- Let $I = (i_1, \ldots, i_d)$ be a multi-index
- Let $\pi(x) \propto e^{-\beta H(x)}$,

  where $x = (x_1, \ldots, x_d)$

  $x_i \in \mathbb{R}^m$
- Let $\phi_1, \ldots, \phi_n$ be O.N.B. for $L^2(\mathbb{R}^m)$
- Also let $x^1, \ldots, x^n \in \mathbb{R}^m$ be collocation points for the basis, so there is a matrix A_{ij} such that if $f = \sum a_i \phi_i$, then

$$a_i = \sum_j A_{ij} f(x_j)$$

Will represent

$$\sqrt{\pi(x)} \approx \psi(x) := \sum \underbrace{c_{i_1 \cdots i_d}}_{c_I} \underbrace{\phi_{i_1} \otimes \cdots \otimes \phi_{i_d}}_{\phi_I}$$

① How to obtain c_I?

② Given c_I, how to sample exactly from $\psi(x)^2 \approx \pi(x)$? (and obtain exact likelihoods).

① We can evaluate $\sqrt{\pi(x)}$ on grid points

$$(x^{i_1}, \dots, x^{i_d})$$

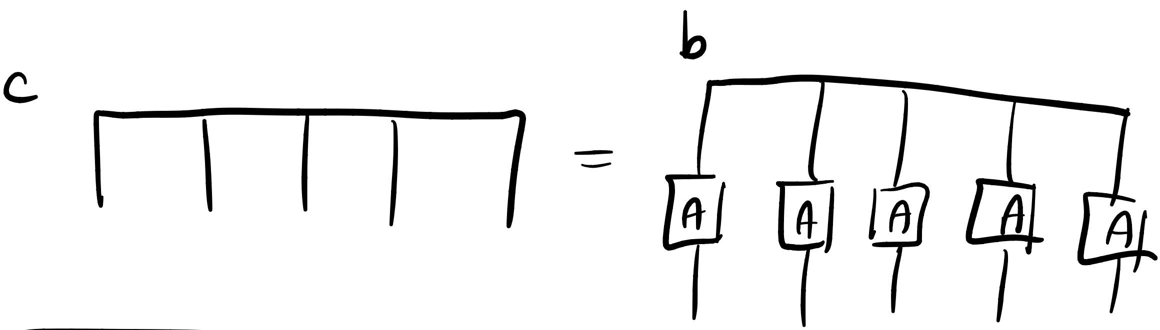
$$\xRightarrow{\text{TT cross}} \; b_{i_1 \cdots i_d} \approx \sqrt{\pi(x^{i_1}, \dots, x^{i_d})}$$

$b_I =$

TT

obtain TT c_I from b_I

using collocation matrix A



② Evaluating $\psi(x)^2$ is easy.

Just evaluate $\psi(x)$ and square

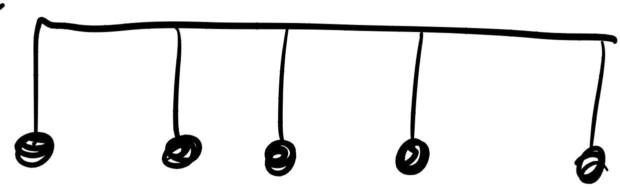
(Here $x = (x_1, \dots, x_d)$ not necessarily a grid point)

To evaluate $\psi(x)$,

observe

$$\psi(x) = \sum_{I} C_I \, \phi_{i_1}(x_1) \cdots \phi_{i_d}(x_d)$$

$= \quad C$



k-th

vector

$(\phi_1(x_k), \ldots, \phi_n(x_k))$

Remains to discuss sampling

First sample $x_1 \sim p_1(x_1)$ (marginal of ψ^2)

$$\psi(x)^2 = \sum_{I,J} c_I c_J \phi_I \phi_J$$

$$P_1 = \sum_{I,J} \int c_I c_J (\phi_{i_1}\phi_{j_1}) \otimes \cdots \otimes (\phi_{i_d}\phi_{j_d}) \, dx_2 \cdots dx_d$$

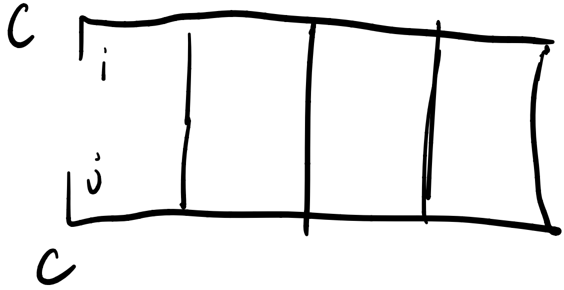
$$P_1 = \sum_{I,J} c_I c_J \phi_{i_1}\phi_{j_1} \langle \phi_{i_2}, \phi_{j_2} \rangle \cdots \langle \phi_{i_d}, \phi_{j_d} \rangle$$

by orthonormality, $\langle \phi_i, \phi_j \rangle = \delta_{ij}$

so

$$P_1 = \sum_{I,J} \underbrace{c_{i_1 i_2 \cdots i_d} c_{j_1 i_2 \cdots i_d}}_{B_{i_1 j_1}} \phi_{i_1} \phi_{j_1}$$

$B_{ij} =$



(form matrix B like so (PSD by construction))

then $$p_1(x) = \sum_{ij} B_{ij}\, \phi_i(x)\, \phi_j(x)$$

(≥ 0 by construction)

can evaluate on ultrafine grid to do essentially exact sampling

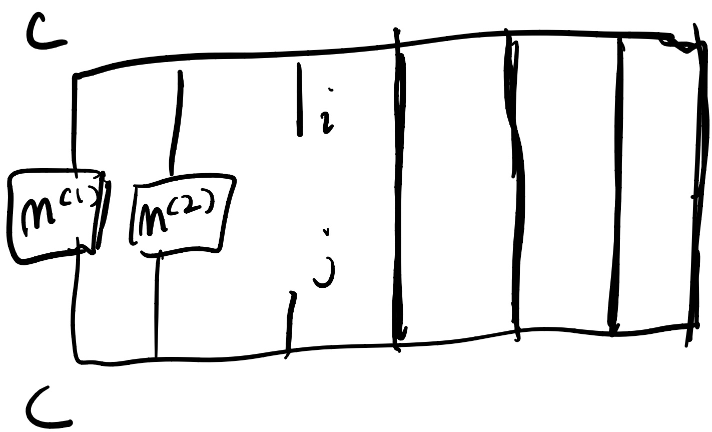
Remains to show how to sample

$$x_k \mid x_1, \ldots, x_{k-1}$$

$$p_k(\cdot \mid x_1, \ldots, x_{k-1}) =$$

$$\sum_{IJ} c_{i_1 \cdots i_k i_{k+1} \cdots i_d} \, c_{j_1 \cdots j_k i_{k+1} \cdots i_d} \, \phi_{i_1}(x_1)\phi_{j_1}(x_1) \cdots \phi_{i_{k-1}}(x_{k-1}) \phi_{j_{k-1}}(x_{k-1}) \, \phi_{i_k} \phi_{j_k}$$

$$= \sum_{i_k j_k} B^{(k)}_{i_k j_k} \phi_{i_k} \phi_{j_k} \quad (\star)$$

$B^{(k)}_{ij} =$ 

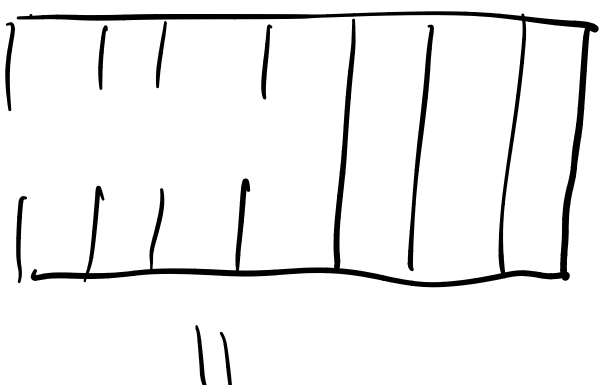
Here, $M^{(\ell)}_{ij} = \phi_i(x_\ell)\,\phi_j(x_\ell)$

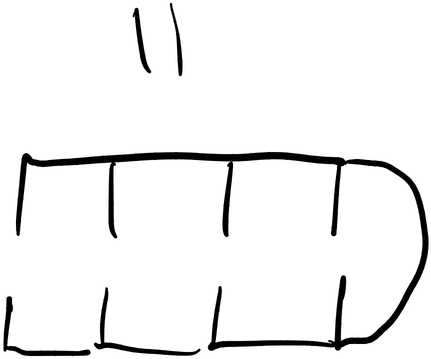
for $\ell = 1, \dots, k-1$

Again form $(\star)$ on ultrafine grid to do essentially exact sampling.

This completes the discussion.

Computationally, if you have put the TT C into "right-left QR" form, then





← same cores

Thus by putting C into this form as a preprocessing step, and "zipping" up with the matrices $M^{(\ell)}$ from the left that we construct by conditional sampling, it is easy to get an exact sample in $O(d)$ time!