Model Predictive Control

1. Basics

Griasde

Compact set: Subset of Euclidean space being closed (*i.e., containing all its limit points*) and bounded (*i.e., having all its points lie within some fixed distance of each other*).

Matrix Inversion: $A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj}(A)$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

2. Introduction

2.1. System Class

Time-invariant discrete-time dynamical control system
$x(k+1) = f(x(k), u(k)); \quad k = 0, 1, 2, \dots \quad x(0) = x_0$
with state $x(k) \in \mathbb{R}^n$ and control input $u(k) \in \mathbb{R}^m$.

Notation:
$\underline{u}^N := \{u(0), u(1), \dots u(N-1)\}$
$\underline{x}^N_{\underline{u}}(x_0) := \{x_0, x_{\underline{u}}(1, x_0), \dots x_{\underline{u}}(N, x_0)\}$
or if context is clear for brevity:
$\underline{x}^N_{\underline{u}}(x_0) := \{x_0, x(1), \dots x(N)\}$

2.2. Cost Function

Infinite Horizon:

$$J_\infty(x_0, \underline{u}^\infty) = \sum_{k=0}^{\infty} l\left(x_{\underline{u}}(k, x_0), u(k)\right)$$

Finite Horizon:

$$J_N\left(x_0, \underline{u}^N\right) = \sum_{k=0}^{N-1} l\left(x_{\underline{u}}(k, x_0), u(k)\right) + J_f\left(x_{\underline{u}}(N, x_0)\right)$$

with stage cost $l(x, u)$ and target cost $J_f(x)$.

2.3. Constraints

Input constraints: $u(k) \in \mathbb{U}$
State constraints: $x(k) \in \mathbb{X}, k = 1, 2, 3, \dots;\ x(N) \in \mathbb{X}_f$

Admissible Controls: $\mathcal{U}_N(x_0) := \{\underline{u} | (x_0, \underline{u}) \in \mathbb{Z}\}$
Feasible Initial Values: $\mathcal{X}_N := \{x_0 \in \mathbb{X} | \mathcal{U}_N(x_0) \neq \emptyset\}$
with

$$\mathbb{Z}_N := \Big\{ (x_0, \underline{u}) \,|\, u(k) \in \mathbb{U}, x_{\underline{u}}(k, x_0) \in \mathbb{X}, \\ k = 0, 1, \dots, N-1; x_{\underline{u}}(N, x_0) \in \mathbb{X}_f \Big\}$$

2.4. Optimization Problem

$$\mathbb{P}_N(x_0) : J_N^*(x_0) = \min_{\underline{u}} \{J_N(x_0, \underline{u}) \,|\, \underline{u} \in \mathcal{U}_N(x_0)\}$$

Assumption 1:
$f,\ l,\ J_f$ are continuous with $f(0,0) = 0,\ l(0,0) = 0,\ J_f(0) = 0$.

Assumption 2:
$\mathbb{X}$ is closed, $\mathbb{X}_f$ and $\mathbb{U}$ are compact and all sets contain the origin.

Under *Assumption 1* and *Assumption 2*, the optimization problem $\mathbb{P}_N(x_0)$ has a solution for all $x_0 \in \mathcal{X}_N$.
($\rightarrow$ Theorem of Weierstrass: Sets are sequentially compact *and* every bounded sequence of complex numbers contains at least one convergent subsequence.)

2.5. Controller

$\kappa_N(x_0) = u^*(0, x_0)$
with optimal control input $u^*(0, x_0)$ from solution of $\mathbb{P}_N(x_0)$,
$\underline{u}^* = \{u^*(0, x_0), u^*(1, x_0), \dots u^*(N-1, x_0)\}$.

2.6. Basic time-invariant MPC algorithm

System: $x^+ = f(x, u)$

Cost: $J(x, \underline{u}) = \sum_{k=0}^{N-1} l(x(k), u(k)) + J_f(x(N))$

Constraints: $x(k) \in \mathbb{X}, u(k) \in \mathbb{U}$ for all $k \in \mathbb{N}_0$ and $x(N) \in \mathbb{X}_f$
where N is the prediction horizon.

- Measure x, determine $\mathcal{U}_N(x)$
- Solve $\mathbb{P}_N(x)$ and obtain $\underline{u}^*(x)$
- Control with $\kappa_N(x)$ such that $x^+ = f(x, \kappa_N(x))$
- Repeat for $x := x^+$

2.7. Constrained Optimization in a nutshell

Cost function: $\min F(z)$
Equality constraints: $g(z) = 0$
Inequality constraints: $h(z) \leq 0$

a) unconstrained
If z^* is minimum, then $\nabla F(z^*) = 0$.
b) equality constrained
Lagrange function $L = F(z) + \lambda^\top g(z)$ with Lagrange multiplier λ.
If z^* is minimum, then $\nabla_z L(z^*, \lambda^*) = 0$ and $\nabla_\lambda L(z^*, \lambda^*) = 0$.
c) inequality constrained
Lagrange function $L = F(z) + \mu^\top h(z)$ with KKT multiplier μ.
If z^* is minimum then $\nabla_z L(z^*, \mu^*) = 0$ and $\mu^* \geq 0, h(z^*) \leq 0$ and $\mu_i^* h_i(z^*) = 0$ for all i.

3. Dynamic Programming

3.1. Problem Statement

Time-invariant discrete-time dynamical control system
$x(k+1) = f(x(k), u(k)); \quad k = 0, 1, 2, \dots \quad x(0) = x_0$

Cost: $V(x_0, \underline{u}) = \sum_{k=0}^{N-1} l(x(k), u(k)) + V_f(x(N))$

State constraints: $x(k) \in \mathbb{X},\ k = 0, 1, \dots, N-1,\ x(N) \in \mathbb{X}_f$
Input constraints: $u(k) \in \mathbb{U}, k = 1, 2, \dots, N-1$

Find $\underline{u} = \{u(0), u(1), \dots, u(N-1)\}$ where $u(k) = \mu_k(x(k))$ are control laws.

3.2. Notation

Cost-to-go: $V_i\left(x, \underline{u}^i\right) = \sum_{k=i}^{N-1} l(x(k), u(k)) + V_f(x(N))$

with $\underline{u}^i = \{u(i), u(i+1), \dots, u(N-1)\}$

Optimal cost-to-go: $V_i^*(x) = \min_{\underline{u}^i \in \Upsilon_i(x)} V_i\left(x, \underline{u}^i\right)$
with

$$\Upsilon_i(x) := \\ \Big\{\underline{u}^i \,|\, \text{for initial state } x(i) = x : u(k) \in \mathbb{U}, k = i, \dots, N-1; \\ x(k) \in \mathbb{X}, k = i+1, \dots, N-1; x(N) \in \mathbb{X}_f \Big\}$$

and $\Xi_i := \{x \in \mathbb{X} | \Upsilon_i(x) \neq \emptyset\}$

Recursive construction of feasible sets Ξ_i from behind:
$\Xi_N = \mathbb{X}_f$
$\Xi_i = \{x(i) \in \mathbb{X} | x(i+1) \in \Xi_{i+1} \text{ with } u(i) \in \mathbb{U}\}$,
$i = N-1, N-2, \dots 0$

3.3. Bellman Recursion

Recursive calculation of optimal-cost-to-go V_i^* from behind:
$V_N^*(x(N)) = V_f(x(N))$

$$V_i^*(x(i)) = \min_{u(i)} \Big\{ l(x(i), u(i)) + V_{i+1}^*(f(x(i), u(i))) \ | \\ u(i) \in \mathbb{U}, x(i) \in \mathbb{X}, f(x(i), u(i)) \in \Xi_{i+1} \Big\}, \\ i = N-1, N-2, \dots, 0$$

delivers $u(i) = \mu_i^*(x(i))$

4. Stability

4.1. Stability Concepts

System class: $x^+ = \tilde{f}(x)$
Equilibrium point: $x_{eq} = \tilde{f}(x_{eq})$

Assumption:
If $\tilde{f}$ is not continuous, it is at least locally bounded.

Definition of stability in the sense of Lyapunov:
The equilibrium point $x_{\text{eq}} = 0$ of $x^+ = \tilde{f}(x)$ is locally stable, if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $\|x(0)\| \leq \delta(\varepsilon)$, it holds that $\|x(k)\| \leq \varepsilon$ for all $k > 0$.

- Locally asymptotically stable, if in addition $\lim_{k\to\infty} \|x(k)\| = 0$ for $x(0)$ close to the origin.
- Globally asymptotically stable, if in addition $\lim_{k\to\infty} \|x(k)\| = 0$ for all $x(0) \in \mathbb{R}^n$.
- Asymptotically stable in $\mathcal{X}$, if in addition $\lim_{k\to\infty} \|x(k)\| = 0$ for all $x(0) \in \mathcal{X}$, where $\mathcal{X}$ is positive.

Definition of positive invariance:
$\mathcal{X}$ is positive invariant for $x^+ = \tilde{f}(x)$, if $\tilde{f}(x) \in \mathcal{X}$ for all $x \in \mathcal{X}$.
(*Once a system trajectory enters set $\mathcal{X}$, it will never leave it again.*)

Definition of comparison functions:

- A function α is a class $\mathcal{K}$ function, if it is continuous and strictly increasing with $\alpha(0) = 0$.
- A function α is a class $\mathcal{K}_\infty$ function, if it a class $\mathcal{K}$ function and in addition unbounded.
- A function α is a class $\mathcal{PD}$ (*positive definite*) function, if it is continuous with $\alpha(0) = 0$ and $\alpha(x) > 0$ for all $x \neq 0$.

Lyapunov's direct method:
A function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is a global Lyapunov function for the equilibrium of $x^+ = \tilde{f}(x)$, if $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\alpha_3 \in \mathcal{PD}$ exist, such that for all $x \in \mathbb{R}^n$:
(1) Bounded: $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$
(2) Descent: $V(f(x)) - V(x) \leq -\alpha_3(\|x\|)$
If V is a global Lyapunov function for the equilibrium of $x^+ = \tilde{f}(x)$, then the equilibrium is globally asymptotically stable.

Lyapunov's direct method (constrained):
A function $V : \mathcal{X} \rightarrow \mathbb{R}$ with $\mathcal{X}$ invariant is a Lyapunov function for the equilibrium of $x^+ = \tilde{f}(x)$ on $\mathcal{X}$, if $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\alpha_3 \in \mathcal{PD}$ exist, such that for all $x \in \mathcal{X}$:
(1) Bounded: $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$
(2) Descent: $V(f(x)) - V(x) \leq -\alpha_3(\|x\|)$
If V is a Lyapunov function for the equilibrium of $x^+ = \tilde{f}(x)$ on $\mathcal{X}$, then the equilibrium is asymptotically stable on $\mathcal{X}$.

4.2. Assumptions for Stability

Assumption 3:
$l(x, u) > \alpha_l(\|x\|) \quad \forall x \in \mathcal{X}_N, \forall u \in \mathbb{U}$
$J_f(x) \leq \alpha_f(\|x\|) \quad \forall x \in \mathbb{X}_f$ where α_l, α_f are class $\mathcal{K}_\infty$ functions.

Assumption 4:
J_f is a Control Lyapunov Function (CLF), that means $J_f(0) = 0$, $J_f > 0$ and there exists $u \in \mathbb{U}$ such that $J_f(f(x, u)) - J_f(x) \leq -l(x, u)\ \forall x \in \mathbb{X}_f$.
If system is CLF, it is feedback stabilizable.

Assumption 5:
$\mathbb{X}_f$ is control invariant, that means, if $x \in \mathbb{X}_f$ then there exists $u \in \mathbb{U}$ such that $f(x, u) \in \mathbb{X}_f$.

4.3. Stability of MPC

Under Assumptions 1 to 5, the equilibrium x_{eq} is asymptotically stable in $\mathcal{X}_N$ for $x^+ = f(x, \kappa_N(x))$.
Proof: Choose Lyapunov function $V_N(x) = J_N(x, \underline{u}^*)$ and show property (1) and (2) of Lyapunov direct method.

4.4. Recursive Feasibility

Definition: MPC is said to be recursively feasible, if one can assure that there is a solution to $\mathbb{P}_N(x^+)$ having a solution of $\mathbb{P}_N(x)$.

Recursive Feasibility:
If $\mathbb{X}_f$ is control invariant, then

- $\mathcal{X}_{j-1} \subseteq \mathcal{X}_{j-1},\ j = 1, \dots, N$
- $\mathcal{X}_{j-1}$ is control invariant, $j = 1, \dots, N$
- MPC is recursively feasible

Notes:

5. MPC for Linear Systems

5.1. System Class for Linear MPC

System class: $x^+ = Ax + Bu$

Cost: $J_N(x, \underline{u}) = \frac{1}{2}\sum_{k=0}^{N-1} \|x(k)\|_Q^2 + \|u(k)\|_R^2 + \frac{1}{2}\|x(N)\|_{P_f}^2$

Constraints: $x(k) \in \mathbb{X}, u(k) \in \mathbb{U}, x(N) \in \mathbb{X}_f$, where $\mathbb{X}$, $\mathbb{U}$ and $\mathbb{X}_f$ are convex polytopes

5.2. LQ Control (no constraints, non-receding finite horizon)

System Class: $x^+ = Ax + Bu$

Cost:

$$V_0(x_0, u) = \frac{1}{2}\sum_{k=1}^{N-1} \underbrace{\|x(k)\|_Q^2}_{\substack{\text{penalize}\\\text{bad performance}}} + \underbrace{\|u(k)\|_R^2}_{\substack{\text{penalize}\\\text{actuator effort}}} + \frac{1}{2}\|x(N)\|_{P_f}^2$$

Control law: $u(k) = K(k)x(k)$

while for $k = 0, \dots, N-1$, with $P(N) = P_f$

$$K(k) = -\left(B^\top P(k+1)B + R\right)^{-1} B^\top P(k+1)A$$

and

$$P(k) = A^\top P(k+1)A + Q - A^\top P(k+1)B$$
$$\left(B^\top P(k+1)B + R\right)^{-1} B^\top P(k+1)A$$

Recursion for the Riccati Matrix:

$$P(k) = A^\top P(k+1)A + Q - \boldsymbol{K}(k)^\top B^\top P(k+1)A$$

In addition:
$V_0^*(x_0) = \frac{1}{2}x_0^\top P(0)x_0$

5.3. LQ Control (no constraints, infinite horizon)

System Class: $x^+ = Ax + Bu$

Cost: $V(x, \underline{u}) = \frac{1}{2}\sum_{k=0}^{\infty} \|x(k)\|_Q^2 + \|u(k)\|_R^2$

Control Law: $u(k) = K_\infty x(k)$, while

$K_\infty = -\left(B^\top P_\infty B + R\right)^{-1} B^\top P_\infty A$ and Riccati Equation:

$P_\infty = Q + K_\infty^\top R K_\infty + (A + BK_\infty)^\top P_\infty (A + BK_\infty)$

Stationary Riccati Matrix:

$$P_\infty = A^\top P_\infty A + Q + \boldsymbol{K}_\infty^\top B^\top P_\infty A$$

5.4. MPC (constrained, receding horizon)

Stability of equilibrium $x_{eq} = 0$ under MPC, if
unconstrained: $P_f = P_\infty$
constrained:
1.) $P_f = P_\infty$
2.) constraint admissibility: $\mathbb{X}_f \subseteq \{x \in \mathbb{X} | Kx \in \mathbb{U}\}$
3.) positive invariance: $x \in \mathbb{X}_f \Rightarrow x^+ = (A + BK_\infty)x \in \mathbb{X}_f$

5.5. Underlying Optimization Problem (QP)

Using previews $x(k) = A^k x_0 + A^{k-1}Bu(0) + \dots + Bu(k-1)$ for all k in horizon in cost function

$$J_N(x_0, \underline{u}) = \frac{1}{2}\sum_{k=0}^{N-1} \|x(k)\|_Q^2 + \|u(k)\|_R^2 + \frac{1}{2}\|x(N)\|_{P_f}^2$$

allows to transform the cost into
$J_N(x, \underline{u}) = \frac{1}{2}\underline{u}^\top H(x_0)\underline{u} + c(x_0)^\top \underline{u} + d(x_0)$ with $\underline{u} \in \mathcal{U}_N(x_0)$ polytopes QP problem (=Quadratic cost with linear constraints) allows for efficient numerics.

6. Generalized Predictive Control (GPC)

6.1. System Class

System class:

$$A\left(z^{-1}\right)y(t) = B\left(z^{-1}\right)z^{-d}u(t-1) + C\left(z^{-1}\right)\frac{e(t)}{\Delta}$$

- Denominator: $A\left(z^{-1}\right) = 1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_n z^{-n}$
- Denumerator: $B\left(z^{-1}\right) = 1 + b_1 z^{-1} + \dots + b_m z^{-m}$, $m < n$
- Shift operator: $z^{-k}y(t) = y(t-k)$
- Dead time: z^{-d}, in the following $d = 0$
- $e(t)$: white noise with zero mean
- $\Delta = 1 - z^{-1}$ (for $u(t-1)$. for $u(t)$: $\Delta = 1$)
- $C\left(z^{-1}\right)$ for colored noise, in the following $C\left(z^{-1}\right) = 1$

Cost:

$$J = \sum_{j=1}^{N} \delta(j)(\underbrace{\hat{y}(t+j|t)}_{\substack{\text{predicted}\\\text{output}}} - \underbrace{w(t+j)}_{\substack{\text{future ref.}\\\text{trajectory}}})^2 + \sum_{j=1}^{M} \lambda(j)(\Delta u(t+j-1))$$

- Horizons: N is prediction horizon, M is control horizon, in the following $M = N$
- Weighting $\delta(j)$, $\lambda(j)$
- Control input $\Delta u = u(t) - u(t-1)$
- Prediction from time t to $t+j$:
  $\hat{y}(t+j|t) = G_j(z^{-1})\Delta u(t+j-1) + F_j(z^{-1})y(t)$

6.2. Diophantine Equation

→ Fewer equations than unknown variables
$1 = E_j z^{-1}\tilde{A}\left(z^{-1}\right) + z^{-j}F_j\left(z^{-1}\right)$ with $\tilde{A} = \Delta A$
$E_j\left(z^{-1}\right)$: polynomial of degree $j-1$
$F_j\left(z^{-1}\right)$: polynomial of degree of $\tilde{A}$

Prediction:
$\underline{y} = \underline{Gu} + \underline{p}$ ($\underline{p}$ often denoted as f)
where the choice of $\underline{y}$ and $\underline{u}$ as:

$$\underline{y} = [\hat{y}(t+1|t), \hat{y}(t+2|t), \dots, \hat{y}(t+N|t)]^\top$$
$$\underline{u} = [\Delta u(t), \Delta u(t+1), \dots, \Delta u(t+N-1)]^\top$$

defines G and $\underline{p}$:

$$G = \begin{bmatrix} g_0 & 0 & \dots & 0 \\ g_1 & g_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ g_{N-1} & g_{N-2} & \dots & g_0 \end{bmatrix}$$

$$\underline{p} = \underline{F}\left(z^{-1}\right)y(t) + \underline{G'}\left(z^{-1}\right)\Delta u(t-1)$$

with

$$\underline{G'}\left(z^{-1}\right) = \begin{bmatrix} \left(G_1\left(z^{-1}\right) - g_0\right)z \\ \left(G_2\left(z^{-1}\right) - g_0 - g_1 z^{-1}\right)z^2 \end{bmatrix}$$

$\underline{F}\left(z^{-1}\right) = \left[F_1\left(z^{-1}\right), F_2\left(z^{-1}\right), \dots, F_N\left(z^{-1}\right)\right]^\top$
$G_j\left(z^{-1}\right) = B\left(z^{-1}\right)E_j\left(z^{-1}\right)$
and g_j j-th coefficients of polynomial G_j.

6.3. QP Problem

Cost:

$$J = u\left(G^T QG + R\right)\underline{u} + 2(\underline{p} - \underline{w})^T QG\underline{u} + (\underline{p} - \underline{w})^T Q(\underline{p} - \underline{w})$$

is minimized by control sequence of future controls
$\underline{u} = -\left(G^T QG + R\right)^{-1} G^T Q(\underline{p} - w)$
where only $\underline{u}_1 = \Delta u(t)$ is applied as control.

7. Numerics

7.1. Nonlinear Programming (NP)

Cost function: $F(z)$
Equality constraints: $g(z) = 0$
Inequality constraints: $h(z) \le 0$

Necessary conditions for a minimum:
If z^* is feasible minimum, then $\nabla_z L(z^*, \lambda^*, \mu^*) = 0$, $\nabla_\lambda L(z^*, \lambda^*, \mu^*) = 0$ and $\mu^* \ge 0$, $h(z^*) \le 0$ and $\mu_i^* h_i(z^*) = 0$ for all i with Lagrange function $L = F(z) + \lambda^\top g(z) + \mu^\top h(z)$ with multipliers λ and μ.
Active Set: $\mathcal{A} = \{j | \mu_j > 0\}$

7.2. Unconstrained Minimization: Newton's Method

Find minimum z^* numerically as follows

- Initialize: Guess $z^{(0)}$ close to z^* for $k = 0, 1, 2, \dots$
- Update (calculate tangent and corresponding root value):
  $z^{(k+1)} = z^{(k)} + \alpha^{(k)}d^{(k)}$ where
  $d^{(k)} = -\left(\nabla^2 F\left(z^{(k)}\right)\right)^{-1}\nabla F\left(z^{(k)}\right)$ (search direction)
  and $\alpha^{(k)} = \operatorname{argmin} F(z^{(k)} + \alpha d^{(k)})$ (line search)
- Stop if $\|\nabla F(z^{(k)})\| < \epsilon_{tol}$

7.3. Constrained Minimization: Quadratic Programming

Applies for QP problem class:
Quadratic cost function: $F(z) = \frac{1}{2}z^\top Hz + c^\top z$
Linear equality constraints: $g(z) = Ez + e = 0$
Liner inequality constraints: $h(z) = Iz + i \le 0$

Find minimum z^* numerically as follows

- Initialize: Guess initial active set $\mathcal{A}_0$ for $k = 0, 1, 2, \dots$
- Update optimization variables:

$$\begin{pmatrix} H & E^\top & \left(I^{(k)}\right)^\top \\ E & 0 & 0 \\ I^{(k)} & 0 & 0 \end{pmatrix}\begin{pmatrix} z^{(k+1)} \\ \lambda^{(k+1)} \\ \mu^{(k+1)} \end{pmatrix} = \begin{pmatrix} -c \\ -e \\ -i^{(k)} \end{pmatrix}$$

where $I^{(k)}z + i^{(k)} \le 0$ are inequality constraints from active set $\mathcal{A}_k$ that are treated as equality constraints.

- Update active set:
  When for $j \in \mathcal{A}_k : \mu_j^{(k+1)} < 0$, delete const. j from active set
  When for $j \notin \mathcal{A}_k I_j z^{(k+1)} + i_j \ge 0$, add constraint j to active set and thus get $\mathcal{A}_{k+1}$

7.4. Constrained Minimization: Sequential Quadratic Programming

Combine Newton's Methods (linearization of NP problem to obtain QP problem) and QP method (choice of feasible active set of linearized problem).

7.5. Constrained Minimization: Interior Point

Cost function: $F(z)$
Equality constraints: $g(z) = 0$
Former inequality constraints, now equality constraints: $h(z) + s = 0$
New inequality constraints: $s \ge 0$

Lagrange Function $L = F(z) + y^\top g(z) + w^\top(h(z) + s) - \mu^\top s$
Necessary Conditions that are related to inequality constraints:
$s \ge 0$, $\mu \ge 0$ and $s^\top \mu = 0$

Relax complementary conditions $s^\top \mu = 0$ by introducing barrier parameter ϵ to $s^\top \mu = \epsilon$ and solve with $\epsilon \to 0$.

8. Robust MPC

8.1. Types of Uncertainties

Parametric uncertainty, modeling errors, measurement noise, etc.
E.g. additive disturbance: $x^+ = f(x, u) + w$ where w is disturbance and $w \in \mathbb{W}$, where $\mathbb{W}$ is bounded.

8.2. Robust Stability

Under bounded small disturbances, does the state reach a small vicinity of the origin.
Compare formal definitions RGAS or Practical Stability in the literature.

Nominal Robust Stability only if Lyapunov Function is continuous (e.g. for linear MPC)!

8.3. Tube base MPC for Linear Systems

Nominal model: $z^+ = Az + Bu$
Disturbed (real) model: $x^+ = Ax + Bu + w$
Use tube $S(k)$, $k = 0, 1, 2, \dots$ to check constraint admissibility: $S(k)$ is a set such that all possible (apply all possible disturbances!) trajectories $x(k)$ of the disturbed system fulfill $x(k) \in \{z(k) \oplus S(k)\}$
MPC feedback control: $u^* = v^* + K(x - z)$, where K is found offline.
Appropiate choice of K reduces the size of the tube.
Optimize cost to find v^* such that x^* is constraint admissible by constraint tightening using the tube.

Notes: