,

just after the production site, λ represents how many items (final products) we sell by unit of time. If considered at the beginning of the chain, it corresponds to how many items (raw materials) are needed by our production site. This coefficient can be computed as

$$\lambda = \frac{\text{nb of items leaving the warehouse}}{\text{observed time}}$$

To fulfill that demand, we need to order a certain quantity of good, which will cost us some amount of money. In fact, the costs are of three kinds :

- A **fixed cost** which we have to pay for each order we make to our supplier. Let K denote this cost, expressed as *euros/order*
- An **inventory cost** which we pay for storing goods in our warehouse, expressed as *euros/(time \times item)*. Let h be the inventory cost.
- A **unitary cost** which corresponds to the price of each item per se. We will refer to this cost as c , expressed as *euros/item*

For sure, it is easy to see that choosing the right time and quantity to order goods is a matter of balancing between the fixed cost we pay for ordering an independent amount of product and the inventory cost which we pay for storing the goods. If storing products in our warehouse is very expensive for us, we might prefer to order often and pay the fixed cost multiple times rather than paying it just once and be left with a lot of goods we pay a fortune for storing. If, however, the inventory cost is very low (which means that we can store many items for a small amount of money), we'd rather order a lot of products and store them for a long time if needed. So, how do we compute the total cost ?

Let T be the period of time between the moment we order some goods and the moment our storage goes empty. Note that since the demand is deterministic and known in advance there is no risk of "unpredicted" events which could lead us to be unable to fulfill the demand nor to order too many items in such a way that some would remain after some time. Finally, let Q be the number of items we will buy at the beginning of that period. The figure (1.2) sums up some of the introduced variables.

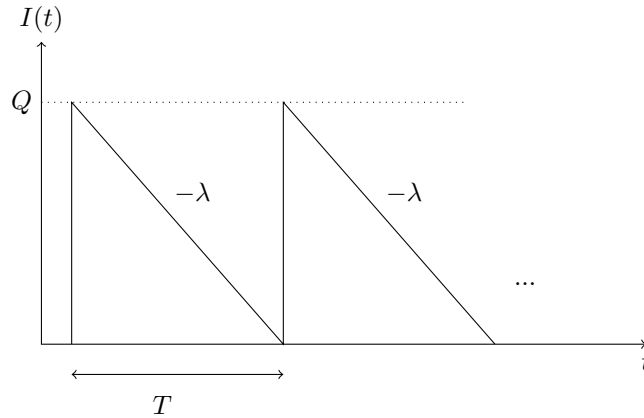


Figure 1.2: Inventory level in the deterministic model

For a given period of time T , we can then compute the total cost as

$$J_T = \underbrace{K}_{\text{fixed cost}} + \underbrace{cQ}_{\text{price for } Q \text{ items}} + \underbrace{\int_0^T hI(t)dt}_{\text{storage cost}}$$

where the integrated part is simply given by $\frac{QT}{2}$ (area of a triangle¹).

¹ $\int_0^T hI(t)dt = \int_0^T h \times (Q - \lambda t)dt = h[Qt - \lambda \frac{t^2}{2}]_0^T = h \frac{2QT - \lambda T^2}{2}$ and since $Q = \lambda T$ we get $\frac{QT}{2}$ by substituting λ

For sure, we want to minimize the cost for a given period of time. That cost is given by

$$J = \frac{1}{T} \left(K + cQ + \frac{hQT}{2} \right) = \frac{K\lambda}{Q} + c\lambda + \frac{hQ}{2}$$

Note that we used the fact that $Q = \lambda T$ holds. Regarded as a function of Q , we can analyse its variations by derivation (since $Q = 0$ does not make any sense for our purpose, it is for sure differentiable) :

$$\begin{cases} \frac{dJ}{dQ} &= -\frac{K\lambda}{Q^2} + \frac{h}{2} \\ \frac{d^2J}{dQ^2} &= \frac{K\lambda}{Q^3} \end{cases}$$

Solving the optimality condition $\frac{dJ}{dQ} = 0$ we get that

$$Q^* = \sqrt{\frac{2K\lambda}{h}}$$

can be regarded as a critical point of J . And, since the second derivative of J is positive, the value Q^* corresponds to a minimum of our cost function. This quantity is known as the **Economic Order Quantity** (EOQ) and corresponds to the number of items we should order every Q/λ unit of time in order to minimize the total cost.

We may discuss the obtained formula noting that : (1) if the fixed price for order is very large with respect to the inventory cost, we will buy a large amount in once and store it for a long period of time ; (2) if the demand is high, we buy more ; (3) if the inventory cost is large with respect to the fixed cost per order, we will prefer making multiple orders in order to store as less as possible.

The moment at which we should order that quantity Q^* of goods is given by the characteristics of our supplier. Let τ be the lead-time of our supplier (difference between the moment we make an order and the moment we actually receive the products). We define the so called **reorder point** as $R = \lambda\tau$ which corresponds to the inventory level at which we should call the supplier to make an order².

1.2 Stochastic model

In the previous model, we supposed that we could both know perfectly the demand and the lead-time of our supplier. In real life however, such quantities are often subject to variations and are found by estimation. That being said, we can not afford to trust a deterministic formula since we know that life is full of unpredicted and unpredictable events. Let x be a random variable such that $x \sim \mathcal{N}(0, 1)$. We now model the demand (denoted d by convention in the stochastic model) by a random variable given by $d = \bar{d} + \sigma_d x$ and the lead-time by $\tau = \bar{\tau} + \sigma_\tau x$.

Actually, the stochastic model is a generalisation of the deterministic model and a lot of common aspects will remain true in this model (in the sense that we can derive the deterministic case from the stochastic case). But, since the demand now varies with time, we can no longer rely on an exact formula giving us *exactly* how many goods we have to order in order to have a 0-inventory-level when we re-order some other goods. Instead, we'd rather save some goods to prevent the fact of being short on stock. That amount of goods is called the **safety stock** and will be denoted by the variable $s.stock$. Given these informations, what is reasonable to expect ?

The expected value of the demand (resp. the leadtime) is \bar{d} (resp. $\bar{\tau}$). The reorder point is thus expected to be given by $\bar{d}\bar{\tau}$ which, in average, would bring the inventory level to 0 before re-ordering. To this quantity, we add the safety stock to prevent us from delays of our supplier.

$$R = \bar{d}\bar{\tau} + s.stock$$

²Note that in the deterministic case, directly using the lead-time is strictly equivalent to using the reorder point, however, three arguments to use the re-order point rather than the lead-time : (1) the reorder point is linked to a quantity that the warehouse can directly control, (2) the deterministic case is a model, which can or can not reflect the reality, using the reorder point assures us that if, for unexpected reasons, we get to the reorder point sooner than planned, we can still make an order at the right time for the new period of time, (3) we will keep using that quantity in the stochastic case.

In average, the stochastic model should be exactly the deterministic model shifted up by precisely *s.stock* goods in the inventory. The figure (1.3) represents two realisations of the demand while figure (1.4) represents the expected value of our inventory level.

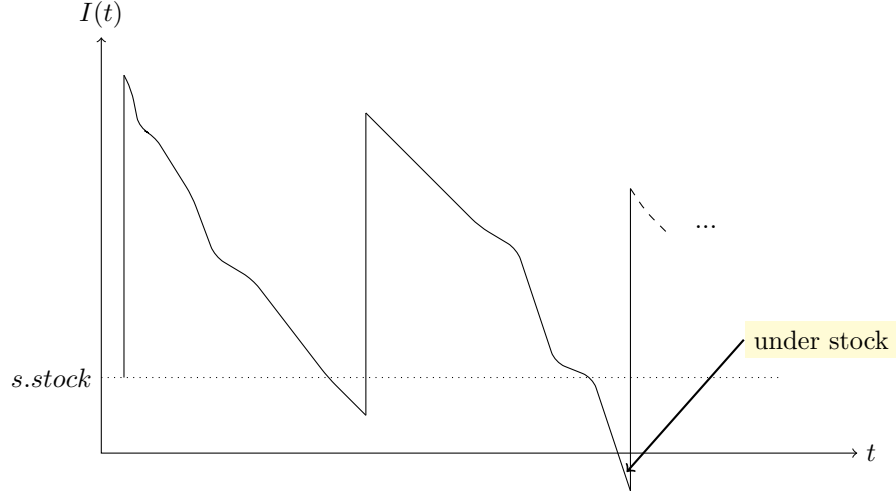


Figure 1.3: Two realisations of the demand

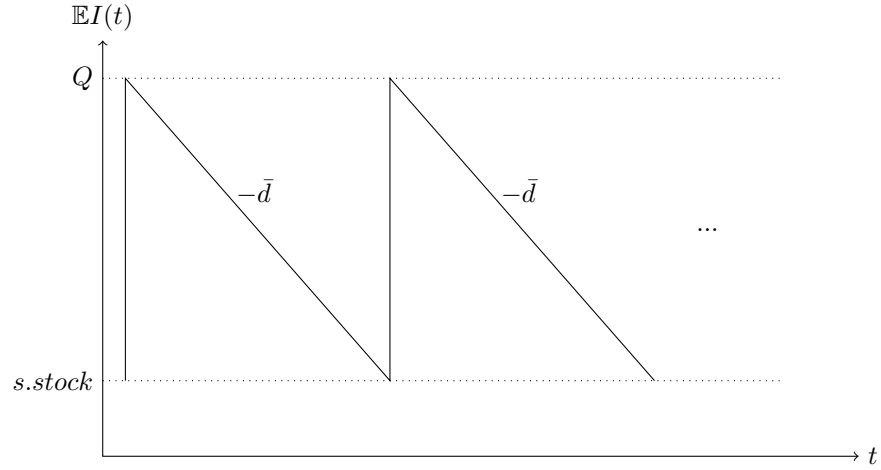


Figure 1.4: Expected inventory level

The expression of the cost must now be updated since we no longer only pay for ordering and storing the goods which are used to fulfill the demand but also for storing some goods as a safety-stock. The total cost is now given by :

$$J = \frac{K\bar{d}}{Q} + c\bar{d} + \frac{hQ}{2} + h \times s.stock$$

This function of two variables (Q and $s.stock$) is actually well splittable into two functions of one variable denoted by $f(Q)$ and $Q(s.stock)$. It is clear that minimizing the total cost can now be done by minimizing both functions separately. Regarding the first function f , the computations remains the same as in the previous section (deterministic model) and we still have $Q^* = \sqrt{2K\bar{d}h}$. Minimizing g how ever, requires some more analysis. Of course, g is linear with respect to $s.stock$ so the global minimum is 0 since it cannot be negative. But setting it to 0 would mean risking to be short on stock at some point. We, somehow, want to control the risks we take and two strategies can be opted for :

1. Let d_τ be the demand over the lead-time τ . It is clear that d_τ is *linked* to both d and τ but it is a rather strong assumption to say that these two random variables are *dependent*. Indeed, the realisation of the leadtime of a supplier is not dependent on the global demand³. d_τ is thus given by $d_\tau = \bar{d}_\tau + \sigma_{d_\tau}x$ where $\bar{d}_\tau = \bar{d}\bar{\tau}$ and $\sigma_{d_\tau} = \sqrt{\sigma_d^2\bar{\tau} + \sigma_\tau^2\bar{d}^2}$. One idea to select effectively the safety stock is to make sure that the probability of being short on stock is lesser than a certain number which we control. Being short on stock is realised when the demand over the lead-time is greater than the reorder point we have chosen. This strategy implies choosing $s.stock$ so that

$$\mathbb{P}(d_\tau \leq R) \geq \alpha = \mathbb{P}\left(x \leq \frac{R - \bar{d}_\tau}{\sigma_{d_\tau}}\right) \geq \alpha$$

where α is called the "fill rate". Using this strategy can be done as follow :

- (a) Choose an acceptable probability α
 - (b) Solve the equation for X and let \bar{X} be the solution
 - (c) Substitute X by \bar{X} in $s.stock = \bar{d}_\tau + \sigma_{d_\tau}X$
2. As it has been previously said, it holds that we are short on stock when $d_\tau > R$ (more demand during the lead-time than the reorder point). Let $n(R)$ be the average number of unused items (items remaining in the warehouse when we re-order again). We can compute this quantity with the following formula⁴ :

$$n(R) = \int_R^\infty (d_\tau - R)f(d_\tau)d(d_\tau)$$

However, it is analytically hard to compute. To deal with that problem, we introduce the function L which is called the loss function expressed as

$$L(X) = \int_X^\infty (t - X)f(t)dt \text{ where } f \text{ is the density of the normal law } \mathcal{N}(0, 1)$$

so that

$$n(R) = \sigma_{d_\tau} L(X)$$

Finally, by defining the average percentage of unused items with respect to the total number of goods we buy, denoting it by $\beta = \frac{n(R)}{Q^*}$ we can find $s.stock$ so that

$$L(X) = \frac{\beta Q^*}{\sigma_{d_\tau}}$$

where β should of course be defined *a priori* depending of the market requirements, of course, the lowest β is, the greatest is the service.

Using this second strategy can be done like so :

- (a) Compute σ_{d_τ}
- (b) Compute $\frac{\beta Q^*}{\sigma_{d_\tau}}$ and use the table to find the corresponding X denoted \bar{X}
- (c) Substitute X by \bar{X} in $s.stock = \bar{d}_\tau + \sigma_{d_\tau}X$

³it is surely linked, for example, during peak demands of a given good, one supplier may have difficulties to satisfy all of his clients, but we cannot consider that a truck getting lost, or breaking down depends on the demand

⁴For any random variable X with density function f , the average value of X is given by $\int_0^\infty xf(x)dx$

Chapter 2

Periodic review

In the previous section, time was supposed to be continuous. In this new chapter however, we will regard time as a discrete variable meaning that we consider fixed size time intervals for our inventory management study. To each of these periods corresponds a certain demand, denoted d_i for period i , and a decision (called "action") that we take in order to satisfy the demand, denoted a_i . We represent this situation with plots similar to figure (2.1) where each box stands for a period of time. The question we have to answer in this chapter is : "What actions do I have to take in order to fulfill the demand minimizing the total cost ?", or similarly : "How many items a_i should I buy in period i to minimize the cost ?"

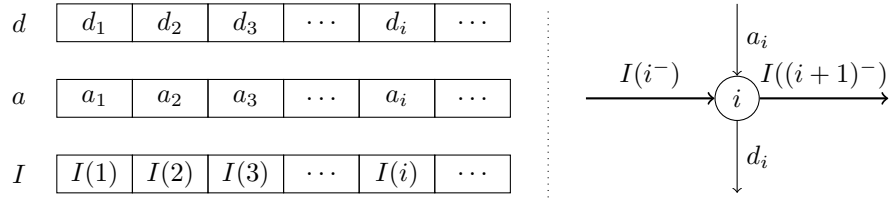


Figure 2.1: A representation of the periodic review model

With this representation, it becomes clear that, at the end of the i -th period, the inventory level, denoted by $I((i+1)^-)$, corresponds to the inventory level we had at the beginning of that period, $I(i^-)$, to which we should add the number of items we have ordered, a_i , minus the demand, d_i , which corresponds to goods which have been sold out. Hence,

$$I((i+1)^-) = I(i^-) + a_i - d_i$$

Thus, by denoting by h the inventory cost and by K the fixed cost per order as we have been doing so far, the total cost can be computed as :

$$J = \sum_{i=1}^N (K\delta(i) + hI((i+1)^-)) \text{ where } \delta(i) = \begin{cases} 1 & \text{if } a_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

Our goal is to minimize this function by choosing relevant a_i s. For example, we need to discuss whether it is more profitable to order goods for two periods in a row instead or ordering period per period with respect to the demand. This can be solved by mixed integer linear programming models in an exact way. However, the cost of these methods are not worth it since efficient Heuristics have been developed which are very simple to implement. In this chapter, we will focus on the "Silver-Mill Heuristic".

Even though it appears clear, note that, if the inventory level in period 0 is 0 and if we suppose knowing in

advance the exact demand for each period, then it holds that

$$\sum_{i=0}^N a_i \geq \sum_{i=0}^N d_i$$

This can easily be demonstrated by recursivity. Let's suppose that $I_0 = 0$. Then, and because $I_i \geq 0, \forall i$ in the deterministic model, $I_1 = a_1 - d_1 \geq 0$ which means that $a_1 \geq d_1$. So the property is true for the first rank. Let's now suppose, that, for a given rank k , we have $\sum^k a_i \geq \sum^k d_i$. Considering I_{k+1} , it holds that $I_{k+1} = I_k + a_{k+1} - d_{k+1}$ with $I_k = \sum^k a_i - \sum^k d_i$ so that in definitive $\sum^{k+1} a_i \geq \sum^{k+1} d_i$.

2.1 The Wagner-Within Theorem

The Wagner-Within Theorem tells us about the shape of the solution to our problem. It stipulates that the optimal solution is either that (1) we buy exactly the demand of multiple time periods or (2) we buy exactly the current period demand, but in no case we would buy a fraction of the demand for one time period and buy the rest in another period. The optimal solution cannot be splitted into periods of time. For instance, the optimal solution cannot be to buy 70% of the forecasted demand of November in September and 30% of that same demand in October, but rather buy all the corresponding demand for November in September or all the corresponding demand for November in October. We do not split the orders for one time period. So that, considering a_1 for example, we now know that

$$a_1 \in \{d_1, d_1 + d_2, d_1 + d_2 + d_3, \dots\}$$

In this section we will demonstrate this theorem for the case of two time periods **WLOG**.

Let's consider a time horizon of two periods and, since the demand is supposed to be known in a deterministic way, we can impose that the inventory level at the end of the horizon is zero, $I_2 = 0$. As well, we'll assume that I_0 is zero (no initial inventory level). The inventory level at the end of this time horizon is given by $I_2 = a_1 + a_2 - d_1 - d_2 = 0$ which leads to the equality

$$a_1 + a_2 = d_1 + d_2 \quad (2.1)$$

In the first time period, we at least have to order enough goods to fulfill the demand of the first period, so that $a_1 \geq d_1$ which we'll rather write as $a_1 = d_1 + \epsilon$ where ϵ is a positive number (which in any case should be less than d_2 because it would violate our will to end with an inventory level of zero). From equation (2.1), it holds that $a_2 = d_2 - \epsilon$, and so, at the end of each period we end with :

Inventory level at the end of each periods :	ϵ	0
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This allows to compute the total cost like so :

$$J = K + h\epsilon + \begin{cases} K & \text{if } \epsilon = 0 \\ 0 & \text{if } \epsilon = d_2 \\ K & \text{if } \epsilon < d_2 \end{cases} = \begin{cases} 2K + h\epsilon & \text{if } \epsilon < d_2 \\ K + hd_2 & \text{if } \epsilon = d_2 \end{cases}$$

Indeed, we always pay at least one order (which corresponds to the first order in period one) and the inventory cost for the remaining items after the first periods (which is ϵ). In the case where $\epsilon = d_2$, we already have ordered enough products to fulfill both the demand of the first period and the demand of the second period, so we don't have to make another order and we don't pay for ordering nothing. Otherwise, we have to order again, so a new K cost has to be paid. We then have two cases :

- if $\epsilon < d_2$, then clearly, the minimum is reached by choosing $\epsilon = 0$
- if $\epsilon = d_2$, then the minimum is $K + hd_2$

What this discussion means, is that, in order to choose a value for ϵ in order to minimize our cost, the only "good" decisions we can make are those of buying once for two periods or buying once at each period but in no case should we buy more in period one and buy again a smaller amount of goods in period two. That's the meaning of the Wagner-Within theorem.

2.2 The Silver & Mill Heuristic

The Silver-Mill heuristic is a very simple algorithm which relies on the Wagner-Within theorem. It often drives a solution near the real solution which in most cases is acceptable. In fact, the algorithm finds a local minimum of the cost function (instead of a global minimum). Simply summed up, this heuristic considers incrementing periods of time and looks at whether or not it would be a good idea to buy more than what is needed for that specific period of time. We know from the Wagner-Within theorem that if we buy more than needed, then we should buy exactly the amount of good sufficient to satisfy the demand of the next period (otherwise it's not worth it). The algorithm computes a series of costs making this kind of assumptions until a local minimum is reached. Here is how it works : let $c(i)$ be the cost we would pay if, starting from the first period, we were to buy as many products as needed to satisfy up to the i -th period's demand. Clearly, the cost $c(1)$ is simply K since we only make one order and don't pay for inventory since all our order is consumed by the demand in the first period ($a_1 = d_1$). For the second period, the cost $c(2)$ is now the fixed price K , since we order once for two periods, to which we have to add the inventory cost we pay for the remaining inventory level after period one which corresponds exactly to hd_2 . In order to compare $c(1)$ and $c(2)$, we'll focus on the *average* cost that we pay per period of time instead of the absolute value. It holds then that $c(2) = \frac{K+hd_2}{2}$. Similarly, we can compute $c(3)$ as $c(3) = \frac{K+h(d_2+d_3)+hd_3}{3} = \frac{K+hd_2+2hd_3}{3}$ since we order once for the three subsequent periods of time and that we have to pay for inventory in period one (whose inventory level is then $d_2 + d_3$) and in period two (whose inventory level is then d_3). Continuing like so, and searching for a local minimum, we get the Silver-Mill heuristic for choosing our actions.

The general formula (which can be recursively demonstrated) for computing the coefficients $c(i)$ s is given by

$$c(i) = \frac{K + hd_2 + 2hd_3 + \dots + (i-1)d_i}{i}$$

Indeed, we assume paying once in period one, so we pay the order price K just once, then we have to deal with inventory costs. The quantity d_k will remain in the inventory until period k is reached. So the associated inventory cost is given by $(k-1)hd_k$ since we pay for having it in store in every preceding periods of time.

The figure (2.2) shows how the heuristic works. The algorithm continues to compute total costs (incrementing the number of periods considered) until a local minimum is found. In this example, our action will be to order in the first period for three periods of time since $c(3)$ is a local minimum of our function J . Note that, if we had continued to search for lower values we may have eventually found that (in this case), $c(5)$ was more advantageous from an economical point of view and we would have better done to buy all the products for 5 periods instead of four. But in general, this approach gives a really good approximation of the optimal value for the problem and, with respect to the simplicity of the algorithm, it often doesn't worth it to use more advanced and complex models.

2.3 Analysis of the method

In this last section of this chapter, we would like to discuss on the accuracy, or rather plausibility, of this method. In particular, we will have a look at what happens if the demand for each period of time is homogeneous. Let d be

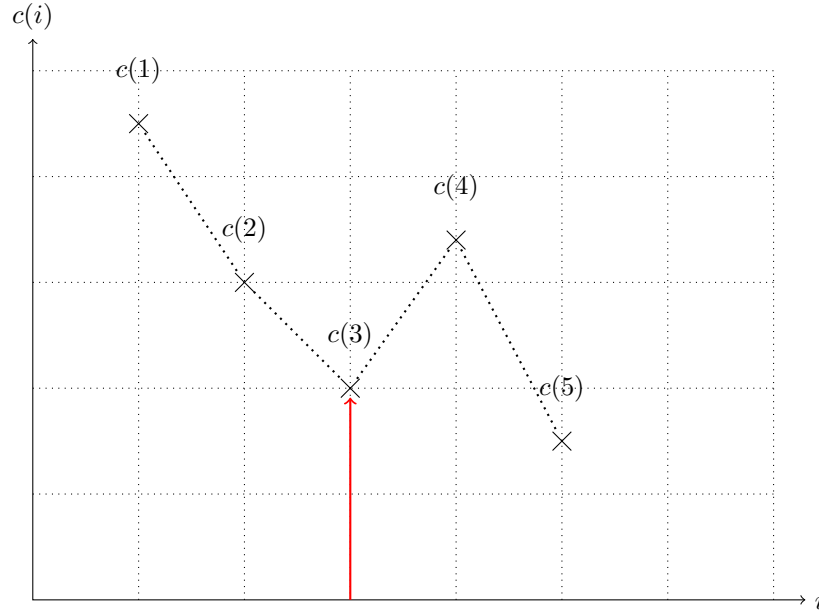


Figure 2.2: Silver-Mill heuristic example

that demand. Using the previously established formula, it holds that :

$$\begin{aligned}
 c(n) &= \frac{K + hd + 2hd + \dots + (n-1)hd}{n} \\
 &= \frac{K + \sum_{i=1}^{n-1} ihd}{n} \\
 &= \frac{K + \frac{n(n-1)}{2}}{n} \\
 &= \frac{K}{n} + \frac{nhd}{2} - \frac{hd}{2}
 \end{aligned}$$

and let us relax our problem considering the function $c(x)$ where x is no longer discrete but continuous. Derivating this function, we get

$$\begin{cases} \frac{dc}{dx} &= -\frac{K}{n^2} + \frac{hd}{2} \\ \frac{d^2c}{dx^2} &= \frac{K}{n^3} > 0, \forall x \in \mathbb{R} \end{cases}$$

The second derivative being greater than zero for any x means that the function c is convex. Knowing that, it is legit to compute the minimum of this function by truncating the solution of $\nabla c = 0$ to find the actual minimum of the non-relaxed problem (where $x \in \mathbb{N}$). Solving this equation we get :

$$\frac{dc}{dx} = 0 \Leftrightarrow x^* = \sqrt{\frac{2K}{hd}}$$

And since $Q = dn^*$, it holds that

$$Q = dx^* = \sqrt{\frac{2kd}{h}}$$

which is the exact same formula which we found in the first part of this course in the deterministic model.

Part II

Bill of materials

Chapter 3

Model without set-up times

Looking back at figure (1), we realise that we have only been focusing on one part of the global schema. Indeed, we have only dealt with inventory management so far. This part of the course will now focus on what is called "Bill of materials" in which we look at how we can manage the relations between raw materials and final product production.

3.1 Model representation and explosion of bill of materials

The Bill of materials uses a standard representation in terms of graph. The nodes are a set of products : raw materials, final products or temporary products. Temporary products are products which are used to make either other temporary products or final products from either other temporary products or raw materials. The arcs of the graph represents transformations. In figure (3.1), the first case where a node i is linked to a node j represents the fact that, somehow, product i is transformed into product j , for instance, if we paint a piece of wood (product i) it becomes a painted piece of wood (product j), or again, if we cut a large piece of iron (product i) we transforme it into *several* smaller pieces of iron. The number n_{ij} valuating the arc tells us how many product i we need to produce a single product j . If we consider the painting operation, then $n_{ij} = 1$, whereas if we consider cutting an iron piece, $n_{ij} \geq 2$ (depending on the situations). The second case represents the fact of assembling products i and j in order to make product k . Again, n_{ik} represents the number of product i needed to make one product k .

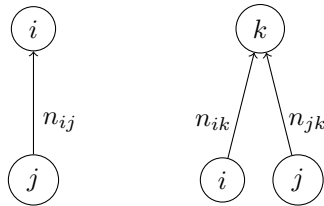


Figure 3.1: Bill of materials representation in terms of graph

Let's now consider figure (3.3) which shows you a first example of bill of materials and let's extend the notation n_{ij} to any nodes i and j as long as there exists a path from i to j in the graph. Then, n_{2f} represents the number product 2 needed to produce 1 final product f . How can we compute that value ? It is rather simple : we need 3 units of product 1 to produce 1 unit of product f , but we need 2 unit of product 2 to produce 1 unit of product 1 ; the quantity n_{2f} is then given by $n_{21} \times n_{1f} = 2 \times 3 = 6$, we need 6 units of procut 2 to produce 1 unit of product f . In general, the following formula holds :

$$n_{kf} = \prod_{(i,j) \in Path(k \rightarrow f)} n_{ij}$$

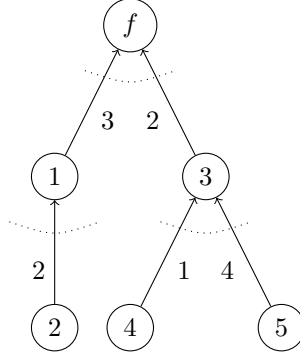


Figure 3.2: First example of Bill of Materials

It is clear now that if we want to produce a certain number of product f denoted X_f , we need to produce at least $X_f n_{if}$ product i so that $X_i \geq X_f n_{if}$. In the following discussion, we will always choose $X_i = X_f n_{if}$ which corresponds to the "conservation of flow" in our system (even though it is not a necessary condition). An other assumption we will make in this chapter, as well as in chapter 2, is that every machine can produce only one kind of product. We will then denote by i both the product and the machine producing it.

Considering machine i , let μ_i be the maximum speed for producing one product, expressed as items/unit of time, and we define u_i as the "utilization of the resource i " which equals :

$$u_i = \frac{X_f n_{if}}{\mu_i} \leq 1, \forall i$$

Using that definition, it holds that

$$X_f \leq \frac{\mu_i}{n_{if}}, \forall i \Leftrightarrow X_f \leq \min_i \left(\frac{\mu_i}{n_{if}} \right) \triangleq X_f^{max}$$

where X_f^{max} is the maximum rate at which we can produce product f with respect to the machines we use to produce it. The bottleneck machine(s) can be computed as $\text{argmin}_i \left(\frac{\mu_i}{n_{if}} \right)$. This (or these) machine(s) is (are) the one(s) which prevents us from producing more, at a higher rate.

3.2 Production time

Given a machine i , we can compute the operation time needed to produce one item of product i which is simple

$$T_{oi} = \frac{1}{\mu_i}$$

We then enhance figure (3.3) with those operation times as a valuation of the arcs. Figure (??) shows such a representation. The operation time has been added as the second element of a couple. The first element will be dealt with in chapter two of this section. The maximum rate X_f^{max} at which we can produce our final product is then given by

$$X_f^{max} = \min_i \left(\frac{1}{T_{oi} n_{if}} \right)$$

Let's start by computing the n_{if} quantities for all i , we get :

$$\begin{aligned} n_{1f} &= 3 & n_{3f} &= 2 & n_{5f} &= 4 \times 2 = 8 \\ n_{2f} &= 2 \times 3 = 6 & n_{4f} &= 1 \times 2 = 2 \end{aligned}$$

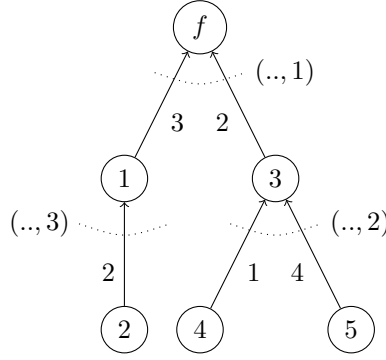


Figure 3.3: First example of Bill of Materials

then we can find X_f^{max} as well as the bottleneck machine by doing the following

$$\begin{aligned}
 X_f^{max} &= \min_i \left(\frac{1}{T_{oi} n_{if}} \right) \\
 &= \min \left(\frac{1}{1 \times 3}, \frac{1}{3 \times 6}, \frac{1}{1 \times 2}, \frac{1}{2 \times 2}, \frac{1}{2 \times 4} \right) \\
 &= \min \left(\frac{1}{3}, \frac{1}{18}, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \right) \\
 &= \frac{1}{18}
 \end{aligned}$$

$$bottleneck = \arg \min_i \left(\frac{1}{T_{oi} n_{if}} \right) = 2$$

Note that, in this case, the bottleneck machine is the one with the highest operation time. This is NOT a general rule, and it's easy to find a counter example : consider a final product f which is made by assembling a hundred small pieces with one big one. Clearly, even if the machine which produces the small pieces is efficient, it is *possible* that making a hundred of these pieces may be longer than producing one other piece, even bigger. But again, this isn't always the case. The only way to solve for the bottleneck machine problem is to look at both the operation time and the number of products required to produce the final product.

3.3 Batch production

A batch b_i can be defined as the number of items i the machine produces without interruption before releasing it to other machines. Let's suppose that we want to produce a batch of final product b_f , how much time do we need to do it ? In this section, we will try to answer that question considering that our batch sizes are $b_i = b_f n_{if}$, which is not necessary, but simpler. In general however, you often have external constraints (linked to the technology you use for example) which imposes a minimum batch size or a maximum batch size, so that :

$$b_i^{min} \leq b_i \leq b_i^{max}, \forall i$$

From that constraint, we can deduce the following

$$\begin{aligned}
 &\Rightarrow b_i^{min} \leq b_f n_{if} \leq b_i^{max}, \forall i \\
 &\Rightarrow \frac{b_i^{min}}{n_{if}} \leq b_i \leq \frac{b_i^{max}}{n_{if}}, \forall i \\
 &\Rightarrow \max_i \left(\frac{b_i^{min}}{n_{if}} \right) \leq b_f \leq \min_i \left(\frac{b_i^{max}}{n_{if}} \right)
 \end{aligned}$$

Since the batch size b_f has to be an integer, we have to make sure that this interval contains at least one integer value, otherwise, we cannot use $b_i = b_f n_{if}$ as a batch size.

Considering that relation to hold however, we can compute the production time for a given batch as

$$T_{i,prod}(b_i) = T_{oi}b_i = T_{oi}b_f n_{if}$$

which now allows us to compute the time it takes to go from one raw material (a leaf of the tree in our graph representation) to the final product (root of tree). Considering \mathcal{P}_k as the path going from the raw material k to the final product in our graph, it holds that

$$T_{\mathcal{P}_k} = \sum_{j \in \mathcal{P}_k} T_{j,prod}(b_j) = b_f \sum_{j \in \mathcal{P}_k} T_{oj} n_{jf}$$

This relation tells us about how much time we would need to produce a batch of final product b_f since

$$T_{prod}(b_f) = \max_{\mathcal{P}_k} (T_{\mathcal{P}_k}(b_f))$$

Up to now, we have considered choosing a batch size b_f and compute the time needed to produce it. But we can as well use the established formulas in the other way to compute the maximum batch size allowed in order respect a certain constraint on production time. For instance, let's say we want to respect the condition $T_{prod}(b_f) \leq T^{max}$ meaning that we do not want to produce more than T^{max} units of time. Then it still holds that

$$\begin{aligned} T_{prod}(b_f) \leq T^{max} &\Leftrightarrow b_f \max_{\mathcal{P}_k} \left(b_f \sum_{j \in \mathcal{P}_k} T_{oj} n_{jk} \right) \leq T^{max} \\ &\Leftrightarrow b_f \leq \frac{T^{max}}{\max_{\mathcal{P}_k} \left(\sum_{j \in \mathcal{P}_k} T_{oj} n_{jk} \right)} \end{aligned}$$

which tells us about the maximum size we can use for a batch in order to fulfill that constraint.

One final consideration is to be taken regarding the rotation of batches. Let's say we want to produce $X_f^* \leq X_f^{max}$ final products. We introduce the variable D as the difference in time between the moment a batch is produced and the moment a subsequent batch is produced so that

$$X_f^* = \frac{b_f}{D}$$

(number of items of the final product over the time of a "period" of production) The figure (3.4) illustrates this notion.



Figure 3.4: GANT diagram with production cycle

Chapter 4

Model with set-up time considerations

Up to now, our considerations were only focused on the operation times for making products and the recipe of the final product. In this chapter, we will start complexifying our model by adding the setup time for machines. The setup time corresponds to the time in which the machine isn't used to actually produce one good but rather to "get ready" to do so. For instance, one machine has a "cold start" or may need some preparation like changing drills or configurations. We will see in this chapter that the setup time is actually a critical aspect of the production plant and that it drives the batch sizes in a very precise way. The figure (4.1) shows how the setup time impacts the production of a given batch.

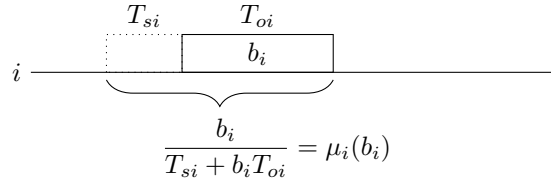


Figure 4.1: Simple example of batch production with setup time in GANT representation

Concerning the rate of production of machine i as depicted in figure (4.1), a few considerations are to be made : (1) of course, if we set the setup time to be zero we get back with our previous formula $\mu_i = 1/T_{oi}$; (2) when $b_i \rightarrow \infty$, we also get that $\mu_i(b_i) \rightarrow \frac{1}{T_{oi}}$, this can be simply explain as the fact that, if we produce more and more products in one batch, the setup time (which is paid just once), becomes less and less significant with respect to the production time.

As far as the representation in terms of graph (bill of materials) is concerned, we will now fill the first element of the couple valuating transformations with the setup times for each machine as you can see in figure (4.2).

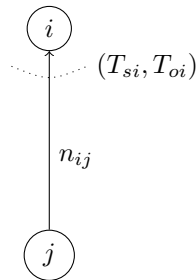


Figure 4.2: Simple example of batch production with setup time in GANT representation

4.1 Finding the bottleneck machine(s)

But let's get back to our prior concerns : production rates. Of course the following fomrula still holds :

$$X_f^{max} = \min_i \left(\frac{\mu_i}{n_{if}} \right)$$

As we did in the last chapter, we will consider setting the batch size as exactly $b_i = n_{if}b_f$ so that

$$X_f^{max} = \min_i \left(\frac{b_f}{T_{si} + T_{oi}b_fn_{if}} \right)$$

which in trun is no longer a constant but rather a function of b_f . We will denote it as $X_f^{max}(b_f)$ not to forget it. The meaning of this is that now the maximum rate of production depends on the batch size we use with respect to the characteristics of the machines we use. And it is somehow logical since we know that producing batches of 1 item would mean "paying" for the setup time for 1 operation time whereas using batches of 10 items would mean paying once the setup time for 10 operation times ! As well, the overall production time is influenced by the setup times in a very simple way :

$$T_{prod}(b_f) = \max_{\mathcal{P}_k} \left(\sum_{i \in \mathcal{P}_k} (T_{si} + T_{oi}b_fn_{if}) \right)$$

Let's have an example to better understand what it means to have a maximum production rate which depends on the batch size. Let's consider the bill of materials of figure (4.3) and let's compute X_f^{max} . Note that when no n_{if} coefficient is written, it is assumed to be 1.

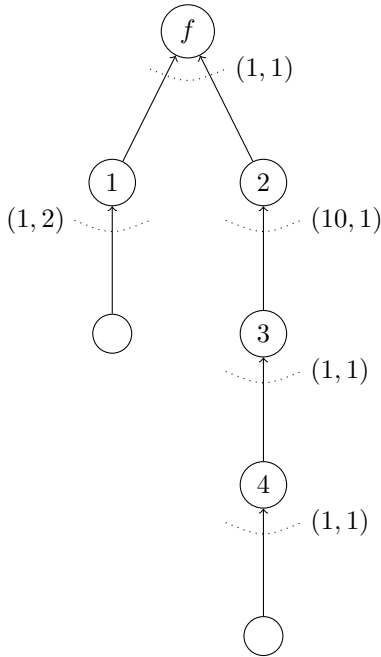


Figure 4.3: Bill of material with setup time example

Applying the given formula, we gen that

$$\begin{aligned} X_f^{max} &= \min_i \left(\frac{b_f}{1 + 1.b_f}; \frac{b_f}{1 + 2.b_f}; \frac{b_f}{10 + 1.b_f}; \frac{b_f}{1 + 1.b_f}; \frac{b_f}{1 + 1.b_f} \right) \\ &= \min_i \left(\frac{b_f}{1 + 1.b_f}; \frac{b_f}{1 + 2.b_f}; \frac{b_f}{10 + 1.b_f} \right) \\ &= \min_i \left(\frac{1}{1 + 2.b_f}; \frac{1}{10 + 1.b_f} \right) \end{aligned}$$

You may notice that in line 2 of these equations, we got rid of the first value because, in any case, we knew it was lower than the other ones since $1 + 1.b_f < 1 + 2.b_f$ and $1 + 1.b_f < 10 + 1.b_f$. The discussion now continues with respect to the machine 1 and 2 and we should look at figure (4.4) which plots the functions $b_f \mapsto 1 + 2b_f$ and $b_f \mapsto 10 + b_f$. The bottleneck machine is then given by $\arg \min_i \left(\frac{1}{1+2.b_f}; \frac{1}{10+1.b_f} \right) = \arg \max_i (1 + 2.b_f; 10 + 1.b_f)$ which corresponds to the second machine (machine 2) as long as the batch size is less than 9 and corresponds to the first machine (machine 1) when $b_f > 9$.

Similarly, the production time for producing a batch of size b_f can be computed as

$$\begin{aligned} T_{prod}(b_f) &= \max((1 + 2.b_f) + (1 + 1.b_f); (1 + 1.b_f) + (1 + 1.b_f) \\ &\quad + (10 + 1.b_f) + (1 + 1.b_f)) \\ &= \max(2 + 3.b_f; 13 + 4.b_f) \end{aligned}$$

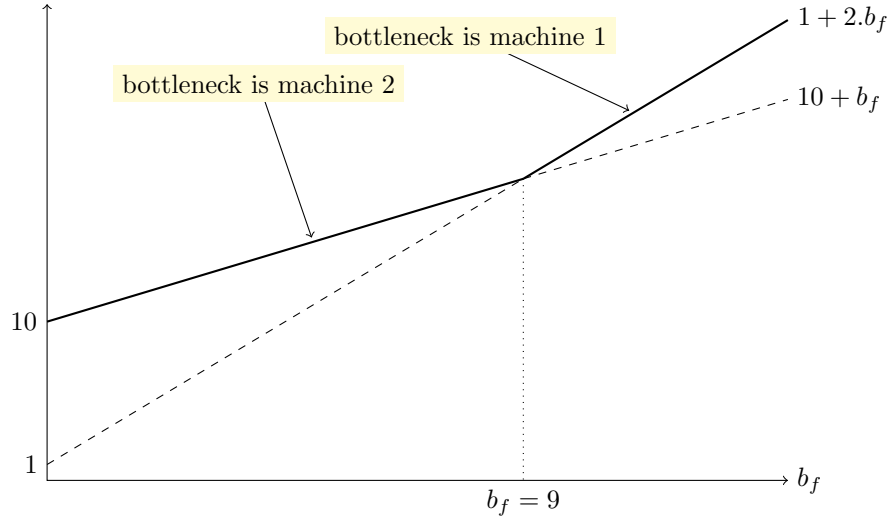


Figure 4.4: Bottleneck machine depending on the batch size

4.2 Controlling the production rate

Let's get deeper with the interpretation of what have been said by looking at both figure (4.8) and (4.9). The first figure shows a situation in which we would like compute the batch size of our production by imposing a certain production rate X_f^* so that $X_f \geq X_f^*$. Solving this problem is, in fact, rather simple and can be done like so

$$\begin{aligned}
 X_f &\geq X_f^* \Leftrightarrow \min_i \left(\frac{b_f}{T_{si} + b_f T_{oi} n_{if}} \right) \geq X_f^* \\
 &\Leftrightarrow \frac{b_f}{T_{si} + b_f T_{oi} n_{if}} \geq X_f^*, \forall i \\
 &\Leftrightarrow b_f \geq X_f^* (T_{si} + T_{oi} b_f n_{if}) \\
 &\Leftrightarrow b_f \geq \frac{X_f^* T_{si}}{1 - X_f^* T_{oi} n_{if}}, \forall i \\
 &\Leftrightarrow b_f \geq \max_i \left(\frac{X_f^* T_{si}}{1 - X_f^* T_{oi} n_{if}} \right) = b_f^*
 \end{aligned}$$

Notice that we used the fact that $1 - X_f^* T_{oi} n_{if} \geq 0$ to go from the third line to the fourth one. This is true because it is equivalent to $X_f^* \geq \frac{1}{T_{oi} n_{if}}$ which has to hold, otherwise, imposing X_f^* would not be feasible.

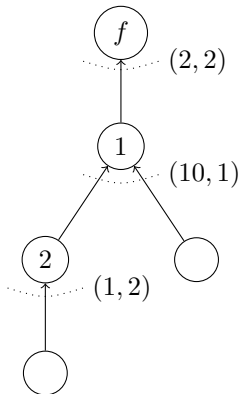


Figure 4.5: Bill of material

Let's take an example in order to see how we can play with these equations. Let's consider the bill of materials presented in figure (4.5) and let us assume that we want to achieve a production rate of at least $X_f^* = 0.9 X_f^{max}(b_f^{max})$. For this example, we will assume that there is no upper bound for batch sizing so that $b_f^{max} = \infty$. It holds that

$$X_f^{max} = \min_i \left(\frac{1}{T_{oi} n_{if}} \right) = \min \left(\frac{1}{2}; \frac{1}{1}; \frac{1}{2} \right) = \frac{1}{2}$$

which tells us that the bottleneck machines are the one denoted by f and by 2, and that the production we actually want to achieve is $X_f^* = .45$. That being said, we can now compute the minimum batch size we can use in order to fulfill this constraint. This is done like so :

$$b_f^* = \max_i \left(\frac{.45 \times 2}{1 - .45 \times 2}; \frac{.45 \times 10}{1 - .45 \times 1}; \frac{.45 \times 1}{1 - .45 \times 2} \right) = 9$$

Let's assume that we choose to use exactly b_f^* as our batch size (any greater size is actually also valid) to produce at rate X_f^* , exactly. Then, one cycle should last

$$D^* = \frac{b_f}{X_f^*} = \frac{9}{.45} = 20$$

And let's have a look at the GANT diagram given by these parameters in figure (4.6). We can see that the bottleneck machine is the machine f and that the machine 2 is no more a bottleneck as we had originally planned. That is because the bottleneck machine actually depends on the batch size we use. When we firstly computed the bottleneck machines (when we computed X_f^{max}), we did it assuming that the batch size was the maximum one, which is no longer true. To make this even more clear, let's deal with the case where we want, with the same production plant, produce at a rate of $X_f^* = .3$ instead of 0.45. But let us still use batches of 9.

First, let's make sure that it is possible to use $b_f = 9$ by computing the lowest batch size we can use with respect to the constraint $X_f \geq .3$:

$$\begin{aligned} b_f^* &= \max \left(\frac{.3 \times 2}{1 - .3 \times 2}; \frac{.3 \times 10}{1 - .3 \times 1}; \frac{.3 \times 1}{1 - .3 \times 2} \right) \\ &= \max \left(\frac{3}{2}; \frac{30}{7}; \frac{3}{4} \right) \\ &= \frac{30}{7} \approx 4 \end{aligned}$$

This tells us two things : (1) that the batch size has to be greater than 4 (so we can use 9 while respecting our constraints) ; (2) that if we use $b_f = b_f^*$, the bottleneck machine will be machine 1. However, and as discussed, the GANT diagram in figure (4.7), the bottleneck machine is actually still the machine f if we use 9 as our batch size. The production cycle is computed as always by $D^* = \frac{9}{.3} = 30$.

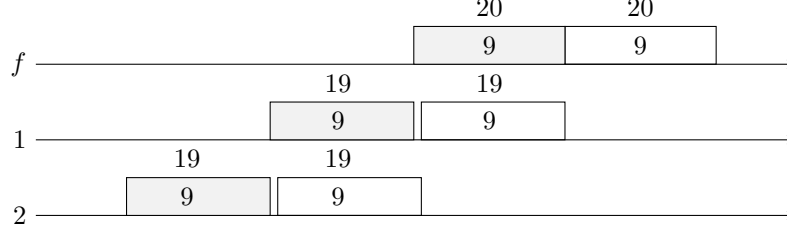


Figure 4.6: GANT diagram when $X_f^* = 0.45$ and batch size is $b_f = b_f^* = 9$

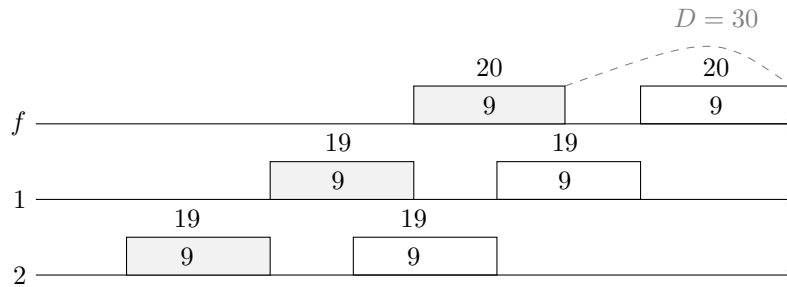


Figure 4.7: GANT diagram when $X_f^* = 0.3$ and batch size is $b_f = 9$

4.3 Controlling the production time

We now can satisfy constraints on the production rate, which is a good thing. However, production plants often encounter another kind of constraints which is linked to the production time. In this section, we will focus on what

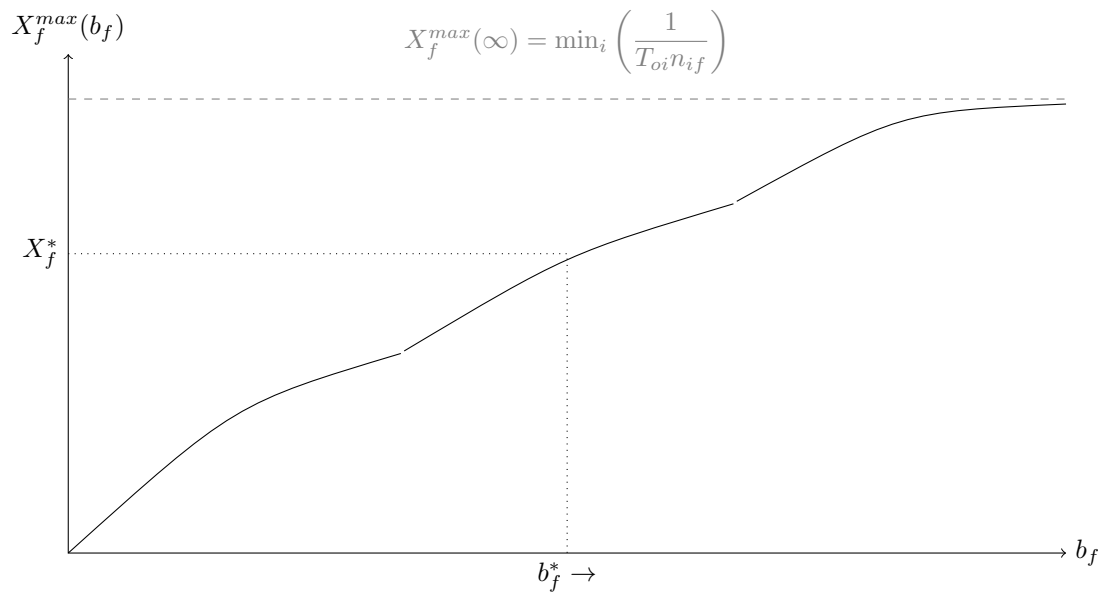


Figure 4.8: How the production rate depends on the batch size

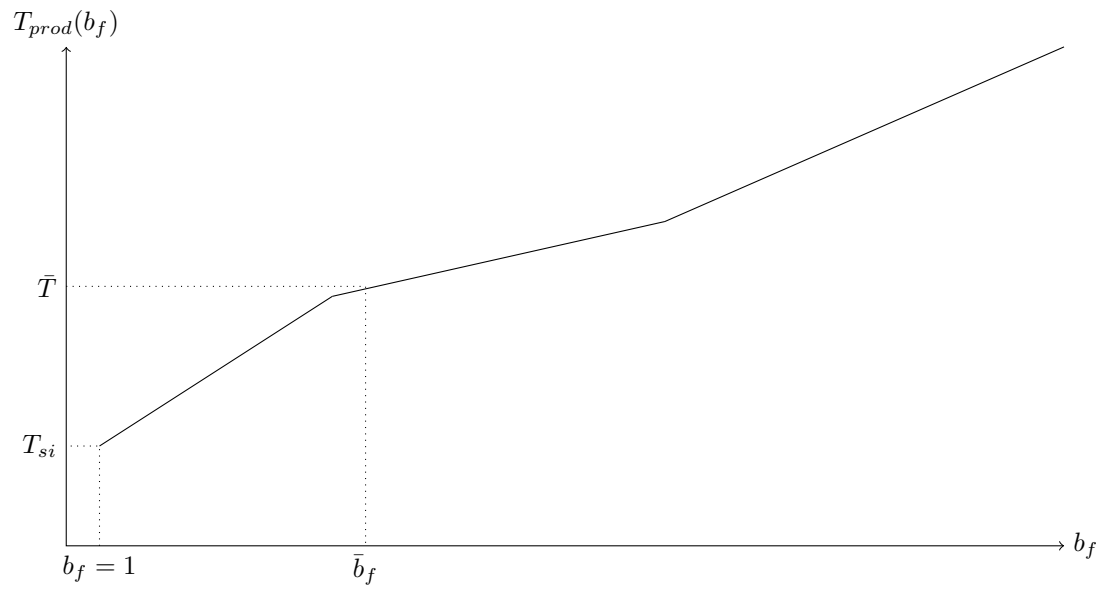


Figure 4.9: How the production time depends on the batch size

happens when we impose that the overall production time must be lower than a certain duration. More formally, we impose that $T_{prod} \leq \bar{T}$. The computation is similar to what have been done so far and is given by :

$$\begin{aligned}
T_{prod} \leq \bar{T} &\Leftrightarrow \max_{\mathcal{P}_k} \left(\sum_{i \in \mathcal{P}_k} (T_{si} + T_{oi} b_f n_{if}) \right) \leq \bar{T} \\
&\Leftrightarrow \sum_{i \in \mathcal{P}_k} T_{si} + b_f \sum_{i \in \mathcal{P}_k} T_{oi} n_{if} \leq \bar{T}, \forall \mathcal{P}_k \\
&\Leftrightarrow b_f \leq \frac{\bar{T} - \sum_{i \in \mathcal{P}_k} T_{si}}{\sum_{i \in \mathcal{P}_k} T_{oi} n_{if}}, \forall \mathcal{P}_k \\
&\Leftrightarrow b_f \leq \min_{\mathcal{P}_k} \left(\frac{\bar{T} - \sum_{i \in \mathcal{P}_k} T_{si}}{\sum_{i \in \mathcal{P}_k} T_{oi} n_{if}} \right) = \bar{b}_f
\end{aligned}$$

Let's take another example considering figure (4.10) to introduce how we can handle situations in which we both have a condition on production time and on production rate. To do that, consider the following problem : finding b_f subject to the constraints :

$$\begin{cases} X_f \geq X_f^* \\ T_{prod} \leq \bar{T} \end{cases} \quad \text{with } X_f^* = 0.8 \times X_f^{max}(b_f^{max}) \text{ and } \bar{T} = 10 \times T_{prod}(1)$$

Let's first compute the inputs of our problem which are $X_f^{max}(b_f^{max})$ and $T_{prod}(1)$. Since no maximum batch size has been given, we will consider that $b_f^{max} = \infty$ so that

$$X_f^{max}(b_f^{max}) = \min_i \left(\frac{1}{T_{oi} n_{if}} \right) = \min \left(\frac{1}{1}; \frac{1}{2} \right) = \frac{1}{2}$$

and

$$T_{prod}(1) = \max_{\mathcal{P}_k} (25; 5) = 25$$

which gives us the values we wanted : $X_f^* = 0.4$ and $T_{prod} = 250$.

Considering the first constraint (on the production rate), we can compute a lower bound for our batch size :

$$\begin{aligned}
b_f^* &= \max_i \left(\frac{X_f^* T_{si}}{1 - X_f^* T_{oi} n_{if}} \right) = \max \left(\frac{.4 \times 1}{1 - .4 \times 1 \times 1}; \frac{.4 \times 20}{1 - .4 \times 1 \times 1}; \frac{.4 \times 1}{1 - .4 \times 1 \times 1}; \frac{.4 \times 1}{1 - .4 \times 2 \times 1} \right) \\
&= \max \left(\frac{2}{3}, \frac{40}{3}, \frac{2}{3}; 2 \right) \approx 13.33
\end{aligned}$$

Using the other constraint (on the production time), we can then compute an upper bound for the batch size :

$$\bar{b}_f = \min_{\mathcal{P}_k} \left(\frac{\bar{T} - \sum_{i \in \mathcal{P}_k} T_{si}}{\sum_{i \in \mathcal{P}_k} T_{oi} n_{if}} \right) = \min \left(\frac{250 - 22}{3}; \frac{250 - 2}{3} \right) = 76$$

Which gives us the final solution :

$$b_f^* = 14 \leq b_f \leq 76 = \bar{b}_f$$

Of course, it is possible to write infeasible constraints which would lead to having $\bar{b}_f < b_f^*$. This situation is depicted in figure (4.11). If such a situation were to happen, the way of solving it is to decrease the setup time which means either (1) numerically decreasing the setup time (buying new machines for example) or buy masking the setup time.

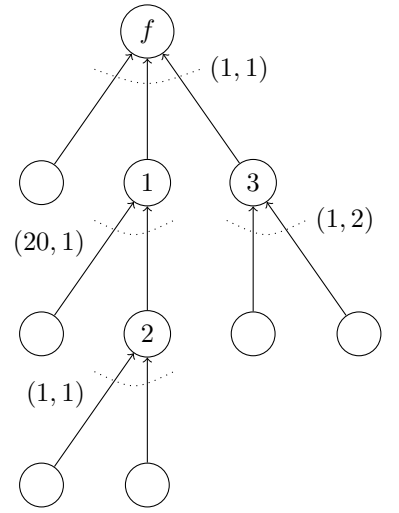


Figure 4.10: Bill of materials

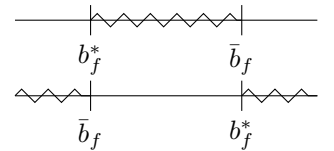


Figure 4.11: Batch size

Chapter 5

Giving up $b_i = b_f n_{if}$