Math221 Linear Algebra

What might make this course different from most other mathematics courses you have taken so far…

Linear Algebra is one of the first courses where you will get a glimpse of *what mathematics is*, *how mathematics is done* and *what applications it has*.

Asking the "Right" Questions—What (fundamental) questions will you learn to ask when dealing with this discipline?

Linear Algebra is a highly structured mathematical system based on definitions, axioms, theorems and proofs, all of which are of fundamental importance to mathematics and mathematical activities. It has many applications to other branches of mathematics, natural sciences and social sciences.

Throughout the course, using the *"Right" Tools*, you will learn to ask the following questions:

<u>About Definitions:</u>

- Why is defining this object necessary?
- How does the definition emerge?
- Why is it defined this way?
- Are there other equivalent ways of defining it?
- Why is the choice of the words, are they precise?
- What happens if certain changes are made to the definition?
- What are some examples (from previous studied mathematical notions) of this object defined?
- How is this object related to other concepts?
- Can you find any real-world applications which can be modeled by the object defined?

<u>About Axioms:</u>

For most of you, it is the first time that you will see structures defined by a set of carefully chosen axioms. While these are important concepts of crucial significance to mathematics (e.g. General Vector Space, Inner Product Space, etc.), the idea of the axioms and using axioms to define objects might be fresh and foreign and usually difficult for you to comprehend and grasp. Learning to ask the following questions will help.

- What is an axiom? What does it mean to designate a statement as an axiom?
- Why do we use axioms to define a structure?
- When defining a specific structure, why is this set of axioms chosen? (How did people come up with them? Why these and not others? What if some axioms were eliminated, or changed? Is this the best choice?)
- Give some examples of objects that satisfy all the axioms, some of the axioms and not all.
- Can some of the axioms be deleted or added so as to describe more general or restricted objects?

<u>About Theorems:</u>

Theorems are central to mathematics. While typically you have encountered theorems in your previous experiences (such as Geometry and Calculus), in Linear Algebra, it is probably the first time you will be asked to fully understand every aspect and detail of a theorem. The following set of questions offers a starting point to guide the students. As you mature as a thinker of mathematics, you will ask more and more questions about a theorem.

About the conditions (hypotheses) and conclusions:

- What are the conditions of the theorem?
- What are the conclusions of the theorem? Can the conclusions be stated in an equivalent but different way?
- Are the conditions necessary, sufficient or both?

- Can one change/delete some of the conditions? How does that affect the conclusions? Do we obtain a meaningful conclusion? Do we obtain a stronger, or weaker or the same conclusion?
- Are there other (similar) conditions leading to the same conclusion?

About some related statements:

- What is the converse of the theorem?
- Does the converse always, or sometimes or never hold?
- If the converse is always true, can you prove it? If it is sometimes true, can you determine when it is true? If it never holds, why? Can you provide examples when it does not hold?

- What is the contrapositive? (Note: the contrapositive of a statement holds whenever the statement holds).
- When is it easier to apply the theorem in its contrapositive form?

About the significance of the statement:
Being a textbook for undergraduates, the book adopted for Linear Algebra calls almost every statement proven a "theorem." In real mathematical activities however, mathematicians make a clear and necessary distinction between propositions and theorems, lemmas and theorems. Being able to evaluate the significance of a statement and distinguish between major theorems and minor facts is a sign of gaining deeper understanding and mathematical maturity.

- Is this theorem important?
- Why or why not is the theorem important? What is its significance? How is it related to other facts and concepts?

- Can you generalize the theorem? What are some meaningful generalizations?
- What are some related questions that worth investigating? Can you think of some ways to approach them?

<u>About Proofs:</u>
In Linear Algebra, you will be asked to study all the proofs of the theorems encountered, memorize and be able to reproduce them. You will also be asked to prove statements in homework and tests.

When studying an existing proof:

- What is the main idea in the proof?
- How does one think of the ideas?
- How are the conditions used?
- How is the argument written and why is it written this way?

When proving a statement:

- What is the main idea?
- Can it be proven directly?
- Can it be proven indirectly? Can I argue by contradiction?
- Can I reduce it to some other statements to prove?
- Can I dissect the conclusions to cases and prove each case?
- Can I get an approach by starting backwards, from the conclusions?
- Are there different proofs? Which approach is better and why?
- Can a more general statement be proven?
- How do I structure the written proof? Are the statements logical, precise and succinct? Can I write it better and clearer?

<u>About the Role of Linear Algebra:</u>
Linear Algebra requires a student to think abstractly. Many key notions are obtained by extracting the common features from other objects in mathematics and models in natural and social sciences. Asking the following

questions will help one to see the connections and appreciate the role of Linear Algebra in mathematics and its applications.

- What previously studied objects in other fields of mathematics and/or natural and social sciences does this concept remind you of?
- What are some known applications? How is Linear Algebra applied?
- What more knowledge/knowhow in Linear Algebra is desirable in order to further its applications?
- Can you think of applications not mentioned in the book and maybe not yet thought of by others?

Using the "Right" Tools—What methods of analysis and argument will students learn to help them investigate questions in this discipline?

You will learn to read a mathematics text critically. Guided by the questions listed above, you will study your textbooks, "chew on" definitions and investigate theorems and their proofs in depth.

You will understand what a mathematical proof is, study and practice different proof techniques.

You will practice *thinking specifically(concretely)*, for example, giving examples, counterexamples, imagining how the concepts apply to a specific situation.

You will practice *thinking generally*, for example, generalizing a theorem, a concept, or a situation.

You will practice *thinking abstractly*, for example, observing how seemingly different objects in mathematics share certain commonalities, identifying these common features and finding a way to describe them both comprehensively and precisely.

Following your interests, you will search for and investigate various applications of Linear Algebra.

Meta-Level Reflection – How will students be asked to reflect upon, question, and appreciate the ways of inquiry used in this discipline?

You are required to keep a journal and make at least one entry every week. The entry can be your reflection on anything related to the course. For example, it can be your thoughts on the questions raised above; or the most impressive thing you have discovered/realized/understood in the past two weeks; or the most puzzling concept/theorem/proof you have encountered recently and why it is so; or a question not raised in the book but asked by the students and why it is a meaningful and interesting question.

Connections to Something Bigger—What specific "real-world" questions, interdisciplinary connections, or ethical issues will students explore in an attempt to deepen their understanding and appreciation of the class content?

Linear Algebra has many applications in other fields of mathematics, natural sciences and social sciences. At the end of the course there will be a bonus assignment on presenting an application of your choice. For example,

- In economics, the Leontief Input-Output Models developed by the 1973 Nobel prize winner in economics, Wassily Leontief, to study the relationships between different sectors in an economy by using matrix methods.
- Dynamical systems and Markov chains, which uses matrix methods to analyze the behavior of physical systems that evolve over time, such as problems in business, ecology, demographics, sociology and most of the physical sciences.

- Internet search engines (applications of matrices)
- Games of strategy, game theory (application of matrix multiplication and basic probability concepts)
- Computer graphics (application of matrix algebra and analytic geometry)
- Computed Tomography such as X-ray imaging (application of linear systems, natural logarithms and Euclidean space)
- Fractals and Chaos (application of geometry of linear operators, eigenvalues and eigenvectors, limits and continuity)
- Cryptography (application of matrices, Gaussian elimination, matrix operations, linear independence, linear transformations)
- Genetics, investigate the propagation of an inherited trait in successive generations (application of eigenvalues and eigenvectors, diagonalization of a matrix, limits)

<u>To succeed in this course, you are expected to do the following:</u>

- **Before class**, guided by the Reading Guide you should study each section. You must digest the definitions, follow the examples and try to understand the theorem statements and their proofs. Furthermore, you should attempt the easier problems at the beginning of the exercise section. Write down any questions that you have.
- **In class**, listen to the presentation carefully. Actively participate by taking notes, asking and answering questions.
- **After class,** study the textbook and your class notes. If you have more questions, ask your instructor or another student. Complete the homework as soon as you can. Ask questions if you run into difficulties.
- **Write your journal at least once a week,** whenever you have any insight related to the course material.
- **After getting graded homework back,** check the solutions and correct any mistakes you might have made. Ask questions if you don't understand.
- **Forming study groups** is a very good idea.