Math 128B, Spring 2021.

Homework 1, due January 30.

Prob 1. Verify that the function

$$\|x\|_3 := \left(\sum_{j=1}^{n} |x_j|^3 \right)^{1/3}$$

is a norm on $\mathbb{C}^n$. As we know, it must be equivalent to the 1-norm $\|\cdot\|_1$. Find, with proof, at least one pair of constants $(0 <) c < C$ such that

$$c\|x\|_1 \leq \|x\|_3 \leq C\|x\|_1 \qquad \forall x \in \mathbb{C}^n.$$

Prob 2. Using MATLAB, plot the unit sphere for the norms $\|\cdot\|_1$ and $\|\cdot\|_3$ in the space $\mathbb{R}^3$. First plot them separately then, if you are feeling adventurous, together, in different colors.

Prob 3. For each of the following matrices, find its ∞-norm and its 2-norm, using MATLAB or by hand:

$$(a)\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}, \qquad (b)\begin{bmatrix} -2 & 3 \\ 3 & -2 \end{bmatrix}, \qquad (c)\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

Prob 4. Show that if a matrix A is symmetric, then its 2-norm is equal to its spectral radius.

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Homework 2, due February 6.

For all problems, turn in your code (and MATLAB diaries when needed).

Prob 1. Create a MATLAB function that inputs a function A, vectors b and $x^{(0)}$ and a tolerance tol, and finds an approximate solution to $Ax = b$ using the Jacobi method.

Prob 2. Same as in Prob. 1 for the Gauss-Seidel method.

Prob 3. Pick three linear systems of your choice and analyze the performance of the algorithms from Prob. 1–2. If you observe differences in convergence, try to explain why that happens.

Prob 4. Let A be a square matrix and let $\|\cdot\|$ be any matrix norm (not necessarily natural/induced). Prove that $\lim_{k\to\infty} \|A^k\| = 0$ if $\rho(A) < 1$. Hint: without loss of generality, it is enough to prove this fact for a matrix in its Jordan normal form – and a particular matrix norm. (Why?)

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Homework 3, due February 13.

Prob 1. Find the first two iterations of the SOR method with $\omega = 1.2$ for the following linear system:

$$\begin{aligned} 10x_1 - x_2 &= 9 \\ -x_1 + 10x_2 - 2x_3 &= 7 \\ -2x_2 + 10x_3 &= 6. \end{aligned}$$

Is the corresponding matrix T_ω convergent?

Prob 2. Let $\kappa(A)$ denote the condition number of a (square) matrix A. Show that any singular matrix B satisfies the inequality

$$\frac{1}{\kappa(A)} \leq \frac{\|A-B\|}{\|A\|}.$$

Hint: The null space of B contains a vector of norm 1.

Prob 3. Use Gaussian elimination and three-digit rounding arithmetic to find an approximate solution to

$$\begin{aligned} 0.03x_1 + 58.9x_2 &= 59.2 \\ 5.31x_1 - 6.1x_2 &= 47.0. \end{aligned}$$

Then use one iteration of iterative refinement. Compare both approximations to the exact solution.

Prob 4. The linear system $Ax = b$ with

$$A = \begin{bmatrix} 1 & 2 \\ 1.00001 & 2 \end{bmatrix}, \qquad b = \begin{bmatrix} 3 \\ 3.00001 \end{bmatrix}$$

has (exact) solution $[1, 1]^T$. Use seven-digit rounding arithmetic to solve the perturbed system with

$$\tilde{A} = \begin{bmatrix} 1 & 2 \\ 1.000011 & 2 \end{bmatrix}, \qquad \tilde{b} = \begin{bmatrix} 3.00001 \\ 3.00003 \end{bmatrix}$$

and compare the actual error to the Perturbation Estimate from class (formula (7.25) in our main textbook).

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Homework 4, due February 20.

Prob 1. Create a MATLAB function that inputs a matrix A, vectors b and $x^{(0)}$ and a tolerance tol and finds an approximate solution to $Ax = b$ using the conjugate gradient method (without preconditioning). The algorithm should terminate after n steps and should output an error message in case the desired precision was not reached. Run the algorithm on a linear system of your choice.

Prob 2. Same as in Prob 1, with an additional (input) matrix C used for preconditioning. Run an example for a linear system and several matrices C. What are good and bad choices of C? Discuss.

Prob 3. Use the Gershgorin Circle theorem to determine bounds for the eigenvalues and the spectral radius of the following matrices:

$$(a)\ \begin{bmatrix} 4 & 0 & 1 & 3 \\ 0 & 4 & 2 & 1 \\ 1 & 2 & -2 & 0 \\ 3 & 1 & 0 & 4 \end{bmatrix}, \qquad (b)\ \begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 2 & -1 & 1 \\ 0 & 1 & 3 & -2 \\ 1 & 0 & 1 & 4 \end{bmatrix}$$

Prob 4. Suppose that A and B are nonsingular $n \times n$ matrices. Show that AB is similar to BA.

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Homework 5, due February 27.

Prob 1. Create a MATLAB function that inputs a matrix A, a vector $x^{(0)}$, a tolerance bound tol and uses the power method with the stopping criterion $\|x^{(k)} - x^{(k-1)}\|_\infty < tol$ to obtain an approximate eigenpair (λ, x). Run your algorithm with

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad x^{(0)} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \qquad tol = 0.01.$$

Prob 2. Let

$$A = \begin{bmatrix} 5 & 2 & 0 & 0 \\ 1 & 4 & -1 & 0 \\ 0 & -1 & 4 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}.$$

Use the Power Method, Wielandt deflation, and the Inverse Power method to approximate the eigenvalues and eigenvectors of A.

Prob 3. Use Householder's method to place the following matrix in tridiagonal form:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Prob 4. Modify Householder's algorithm to find a Hessenberg matrix similar to the matrix

$$A = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 1 & 4 & 0 \\ 1 & 1 & 1 & 4 \end{bmatrix}.$$

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Homework 6, due March 6.

Prob 1. Create a MATLAB function that implements the QR algorithm with shifts as discussed on pp. 616-620 of our (main) textbook. Use it to determine, to within 10^{-5}, all eigenvalues of the matrix

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & -1 & -2 \\ 0 & -2 & 3 \end{bmatrix}.$$

Prob 2. Determine the singular values of the following matrices:

$$(a) \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad (b) \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Prob 3. Show that if A is an $n \times n$ nonsingular matrix with singular values s_1, s_2, $\ldots$, s_n, then its ℓ_2-condition number is equal to $\kappa_2(A) = s_1/s_n$.

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Homework 7, due March 27.

Prob 1. Find the least squares polynomial approximation of degrees 2 and 3 to the function $f(x) = e^x$ on the interval $[-1, 1]$.

Prob 2. Show that for any positive integers $i > j$, the following identity holds:

$$T_i(x)T_j(x) = \frac{1}{2}\left[T_{i+j}(x) + T_{i-j}(x)\right].$$

Prob 3. Derive the 3-term recurrence relation for the Laguerre polynomials L_n, which are orthogonal with respect to the weight function $w(x) = e^{-x}$ on the interval $(0, \infty)$. Plot the polynomials L_0, L_1, L_2 and L_3 on an interval containing all their zeros. Do you observe interlacing? Discuss.

Prob 4. Determine the Padé approximation of degree 6 with $n = 2$, $m = 4$ to the function $f(x) = \sin x$.

Prob 5. Express the following rational functions as continued fractions:

$$(a)\ \ \frac{4x^2+3x-7}{2x^3+x^2-x+5}, \qquad\qquad (b)\ \ \frac{2x^3+x^2+3x-1}{3x^3+x^2-x+1}.$$

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Homework 8, due April 10.

Prob 1. Find the continuous least squares trigonometric polynomial S_n (following the book notation) for $f(x) = e^x$ on the interval $[-\pi, \pi]$.

Prob 2. Determine the discrete least squares trigonometric polynomial S_4 for $f(x) = e^x$ on the interval $[-\pi, \pi]$, and compute the error $E(S_4)$ of this approximation.

Prob 3. Determine the trigonometric interpolating polynomial of degree 4 for $f(x) = x(\pi - x)$ on the interval $[-\pi, \pi]$ using (a) direct calculation; (b) FFT.

Prob 4. (a) Show that $c_0, \ldots, c_{2m-1}$ in Algorithm 8.3 are given by

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{2m-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{2m-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{4m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{2m-1} & \omega^{4m-2} & \cdots & \omega^{(2m-1)^2} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{2m-1} \end{bmatrix},$$

where $\omega := e^{\pi \mathrm{i}/m}$. (b) What are the eigenvalues of this matrix?

Math 128B, Spring 2021.

Homework 9, due April 17.

Prob 1. Use fixed-point iteration to find all solutions to the following nonlinear system, accurate to within 10^{-5}, using the ℓ_∞-norm. Justify that you have indeed found all solutions.

$$\begin{aligned} x_1^2 + x_2^2 - x_1 &= 0 \\ x_1^2 - x_2^2 - x_2 &= 0. \end{aligned}$$

Prob 2. Let A be an $n \times n$ matrix and let $F : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $F(x) := Ax$. Show that F is continuously differentiable on $\mathbb{R}^n$. What is its Jacobian at an arbitrary point x?

Prob 3. Use Newton's method with $x^{(0)} = 0$ to compute $x^{(1)}$ and $x^{(2)}$ for the following nonlinear system:

$$\begin{aligned} x_1^2 + x_2 - 37 &= 0 \\ x_1 - x_2^2 - 5 &= 0 \\ x_1 + x_2 + x_3 - 3 &= 0. \end{aligned}$$

Prob 4. The nonlinear system

$$\begin{aligned} 4x_1 - x_2 + x_3 &= x_1x_4 \\ -x_1 + 3x_2 - 2x_3 &= x_2x_4 \\ x_1 - 2x_2 + 3x_3 &= x_3x_4 \\ x_1^2 + x_2^2 + x_3^2 &= 1 \end{aligned}$$

has six solutions. (a) Show that if $(x_1, x_2, x_3, x_4)^T$ is a solution, then so is $(-x_1, -x_2, -x_3, x_4)^T$.
(b) Use Broyden's method three times to approximate each solution. Iterate until $\|x^{(k)} - x^{(k-1)}\|_\infty < 10^{-5}$.

Math 128B, Spring 2021.

Homework 10 / final project.

For each problem, turn in your code and MATLAB diaries if necessary. One of the goals of this assignment is to write elegant MATLAB code. Please do not submit code that merely reproduces Algorithm 11.2, 11.4, 12.1, 12.3, or 12.4. Instead, study the theoretical description of each algorithm and implement it using available matrix functions in MATLAB. Your code for each problem below should be only a few lines long.

Prob 1. Create a MATLAB function for the nonlinear shooting algorithm and run it with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.01$ to approximate the solution to the boundary value problem

$$y'' = -(y')^2 - y + \ln x, \qquad 1 \le x \le 2, \quad y(1) = 0, \quad y(2) = \ln 2.$$

Plot your results against the exact solution $y = \ln x$. Discuss.

Prob 2. Same as in Problem 1 for the finite difference method.

Prob 3. Implement the Poisson equation finite-difference method in MATLAB and use your code to solve any of the subproblems to problem 3 on p. 742.

Prob 4. Implement the Crank-Nicholson method in MATLAB; use your code to solve problem 2 on p. 754.

Prob 5. Implement the Wave equation finite-difference method in MATLAB; use your code to solve problem 2 on p. 763.

MATH 128B, Spring 2021, midterm test.

All the necessary work to justify an answer and all the necessary steps of a proof must be shown clearly to obtain full credit. Partial credit **may** be given but only for significant progress towards a solution. Cross out all work you do not wish considered. Books and notes are allowed. Electronic devices are not allowed during the test.

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1. (10pts.) Determine, with proof, the limit $\lim_{n\to\infty} A^n v$ where

$$\begin{bmatrix} 2/3 & -3 & 12 \\ 0 & 1/2 & 1/2 \\ 0 & 1/3 & 1/2 \end{bmatrix}, \qquad v = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

2. (10pts.) Determine, with proof, the ℓ_2-norm and the 2-condition number of the matrix

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}.$$

3. (10pts.) Using the initial vector $x^{(0)} = 0$, find two iterations of the Jacobi method for the linear system

$$\begin{aligned} -2x_1 + x_2 + \frac{1}{2}x_3 &= 4 \\ x_1 - 2x_2 - \frac{1}{2}x_3 &= 2 \\ x_2 + 2x_3 &= 0. \end{aligned}$$

4. (10pts.) Find all eigenvalues of a real $n \times n$ Householder matrix $I - 2w^t w$ where $\|w\|_2 = 1$.

MATH 128B, Spring 2021, final exam.

All the necessary work to justify an answer and all the necessary steps of a proof must be shown clearly to obtain full credit. Partial credit **may** be given but only for significant progress towards a solution. Cross out all work you do not wish considered. Books and notes are allowed. Electronic computing devices are not allowed during the test.

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1. (10pts.) Define the following matrix function on the space of $n \times n$ complex matrices:

$$\|A\|_{sum} := \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|.$$

Show that $\| \cdot \|_{sum}$ is a matrix norm.

2. (10pts.) Show that the following formula describes a family of orthonormal polynomials for the weight function $w(x) \equiv 1$ on the interval $[-1, 1]$:

$$P_n(x) := \frac{(n + \frac{1}{2})^{1/2}}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n.$$

3. (14pts.) Consider the trigonometric sum $U_n(x) := {\sum'}_{j=-n}^{n} c_j e^{\mathrm{i}jx}$, where

$$ {\sum'}_{j=-n}^{n} a_j := \sum_{j=-n}^{n} a_j - \frac{1}{2}\left(a_n + a_{-n}\right). $$

Show that the solution to the interpolation problem $U_n(x_k) = f(x_k)$, $k = -n, \ldots, 0, \ldots, n$, for the equally spaced points $x_k := k\pi/n$ satisfies

(a) $c_j = \frac{1}{2n} {\sum'}_{k=-n}^{n} f(x_k) e^{-\mathrm{i}jx_k}, \quad j = -n, \ldots, n;$ $\qquad$ (b) $\frac{1}{2n} {\sum'}_{k=-n}^{n} |f(x_k)|^2 = {\sum'}_{j=-n}^{n} |c_j|^2.$

4. (10pts.) Suppose that p, q, and r are continuous functions and, moreover, that $q(x) \geq 0$ on $[a, b]$. Let $h := (b-a)/(N+1)$ for some positive integer N, and consider the finite-difference method applied to the linear boundary value problem

$$y'' = p(x)y' + q(x)y + r(x) \quad \text{for } x \in [a, b], \quad y(a) = \alpha,\ y(b) = \beta.$$

Use Gershgorin's theorem to prove that the resulting tridiagonal linear system (11.19) has a unique solution provided $h < 2/L$ where $L := \max_{x \in [a,b]} |p(x)|$.

5. (10pts.) Suppose the $n \times n$ matrix A has eigenvalues λ_j, $j = 1, \ldots, n$, such that

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|,$$

and suppose the power method is applied to an initial vector of the form

$$x^{(0)} = \alpha_2 v^{(2)} + \cdots + \alpha_n v^{(n)}$$

where $\alpha_2 \neq 0$ and where $v^{(j)}$ denotes an eigenvector corresponding to eigenvalue λ_j, for each j. Determine, with proof, the limit of the sequence $\{\mu^{(m)}\}$ described in Algorithm 9.1.

6. (6pts.) Let $x \in \mathbb{R}^n$, $x \neq 0$, $n \geq 2$. Let P be a Householder matrix such that $Px = \pm\|x\|_2 e_1$. Let $G_{k,\ell}$ denote a Givens rotation matrix which is equal to the identity except for the rotation submatrix in rows and columns k, ℓ. Suppose $G_{1,2}, \ldots, G_{n-1,n}$ are such Givens rotations that

$$Qx = Px \qquad \text{for } Q := G_{1,2} G_{2,3} \cdots G_{n-1,n}.$$

True or false: $P = Q$? Justify your answer.