Symbolic linear algebra in R with `caracas`

Søren Højsgaard

Contents

Define the following matrices

```
L1 <- as_sym(matrix(c(1,1,1,1), byrow=T))
L2 <- as_sym(matrix(c(1,0,1,0,0,1,0,1), byrow=T, ncol=2))
L <- cbind(L1, L2)
```

$$L1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad L2 = \begin{bmatrix}1 & 0\\1 & 0\\0 & 1\\0 & 1\end{bmatrix}, \quad L = \begin{bmatrix}1 & 1 & 0\\1 & 1 & 0\\1 & 0 & 1\\1 & 0 & 1\end{bmatrix}$$

Basis

The span (columnspace) of a matrix A is denoted enoted $C(A)$. A basis for $C(L)$ is denoted B_L while w is the generic form of vectors in L:

```
B_L <- columnspace(L)
v <- vec(B_L, "v")
```

$$B_L = \begin{bmatrix}1 & 1\\1 & 1\\1 & 0\\1 & 0\end{bmatrix}, \quad v = \begin{bmatrix}v_1 + v_2\\v_1 + v_2\\v_1\\v_1\end{bmatrix}$$

Basis for orthogonal complement

A basis for the orthogonal complement $W = C(L)^\perp$ of L can be found as follows: W consists of the vectors v that are orthogonal to any vector in $C(L)$; that is to any vector of the form Lx. In matrix notation, $W = \{v | v'Lx = 0 \quad \forall x\}$. That is $x'L'v = 0$. For this to be satisfied for all x we must have $L'v = 0$, that is v is in the null space of L':

```
B_W <- nullspace(t(L))
w <- vec(B_W, "w")
```

$$B_W = \begin{bmatrix}-1 & 0\\1 & 0\\0 & -1\\0 & 1\end{bmatrix}, \quad w = \begin{bmatrix}-w_1\\w_1\\-w_2\\w_2\end{bmatrix}$$

Vectors in L and V are indeed orthogonal:

```
sum(v * w)
```

```
## c: 0
```

Null space

```
N <- nullspace(L); N
```

```
## c: [[-1, 1, 1]]^T
```

```
vec(N)
```

```
## c: [[-v1, v1, v1]]^T
```

```
N <- nullspace(L2); N
```

```
## NULL
```

```
vec(N)
```

The null space of L1 is spanned by

```
N <- nullspace(t(L1))
b <- vector_sym(ncol(N), "b")
```

$$N = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, n = \begin{bmatrix} -b_1 - b_2 - b_3 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}, n = \begin{bmatrix} -n_1 - n_2 - n_3 \\ n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

Intersection of two spaces

Let U and V be bases and matrices. A basis for the intersection can be found as: We look for $C(U) \cap C(V)$ ie vectors z of the form

$$z = Ux = Vy$$

for (x, y) non-zero. That is, we find the null space of $A = [U| - V]$ because if $A(x, y)t = 0$ then $Ux = Vy$. For example

```
U = nullspace(t(L1))
V = L2
A = cbind(U, -V)
N_A <- nullspace(A)
```

$$A = \begin{bmatrix} -1 & -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}, \quad N_A = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

So the vectors in the intersection $U \cap V$ have the form:

```
x <- N_A[1:3,]
y <- N_A[4:5,]
U %*% x
```

```
## c: [[-1, -1, 1, 1]]^T
```

```
V %*% y
```

```
## c: [[-1, -1, 1, 1]]^T
```

```
z <- vec(U %*% x, "z")
z
```

```
## c: [[-z1, -z1, z1, z1]]^T
```

$$U = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad V = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}; \quad z = \begin{bmatrix} -z_1 \\ -z_1 \\ z_1 \\ z_1 \end{bmatrix}$$

projections

```
II <- diag_(1, 4)
P_L1 <- L1 %*% inv(t(L1) %*% L1) %*% t(L1)
R <- II-P_L1
```