MATH 128B, Spring 2021, final exam.

All the necessary work to justify an answer and all the necessary steps of a proof must be shown clearly to obtain full credit. Partial credit **may** be given but only for significant progress towards a solution. Cross out all work you do not wish considered. Books and notes are allowed. Electronic computing devices are not allowed during the test.

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1. (10pts.) Define the following matrix function on the space of $n \times n$ complex matrices:

$$\|A\|_{sum} := \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|.$$

Show that $\| \cdot \|_{sum}$ is a matrix norm.

2. (10pts.) Show that the following formula describes a family of orthonormal polynomials for the weight function $w(x) \equiv 1$ on the interval $[-1, 1]$:

$$P_n(x) := \frac{(n + \frac{1}{2})^{1/2}}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n.$$

3. (14pts.) Consider the trigonometric sum $U_n(x) := {\sum'}_{j=-n}^{n} c_j e^{\mathrm{i}jx}$, where

$$
{\sum'}_{j=-n}^{n} a_j := \sum_{j=-n}^{n} a_j - \frac{1}{2}\left(a_n + a_{-n}\right).
$$

Show that the solution to the interpolation problem $U_n(x_k) = f(x_k)$, $k = -n, \ldots, 0, \ldots, n$, for the equally spaced points $x_k := k\pi/n$ satisfies

(a) $c_j = \dfrac{1}{2n} {\sum'}_{k=-n}^{n} f(x_k) e^{-\mathrm{i}jx_k}, \quad j = -n, \ldots, n;$ $\qquad (b)$ $\dfrac{1}{2n} {\sum'}_{k=-n}^{n} |f(x_k)|^2 = {\sum'}_{j=-n}^{n} |c_j|^2.$

4. (10pts.) Suppose that p, q, and r are continuous functions and, moreover, that $q(x) \geq 0$ on $[a, b]$. Let $h := (b-a)/(N+1)$ for some positive integer N, and consider the finite-difference method applied to the linear boundary value problem

$$y'' = p(x)y' + q(x)y + r(x) \quad \text{for } x \in [a, b], \quad y(a) = \alpha, \; y(b) = \beta.$$

Use Gershgorin's theorem to prove that the resulting tridiagonal linear system (11.19) has a unique solution provided $h < 2/L$ where $L := \max_{x \in [a,b]} |p(x)|$.

5. (10pts.) Suppose the $n \times n$ matrix A has eigenvalues λ_j, $j = 1, \dots, n$, such that

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|,$$

and suppose the power method is applied to an initial vector of the form

$$x^{(0)} = \alpha_2 v^{(2)} + \cdots + \alpha_n v^{(n)}$$

where $\alpha_2 \neq 0$ and where $v^{(j)}$ denotes an eigenvector corresponding to eigenvalue λ_j, for each j. Determine, with proof, the limit of the sequence $\{\mu^{(m)}\}$ described in Algorithm 9.1.

6. (6pts.) Let $x \in \mathrm{I\!R}^n$, $x \neq 0$, $n \geq 2$. Let P be a Householder matrix such that $Px = \pm\|x\|_2 e_1$. Let $G_{k,\ell}$ denote a Givens rotation matrix which is equal to the identity except for the rotation submatrix in rows and columns k, ℓ. Suppose $G_{1,2}, \ldots, G_{n-1,n}$ are such Givens rotations that

$$Qx = Px \qquad \text{for } Q := G_{1,2} G_{2,3} \cdots G_{n-1,n}.$$

True or false: $P = Q$? Justify your answer.