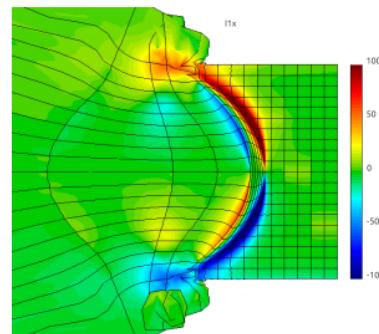
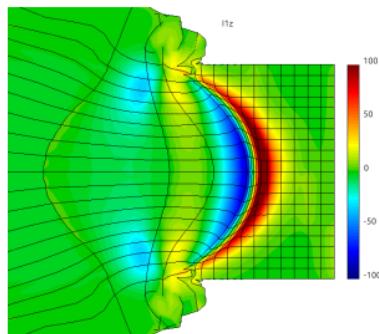
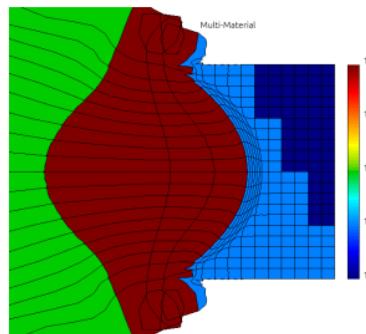


Serious Numerics for Serious Physics: Nonlocal Transport Hydrodynamic Model on Curvilinear Meshes

Milan Holec



Center for Applied Scientific Computing,
Lawrence Livermore National Laboratory,
Livermore, CA, USA
January 16, 2018

- 1** Introduction to Nonlocal Transport Hydrodynamic model (NTH)
 - Thermal radiative transfer
 - Nonlocal electron transport
- 2** High-order discontinuous Galerkin BGK transport scheme for hydrodynamics (DG-BGK&T)
 - DG-BGK&T finite element transport scheme
 - Nonlocal transport on curvilinear meshes
 - DG-BGK&T + LAGHOS
- 3** Application of NTH in high-power laser physics
 - Nonlocal transport hydrodynamic model for laser-heated plasmas
 - Measurement of preheat due to nonlocal electron transport in warm dense matter
 - The effect of pre-plasma formation under nonlocal transport conditions for ultra-relativistic laser-plasma interaction
- 4** Extras...
- 5** Conclusions

Thermal Radiative Transfer

Radiation transport equation

The radiation intensity $I_\nu(\mathbf{x}, \mathbf{n})$ representing photons of frequency ν obeys the equation

$$\mathbf{n} \cdot \nabla I_\nu = \eta_\nu - \chi_\nu I_\nu,$$

where *emmisivity* $\eta_\nu(\rho, T)$ and absorptivity $\chi_\nu(\rho, T)$ are considered to be isotropic.

$$\nabla \cdot \mathbf{q}_\nu = 4\pi\eta_\nu - \chi_\nu E_\nu$$

$$\nabla \cdot \mathbf{P}_\nu = -\frac{\chi_\nu}{c} \mathbf{q}_\nu$$

Radiation diffusion

In the case of low anisotropy pressure tensor can be approximated as $\mathbf{P}_\nu = \mathbf{I} f_\nu E_\nu$ and the radiation field can be modeled by

$$\nabla \cdot \left(\frac{c}{\chi_\nu} \nabla (f_\nu E_\nu) \right) = \chi_\nu E_\nu - 4\pi\eta_\nu,$$

where the lowest anisotropy approximation $\tilde{I}_\nu(\mathbf{x}, \mathbf{n}) = I_\nu^0(\mathbf{x}) + \mathbf{n} \cdot \mathbf{I}_\nu^1(\mathbf{x})$ corresponds to the *variable Eddington factor* $f_\nu = \frac{1}{3}$.

¹D. Mihalas and B. Mihalas, Foundations of Radiation Hydrodynamics (Oxford University Press, New York, 1985).

Boltzmann transport equation

$$\frac{\partial \mathbf{f}}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} \mathbf{f} + \frac{q_e}{m_e} \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \cdot \nabla_{\mathbf{v}} \mathbf{f} = \sigma \nabla_{\mathbf{v}} \cdot \int \frac{|\mathbf{v} - \mathbf{v}'|^2 \mathbf{l} - (\mathbf{v} - \mathbf{v}') \otimes (\mathbf{v} - \mathbf{v}')}{|\mathbf{v} - \mathbf{v}'|^3} (\nabla_{\mathbf{v}} \mathbf{f}(\mathbf{v}) f(\mathbf{v}') - \mathbf{f}(\mathbf{v}) \nabla_{\mathbf{v}} f(\mathbf{v}')) d\mathbf{v}'$$

Fluid equations

$$\begin{aligned}\frac{d\rho}{dt} &= -\rho \nabla_{\mathbf{x}} \cdot \mathbf{U} \\ \rho \frac{d\mathbf{U}}{dt} &= \nabla_{\mathbf{x}} \cdot \sigma \\ \rho \frac{d\varepsilon}{dt} &= \sigma : \nabla_{\mathbf{x}} \mathbf{U} - \nabla_{\mathbf{x}} \cdot (\mathbf{q}_e + \mathbf{q}_R)\end{aligned}$$

Microscopic closure

$$\begin{aligned}\sigma &= -\rho \int (\mathbf{v} - \mathbf{U}) \otimes (\mathbf{v} - \mathbf{U}) \mathbf{f} d\mathbf{v}^3 \approx -\mathbf{l} p + \tilde{\sigma}(\nabla \mathbf{U}) \\ \mathbf{q}_e &= \frac{\rho}{2} \int |\mathbf{v} - \mathbf{U}|^2 (\mathbf{v} - \mathbf{U}) \mathbf{f} d\mathbf{v}^3 \approx \frac{\rho}{2} \int_{4\pi} \mathbf{n} \int_0^\infty |\mathbf{v}|^5 \mathbf{f} d|\mathbf{v}| d\mathbf{n} \approx -\kappa(T^{2.5}) \nabla T \\ \mathbf{q}_R &= \int_{4\pi} \mathbf{n} \int_\nu I_\nu^p d\nu d\mathbf{n} \approx - \int_\nu \frac{c}{\chi_\nu} \nabla(f_\nu E_\nu) d\nu\end{aligned}$$

BGK collision operator

$$\begin{aligned}
 C_{ei}(f^e, f^i) &\stackrel{m_e/m_i}{\approx} \sigma \nabla_v \cdot \left(\frac{1}{v} \left(\mathbf{I} - \frac{\mathbf{v} \otimes \mathbf{v}}{v^2} \right) \cdot \nabla_v f^e \right) \\
 &\stackrel{\text{spher}}{\approx} \frac{\sigma}{v^3} \frac{1}{\sin(\phi)} \left[\frac{\partial}{\partial \phi} \left(\sin(\phi) \frac{\partial f^e}{\partial \phi} \right) + \frac{1}{\sin(\phi)} \frac{\partial^2 f^e}{\partial \theta^2} \right] \\
 &\stackrel{f^0 + f^1 \cos(\phi)}{\approx} -\frac{2\sigma}{v^3} f_1^e \cos(\phi) = 2\nu_{ei} (f_0^e - f^e)
 \end{aligned}$$

Then, the electron distribution function can be governed by the BGK Boltzmann transport equation

$$\mathbf{n} \cdot \nabla_x f^e + \frac{q_e}{m_e |\mathbf{v}|} \mathbf{E} \cdot \nabla_v f^e = \frac{2(f_{MB}(|\mathbf{v}|, T_e) - f^e)}{\lambda_{ei}},$$

where f_{MB} is the Maxwell-Boltzmann equilibrium distribution and λ_{ei} the electron mean free path when scattered on ions.

Chapman-Enskog approximation in small parameter λ_{ei}

$$\begin{aligned}
 f^e &= f_0 + \lambda_{ei} f_1 + O(\lambda_{ei}^2) \approx f_{MB}(|\mathbf{v}|, T_e) - f_{MB}(|\mathbf{v}|, T_e) g(\bar{Z}) \left(\frac{|\mathbf{v}|^2}{2v_{T_e}^2} - 4 \right) \mathbf{n} \cdot \frac{\lambda_{ei} \nabla T_e}{T_e} \\
 &\rightarrow \mathbf{q}_{SH} = \kappa_{SH} T_e^{\frac{5}{2}} \nabla T_e
 \end{aligned}$$

$$\frac{\lambda_{ei} f_1}{f_0} = 0.25 \left(\frac{|\mathbf{v}|^2}{2v_{T_e}^2} - 4 \right) \mathbf{n} \cdot \frac{\lambda_{ei} (|\mathbf{v}|^4) \nabla T_e}{T_e} < 0.1 \quad \longrightarrow \quad \text{Kn}^e = \frac{\lambda_{ei} \nabla T_e}{T_e} < 7.5 \times 10^{-4}$$

Nonlocal Transport Hydrodynamic model

$$\begin{aligned}
 \frac{d\rho}{dt} &= -\rho \nabla \cdot \mathbf{u}, \\
 \rho \frac{d\mathbf{u}}{dt} &= -\nabla(p_i + p_e), \\
 \rho \left(\frac{\partial \varepsilon_i}{\partial T_i} \frac{dT_i}{dt} + \frac{\partial \varepsilon_i}{\partial \rho} \frac{d\rho}{dt} \right) &= -p_i \nabla \cdot \mathbf{u} - G(T_i - T_e), \\
 \rho \left(\frac{\partial \varepsilon_e}{\partial T_e} \frac{dT_e}{dt} + \frac{\partial \varepsilon_e}{\partial \rho} \frac{d\rho}{dt} \right) &= -p_e \nabla \cdot \mathbf{u} - \nabla \cdot (\mathbf{q}_e + \mathbf{q}_R) + G(T_i - T_e) + Q_{IB}(\mathbf{E}_L),
 \end{aligned}$$

the quantities $\frac{\partial \varepsilon}{\partial \rho} = \frac{\partial f}{\partial \rho} - T \frac{\partial^2 f}{\partial \rho \partial T}$, $\frac{\partial \varepsilon}{\partial T} = -T \frac{\partial^2 f}{\partial T^2}$, $p = \rho^2 \frac{\partial f}{\partial \rho}$, $G = \rho \frac{\partial \varepsilon_e}{\partial T_e} \nu_{ei}$ provides our HerEOS equation of state.

Nonlocal transport of photon intensity

$$I^p = \int_\nu f^p \frac{h^4 \nu^3}{c^2} d\nu$$

$$\mathbf{n} \cdot \nabla I^p = \frac{a T_e^4 - I^p}{\lambda^p}$$

Radiation closure relations

$$\mathbf{q}_R = \int_{4\pi} \mathbf{n} I^p d\mathbf{n} \xrightarrow{\text{diffusive}} \nabla \cdot \mathbf{q}_R = -\nabla \cdot \left(\frac{c}{3\chi_R} \nabla E \right)$$

Nonlocal transport of electron intensity $I^e = \int_\nu f^e \frac{m_e |\mathbf{v}|^5}{2} d|\mathbf{v}|$

$$\mathbf{n} \cdot \nabla I^e = k_B \left(n_i \frac{d\bar{Z}}{dt} \frac{3}{8\pi} + \frac{n_e \sqrt{2}}{8.51 \lambda^{ei} \pi^{\frac{3}{2}}} v_{T_e} \right) T_e - \frac{I^e}{8.51 \lambda^{ei}}$$

Electron closure relations

$$\mathbf{q}_e = \int_{4\pi} \mathbf{n} I^e d\mathbf{n} \xrightarrow{\text{diffusive}} \nabla \cdot \mathbf{q}_e = -\nabla \cdot \left(\kappa_{SH} T_e^{\frac{5}{2}} \nabla T_e \right)$$

Planar geometry - transport equation

$$\cos(\Phi) \frac{\partial I}{\partial z} = S_T T_e - kI,$$

$$I(t, z, \Phi) = (\omega_\Phi \otimes \psi_z)^T \cdot I^{n+1}, T_e(t, z) = \phi_z^T \cdot T_e^{n+1}$$

Composition of the multi-dimensional interpolation based on *outerproduct* \otimes

$$\omega_\Phi = [\omega_1(\Phi), \dots, \omega_{M_\Phi}(\Phi)]^T, \psi_z = [\psi_1(z), \dots, \psi_{N_f}(z)]^T, \phi_z = [\phi_1(z), \dots, \phi_{N_e}(z)]^T$$

$$\int_{\Omega_\Phi} \int_{\Omega_z} (\omega_\Phi \otimes \psi_z) \otimes \left[\cos(\Phi) \left(\omega_\Phi \otimes \frac{\partial \psi_z}{\partial z} \right)^T \cdot I^{n+1} + k (\omega_\Phi \otimes \psi_z)^T \cdot I^{n+1} - (\omega_\Phi \otimes \phi_z)^T \cdot S_T \cdot T_e^{n+1} \right] d\Omega_z \sin(\Phi) d\Phi = \\ \int_{\Omega_\Phi} \int_{\Gamma_n, n_\Gamma < 0} (\omega_\Phi \otimes \psi_z) \otimes \left[(\omega_\Phi \otimes \psi_z)^T \cdot I^{n+1} - (\omega_\Phi \otimes \tilde{\psi}_z)^T \cdot \tilde{I} \right] (\cos(\Phi) n_\Gamma^z) d\Gamma_z \sin(\Phi) d\Phi$$

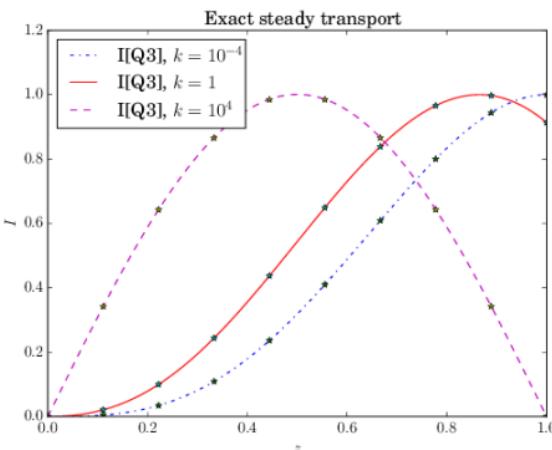
Discrete DG transport equation - transport operator inversion $I^{n+1} = A \cdot T_e^{n+1} + b(\bar{I})$

$$D \cdot I^{n+1} + k M \cdot I^{n+1} - B \cdot I^{n+1} = S \cdot T_e^{n+1} - \tilde{B} \cdot \tilde{I}$$

Exact steady state "given direction" transport

$$\cos(\Phi_0) \frac{dI(z, \Phi_0)}{dz} = k (\sin(\pi z) - I(z, \Phi_0))$$

Three different values of $k = 10^{-4}, 1, 10^4$, which corresponds to free-streaming, nonlocal, and diffusive transport ($\Phi_0 = \pi/4$).



element	cells	$E_{L1}^{k=10^{-4}}$	$q_{L1}^{k=10^{-4}}$	$E_{L1}^{k=1}$	$q_{L1}^{k=1}$	$E_{L1}^{k=10^4}$	$q_{L1}^{k=10^4}$
I[Q1]	10	2.7e-07		2.3e-03		8.3e-03	
I[Q1]	20	4.9e-08	2.5	4.3e-04	2.4	1.6e-03	2.4
I[Q1]	40	1.2e-08	2.0	1.1e-04	2.0	3.7e-04	2.1
I[Q1]	80	2.9e-09	2.0	2.6e-05	2.0	9.0e-05	2.1
I[Q2]	10	4.6e-09		4.1e-05		3.5e-07	
I[Q2]	20	4.1e-10	3.5	3.5e-06	3.5	5.2e-08	2.8
I[Q2]	40	4.5e-11	3.2	4.0e-07	3.2	1.2e-08	2.1
I[Q2]	80	5.4e-12	3.1	4.7e-08	3.1	2.8e-09	2.1
I[Q3]	10	7.3e-11		2.6e-07		2.3e-06	
I[Q3]	20	2.8e-12	4.7	8.4e-09	5.0	1.0e-07	4.5
I[Q3]	40	1.5e-13	4.2	4.3e-10	4.3	5.6e-09	4.2
I[Q3]	80	8.9e-15	4.1	2.4e-11	4.1	3.3e-10	4.1

It is worth mentioning that the method works well also in diffusive limit $k = 10^4$ ($Kn \approx 10^{-4}$).

²M. Holec, J. Limpouch, R. Liska and S. Weber, *Int. J. Numer. Meth. F.*, 83, 779 (2017).

Exact steady state "full" transport

Model steady equation

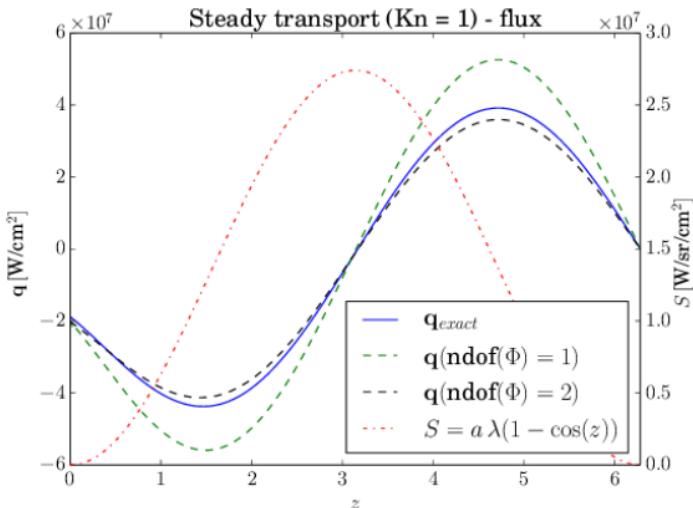
$$\cos(\Phi) \frac{dI(z, \Phi)}{dz} = k(S(z) - I(z, \Phi))$$

Energy density flux

$$\mathbf{q}(z) = 2\pi \int_0^\pi \cos(\Phi) I(z, \Phi) \sin(\Phi) d\Phi$$

Divergence of energy density flux

$$\nabla \cdot \mathbf{q}(z) = 2\pi \int_0^\pi \cos(\Phi) \frac{\partial I(z, \Phi)}{\partial z} \sin(\Phi) d\Phi$$



Relative L1 error convergence in polar angle Φ of $\nabla \cdot \mathbf{q}$

transport regime	$\text{Kn}=\lambda/L - \text{ndof}(\Phi)$	1	2	3	4	5	20	40
transparent/ballistic	100.0	1.5e-01	4.1e-01	9.5e-02	2.4e-01	7.1e-02	5.1e-02	2.2e-02
nonlocal/highly anisotropic	10.0	2.0e-01	3.7e-01	1.1e-01	2.0e-01	7.8e-02	1.3e-02	1.1e-03
nonlocal/anisotropic	1.0	3.5e-01	8.8e-02	2.6e-02	1.1e-02	2.7e-03	1.1e-05	9.6e-07
nonlocal/almost isotropic	0.1	1.5e-01	9.6e-03	4.8e-03	3.3e-03	6.9e-04	5.7e-07	4.6e-07
diffusive/isotropic	0.01	1.4e-01	1.0e-02	5.9e-03	4.5e-03	2.0e-03	2.9e-05	2.9e-05

²M. Holec, J. Limpouch, R. Liska and S. Weber, *Int. J. Numer. Meth. F.*, 83, 779 (2017).

Planar geometry - transport equation

$$\cos(\Phi) \frac{\partial I}{\partial z} = S_T T_e - (k + \sigma + \cos(\Phi) E_z) I + \sigma I_0, \quad I_0 = \frac{1}{2} \int_{\Omega_\Phi} I \sin(\Phi) d\Phi$$

$$I(t, z, \Phi) = (\omega_\Phi \otimes \psi_z)^T \cdot \textcolor{blue}{I^{n+1}}, \quad T_e(t, z) = \phi_z^T \cdot \textcolor{red}{T_e^{n+1}}$$

Composition of the multi-dimensional interpolation based on *outerproduct* \otimes

$$\omega_\Phi = [\omega_1(\Phi), \dots, \omega_{M_\Phi}(\Phi)]^T, \quad \psi_z = [\psi_1(z), \dots, \psi_{N_f}(z)]^T, \quad \phi_z = [\phi_1(z), \dots, \phi_{N_e}(z)]^T$$

$$\begin{aligned} \int_{\Omega_\Phi} \int_{\Omega_z} (\omega_\Phi \otimes \psi_z) \otimes & \left[\cos(\Phi) \left(\omega_\Phi \otimes \frac{\partial \psi_z}{\partial z} \right)^T \cdot \textcolor{blue}{I^{n+1}} + (k + \cos(\Phi) E_z) (\omega_\Phi \otimes \psi_z)^T \cdot \textcolor{blue}{I^{n+1}} - \frac{\sigma}{2} \int_{\Omega_\Phi} \omega_\Phi^T \sin(\Phi) d\Phi \otimes \psi_z^T \cdot \textcolor{blue}{I^{n+1}} \right. \\ & \left. - (\omega_\Phi \otimes \phi_z)^T \cdot \mathbf{S}_T \cdot \textcolor{red}{T_e^{n+1}} \right] d\Omega_z \sin(\Phi) d\Phi = \\ \int_{\Omega_\Phi} \int_{\Gamma_{n \cdot n_\Gamma < 0}} (\omega_\Phi \otimes \psi_z) \otimes & \left[(\omega_\Phi \otimes \psi_z)^T \cdot \textcolor{blue}{I^{n+1}} - (\omega_\Phi \otimes \tilde{\psi}_z)^T \cdot \textcolor{red}{\tilde{I}} \right] (\cos(\Phi) \mathbf{n}_\Gamma^z) d\Gamma_z \sin(\Phi) d\Phi \end{aligned}$$

Discrete DG transport equation - transport operator inversion $I^{n+1} = \mathbf{A} \cdot \textcolor{red}{T_e^{n+1}} + \mathbf{b}(\tilde{I})$

$$\mathbf{D} \cdot \textcolor{blue}{I^{n+1}} + ((k + \sigma) \mathbf{M} + E_z \mathbf{M}_{\cos(\Phi)} - \sigma \mathbf{M} \mathbf{I}) \cdot \textcolor{blue}{I^{n+1}} - \mathbf{B} \cdot \textcolor{blue}{I^{n+1}} = \mathbf{S} \cdot \textcolor{red}{T_e^{n+1}} - \tilde{\mathbf{B}} \cdot \textcolor{red}{\tilde{I}}$$

²M. Holec, J. Limpouch, R. Liska and S. Weber, *Int. J. Numer. Meth. F.*, 83, 779 (2017).

Planar geometry - energy equation equipped with the nonlocal transport

$$a \frac{dT_e}{dt} + G_{ei}(T_e - T_i) + \int_{4\pi} \cos(\Phi) \frac{\partial I}{\partial z} \sin(\Phi) d\Phi d\Theta = P_e$$

$$I(t, z, \Phi) = (\omega_\Phi \otimes \psi_z)^T \cdot I^{n+1}, T_e(t, z) = \phi_z^T \cdot T_e^{n+1}, T_i(t, z) = \tilde{\phi}_z^T \cdot T_i^{n+1}$$

$$\begin{aligned} \int_{\Omega_z} \phi_z \otimes \left[a \phi_z^T \cdot \frac{T_e^{n+1} - T_e^n}{\Delta t} + G_{ei} \left(\phi_z^T \cdot T_e^{n+1} - \tilde{\phi}_z^T \cdot T_i^{n+1} \right) + 2\pi \int_{\Omega_\Phi} \cos(\Phi) \left(\omega_\Phi \otimes \frac{\partial \psi_z}{\partial z} \right)^T \cdot I^{n+1} \sin(\Phi) d\Phi \right. \\ \left. - \phi_z^T \cdot P_e \right] d\Omega_z = \mathbf{0}. \end{aligned}$$

$$a \mathbf{M} \cdot \frac{T_e^{n+1} - T_e^n}{\Delta t} + \mathbf{G}_{ei} \cdot T_e^{n+1} - \tilde{\mathbf{G}}_{ei} \cdot T_i^{n+1} + \mathbf{D}I \cdot I^{n+1} = \mathbf{P}_e$$

DG-BGK&Ts scheme, where $I^{n+1} = \mathbf{A} \cdot T_e^{n+1} + \mathbf{b}$

$$\mathbf{A}_{T_e}(\mathbf{A}) \cdot T_e^{n+1} + \tilde{\mathbf{G}}_{ei} \cdot T_i^{n+1} = \mathbf{b}_{T_e}(\mathbf{b})$$

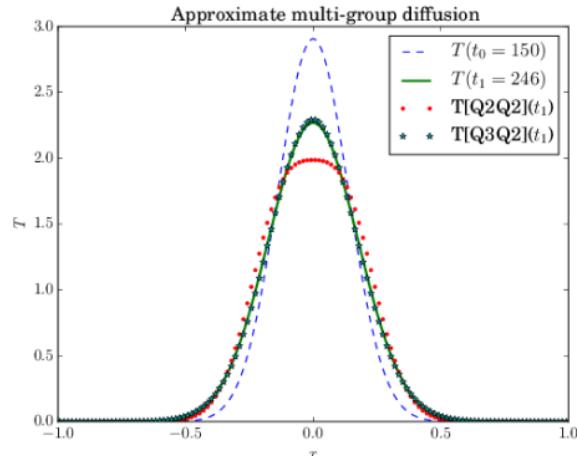
M. Holec, J. Limpouch, R. Liska and S. Weber, *Int. J. Numer. Meth. F.*, 83, 779 (2017).

Approximate multi-group diffusion test

Two energy groups transport model

$$\cos(\Phi) \frac{\partial I_{g_j}}{\partial z} = k_{g_j} (S_T T - I_{g_j})$$

$$a \frac{\partial T}{\partial t} = - \sum_{j=1,2} \frac{2\pi}{\Delta_{g_j}} \int_0^\pi \cos(\Phi) \frac{\partial I_{g_j}}{\partial z} \sin(\Phi) d\Phi ,$$



Diffusive(local) asymptotic behavior of the two energy groups transport model

$$I_{g_j} \approx S_T \left(T - \frac{\cos(\Phi)}{k_{g_j}} \frac{\partial T}{\partial z} \right) \xrightarrow{\text{local}} a \frac{\partial T}{\partial t} \approx \frac{4\pi S_T}{3} \left(\frac{1}{k_{g_1} \Delta_{g_1}} + \frac{1}{k_{g_2} \Delta_{g_2}} \right) \frac{\partial^2 T}{\partial z^2} .$$

cells	32	64	128	256	512
T[Q1Q1]	2.2e-01 [5]	2.4e-01 (-0.1)	2.3e-01 (0.0)	2.3e-01 (0.0)	2.3e-01 (0.0)
T[Q2Q2]	8.7e-02 [4]	9.5e-02 (-0.1)	9.8e-02 (-0.0)	9.6e-02 (0.0)	9.1e-02 (0.1)
T[Q3Q2]	1.2e-01 [4]	5.7e-02 (1.1)	1.0e-02 (2.5)	1.3e-03 (2.9)	-
T[Q3Q3]	7.6e-02 [4]	4.6e-02 (0.7)	9.8e-03 (2.2)	1.3e-03 (2.9)	-
T[Q4Q4]	2.9e-02 [4]	9.6e-03 (1.6)	1.3e-03 (2.9)	-	-
T[Q5Q5]	2.3e-03 [4]	8.2e-05 (4.8)	-	-	-
T[Q6Q6]	1.7e-04 [4]	-	-	-	-

²M. Holec, J. Limpouch, R. Liska and S. Weber, *Int. J. Numer. Meth. F.*, 83, 779 (2017).

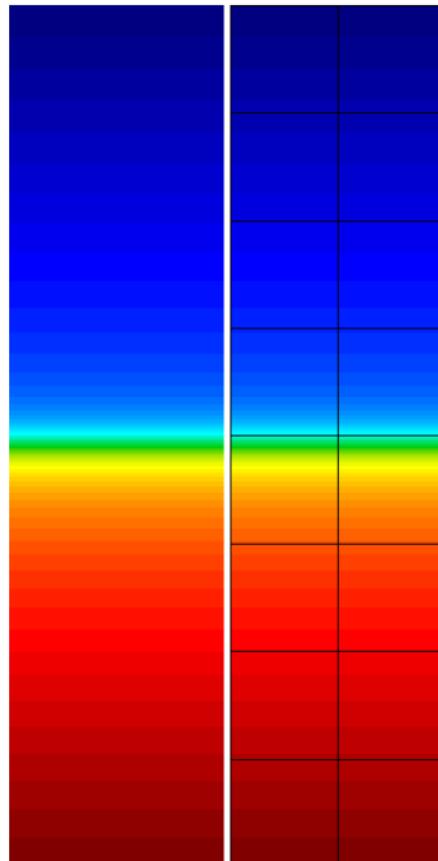
Lagrangian High-Order Curvilinear Framework

$$\mathbf{M}_v \cdot \frac{d\mathbf{v}}{dt} = -\mathbf{F} \cdot \mathbf{I}$$

$$\mathbf{M}_e \cdot \frac{d\mathbf{e}}{dt} = \mathbf{F}^T \cdot \mathbf{v}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}$$

[Dobrev, Kolev, Rieben, SIAM JSC 34, B606 (2012)]



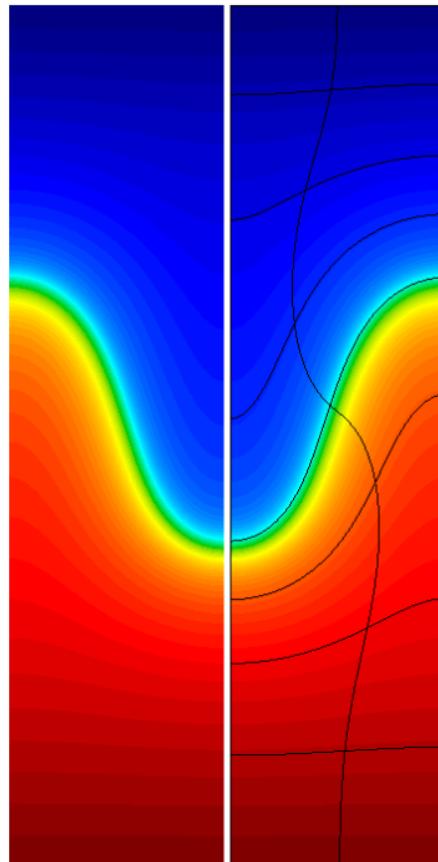
Lagrangian High-Order Curvilinear Framework

$$\mathbf{M}_v \cdot \frac{d\mathbf{v}}{dt} = -\mathbf{F} \cdot \mathbf{I}$$

$$\mathbf{M}_e \cdot \frac{d\mathbf{e}}{dt} = \mathbf{F}^T \cdot \mathbf{v}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}$$

[Dobrev, Kolev, Rieben, SIAM JSC 34, B606 (2012)]



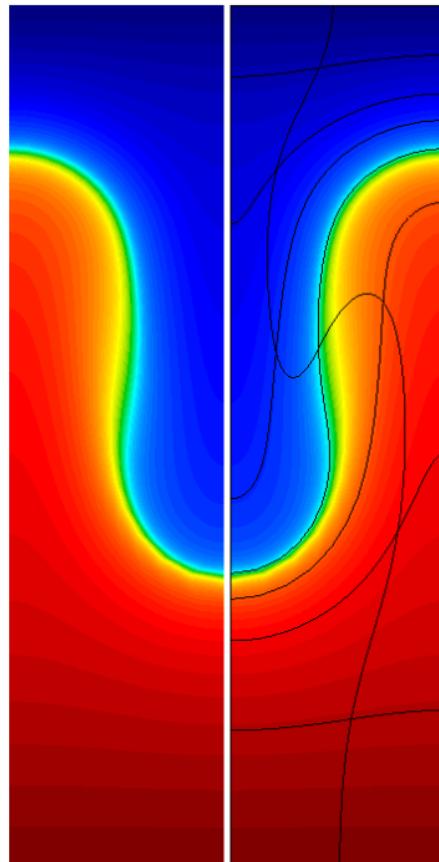
Lagrangian High-Order Curvilinear Framework

$$\mathbf{M}_v \cdot \frac{d\mathbf{v}}{dt} = -\mathbf{F} \cdot \mathbf{I}$$

$$\mathbf{M}_e \cdot \frac{d\mathbf{e}}{dt} = \mathbf{F}^T \cdot \mathbf{v}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}$$

[Dobrev, Kolev, Rieben, SIAM JSC 34, B606 (2012)]



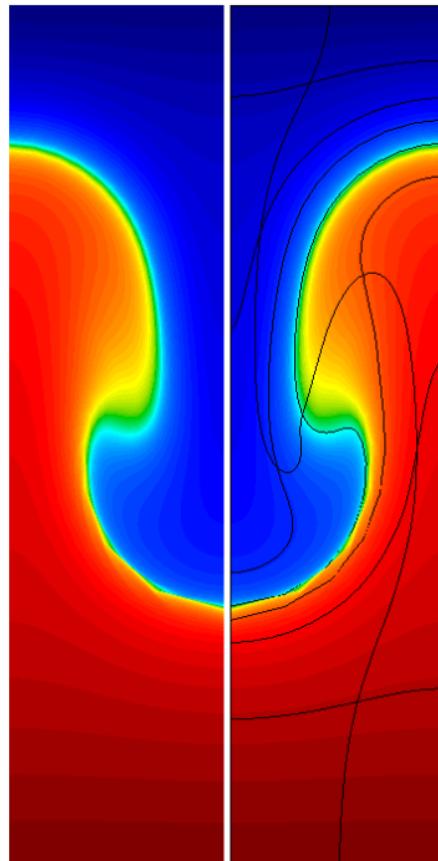
Lagrangian High-Order Curvilinear Framework

$$\mathbf{M}_v \cdot \frac{d\mathbf{v}}{dt} = -\mathbf{F} \cdot \mathbf{I}$$

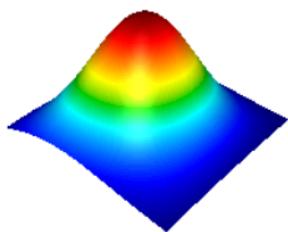
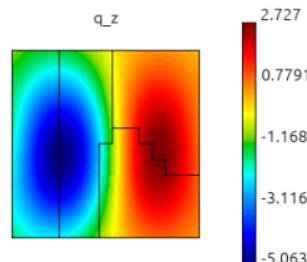
$$\mathbf{M}_e \cdot \frac{d\mathbf{e}}{dt} = \mathbf{F}^T \cdot \mathbf{v}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}$$

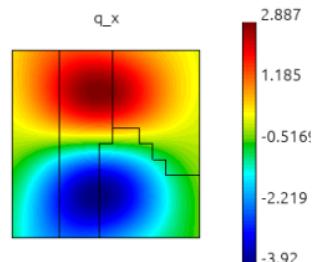
[Dobrev, Kolev, Rieben, SIAM JSC 34, B606 (2012)]



$$\begin{aligned}
 \mathbf{M}_v \cdot \frac{d\mathbf{v}}{dt} &= -\mathbf{F} \cdot \mathbf{I} \\
 k_B \mathbf{M}_e \cdot \frac{dT}{dt} &= \mathbf{F}^T \cdot \mathbf{v} - \int_{4\pi} \mathbf{D} \cdot \mathbf{I} d\mathbf{n} \\
 \frac{dx}{dt} &= \mathbf{v} \\
 \mathbf{D} \cdot \mathbf{I} &= \mathbf{S} \cdot \mathbf{T} - ((k + \sigma)\mathbf{M} + \mathbf{E} \cdot \mathbf{M}_n - \sigma \mathbf{M} \mathbf{I}) \cdot \mathbf{I} + \mathbf{B} \cdot \tilde{\mathbf{I}} - \tilde{\mathbf{B}} \cdot \mathbf{I}
 \end{aligned}$$

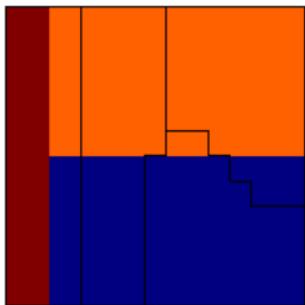
source $\sin(z) \sin(x)$ 

z flux component

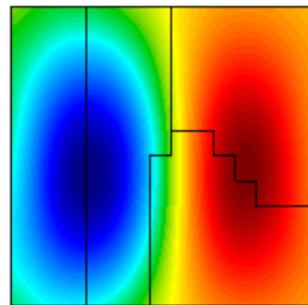
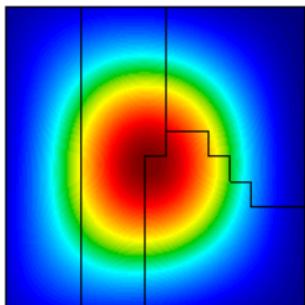
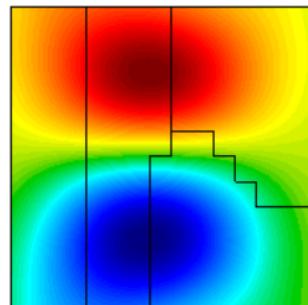


x flux component

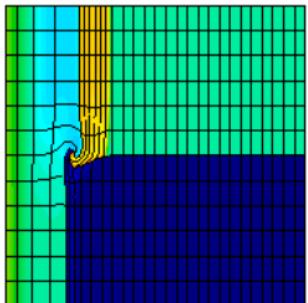
Temperature



2

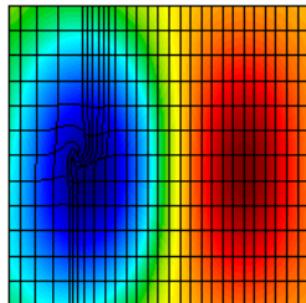
1.563
1.125
0.6875
0.25 q_z 2.727
0.7791
-1.168
-3.116
-5.063I₀27.13
20.46
13.78
7.103
0.4267 q_x 2.887
1.185
-0.516
-2.219
-3.92

Temperature



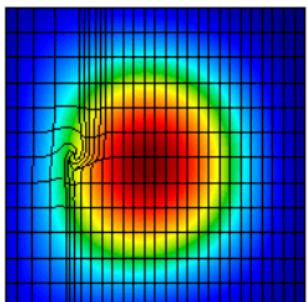
3.492
2.675
1.858
1.041
0.224

q_z



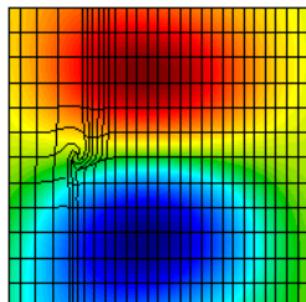
2.727
0.7787
-1.169
-3.117
-5.065

I_0



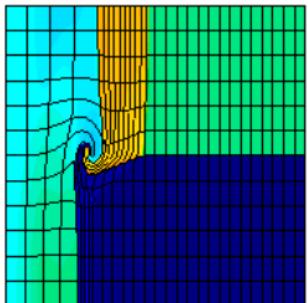
27.13
20.45
13.78
7.103

q_x



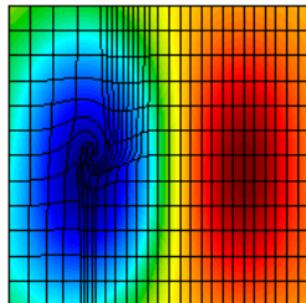
2.887
1.185
-0.517
-2.219

Temperature



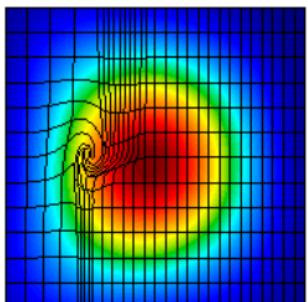
3.364
2.582
1.8
1.018
0.2359

q_z



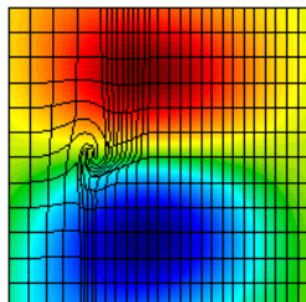
2.727
0.7789
-1.169
-3.116
-5.064

I_0



27.13
20.45
13.78
7.103

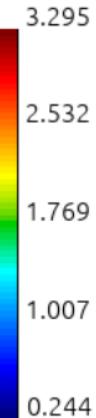
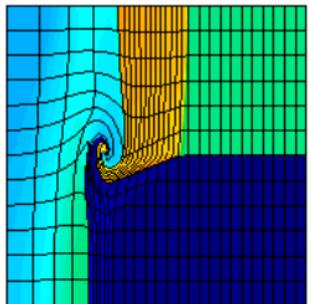
q_x



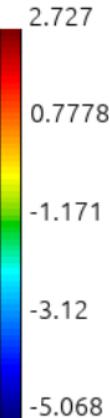
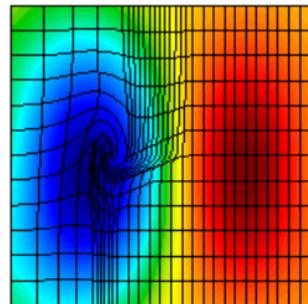
2.886
1.185
-0.516
-2.218



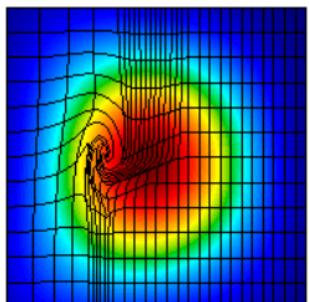
Temperature



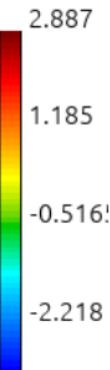
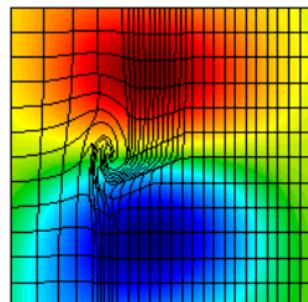
q_z



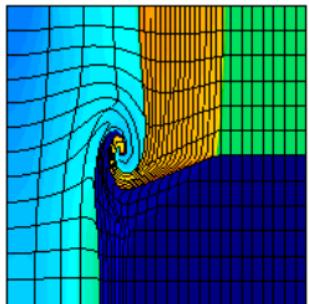
I_0



q_x

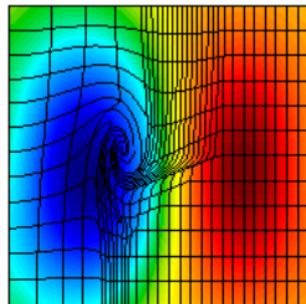


Temperature



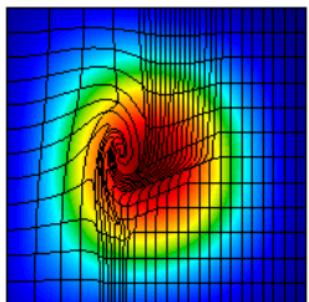
3.232
2.479
1.725
0.9718
0.2183

q_z



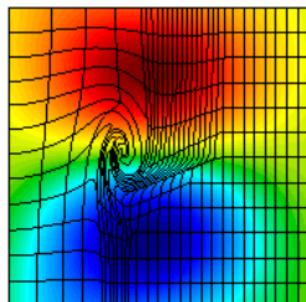
2.727
0.7788
-1.169
-3.117
-5.065

I_0



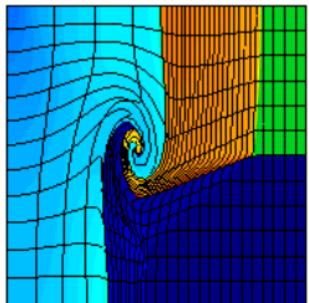
27.13
20.46
13.78
7.103

q_x



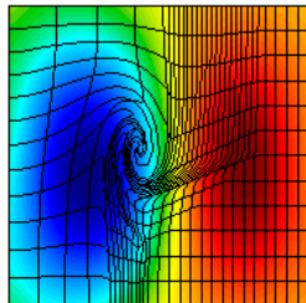
2.887
1.185
-0.5168
-2.219

Temperature



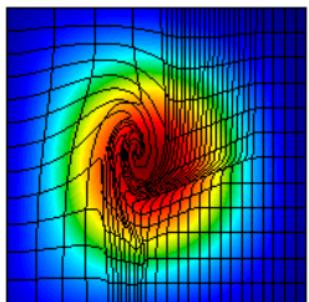
3.086
2.373
1.66
0.9474
0.2345

q_z



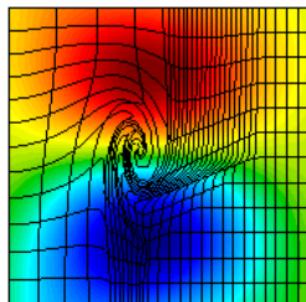
2.727
0.7796
-1.167
-3.114
-5.061

I_0



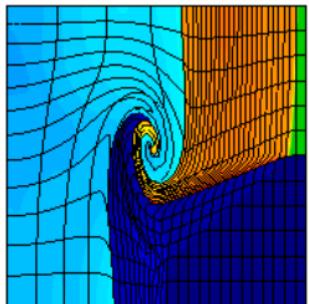
27.12
20.44
13.77
7.099

q_x



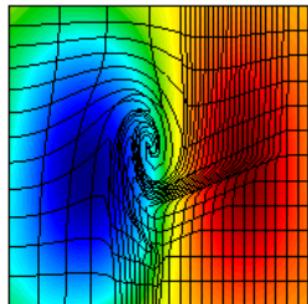
2.886
1.185
-0.517
-2.219

Temperature



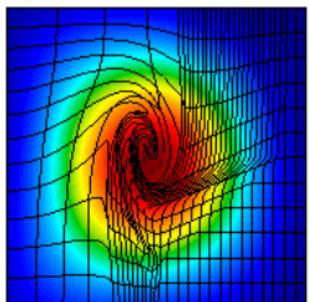
3.008
2.31
1.612
0.9143
0.2165

q_z



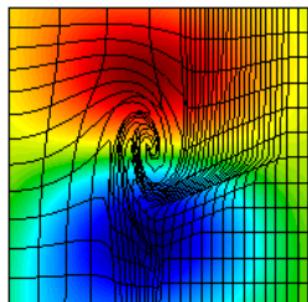
2.726
0.7798
-1.167
-3.113
-5.06

I_0



27.16
20.48
13.8
7.111

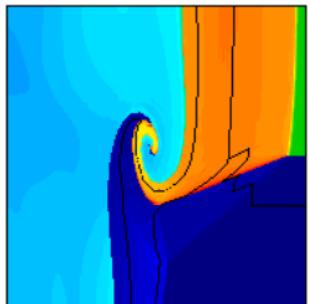
q_x



2.884
1.183
-0.518
-2.219

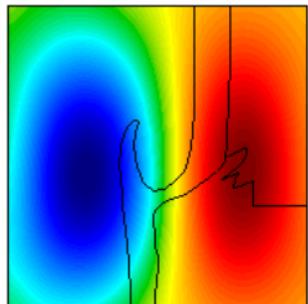


Temperature



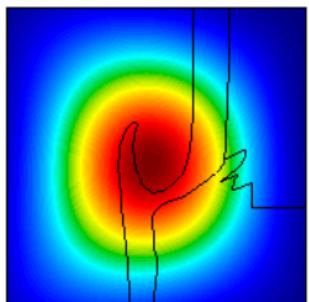
3.008
2.31
1.612
0.9143
0.2165

q_z



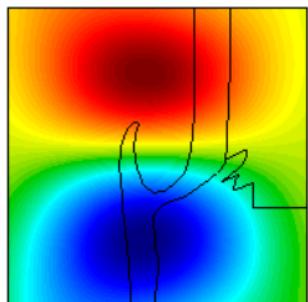
2.726
0.7798
-1.167
-3.113
-5.06

I_0



27.16
20.48
13.8
7.111

q_x



2.884
1.183
-0.518
-2.219

Curvilinear Framework of Nonlocal Transport

Axisymmetric transport equation

$$\begin{aligned}\sin(\phi) \left(\cos(\theta) \frac{\partial I}{\partial r} - \frac{\sin(\theta)}{r} \frac{\partial I}{\partial \theta} \right) + \cos(\phi) \frac{\partial I}{\partial z} &= S_T T_e - (k + \sigma - \sin(\phi) \cos(\theta) E_r - \cos(\phi) E_z) I + \sigma I_0 \\ \mathbf{D} \cdot \mathbf{I} &= \mathbf{S} \cdot \mathbf{T} - ((k + \sigma) \mathbf{M} + \mathbf{E} \cdot \mathbf{M}_n - \sigma \mathbf{M} \mathbf{I}) \cdot \mathbf{I} + \mathbf{B} \cdot \mathbf{I} - \tilde{\mathbf{B}} \cdot \tilde{\mathbf{I}}\end{aligned}$$

3

- high-order curvilinear divergence matrix

$$\begin{aligned}\mathbf{D} = \int_0^\pi \int_0^\pi \int_{\Omega} (\omega_\theta \otimes \omega_\phi \otimes \psi) \otimes \\ \left(\sin(\phi) \omega_\phi \otimes \left(\cos(\theta) \omega_\theta^T \otimes \frac{\partial \psi^T}{\partial r} - \frac{\sin(\theta)}{r} \frac{\partial \omega_\theta^T}{\partial \theta} \otimes \psi^T \right) + \omega_\theta^T \otimes \cos(\phi) \omega_\phi^T \otimes \frac{\partial \psi^T}{\partial z} \right) r \sin(\phi) d\Omega d\phi d\theta\end{aligned}$$

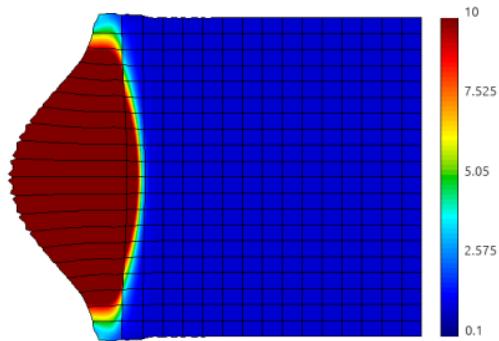
- high-order curvilinear numerical flux matrix

$$\mathbf{B} = \int_0^\pi \int_0^\pi \int_{\Gamma_n \cdot \mathbf{n}_\Gamma < 0} (\omega_\theta \otimes \omega_\phi \otimes \psi) \otimes (\omega_\theta \otimes \omega_\phi \otimes \psi)^T (\sin(\phi) \cos(\theta) n_{\Gamma_r} + \cos(\phi) n_{\Gamma_z}) r \sin(\phi) d\Gamma d\phi d\theta$$

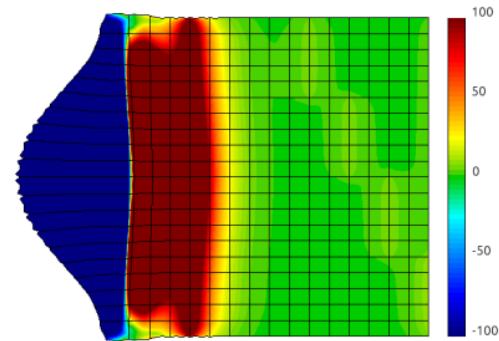
- proper treatment of Lorentz force

$$\mathbf{E} \cdot \mathbf{n} \frac{\partial I}{\partial |\mathbf{v}|} + (\mathbf{E} + \mathbf{n} \times \mathbf{B}) \cdot \nabla_{\mathbf{n}} I = \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}^T \cdot \begin{bmatrix} \cos(\theta) \sin(\phi) \\ \sin(\theta) \sin(\phi) \\ \cos(\phi) \end{bmatrix} \frac{\partial I}{\partial |\mathbf{v}|} + \left(\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} + \begin{bmatrix} \cos(\theta) \sin(\phi) \\ \sin(\theta) \sin(\phi) \\ \cos(\phi) \end{bmatrix} \times \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} \right)^T \cdot \begin{bmatrix} \cos(\phi) \frac{\partial I}{\partial \phi} \\ \frac{1}{\sin(\phi)} \frac{\partial I}{\partial \theta} \\ -\sin(\phi) \frac{\partial I}{\partial \phi} \end{bmatrix}$$

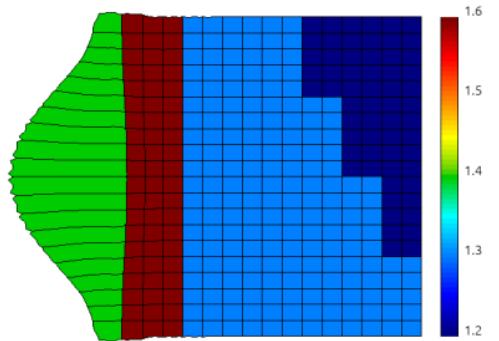
Temperature



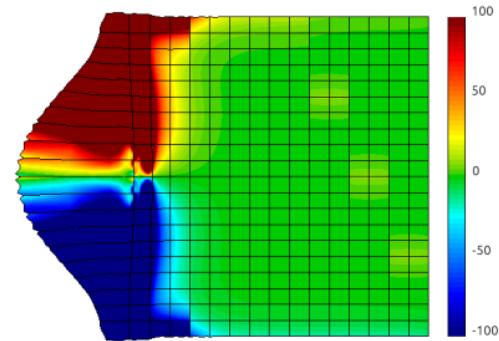
Hz

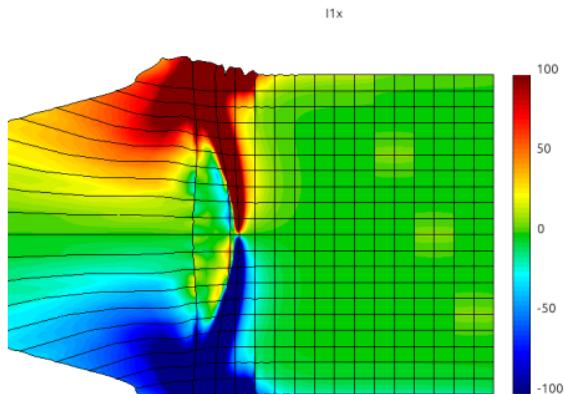
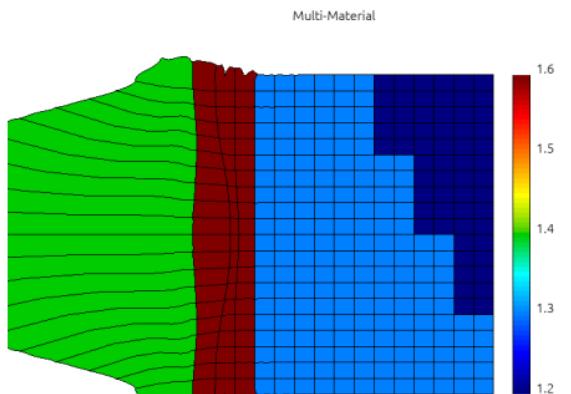
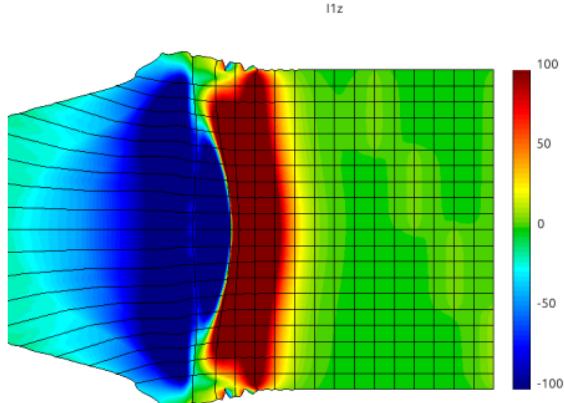
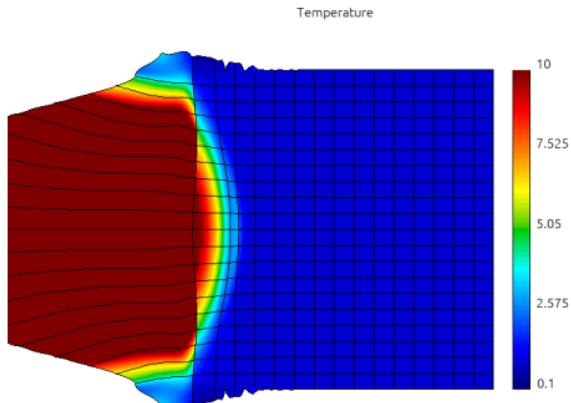


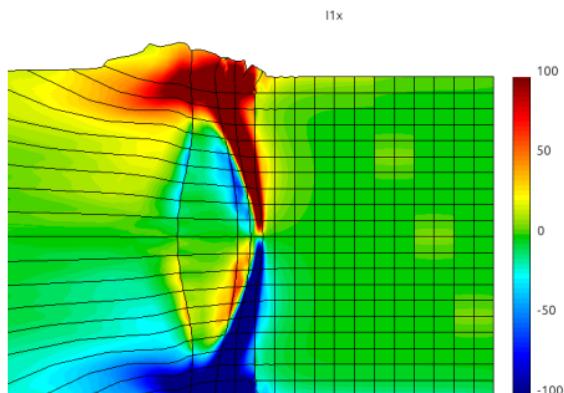
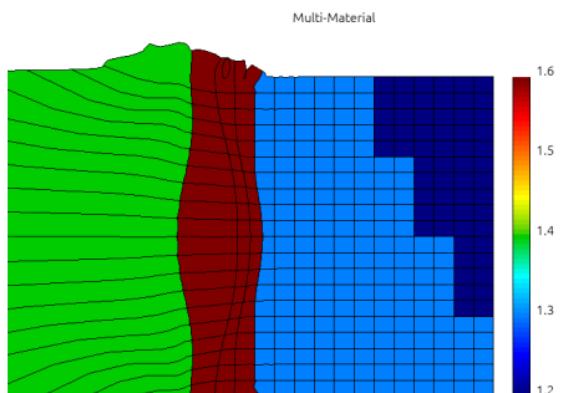
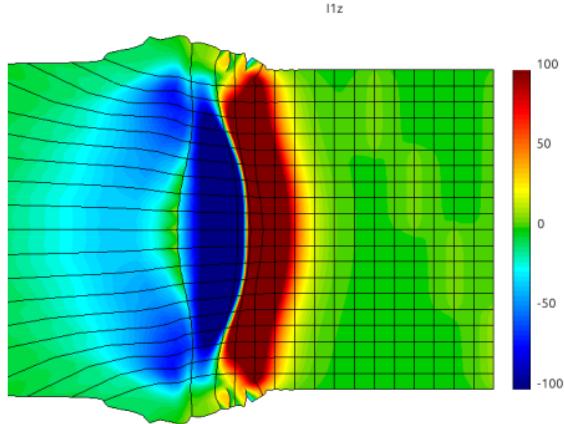
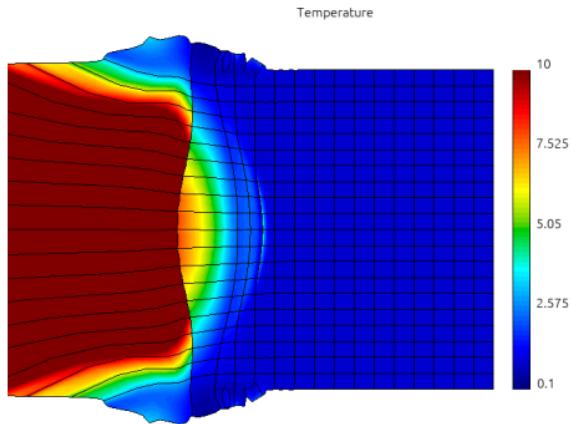
Multi-Material

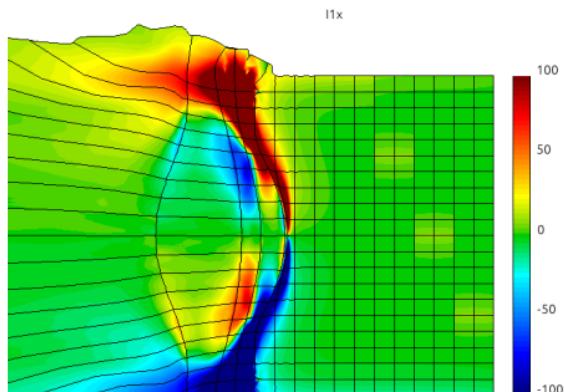
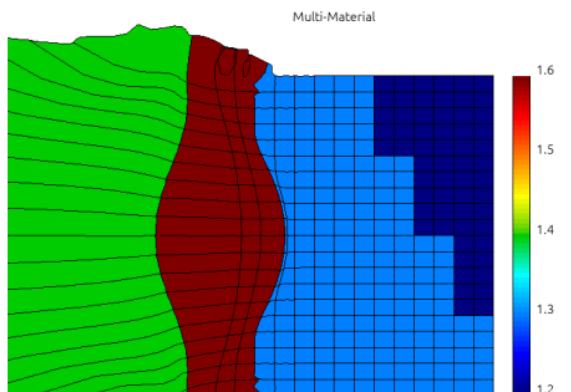
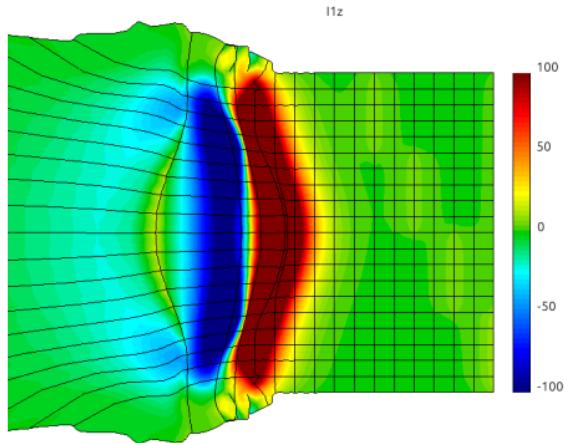
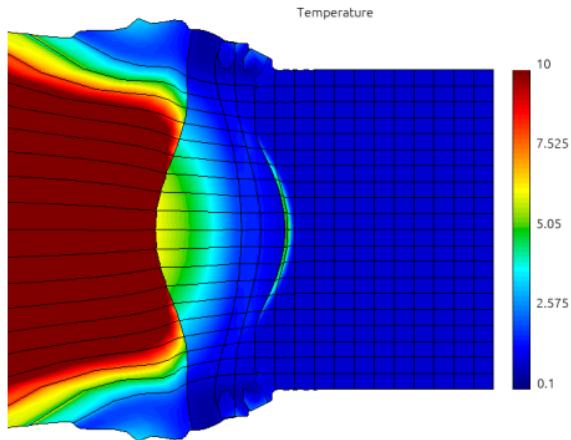


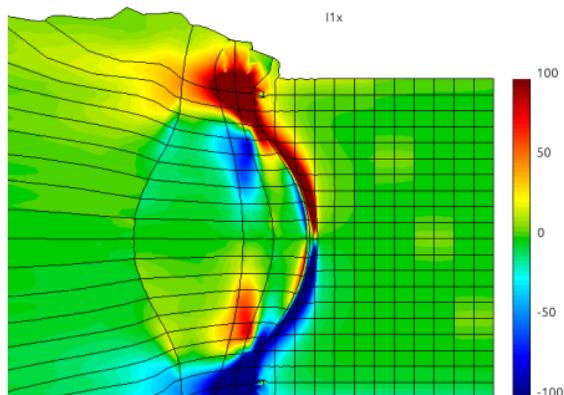
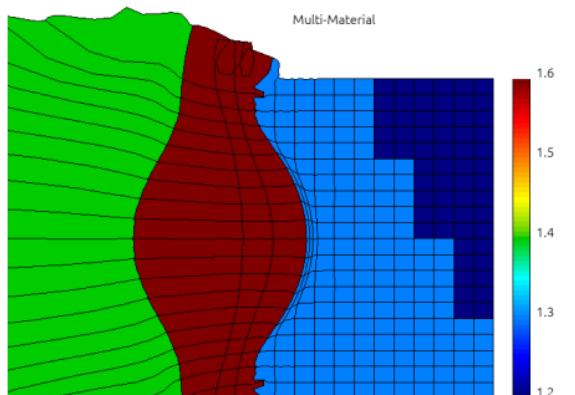
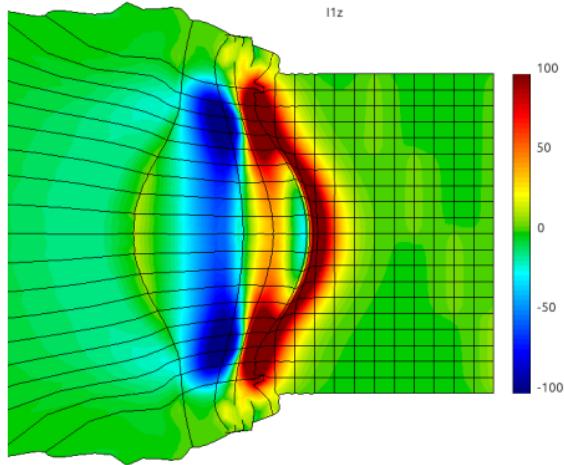
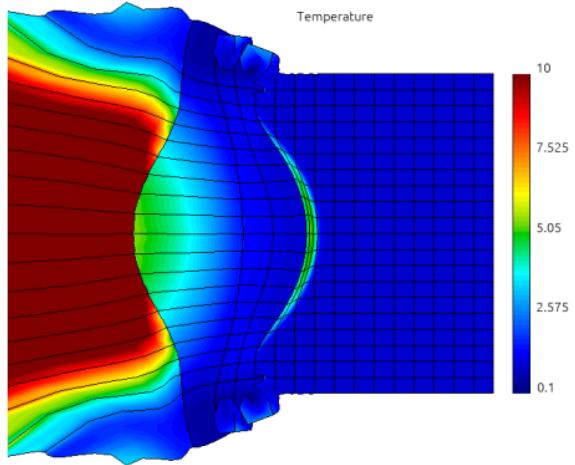
Hz

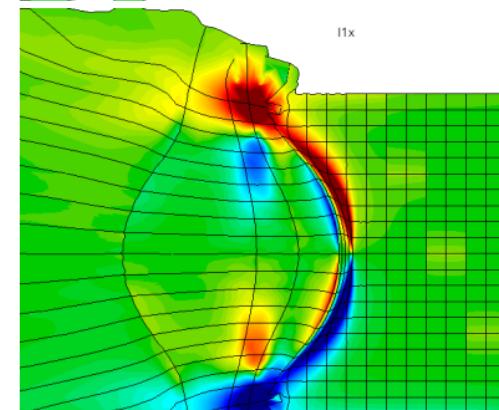
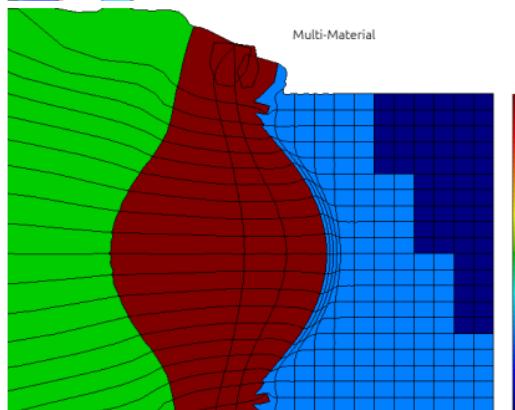
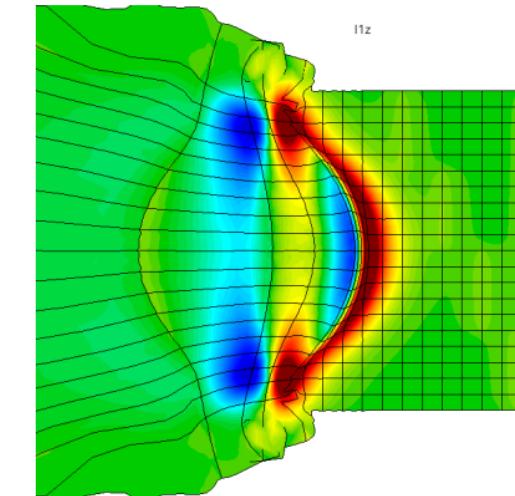
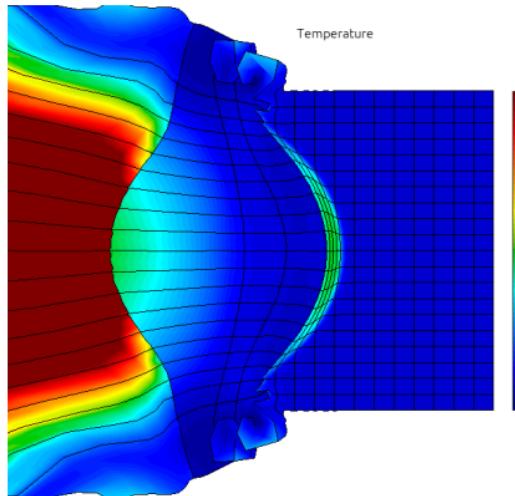


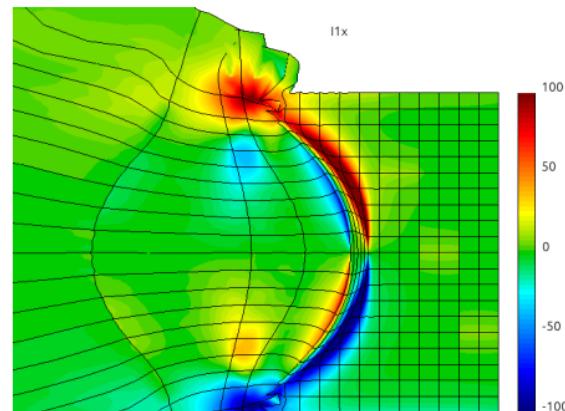
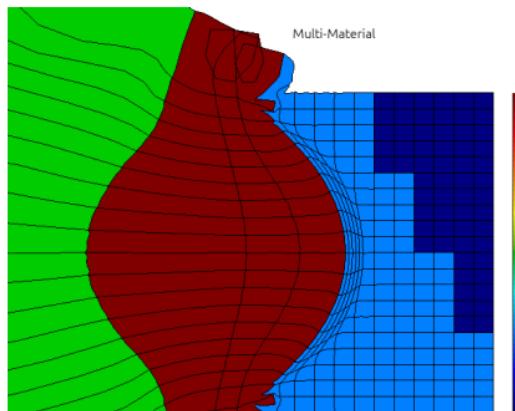
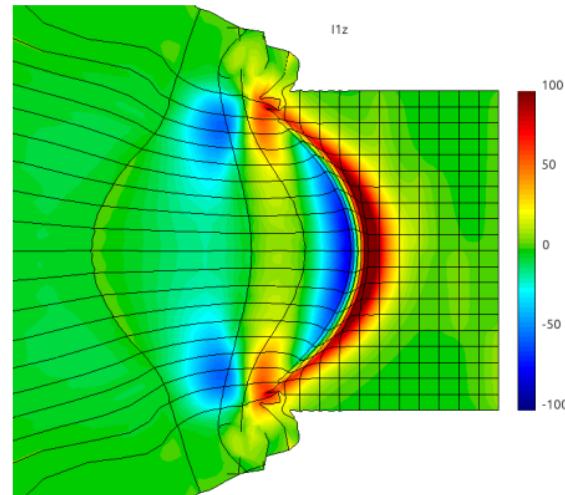
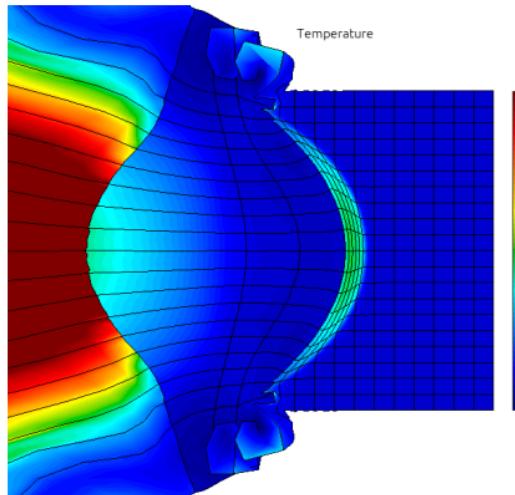


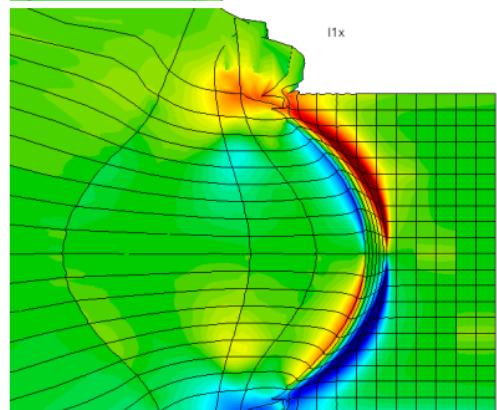
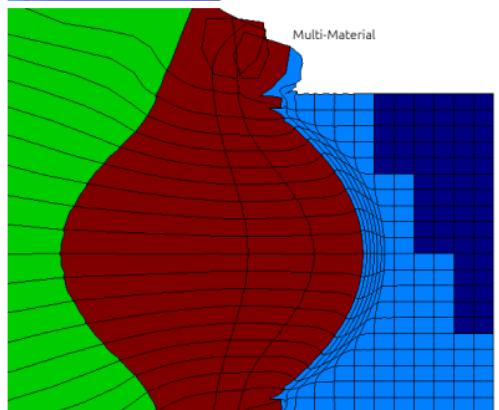
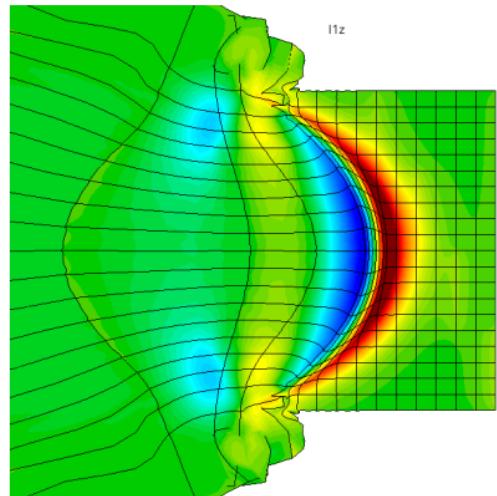
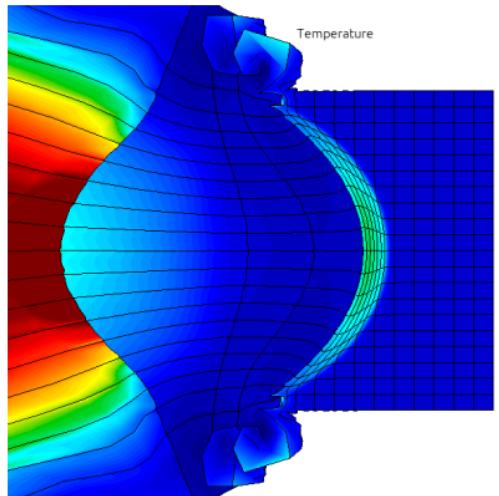


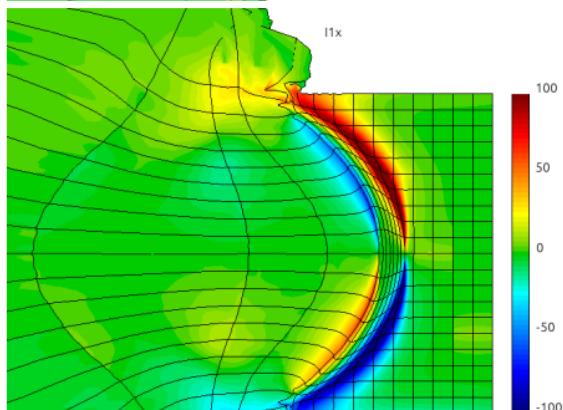
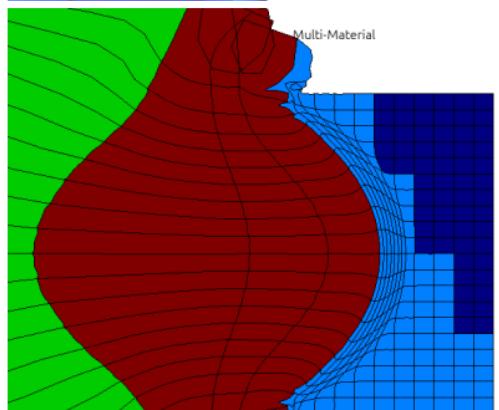
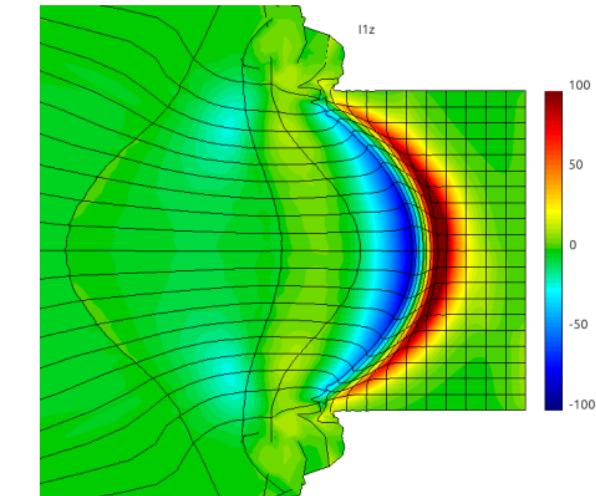
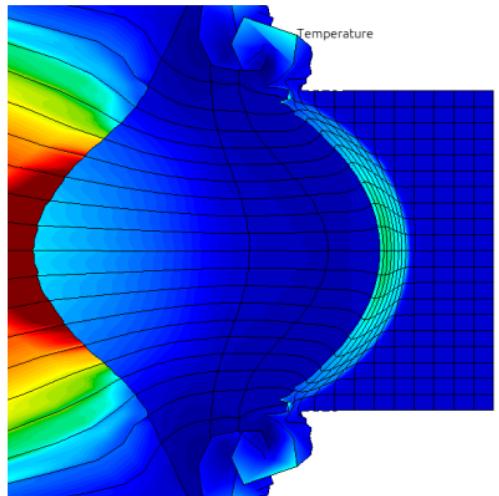


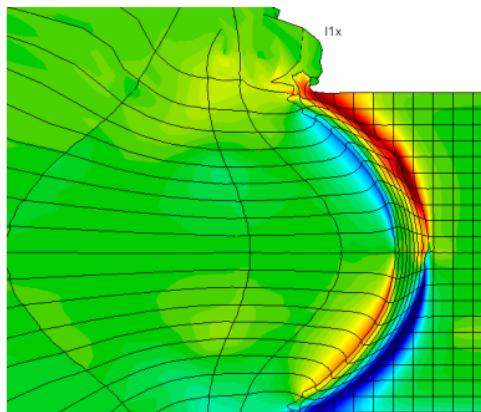
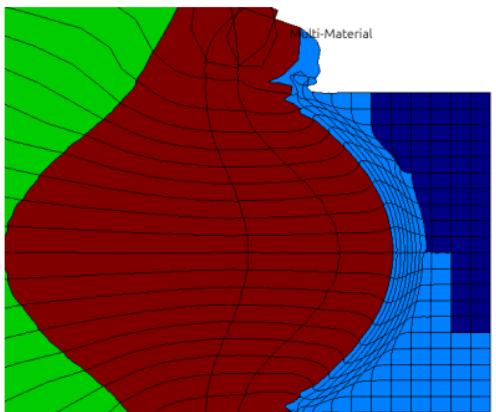
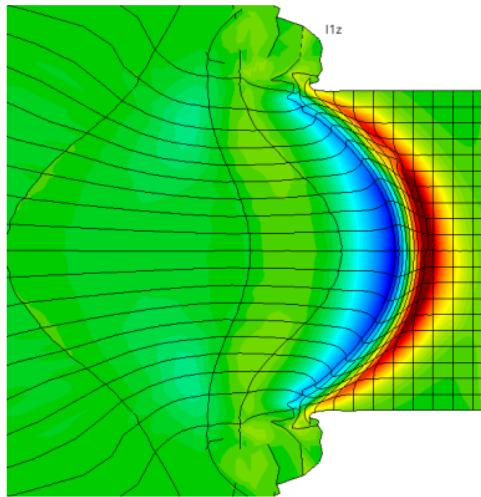
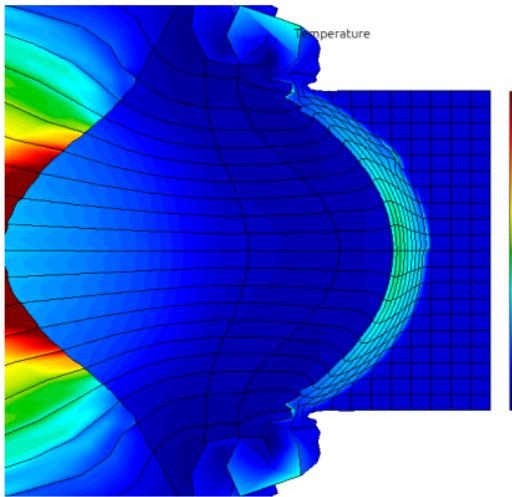




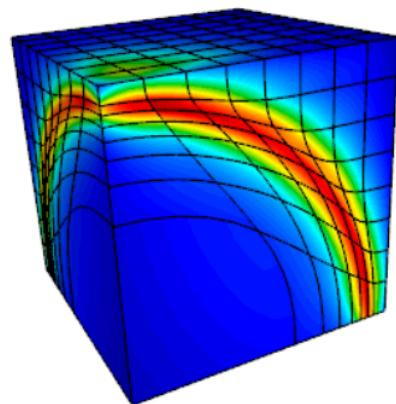
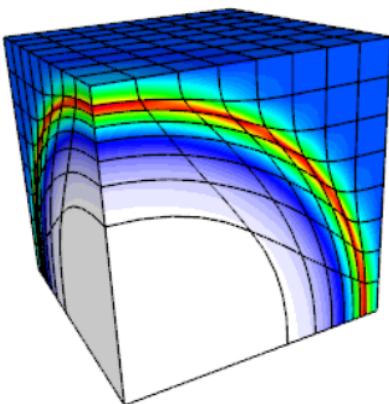








Laghos + DG-BGK&T



4

⁴<https://github.com/homijan/NTH-Laghos>

Hotspot relaxation

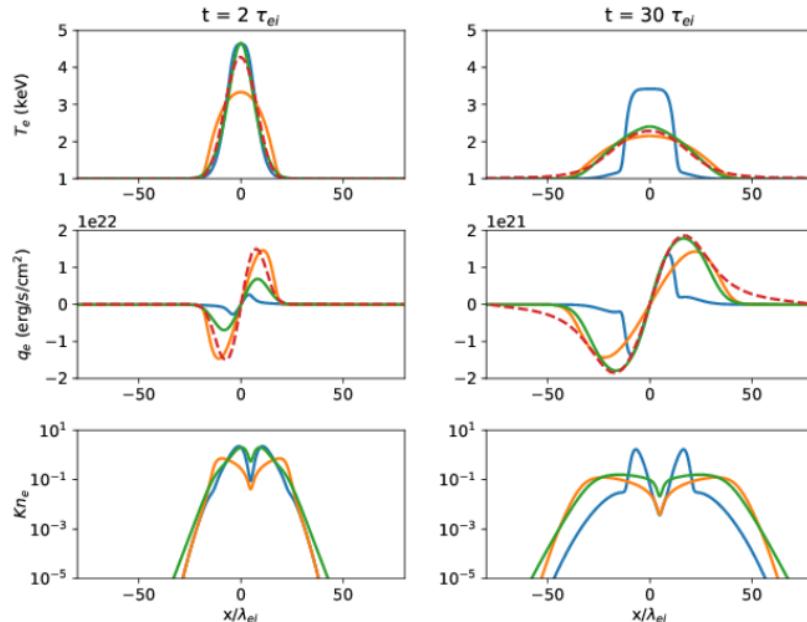
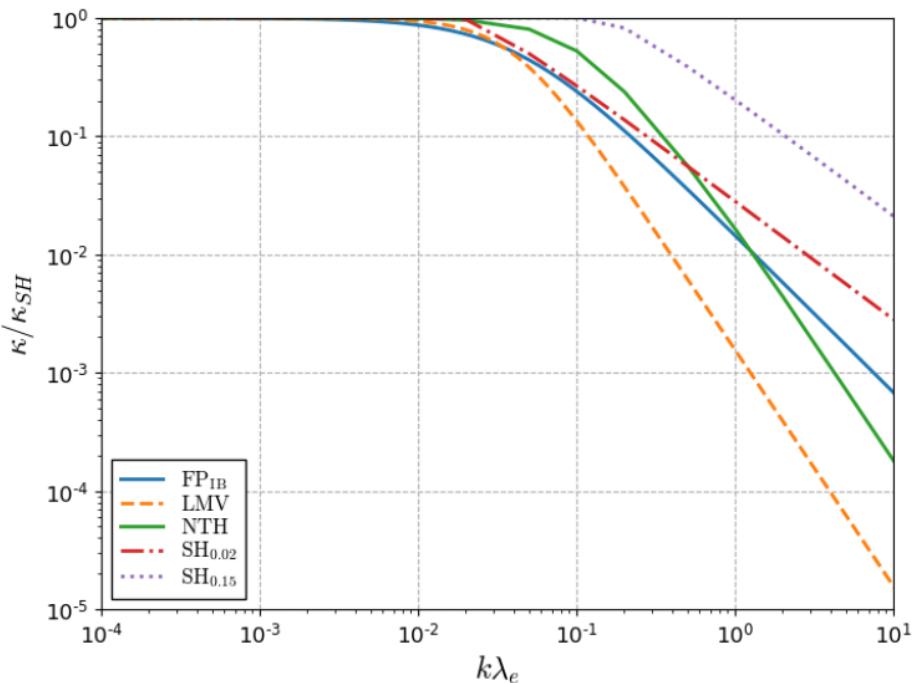


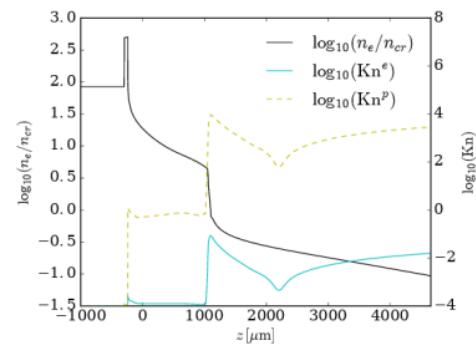
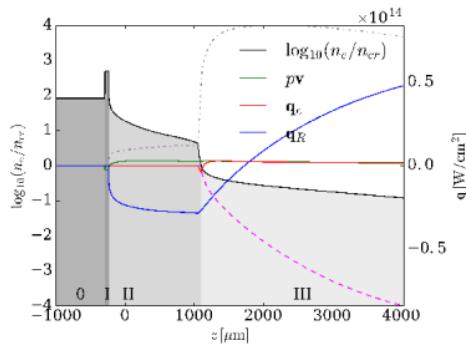
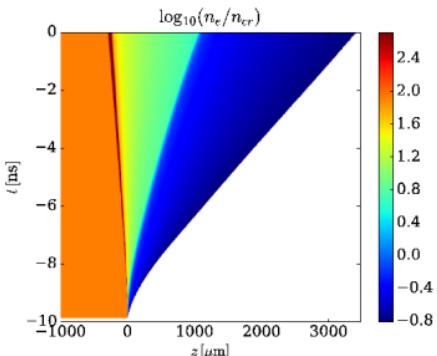
Figure: Green - NTH, red - Fokker-Planck, yellow - SH, blue - flux limited SH.

Short-Epperlein test



⁵M. Holec, J. Nikl, and S. Weber, "Nonlocal Transport Hydrodynamic Model for Laser Heated Plasmas", *Phys. Plasmas*, submitted (2017)

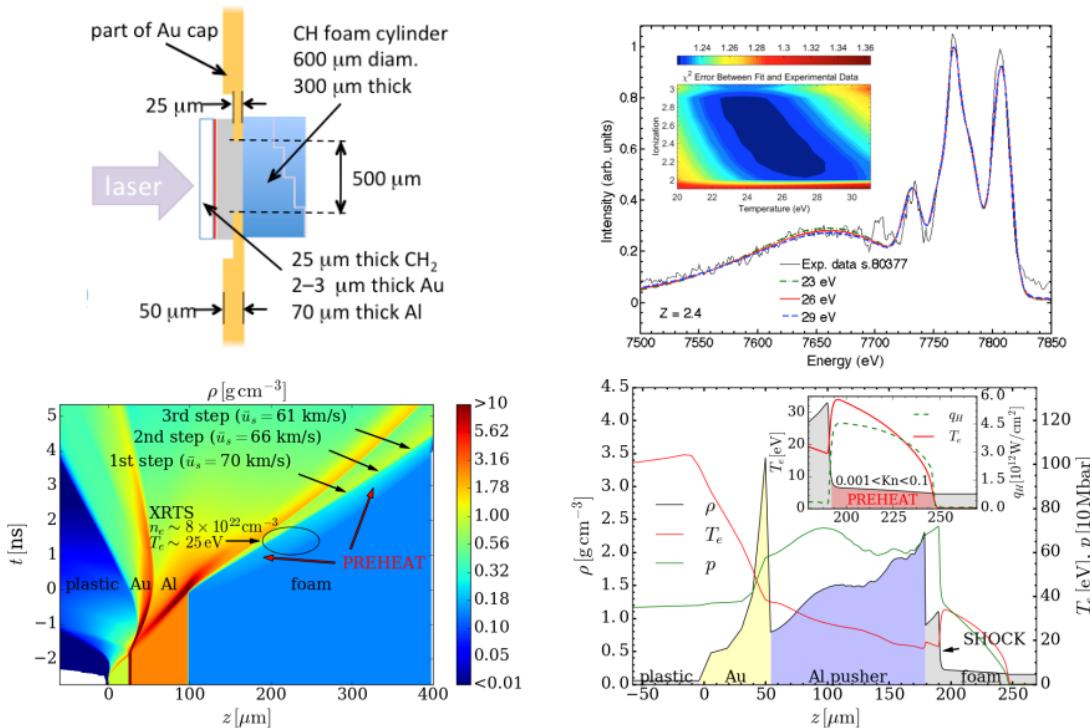
Planar plastic target irradiated by NIF laser pulse



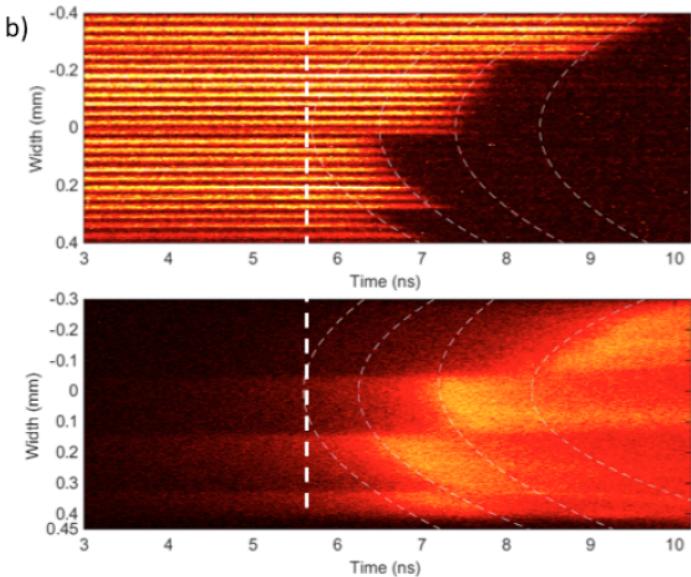
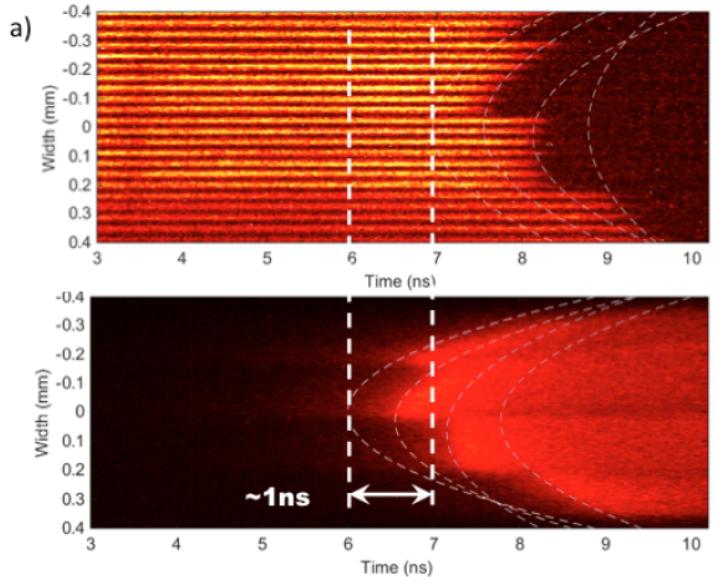
The nonlocal transport of energy leads to generation of *conduction zone* (turquoise color), whose thickness is one of the main parameters for the seeding of Rayleigh-Taylor and Richtmyer-Meshkov instabilities.

⁵M. Holec, J. Nikl, and S. Weber, "Nonlocal Transport Hydrodynamic Model for Laser Heated Plasmas", *Phys. Plasmas*, submitted (2017)

Preheat observed in a shocked CH foam at OMEGA facility



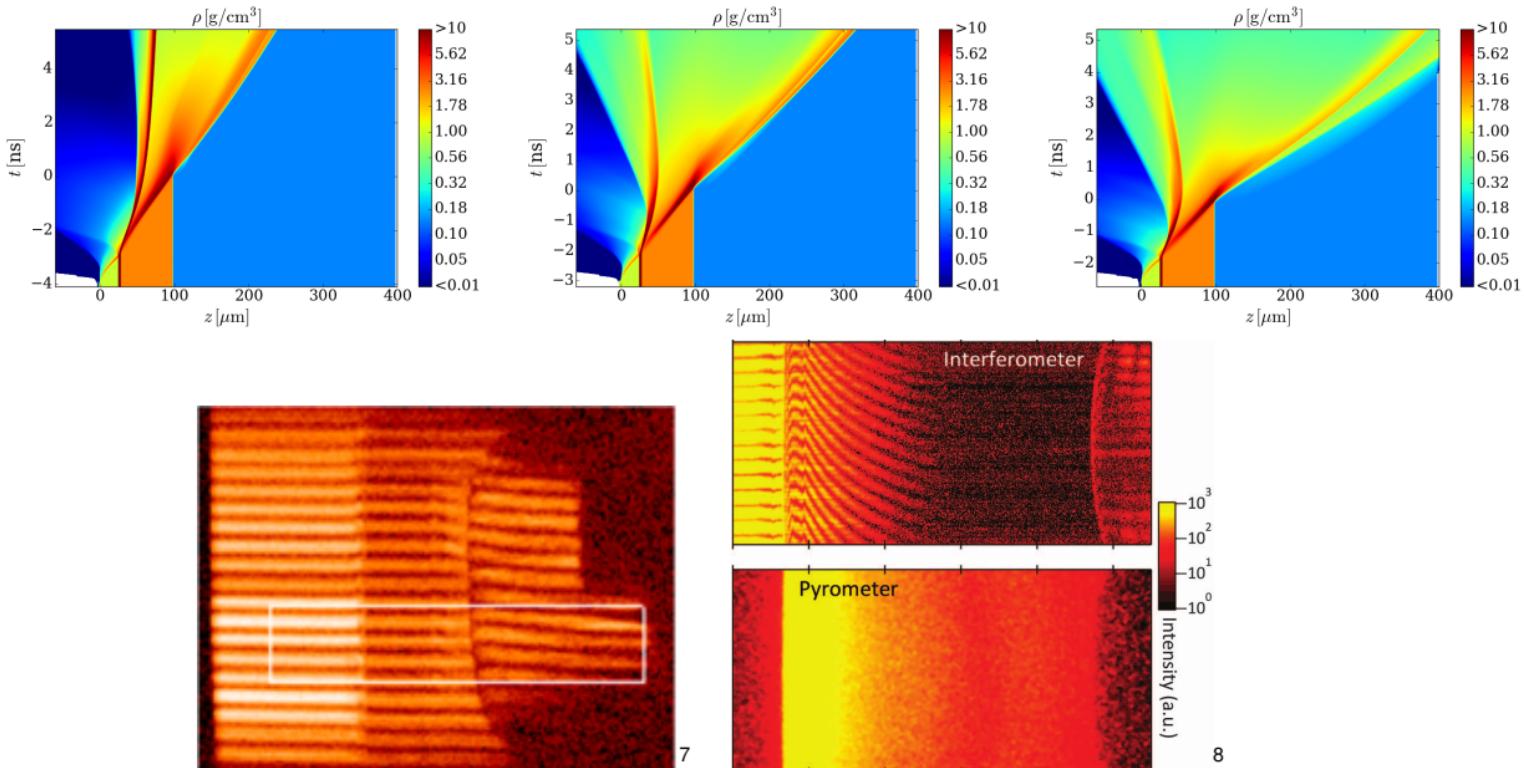
flat 2 ns laser pulse, $8 \times 10^{14} \text{ W cm}^2$, 300 μm thick foam ($\rho = 0.13 \text{ g/cm}^3$)



a) recent experiment with new phase-plates, b) previous experiment

⁶K. Falk, M. Holec, C. J. Fontes et al, "Measurement of preheat due to nonlocal electron transport in warm dense matter", *Phys. Rev. Lett.*, 120, 025002 (2018).

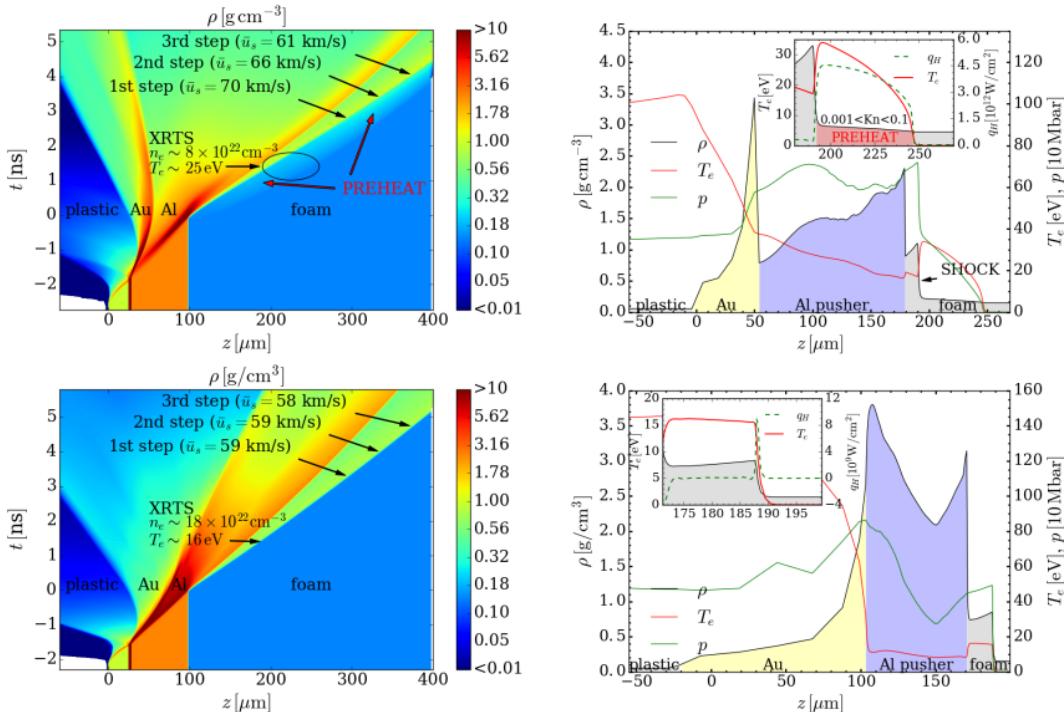
VISAR oscillations



⁷K. Falk et al, *Phys. Plas.*, 21, 056309 (2014).

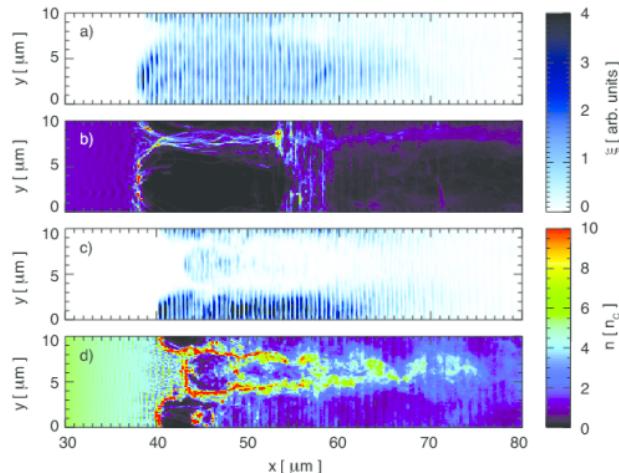
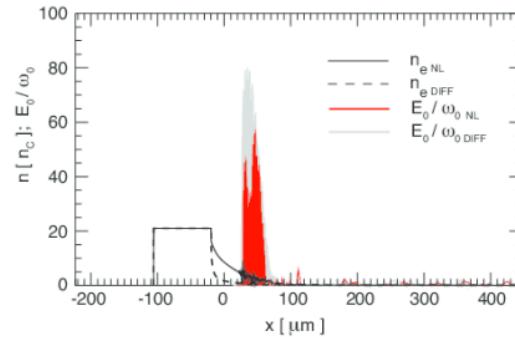
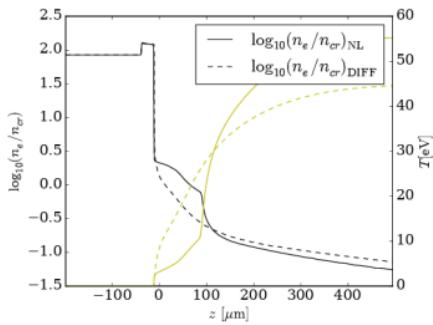
⁸R. S. Williams et al, *Science*, 338, 1330 (2012).

Nonlocal vs. SH diffusion



Effective mean free path was $4.1 \times v_T$, according to LANL code ATOMIC the mean free path in WDM foam increases $\approx 30\%$.

⁸K. Falk, M. Holec, C. J. Fontes et al, "Measurement of preheat due to nonlocal electron transport in warm dense matter", *Phys. Rev. Lett.* 120, 025002 (2018).



9

⁹M. Holec, J. Nikl, and M. Vranic, "The effect of pre-plasma formation under nonlocal transport conditions for ultra-relativistic laser-plasma interaction", *Plasma Phys. Control. Fusion*, submitted (2017)

Example of moment methods

AWBS Boltzmann transport equation

$$\mathbf{n} \cdot \nabla f + \frac{1}{v} (\mathbf{E} + v \mathbf{n} \times \mathbf{B}) \cdot \nabla_v f = \frac{v}{\lambda} \frac{\partial}{\partial v} (f - f_M) + \frac{1}{2\lambda_{ei}} \frac{\partial}{\partial \mu} \left((1 - \mu^2) \frac{\partial f}{\partial \mu} \right).$$

M1-AWBS transport model

$$\begin{aligned}\nu_e v \frac{\partial}{\partial v} (f_0 - f_M) &= v \nabla \cdot \mathbf{f}_1 + \frac{q_e}{m_e v^2} \mathbf{E} \cdot \frac{\partial}{\partial v} (v^2 \mathbf{f}_1), \\ \nu_e v \frac{\partial}{\partial v} \mathbf{f}_1 - \nu_t \mathbf{f}_1 &= v \nabla \cdot (\mathbf{A} f_0) + \frac{q_e}{m_e v^2} \mathbf{E} \cdot \frac{\partial}{\partial v} (v^2 \mathbf{A} f_0) + \frac{q_e}{m_e v} \mathbf{E} \cdot (\mathbf{A} - \mathbf{I}) f_0 + \frac{q_e}{m_e c} \mathbf{B} \times \mathbf{f}_1.\end{aligned}$$

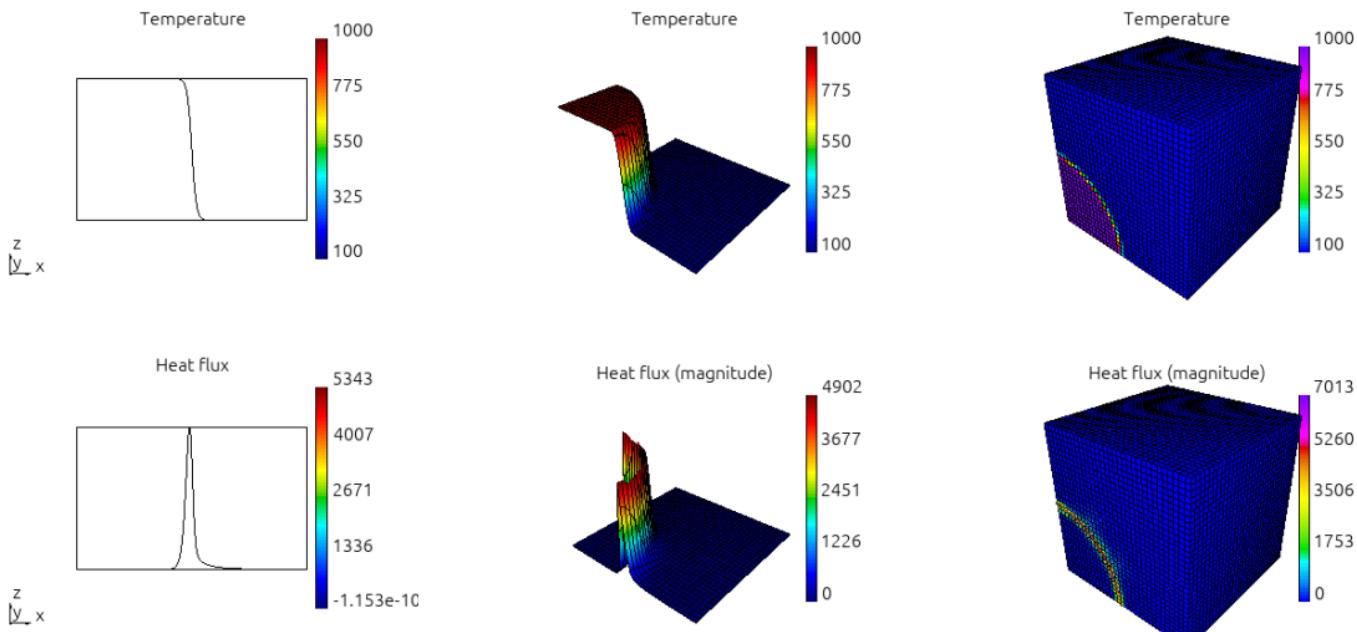
The second moment closure uses the M1 form $\mathbf{A}(f_0, \mathbf{f}_1)$, i.e. is nonlinear and degenerates to $\mathbf{I} \frac{1}{3}$ with decreasing anisotropy.

High-order finite element bilinear form integrators

$$\mathcal{M}_{(g)}^0 = \int_{\Omega} \phi \otimes \phi^T g \, d\Omega, \quad \mathcal{M}_{(g)}^1 = \int_{\Omega} \mathbf{w} \cdot \mathbf{w}^T g \, d\Omega, \quad \mathcal{D}_{(\mathbf{G})} = \int_{\Omega} \mathbf{G} : \nabla \mathbf{w} \otimes \phi^T \, d\Omega,$$

$$\mathcal{V}_{(g)} = \int_{\Omega} \mathbf{w} \cdot \mathbf{g} \otimes \phi^T \, d\Omega, \quad \mathcal{B}_{(g)} = \int_{\Omega} \mathbf{w} \cdot \mathbf{g} \times \mathbf{w}^T \, d\Omega,$$

$$\begin{aligned}
\left(\mathcal{M}_{(\nu_e)}^0 - \mathcal{M}_{\left(\frac{1}{\nu f_0^n} \tilde{\mathbf{E}}^T \cdot \mathbf{f}_1^n\right)}^0 \right) \cdot \frac{d\mathbf{f}_0}{dv}^* &= \mathcal{D}_{(\mathbf{I})}^T \cdot \mathbf{f}_1^n + \mathcal{V}_{(2\tilde{\mathbf{E}}/\nu^2)}^T \cdot \mathbf{f}_1^n + \mathcal{M}_{(\nu_e)}^0 \cdot \frac{\partial \mathbf{f}_M}{\partial v} \\
\mathcal{M}_{(\nu_e)}^1 \cdot \frac{d\mathbf{f}_1}{dv} &= -\mathcal{D}_{(\mathbf{A})} \cdot \mathbf{f}_0^n + \mathcal{V}_{(\mathbf{A} \cdot \tilde{\mathbf{E}}/\nu)} \cdot \frac{d\mathbf{f}_0}{dv}^* + \mathcal{V}_{((3\mathbf{A}-\mathbf{I}) \cdot \tilde{\mathbf{E}}/\nu^2)} \cdot \mathbf{f}_0^n + \mathcal{B}_{(\tilde{\mathbf{B}}/\nu)} \cdot \mathbf{f}_1^n + \mathcal{M}_{(\nu_t/\nu)}^1 \cdot \mathbf{f}_1^n
\end{aligned}$$



Nonlocal Transport Magneto-Hydrodynamic model

$$\mathbf{n} \cdot \nabla_{\mathbf{x}} f + \frac{1}{|\mathbf{v}|} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = \frac{f_{MB}(|\mathbf{v}|, T_e) - f}{\lambda_{ei}(|\mathbf{v}|^4)} \rightarrow \mathbf{D} \cdot \mathbf{f} + \frac{1}{|\mathbf{v}|} \mathbf{E} \cdot \mathbf{V}_E \cdot \mathbf{f} + \mathbf{B} \times \mathbf{V}_B \cdot \mathbf{f} = \mathbf{S} - k_{|\mathbf{v}|} \mathbf{M} \cdot \mathbf{f}$$

NTH Electric field vs. generalized Ohm's law

$$\begin{aligned}\mathbf{E} &= \frac{1}{en_e} (\mathbf{R}_T - \nabla p_e) + \frac{\mathbf{j}}{en_e \sigma} + \frac{1}{en_e c} \mathbf{j} \times \mathbf{B} \\ \mathbf{E} \cdot \sum_{\Delta v^g} \int_{4\pi} \mathbf{V}_E \cdot \mathbf{f} &= - \sum_{\Delta v^g} \int_{4\pi} \mathbf{D} \cdot |\mathbf{v}| \mathbf{f} - \sum_{\Delta v^g} \int_{4\pi} k_{|\mathbf{v}|} \mathbf{M} \cdot |\mathbf{v}| \mathbf{f} - \sum_{\Delta v^g} \int_{4\pi} \mathbf{B} \times \mathbf{V}_B \cdot |\mathbf{v}| \mathbf{f}\end{aligned}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \quad (\text{life of magnetic field } \mathbf{B})$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} (\mathbf{j} + \tilde{\mathbf{j}}) \quad (\text{quasi-neutrality } \nabla \cdot (\mathbf{j} + \tilde{\mathbf{j}}) = 0)$$

Applying generalized Ohm's and Ampere's laws, we get

$$\nabla \times \mathbf{E} = \nabla \times \left(\frac{1}{en_e} (\mathbf{R}_T - \nabla p_e) + \frac{c}{en_e \sigma 4\pi} \nabla \times \mathbf{B} - \frac{\tilde{\mathbf{j}}}{en_e \sigma} + \frac{1}{en_e c} \mathbf{j} \times \mathbf{B} \right)$$

Maxwell Equations for Hydrodynamics - dynamo equation for nonlocal magnetic field source

$$\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \frac{1}{en_e c} \mathbf{j} \times \mathbf{B} - \nabla \times \frac{c}{en_e \sigma 4\pi} \nabla \times \mathbf{B} - \nabla \times \left(\frac{\sum_{\Delta v^g} \int_{4\pi} \mathbf{D} \cdot |\mathbf{v}| \mathbf{f}}{\sum_{\Delta v^g} \int_{4\pi} \mathbf{V}_E \cdot \mathbf{f}} + \frac{\tilde{\mathbf{j}}}{en_e \sigma} \right)$$

Nonlocal Transport Magneto-Hydrodynamics

$$\begin{aligned}
 \frac{d\rho}{dt} &= -\rho \nabla \cdot \mathbf{u}, \\
 \rho \frac{d\mathbf{u}}{dt} &= -\nabla(p_i + p_e), \\
 \rho \left(\frac{\partial \varepsilon_i}{\partial T_i} \frac{dT_i}{dt} + \frac{\partial \varepsilon_i}{\partial \rho} \frac{d\rho}{dt} \right) &= -p_i \nabla \cdot \mathbf{u} - G(T_i - T_e), \\
 \rho \left(\frac{\partial \varepsilon_e}{\partial T_e} \frac{dT_e}{dt} + \frac{\partial \varepsilon_e}{\partial \rho} \frac{d\rho}{dt} \right) &= -p_e \nabla \cdot \mathbf{u} - \nabla \cdot (\mathbf{q}_e + \mathbf{q}_R) + G(T_i - T_e) + Q_{IB}(\mathbf{E}_L),
 \end{aligned}$$

the quantities $\frac{\partial \varepsilon}{\partial \rho} = \frac{\partial f}{\partial \rho} - T \frac{\partial^2 f}{\partial \rho \partial T}$, $\frac{\partial \varepsilon}{\partial T} = -T \frac{\partial^2 f}{\partial T^2}$, $p = \rho^2 \frac{\partial f}{\partial \rho}$, $G = \rho \frac{\partial \varepsilon_e}{\partial T_e} \nu_{ei}$ provides our HerEOS equation of state.¹⁰

Nonlocal transport of photon intensity

$$I^p = \int_{\nu} f^p \frac{h^4 \nu^3}{c^2} d\nu$$

$$\mathbf{n} \cdot \nabla I^p = \frac{a T_e^4 - I^p}{\lambda^p}$$

Radiation closure relations

$$\mathbf{q}_R = \int_{4\pi} \mathbf{n} I^p d\mathbf{n} \xrightarrow{\text{diffusive}} \nabla \cdot \mathbf{q}_R = -\nabla \cdot \left(\frac{c}{3\chi_R} \nabla E \right)$$

Nonlocal transport of electron intensity $I^e = \int_{\nu} f^e \frac{m_e |\mathbf{v}|^5}{2} d|\mathbf{v}|$

$$\mathbf{n} \cdot \nabla f^e + (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f^e = \frac{k_B n_e \frac{\sqrt{2}}{3} v_{th} T_e - f^e}{\frac{\pi^2}{\lambda^e}}$$

Electron closure relations

$$\mathbf{q}_e = \int_{4\pi} \mathbf{n} f^e d\mathbf{n} \xrightarrow{\text{diffusive}} \nabla \cdot \mathbf{q}_e = -\nabla \cdot \left(\kappa_{SH} T_e^{5/2} \nabla T_e \right)$$

$$\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \frac{1}{en_e c} \mathbf{j} \times \mathbf{B} - \nabla \times \frac{c}{en_e \sigma 4\pi} \nabla \times \mathbf{B} - \nabla \times \left(\frac{\sum_{\Delta \nu g} \int_{4\pi} \mathbf{D} \cdot |\mathbf{v}| \mathbf{f}}{\sum_{\Delta \nu g} \int_{4\pi} \mathbf{V}_E \cdot \mathbf{f}} + \frac{\tilde{\mathbf{j}}}{en_e \sigma} \right)$$

¹⁰M. Zeman, M. Holec, P. Vachal, "HerEOS: a Framework for Consistent Treatment of the Equation of State in ALE Hydrodynamics", *Comp. Math. Appl.*, submitted (2017).

Conclusions

Plasma Euler and Transport Equations hydro code PETE

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u}),$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} = -\nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + p \mathbf{I}),$$

$$\frac{\partial E}{\partial t} = -\nabla \cdot (E \mathbf{u} + p \mathbf{u} + \mathbf{q}_L + \mathbf{q}_e + \mathbf{q}_R),$$

density ρ , fluid velocity \mathbf{u} , total energy $E(T)$, pressure $p(\rho, T)$, laser energy flux \mathbf{q}_L , electron heat flux \mathbf{q}_e , radiation flux \mathbf{q}_R .

$$\mathbf{n} \cdot \nabla I^p = \frac{\sigma^p T_e - I^p}{\lambda^p},$$

$$\mathbf{q}_R = \int_{4\pi} \mathbf{n} I^p d\mathbf{n},$$

$$\mathbf{n} \cdot \nabla I^e = \frac{\sigma^e T_e - I^e}{\lambda^e},$$

$$\mathbf{q}_e = \int_{4\pi} \mathbf{n} I^e d\mathbf{n},$$

- PETE simulations: Lagrangian frame, 2T single fluid, IB laser deposition, SESAME
- High-order finite element scheme for 6D nonlocal radiation and electron transport
- Inherent coupling of nonlocal transport and energy equations via $I = a(\mathbf{x}, \mathbf{n}) T_e + b(\mathbf{x}, \mathbf{n})$, which leads to a temperature dependence of energy fluxes $\mathbf{q}_e + \mathbf{q}_R = \mathbf{A} T_e + \mathbf{B}$
- What about a Multi-Grid approach for an efficient solution of DG-BGK&T?
- What about an implementation of the spherical harmonics basis in MFEM?

